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On a sum involving the number of distinct prime factors function related to the integer part function

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Abstract: In this paper, we obtain asymptotic formula on the sum $\sum_{n\leq x} \omega\left(\left\lfloor \frac{x}{n} \right\rfloor\right)$, where $\omega(n)$ denote the number of distinct prime divisors of n and $\lfloor t \rfloor$ denotes the integer part of t.
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1 Introduction

Let, as usual, for an integer $n \geq 1$, $\omega(n) := \sum_{p|n} 1$ denote the the number of distinct prime divisors of n. Many authors investigated the properties of this function. In 1917, G. H. Hardy and S. Ramanujan [4] proved the classical result,

$$\sum_{n\leq x} \omega(n) = x \log\log x + Bx + O\left(\frac{x}{\log x}\right), \tag{1}$$

such that $B = \gamma + \sum_{p} (\log(1 - 1/p) + 1/p)$ and γ is Euler's constant. The result (1) was generalized in 1970 [6] and in 1976 [3] by the following formula

$$\sum_{k\leq n}\omega\left(k\right)=n\log\log n+Bn+\sum_{j=1}^{m}\frac{na_j}{\left(\log n\right)^j}+O\left(\frac{n}{\left(\log n\right)^{m+1}}\right),\tag{2}$$

for all integer $m\geq 1$, with

$$a_j=-\int_1^{\infty}\frac{\{t\}}{t^2}\left(\log t\right)^{j-1}dt.$$

In [5], we find another interesting result

$$\sum_{n\leq x}\omega\left(d(n)\right)=cx+O\left(x^{1/2}\log^5 x\right),\tag{3}$$

such that $d(n)$ is the number of divisors of n and $c>0$ it's a constant. It is easy to show that the following relationship is correct for all real $x\geq 1$

$$\sum_{n\leq x}d(n)=\sum_{n\leq x}\left\lfloor\frac{x}{n}\right\rfloor,$$

where $\lfloor t\rfloor$ denotes the integer part of any $t\in\mathbb{R}$ (see [2, example 4.18]).

The possible question is what are the similarities between the mean values of the functions $\omega\left(d(n)\right)$ and $\omega\left(\left\lfloor\frac{x}{n}\right\rfloor\right)$? Since, the sum is on a less dense set than the first, it is obvious that the result will be at least with an error term lower, than what is given in the formula (3).

2 Main result

In this section, we establish a result concerning the mean value of the function $\omega\left(\left\lfloor\frac{x}{n}\right\rfloor\right)$. More precisely, we prove the following theorem:

Theorem 1. *For all $x\geq 1$ large enough, we have*

$$\sum_{n\leq x}\omega\left(\left\lfloor\frac{x}{n}\right\rfloor\right)=Cx+O\left(x^{1/2}\log x\right).$$

Such that $C\approx 0.5918\cdots$.

The proof of this result is based on the following lemmas:

Lemma 1. *Let $x\geq 1$ be real number. For any arithmetic function f we have*

$$\sum_{n\leq x}f\left(\left\lfloor\frac{x}{n}\right\rfloor\right)=\sum_{n\leq x}f\left(n\right)\left(\left\lfloor\frac{x}{n}\right\rfloor-\left\lfloor\frac{x}{n+1}\right\rfloor\right).$$

Proof. If we pose $\left\lfloor\frac{x}{n}\right\rfloor=k$, then we have the following equivalents:

$$\left\lfloor\frac{x}{n}\right\rfloor=k\Longleftrightarrow x/n-1<k\leq x/n\Longleftrightarrow x/\left(k+1\right)<n\leq x/k.$$

Using that, we get

$$\begin{aligned}
\sum_{n\le x} f\left(\left\lfloor \frac{x}{n}\right\rfloor\right) &= \sum_{\substack{n\le x\\ x/n-1<k\le x/n}} f(k)\\
&= \sum_{\substack{k\le x\\ x/(k+1)<n\le x/k}} f(k)\\
&= \sum_{\substack{k\le x\\ n\le x/k}} f(k) - \sum_{\substack{k\le x\\ n\le x/(k+1)}} f(k)\\
&= \sum_{k\le x}\sum_{n\le x/k} f(k) - \sum_{k\le x}\sum_{n\le x/(k+1)} f(k)\\
&= \sum_{k\le x} f(k) \sum_{n\le x/k} 1 - \sum_{k\le x} f(k) \sum_{n\le x/(k+1)} 1\\
&= \sum_{k\le x} f(k)\left(\left\lfloor \frac{x}{k}\right\rfloor - \left\lfloor \frac{x}{k+1}\right\rfloor\right). \qquad \square
\end{aligned}$$

Lemma 2. *Let $n \in \mathbb{Z}_{\ge 0}$ and $\delta > 0$ real. For all real $x \ge 1$, we have*

$$\int_x^{+\infty} e^{-\delta t} (\log t)^n \, dt \le \frac{n!}{\delta} e^{-\delta x} \left(\log x + \frac{1}{\delta x}\right)^n.$$

Proof. We put $I_n = \int_x^{+\infty} e^{-\delta t} (\log t)^n \, dt$ and we use integration by parts, so

$$I_n = \frac{e^{-\delta x}}{\delta} (\log x)^n + \frac{n}{\delta} \int_x^{+\infty} \frac{(\log t)^{n-1}}{t e^{\delta t}} dt \le \frac{e^{-\delta x}}{\delta} (\log x)^n + \frac{n}{\delta x} I_{n-1}.$$

And by recurrence, we get

$$\begin{aligned}
I_n &\le \frac{e^{-\delta x}}{\delta} \sum_{k=0}^{n} k! \binom{n}{k} \frac{(\log x)^{n-k}}{(\delta x)^k}\\
&\le \frac{n!}{\delta} e^{-\delta x} \left(\log x + \frac{1}{\delta x}\right)^n. \qquad \square
\end{aligned}$$

Lemma 3. *Let x be sufficiently large, there is a constant $C > 0$ such that*

$$\sum_{n\le x} \frac{\omega(n)}{n(n+1)} = C + O\left(\frac{\log\log x}{x}\right), \tag{4}$$

such that $C \approx 0.5918\ldots$.

Proof. Let $x \ge 2$, we have

$$\sum_{n\le x} \frac{\omega(n)}{n(n+1)} = \sum_{n\ge 1} \frac{\omega(n)}{n(n+1)} - \sum_{n>x} \frac{\omega(n)}{n(n+1)}. \tag{5}$$

Now the well-known trivial bound of $\omega(n)$, applied to the first sum on the right-hand side of (5), implies that

$$\sum_{n\geq 1}\frac{\omega(n)}{n(n+1)}\leq\frac{1}{\log 2}\sum_{n\geq 1}\frac{\log n}{n(n+1)}.$$

We deduce that the series $\sum\limits_{n\geq 1}\dfrac{\omega(n)}{n(n+1)}$ is convergent, and with a numerical calculation, we find

$$C=\sum_{n\geq 1}\frac{\omega(n)}{n(n+1)}\approx 0.5918\ldots. \tag{6}$$

In the last sum in (5), we put $g(t)=\dfrac{1}{t(t+1)}$, and by partial summation, we have

$$\begin{aligned}\sum_{n>x}\frac{\omega(n)}{n(n+1)} &= -g(x)\sum_{n\leq x}\omega(n)-\int_x^{+\infty}g'(t)\left(\sum_{x<n\leq t}\omega(n)\right)dt\\ &= \frac{-1}{x(x+1)}\sum_{n\leq x}\omega(n)+\int_x^{+\infty}\frac{2t-1}{t^2(t+1)^2}\left(\sum_{x<n\leq t}\omega(n)\right)dt.\end{aligned}$$

And from (1) we obtain,

$$\left|\sum_{n>x}\frac{\omega(n)}{n(n+1)}\right|\leq\frac{\log\log x}{x}+\frac{B}{x}+O\left(\frac{1}{x\log x}\right)+O\left(\int_x^{+\infty}\frac{\log\log t}{t^2}dt\right). \tag{7}$$

So, by Lemma 2 $(n=1,\ \delta=1)$, and using a variable change, we find

$$\begin{aligned}\int_x^{+\infty}\frac{\log\log t}{t^2}dt &\leq \frac{\log\log x}{x}+\frac{1}{x\log x}\\ &= O\left(\frac{\log\log x}{x}\right).\end{aligned}$$

Finally, using the last estimate in (7), we get

$$\sum_{n>x}\frac{\omega(n)}{n(n+1)}=O\left(\frac{\log\log x}{x}\right), \tag{8}$$

and collecting (8), (6) and (5), we get the following desired result. □

Lemma 4. *For all $x\geq 1$, we have*

$$\sum_{n\geq 0}\left|\left\{\frac{x}{n+1}\right\}-\left\{\frac{x}{n+2}\right\}\right|=\frac{2}{\pi}\zeta(3/2)\,x^{1/2}+O\left(x^{2/5}\right),$$

where $\{t\}$ denotes the fractional part of any $t\in\mathbb{R}$.

Proof. The proof of this result is found in the paper [1]. □

Proof of the theorem. For all $x\geq 1$, by Lemma 1, we have

$$\begin{aligned}\sum_{n\leq x}\omega\left(\left\lfloor\frac{x}{n}\right\rfloor\right) &= \sum_{n\leq x}\omega(n)\left(\left\lfloor\frac{x}{n}\right\rfloor-\left\lfloor\frac{x}{n+1}\right\rfloor\right)\\ &= x\sum_{n\leq x}\frac{\omega(n)}{n(n+1)}+\sum_{n\leq x}\omega(n)\left(\left\{\frac{x}{n+1}\right\}-\left\{\frac{x}{n}\right\}\right),\end{aligned}$$

on the other hand, by trivial bound of $\omega(n)$ and Lemma 4, we have

$$\begin{aligned}\left|\sum_{n\le x}\omega(n)\left(\left\{\frac{x}{n}\right\}-\left\{\frac{x}{n+1}\right\}\right)\right| &\le \sum_{n\le x}\omega(n)\left|\left\{\frac{x}{n}\right\}-\left\{\frac{x}{n+1}\right\}\right| \\ &\le \frac{\log x}{\log 2}\sum_{n\ge 0}\left|\left\{\frac{x}{n+1}\right\}-\left\{\frac{x}{n+2}\right\}\right| \\ &= \frac{\log x}{\log 2}\left(\frac{2}{\pi}\zeta(3/2)\,x^{1/2}+O\left(x^{2/5}\right)\right) \\ &= \frac{2\zeta(3/2)}{\pi\log 2}x^{1/2}\log x+O\left(x^{2/5}\log x\right).\end{aligned}$$

Therefore,

$$\sum_{n\le x}\omega(n)\left(\left\{\frac{x}{n}\right\}-\left\{\frac{x}{n+1}\right\}\right)=O\left(x^{1/2}\log x\right).$$

Finally, by Lemma 3, we obtain

$$\begin{aligned}\sum_{n\le x}\omega\left(\left\lfloor\frac{x}{n}\right\rfloor\right) &= Cx+O(\log\log x)+O\left(x^{1/2}\log x\right) \\ &= Cx+O\left(x^{1/2}\log x\right).\end{aligned}$$ □

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