Notes on Number Theory and Discrete Mathematics
Print ISSN 1310–5132, Online ISSN 2367–8275
Vol. 25, 2019, No. 3, 97–101
DOI: 10.7546/nntdm.2019.25.3.97-101

A note on generalized Leonardo numbers

A. G. Shannon

Warrane College, The University of New South Wales,
Kensington, NSW 2033, Australia
e-mail: tshannon38@gmail.com, t.shannon@warrane.unsw.edu.au

Received: 27 May 2019 **Accepted:** 29 July 2019

Abstract: This is essentially an expository paper which sheds new light on existing knowledge due to Asveld and Horadam and suggests ideas for extension and generalization based on the approaches of these authors.
Keywords: Fibonacci sequence, Pell sequence, Homogeneous and inhomogeneous recurrence relations, Pascal's triangle.
2010 Mathematics Subject Classification: 11B37, 11B39.

1 Introduction

In a recent paper Catarino and Borges introduced some properties of the Leonardo numbers $\{Le_n\}$ [1]. They defined these by the second order inhomogeneous recurrence relation

$$Le_n = Le_{n-1} + Le_{n-2} + 1, \quad n \geq 2, \tag{1.1}$$

with initial conditions $Le_0 = Le_1 = 1$ [2].

The Leonardo number properties are analogues of Fibonacci properties and related to the better known Fibonacci numbers.

In that spirit, we extend some of these to generalized Leonardo numbers defined in the spirit of Alwyn Horadam's generalized Fibonacci sequences $\{H_n\}$ [3] and $\{w_n\}$ [4]. To do this, we first look at a generalization of $\{H_n\}$ [5] and then a generalization of $\{w_n\}$ [6]. In both cases, we shall repeat some of their material albeit in slightly different ways and altered notation.

2 Asveld's extension

Asveld [5] in effect considered this inhomogeneous extension of (1.1):

$$H_n = H_{n-1} + H_{n-2} + \sum_{j=0}^{k} \alpha_j n^j , \tag{2.1}$$

in which $\{k, \alpha_0, \alpha_1, \ldots, \alpha_k\} \in \mathbb{Z}$ and $H_0 = H_1 = 1$. The total solution of this is

$$H_n = H_n^{(1)} + H_n^{(2)}, \tag{2.2}$$

with the homogeneous part given by

$$H_n^{(1)} = C_1\phi_1^n + C_2\phi_2^n, \tag{2.3}$$

where, as usual, ϕ_1, ϕ_2 are the zeroes of the characteristic equation associated with (2.1) and C_1, C_2 arise from the initial conditions. We can represent a particular solution by

$$H_n^{(2)} = \sum_{i=0}^{k} A_i n^i,, \tag{2.4}$$

so that for $m \geq i$,

$$\begin{aligned}
0 &= \sum_{i=0}^{k} A_i n^i - \sum_{i=0}^{k} A_i (n-1)^i - \sum_{i=0}^{k} A_i (n-2)^i - \sum_{i=0}^{k} \alpha_i n^i \\
&= \sum_{i=0}^{k} A_i n^i - \sum_{i=0}^{k}\left(\sum_{l=0}^{j} A_i \binom{i}{l} (-1)^{i-l}\left(1+2^{i-l}\right) n^l \right) - \sum_{i=0}^{k} \alpha_i n^i \\
&= \sum_{i=0}^{k} A_i n^i - \sum_{i=0}^{k}\left(\sum_{m=i}^{k} A_m \binom{m}{i} (-1)^{m-i}\left(1+2^{m-i}\right) \right) n^i - \sum_{i=0}^{k} \alpha_i n^i
\end{aligned}$$

so that, continuing to follow Asveld,

$$\begin{aligned}
A_i &= \alpha_i + \sum_{m=i}^{k} \binom{m}{i} (-1)^{m-i}\left(1+2^{m-i}\right) A_m \\
&= \alpha_i + \sum_{m=i}^{k} \beta_{im} A_m \\
&= -\sum_{j=i}^{k} a_{ij}\alpha_j
\end{aligned}$$

for notational convenience, and with

$$\beta_{im} = \binom{m}{i} (-1)^{m-i}\left(1+2^{m-i}\right)$$

and $\beta_{ii} = 2$; then $a_{ii} = 1$, and if $j > i$, then

$$a_{ij} = -\sum_{m=i+1}^{j} \beta_{im} a_{mj}$$

from which Asveld obtained

$$H_n = (1+\Lambda_k)F_n + \lambda_k F_{n-1} - \sum_{j=0}^{k} \alpha_j p_j(n), \tag{2.5}$$

where

- $\{F_n\}$ is the sequence of Fibonacci numbers,
- Λ_k is a linear combination of $\alpha_0, \alpha_1, \ldots, \alpha_k : \Lambda_k = \sum_{j=0}^{k} a_{oj}\alpha_j$,
- λ_k is a linear combination of $\alpha_1, \alpha_2, \ldots, \alpha_k$, $\lambda_k = \sum_{j=1}^{k}\left(\sum_{i=1}^{j} a_{ij} \right)\alpha_j$,
- for each j $(0 \leq j \leq k)$, $p_j(n)$ is a polynomial of degree j: $p_j(n) = \sum_{i=0}^{j} a_{i,j} n^i$,

for each of which Asveld developed specific examples. For our purpose here it is sufficient to point out that (2.5) is effectively a generalization of Proposition 2.2 of [1].

3 Horadam's generalized sequence

An inhomogeneous Horadam extention of (1.1) is then:

$$w_n = pw_{n-1} - qw_{n-2} + (p-q-1)\sum_{j=0}^{k} \alpha_j n^j, \tag{3.1}$$

in which $\{k, \alpha_0, \alpha_1, \ldots, \alpha_k\} \in \mathbb{Z}$ and $w_0 = b, w_1 = pb - qa$. The total solution of this in following Asveld's steps is

$$w_n = (1+\Lambda_k)U_n + (\lambda_k - qa)U_{n-1} - \sum_{j=0}^{k} \alpha_j p_j(n), \tag{3.2}$$

where λ_k is now a linear combination of $\alpha_1, \alpha_2, \ldots, \alpha_k$:

$$\lambda_k = \sum_{j=1}^{k}\left(\sum_{i=1}^{j} a_{ij}\right)\alpha_j - (p-1)\sum_{j=0}^{k} a_{0j}\alpha_j,$$

see [6]. Equation (3.2) corresponds to (2.5) above and is also an analogue of Proposition (2.2) of [1], and in which $\{U_n\}$ satisfies the recurrence relation

$$U_n - pU_{n-1} - qU_{n-1}, n \geq 2,$$

with initial conditions $U_0 = 1$, $U_1 = p$. Pell numbers were then considered in [6] ($p = q = -1$, initial values, 1 and 2) with Tables 1 and 2 arising for the Fibonacci and Pell cases, respectively.

$n \backslash m$	0	1	2	3	4	5	6	7	...
0	1	3	13	81	673	6993	87193	1268361	
1		1	6	39	324	3365	41958	610351	
2			1	9	78	810	10095	146853	
3				1	12	130	1620	23555	
4					1	15	195	2835	
5						1	18	273	
6							1	21	
7								1	
...									...

Table 1. $a_{n,m}$ for Fibonacci numbers

$n \backslash m$	0	1	2	3	4	5	6	7	...
0	1	4	26	250	3206	51394	988646	221887890	
1		1	8	78	1000	16030	308364	6920522	
2			1	12	156	2500	48090	1079274	
3				1	16	260	5000	112210	
4					1	20	390	8750	
5						1	24	546	
6							1	28	
7								1	
...									...

Table 2. $a_{n,m}$ for Pell numbers

Although the authors of [5, 6] did not discern patterns, they do appear with the aid of Table 3 which is an array formed by moving the Pascal triangle though 45° – a Pascal matrix with the usual Pascal rows as the diagonals [7]. The patterns fall out in Tables 1 and 2 when they are displayed this way as we illustrate in Tables 4 and 5.

i\i	1	2	3	4	5	6	7	...
1	1	1	1	1	1	1	1	
2	1	2	3	4	5	6	7	
3	1	3	6	10	15	21	28	
4	1	4	10	20	35	56	84	
5	1	5	15	35	70	126	210	
6	1	6	21	56	126	252	468	
7	1	7	28	84	216	468	936	
...								...

Table 3. Pascal Matrix

The patterns among the sequences are now set out in Table 4 and 5.

$a_{i,i}$	1	1	1	1	1	1	1	1	$\equiv$	$a_{0,0} \times M_{1,i}$
$a_{i,i+1}$		3	6	9	12	15	18	21	$\equiv$	$a_{0,1} \times M_{2,i}$
$a_{i,i+2}$			13	39	78	130	195	273	$\equiv$	$a_{0,2} \times M_{3,i}$
$a_{i,i+3}$				81	324	810	1620	2835	$\equiv$	$a_{0,3} \times M_{4,i}$
$a_{i,i+4}$					673	3365	10095	2355	$\equiv$	$a_{0,4} \times M_{5,i}$
$a_{i,i+5}$						6993	41958	146853	$\equiv$	$a_{0,5} \times M_{6,i}$
$a_{i,i+6}$							87193	610351	$\equiv$	$a_{0,6} \times M_{7,i}$
$a_{i,i+7}$								1268361	$\equiv$	$a_{0,7} \times M_{8,i}$

Table 4. $\{a_{n,m}\}$ for Fibonacci numbers from Table 1

$a_{i,i}$	1	1	1	1	1	1	1	1	$\equiv$	$a_{0,0} \times M_{1,i}$
$a_{i,i+1}$		4	8	12	16	20	24	28	$\equiv$	$a_{0,1} \times M_{2,i}$
$a_{i,i+2}$			26	78	156	260	390	546	$\equiv$	$a_{0,2} \times M_{3,i}$
$a_{i,i+3}$				250	1000	2500	5000	8750	$\equiv$	$a_{0,3} \times M_{4,i}$
$a_{i,i+4}$					3206	16030	48090	112210	$\equiv$	$a_{0,4} \times M_{5,i}$
$a_{i,i+5}$						51394	308364	1079274	$\equiv$	$a_{0,5} \times M_{6,i}$
$a_{i,i+6}$							988646	6920522	$\equiv$	$a_{0,6} \times M_{7,i}$
$a_{i,i+7}$								221887890	$\equiv$	$a_{0,7} \times M_{8,i}$

Table 4. $\{a_{n,m}\}$ for Pell numbers from Table 2

References

[1] Catarino, P., & Borges, A. (2019). On Leonardo numbers. A*cta Mathematica Universitatis Comenianae*, 1–12. Available online at: `http://www.iam.fmph.uniba.sk/amuc/ojs/index.php/amuc/article/view/1005/650`.

[2] Sloane, N. J. A. (2019). *The on-line encyclopedia of integer sequences.* The OEIS Foundation Inc. Available online at: `http://oeis.org`.

[3] Horadam, A. F. (1961). A Generalized Fibonacci sequence. *American Mathematical Monthly*, 68 (5), 455–459.

[4] Horadam, A. F. (1965). Basic properties of a certain generalized sequence of numbers. *The Fibonacci Quarterly*, 3 (3), 161–176.

[5] Asveld, P. R. J. (1987). A family of Fibonacci-like sequences. *The Fibonacci Quarterly*, 25 (1), 81–83.

[6] Horadam, A. F., & Shannon, A. G. (1988). Asveld's polynomials. In Philippou, A.N., Horadam, A.F. & Bergum, G. E. (eds). *Applications of Fibonacci Numbers, Volume 2*. Dordrecht: Kluwer, 163–176.

[7] Bondarenko, B. A. (1993). *Generalized Pascal Triangles and Pyramids: Their Fractals, Graphs and Applications. (Translated by R. C. Bollinger.)* Santa Clara, CA: The Fibonacci Association.