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Bi-unitary multiperfect numbers, IV(a)

Pentti Haukkanen[1] **and Varanasi Sitaramaiah** [2,*]

[1] Faculty of Information Technology and Communication Sciences,
FI-33014 Tampere University, Finland
e-mail: pentti.haukkanen@tuni.fi

[2] 1/194e, Poola Subbaiah Street, Taluk Office Road, Markapur
Prakasam District, Andhra Pradesh, 523316 India

Dedicated to the memory of Prof. D. Suryanarayana

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Abstract: A divisor d of a positive integer n is called a unitary divisor if $\gcd(d, n/d) = 1$; and d is called a bi-unitary divisor of n if the greatest common unitary divisor of d and n/d is unity. The concept of a bi-unitary divisor is due to D. Surynarayana (1972). Let $\sigma^{**}(n)$ denote the sum of the bi-unitary divisors of n. A positive integer n is called a bi-unitary multiperfect number if $\sigma^{**}(n) = kn$ for some $k \geq 3$. For $k = 3$ we obtain the bi-unitary triperfect numbers.

Peter Hagis (1987) proved that there are no odd bi-unitary multiperfect numbers. The present paper is Part IV(a) in a series of papers on even bi-unitary multiperfect numbers. In parts I, II and III we found all bi-unitary triperfect numbers of the form $n = 2^a u$, where $1 \leq a \leq 6$ and u is odd. There exist exactly ten such numbers. In this part we solve partly the case $a = 7$. We prove that if n is a bi-unitary triperfect number of the form $n = 2^7.5^b.17^c.v$, where $(v, 2.5.17) = 1$, then $b \geq 2$. We then confine ourselves to the case $b = 2$. We prove that in this case we have $c = 1$ and further show that $n = 2^7.3^2.5^2.7.13.17 = 44553600$ is the only bi-unitary triperfect number of this form.

Keywords: Perfect numbers, Triperfect numbers, Multiperfect numbers, Bi-unitary analogues.
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1 Introduction

Throughout this paper, all lower case letters denote positive integers; p and q denote primes. The letters u, v and w are reserved for odd numbers.

A divisor d of n is called a unitary divisor if $\gcd(d, n/d) = 1$. If d is a unitary divisor of n, we write $d \| n$. A divisor d of n is called a *bi-unitary* divisor if $(d, n/d)^{**} = 1$, where the symbol $(a, b)^{**}$ denotes the greatest common unitary divisor of a and b. The concept of a bi-unitary divisor is due to D. Suryanarayana (cf. [6]). Let $\sigma^{**}(n)$ denote the sum of bi-unitary divisors of n. The function $\sigma^{**}(n)$ is multiplicative, that is, $\sigma^{**}(1) = 1$ and $\sigma^{**}(mn) = \sigma^{**}(m)\sigma^{**}(n)$ whenever $(m, n) = 1$. If p^α is a prime power and α is odd, then every divisor of p^α is a bi-unitary divisor; if α is even, each divisor of p^α is a bi-unitary divisor except for $p^{\alpha/2}$. Hence

$$\sigma^{**}(p^\alpha) = \begin{cases} \sigma(p^\alpha) = \dfrac{p^{\alpha+1} - 1}{p - 1} & \text{if } \alpha \text{ is odd,} \\ \sigma(p^\alpha) - p^{\alpha/2} & \text{if } \alpha \text{ is even.} \end{cases} \tag{1.3}$$

If α is even, say $\alpha = 2k$, then $\sigma^{**}(p^\alpha)$ can be simplified to

$$\sigma^{**}(p^\alpha) = \left(\frac{p^k - 1}{p - 1}\right).(p^{k+1} + 1). \tag{1.4}$$

From (1.3), it is not difficult to observe that $\sigma^{**}(n)$ is odd only when $n = 1$ or $n = 2^\alpha$.

The concept of a bi-unitary perfect number was introduced by C. R. Wall [7]; a positive integer n is called a bi-unitary perfect number if $\sigma^{**}(n) = 2n$. C. R. Wall [7] proved that there are only three bi-unitary perfect numbers, namely $6, 60$ and 90. A positive integer n is called a bi-unitary multiperfect number if $\sigma^{**}(n) = kn$ for some $k \geq 3$. For $k = 3$ we obtain the bi-unitary triperfect numbers.

Peter Hagis [1] proved that there are no odd bi-unitary multiperfect numbers. Our present paper is Part IV(a) in a series of papers on even bi-unitary multiperfect numbers. In Part I (see [2]), we found all bi-unitary triperfect numbers of the form $n = 2^a u$, where $1 \leq a \leq 3$ and u is odd. We proved that if $1 \leq a \leq 3$ and $n = 2^a u$ is a bi-unitary triperfect number, then $a = 3$ and $n = 120 = 2^3.3.5$. In Part II (see [3]), we considered the cases $a = 4$ and $a = 5$. We proved that if $n = 2^4 u$ is a bi-unitary triperfect number, then $n = 2160 = 2^4.3^3.5$, and that if $n = 2^5 u$ is a bi-unitary triperfect number, then $n = 672 = 2^5.3.7$, $n = 10080 = 2^5.3^2.5.7$, $n = 528800 = 2^5.3.5^2.13$ or $n = 22932000 = 2^5.3^2.5^3.7^2.13$. In Part III (see [4]) we showed that the bi-unitary triperfect numbers of the form $n = 2^6 u$ are $n = 22848 = 2^6.3.7.17$, $n = 342720 = 2^6.3^2.5.7.17$, $n = 51979200 = 2^6.3.5^2.7^2.13.17$ and $n = 779688000 = 2^6.3^2.5^3.7^2.13.17$. In the present part we consider the case $a = 7$; we solve it partly. We prove that if n is a bi-unitary triperfect number of the form $n = 2^7.5^b.17^c.v$, where $(v, 2.5.17) = 1$, then $b \geq 2$. We then confine ourselves to the case $b = 2$. We prove that in this case c has to equal 1 and further show that $n = 2^7.3^2.5^2.7.13.17 = 44553600$ is the only bi-unitary triperfect number of the form considered here. We will continue the study of the case $a = 7$ in future papers.

For a general account on various perfect-type numbers, we refer to [5].

2 Preliminaries

We assume that the reader has Part I (see [2]) available. We, however, recall Lemmas 2.1 to 2.4 from Part I, because they are so important also here.

Lemma 2.1. (I) *If α is odd, then*

$$\frac{\sigma^{**}(p^{\alpha})}{p^{\alpha}} > \frac{\sigma^{**}(p^{\alpha+1})}{p^{\alpha+1}}$$

for any prime p.
(II) *For any $\alpha \geq 2\ell - 1$ and any prime p,*

$$\frac{\sigma^{**}(p^{\alpha})}{p^{\alpha}} \geq \left(\frac{1}{p-1}\right)\left(p - \frac{1}{p^{2\ell}}\right) - \frac{1}{p^{\ell}} = \frac{1}{p^{2\ell}}\left(\frac{p^{2\ell+1}-1}{p-1} - p^{\ell}\right).$$

(III) *If p is any prime and α is a positive integer, then*

$$\frac{\sigma^{**}(p^{\alpha})}{p^{\alpha}} < \frac{p}{p-1}.$$

Remark 2.1. (I) and (III) of Lemma 2.1 are mentioned in C. R. Wall [7]; (II) of Lemma 2.1 has been used by him [7] without explicitly stating it.

Lemma 2.2. *Let $a > 1$ be an integer not divisible by an odd prime p and let α be a positive integer. Let r denote the least positive integer such that $a^r \equiv 1\,(mod\ p^{\alpha})$; then r is usually denoted by $ord_{p^{\alpha}}\,a$. We have the following properties.*
(i) *If r is even, then $s = r/2$ is the least positive integer such that $a^s \equiv -1\,(mod\ p^{\alpha})$. Also, $a^t \equiv -1\,(mod\ p^{\alpha})$ for a positive integer t if and only if $t = su$, where u is odd.*
(ii) *If r is odd, then $p^{\alpha} \nmid a^t + 1$ for any positive integer t.*

Remark 2.2. Let a, p, r and $s = r/2$ be as in Lemma 2.2 ($\alpha = 1$). Then $p|a^t - 1$ if and only if $r|t$. If t is odd and r is even, then $r \nmid t$. Hence $p \nmid a^t - 1$. Also, $p|a^t + 1$ if and only if $t = su$, where u is odd. In particular if t is even and s is odd, then $p \nmid a^t + 1$. In order to check the divisibility of $a^t - 1$ (when t is odd) by an odd prime p, we can confine to those p for which $ord_p\,a$ is odd. Similarly, for examining the divisibility of $a^t + 1$ by p when t is even we need to consider primes p with $s = ord_p\,a/2$ even.

Lemma 2.3. *Let k be odd and $k \geq 3$. Let $p \neq 5$.*
(a) *If $p \in [3, 2520] - \{11, 19, 31, 71, 181, 829, 1741\}$, $ord_p 5$ is odd and $p|5^k - 1$, then we can find a prime p' (depending on p) such that $p'|\frac{5^k-1}{4}$ and $p' \geq 2521$.*
(b) *If $q \in [3, 2520] - \{13, 313, 601\}$, $s = \frac{1}{2} ord_q 5$ is even and $q|5^{k+1} + 1$, then we can find a prime q' (depending on q) such that $q'|\frac{5^{k+1}+1}{2}$ and $q' \geq 2521$.*

Lemma 2.4. *Let k be odd and $k \geq 3$. Let $p \neq 7$.*
(a) *If $p \in [3, 2520] - \{3, 19, 37, 1063\}$, $r = ord_p 7$ is odd and $p|7^k - 1$, then we can find a prime p' (depending on p) such that $p'|\ \frac{7^k-1}{6}$ and $p' \geq 2521$.*
(b) *If $q \in [3, 1193] - \{5, 13, 181, 193, 409\}$, $s = \frac{1}{2} ord_q 7$ is even and $q|7^{k+1} + 1$, then we can find a prime q' (depending on q) such that $q'|\frac{7^{k+1}+1}{2}$ and $q' > 1193$.*

Lemma 2.5. *Let k be odd and $k \geq 3$. Let $p \neq 13$.*
(a) *If $p \in [3, 293] - \{3, 61\}$, $r = ord_p 13$ is odd and $p | 13^k - 1$, then we can find a prime p' (depending on p) such that $p' | \frac{13^k - 1}{12}$ and $p' \geq 293$.*
(b) *If $q \in [3, 293] - \{5, 17\}$, $s = \frac{1}{2} ord_q 13$ is even and $q | 13^{k+1} + 1$, then we can find a prime q' (depending on q) such that $q' | \frac{13^{k+1} + 1}{2}$ and $q' > 293$.*

Proof. (a) Let $p | 13^k - 1$. If $r = ord_p 13$, that is, r is the least positive integer such that $13^r \equiv 1 \pmod{p}$, then $r | k$. Since k is odd, r must be odd. Also, $13^r - 1 | 13^k - 1$. Let

$$S_{13} = \{(p, r) \ : p \neq 13, p \in [3, 293] \text{ and } r = ord_p 13 \ \text{ is odd}\}.$$

From Appendix A, we have

$$\begin{aligned} S_{13} = \{&(3,1), (23,11), (43,21), (53,13), (61,3), (79,39), (103,17), \\ &(107,53), (127,63), (131,65), (139,69), (179,89), (181,45), \\ &(191,95), (199,99), (211,35), (251,125), (263,131), (283,141)\}. \end{aligned}$$

Let $p | 13^k - 1$ and $p \in [3, 293] - \{3, 61\}$. Then $(p, r) \in S_{13} - \{(3, 1), (61, 3)\}$, where $r = ord_p 13$. Also, $13^r - 1 | 13^k - 1$. To prove (a), it is enough to show that $\frac{13^r - 1}{12}$ is divisible by a prime $p' \geq 293$. From Appendix C, we know the factors of $13^r - 1$. By examining the factors of $13^r - 1$ for $r \notin \{1, 3\}$, which correspond to the primes 3 and 61, we infer that we can find a prime $p' | \frac{13^r - 1}{12} | \frac{13^k - 1}{12}$ satisfying $p' \geq 293$. This proves (a).

For example, if $p = 43$, then $r = 21$. Also,

$$13^{21} - 1 = \{\{2, 2\}, \{3, 2\}, \{43, 1\}, \{61, 1\}, \{337, 1\}, \{547, 1\}, \{2714377, 1\}, \{5229043, 1\}\}.$$

We can take $p' = 337$.

(b) Let $q | 13^{k+1} + 1$ and $q \in [3, 293] - \{5, 17\}$. Let $r = ord_q 13$. If r is odd, then $q \nmid 13^{k+1} + 1$ (see Remark 2.2 ($a = 13$)). We may assume that r is even. Let $s = r/2$. Then s is the least positive integer such that $q | 13^s + 1$. Again from Remark 2.2 ($a = 13$), $q \nmid 13^{k+1} + 1$ if s is odd. Since $q | 13^{k+1} + 1$, we have that s is even. Also, $k + 1 = su$, where u is odd. This implies that $13^s + 1 | 13^{k+1} + 1$. Let

$$T_{13} = \{(q, s) \ : q \neq 13, q \in [3, 293] \text{ and } s = \frac{1}{2} ord_q 13 \text{ even}\}.$$

From Appendix A, we have

$$\begin{aligned} T_{13} = \{&(5,2), (17,2), (37,18), (41,20), (73,36), (89,44), \\ &(97,48), (109,54), (113,28), (137,68), (149,74), (193,32), \\ &(197,98), (229,38), (233,58), (241,120), (257,64), (281,140), (293,146)\}. \end{aligned}$$

Let $q | 13^{k+1} + 1$ and $q \in [3, 293] - \{5, 17\}$. Then $(q, s) \in T_{13} - \{(5, 2), (17, 2)\}$, where $s = \frac{1}{2} ord_q 13$. To prove (b), it is enough to show that $\frac{13^s + 1}{2}$ is divisible by a prime $q' > 293$

for all $s \in T'_{13} = \{s : (q, s) \in T_{13} - \{(5, 2), (17, 2)\}$. This follows by examining the factors of $13^s + 1$ given in Appendix D.

For example, if $q = 37$, then $s = 18$. Also,

$$13^{18} + 1 = \{\{2, 1\}, \{5, 1\}, \{17, 1\}, \{37, 1\}, \{28393, 1\}, \{428041, 1\}, \{1471069, 1\}\}.$$

We can take $q' = 28393$. □

Lemma 2.6. *Let k be odd and $k \geq 3$. Let $p \neq 17$.*
(a) *If $p \in [3, 519] - \{307\}$, $r = ord_p 17$ is odd and $p|17^k - 1$, then we can find a prime p' (depending on p) such that $p'|\frac{17^k-1}{16}$ and $p' > 519$.*
(b) *If $q \in [3, 519] - \{5, 29\}$, $s = \frac{1}{2} ord_q 17$ is even and $q|17^{k+1} + 1$, then we can find a prime q' (depending on q) such that $q'|\frac{17^{k+1}+1}{2}$ and $q' > 519$.*

Proof. (a) Let $p|17^k - 1$. If $r = ord_p 17$, that is, r is the least positive integer such that $17^r \equiv 1 \pmod p$, then $r|k$. Since k is odd, r must be odd. Also, $17^r - 1|17^k - 1$. Let

$$S_{17} = \{(p, r) : p \neq 17, p \in [3, 519] \text{ and } r = ord_p 17 \text{ is odd}\}.$$

From Appendix B, we have

$$\begin{aligned} S_{17} = \{&(19, 9), (43, 21), (47, 23), (59, 29), (67, 33), (83, 41), (103, 51),\\ &(127, 63), (149, 37), (151, 75), (157, 39), (179, 89), (191, 95), (223, 37),\\ &(229, 19), (239, 119), (263, 131), (271, 135), (293, 73), (307, 3), (331, 165),\\ &(359, 179), (383, 191), (389, 97), (409, 51), (433, 27), (443, 221),\\ &(463, 231), (467, 233), (491, 49), (509, 127)\}. \end{aligned}$$

Let $p|17^k - 1$ and $p \in [3, 519] - \{307\}$. Then $(p, r) \in S_{17} - \{(307, 3)\}$, where $r = ord_p 17$. Also, $17^r - 1|17^k - 1$. To prove (a), it is enough to show that $\frac{17^r-1}{16}$ is divisible by a prime $p' \geq 519$. From Appendix E, we know the factors of $17^r - 1$. By examining the factors of $17^r - 1$ for $r \notin \{3\}$, which corresponds to the prime 307, we infer that we can find a prime $p'|\frac{17^r-1}{16}|\frac{17^k-1}{16}$ satisfying $p' > 519$. This proves (a).

For example, if $p = 19$, then $r = 9$. Also,

$$17^9 - 1 = \{\{2, 4\}, \{19, 1\}, \{307, 1\}, \{1270657, 1\}\}.$$

We can take $p' = 1270657$.

(b) Let $q|17^{k+1} + 1$ and $q \in [3, 519] - \{5, 29\}$. Let $r = ord_q 17$. If r is odd, then $q \nmid 17^{k+1} + 1$ (see Remark 2.2 ($a = 17$)). We may assume that r is even. Let $s = r/2$. Then s is the least positive integer such that $q|17^s + 1$. Again from Remark 2.2 ($a = 17$), $q \nmid 17^{k+1} + 1$ if s is odd. Since $q|17^{k+1} + 1$, we have that s is even. Also, $k + 1 = su$, where u is odd. This implies that $17^s + 1|17^{k+1} + 1$. Let

$$T_{17} = \{(q, s) : q \neq 17, q \in [3, 519] \text{ and } s = \frac{1}{2} ord_q 17 \text{ even}\}.$$

From Appendix B, we have

$$T_{17} = \{(5,2),(29,2),(37,18),(41,20),(61,30),(73,12),(89,22),(97,48),(109,18),$$
$$(113,56),(137,34),(173,86),(181,18),(193,96),(197,98),(233,116),(241,40),$$
$$(257,16),(269,134),(277,138),(281,70),(313,156),(317,158),(337,56),$$
$$(353,44),(397,66),(401,200),(449,224)\}.$$

Let $q|17^{k+1}+1$ and $q \in [3,519]-\{5,29\}$. Then $(q,s) \in T_{17}-\{(5,2),(29,2)\}$, where $s = \frac{1}{2}\, ord_q 17$. To prove (b), it is enough to show that $\frac{17^s+1}{2}$ is divisible by a prime $q' > 519$ for all $s \in T'_{17} = \{s : (q,s) \in T_{17} - \{(5,2),(29,2)\}$. This follows by examining the factors of 17^s+1 given in Appendix F.

For example if $q = 37$, then $s = 18$. Also,

$$17^{18}+1 = \{\{2,1\},\{5,1\},\{29,1\},\{37,1\},\{109,1\},\{181,1\},$$
$$\{2089,1\},\{83233,1\},\{382069,1\}\}.$$

We can take $q' = 2089$. □

3 Partial results on bi-unitary triperfect numbers of the form $n = 2^7u$

Let n be a bi-unitary triperfect number divisible unitarily by 2^7 so that $\sigma^{**}(n) = 3n$ and $n = 2^7.u$, where u is odd. Since $\sigma^{**}(2^7) = 2^8 - 1 = 255 = 3.5.17$, using $n = 2^7u$ in $\sigma^{**}(n) = 3n$, we get the following equations:

$$n = 2^7.5^b.17^c.v, \tag{3.1a}$$

and

$$2^7.5^{b-1}.17^{c-1}.v = \sigma^{**}(5^b).\sigma^{**}(17^c).\sigma^{**}(v), \tag{3.1b}$$

where $(v, 2.5.17) = 1$. Considering the parity of the function values of σ^{**} and applying multiplicativity of σ^{**} we conclude that v has not more than five odd prime factors. Also note that $b, c \geq 1$.

In this paper we show that $b \geq 2$ in $(3.1a)$ and consider completely the case $b = 2$. We will examine the case $b \geq 3$ in future papers.

Theorem 3.1. *(a) If n is as in $(3.1a)$ and n is a bi-unitary triperfect number, then $b \geq 2$.*
(b) If $b = 2$, then $c = 1$ and $n = 44553600 = 2^7.3^2.5^2.7.13.17$.

Proof. (a) We assume that $b = 1$ and obtain a contradiction. Since $\sigma^{**}(5) = 6$, taking $b = 1$ in $(3.1b)$, after simplification we get

$$2^6.17^{c-1}.v = 3.\sigma^{**}(17^c).\sigma^{**}(v). \tag{3.2}$$

From (3.2), $3|v$. Let $v = 3^d.w$, where $(w, 2.3.5.17) = 1$.

From $(3.1a)$ we have

$$n = 2^7.5.17^c.3^d.w, \tag{3.2a}$$

and from (3.2),

$$2^6.17^{c-1}.3^{d-1}.w = \sigma^{**}(17^c).\sigma^{**}(3^d).\sigma^{**}(w), \tag{3.2b}$$

$$w \text{ has not more than four odd prime factors and } (w, 2.3.5.17) = 1. \tag{3.2c}$$

If $d = 1$, from $(3.2a)$, we have, by (1.3),

$$3 = \frac{\sigma^{**}(n)}{n} \geq \frac{255}{128}.\frac{6}{5}.\frac{4}{3} = 3.1875 > 3,$$

a contradiction.

Taking $d = 2$ in $(3.2b)$, since $\sigma^{**}(3^2) = 10$, we see that $5|w$. But this is false. Hence $d \neq 2$.

Thus we may assume that $d \geq 3$. By Lemma 2.1, $\frac{\sigma^{**}(3^d)}{3^d} \geq \frac{112}{81}$. Hence from $(3.2a)$,

$$3 = \frac{\sigma^{**}(n)}{n} \geq \frac{255}{128}.\frac{6}{5}.\frac{112}{81} = 3.3 > 3,$$

a contradiction.

Hence $b = 1$ is not admissible. Hence $b \geq 2$.

The proof of (a) is complete.

(b) Since $\sigma^{**}(5^2) = 26 = 2.13$, taking $b = 2$ in $(3.1b)$, we find that $13|v$. Let $v = 13^d.w$, where $(w, 2.5.13.17) = 1$. It now follows from $(3.1a)$ and $(3.1b)$ that

$$n = 2^7.5^2.17^c.13^d.w, \tag{3.3a}$$

and

$$2^6.5.17^{c-1}.13^{d-1}.w = \sigma^{**}(17^c).\sigma^{**}(13^d).\sigma^{**}(w), \tag{3.3b}$$

where

$$w \text{ has not more than four odd prime factors and } (w, 2.5.13.17) = 1. \tag{3.3c}$$

The rest of the proof of (b) of Theorem 3.1 depends on the following lemmas:

Lemma 3.1. *Assume that n given in $(3.3a)$ is a bi-unitary triperfect number.*
(i) If $c = 1$, then $3^2 \| n$.
(ii) If $c = d = 1$, then $n = 44553600 = 2^7.3^2.5^2.7.13.17$.

Proof. (i) Since $\sigma^{**}(17) = 18 = 2.3^2$, taking $c = 1$ in $(3.3b)$, we obtain

$$2^5.5.13^{d-1}.w = 3^2.\sigma^{**}(13^d).\sigma^{**}(w). \tag{3.3d}$$

Hence $3^2|w$ so that $w = 3^e.w'$, where $e \geq 2$ and $(w', 2.3.5.13.17) = 1$. From $(3.3a)$ and $(3.3d)$, we have

$$n = 2^7.5^2.17.13^d.3^e.w', \quad (e \geq 2) \tag{3.4a}$$

and

$$2^5.5.13^{d-1}.3^{e-2}.w' = \sigma^{**}(13^d).\sigma^{**}(3^e).\sigma^{**}(w'), \tag{3.4b}$$

where

$$w' \text{ has not more than three odd prime factors and } (w', 2.3.5.13.17) = 1. \tag{3.4c}$$

When $e \geq 3$, by Lemma 2.1, $\frac{\sigma^{**}(3^e)}{3^e} \geq \frac{112}{81}$. Using this from $(3.4a)$, we obtain

$$3 = \frac{\sigma^{**}(n)}{n} \geq \frac{255}{128}.\frac{26}{25}.\frac{18}{17}.\frac{112}{81} = 3.033 > 3,$$

a contradiction.

Hence $e \geq 3$ is not possible. Since $e \geq 2$, we must have $e = 2$. Thus $3^2 \| n$. This proves (i).

Note 3.1. Taking $e = 2$, in $(3.4a)$ and $(3.4b)$, we obtain

$$n = 2^7.5^2.17.13^d.3^2.w', \tag{3.5a}$$

and

$$2^4.13^{d-1}.w' = \sigma^{**}(13^d).\sigma^{**}(w'), \tag{3.5b}$$

where

$$w' \text{ has not more than three odd prime factors and } (w'', 2.3.5.13.17) = 1. \tag{3.5c}$$

(ii) From (i), $c = 1$ implies $e = 2$. Taking $d = 1$ in $(3.5b)$, since $\sigma^{**}(13) = 14 = 2.7$, we find that $7|w'$ so that $w' = 7^f.w''$. Using these results in $(3.5b)$ and $(3.4a)$ $(d = 1)$, we obtain

$$n = 2^7.5^2.17.13.3^2.7^f.w'', \tag{3.6a}$$

and

$$2^3.7^{f-1}.w'' = \sigma^{**}(7^f).\sigma^{**}(w''), \tag{3.6b}$$

where

$$w'' \text{ has not more than two odd prime factors and } (w'', 2.3.5.7.13.17) = 1. \tag{3.6c}$$

Let $f = 1$. From $(3.6b)$, we have $w'' = 1$. Hence form $(3.5a)$, $n = 7.5^2.17.13.3^2.7 = 44553600$.

If $f = 2$, then since $\sigma^{**}(7^2) = 50 = 2.5^2$, from $(3.6b)$ $(f = 2)$, it follows that $5|w''$. But w'' is prime to 5.

We may assume that $f \geq 3$. From Lemma 2.1, $\frac{\sigma^{**}(7^f)}{7^f} \geq \frac{2752}{2401}$. Hence from $(3.6a)$, we obtain

$$3 = \frac{\sigma^{**}(n)}{n} \geq \frac{255}{128}.\frac{26}{25}.\frac{18}{17}.\frac{14}{13}.\frac{10}{9}.\frac{2752}{2401} = 3.008746356 > 3,$$

a contradiction. This proves (ii) and the proof of Lemma 3.1 is complete. □

Note 3.2. If $c = 1$ and $d = 2$, since $\sigma^{**}(13^2) = 170$, it follows from $(3.5b)$ $(d = 2)$ that 17 divides its left-hand side. But this is not possible. Hence we may assume that $d \geq 3$ (the case $c = d = 1$ is settled in (ii) of Lemma 3.1).

Lemma 3.2. *Let n be as given in $(3.5a)$ with $d \geq 3$. If n is a bi-unitary triperfect number then $7 \nmid n$.*

Proof. By our assumption, $(3.5b)$ and $(3.5c)$ are valid. Suppose that $7|n$. We arrive at a contradiction as follows.

From $(3.5a)$, $7|w'$. Let $w' = 7^f.w''$; using this in $(3.5a)$ and $(3.5b)$, we get

$$n = 2^7.5^2.17.13^d.3^2.7^f.w'' \quad (d \geq 3), \tag{3.7a}$$

and

$$2^4.13^{d-1}.7^f.w'' = \sigma^{**}(13^d).\sigma^{**}(7^f).\sigma^{**}(w''), \tag{3.7b}$$

where

$$w'' \text{ has not more than three odd prime factors and } (w'', 2.3.5.7.13.17) = 1. \tag{3.7c}$$

Since $d \geq 3$, by Lemma 2.1, $\frac{\sigma^{**}(13^d)}{13^d} \geq \frac{30772}{28561}$; also, for $f \geq 3$, $\frac{\sigma^{**}(7^f)}{7^f} \geq \frac{2752}{2401}$. From $(3.7a)$, for $f \geq 3$, we have

$$3 = \frac{\sigma^{**}(n)}{n} \geq \frac{255}{128}.\frac{26}{25}.\frac{18}{17}.\frac{30772}{28561}.\frac{10}{9}.\frac{2752}{2401} = 3.0110115835 > 3,$$

a contradiction.

Hence $f = 1$ or $f = 2$.

Let $f = 1$. From $(3.7a)$ $(f = 1)$, we have

$$3 = \frac{\sigma^{**}(n)}{n} \geq \frac{255}{128}.\frac{26}{25}.\frac{18}{17}.\frac{30772}{28561}.\frac{10}{9}.\frac{8}{7} = 3.00136598 > 3,$$

a contradiction.

Let $f = 2$. Since $\sigma^{**}(7^2) = 50$, taking $f = 2$ in $(3.7b)$, we find that $5|w''$, which is false. Thus $7 \nmid n$.

The proof of Lemma 3.2 is complete. □

Lemma 3.3. *Let n be as given in $(3.5a)$, and let n be a bi-unitary triperfect number.*
(a) Then d can neither be odd nor $4|d$.
(b) Let $d = 2k$ where k is odd and $k \geq 3$. We have

$$\sigma^{**}(13^d) = \left(\frac{13^k - 1}{12}\right).\left(13^{k+1} + 1\right).$$

Here,

(i) $\frac{13^k - 1}{12}$ *is divisible by a prime $p'|w'$ and $p' > 61$,*

(ii) $\frac{13^{k+1} + 1}{2}$ *is divisible by a prime $q'|w'$ and $q' > 61$.*

Proof. We assume that n is a bi-unitary triperfect number. Thus $(3.5b)$ and $(3.5c)$ are valid.
(a) If d is odd or $4|d$, then $7|\sigma^{**}(13^d)$. It follows from $(3.5b)$ that $7|w'|n$. By Lemma 3.2, $7 \nmid n$. This proves (a).

(b) Let $d = 2k$, where k is odd. Since $d \geq 3$, we have $k \geq 3$.

(i) Let

$$S'_{13} = \{p|13^k - 1 \ : \ p \in [3, 61] - \{3, 61\} \text{ and } ord_p 13 \text{ is odd}\}.$$

Let us replace the interval $[3, 293]$ by $[3, 61]$ in Lemma 2.5(a). Then it follows quickly that (i) is true when S'_{13} is non-empty.

We may assume that S'_{13} is empty. Since $p \nmid 13^k - 1$ if $ord_p 13$ is even, it follows that $13^k - 1$ is not divisible by any prime $p \in [3, 61]$ except for possibly $p = 3, 61$; but from $(3.5b)$, $\dfrac{13^k - 1}{12} |\sigma^{**}(13^d)$ is not divisible by 3. We may note that $9|13^k - 1 \iff k = 3u \iff 61|13^k - 1$. Since $13^k - 1$ is not divisible by 3, it is not divisible by 61 either. It now follows that $\dfrac{13^k - 1}{12}$ is not divisible by any prime in $[3, 61]$. Since $\dfrac{13^k - 1}{12}$ is odd and > 1, we can find an odd prime $p'| \dfrac{13^k - 1}{12}$. Clearly, $p' > 61$ and from $(3.5b)$, $p'|w'$. This proves (i).

(ii) Let

$$T'_{13} = \{q|13^{k+1} + 1 \ : \ q \in [3, 61] - \{5, 17\} \text{ and } s = \frac{1}{2} ord_p 13 \text{ is even }\}.$$

Replacing the interval $[3, 293]$ in Lemma 2.5 (b) by $[3, 61]$, we infer that (ii) holds if T'_{13} is non-empty.

Suppose that T'_{13} is empty. Since $q \nmid 13^{k+1} + 1$ if $s = \frac{1}{2} ord_p 13$ is odd, it follows that $\dfrac{13^{k+1} + 1}{2}$ is not divisible by any prime $q \in [3, 61]$ except for possibly $q = 5$ or $q = 17$.

It may be noted that $5|13^{k+1} + 1 \iff k + 1 = 2u \iff 17|13^{k+1} + 1$. From $(3.5b)$, 5 is not a factor of its left-hand side and so $5 \nmid 13^{k+1} + 1|\sigma^{**}(13^d)$. Hence $17 \nmid \dfrac{13^{k+1} + 1}{2}$. Thus $\dfrac{13^{k+1} + 1}{2}$ is odd, > 1 and not divisible by any prime in $[3, 61]$. Let $q'| \dfrac{13^{k+1} + 1}{2}$. Then $q' > 61$ and $q'|w'$ by $(3.5b)$. This proves (ii).

The proof of Lemma 3.3 is complete. □

Lemma 3.4. *Let n be as given in $(3.5a)$ with $d \geq 3$. Then n cannot a bi-unitary triperfect number.*

Proof. On the contrary, assume that n is a bi-unitary triperfect number.

By Lemma 3.2, $7 \nmid n$. Hence from $(3.5a)$ each prime factor of w' can be assumed to be ≥ 11. By Lemma 3.3, w' is divisible by two distinct odd prime factors $p' > 61$ and $q' > 61$. We may assume without loss of generality that $p' \geq 67$ and $q' \geq 71$. By $(3.5c)$, w' cannot have not more than three odd prime factors. If y denotes a possible third prime factor of w' we may assume that $y \geq 11$ and $w' = p'^f.q'^g.y^h$. From $(3.5a)$, we have $n = 2^7.5^2.17.13^d.3^2.p'^f.q'^g.y^h$. Hence

$$3 = \frac{\sigma^{**}(n)}{n} < \frac{255}{128}.\frac{26}{25}.\frac{18}{17}.\frac{13}{12}.\frac{10}{9}.\frac{67}{66}.\frac{71}{70}.\frac{11}{10} = 2.990822173 < 3,$$

a contradiction. This proves Lemma 3.4. □

Remark 3.1. Thus we have proved that when $b = 2$, the case (i) $c = 1, d = 1$ yields the bi-unitary perfect number $n = 44553600$. The cases (ii) $c = 1, d = 2$ and (iii) $c = 1, d \geq 3$ lead to a contradiction. So when $b = 2$ we may assume that $c \geq 2$.

Remark 3.2. Let $b = 2$ and $c \geq 2$. If $n = 2^7.5^2.17^c.13^d.w$, where $(w, 2.5.13.17) = 1$, $3^e \| n$ and n is a bi-unitary triperfect number, then taking $w = 3^e.w'$ in $(3.3a)$ and $(3.3b)$, we obtain the following:

$$n = 2^7.5^2.17^c.13^d.3^e.w', \quad (c \geq 2), \tag{3.8a}$$

and

$$2^6.5.17^{c-1}.13^{d-1}.3^e.w' = \sigma^{**}(17^c).\sigma^{**}(13^d).\sigma^{**}(3^e).\sigma^{**}(w'), \tag{3.8b}$$

where

$$w' \text{ has not more than three odd prime factors and } (w', 2.3.5.13.17) = 1. \tag{3.8c}$$

Lemma 3.5. *Let* $n = 2^7.5^2.17^c.13^d.3^e.w'$ $(c \geq 2)$ *be as in* $(3.8a)$ *and* $(w', 2.3.5.13.17) = 1$. *Then* n *cannot be a bi-unitary triperfect number if* $c \geq 3$ *and* $e \geq 3$.

Proof. Let $c \geq 3$ and $e \geq 3$. We assume that n is a bi-unitary triperfect number and obtain a contradiction.

By Lemma 2.1, for $c \geq 3$, $\frac{\sigma^{**}(17^c)}{17^c} \geq \frac{88452}{83521}$ and for $e \geq 3$, $\frac{\sigma^{**}(3^e)}{3^e} \geq \frac{112}{81}$. Hence from $(3.8a)$ for $c \geq 3$ and $e \geq 3$,

$$3 = \frac{\sigma^{**}(n)}{n} \geq \frac{255}{128}.\frac{26}{25}.\frac{88452}{83521}.\frac{112}{81} = 3.033950743 > 3,$$

a contradiction. This proves Lemma 3.5. □

Remark 3.3. In order to prove that n given in $(3.8a)$ is not a bi-unitary triperfect number, in view of Lemma 3.5, it remains to examine the cases (I) $c = 2$, $e \geq 3$, (II) $c \geq 3$, $e = 1$ or 2, (III) $c = 2$, $e = 1$ or 2.

In the following Lemmas 3.6 to 3.8, we deal with the three cases mentioned in Remark 3.3.

Lemma 3.6. *The number* n *given in* $(3.8a)$ *with* $c = 2$ *and* $e \geq 3$ *cannot be a bi-unitary triperfect number.*

Proof. Assume that n in $(3.8a)$ with $c = 2$ and $e \geq 3$ is a bi-unitary triperfect number. We can use $(3.8b)$ and $(3.8c)$. Since $\sigma^{**}(17^2) = 290 = 2.5.29$, taking $c = 2$ in $(3.8b)$, we get after simplification,

$$2^5.17.13^{d-1}.3^e.w' = 29.\sigma^{**}(13^d).\sigma^{**}(3^e).\sigma^{**}(w'). \tag{3.8d}$$

From $(3.8d)$, it follows that $29 | w'$. Let $w' = 29^f.w''$. From $(3.8a)$ and $(3.8d)$, we have

$$n = 2^7.5^2.17^2.13^d.3^e.29^f.w'', \tag{3.9a}$$

and

$$2^5.17.13^{d-1}.3^e.29^{f-1}.w'' = \sigma^{**}(13^d).\sigma^{**}(3^e).\sigma^{**}(29^f).\sigma^{**}(w''), \tag{3.9b}$$

where

$$w'' \text{ has not more than two odd prime factors and } (w'', 2.3.5.13.17.29) = 1. \tag{3.9c}$$

By Lemma 2.1, for $d \geq 3$, $\frac{\sigma^{**}(13^d)}{13^d} \geq \frac{30772}{28561}$; also, for $e \geq 3$, $\frac{\sigma^{**}(3^e)}{3^e} \geq \frac{112}{81}$. Hence from $(3.9a)$, for $d \geq 3$, we have

$$3 = \frac{\sigma^{**}(n)}{n} \geq \frac{255}{128}.\frac{26}{25}.\frac{290}{289}.\frac{30772}{28561}.\frac{112}{81} = 3.097269697 > 3,$$

a contradiction.

Hence $d = 1$ or $d = 2$.

When $d = 1$, again from $(3.9a)$ $(d = 1)$, we have

$$3 = \frac{\sigma^{**}(n)}{n} \geq \frac{255}{128}.\frac{26}{25}.\frac{290}{289}.\frac{14}{13}.\frac{112}{81} = 3.095860566 > 3,$$

a contradiction.

Let $d = 2$. Since $\sigma^{**}(13^2) = 170 = 2.5.17$, taking $d = 2$ in $(3.9b)$, we find that 5 is a factor of the left-hand side of $(3.9b)$. This is false. This completes the proof of Lemma 3.6. □

Lemma 3.7. *The number n given in $(3.8a)$ with $c \geq 3$ and $e = 1$ or 2 cannot be a bi-unitary triperfect number.*

Proof. We assume that n is a bi-unitary triperfect number and obtain a contradiction. Since $c \geq 3$, by Lemma 2.1, $\frac{\sigma^{**}(17^c)}{17^c} \geq \frac{88452}{83521}$. Also, for $d \geq 3$, $\frac{\sigma^{**}(13^d)}{13^d} \geq \frac{30772}{28561}$.

Let $e = 1$. Hence from $(3.8a)$ $(e = 1)$, for $d \geq 3$,

$$3 = \frac{\sigma^{**}(n)}{n} \geq \frac{255}{128}.\frac{26}{25}.\frac{88452}{83521}.\frac{30772}{28561}.\frac{4}{3} = 3.152075221 > 3,$$

a contradiction.

Hence $d = 1$ or $d = 2$.

If $d = 1$, from $(3.8a)$ $(d = 1, e = 1)$, we have

$$3 = \frac{\sigma^{**}(n)}{n} \geq \frac{255}{128}.\frac{26}{25}.\frac{14}{13}.\frac{30772}{28561}.\frac{4}{3} = 3.150614156 > 3,$$

a contradiction.

Let $d = 2$ (already $e = 1$). We have from $(3.8a)$

$$n = 2^7.5^2.17^c.13^2.3.w', \quad (c \geq 3) \tag{3.10a}$$

and from $(3.8b)$ $(d = 2, e = 1)$, since $\sigma^{**}(13^2) = 2.5.17$, we get after simplification

$$2^3.17^{c-2}.13.3.w' = \sigma^{**}(17^c).\sigma^{**}(w'), \tag{3.10b}$$

$$w' \text{ has not more than two odd prime factors and } (w', 2.3.5.13.17) = 1. \tag{3.10c}$$

Let c be odd. Then $17^2 - 1 = 288|17^{c+1} - 1$. Hence $9|\frac{17^{c+1}-1}{16} = \sigma^{**}(17^c)$. From $(3.10b)$, it follows that $3|w'$. This is not possible.

Let c be even, say $c = 2k$. We have

$$\sigma^{**}(17^c) = \left(\frac{17^k - 1}{16}\right).(17^{k+1} + 1).$$

If k is even, then $9|\frac{17^k-1}{16}|\sigma^{**}(17^c)$. This leads to a contradiction as before.

We may assume that $c = 2k$ and k is odd. Since $c \geq 3$, we have $k \geq 3$. We prove that:

(I) $\frac{17^k - 1}{16}$ is divisible by an odd prime $p'|w'$ and $p' > 127$,

(II) $\frac{17^{k+1} + 1}{2}$ is divisible by an odd prime $q'|w'$ and $q' > 127$.

- Proof of (I). Let

$$S'_{17} = \{p|17^k - 1 \,: p \in [3, 127] \text{ and } ord_p 17 \text{ is odd}\}.$$

By Lemma 2.6(a), if S'_{17} is non-empty, then (I) holds. Suppose that S'_{17} is empty. Since $p \nmid 17^k - 1$ if $ord_p 17$ is even (and k is odd), it follows that $\frac{17^k - 1}{16}$ is not divisible by any prime in $[3, 127]$. Since $\frac{17^k - 1}{16}$ is odd, > 1, it must be divisible by an odd prime p' and clearly $p' > 127$. Also, from $(3.10b)$, $p'|w'$. This proves (I).

- Proof of (II). Let

$$T'_{17} = \{q|17^{k+1} + 1 \,: q \in [3, 127] - \{5, 29\} \text{ and } s = \frac{1}{2} ord_q 17 \text{ is even }\}.$$

By Lemma 2.6(b), if T'_{17} is non-empty, (II) holds. So we may assume that T'_{17} is empty. Since $s = \frac{1}{2} ord_q 17$ is not even implies that $q \nmid 17^{k+1} + 1$ it follows that $\frac{17^{k+1} + 1}{2}$ is divisible by none of the primes in $[3, 127]$ except for possibly 5 or 29.

We may note that $5|17^{k+1} + 1 \iff k + 1 = 2u \iff 29|17^{k+1} + 1$. Let $5|17^{k+1} + 1$. Since $17^{k+1} + 1|\sigma^{**}(17^c)$, it follows from $(3.10b)$ that 5 is a factor of the left-hand side of it. But this is false. Hence $5 \nmid 17^{k+1} + 1$ and hence $29 \nmid 17^{k+1} + 1$. Thus $\frac{17^{k+1} + 1}{2}$ is not divisible by any prime in $[3, 127]$. Since $\frac{17^{k+1} + 1}{2}$ is > 1 and odd, we can find an odd prime $q'|\frac{17^{k+1} + 1}{2}$. Clearly $q' > 127$ and $q'|w'$ from $(3.10b)$. This proves (II).

Since $\frac{17^k - 1}{16}$ and $\frac{17^{k+1} + 1}{2}$ are relatively prime, we have $p' \neq q'$. We may assume that $p' \geq 131$ and $q' \geq 137$. By $(3.10c)$, $w' = (p')^f.(q')^g$. Hence from $(3.10a)$, $n = 2^7.5^2.17^c.13^2.3.(p')^f.(q')^g$ and so

$$3 = \frac{\sigma^{**}(n)}{n} < \frac{255}{128}.\frac{26}{25}.\frac{17}{16}.\frac{170}{169}.\frac{4}{3}.\frac{131}{130}.\frac{137}{136} = 2.997112495 < 3,$$

a contradiction. This proves that $e = 1$ is not possible.

Let $e = 2$. Since $\sigma^{**}(3^2) = 10 = 2.5$, taking $e = 2$ in $(3.8a)$ and $(3.8b)$, we obtain

$$n = 2^7.5^2.17^c.13^d.3^2.w', \quad (c \geq 3) \tag{3.11a}$$

and

$$2^5.17^{c-1}.13^{d-1}.3^2.w' = \sigma^{**}(17^c).\sigma^{**}(13^d).\sigma^{**}(w'), \tag{3.11b}$$

where

$$w' \text{ has not more than three odd prime factors and } (w', 2.3.5.13.17) = 1. \tag{3.11c}$$

Since $\sigma^{**}(13^2) = 170 = 2.5.17$, taking $d = 2$ in $(3.11b)$, we see that 5 is a factor of its left-hand side. But this is not so. Hence we may assume that $d \neq 2$.

Let $d = 1$. Since $\sigma^{**}(13) = 14 = 2.7$, taking $d = 1$ in $(3.11b)$, we see that $7|w'$. Let $w' = 7^f.w''$. From $(3.11a)$ and $(3.11b)$, we have

$$n = 2^7.5^2.17^c.13^d.3^2.7^f.w'', \quad (c \geq 3) \tag{3.12a}$$

and

$$2^4.17^{c-1}.3^2.7^{f-1}.w'' = \sigma^{**}(17^c).\sigma^{**}(7^f).\sigma^{**}(w''), \tag{3.12b}$$

where

$$w'' \text{ has not more than two odd prime factors and } (w', 2.3.5.7.13.17) = 1. \tag{3.12c}$$

By Lemma 2.1, for $f \geq 3$ $\frac{\sigma^{**}(7^f)}{7^f} \geq \frac{2752}{2401}$ and since $c \geq 3$ $\frac{\sigma^{**}(17^c)}{17^c)} \geq \frac{88452}{83521}$. Hence from $(3.12a)$,

$$3 = \frac{\sigma^{**}(n)}{n} \geq \frac{255}{128}.\frac{26}{25}.\frac{88452}{83521}.\frac{14}{13}.\frac{10}{9}.\frac{2752}{2401} = 3.009358761 > 3,$$

a contradiction.

Hence $f = 1$ or $f = 2$ when $d = 1$.

If $f = 1$, from $(3.12a)$ $(f = 1)$, we have

$$3 = \frac{\sigma^{**}(n)}{n} \geq \frac{255}{128}.\frac{26}{25}.\frac{88452}{83521}.\frac{14}{13}.\frac{10}{9}.\frac{8}{7} = 3.000610625 > 3,$$

a contradiction.

Let $f = 2$. Since $\sigma^{**}(7^2) = 50 = 2.5^2$, taking $f = 2$ in $(3.12b)$, we find that 5 is a factor of its left-hand side which is false. Thus $d = 1$ is not admissible. Since $d \neq 2$, we may assume that $d \geq 3$.

Thus $c \geq 3, d \geq 3$ and $e = 2$. The relevant equations are $(3.11a)$ to $(3.11c)$.

We now show that $7 \nmid n$ when n is as given in $(3.11a)$. On the contrary, assume that $7|n$. Hence $7|w'$ and let $w' = 7^f.w''$. From $(3.11a)$ and $(3.11b)$, we get

$$n = 2^7.5^2.17^c.13^d.3^2.7^f.w'', \quad (c \geq 3,\ d \geq 3) \tag{3.13a}$$

and

$$2^5.17^{c-1}.13^{d-1}.3^2.7^f.w'' = \sigma^{**}(17^c).\sigma^{**}(13^d).\sigma^{**}(7^f).\sigma^{**}(w''), \tag{3.13b}$$

where

$$w'' \text{ has not more than two odd prime factors and } (w'', 2.3.5.7.13.17) = 1. \tag{3.13c}$$

Since c and d are ≥ 3, we have by Lemma 2.1, $\frac{\sigma^{**}(17^c)}{17^c} \geq \frac{88452}{83521}$ and $\frac{\sigma^{**}(13^d)}{13^d)} \geq \frac{30772}{28561}$. Also, for $f \geq 3$, we have $\frac{\sigma^{**}(7^f)}{7^f} \geq \frac{2752}{2401}$. Using these results, from $(3.13a)$, we obtain for $f \geq 3$,

$$3 = \frac{\sigma^{**}(n)}{n} \geq \frac{255}{128}.\frac{26}{25}.\frac{88452}{83521}.\frac{30772}{28561}.\frac{10}{9}.\frac{2752}{2401} = 3.010728519 > 3,$$

a contradiction. Hence $f = 1$ or $f = 2$.

If $f = 1$, from $(3.13a)$ $(f = 1)$,

$$3 = \frac{\sigma^{**}(n)}{n} \geq \frac{255}{128}.\frac{26}{25}.\frac{88452}{83521}.\frac{30772}{28561}.\frac{10}{9}.\frac{8}{7} = 3.001976401 > 3,$$

a contradiction.

Let $f = 2$. Since $\sigma^{**}(7^2) = 50 = 2.5^2$, taking $f = 2$ in $(3.13b)$, we see that 5 is a factor of its left-hand side and this is false. Hence $7 \nmid n$.

We return to the equations $(3.11a) - (3.11c)$ in which $7 \nmid n$ or equivalently $7 \nmid w'$. We can assume that each prime factor of w' in $(3.11a) - (3.11c)$ is at least 11.

We examine the factors of $\sigma^{**}(13^d)$ and obtain a contradiction when $e = 2$.

If d is odd or $4|d$, then $\sigma^{**}(13^d)$ is divisible by 7. From $(3.11b)$ it follows that $7|w'|n$. But $7 \nmid n$.

Hence we may assume that $d = 2k$, where k is odd and $k \geq 3$, since $d \geq 3$. We have

$$\sigma^{**}(13^d) = \left(\frac{13^k - 1}{12}\right).(13^{k+1} + 1).$$

We prove that:

(III) $\dfrac{13^k - 1}{12}$ is divisible by a prime $p'|w'$ and $p' > 127$,

(IV) $\dfrac{13^{k+1} + 1}{2}$ is divisible by a prime $q'|w'$ and $q' > 127$.

- Proof of (III). Let

$$S'_{13} = \{p|13^k - 1 \, : p \in [3, 127] - \{3, 61\} \text{ and } ord_p 13 \text{ is odd } \}.$$

If we replace the interval $[3, 293]$ in Lemma 2.5(a), by $[3, 127]$, it follows that (III) holds if S'_{13} is non-empty.

Suppose that S'_{13} is empty. Since $p \nmid 13^k - 1$ if $ord_p 13$ is even, it follows that $\dfrac{13^k - 1}{12}$ is not divisible by any prime in $[3, 127]$ except for possibly 3 and 61.

Clearly $3|13^k - 1$. We show that $27 \nmid 13^k - 1$. On the contrary, suppose that $27|13^k - 1$. This is equivalent to $9|k$. Hence $13^9 - 1|13^k - 1$. Also, $13^9 - 1 = 2^2.3^3.61.1609669$. Hence 61 and 1609669 are factors of w' and by $(3.11c)$, $w' = 61^f.(1609669)^g.w''$, where $w'' = 1$ or $w'' = p^\alpha$ where $p \geq 11$. Hence $\sigma^{**}(w'')/w'' < 11/10$. From $(3.11a)$,

$$n = 2^7.5^2.17^c.13^d.3^2.61^f.(1609669)^g.w''$$

so that

$$3 = \frac{\sigma^{**}(n)}{n} < \frac{255}{128}.\frac{26}{25}.\frac{17}{16}.\frac{13}{12}.\frac{10}{9}.\frac{61}{60}.\frac{1609669}{1609668}.\frac{11}{10} = 2.963354615 < 3,$$

a contradiction. Hence $27 \nmid 13^k - 1$.

We may note that $9|13^k - 1 \iff 3|k \iff 61|13^k - 1$.

Assume that $9 \nmid 13^k - 1$. Then $61 \nmid 13^k - 1$. Thus $\dfrac{13^k - 1}{12} > 1$, odd and not divisible by 3 and 61; and so not divisible by any prime in $[3, 127]$. If $p'| \dfrac{13^k - 1}{12}$, then $p' > 127$ and $p'|w'$ by $(3.11b)$. This proves (III) in this case.

Suppose that $9|13^k-1$. Then $61|13^k-1$. Also, $\frac{13^k-1}{36}>1$, odd and not divisible by 3 but divisible by 61. We show that $\frac{13^k-1}{36}$ must be divisible by an odd prime other than 61. If this is not so let $\frac{13^k-1}{36}=61^\alpha$, where α is a positive integer. If $\alpha \geq 2$, then $61^2|13^k-1$. This holds if and only if $183|k$. Hence, $367|13^{183}-1|13^k-1$. Hence, $367|\ \frac{13^k-1}{36}=61^\alpha$. This is not possible and so $\alpha=1$. Thus $\frac{13^k-1}{36}=61$ or $k=3$ or $d=6$.

We now show that $d=6$ is not admissible. We have $\sigma^{**}(13^6)=2.3.61.14281$. Taking $d=6$ in $(3.11b)$, we see that w' is divisible by 61 and 14281. From $(3.11c)$, $w'=61^f.(14281)^g.w''$, where $w''=1$ or p^α, where $p\geq 11$. Hence

$$n=2^7.5^2.17^c.13^6.3^2.61^f.(14281)^g.w'',$$

so that

$$3=\frac{\sigma^{**}(n)}{n}<\frac{255}{128}.\frac{26}{25}.\frac{17}{16}.\frac{13}{12}.\frac{10}{9}.\frac{61}{60}.\frac{14281}{14280}.\frac{11}{10}=2.963560292<3,$$

a contradiction.

It now follows that $\frac{13^k-1}{36}$is not divisible by 61 alone. Let $p'|\ \frac{13^k-1}{36}$ and $p'\neq 61$. It follows that $p'>127$ and from $(3.11b)$, $p'|w'$. This proves (III).

- Proof of (IV). Let

$$T'_{13}=\{q|13^{k+1}+1\ \ : q\in[3,127]-\{5,17\} \text{ and } s=\frac{1}{2}ord_q 13 \text{ is even }\}.$$

By Lemma 2.5 (b), (IV) holds if T'_{13} is non-empty. We may assume that T'_{13} is empty. Since $q\nmid 13^{k+1}+1$ if $s=\frac{1}{2}ord_q 13$ is not even, it follows that $\frac{13^{k+1}+1}{2}$ is not divisible by any prime in $[3,127]$ except for possibly 5 and 17.

If $5|13^{k+1}+1$, then $5|\sigma^{**}(13^d)$ and from $(3.11b)$, it follows that 5 divides its left-hand side. This is false. Hence $5\nmid 13^{k+1}+1$. Since $5\nmid 13^{k+1}+1 \iff k+1=2u \iff 17\nmid 13^{k+1}+1$, we conclude that $17\nmid 13^{k+1}+1$.

Thus $\frac{13^{k+1}+1}{2}>1$ and is odd, and it is not divisible by any prime in $[3,127]$. Let $q'|\ \frac{13^{k+1}+1}{2}$. Then $q'>127$ and $q'|w'$ from $(3.11b)$.

This proves (IV).

From (III), (IV) and $(3.11c)$, $w'=(p')^f.(q')^g.t^h$, where t is the possible third prime factor of w' and $t\geq 11$. We can assume that $p'\geq 131$ and $q'\geq 137$.

From $(3.11a)$, $n=2^7.5^2.17^c.13^d.3^2.(p')^f.(q')^d.t^h$, and so

$$3=\frac{\sigma^{**}(n)}{n}<\frac{255}{128}.\frac{26}{25}.\frac{17}{16}.\frac{13}{12}.\frac{10}{9}.\frac{131}{130}.\frac{137}{136}.\frac{11}{10}=2.958791572<3,$$

a contradiction. This completes the case $e=2$ and the proof of Lemma 3.7. □

Lemma 3.8. *The number* n *given in* $(3.8a)$ *with* $c = 2$ *and* $e = 1$ *or* 2 *cannot be a bi-unitary triperfect number.*

Proof. We assume that n given in (3.8) is a bi-unitary triperfect number.

Let $c = 2$ and $e = 1$. Since $c = 2$, we can use the equations $(3.9a)$ to $(3.9c)$. Taking $e = 1$ in $(3.9a)$ and $(3.9b)$, we get after simplification

$$n = 2^7.5^2.17^2.13^d.3.29^f.w'', \tag{3.14a}$$

and

$$2^3.17.13^{d-1}.3.29^{f-1}.w'' = \sigma^{**}(13^d).\sigma^{**}(29^f).\sigma^{**}(w''), \tag{3.14b}$$

where

$$w'' = 1 \text{ or } \text{ a prime power.} \tag{3.14c}$$

Let $d = 1$. Since $\sigma^{**}(13) = 14 = 2.7$, taking $d = 1$ in $(3.14b)$, we see that $7|w''$. Hence form $(3.14c)$, $w'' = 7^g$. From $(3.14a)$ and $(3.14b)$ we have

$$n = 2^7.5^2.17^2.13.3.29^f.7^g, \tag{3.15a}$$

and

$$2^2.17.3.29^{f-1}.7^{g-1} = \sigma^{**}(29^f).\sigma^{**}(7^g). \tag{3.15b}$$

Since $\sigma^{**}(7) = 8$, taking $g = 1$ in $(3.15b)$, we find that 2^4 divides the right-hand side of it, whereas 2^2 is a unitary divisor of its left-hand side. Hence $g = 1$ is not possible.

Since $\sigma^{**}(7^2) = 50$, taking $g = 2$ in $(3.15b)$, we see that 5 divides its right-hand side but this is not true with respect to its left-hand side. Hence $g = 2$ is also not possible.

We may assume that $g \geq 3$ so that $\frac{\sigma^{**}(7^g)}{7^g} \geq \frac{2752}{2401}$. From $(3.14a)$, we have

$$3 = \frac{\sigma^{**}(n)}{n} \geq \frac{255}{128}.\frac{26}{25}.\frac{290}{289}.\frac{14}{13}.\frac{4}{3}.\frac{2752}{2401} = 3.421711542 > 3,$$

a contradiction. Hence $d = 1$ is not admissible.

Let $d = 2$ in $(3.14b)$. Since $\sigma^{**}(13^2) = 170 = 2.5.17$, we find that 5 is a factor of its left-hand side. This cannot happen. Hence $d = 2$ is not possible. We may assume that $d \geq 3$.

Let $f = 1$ in $(3.14b)$. Since $\sigma^{**}(29) = 30 = 2.3.5$, we see that 5 is a factor of its left-hand side which is not true.

Let $f = 2$. We have $\sigma^{**}(29^2) = 842 = 2.421$. From $(3.14c)$, $w'' = (421)^g$. Hence from $(3.14a)$ $(f = 2)$, we have

$$n = 2^7.5^2.17^2.13^d.3.29^2.(421)^g, \tag{3.16a}$$

and from $(3.14b)$ $(f = 2)$, we obtain

$$2^2.17.13^{d-1}.3.29.(421)^{g-1} = \sigma^{**}(13^d).\sigma^{**}(29^f).\sigma^{**}((421)^g). \tag{3.16b}$$

We obtain a contradiction by examining the factors of $\sigma^{**}(13^d)$.

If d is odd or $4|d$, then $7|\sigma^{**}(13^d)$. From $(3.16b)$, we find that 7 divides the left-hand side of it. This cannot happen.

We may assume that $d = 2k$ where k is odd and $k \geq 3$, since $d \geq 3$. We have

$$\sigma^{**}(13^d) = \left(\frac{13^k - 1}{12}\right).(13^{k+1} + 1).$$

We prove that $\frac{13^k - 1}{12}$ is not divisible by $2, 3, 5, 17, 29$, and 421. This leads to a contradiction.

(i) Clearly $4|13^k - 1$ but $8 \nmid 13^k - 1$, since k is odd. Hence $\frac{13^k - 1}{12}$ is odd.

(ii) Clearly $3|13^k - 1$. We note that $9|13^k - 1 \iff 3|k \iff 61|13^k - 1$. Suppose $9|13^k - 1$. Hence $61|13^k - 1$ and so $61|\frac{13^k - 1}{12}|\sigma^{**}(13^d)$. Thus 61 is a factor of the left-hand side of $(3.16b)$. This is false. Hence $9 \nmid 13^k - 1$. Hence $\frac{13^k - 1}{12}$ is not divisible by 3.

(iii) We have $17|13^k - 1 \iff 4|k;\ 29|13^k - 1 \iff 14|k$; and $421|13^k - 1 \iff 20|k$. But $4, 14, 20$ cannot divide k since k is odd. Hence $\frac{13^k - 1}{12}$ is not divisible by $17, 29, 421$.

From (i), (ii) and (iii), $\frac{13^k - 1}{12}$ is not divisible by $2, 3, 17, 29, 421$ and trivially not divisible by 13. This cannot happen in view of $(3.16b)$. Thus $f = 2$ is not possible.

We may assume that $f \geq 3$. Since $d \geq 3$ and $f \geq 3$, by Lemma 2.1, $\frac{\sigma^{**}(13^d)}{13^d} \geq \frac{30772}{28561}$ and $\frac{\sigma^{**}(29^f)}{29^f} \geq \frac{731700}{707281}$. From $(3.16a)$, we obtain

$$3 = \frac{\sigma^{**}(n)}{n} \geq \frac{255}{128}.\frac{26}{25}.\frac{290}{289}.\frac{30772}{28561}.\frac{4}{3}.\frac{731700}{707281} = 3.089767636 > 3,$$

a contradiction.

This completes the case $c = 2$ and $e = 1$.

Let $c = 2$ and $e = 2$. Taking $e = 2$ in $(3.9b)$, we see that 5 is a factor of its right-hand side but it cannot divide its left-hand side. This completes the proof of Lemma 3.8. □

We return to the equations $(3.3a) - (3.3c)$. In Lemmas 3.5 to 3.8, we proved that if $3|n$, $5^2\|n$ and $17^2|n$, then n cannot be a bi-unitary triperfect number. In what follows we will be prove that n cannot be a bi-unitary triperfect number if $3 \nmid n$, $5^2\|n$ and $17^2|n$.

Lemma 3.9. *Let n be as in $(3.3a)$ with $c \geq 2$. Assume that $3 \nmid n$.*
(a) *If $7 \nmid n$, then n is not a bi-unitary triperfect number.*
(b) *Assume that n is a bi-unitary triperfect number. If $7|n$, then n is not divisible by s where $s \in \{11, 13, 19, 23\}$.*

Proof. (a) Suppose $7 \nmid n$ and n is a bi-unitary triperfect number. From the hypothesis, $n = 2^7.5^2.17^c.13^d.w$, $(c \geq 2)$, where w is prime to $2.5.7.13.17$. Also, by $(3.3c)$, w cannot have more than four odd prime factors. If p_1, p_2, p_3 and p_4 denote the four odd prime factors of w, we can assume that $p_1 \geq 11$, $p_2 \geq 19$, $p_3 \geq 23$ and $p_4 \geq 29$. We have from $(3.3a)$, $n = 2^7.5^2.17^c.13^d.p_1^e.p_2^f.p_3^g.p_4^h$, and by Lemma 2.1,

$$3 = \frac{\sigma^{**}(n)}{n} < \frac{255}{128}.\frac{26}{25}.\frac{17}{16}.\frac{13}{12}.\frac{11}{10}.\frac{19}{18}.\frac{23}{22}.\frac{29}{28} = 2.998289044 < 3,$$

a contradiction. Hence n is not a bi-unitary triperfect number. This proves (a).

(b) Assume that $7|n$ so that $7|w$. Let $w = 7^e.w'$. Using this in $(3.3a)$ and $(3.3b)$, we obtain

$$n = 2^7.5^2.17^c.13^d.7^e.w', \quad (c \geq 2) \tag{3.17a}$$

and

$$2^6.5.17^{c-1}.13^{d-1}.7^e.w' = \sigma^{**}(17^c).\sigma^{**}(13^d).\sigma^{**}(7^e).\sigma^{**}(w'), \tag{3.17b}$$

where

$$w' \text{ has not more than three odd prime factors and } (w', 2.3.5.7.13.17) = 1. \tag{3.17c}$$

We next show that $11 \nmid n$. Suppose that $11|n$. Hence $11|w'$. Let $w' = 11^f.w''$, where $(w'', 2.3.5.7.11.13.17) = 1$. From $(3.17a)$ and $(3.17b)$, we have

$$n = 2^7.5^2.17^c.13^d.7^e.11^f.w'', \quad (c \geq 2) \tag{3.18a}$$

and

$$2^6.5.17^{c-1}.13^{d-1}.7^e.11^f.w'' = \sigma^{**}(17^c).\sigma^{**}(13^d).\sigma^{**}(7^e).\sigma^{**}(11^f)\sigma^{**}(w''), \tag{3.18b}$$

where

$$w'' \text{ has not more than two odd prime factors and } (w'', 2.3.5.7.11.13.17) = 1. \tag{3.18c}$$

Let $e = 1$. Since $\sigma^{**}(7) = 8$, from $(3.18b)$ it follows that $w'' = 1$. In this case $n = 2^7.5^2.17^c.13^d.7^e.11^f$. Hence

$$3 = \frac{\sigma^{**}(n)}{n} < \frac{255}{128}.\frac{26}{25}.\frac{17}{16}.\frac{13}{12}.\frac{8}{7}.\frac{11}{10} = 2.998052455 < 3,$$

a contradiction. We have

$$\sigma^{**}(7^e) = \begin{cases} 50 & \text{when } e = 2, \\ 400 & \text{when } e = 3, \\ 2^6.43 & \text{when } e = 4. \end{cases}$$

If $e = 2$ or $e = 3$, $5^2|\sigma^{**}(7^e)$. Hence 5^2 is a factor of the left-hand side of $(3.18b)$. This is not possible. When $e = 4$, $2^6|\sigma^{**}(7^e)$. Thus 2^9 is a factor of the right-hand side of $(3.18b)$, whereas 2^6 is a unitary divisor of its left-hand side.

Thus we may assume that $e \geq 5$.

We now prove that $c = 2$ is not admissible in $(3.18b)$. We assume that $c = 2$ and obtain a contradiction by examining the prime factors of $\sigma^{**}(7^e)$. Since $\sigma^{**}(17^2) = 290 = 2.5.29$, taking $c = 2$ in $(3.18b)$, we find that $29|w''$. Let $w'' = 29^g.w^*$. Using this in $(3.18a)$ $(c = 2)$ and $(3.18b)$ $(c = 2)$, we get

$$n = 2^7.5^2.17^2.13^d.7^e.11^f.29^g.w^*, \tag{3.19a}$$

and

$$2^5.17.13^{d-1}.7^e.11^f 29^{g-1}.w^* = \sigma^{**}(13^d).\sigma^{**}(7^e).\sigma^{**}(11^f).\sigma^{**}(29^g).\sigma^{**}(w^*), \tag{3.19b}$$

where

$$w^* \text{ has not more than one odd prime factors and } (w^*, 2.3.5.7.11.13.17) = 1. \tag{3.19c}$$

If e is odd or $4|e$, then $8|\sigma^{**}(7^e)$; from $(3.19b)$ we find that this results in an imbalance of powers of 2 between both sides of $(3.19b)$. Hence we may assume that $e = 2k$, where k is odd and $k \geq 3$, since $e \geq 5$. We have

$$\sigma^{**}(7^e) = \left(\frac{7^k - 1}{6}\right).(7^{k+1} + 1).$$

We obtain a contradiction by showing that $\frac{7^k - 1}{6}$ and $\frac{7^{k+1} + 1}{2}$ are divisible by two distinct odd primes p' and q', respectively, which are also factors of w^*. This would contradict $(3.19c)$.

In Lemma 2.4 (a), if we replace the interval $[3, 2520]$ by $[3, 31]$ we have the following conclusion:

(I) If $p|7^k - 1$, where $p \in [3, 31] - \{3, 19\}$ and $ord_p 7$ is odd, then we can find an odd prime $p'| \frac{7^k - 1}{6}$ and $p' > 31$. By $(3.19b)$, $p'|w^*$.

Let

$$S'_7 = \{p|7^k - 1 \ : \ p \in [3, 31] - \{3, 19\} \text{ and } ord_p 7 \text{ is odd}\}.$$

If S'_7 is non-empty, we can conclude from (I) that w^* is divisible by an odd prime $p'|\frac{7^k - 1}{6}$.

Suppose that S'_7 is empty. Since $p \nmid 7^k - 1$ if $ord_p 7$ is even, it follows that $7^k - 1$ is not divisible by any prime in $[3, 31]$ except for possibly 3 and 19.

We have $19|7^k - 1 \iff 3|k \iff 9|7^k - 1$. Also, $9|7^k - 1 \iff 3| \frac{7^k - 1}{6}$. Further from $(3.19b)$, $3|\frac{7^k - 1}{6}\sigma^{**}(7^e)$ implies that 3 is a factor of the left-hand side of $(3.19b)$ and so $3|w^*$. This cannot happen. Hence $9 \nmid 7^k - 1$ and consequently $19 \nmid 7^k - 1$.

Thus $\frac{7^k - 1}{6}$ is not divisible by 3 and 19. It follows that $\frac{7^k - 1}{6}$ is not divisible by any prime in $[3, 31]$; also, $\frac{7^k - 1}{6} > 1$ and odd. Let $p'|\frac{7^k - 1}{6}$. Then $p' > 31$ and from $(3.19b)$, $p'|w^*$.

In Lemma 2.4 (b), if we replace the interval $[3, 1193]$ by $[3, 31]$ we have the following conclusion:

(II) If $q|7^{k+1} + 1$, $q \in [3, 31] - \{5, 13\}$ and $s = \frac{1}{2} ord_p 7$ is even, then we can find an odd prime $q'|\frac{7^{k+1} + 1}{2}$ and $q' > 31$. By $(3.19b)$, $q'|w^*$.

Let

$$T'_7 = \{q|7^{k+1} + 1 \ : \ q \in [3, 31] - \{5, 13\} \text{ and } s = \frac{1}{2} ord_p 7 \ \text{ is even } \}.$$

If T'_7 is non-empty, by (II) above, we can find an odd prime $q'|\frac{7^{k+1} + 1}{2}$ and $q' > 31$. Also, $q'|w^*$. This is what we require.

Suppose that T'_7 is empty. Since $q \nmid 7^{k+1} + 1$ if $s = \frac{1}{2} ord_p 7$ is not even, it follows that $\frac{7^{k+1} + 1}{2}$ is not divisible by any prime in $[3, 31]$ except for possibly 5 and 13.

If $5|\frac{7^{k+1} + 1}{2}|\sigma^{**}(7^e)$, then from $(3.19b)$ it follows that 5 divides its left-hand side. This is not possible. Hence $5 \nmid 7^{k+1} + 1$.

Suppose $13|7^{k+1}+1$. This is equivalent to $k+1=6u$. Hence $7^6+1|7^{k+1}+1$. Also, $7^6+1=2.5^2.13.181$. Hence $5|7^6+1|7^{k+1}+1$. But this is false.

Thus $7^{k+1}+1$ is not divisible by 5 and 13 and so $\dfrac{7^{k+1}+1}{2}$ is not divisible by any prime in $[3,31]$. Since $\dfrac{7^{k+1}+1}{2}>1$ and odd, let $q'|\dfrac{7^{k+1}+1}{2}$. Then $q'>31$ and from $(3.19b)$, $q'|w^*$. It follows that w^* is divisible by two distinct odd primes p' and q'. But this is not possible by $(3.19c)$.

This proves that $c=2$ is not admissible. We may assume that $c\geq 3$. The relevant equations are $(3.18a)$ and $(3.18b)$ with $c\geq 3$ and $e\geq 5$. By Lemma 2.1, we have $\frac{\sigma^{**}(17^c)}{17^c}\geq\frac{88452}{83521}$ $(c\geq 3)$, $\frac{\sigma^{**}(13^d)}{13^d}\geq\frac{30772}{28561}$ $(d\geq 3)$, $\frac{\sigma^{**}(7^e)}{7^e}\geq\frac{136914}{117649}$ $(e\geq 5)$ and $\frac{\sigma^{**}(11^f)}{11^f}\geq\frac{15984}{14641}$ $(f\geq 3)$.

If $d\geq 3$ and $f\geq 3$, from $(3.18a)$, we have

$$3=\frac{\sigma^{**}(n)}{n}\geq\frac{255}{128}.\frac{26}{25}.\frac{88452}{83521}.\frac{30772}{28561}.\frac{136914}{117649}.\frac{15984}{14641}=3.00353146>3, \qquad (3.19d)$$

a contradiction. Hence $d\geq 3$ and $f\geq 3$ cannot hold together. So we have the following cases:

- *Case 1:* $\{d\geq 3,\ f=1,\ 2\}$
- *Case 2:* $\{d=1,\ 2,\ f\geq 3\}$, and
- *Case 3:* $\{d=1,\ 2, f=1,\ 2\}$.

Let $f=1$. Taking $f=1$ in $(3.18b)$, since $\sigma^{**}(11)=12$, it follows that 3 is a factor of the left-hand side of $(3.18b)$. Since $3\nmid n$, by our assumption, this is not possible.

Let $f=2$. We have $\sigma^{**}(11^2)=122=2.61$. From $(3.18b)$ $(f=2)$, $61|w''$. Let $w''=61^g.w^*$. From $(3.18a)$ $(f=2)$ and $(3.18b)$ $(f=2)$, we obtain

$$n=2^7.5^2.17^c.13^d.7^e.11^2.61^g.w^*,\quad (c\geq 3) \qquad (3.20a)$$

and

$$2^5.5.17^{c-1}.13^{d-1}.7^e.11^2.61^{g-1}.w^*=\sigma^{**}(17^c).\sigma^{**}(13^d).\sigma^{**}(7^e).\sigma^{**}(61^g).\sigma^{**}(w^*), \qquad (3.20b)$$

where

$$w^*=1 \text{ or a prime power with } (w^*,2.3.5.7.11.13.17)=1. \qquad (3.20c)$$

We now show that $19\nmid w^*$. On the contrary, suppose that $19|w^*$ so that $w^*=19^h$. Using this in $(3.20a)$ and $(3.20b)$, we get

$$n=2^7.5^2.17^c.13^d.7^e.11^2.61^g.w^*,\quad (c\geq 3) \qquad (3.21a)$$

and

$$2^5.5.17^{c-1}.13^{d-1}.7^e.11^2.61^g.19^h=\sigma^{**}(17^c).\sigma^{**}(13^d).\sigma^{**}(7^e).\sigma^{**}(61^g).\sigma^{**}(19^h). \qquad (3.21b)$$

If e is odd or $4|e$, then $8|\sigma^{**}(7^e)$; this brings an imbalance in powers of 2 between the two sides of $(3.21b)$. We may thus assume that $e=2k$, where k is odd and $k\geq 3$ (since $e\geq 5$). So, $\sigma^{**}(7^e)=\Big(\dfrac{7^k-1}{6}\Big).(7^{k+1}+1)$.

Since k is odd and $k \geq 3$, $\frac{7^k-1}{6} > 1$ and odd. Also, $7^k - 1$ is not divisible by $5, 11, 13, 17$ and 61, since k is odd.

Further, $9|7^k - 1 \iff 3|k \iff 19|7^k - 1$. But $9|7^k - 1$ implies that $3|\frac{7^k-1}{6}|\sigma^{**}(7^e)$ and from $(3.21b)$, we find that this is not possible. Hence $9 \nmid 7^k - 1$ and consequently $19 \nmid 7^k - 1$.

Thus $\frac{7^k-1}{6} > 1$, is odd and not divisible by $5, 7, 11, 13, 17, 19$ and 61 (divisible by none of these primes). From $(3.21b)$, this is not possible. This contradiction shows that $19 \nmid w^*$.

From $(3.20c)$, we may assume that $w^* = p^h$, where $p \geq 23$. From $(3.20a)$, we have $n = 2^7.5^2.17^c.13^d.7^e.11^2.61^g.p^h$. Hence

$$3 = \frac{\sigma^{**}(n)}{n} < \frac{255}{128}.\frac{26}{25}.\frac{17}{16}.\frac{13}{12}.\frac{7}{6}.\frac{122}{121}.\frac{61}{60}.\frac{23}{22} = 2.981670063 < 3,$$

a contradiction.

Hence $f = 2$ is not admissible.

Let $d = 1$ and $f \geq 3$. Already $c \geq 3$ and $e \geq 5$. From $(3.18a)$ $(d = 1)$, we have

$$3 = \frac{\sigma^{**}(n)}{n} \geq \frac{255}{128}.\frac{26}{25}.\frac{88452}{83521}.\frac{14}{13}.\frac{15984}{14641}.\frac{136914}{117649} = 3.002164976 > 3,$$

a contradiction.

Let $d = 2$. Since $\sigma^{**}(13^2) = 170 = 2.5.17$. Taking $d = 2$ in $(3.18a)$ and $(3.18b)$, we obtain

$$n = 2^7.5^2.17^c.13^2.7^e.11^f.w'', \tag{3.22a}$$

and

$$2^5.17^{c-2}.13.7^e.11^f.w'' = \sigma^{**}(17^c).\sigma^{**}(7^e).\sigma^{**}(11^f).\sigma^{**}(w''), \tag{3.22b}$$

where

$$w'' \text{ has not more than two odd prime factors and } (w'', 2.3.5.7.11.13.17) = 1. \tag{3.22c}$$

By applying Lemma 2.4, we show that $\sigma^{**}(7^e)$ is divisible by two distinct odd primes each greater than 67 when e is even and $4 \nmid e$.

If e is odd or $4|e$, then $8|\sigma^{**}(7^e)$. From $(3.22b)$, it follows that $w'' = 1$. Hence from $(3.22a)$, $n = 2^7.5^2.17^c.13^2.7^e.11^f$ and so

$$3 = \frac{\sigma^{**}(n)}{n} < \frac{255}{128}.\frac{26}{25}.\frac{17}{16}.\frac{170}{169}.\frac{7}{6}.\frac{11}{10} = 2.841804387 < 3,$$

a contradiction.

We may assume that $e = 2k$, where k is odd and ≥ 3. We have

$$\sigma^{**}(7^e) = \left(\frac{7^k-1}{6}\right).(7^{k+1}+1).$$

We show (by using Lemma 2.4) that:

(III) $\frac{7^k-1}{6}$ is divisible by an odd prime $p'|w''$ and $p' > 67$,

(IV) $\frac{7^{k+1}+1}{2}$ is divisible by an odd prime $q'|w''$ and $q' > 67$.

(III) Proof of (III). If we replace the interval $[3, 2520]$ by the interval $[3, 67]$ in Lemma 2.4 (a), then we have the following conclusion:

(A) If $p \in [3, 67] - \{3, 19, 37\}$, $p|7^k - 1$ and $ord_7 p$ is odd, then we can find an odd prime $p'|\dfrac{7^k-1}{6}$ and $p' > 67$.

Let
$$S'_7 = \{p|7^k - 1 \ : p \in [3, 67] - \{3, 19, 37\} \text{ and } ord_p 7 \text{ is odd } \}.$$

By (A), if S'_7 is non-empty, then (III) holds.

Suppose that S'_7 is empty. Since $p \nmid 7^k - 1$ if $ord_7 p$ is even, it follows that $\dfrac{7^k-1}{6}$ is not divisible by any prime in $[3, 67]$ except for possibly $3, 19$ and 37. We have the following:

(i) $9|7^k - 1 \Longleftrightarrow 3|\ \dfrac{7^k-1}{6}$. Hence if $9|7^k - 1$, then 3 is a factor of the left-hand side of $(3.22b)$ which is not the case. Hence $9 \nmid 7^k - 1$ and so $\dfrac{7^k-1}{6}$ is not divisible by 3.

(ii) $19|7^k - 1 \Longleftrightarrow 3|k \Longleftrightarrow 9|7^k - 1$. By (i), $19 \nmid 7^k - 1$.

(iii) Suppose $37|7^k - 1$. Hence $9|k$ and so $7^9 - 1|7^k - 1$. Also, $7^9 - 1 = 2.3^3.19.37.1063$. Hence $19|7^9 - 1|7^k - 1$. By (i) this is false. Hence $37 \nmid 7^k - 1$.

(iv) Since k is odd and ≥ 3, $\dfrac{7^k-1}{6}$ is odd and > 1.

From the above discussion, it is clear that $\dfrac{7^k-1}{6} > 1$, is odd and is not divisible by any prime in $[3, 67]$. Let $p'|\dfrac{7^k-1}{6}$. Then $p' > 67$ and by $(3.22b)$, $p'|w''$. This proves (III).

(IV) Proof of (IV). If we replace the interval $[3, 1193]$ by the interval $[3, 67]$ in Lemma 2.4 (b), then we have the following conclusion:

(B) If $q \in \ [3, 67] - \{5, 13\}, q|7^{k+1} + 1$ and $s = \frac{1}{2} ord_7 q$ is even, then we can find an odd prime $q'|\dfrac{7^{k+1}+1}{2}$ and $q' > 67$.

Let
$$T'_7 = \{q|7^{k+1} + 1 \ : q \in [3, 67] - \{5, 13\} \text{ and } s = \frac{1}{2} ord_q 7 \text{ is even} \}.$$

By (B), if T'_7 is non-empty, then (III) holds.

Suppose that T'_7 is empty. Since $q \nmid 7^{k+1} + 1$ if $s = \frac{1}{2} ord_q 7$ is not even, it follows that $\dfrac{7^{k+1}+1}{2}$ is not divisible by any prime in $[3, 67]$ except for possibly by 5 and 13.

From $(3.22b)$ it follows that $5 \nmid \sigma^{**}(7^e)$, since 5 is not a factor of the left-hand side of $(3.22b)$.

Note that $5|7^{k+1} + 1$ implies that $5^2|\sigma^{**}(7^e)$; also, $13|7^{k+1} + 1$ implies that $k + 1 = 6u$ and so $5^2|7^6 + 1|7^{k+1} + 1|\sigma^{**}(7^e)$. In both the cases $5^2|\sigma^{**}(7^e)$ which is false. Thus $7^{k+1} + 1$ is divisible by neither 5 nor 13.

It follows that $\frac{7^{k+1}+1}{2}$ is not divisible by any prime in $[3, 67]$. Also, $\dfrac{7^{k+1}+1}{2} > 1$ and is odd. Let $q'|\ \dfrac{7^{k+1}+1}{2}$. Then $q' > 67$ and $q'|w''$ by $(3.22b)$.

This proves (IV).

From $(3.22c)$, we have $w'' = (p')^g.(q')^h$. Hence from $(3.22a)$, we have

$$n = 2^7.5^2.17^c.13^2.7^e.11^f.(p')^g.(q')^h$$

so that

$$3 = \frac{\sigma^{**}(n)}{n} < \frac{255}{128}.\frac{26}{25}.\frac{17}{16}.\frac{170}{169}.\frac{7}{6}.\frac{11}{10}.\frac{71}{70}.\frac{73}{72} = 2.922434948 < 3,$$

a contradiction.

Hence $d = 2$ is not possible. Thus we proved the non-admissibility of $f = 1, 2$ and $d = 1, 2$. These cover the three cases (Case 1, Case 2 and Case 3) mentioned below $(3.19d)$.

This completes the proof of $11 \nmid n$.

Let $s = 19$ or 23. We show that $s \nmid n$ which is same as $s \nmid w'$ where n is as in $(3.17a)$. On the contrary, we assume that $s|w'$ so that $w' = s^f.w''$. From $(3.17a)$ and $(3.17b)$, we have

$$n = 2^7.5^2.17^c.13^d.7^e.s^f.w'', \quad (c \geq 2) \tag{3.23a}$$

and

$$2^6.5.17^{c-1}.13^{d-1}.7^e.s^f.w'' = \sigma^{**}(17^c).\sigma^{**}(13^d).\sigma^{**}(7^e).\sigma^{**}(s^f).\sigma^{**}(w''), \tag{3.23b}$$

where

$$w'' \text{ has not more than two odd prime factors and } (w', 2.3.5.7.11.13.17.s) = 1. \tag{3.23c}$$

We obtain a contradiction by examining $\sigma^{**}(7^e)$. If e is odd or $4|e$, then $8|\sigma^{**}(7^e)$. From $(3.23b)$, we infer that $w'' = 1$. From $(3.23a)$, $n = 2^7.5^2.17^c.13^d.7^e.s^f$ where $s \geq 19$; and so

$$3 = \frac{\sigma^{**}(n)}{n} < \frac{255}{128}.\frac{26}{25}.\frac{17}{16}.\frac{13}{12}.\frac{7}{6}.\frac{19}{18} = 2.936854836 < 3,$$

a contradiction.

We may assume that $e = 2k$, where k is odd, and since $e \neq 2,\ k \geq 3$. Also,

$$\sigma^{**}(7^e) = \left(\frac{7^k - 1}{6}\right).(7^{k+1} + 1) \quad (k \text{ odd, } k \geq 3).$$

We show that:

(III)′ $\frac{7^k - 1}{6}$ is divisible by a prime $p' > 89$ and $p'|w''$,

(IV)′ $\frac{7^{k+1} + 1}{2}$ is divisible by a prime $q' > 89$ and $q'|w''$.

The left-hand side of $(3.23b)$) is neither divisible by 3 nor 5^2. The proofs of (III)′ and (IV)′ are similar to those of (III) and (IV); we need to apply Lemma 2.4 by replacing the intervals $[3, 2520]$ and $[3, 1193]$ by $[3, 89]$. We omit the details.

From $(3.23c)$, we have $w'' = (p')^g.(q')^h$. We may assume that $p' \geq 97$ and $q' \geq 101$. From $(3.23a)$, we have $n = 2^7.5^2.17^c.13^d.7^e.s^f.(p')^g.(q')^h$, so that

$$3 = \frac{\sigma^{**}(n)}{n} < \frac{255}{128}.\frac{26}{25}.\frac{17}{16}.\frac{13}{12}.\frac{7}{6}.\frac{19}{18}.\frac{97}{96}.\frac{101}{100} = 2.997121544 < 3,$$

a contradiction. Hence $s \nmid w''$.

Thus if n is as in $(3.3a)$, $3 \nmid n$ and $7|n$, then n is divisible by none of $11, 19$ and 23. The proof of (b) of Lemma 3.9 is complete. □

Lemma 3.10. *Let n be as in $(3.3a)$ with $c \geq 2$. Assume that $3 \nmid n$. If $7|n$, then n cannot be a bi-unitary triperfect number.*

Proof. Suppose that $7|n$ and n is a bi-unitary triperfect number. Then n satisfies the equations $(3.17a) - (3.17c)$. We obtain a contradiction by examining the factors of $\sigma^{**}(7^e)$ in $(3.17b)$.

If e is odd or $4|e$, then $8|\sigma^{**}(7^e)$. From $(3.17b)$, we at once have $w' = p^f$, where $p \geq 29$ by Lemma 3.9 (b). Hence from $(3.17a)$, $n = 2^7.5^2.17^c.13^d.7^e.p^f$ and so by Lemma 2.1,

$$3 = \frac{\sigma^{**}(n)}{n} < \frac{255}{128}.\frac{26}{25}.\frac{17}{16}.\frac{13}{12}.\frac{7}{6}.\frac{29}{28} = 2.881650798 < 3,$$

a contradiction.

Let $e = 2k$, where k is odd and $k \geq 3$ (as $e \neq 2$). We note that the left-hand side of $(3.17b)$ is neither divisible by 3 nor 5^2. As in (III)$'$ and (IV)$'$ of Lemma 3.9 (b), $\frac{7^k - 1}{6}$ and $\frac{7^{k+1}+1}{2}$ are divisible by odd primes $p' > 89$ and $q' > 89$, respectively.

Further, w' is divisible by p' and q'. We may assume that $p' \geq 97$ and $q' \geq 101$. Assuming that y is a possible third prime factor of w' by $(3.17c)$, by Lemma 3.9 (b), we have $y \geq 29$ and $w' = (p')^f.(q')^g.y^h$. By $(3.17c)$ and $(3.17a)$, $n = 2^7.5^2.17^c.13^d.7^e.(p')^f.(q')^g.y^h$ and by Lemma 2.1, we have

$$3 = \frac{\sigma^{**}(n)}{n} < \frac{255}{128}.\frac{26}{25}.\frac{17}{16}.\frac{13}{12}.\frac{7}{6}.\frac{97}{96}.\frac{101}{100}.\frac{29}{28} = 2.940784673 < 3,$$

a contradiction. Hence n cannot be a bi-unitary triperfect number. The proof of Lemma 3.10 is complete. □

Completion of Proof of Theorem 3.1(b). Theorem 3.1(b) follows from Lemmas 3.1–3.10. □

Remark 3.4. Let n be as given in $(3.1a)$ and $b \geq 3$. Assume that n is a bi-unitary triperfect number. Then $(3.1b)$ is valid. Further suppose that n is not divisible by 3. If b is odd or $4|b$, then $3|\sigma^{**}(5^b)$. Also, if c is odd or $4|c$, then $9|\sigma^{**}(17^c)$. These are not possible in $(3.1b)$, and therefore it follows that $b = 2k$ and $c = 2\ell$, where $k \geq 3$ and ℓ are odd. Hence $b \geq 6$ and $c \geq 2$. We will consider the case $3 \nmid n$ in more detail in future.

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Appendix A Tables of $ord_p 13$

Let p denote an odd prime $\neq 13$. In the following table, r denotes the smallest positive integer such that $13^r \equiv 1 \pmod{p}$; that is, $r = ord_p 13$; s denotes the smallest positive integer such that $13^s \equiv -1 \pmod{p}$ if s exists; if s does not exist, that is, if $13^t + 1$ is not divisible by p for any positive integer t, the entry in column s will be filled up by dash sign. If r is even, then $s = r/2$, and if r is odd s does not exist.

$SL.No$	p	r	s	$SL.No$	p	r	s	$SL.No$	p	r	s
1	3	1	–	37	163	54	27	73	373	62	31
2	5	4	2	38	167	166	83	74	379	378	189
3	7	2	1	39	173	86	43	75	383	382	191
4	11	10	5	40	179	89	–	76	389	97	–
5	13	–	–	41	181	45	–	77	397	396	198
6	17	4	2	42	191	95	–	78	401	400	200
7	19	18	9	43	193	64	32	79	409	136	68
8	23	11	–	44	197	196	98	80	419	11	–
9	29	14	7	45	199	99	–	81	421	20	10
10	31	30	15	46	211	35	–	82	431	430	215
11	37	36	18	47	223	74	37	83	433	216	108
12	41	40	20	48	227	226	113	84	439	219	–
13	43	21	–	49	229	76	38	85	443	17	–
14	47	46	23	50	233	116	58	86	449	448	224
15	53	13	–	51	239	238	119	87	457	456	228
16	59	58	29	52	241	240	120	88	461	92	46
17	61	3	–	53	251	125	–	89	463	42	21
18	67	66	33	54	257	128	64	90	467	233	–
19	71	70	35	55	263	131	–	91	479	478	239
20	73	72	36	56	269	134	67	92	487	486	243
21	79	39	–	57	271	18	9	93	491	245	–
22	83	82	41	58	277	46	23	94	499	166	83
23	89	88	44	59	281	280	140	95	503	251	–
24	97	96	48	60	283	141	–	96	509	508	254
25	101	50	25	61	293	292	146	97	521	260	130
26	103	17	–	62	307	306	153	98	523	261	–
27	107	53	–	63	311	31	–	99	541	540	270
28	109	108	54	64	313	156	78	100	547	21	–
29	113	56	28	65	317	316	158	101	557	556	278
30	127	63	–	66	331	66	33	102	563	281	–
31	131	65	–	67	337	21	–	103	569	284	142
32	137	136	68	68	347	173	–	104	571	285	–
33	139	69	–	69	349	348	174	105	577	576	288
34	149	148	74	70	353	352	176	106	587	586	293
35	151	150	75	71	359	358	179	107	593	592	296
36	157	6	3	72	367	183	–	108	599	299	–

Appendix B Tables of ord_p17

Let p denote an odd prime $\neq 17$. In the following table, r denotes the smallest positive integer such that $17^r \equiv 1 \pmod{p}$; that is, $r = ord_p17$; s denotes the smallest positive integer such that $17^s \equiv -1 \pmod{p}$ if s exists; if s does not exist, that is, if $17^t + 1$ is not divisible by p for any positive integer t, the entry in column s will be filled up by dash sign. If r is even, then $s = r/2$, and if r is odd s does not exist.

$SL.No$	p	r	s	$SL.No$	p	r	s	$SL.No$	p	r	s
1	3	2	1	33	139	138	69	65	317	316	158
2	5	4	2	34	149	37	–	66	331	165	–
3	7	6	3	35	151	75	–	67	337	112	56
4	11	10	5	36	157	39	–	68	347	346	173
5	13	6	3	37	163	54	27	69	349	58	29
6	17	–	–	38	167	166	83	70	353	88	44
7	19	9	–	39	173	172	86	71	359	179	–
8	23	22	11	40	179	89	–	72	367	366	183
9	29	4	2	41	181	36	18	73	373	62	31
10	31	30	15	42	191	95	–	74	379	378	189
11	37	36	18	43	193	192	96	75	383	191	–
12	41	40	20	44	197	196	98	76	389	97	–
13	43	21	–	45	199	66	–	77	397	132	66
14	47	23	–	46	211	210	105	78	401	400	200
15	53	26	13	47	223	37	–	79	409	51	–
16	59	29	–	48	227	226	113	80	419	418	209
17	61	60	30	49	229	19	–	81	421	210	105
18	67	33	–	50	233	232	116	82	431	430	215
19	71	10	5	51	239	119	–	83	433	27	–
20	73	24	12	52	241	80	40	84	439	438	219
21	79	26	13	53	251	125	–	85	443	221	–
22	83	41	−1	54	257	32	16	86	449	448	224
23	89	44	22	55	263	131	–	87	457	38	19
24	97	96	48	56	269	268	134	88	461	230	115
25	101	10	5	57	271	135	–	89	463	231	–
26	103	51	–	58	277	276	138	90	467	233	–
27	107	106	53	59	281	140	70	91	479	478	239
28	109	36	18	60	283	282	141	92	487	486	243
29	113	112	56	61	293	73	–	93	491	49	–
30	127	63	–	62	307	3	–	94	499	498	249
31	131	130	65	63	311	310	155	95	503	502	251
32	137	68	34	64	313	312	156	96	509	127	–

Appendix C Factors of $13^t - 1$

$$13^{11} - 1 = \{\{2,2\},\{3,1\},\{23,1\},\{419,1\},\{859,1\},\{18041,1\}\}$$
$$13^{13} - 1 = \{\{2,2\},\{3,1\},\{53,1\},\{264031,1\},\{1803647,1\}\}$$
$$13^{17} - 1 = \{\{2,2\},\{3,1\},\{103,1\},\{443,1\},\{15798461357509,1\}\}$$
$$13^{21} - 1 = \{\{2,2\},\{3,2\},\{43,1\},\{61,1\},\{337,1\},\{547,1\},\{2714377,1\},\{5229043,1\}\}$$
$$13^{35} - 1 = \{\{2,2\},\{3,1\},\{211,1\},\{30941,1\},\{5229043,1\},\ldots\}$$
$$13^{39} - 1 = \{\{2,2\},\{3,2\},\{53,1\},\{61,1\},\{79,1\},\{1093,1\},\{4603,1\},\ldots\}$$
$$13^{45} - 1 = \{\{2,2\},\{3,3\},\{61,1\},\{181,1\},\{4651,1\},\{30941,1\},\{161971,1\},\ldots\}$$
$$13^{63} - 1 = \{\{2,2\},\{3,3\},\{43,1\},\{61,1\},\{127,1\},\{337,1\},\{547,1\},\ldots\}$$
$$13^{65} - 1 = \{\{2,2\},\{3,1\},\{53,1\},\{131,1\},\{1171,1\},\ldots\}$$
$$13^{69} - 1 = \{\{2,2\},\{3,2\},\{61,1\},\{139,1\},\{1381,1\},\{10903,1\},\ldots\}$$
$$13^{89} - 1 = \{\{2,2\},\{3,1\},\{179,1\},\{9257,1\},\ldots\}$$
$$13^{95} - 1 = \{\{2,2\},\{3,1\},\{191,1\},\{27361,1\},\{30941,1\},\ldots\}$$
$$13^{99} - 1 = \{\{2,2\},\{3,3\},\{23,1\},\{61,1\},\{199,1\},\{419,1\},\{859,1\},\{3169,1\},\ldots\}$$
$$13^{125} - 1 = \{\{2,2\},\{3,1\},\{251,1\},\{701,1\},\{9851,1\},\ldots\}$$
$$13^{131} - 1 = \{\{2,2\},\{3,1\},\{263,1\},\{135979,1\},\ldots\}$$
$$13^{141} - 1 = \{\{2,2\},\{3,2\},\{61,1\},\{283,1\},\{1693,1\},\{183959,1\},\ldots\}$$

Appendix D Factors of $13^t + 1$

$$13^{18} + 1 = \{\{2,1\},\{5,1\},\{17,1\},\{37,1\},\{28393,1\},\{428041,1\},\{1471069,1\}\}$$
$$13^{20} + 1 = \{\{2,1\},\{41,1\},\{14281,1\},\{29881,1\},\{543124566401,1\}\}.$$
$$13^{28} + 1 = \{\{2,1\},\{113,1\},\{14281,1\},\{480337846084945968040633 7,1\}\}$$
$$13^{32} + 1 = \{\{2,1\},\{193,1\},\{1601,1\},\{10433,1\},\{6867512045613988148256 2689,1\}\}$$
$$13^{36} + 1 = \{\{2,1\},\{73,1\},\{4177,1\},\{14281,1\},\ldots\}$$
$$13^{38} + 1 = \{\{2,1\},\{5,1\},\{17,1\},\{229,1\},\{94621,1\},\ldots\}$$
$$13^{44} + 1 = \{\{2,1\},\{89,1\},\{6073,1\},\{14281,1\},\ldots\}$$
$$13^{48} + 1 = \{\{2,1\},\{97,1\},\{2657,1\},\{88993,1\},\{441281,1\},\ldots\}$$
$$13^{54} + 1 = \{\{2,1\},\{5,1\},\{17,1\},\{37,1\},\{109,1\},\{28393,1\},\ldots\}$$
$$13^{58} + 1 = \{\{2,1\},\{5,1\},\{17,1\},\{233,1\},$$
$$\{1025438434909702346128619902547481080256923768726946695435273,1\}\}$$
$$13^{64} + 1 = \{\{2,1\},\{257,1\},\{3230593,1\},\ldots\}$$
$$13^{68} + 1 = \{\{2,1\},\{137,1\},\{409,1\},\{14281,1\},\ldots\}$$
$$13^{98} + 1 = \{\{2,1\},\{5,1\},\{17,1\},\{197,1\},\{2710681,1\},\ldots\}$$
$$13^{120} + 1 = \{\{2,1\},\{241,1\},\{1009,1\},\{407865361,1\},\ldots\}$$
$$13^{140} + 1 = \{\{2,1\},\{41,1\},\{113,1\},\{281,1\},\{14281,1\},\ldots\}$$
$$13^{146} + 1 = \{\{2,1\},\{5,1\},\{17,1\},\{293,1\},\{466462905277,1\},\ldots\}$$

Appendix E Factors of $17^t - 1$

$$
\begin{aligned}
17^{9} - 1 =& \{\{2,4\},\{19,1\},\{307,1\},\{1270657,1\}\} \\
17^{19} - 1 =& \{\{2,4\},\{229,1\},\{1103,1\},\{202607147,1\},\{291973723,1\}\} \\
17^{21} - 1 =& \{\{2,4\},\{43,1\},\{307,1\},\{13567,1\},\{25646167,1\},\{940143709,1\}\} \\
17^{23} - 1 =& \{\{2,4\},\{47,1\},\{265526182192280901629777481,1\} \\
17^{27} - 1 =& \{\{2,4\},\{19,1\},\{307,1\},\{433,1\},\{24733,1\},\{1270657,1\},\ldots\} \\
17^{29} - 1 =& \{\{2,4\},\{59,1\},\{7193,1\},\{6088087,1\},\{11658852700685942029849,1\}\} \\
17^{33} - 1 =& \{\{2,4\},\{67,1\},\{307,1\},\{3697,1\},\{976669,1\},\ldots\} \\
17^{37} - 1 =& \{\{2,4\},\{149,1\},\{223,1\},\{1016919604559540581,1\},\ldots\} \\
17^{39} - 1 =& \{\{2,4\},\{157,1\},\{307,1\},\{212057,1\},\{2919196853,1\},\ldots\} \\
17^{41} - 1 =& \{\{2,4\},\{83,1\},\{892079,1\},\{13365673,1\},\ldots\} \\
17^{49} - 1 =& \{\{2,4\},\{491,1\},\{883,1\},\{25646167,1\},\{474969439337,1\},\ldots\} \\
17^{51} - 1 =& \{\{2,4\},\{103,1\},\{307,1\},\{409,1\},\{10949,1\},\{1749233,1\},\ldots\} \\
17^{63} - 1 =& \{\{2,4\},\{19,1\},\{43,1\},\{127,1\},\{307,1\},\{13567,1\},\ldots\} \\
17^{73} - 1 =& \{\{2,4\},\{293,1\},\{1621745371,1\},\{3038535503,1\},\{319344640907,1\},\ldots\} \\
17^{75} - 1 =& \{\{2,4\},\{151,1\},\{307,1\},\{2551,1\},\{5101,1\},\{5351,1\},\ldots\} \\
17^{89} - 1 =& \{\{2,4\},\{179,1\},\{7121,1\},\{10859,1\},\ldots\} \\
17^{95} - 1 =& \{\{2,4\},\{191,1\},\{229,1\},\{1103,1\},\{88741,1\},\{202607147,1\},\ldots\} \\
17^{97} - 1 =& \{\{2,4\},\{389,1\},\{90976939813,1\},\{6588862794095439173,1\},\ldots\} \\
17^{119} - 1 =& \{\{2,4\},\{239,1\},\{2381,1\},\{3571,1\},\{10949,1\},\{16661,1\},\ldots\} \\
17^{127} - 1 =& \{\{2,4\},\{509,1\},\{2287,1\},\{19813,1\},\{9085073,1\},\ldots\} \\
17^{131} - 1 =& \{\{2,4\},\{263,1\},\{3670565424723533964145519323675507037326022240 \\
& 62626643758058951204255793967401304642571232969455436113641 0 \\
& 4958684168918108427651116351340245898427663672038782 9,1\}\} \\
17^{135} - 1 =& \{\{2,4\},\{19,1\},\{271,1\},\{307,1\},\{433,1\},\{3691,1\},\{24733,1\},\ldots\} \\
17^{165} - 1 =& \{\{2,4\},\{67,1\},\{307,1\},\{331,1\},\{3697,1\},\{46861,1\},\{88741,1\},\ldots\} \\
17^{179} - 1 =& \{\{2,4\},\{359,1\},\{18617,1\},\{121721,1\},\{1108776121,1\},\ldots\} \\
17^{191} - 1 =& \{\{2,4\},\{383,1\},\{3738211891,1\},\ldots\} \\
17^{221} - 1 =& \{\{2,4\},\{443,1\},\{10949,1\},\{151607,1\},\{212057,1\},\{1749233,1\},\ldots\} \\
17^{233} - 1 =& \{\{2,4\},\{467,1\}, \\
& \{66246329122722518021267669707378357816467757525685531863560 2076 \\
& 6775492769444675372228864152379392277413358880893612987903057911321843 \\
& 4295120243879391825646598579562732265304498078847938955250726380620973 \\
& 1783877309393965931437452025559757828597962954773024783566035419681902 \\
& 8096943763,1\}\}
\end{aligned}
$$

Appendix F Factors of $17^t + 1$

$$17^{12}+1 =\{\{2,1\},\{73,1\},\{1321,1\},\{41761,1\},\{72337,1\}\}$$
$$17^{16}+1 =\{\{2,1\},\{257,1\},\{1801601,1\},\{52548582913,1\}\}$$
$$17^{18}+1 =\{\{2,1\},\{5,1\},\{29,1\},\{37,1\},\{109,1\},\{181,1\},\{2089,1\},\{83233,1\},\ldots\}$$
$$17^{20}+1 =\{\{2,1\},\{41,1\},\{41761,1\},\{1186844128302568601,1\}\}$$
$$17^{22}+1 =\{\{2,1\},\{5,1\},\{29,1\},\{89,1\},\{25741,1\},\ldots\}$$
$$17^{30}+1 =\{\{2,1\},\{5,2\},\{29,1\},\{61,1\},\{541,1\},\{21881,1\},\ldots\}$$
$$17^{34}+1 =\{\{2,1\},\{5,1\},\{29,1\},\{137,1\},\{1361,1\},\{2698649,1\},\ldots\}$$
$$17^{40}+1 =\{\{2,1\},\{241,1\},\{18913,1\},\{184417,1\},\ldots\}$$
$$17^{44}+1 =\{\{2,1\},\{353,1\},\{41761,1\},\{4578289,1\},\ldots\}$$
$$17^{48}+1 =\{\{2,1\},\{97,1\},\{257,1\},\{1120513,1\},\{1801601,1\},\{53160769,1\},\ldots\}$$
$$17^{56}+1 =\{\{2,1\},\{113,1\},\{337,1\},\{18913,1\},\{184417,1\},\ldots\}$$
$$17^{66}+1 =\{\{2,1\},\{5,1\},\{29,1\},\{89,1\},\{397,1\},\{19801,1\},\ldots\}$$
$$17^{70}+1 =\{\{2,1\},\{5,2\},\{29,1\},\{281,1\},\{21881,1\},\{63541,1\},\ldots\}$$
$$17^{86}+1 =\{\{2,1\},\{5,1\},\{29,1\},\{173,1\},\{2237,1\},\{26673589,1\},\ldots\}$$
$$17^{96}+1 =\{\{2,1\},\{193,1\},\{1409,1\},\{165569,1\},\{2533128442908097,1\},\ldots\}$$
$$17^{98}+1 =\{\{2,1\},\{5,1\},\{29,1\},\{197,1\},\{578789,1\},\{5766433,1\},\ldots\}$$
$$17^{116}+1 =\{\{2,1\},\{233,1\},\{41761,1\},\{244297,1\},\ldots\}$$
$$17^{134}+1 =\{\{2,1\},\{5,1\},\{29,1\},\{269,1\},\{522580700249,1\},\ldots\}$$
$$17^{138}+1 =\{\{2,1\},\{5,1\},\{29,1\},\{277,1\},\{83233,1\},\{102121,1\},\ldots\}$$
$$17^{156}+1 =\{\{2,1\},\{73,1\},\{313,1\},\{1321,1\},\{41761,1\},\{72337,1\},\ldots\}$$
$$17^{158}+1 =\{\{2,1\},\{5,1\},\{29,1\},\{317,1\},\{6637,1\},\{155473,1\},\ldots\}$$
$$17^{200}+1 =\{\{2,1\},\{241,1\},\{401,1\},\{18913,1\},\{184417,1\},\{3583912721,1\},\ldots\}$$
$$17^{224}+1 =\{\{2,1\},\{449,1\},\{1409,1\},\{165569,1\},\ldots\}$$