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On applications of blending generating functions of q-Apostol-type polynomials

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Abstract: Motivated by Kurt's blending generating functions of q-Apostol polynomials [16], we investigate some new identities and relations. We also aim to derive several new connections between these polynomials and generalized q-Stirling numbers of the second kind. Additionally, by making use of the fermionic p-adic integral over the p-adic numbers field, some relationships including unified Apostol-type q-polynomials and classical Euler numbers are obtained.

Keywords: q-calculus, Apostol–Bernoulli polynomials, Apostol–Euler polynomials, Apostol–Genocchi polynomials, Stirling numbers of second kind, Fermionic p-adic integral, p-adic numbers.

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1 Introduction

Special polynomials and numbers possess a lot of importances in many fields of mathematics, physics, engineering and other related disciplines including the topics such as differential equations, mathematical analysis, functional analysis, mathematical physics, quantum mechanics and so on. One of the most considerable polynomials in special polynomials is the Apostol-type polynomials that is firstly considered by Apostol [1] (also extensively investigated by Srivastava in [32]). Since then, these type polynomials and several generalizations of them have been studied and investigated by many mathematicians, see [2–5, 7, 8, 14–21, 23, 26, 27, 30, 35–37]. For example, Ozden [28] gave unification of Genocchi, Bernoulli and Euler polynomials. By the motivation of Ozden's work, Özarslan [26] introduced unified Apostol–Bernoulli, Apostol–Euler and Apostol–Genocchi polynomials. Recently, Kurt [16] also introduced and studied q-Apostol-type polynomials.

Let us now give briefly some definitions and notations.

By means of the following Taylor series expansions about $z = 0$, the Apostol–Bernoulli polynomials $B_n(x;\lambda)$, the Apostol–Euler polynomials $E_n(x;\lambda)$ and the Apostol–Genocchi polynomials $G_n(x;\lambda)$ are defined by

$$\sum_{n=0}^{\infty} B_n(x;\lambda)\frac{z^n}{n!} = \frac{z}{\lambda e^z - 1}e^{xz} \quad (\lambda \in \mathbb{C}; |z| < |\log \lambda|),$$

$$\sum_{n=0}^{\infty} E_n(x;\lambda)\frac{z^n}{n!} = \frac{2}{\lambda e^z + 1}e^{xz} \quad (\lambda \in \mathbb{C}; |z| < |\log(-\lambda)|)$$

and

$$\sum_{n=0}^{\infty} G_n(x;\lambda)\frac{z^n}{n!} = \frac{2z}{\lambda e^z + 1}e^{xz} \quad (\lambda \in \mathbb{C}; |z| < |\log(-\lambda)|)$$

Note that

$$B_n(0;\lambda) := B_n(\lambda),\ E_n(0;\lambda) := E_n(\lambda) \text{ and } G_n(0;\lambda) := G_n(\lambda)$$

are known as, respectively, Apostol–Bernoulli, Apostol–Euler and Apostol–Genocchi numbers. For further information about the aforementioned polynomials, see [3,7,8,18–21,26,35,37].When $\lambda = 1$, these polynomials and numbers reduce to the classical form, look at [10,13,28,29,32–34] for details.

In this paper, the usual notations $\mathbb{C}$, $\mathbb{R}$, $\mathbb{Z}$, $\mathbb{N}$ and $\mathbb{N}_0$ refer to the set of all complex numbers, the set of the all real numbers, the set of the all integers, the set of the all natural numbers and the set of all nonnegative integers, respectively, in the content of this paper.

The ordinary quantum calculus, denoted by q-calculus, has been widely studied and developed for a long while by a lot of mathematicians, economists, engineers and physicists. The development of q-calculus arises from the many applications in several scientific fields such as combinatorics, quantum mechanics, special functions, quantum gravity, umbral calculus and other related fields. One of the significant branches of q-calculus is the q-special numbers and polynomials (see [5, 6, 11, 12, 14–17, 22–25, 30, 31, 33] for more information related these issues).

The following notations about q-calculus are taken from [9].

The q-numbers $[x]_q$ and the q-derivative $D_q f(x)$ are defined as

$$[x]_q = \begin{cases} \frac{1-q^x}{1-q}, & \text{if } q \neq 1, \\ x, & \text{if } q = 1 \end{cases} \quad \text{and } D_q f(x) = \frac{d_q f(x)}{d_q x} = \begin{cases} \frac{f(x)-f(qx)}{(1-q)x} & \text{if } q \neq 1 \text{ and } x \neq 0, \\ f'(x) & \text{if } q = 1, \\ f'(0) & \text{if } x = 0, \end{cases} \tag{1}$$

seeing $x \in \mathbb{R}$ (or $x \in \mathbb{C}$).

The q-binomial coefficients are defined for the positive integers n, k as

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q!\,[n-k]_q!}$$

where $[n]_q! = [1]_q [2]_q [3]_q \cdots [n-1]_q [n]_q \quad (n \in \mathbb{N})$ with $[0]_q! = 1$.

The following expressions can be easily derived using (1):

$$D_q (g(x) f(x)) = f(x) D_q g(x) + g(qx) D_q f(x) = g(x) D_q f(x) + f(qx) D_q g(x) \tag{2}$$

and

$$D_q \left(\frac{g(x)}{f(x)} \right) = \frac{f(qx) D_q g(x) - g(qx) D_q f(x)}{f(x) f(qx)} = \frac{f(x) D_q g(x) - g(x) D_q f(x)}{f(x) f(qx)}. \tag{3}$$

The q-generalization of $(x+y)^n$ is defined by

$$(x+a)_q^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\binom{n-k}{2}} x^k a^{n-k}. \tag{4}$$

The two different types of the q-exponential functions are given by

$$e_q(z) = \sum_{n=0}^{\infty} \frac{z^n}{[n]_q!} \text{ and } E_q(z) = \sum_{n=0}^{\infty} q^{\binom{n}{2}} \frac{z^n}{[n]_q!} \quad (z \in \mathbb{C} \text{ with } |z| < 1) \tag{5}$$

which possess the following features

$$e_{q^{-1}}(x) = E_q(x),\ e_q(x) E_q(-x) = 1, \tag{6}$$

and q-derivative representations

$$D_q e_q(x) = e_q(x) \text{ and } D_q E_q(x) = E_q(qx). \tag{7}$$

For x and y in concujtion with the commuting technique $yx = qxy$, we note that

$$e_q(x+y) = e_q(x) e_q(y). \tag{8}$$

The q-definite integral is defined as

$$\int_0^{\xi} f(x)\, d_q x = (1-q)\, \xi \sum_{k=0}^{\infty} q^k f\left(q^k \xi\right) \text{ with } \int_{\xi}^{\varpi} f(x)\, d_q x = \int_0^{\varpi} f(x)\, d_q x - \int_0^{\xi} f(x)\, d_q x. \tag{9}$$

The Apostol-type q-Bernoulli polynomials $\mathcal{B}_{n,q}^{(\alpha)}(x,y;\lambda)$ of order $\alpha \in \mathbb{N}_0$, the Apostol-type q-Euler polynomials $\mathcal{E}_{n,q}^{(\alpha)}(x,y;\lambda)$ of order $\alpha \in \mathbb{N}_0$ and the Apostol-type q-Genocchi polynomials $\mathcal{G}_{n,q}^{(\alpha)}(x,y;\lambda)$ of order $\alpha \in \mathbb{N}_0$ are defined by the following generating functions:

$$\begin{aligned}
\sum_{n=0}^{\infty} \mathcal{B}_{n,q}^{(\alpha)}(x,y;\lambda)\frac{z^n}{[n]_q!} &= \left(\frac{z}{\lambda e_q(z)-1}\right)^{\alpha} e_q(xz)\,E_q(yz),\,(\ |z|<|\log\lambda|\,,\,1^{\alpha}:=1)\\
\sum_{n=0}^{\infty} \mathcal{E}_{n,q}^{(\alpha)}(x,y;\lambda)\frac{z^n}{[n]_q!} &= \left(\frac{2}{\lambda e_q(z)+1}\right)^{\alpha} e_q(xz)\,E_q(yz),\,(|z|<|\log(-\lambda)|\,,\,1^{\alpha}:=1)\\
\sum_{n=0}^{\infty} \mathcal{G}_{n,q}^{(\alpha)}(x,y;\lambda)\frac{z^n}{[n]_q!} &= \left(\frac{2z}{\lambda e_q(z)+1}\right)^{\alpha} e_q(xz)\,E_q(yz),\,(|z|<|\log(-\lambda)|\,,\,1^{\alpha}:=1)
\end{aligned}$$

where α and λ are suitable (complex or real) parameters and $q \in \mathbb{C}$ with $0<|q|<1$ (see [15,16]). Putting $x=0$ and $y=0$, we have $\mathcal{B}_{n,q}^{(\alpha)}(0,0;\lambda) := \mathcal{B}_{n,q}^{(\alpha)}(\lambda)$, $\mathcal{E}_{n,q}^{(\alpha)}(0,0;\lambda) := \mathcal{E}_{n,q}^{(\alpha)}(\lambda)$ and $\mathcal{G}_{n,q}^{(\alpha)}(0,0;\lambda) := \mathcal{G}_{n,q}^{(\alpha)}(\lambda)$ which are termed, respectively, n-th Apostol-type q-Bernoulli number of order α, n-th Apostol-type q-Euler number of order α and n-th Apostol-type q-Genocchi number of order α.

In the next sections, we shall perform to derive and develop several properties of the family of unified Apostol-type q-Genocchi, q-Euler and q-Bernoulli polynomials of order α. Moreover, some relationships for the generalized q-Stirling numbers of the second kind of order v and unified Apostol-type q-Genocchi, q-Euler and q-Bernoulli polynomials of order α are derived. By making use of the fermionic p-adic integral over the p-adic number fields, formulas between the family of unified Apostol-type q-Genocchi, q-Euler and q-Bernoulli polynomials of order α and classical Euler numbers are derived appropriately.

2 Applications of blending generating functions of q-Apostol-type polynomials

This section provides some properties and identities for unified Apostol-type q-Genocchi, q-Euler and q-Bernoulli polynomials of order α defined by Kurt in [16].

Unified Apostol-type q-Genocchi, q-Euler and q-Bernoulli polynomials of order α are defined by Kurt [16] as follows:

$$\sum_{n=0}^{\infty} \mathcal{P}_{n,\beta,q}^{(\alpha)}(x,y,k,a,b)\frac{z^n}{[n]_q!} = \left(\frac{2^{1-k}z^k}{\beta^b e_q(z)-a^b}\right)^{\alpha} e_q(xz)\,E_q(yz), \tag{10}$$

$$\left(|z|<2\pi \textit{ when } \beta=a;\,|z|<\left|\beta\log\left(\frac{b}{a}\right)\right| \textit{ when } \beta\neq a;\,\alpha,k\in\mathbb{N}_0;\,a,b\in\mathbb{R}\backslash\{0\};\,\beta\in\mathbb{C}\right).$$

When $y=0$ and q goes to 1^-, then the polynomials $\mathcal{P}_{n,\beta,q}^{(\alpha)}(x,y,k,a,b)$ turn out to be the following polynomials defined by Özarslan [26]:

$$\sum_{n=0}^{\infty} \mathcal{P}_{n,\beta}^{(\alpha)}(x;k,a,b)\frac{z^n}{n!} = \left(\frac{2^{1-k}z^k}{\beta^b e^z - a^b}\right)^{\alpha} e^{zx}, \tag{11}$$

$$\left(|z| < 2\pi \textit{ when } \beta = a; |z| < \left|\beta \log\left(\frac{b}{a}\right)\right| \textit{ when } \beta \neq a; \alpha, k \in \mathbb{N}_0; a, b \in \mathbb{R}\backslash\{0\}; \beta \in \mathbb{C}\right).$$

We note that (see [16])

$$\begin{aligned}
\mathcal{P}_{n,\beta,q}^{(1)}(x,y,k,a,b) &= \mathcal{P}_{n,\beta,q}(x,y,k,a,b), \ \mathcal{P}_{n,\lambda,q}^{(\alpha)}(x,y,1,1,1) = \mathcal{B}_{n,q}^{(\alpha)}(x,y;\lambda), \\
\mathcal{P}_{n,\lambda,q}^{(\alpha)}(x,y,0,-1,1) &= \mathcal{E}_{n,q}^{(\alpha)}(x,y;\lambda), \ \mathcal{P}_{n,\frac{\lambda}{2},q}^{(\alpha)}\left(x,y,1,-\frac{1}{2},1\right) = \mathcal{G}_{n,q}^{(\alpha)}(x,y;\lambda).
\end{aligned}$$

Moreover, $\mathcal{P}_{n,\beta,q}^{(\alpha)}(x,y,k,a,b)$ satisfies the following properties (see [16]):

$$\begin{aligned}
D_{q;x}\mathcal{P}_{n,\beta}^{(\alpha)}(x,y,k,a,b) &= [n]_q \mathcal{P}_{n-1,\beta,q}^{(\alpha)}(x,y,k,a,b) \\
D_{q;y}\mathcal{P}_{n,\beta}^{(\alpha)}(x,y,k,a,b) &= [n]_q \mathcal{P}_{n-1,\beta,q}^{(\alpha)}(x,qy,k,a,b)
\end{aligned} \tag{12}$$

and

$$\frac{2^{k-1}[n]_q!}{[n+k]_q!}\left[\beta^b \mathcal{P}_{n+k,\beta,q}^{(\alpha)}(1,y,k,a,b) - a^b \mathcal{P}_{n+k,\beta,q}^{(\alpha)}(0,y,k,a,b)\right] = \mathcal{P}_{n,\beta,q}^{(\alpha-1)}(0,y,k,a,b) \tag{13}$$

$$\frac{2^{k-1}[n]_q!}{[n+k]_q!}\left[\beta^b \mathcal{P}_{n+k,\beta,q}^{(\alpha)}(x,0,k,a,b) - a^b \mathcal{P}_{n+k,\beta,q}^{(\alpha)}(x,-1,k,a,b)\right] = \mathcal{P}_{n,\beta,q}^{(\alpha-1)}(x,-1,k,a,b). \tag{14}$$

Using (8), we obtain the addition property given below.

Theorem 2.1. *The following addition formula is valid for x and u satisfying the commuting method $ux = qxu$:*

$$\mathcal{P}_{n,\beta,q}^{(\alpha)}(x+u,y,k,a,b) = \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_q \mathcal{P}_{j,\beta,q}^{(\alpha)}(x,y,k,a,b)\, u^{n-j}.$$

From the following relation

$$\mathcal{P}_{n,\beta,q}^{(\alpha)}(1,y,k,a,b) = \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_q \mathcal{P}_{j,\beta,q}^{(\alpha)}(0,y,k,a,b) \ \text{(see [16])}$$

and Eq. (13), we get the formula given below.

Theorem 2.2. *We have*

$$\beta^b \sum_{j=0}^{n+k} \begin{bmatrix} n+k \\ j \end{bmatrix}_q \mathcal{P}_{j,\beta,q}^{(\alpha)}(0,y,k,a,b) - a^b \mathcal{P}_{n+k,\beta,q}^{(\alpha)}(0,y,k,a,b) = \frac{[n+k]_q!}{2^{k-1}[n]_q!}\mathcal{P}_{n,\beta,q}^{(\alpha-1)}(0,y,k,a,b). \tag{15}$$

Corollary 2.2.1. *Putting $\alpha = 1$ in Eq. (15) gives the following relation*

$$y^n = \frac{2^{k-1}[n]_q!}{q^{\binom{n}{2}}[n+k]_q!}\left[\beta^b \sum_{j=0}^{n+k} \begin{bmatrix} n+k \\ j \end{bmatrix}_q \mathcal{P}_{j,\beta,q}(0,y,k,a,b) - a^b \mathcal{P}_{n+k,\beta,q}(0,y,k,a,b)\right].$$

The formula in Corollary 2.2.1 seems to be a q-generalization of each of the following known formulas:

$$y^n = \frac{1}{n+1}\sum_{j=0}^{n}\binom{n+1}{j}B_j(y),\, y^n = \frac{1}{2}\sum_{j=0}^{n}\binom{n}{j}E_j(y) + E_n(y) \tag{16}$$

and

$$y^n = \frac{1}{2(n+1)}\sum_{j=0}^{n+1}\binom{n+1}{j}G_j(y) + G_{n+1}(y). \tag{17}$$

Here is a recurrence relation of unified Apostol-type q-Bernouılli, q-Euler and q-Genocchi polynomials as given below.

Theorem 2.3. *The following expression is valid for $\mathcal{P}_{n,\beta,q}(x,y,k,a,b)$:*

$$a^b \mathcal{P}_{n,\beta,q}(x,y,k,a,b) = \beta^b \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_q \mathcal{P}_{j,\beta,q}(x,y,k,a,b) - \frac{[n]_q!}{[n-k]_q!} 2^{1-k}(x+y)_q^{n-k}.$$

Proof. In view of (7) and the identity

$$\frac{a^b}{(\beta^b e_q(z) - a^b)\, e_q(z)} = \frac{\beta^b}{\beta^b e_q(z) - a^b} - \frac{1}{e_q(z)},$$

we can write

$$\begin{aligned} & a^b \sum_{n=0}^{\infty} \mathcal{P}_{n,\beta,q}(x,y,k,a,b)\frac{z^n}{[n]_q!} \\ = \; & \beta^b \sum_{n=0}^{\infty} \mathcal{P}_{n,\beta,q}(x,y,k,a,b)\frac{z^n}{[n]_q!}\sum_{n=0}^{\infty}\frac{z^n}{[n]_q!} - 2^{1-k}\sum_{n=0}^{\infty}(x+y)_q^n \frac{z^{n+k}}{[n]_q!}. \end{aligned}$$

By utilizing the technique of Cauchy product and thereafter matching the coefficients of $\frac{z^n}{[n]_q!}$, we have the asserted result.

Hence, the proof is completed. □

We provide now the following formula for unified Apostol-type q-Genocchi, q-Euler and q-Bernoulli polynomials of order α.

Theorem 2.4. *The unified polynomial $\mathcal{P}^{(\alpha)}_{n,\beta,q}(x,y,k,a,b)$ satisfies the following relation:*

$$\begin{aligned} \mathcal{P}^{(\alpha)}_{n,\beta,q}(x,y,k,a,b) = \; & \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_q \frac{2^{k-1}[n]_q!}{[n+k]_q!}\mathcal{P}^{(\alpha)}_{n-j,\beta,q}(0,0,k,a,b) \\ & \cdot \left[\beta^b \sum_{s=0}^{j+k} \begin{bmatrix} j+k \\ s \end{bmatrix}_q \mathcal{P}_{s,\beta,q}(x,y,k,a,b) - a^b \mathcal{P}_{j+k,\beta,q}(x,y,k,a,b)\right]. \end{aligned}$$

Proof. The proof of this theorem is derived from

$$\mathcal{P}_{n,\beta,q}^{(\alpha)}(x,y,k,a,b)=\sum_{j=0}^{n}\begin{bmatrix}n\\j\end{bmatrix}_q\mathcal{P}_{j,\beta,q}^{(\alpha)}(0,0,k,a,b)(x+y)_q^{n-j}\ \text{(see [16])}$$

and Theorem 2.2. So we omit the proof of this theorem. □

The q-integral representations of $\mathcal{P}_{n,\beta,q}^{(\alpha)}(x,y,k,a,b)$ are presented in the following theorem.

Theorem 2.5. *(Integral representations) We have*

$$\int_u^v\mathcal{P}_{n,\beta,q}^{(\alpha)}(x,y,k,a,b)\,d_qx=\frac{\mathcal{P}_{n+1,\beta,q}^{(\alpha)}(v,y,k,a,b)-\mathcal{P}_{n+1,\beta,q}^{(\alpha)}(u,y,k,a,b)}{[n+1]_q},$$

$$\int_u^v\mathcal{P}_{n,\beta,q}^{(\alpha)}(x,y,k,a,b)\,d_qy=\frac{\mathcal{P}_{n+1,\beta,q}^{(\alpha)}\left(x,\frac{v}{q},k,a,b\right)-\mathcal{P}_{n+1,\beta,q}^{(\alpha)}\left(x,\frac{u}{q},k,a,b\right)}{[n+1]_q}.$$

Proof. Since

$$\int_u^v D_qf(x)\,d_qx=f(v)-f(u)\ \text{(see [9])}\tag{18}$$

using Eqs. (7), (9) and (12), we obtain

$$\begin{aligned}\int_u^v\mathcal{P}_{n,\beta,q}^{(\alpha)}(x,y,k,a,b)\,d_qx&=\frac{1}{[n+1]_q}\int_u^v D_{q;x}\mathcal{P}_{n+1,\beta,q}^{(\alpha)}(x,y,k,a,b:p,q)\,d_qx\\&=\frac{1}{[n+1]_q}\left[\mathcal{P}_{n+1,\beta,q}^{(\alpha)}(v,y,k,a,b)-\mathcal{P}_{n+1,\beta,q}^{(\alpha)}(u,y,k,a,b)\right].\end{aligned}$$

Also, the other q-integral representation can be shown in a similar manner. □

The equations in Theorem 2.5 are q-extensions of the familiar formulas for usual Apostol-type Bernoulli, Euler and Genocchi polynomials (see [32]).

The following theorem involves the recurrence relationship for unified Apostol-type q-Bernoulli, q-Euler and q-Genocchi polynomials of order α.

Theorem 2.6. (Recurrence relationship) *The following equality is true for* $n,k\in\mathbb{N}_0$*:*

$$\begin{aligned}&\beta^b\sum_{j=0}^{n}\begin{bmatrix}n\\j\end{bmatrix}_q m^j\mathcal{P}_{j,\beta,q}^{(\alpha)}(x,0,k,a,b)-a^b\sum_{j=0}^{n}\begin{bmatrix}n\\j\end{bmatrix}_q m^j\mathcal{P}_{j,\beta,q}^{(\alpha)}(x,-1,k,a,b)\\&=\frac{2^{1-k}[n]_q!}{[n-k]_q!}\sum_{j=0}^{n-k}\begin{bmatrix}n-k\\j\end{bmatrix}_q m^{j+k}\mathcal{P}_{j,\beta,q}^{(\alpha-1)}(x,-1,k,a,b).\end{aligned}\tag{19}$$

Proof. The proof of this theorem follows from the following expression:

$$\begin{aligned}&\sum_{n=0}^{\infty}\mathcal{P}_{n,\beta,q}^{(\alpha-1)}(x,-1,k,a,b)m^n\frac{z^n}{[n]_q!}\sum_{n=0}^{\infty}\frac{z^n}{[n]_q!}\\&=\left(\frac{2^{1-k}(mz)^k}{\beta^be_q(mz)-a^b}\right)^{\alpha}\frac{\beta^be_q(mz)-a^b}{2^{1-k}(mz)^k}e_q(mxz)E_q(-mz)e_q(z)\\&=2^{1-k}(mz)^k\left[\beta^b\sum_{n=0}^{\infty}\mathcal{P}_{n,\beta,q}^{(\alpha)}(x,0,k,a,b)m^n\frac{z^n}{[n]_q!}\sum_{n=0}^{\infty}m^{-n}\frac{z^n}{[n]_q!}\right.\\&\qquad\left.-a^b\sum_{n=0}^{\infty}\mathcal{P}_{n,\beta,q}^{(\alpha)}(x,-1,k,a,b)m^n\frac{z^n}{[n]_q!}\sum_{n=0}^{\infty}\frac{z^n}{[n]_q!}\right].\end{aligned}$$

By utilizing the Cauchy product and equating the coefficients $\frac{z^n}{[n]_q!}$ on both sides, we procure the recurrence relation (19). $\square$

Considering the Theorem 2.6, we acquire the following result:

Corollary 2.6.1. *We have*

$$\begin{aligned}\beta^b \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_q m^j \mathcal{P}_{j,\beta,q}(x,0,k,a,b) - a^b \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_q m^j \mathcal{P}_{j,\beta,q}(x,-1,k,a,b) \\ = \frac{2^{1-k}[n]_q!}{[n-k]_q!} \sum_{j=0}^{n-k} \begin{bmatrix} n-k \\ j \end{bmatrix}_q m^{j+k} (x-1)_q^j .\end{aligned} \tag{20}$$

We now state the recurrence relation as follows.

Theorem 2.7. *For $n \in \mathbb{N}_0$ and $x, y \in \mathbb{C}$, the following formulas are valid:*

$$\begin{aligned}\mathcal{P}^{(\alpha)}_{n,\beta,q}(x,y,k,a,b) &= \frac{2^{k-1}[n]_q!}{[n+k]_q!} \sum_{s=0}^{n+k} \begin{bmatrix} n+k \\ s \end{bmatrix}_q \mathcal{P}_{n+k-s,\beta,q}(0,my,k,a,b)\, m^{s-n} \\ &\quad \cdot \left\{ \beta^b \sum_{j=0}^{s} \begin{bmatrix} s \\ j \end{bmatrix}_q m^{s-j} \mathcal{P}^{(\alpha)}_{s-j,\beta,q}(x,0,k,a,b) - a^b \mathcal{P}^{(\alpha)}_{s,\beta,q}(x,0,k,a,b) \right\},\end{aligned} \tag{21}$$

$$\begin{aligned}\mathcal{P}^{(\alpha)}_{n,\beta,q}(x,y,k,a,b) &= \frac{2^{k-1}[n]_q!}{[n+k]_q!} \sum_{s=0}^{n+k} \begin{bmatrix} n+k \\ s \end{bmatrix}_q \mathcal{P}_{n+k-s,\beta,q}(mx,0,k,a,b)\, m^{s-n} \\ &\quad \cdot \left\{ \beta^b \sum_{j=0}^{s} \begin{bmatrix} s \\ j \end{bmatrix}_q \mathcal{P}^{(\alpha)}_{s-j,\beta,q}(0,y,k,a,b)\, m^{-j} - a^b \mathcal{P}^{(\alpha)}_{s,\beta,q}(0,y,k,a,b) \right\}.\end{aligned}$$

Proof. Indeed,

$$\begin{aligned}&\sum_{n=0}^{\infty} \mathcal{P}^{(\alpha)}_{n,\beta,q}(x,y,k,a,b) \frac{z^n}{[n]_q!} \\ =& \left(\frac{2^{1-k} z^k}{\beta^b e_q(z) - a^b} \right)^{\alpha} e_q(xz) \frac{\beta^b e_q\left(\frac{z}{m}\right) - a^b}{2^{1-k}(z/m)^k} \frac{2^{1-k}(z/m)^k}{\beta^b e_q\left(\frac{z}{m}\right) - a^b} E_q\left(my\frac{z}{m}\right) \\ =& \frac{m^k}{2^{1-k}} \sum_{n=0}^{\infty} \sum_{s=0}^{n} \begin{bmatrix} n \\ s \end{bmatrix}_q \left\{ \beta^b \sum_{j=0}^{s} \begin{bmatrix} s \\ j \end{bmatrix}_q \mathcal{P}^{(\alpha)}_{s-j,\beta,q}(x,0,k,a,b)\, m^{-j} - a^b \mathcal{P}^{(\alpha)}_{s,\beta,q}(x,0,k,a,b) \right\} \\ &\cdot \mathcal{P}_{n-s,\beta,q}(0,my,k,a,b)\, m^{s-n} \frac{z^{n-k}}{[n]_q!}.\end{aligned}$$

Matching the coefficients of $\frac{z^n}{[n]_q!}$, we obtain the desired result for the first equation. The other equation can be proved similarly. $\square$

By combining Theorem 2.7 and Eq. (21), the following theorem is given.

Theorem 2.8. *We have*

$$\begin{aligned}
&\mathcal{P}_{n,\beta}^{(\alpha)}(x,y,k,a,b:p,q)\\
&=\frac{2^{k-1}[n]_{p,q}!}{[n+k]_{p,q}!}\sum_{s=0}^{n+k}\begin{bmatrix}n+k\\ s\end{bmatrix}_{p,q}\mathcal{P}_{n+k-s,\beta}(0,my,k,a,b:p,q)\,m^{s-n}\\
&\quad\cdot\left\{\frac{2^{1-k}[s]_{p,q}!}{m^s[s-k]_{p,q}!}\sum_{j=0}^{s-k}\begin{bmatrix}s-k\\ j\end{bmatrix}_{p,q}p^{\binom{s-k-j}{2}}m^{j+k}\mathcal{P}_{j,\beta}^{(\alpha-1)}(x,-1,k,a,b:p,q)\right.\\
&\quad\left.+a^b\sum_{j=0}^{s}\begin{bmatrix}s\\ j\end{bmatrix}_{p,q}p^{\binom{s-j}{2}}m^j\mathcal{P}_{j,\beta}^{(\alpha)}(x,-1,k,a,b:p,q)-a^b\mathcal{P}_{s,\beta}^{(\alpha)}(x,0,k,a,b:p,q)\right\}.
\end{aligned}$$

We now give a special case of Theorem 2.8.

Corollary 2.8.1. *We have*

$$\begin{aligned}
\mathcal{P}_{n,\beta,q}(x,y,k,a,b) &= \frac{2^{k-1}[n]_q!}{[n+k]_q!}\sum_{s=0}^{n+k}\begin{bmatrix}n+k\\ s\end{bmatrix}_q\mathcal{P}_{n+k-s,\beta,q}(0,my,k,a,b)\,m^{s-n}\\
&\quad\cdot\left\{\frac{2^{1-k}[s]_q!}{m^s[s-k]_q!}\sum_{j=0}^{s-k}\begin{bmatrix}s-k\\ j\end{bmatrix}_q m^{j+k}(x-1)_q^j\right.\\
&\quad\left.+a^b\sum_{j=0}^{s}\begin{bmatrix}s\\ j\end{bmatrix}_q m^j\mathcal{P}_{j,\beta,q}(x,-1,k,a,b)-a^b\mathcal{P}_{s,\beta,q}(x,0,k,a,b)\right\}.
\end{aligned}$$

Kurt [16] introduced the generalized q-Stirling numbers of the second kind of order v as:

$$\sum_{n=0}^{\infty}S_q(n,v,a,b,\beta)\frac{z^n}{[n]_q!}=\frac{\left(\beta^b e_q(t)-a^b\right)^v}{[v]_q!}. \tag{22}$$

Also, Kurt [16] gave the following relation for $\mathcal{P}_{n,\beta,q}^{(\alpha)}(x,y,k,a,b)$ and $S_q(n,v,a,b,\beta)$:

$$\mathcal{P}_{n-vk,\beta,q}^{(\alpha)}(x,y,k,a,b)=2^{(k-1)v}\frac{[v]_q![n-vk]_q!}{[n]_q!}\sum_{j=0}^{n}\begin{bmatrix}n\\ j\end{bmatrix}_q\mathcal{P}_{j,\beta,q}^{(v-\alpha)}(x,y,k,a,b)\,S_q(n-j,v;a,b,\beta).$$

By this motivation, we list the following theorems.

Theorem 2.9. *The correlation given below is valid:*

$$\frac{2^{(1-k)v}}{[v]_q!}(x+y)_q^{n-vk}\frac{[n-vk]_q!}{[n]_q!}=\sum_{j=0}^{n}\begin{bmatrix}n\\ j\end{bmatrix}_q\mathcal{P}_{j,\beta,q}^{(v)}(x,y,k,a,b)\,S_q(n-j,v;a,b,\beta). \tag{23}$$

Proof. The right-hand side is obtained by:

$$\begin{aligned}
&\sum_{n=0}^{\infty}S_q(n,v,a,b,\beta)\frac{z^n}{[n]_q!}\sum_{n=0}^{\infty}\mathcal{P}_{n,\beta,q}^{(v)}(x,y,k,a,b)\frac{z^n}{[n]_q!}\\
&\qquad=\sum_{n=0}^{\infty}\sum_{j=0}^{n}\begin{bmatrix}n\\ j\end{bmatrix}_q\mathcal{P}_{j,\beta,q}^{(v)}(x,y,k,a,b)\,S_q(n-j,v;a,b,\beta)\frac{z^n}{[n]_q!},
\end{aligned}$$

the left-hand side is derived as

$$\sum_{n=0}^{\infty} S_q(n,v,a,b,\beta) \frac{z^n}{[n]_q!} \sum_{n=0}^{\infty} \mathcal{P}_{n,\beta,q}^{(v)}(x,y,k,a,b) \frac{z^n}{[n]_q!}$$

$$= \frac{\left(\beta^b e_q(t) - a^b\right)^v}{[v]_q!} \frac{\left(2^{1-k} z^k\right)^v}{\left(\beta^b e_q(z) - a^b\right)^v} e_q(xz) E_q(yz)$$

$$= \frac{2^{(1-k)v}}{[v]_q!} \sum_{n=0}^{\infty} (x+y)_q^n \frac{z^{n+kv}}{[n]_q!}.$$

Equating the coefficients $\frac{z^n}{[n]_q!}$ on both sides yields the asserted result (23). □

Theorem 2.10. *We have*

$$\mathcal{P}_{n,\beta,q}^{(\alpha-v)}(x,y,k,a,b) = 2^{(k-1)v} \frac{[n]_q! \, [v]_q!}{[n+kv]_q!} \sum_{j=0}^{n+kv} \begin{bmatrix} n+kv \\ j \end{bmatrix}_q \mathcal{P}_{n+kv-j,\beta,q}^{(\alpha)}(x,y,k,a,b) \, S_q(j,v;a,b,\beta). \tag{24}$$

Proof. Using (10), it is observed that

$$\left(\frac{\beta^b e_q(t) - a^b}{2^{1-k} z^k}\right)^v \frac{1}{[v]_q!} = \frac{1}{2^{(1-k)v}} \sum_{n=0}^{\infty} S_q(n+kv,v,a,b,\beta) \frac{z^n}{[n+kv]_q!},$$

Then, by (22), with some elementary calculations, we obtain

$$\sum_{n=0}^{\infty} \mathcal{P}_{n,\beta,q}^{(\alpha-v)}(x,y,k,a,b) \frac{z^n}{[n]_q!}$$

$$= \frac{[v]_q!}{z^{kv} 2^{(1-k)v}} \sum_{n=0}^{\infty} S_q(n,v,a,b,\beta) \frac{z^n}{[n]_q!} \sum_{n=0}^{\infty} \mathcal{P}_{n,\beta,q}^{(\alpha)}(x,y,k,a,b) \frac{z^n}{[n]_q!}$$

$$= \frac{[v]_q!}{2^{(1-k)v}} \sum_{n=0}^{\infty} \left\{ \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_q \mathcal{P}_{n-j,\beta,q}^{(\alpha)}(x,y,k,a,b) \, S_q(j,v;a,b,\beta) \right\} \frac{z^{n-kv}}{[n]_q!}.$$

By matching the coefficients $\frac{z^n}{[n]_q!}$ on both sides, we obtain the asserted result (24). □

Theorem 2.11. *We have*

$$\sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_q S_q(n-j,v;a,b,\beta) (x+y)_q^j = \frac{2^{(1-k)v} [n]_q!}{[v]_q! \, [n-vk]_q!} \mathcal{P}_{n-vk,\beta,q}^{(-v)}(x,y,k,a,b). \tag{25}$$

Proof. Considering the Eqs. (10) and (22), we obtain

$$\sum_{n=0}^{\infty} S_q(n,v,a,b,\beta) \frac{z^n}{[n]_q!} e_q(xz) E_q(yz) = \left(\frac{\beta^b e_q(z) - a^b}{2^{1-k} z^k}\right)^v e_q(xz) E_q(yz) \frac{2^{(1-k)v} z^{vk}}{[v]_q!},$$

then

$$\sum_{n=0}^{\infty} S_q(n,v,a,b,\beta) \frac{z^n}{[n]_q!} \sum_{n=0}^{\infty} (x+y)_q^n \frac{z^n}{[n]_q!} = \sum_{n=0}^{\infty} \mathcal{P}_{n,\beta,q}^{(-v)}(x,y,k,a,b) \frac{z^{n+vk}}{[n]_q!} \frac{2^{(1-k)v}}{[v]_q!}.$$

Using the Cauchy product and comparing the coefficients of $\frac{z^n}{[n]_q!}$ on both sides, we arrive at the desired result (25). □

3 $P_{n,\beta,q}^{(\alpha)}(x,y,k,a,b)$ associated with fermionic p-adic integral on $\mathbb{Z}_p$

In this part, we will consider fermionic p-adic integral representation of the polynomials $P_{n,\beta,q}^{(\alpha)}(x,y,k,a,b)$. Therefore, we first state some definitions and notations which will be useful for the sequel of this paper.

The symbols $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ denote the ring of the p-adic integers, the field of the p-adic numbers, and the field of p-adic completion of an algebraic structure of $\mathbb{Q}_p$, respectively, by letting p be an odd prime number. For d an odd positive number with $(p,d)=1$, put

$$X := X_d = \lim_{\overleftarrow{N}} \mathbb{Z}/dp^N\mathbb{Z} \text{ and } X_1 = \mathbb{Z}_p$$

and

$$a + dp^N\mathbb{Z}_p = \left\{x \in X \mid x \equiv a \,(\mathrm{mod}\ dp^N)\right\}$$

where $a \in \mathbb{Z}$ lies in $0 \leq a < dp^N$. The normalized p-adic value is given by $|p|_p = p^{-1}$, *cf* [6,12,13]. For

$$f \in C(\mathbb{Z}_p) = \{f \,|\, f : \mathbb{Z}_p \to \mathbb{C}_p, f \text{ is a continuous function}\}$$

the fermionic p-adic integral on $\mathbb{Z}_p$ of a function $f \in C(\mathbb{Z}_p)$ is originally defined by Kim [12,13], as follows:

$$I_{-1}(f) = \int_{\mathbb{Z}_p} f(x)\, d\mu_{-1}(x) = \lim_{N\to\infty} \sum_{x=0}^{p^N-1} f(x)(-1)^x.$$

We know the following idenitities from Kurt's work (see [16])

$$\begin{aligned} \mathcal{P}_{n,\beta,q}^{(\alpha)}(x,y,k,a,b) &= \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix}_q \mathcal{P}_{n-m,\beta,q}^{(\alpha)}(0,y,k,a,b)\, x^m, & (26) \\ &= \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix}_q q^{\binom{m}{2}} \mathcal{P}_{n-m,\beta,q}^{(\alpha)}(x,0,k,a,b)\, y^m. & (27) \end{aligned}$$

Thus, we now give the following Theorem 3.1.

Theorem 3.1. *The following relationships hold true*

$$\begin{aligned} \int_{\mathbb{Z}_p} \mathcal{P}_{n,\beta,q}^{(\alpha)}(x,y,k,a,b)\, d\mu_{-1}(x) &= \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix}_q \mathcal{P}_{n-m,\beta,q}^{(\alpha)}(0,y,k,a,b)\, E_m, & (28) \\ \int_{\mathbb{Z}_p} \mathcal{P}_{n,\beta,q}^{(\alpha)}(x,y,k,a,b)\, d\mu_{-1}(y) &= \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix}_q q^{\binom{m}{2}} \mathcal{P}_{n-m,\beta,q}^{(\alpha)}(x,0,k,a,b)\, E_m, & (29) \end{aligned}$$

where E_m is the m-th usual Euler number.

Proof. Since

$$E_n = \int_{\mathbb{Z}_p} x^n d\mu_{-1}(x) = \int_{\mathbb{Z}_p} y^n d\mu_{-1}(y) \quad \text{(see [13])}$$

and Eq. (26), we have

$$\begin{aligned} \int_{\mathbb{Z}_p} \mathcal{P}_{n,\beta,q}^{(\alpha)}(x,y,k,a,b)\, d\mu_{-1}(x) &= \int_{\mathbb{Z}_p} \left(\sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix}_q \mathcal{P}_{n-m,\beta,q}^{(\alpha)}(0,y,k,a,b)\, x^m \right) d\mu_{-1}(x) \\ &= \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix}_q \mathcal{P}_{n-m,\beta,q}^{(\alpha)}(0,y,k,a,b) \int_{\mathbb{Z}_p} x^m d\mu_{-1}(x) \\ &= \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix}_q \mathcal{P}_{n-m,\beta,q}^{(\alpha)}(0,y,k,a,b)\, E_m \end{aligned}$$

which is a linear combination of the product $\mathcal{P}_{n-m,\beta,q}^{(\alpha)}(0,y,k,a,b)\, E_m$. Also, by using Eq. (27), we may obtain the other fermionic p-adic integral representation in Theorem 3.1. Thus, we acquire the desired results. □

Corollary 3.1.1. *When $y = 0$ and $x = 0$ in Eqs. (28) and (29), respectively, we obtain*

$$\int_{\mathbb{Z}_p} \mathcal{P}_{n,\beta,q}^{(\alpha)}(x,0,k,a,b)\, d\mu_{-1}(x) = \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix}_q \mathcal{P}_{n-m,\beta,q}^{(\alpha)}(0,0,k,a,b)\, E_m, \tag{30}$$

$$\int_{\mathbb{Z}_p} \mathcal{P}_{n,\beta,q}^{(\alpha)}(0,y,k,a,b)\, d\mu_{-1}(y) = \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix}_q q^{\binom{m}{2}} \mathcal{P}_{n-m,\beta,q}^{(\alpha)}(0,0,k,a,b)\, E_m. \tag{31}$$

Note that taking $y = 0$ (or $x = 0$) in Eq. (25) yields to

$$\sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix}_q S_q(n-m, v; a, b, \beta)\, x^m = \frac{2^{(1-k)v} [n]_q!}{[v]_q! [n-vk]_q!} \mathcal{P}_{n-vk,\beta,q}^{(-v)}(x,0,k,a,b), \tag{32}$$

$$\sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix}_q q^{\binom{m}{2}} S_q(n-m, v; a, b, \beta)\, y^m = \frac{2^{(1-k)v} [n]_q!}{[v]_q! [n-vk]_q!} \mathcal{P}_{n-vk,\beta,q}^{(-v)}(0,y,k,a,b). \tag{33}$$

If the integral $\int_{\mathbb{Z}_p} d\mu_{-1}(y)$ is applied to both sides of Eqs. (32) and (33), by making use of the Eqs. (30) and (31), we have the theorem given below.

Theorem 3.2. *For $n, k \in \mathbb{N}_0$, each of the following relationships holds true:*

$$\begin{aligned} &\sum_{m=0}^{n+vk} \begin{bmatrix} n+vk \\ m \end{bmatrix}_q S_q(n+vk-m, v; a, b, \beta)\, E_m \\ &\qquad = \frac{2^{(1-k)v} [n+vk]_q!}{[v]_q! [n]_q!} \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix}_q \mathcal{P}_{n-m,\beta,q}^{(-v)}(0,0,k,a,b)\, E_m, \end{aligned}$$

$$\begin{aligned} &\sum_{m=0}^{n+vk} \begin{bmatrix} n+vk \\ m \end{bmatrix}_q q^{\binom{m}{2}} S_q(n+vk-m, v; a, b, \beta)\, E_m \\ &\qquad = \frac{2^{(1-k)v} [n+vk]_q!}{[v]_q! [n]_q!} \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix}_q q^{\binom{m}{2}} \mathcal{P}_{n-m,\beta,q}^{(-v)}(0,0,k,a,b)\, E_m. \end{aligned}$$

4 Conclusion

Kurt [16] introduced unified Apostol-type q-Genocchi, q-Euler and q-Bernoulli polynomials of order α and investigated some properties of them. Also, by defining the generalized q-Stirling numbers of the second kind, he derived a relation between these numbers the unified Apostol-type q-polynomials. In the present paper, we have obtained a lot of novel identities for these unified Apostol-type q-polynomials and some new theorems for these generalized Stirling numbers and unified q-polynomials. Moreover, using the fermionic p-adic integral over the p-adic numbers field, we acquire relations between the new and old polynomials.

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