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A remark on the Tribonacci sequences

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Abstract. One of the first extensions of the Fibonacci sequence are the Tribonacci sequences. In the paper, some of their properties are discussed.
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1 Introduction and main result

The well-known Fibonacci sequence is an object of a lot extensions. One of the first of them is the Tribonacci sequence, see [6], introduced in 1963 by Mark Feinberg, a fourteen-year-old student in the ninth grade of the Susquehanna Township Junior High School (USA). He introduced the Tribonacci sequence in three different forms. The first of them is:

1, 1, 1, 3, 5, 9, 17, 31, 57, 105, 193, 355, 653, 1201, 2209, 4063, 7473, 13745, 25281, 46499, 85525, 157305, 289329, 532159, 978793, 1800281, 3311233, 6090307, 11201821, ...

The second form, which is very often used (see, e.g., [9]), is

0, 0, 1, 1, 2, 4, 7, 13, 24, 44, 81, 149, 274, 504, 927, 1705, 3136, 5768, 10609, 19513, 35890, 66012, 121415, 223317, 410744, 755476, 1389537, 2555757, 4700770, 8646064, 15902591, 29249425, 53798080, 98950096, 181997601, 334745777, 615693474, 1132436852,2082876103, 3831006429, ...

In practice, the third form coincides up to the enumeration with the second one:

0, 1, 1, 2, 4, 7, 13, 24, 44, 81, 149, 274, 504, 927, 1705, 3136, 5768, 10609, 19513, 35890, 66012, 121415, 223317, 410744, 755476, 1389537, 2555757, 4700770, 8646064, 15902591, 29249425, 53798080, 98950096, 181997601, 334745777, 615693474, 1132436852, 2082876103, 3831006429, ...

The formula for the n-th number is given by A. Shannon in [8].

Let the elements of each of the three Tribonacci sequence forms be denoted as $\{T_i\}_{i\geq 0}$.

Theorem 1. *For each natural number $n \geq 0$,*

$$T_n + T_{n+13} + T_{n+26} \equiv 0 \pmod{3}.$$

Proof. Let $n = 0, 1, 2$. Then, for the first Tribonacci sequence form we obtain

$$T_0 + T_{13} + T_{26} = 1 + 1201 + 3311233 = 3312435 \equiv 0 \ (\text{mod } 3),$$

$$T_1 + T_{14} + T_{27} = 1 + 2209 + 6090307 = 6092517 \equiv 0 \ (\text{mod } 3),$$

$$T_2 + T_{15} + T_{28} = 1 + 4063 + 11201821 = 11205885 \equiv 0 \ (\text{mod } 3).$$

For the second Tribonacci sequence form we obtain

$$T_0 + T_{13} + T_{26} = 0 + 504 + 1389537 = 1390041 \equiv 0 \ (\text{mod } 3),$$

$$T_1 + T_{14} + T_{27} = 0 + 927 + 2555757 = 2556684 \equiv 0 \ (\text{mod } 3),$$

$$T_2 + T_{15} + T_{28} = 1 + 1705 + 4700770 = 11205885 \equiv 0 \ (\text{mod } 3).$$

For the third Tribonacci sequence form we obtain

$$T_0 + T_{13} + T_{26} = 0 + 927 + 2555757 = 2556684 \equiv 0 \ (\text{mod } 3),$$

$$T_1 + T_{14} + T_{27} = 1 + 1705 + 4700770 = 11205885 \equiv 0 \ (\text{mod } 3),$$

$$T_2 + T_{15} + T_{28} = 1 + 3136 + 8646064 = 8649201 \equiv 0 \ (\text{mod } 3).$$

Let us assume that the assertion is valid for each of the three Tribonacci sequence forms and for the natural numbers less than or equal to $n + 2$, i.e.,

$$T_n + T_{n+13} + T_{n+26} \equiv 0 \ (\text{mod } 3),$$

$$T_{n+1} + T_{n+14} + T_{n+27} \equiv 0 \ (\text{mod } 3),$$

$$T_{n+2} + T_{n+15} + T_{n+28} \equiv 0 \ (\text{mod } 3).$$

Then

$$\begin{aligned} & T_{n+3} + T_{n+16} + T_{n+29} \\ &= (T_n + T_{n+1} + T_{n+2}) + (T_{n+13} + T_{n+14} + T_{n+15}) + (T_{n+26} + T_{n+27} + T_{n+28}) \\ &= (T_n + T_{n+13} + T_{n+26}) + (T_{n+1} + T_{n+14} + T_{n+27}) + (T_{n+2} + T_{n+15} + T_{n+28}) \\ &\equiv 0 \ (\text{mod } 3). \end{aligned}$$

□

It is suitable to re-write the above sequences as follows

1	1	1	3	5	9	17	31	57	105	193	355	653
1201	2209	4063	7473	13745	25281	46499	85525	157305	289329	532159	978793	1800281
...	...	...	...	...	...	...	...	...	...	...	...	...

0	0	1	1	2	4	7	13	24	44	81	149	274
504	927	1705	3136	5768	10609	19513	35890	66012	121415	223317	410744	755476
...	...	...	...	...	...	...	...	...	...	...	...	...

0	1	1	2	4	7	13	24	44	81	149	274	504
927	1705	3136	5768	10609	19513	35890	66012	121415	223317	410744	755476	1389537
...	...	...	...	...	...	...	...	...	...	...	...	...

respectively. Now, we can prove by the above manner that for each natural number $n \geq 0$ for the first sequence are valid:

$$T_{13n+3} \equiv 0 \pmod 3,$$

$$T_{13n+5} \equiv 0 \pmod 9,$$

$$T_{13n+8} \equiv 0 \pmod 3,$$

$$T_{13n+9} \equiv 0 \pmod 3;$$

for the second sequence are valid:

$$T_{13n} \equiv 0 \pmod 9,$$

$$T_{13n+1} \equiv 0 \pmod 9,$$

$$T_{13n+8} \equiv 0 \pmod 3,$$

$$T_{13n+10} \equiv 0 \pmod 9;$$

and for the third sequence are valid:

$$T_{13n} \equiv 0 \pmod 9,$$

$$T_{13n+7} \equiv 0 \pmod 3,$$

$$T_{13n+9} \equiv 0 \pmod 9,$$

$$T_{13n+12} \equiv 0 \pmod 9.$$

2 Conclusion

In a next research, we will check whether the 2-Fibonacci (see [1, 2, 5, 7]), 3-Fibonacci (see [3, 5]) and 2-Tribonacci (see [4, 5]) sequences have similar properties.

References

[1] Atanassov, K., Atanassova, L., & Sasselov, D. (1985). A new perspective to the generalization of the Fibonacci sequence. *The Fibonacci Quarterly*, 23 (1), 21–28.

[2] Atanassov, K. (1986). On a second new generalization of the Fibonacci sequence. *The Fibonacci Quarterly*, 24 (4), 362–365.

[3] Atanassov, K. (1989). On a generalization of the Fibonacci sequence in the case of three sequences. *The Fibonacci Quarterly*, 27 (1), 7–10.

[4] Atanassov, K., Hlebarova, J., & Mihov, S. (1992). Recurrent formulas of the generalized Fibonacci and Tribonacci sequences. *The Fibonacci Quarterly*, 30 (1), 77–79.

[5] Atanassov, K., Atanassova, V., Shannon, A., & Turner, J. (2002). *New Visual Perspectives on Fibonacci Numbers*. World Scientific, New Jersey.

[6] Feinberg, M. (1963). Fibonacci–Tribonacci. *The Fibonacci Quarterly*, 1 (3), 71—74.

[7] Lee, J.-Z.,& Lee, J.-S. (1987). Some properties of the generalization of the Fibonacci sequence. *The Fibonacci Quarterly*, 25 (2), 111–117.

[8] Shannon, A. (1977). Tribonacci numbers and Pascal's pyramid. *The Fibonacci Quarterly*, 15 (3), 268-275.

[9] Spickerman, W. (1982). Binet's formula for the Tribonacci sequence. *The Fibonacci Quarterly*, 20 (2), 118–120.