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Numbers with the same kernel

Rafael Jakimczuk

División Matemática, Universidad Nacional de Luján
Buenos Aires, Argentina
e-mail: jakimczu@mail.unlu.edu.ar

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Abstract: In this article we study functions related to numbers which have the same kernel. We apply the results obtained to the sums $\sum_{n\leq x}\frac{1}{u(n)^s}$, where $s \geq 2$ is an arbitrary but fixed positive integer and $u(n)$ denotes the kernel of n. For example, we prove that

$$\sum_{n\leq x}\frac{1}{u(n)^s}\sim f_s(x),$$

where

$$f_s(x)=\sum_{k=1}^{\infty}\frac{b_{k,s}}{k!}(\log x)^k$$

and the positive coefficients $b_{k,s}$ of the series have a strong connection with the prime numbers. We also prove that

$$\sum_{n\leq x}\frac{1}{u(n)^s}=\exp\left((\log x)^{\beta_s(x)}\right),$$

where $\lim_{x\to\infty}\beta_s(x)=\frac{1}{s+1}$. The methods used are very elementary. The case $s=1$, namely $\sum_{n\leq x}\frac{1}{u(n)}$, was studied, as it is well-known, by N. G. de Bruijn (1962) and W. Schwarz (1965).
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1 Introduction and preliminary notes

A squarefree number (also called quadratfrei number) is a number without square factors, a product of different primes. The first few terms of the integer sequence of squarefree numbers are

$$1,2,3,5,6,7,10,11,13,14,15,17,19,21,22,23,26,29,30,\dots$$

Let us consider the prime factorization of a positive integer $n \geq 2$

$$n = q_1^{s_1} q_2^{s_2} \cdots q_t^{s_t},$$

where $q_1, q_2, \ldots q_t$ are the different primes in the prime factorization.

We have the following two arithmetical functions

$$u(n) = q_1 q_2 \cdots q_t.$$

The arithmetical function $u(n)$ is well-known in the literature, it is called *kernel of* n, or *radical of* n, etc. There are many papers dedicated to this arithmetical function. This function is fundamental in the establishment of the famous ABC conjecture

$$v(n) = \frac{n}{u(n)} = q_1^{s_1-1} q_2^{s_2-1} \cdots q_t^{s_t-1}.$$

We call $v(n)$ the remainder of n. Note that $v(n) = 1$ if and only if n is a squarefree.

In this article we study functions related to numbers which have the same kernel. We apply the results obtained to the sums $\sum_{n \leq x} \frac{1}{u(n)^s}$, where $s \geq 2$ is an arbitrary but fixed positive integer and $u(n)$ denotes the kernel of n. For example, we prove that

$$\sum_{n \leq x} \frac{1}{u(n)^s} \sim f_s(x),$$

where

$$f_s(x) = \sum_{k=1}^{\infty} \frac{b_{k,s}}{k!} (\log x)^k$$

and the positive coefficients $b_{k,s}$ of the series have a strong connection with the prime numbers. We also prove that

$$\sum_{n \leq x} \frac{1}{u(n)^s} = \exp\left((\log x)^{\beta_s(x)} \right),$$

where $\lim_{x \to \infty} \beta_s(x) = \frac{1}{s+1}$. The methods used are very elementary. The case $s = 1$, namely $\sum_{n \leq x} \frac{1}{u(n)}$, was studied, as it is well-known, by N. G. de Bruijn (1962) and W. Schwarz (1965) (see [2]).

We shall need the following well-known lemma (see [3], Chapter XXII).

Lemma 1.1. *Let c_n $(n \geq 1)$ be a sequence of real numbers. Let us consider the function $A(x) = \sum_{n \leq x} c_n$. Suppose that $f(x)$ has a continuous derivative $f'(x)$ on the interval $[1, \infty]$, then the following formula holds*

$$\sum_{n \leq x} c_n f(n) = A(x) f(x) - \int_1^x A(t) f'(t)\, dt.$$

2 Main results

Let $p_1, \ldots, p_k$ be distinct primes fixed. Let us consider the numbers a of the form

$$a = p_1^{s_1} p_2^{s_2} \cdots p_k^{s_k}, \tag{1}$$

where the multiplicities $s_1, \ldots, s_k$ are variables. That is, the numbers a with the same kernel $u(a) = p_1 \cdots p_k$. We have the following theorem.

Theorem 2.1. *Let $k \geq 2$ and s be arbitrary but fixed positive integers. The following asymptotic formula holds.*

$$\begin{aligned} \sum_{a \leq x} \frac{1}{u(a)^s} &= \frac{1}{k!} \frac{1}{(p_1 \cdots p_k)^s \log p_1 \cdots \log p_k} \log^k x \\ - & \frac{1}{2} \frac{1}{(k-1)!} \frac{\log p_1 + \cdots + \log p_k}{(p_1 \cdots p_k)^s \log p_1 \cdots \log p_k} \log^{k-1} x \\ + & o\left(\log^{k-1} x\right). \end{aligned} \tag{2}$$

Proof: The distribution of the a numbers is well-known (see either [1] or [5]). The following asymptotic formula holds

$$\begin{aligned} \sum_{a \leq x} 1 &= \frac{1}{k!} \frac{1}{\log p_1 \cdots \log p_k} \log^k x - \frac{1}{2} \frac{1}{(k-1)!} \frac{\log p_1 + \cdots + \log p_k}{\log p_1 \cdots \log p_k} \log^{k-1} x \\ + & o\left(\log^{k-1} x\right). \end{aligned} \tag{3}$$

Therefore, we obtain

$$\begin{aligned} \sum_{a \leq x} \frac{1}{u(a)^s} &= \frac{1}{u(a)^s} \sum_{a \leq x} 1 = \frac{1}{k!} \frac{1}{(p_1 \cdots p_k)^s \log p_1 \cdots \log p_k} \log^k x \\ - & \frac{1}{2} \frac{1}{(k-1)!} \frac{\log p_1 + \cdots + \log p_k}{(p_1 \cdots p_k)^s \log p_1 \cdots \log p_k} \log^{k-1} x \\ + & o\left(\log^{k-1} x\right), \end{aligned}$$

since $u(a)^s = (p_1 \cdots p_k)^s$. □

Theorem 2.2. *Let $k \geq 2$ and s be arbitrary but fixed positive integers. The following asymptotic formula holds.*

$$\sum_{a \leq x} v(a)^s = \frac{k}{s} \frac{1}{k!} \frac{1}{(p_1 \cdots p_k)^s \log p_1 \cdots \log p_k} x^s \log^{k-1} x + o\left(x^s \log^{k-1} x\right) \tag{4}$$

Proof: For the sake of simplicity, we write equation (3) in the compact form

$$\sum_{a \leq x} 1 = C_1 \log^k x + C_2 \log^{k-1} x + o\left(\log^{k-1} x\right).$$

If we put $A(x) = \sum_{a \le x} 1$ and $f(x) = x^s$, then Lemma 1.1 gives

$$\begin{aligned} & \sum_{a \le x} a^s = C_1 x^s \log^k x + C_2 x^s \log^{k-1} x + o\left(x^s \log^{k-1} x\right) \\ - \quad & s \int_1^x \left(C_1 t^{s-1} \log^k t + C_2 t^{s-1} \log^{k-1} t\right)\, dt + \int_1^x o\left(t^{s-1} \log^{k-1} t\right)\, dt \\ = \quad & \frac{k}{s} C_1 x^s \log^{k-1} x + o\left(x^s \log^{k-1} x\right), \end{aligned} \tag{5}$$

where we have used the formula (integration by parts)

$$\begin{aligned} & \int t^{s-1} \log^k t \, dt = \frac{t^s}{s} \log^k t - \frac{k}{s} \int t^{s-1} \log^{k-1} t \, dt = \frac{t^s}{s} \log^k t \\ - \quad & \frac{k}{s} \left(\frac{t^s}{s} \log^{k-1} t - \frac{k-1}{s} \int t^{s-1} \log^{k-2} t \, dt \right) = \frac{t^s}{s} \log^k t - \frac{k}{s^2} t^s \log^{k-1} t \\ + \quad & \frac{k(k-1)}{s^2} \int t^{s-1} \log^{k-2} t \, dt \end{aligned}$$

the formula (integration by parts)

$$\int t^{s-1} \log^{k-1} t \, dt = \frac{t^s}{s} \log^{k-1} t - \frac{k-1}{s} \int t^{s-1} \log^{k-2} t \, dt,$$

the formula

$$\int_1^x o\left(t^{s-1} \log^{k-1} t\right)\, dt = o\left(x^s \log^{k-1} x\right),$$

and the formula (L'Hospital's rule)

$$\lim_{x \to \infty} \frac{\int_a^x t^b \log^c t \, dt}{\frac{x^{b+1} \log^c x}{b+1}} = 1.$$

In the last formula a, b and c are positive numbers. Equation (5) gives

$$\begin{aligned} & \sum_{a \le x} v(a)^s = \sum_{a \le x} \frac{a^s}{u(a)^s} = \frac{1}{u(a)^s} \sum_{a \le x} a^s = \frac{k}{s} \frac{C_1}{u(a)^s} x^s \log^{k-1} x + o\left(x^s \log^{k-1} x\right) \\ = \quad & \frac{k}{s} \frac{1}{k!} \frac{1}{(p_1 \cdots p_k)^s \log p_1 \cdots \log p_k} x^s \log^{k-1} x + o\left(x^s \log^{k-1} x\right) \end{aligned}$$

That is, equation (4). □

In a previous article [6], the author proved the following limit

$$\lim_{x \to \infty} \frac{\sum_{n \le x} v(n)}{x} = \infty,$$

where n denotes a positive integer. In the following theorem we prove more precise results.

Theorem 2.3. *Let s be an arbitrary but fixed positive integer. For all $\alpha > 0$ and for all $\beta > 0$ the following limits hold*

$$\lim_{x\to\infty} \frac{\sum_{n\le x} \frac{1}{u(n)^s}}{x^\alpha} = 0, \tag{6}$$

$$\lim_{x\to\infty} \frac{\sum_{n\le x} \frac{1}{u(n)^s}}{\log^\beta x} = \infty, \tag{7}$$

$$\lim_{x\to\infty} \frac{\sum_{n\le x} v(n)^s}{x^{s+\alpha}} = 0, \tag{8}$$

$$\lim_{x\to\infty} \frac{\sum_{n\le x} v(n)^s}{x^s \log^\beta x} = \infty. \tag{9}$$

Proof: Let $\alpha > 0$. We have

$$\sum_{n=1}^{\infty} \frac{1}{u(n)n^\alpha} = \prod_p \left(1 + \frac{1}{pp^\alpha} + \frac{1}{p(p^\alpha)^2} + \frac{1}{p(p^\alpha)^3} + \cdots\right)$$
$$= \prod_p \left(1 + \frac{1}{pp^\alpha}\left(\frac{1}{1-\frac{1}{p^\alpha}}\right)\right) = \prod_p \left(1 + \frac{1}{p(p^\alpha - 1}\right). \tag{10}$$

Now, the product $\prod_p \left(1 + \frac{1}{p(p^\alpha - 1}\right)$ converges to a positive number, since the series of positive terms $\sum_p \frac{1}{p(p^\alpha-1)}$ clearly converges. Therefore, the series of positive terms (10) is convergent, that is, we have $\sum_{n=1}^{\infty} \frac{1}{u(n)n^\alpha} = C > 0$. Therefore, if we apply Lemma 1.1 with $f(x) = x^\alpha$, then we obtain

$$\sum_{n\le x} \frac{1}{u(n)} = (C + o(1))x^\alpha - \alpha \int_1^x (C + o(1))t^{\alpha-1}\, dt = o(x^\alpha).$$

Consequently limit (6) holds, since we have the inequality

$$\sum_{n\le x} \frac{1}{u(n)^s} \le \sum_{n\le x} \frac{1}{u(n)}.$$

Besides, limit (8) holds, since we have

$$\sum_{n\le x} v(n)^s = \sum_{n\le x} \frac{n^s}{u(n)^s} \le x^s \sum_{n\le x} \frac{1}{u(n)^s} = o(x^{s+\alpha}).$$

Limits (7) and (9) are an immediate consequence of Theorem 2.1 and Theorem 2.2, since

$$\sum_{n\le x} v(n)^s \ge \sum_{a\le x} v(a)^s, \qquad \sum_{n\le x} \frac{1}{u(n)^s} \ge \sum_{a\le x} \frac{1}{u(a)^s}.$$

This completed the proof. □

In the following theorem, n_k denotes a positive integer with exactly k distinct prime factors, that is, its prime factorization is of the form $n_k = p_1^{s_1} \cdots p_k^{s_k}$, where k is fixed.

Theorem 2.4. *Let k and s be arbitrary but fixed positive integers. The following asymptotic formula holds*

$$\sum_{n_k \leq x} \frac{1}{u(n_k)^s} = \frac{b_{k,s}}{k!} \log^k x + o\left(\log^k x\right), \tag{11}$$

where

$$b_{k,s} = \sum_{p_1 \cdots p_k} \frac{1}{(p_1 \cdots p_k)^s \log p_1 \cdots \log p_k} \leq \frac{a_s^k}{k!} \tag{12}$$

and $a_s = \sum_p \frac{1}{p^s \log p}$.

The symbol $\sum_{p_1 \cdots p_k}$ means that the sum runs on all products of k distinct primes, that is, on all squarefree with k prime factors.

The symbol $\sum_p$ means that the sum runs on all positive primes.

The following limit holds

$$\lim_{k \to \infty} b_{k,s} = 0.$$

Proof: Note that the series $\sum_p \frac{1}{p^s \log p}$ converges. Clearly this fact is true if $s \geq 2$. Also, it is true if $s = 1$ since (prime number theorem) $r_n \sim n \log n$, $\log r_n \sim \log n$ and the series $\sum \frac{1}{n \log^2 n}$ converges (integral criterion). Here $r(n)$ denotes the n-th prime number.

Therefore, the series (12) converges, since (product of convergent series) we have

$$k! \sum_{p_1 \cdots p_k} \frac{1}{(p_1 \cdots p_k)^s \log p_1 \cdots \log p_k} \leq \left(\sum_p \frac{1}{p^s \log p} \right)^k = a_s^k.$$

The sum of this series (12) we have denoted by $b_{k,s}$.

Let q_n be the sequence of squarefree numbers with k prime factors. There exists q_{t+1} such that

$$\sum_{p_1 \cdots p_k \geq q_{t+1}} \frac{1}{(p_1 \cdots p_k)^s \log p_1 \cdots \log p_k} \leq \epsilon \tag{13}$$

On the other hand, we have (see Theorem 2.1 and (12))

$$\begin{aligned}
\sum_{n_k \leq x} \frac{1}{u(n_k)^s} &= \sum_{p_1 \cdots p_k \leq x} \left(\sum_{a \leq x} \frac{1}{u(a)^s} \right) \\
&= \sum_{p_1 \cdots p_k \leq q_t} \left(\frac{\log^k x}{k!(p_1 \cdots p_k)^s \log p_1 \cdots \log p_k} \right) + o\left(\log^k x\right) \\
&+ \sum_{q_{t+1} \leq p_1 \cdots p_k \leq q_x} \left(\frac{1}{(p_1 \cdots p_k)^s} \sum_{a \leq x} 1 \right) = \frac{b_{k,s}}{k!} \log^k x \\
&- \frac{1}{k!} \sum_{p_1 \cdots p_k \geq q_{t+1}} \frac{\log^k x}{(p_1 \cdots p_k)^s \log p_1 \cdots \log p_k} + o\left(\log^k x\right) \\
&+ \sum_{q_{t+1} \leq p_1 \cdots p_k \leq q_x} \left(\frac{1}{(p_1 \cdots p_k)^s} \sum_{a \leq x} 1 \right).
\end{aligned} \tag{14}$$

In equation (14), q_x denotes the greatest squarefree number with k prime factors not exceeding x.

Let us consider the inequality $p_1^{s_1}\cdots p_k^{s_k}\leq x$, where $p_1,\ldots,p_k$ are fixed primes. We have $p_i^{s_i}\leq x(i=1,\ldots,k)$, consequently, s_i can take the values $s_i=1,\ldots,\left[\frac{\log x}{\log p_i}\right](i=1,\ldots,k)$ and, therefore, an upper bound for $\sum_{a\leq x}1$ is

$$\sum_{a\leq x}1\leq\left[\frac{\log x}{\log p_1}\right]\cdots\left[\frac{\log x}{\log p_k}\right]\leq\frac{\log^k x}{\log p_1\cdots\log p_k}. \tag{15}$$

We have (see (13) and (15))

$$\begin{aligned}&\sum_{q_{t+1}\leq p_1\cdots p_k\leq q_x}\left(\frac{1}{(p_1\cdots p_k)^s}\sum_{a\leq x}1\right)\\ \leq&\sum_{q_{t+1}\leq p_1\cdots p_k\leq q_x}\frac{1}{(p_1\cdots p_k)^s}\frac{\log^k x}{\log p_1\cdots\log p_k}\leq\epsilon\log^k x.\end{aligned} \tag{16}$$

Equations (14), (13) and (16) give

$$\left|\frac{\sum_{n_k\leq x}\frac{1}{u(n_k)^s}}{\log^k x}-\frac{b_{k,s}}{k!}\right|\leq 3\epsilon\qquad(x\geq x_\epsilon). \tag{17}$$

Now, $\epsilon>0$ can be arbitrarily small. Consequently, equation (17) can be written in the form

$$\frac{\sum_{n_k\leq x}\frac{1}{u(n_k)^s}}{\log^k x}-\frac{b_{k,s}}{k!}=o(1).$$

That is, equation (11). □

In the following theorem we obtain a stronger inequality than inequality (12).

Theorem 2.5. *Let k and s be arbitrary but fixed positive integers. The following inequalities hold*

$$b_{k,s}\leq\frac{(c_s^s)^k}{(k!)^s},$$

where $c_s=\sum_p\frac{1}{p(\log p)^{1/s}}$.

Proof: We have

$$k!\sum_{p_1\cdots p_k}\frac{1}{p_1\cdots p_k(\log p_1)^{1/s}\cdots(\log p_k)^{1/s}}\leq\left(\sum_p\frac{1}{p(\log p)^{1/s}}\right)^k=c_s^k.$$

Hence,

$$\sum_{p_1\cdots p_k}\frac{1}{p_1\cdots p_k(\log p_1)^{1/s}\cdots(\log p_k)^{1/s}}\leq\frac{c_s^k}{k!}$$

and, consequently,

$$\begin{aligned} b_{k,s} &= \sum_{p_1 \cdots p_k} \frac{1}{(p_1 \cdots p_k)^s \log p_1 \cdots \log p_k} \\ &\leq \left(\sum_{p_1 \cdots p_k} \frac{1}{p_1 \cdots p_k (\log p_1)^{1/s} \cdots (\log p_k)^{1/s}} \right)^s \leq \frac{(c_s^s)^k}{(k!)^s}. \end{aligned}$$

Therefore, the inequality is proved. □

Now, we establish a general theorem.

Theorem 2.6. *Let us consider the inequality*

$$r_1 x_1 + \cdots + r_n x_n \leq x \qquad (x \geq 0),$$

where r_i $(i = 1, \ldots, n)$ are fixed positive real numbers. The number of solutions $(x_1, \ldots, x_n)$ to this inequality, where x_i $(i = 1, \ldots, n)$ are positive integers, will be denoted by $S_n(x)$. The following inequality holds:

$$S_n(x) \leq \frac{1}{n!} \frac{x^n}{r_1 \cdots r_n} \qquad (x \geq 0).$$

Proof: If $x \geq r_1$, then the solutions to the inequality $r_1 x_1 \leq x$ are $x_1 = 1, \ldots, \left\lfloor \frac{x}{r_1} \right\rfloor$ and, consequently, $S_1(x) = \left\lfloor \frac{x}{r_1} \right\rfloor \leq \frac{x}{r_1}$. On the other hand, if $0 \leq x < r_1$, we have $S_1(x) = 0$ and, consequently, also $S_1(x) \leq \frac{x}{r_1}$. Therefore, the theorem is true for $n = 1$. Suppose that the theorem is true for $n - 1 \geq 1$, we shall prove that the theorem is also true for n. Suppose that $x \geq r_1 + \cdots + r_n$, then

$$\begin{aligned} S_n(x) &= \sum_{x_n=1}^{\left\lfloor \frac{x}{r_n} \right\rfloor} S_{n-1}(x - r_n x_n) \leq \frac{1}{(n-1)!} \frac{1}{r_1 \cdots r_{n-1}} \sum_{x_n=1}^{\left\lfloor \frac{x}{r_n} \right\rfloor} (x - r_n x_n)^{n-1} \\ &\leq \frac{1}{(n-1)!} \frac{1}{r_1 \cdots r_{n-1}} \int_0^{\frac{x}{r_n}} (x - r_n x_n)^{n-1} \, dx_n = \frac{1}{n!} \frac{x^n}{r_1 \cdots r_n}. \end{aligned}$$

Note that the function $f(x_n) = (x - r_n x_n)^{n-1}$ is strictly decreasing in the interval $\left[0, \frac{x}{r_n}\right]$ and in this interval the area below the function is greater than the sum of the areas of the $\left\lfloor \frac{x}{r_n} \right\rfloor$ rectangles of base 1 and height $(x - r_n x_n)^{n-1}$, that is, the sum $\sum_{x_n=1}^{\left\lfloor \frac{x}{r_n} \right\rfloor} (x - r_n x_n)^{n-1}$.

On the other hand, if $0 \leq x < r_1 + \cdots + r_n$, then $S_n(x) = 0$ and, consequently, the inequality also holds. □

We have the following immediate corollary (see Theorem 2.1).

Corollary 2.7. *Let k be an arbitrary but fixed positive integer. The following inequalities hold*

$$\sum_{a \leq x} 1 \leq \frac{1}{k!} \frac{1}{\log p_1 \cdots \log p_k} \log^k x \qquad (x \geq 1)$$

$$\sum_{a\le x}\frac{1}{u(a)^s}\le\frac{1}{k!}\frac{1}{(p_1\cdots p_k)^s\log p_1\cdots\log p_k}\log^k x\qquad(x\ge 1)$$

Theorem 2.8. *Let s be an arbitrary but fixed positive integer. The following inequalities hold*

$$\sum_{n\le x}\frac{1}{u(n)^s}\le F_{1,s}(x)\le F_{2,s}(x)\le F_{3,s}(x)\le e^{(s+1)c_s^{s/s+1}\sqrt[s+1]{\log x}}\quad(x\ge 1),\tag{18}$$

where

$$F_{1,s}(x)=\sum_{p_1\cdots p_k\le x}\frac{1}{k!}\frac{1}{(p_1\cdots p_k)^s\log p_1\cdots\log p_k}\log^k x,\tag{19}$$

$$F_{2,s}(x)=\sum_{k=1}^{h}\frac{b_{k,s}}{k!}\log^k x,\tag{20}$$

$$F_{3,s}(x)=\sum_{k=1}^{\infty}\frac{b_{k,s}}{k!}\log^k x.\tag{21}$$

Note that in (19) the positive integer k is a variable, that is, the sum runs on all squarefree not exceeding x. On the other hand, in (20) the positive integer h denotes the greatest number of prime factors of the squarefree number not exceeding x.

Besides, for all $\beta>0$ and all $\alpha>0$ the following limits hold

$$\lim_{x\to\infty}\frac{F_{i,s}(x)}{\log^\beta x}=\infty\qquad(i=1,2,3),\tag{22}$$

$$\lim_{x\to\infty}\frac{F_{i,s}(x)}{x^\alpha}=0\qquad(i=1,2,3),\tag{23}$$

and also the following inequalities hold

$$\sum_{n\le x}v(n)^s\le x^sF_{1,s}(x)\le x^sF_{2,s}(x)\le x^sF_{3,s}(x)\le x^se^{(s+1)c_s^{s/s+1}\sqrt[s+1]{\log x}},\tag{24}$$

where $x\ge 1$.

Proof: First all, we shall prove that the functions $F_{3,s}(x)$ exist for $x\ge 1$. We have (see Theorem 2.5)

$$F_{3,s}(x)=\sum_{k=1}^{\infty}\frac{b_{k,s}}{k!}\log^k x\le\sum_{k=0}^{\infty}\frac{(c_s^s)^k}{(k!)^{s+1}}\log^k x\le\left(\sum_{k=0}^{\infty}\frac{(c_s^{s/s+1}\sqrt[s+1]{\log x})^k}{k!}\right)^{s+1}$$
$$=\quad e^{(s+1)c_s^{s/s+1}\sqrt[s+1]{\log x}}.\tag{25}$$

Note that we have used the power series $e^x=\sum_{k=0}^{\infty}\frac{x^k}{k!}$.

Now, we have (see Corollary 2.7 and (19))

$$\sum_{n\le x}\frac{1}{u(n)^s}=\sum_{p_1\cdots p_k\le x}\left(\sum_{a\le x}\frac{1}{u(a)^s}\right)\le F(1,s)(x)\qquad (x\ge 1).$$

From the definition of $b_{k,s}$ (see equation (12)) and the definitions of $F(2,s)(x)$ and $F(3,s)(x)$, we obtain (18) (see (19), (20) and (21)).

Limit (22) is an immediate consequence of the definitions (19), (20) and (21).

Let $\alpha>0$. Limit (23) is an immediate consequence of (18) and the limit $\lim_{x\to\infty}\frac{e^{s+1\sqrt{\log x}}}{x^\alpha}=0$.

Equation (24) is an immediate consequence of (18) and the trivial inequality

$$\sum_{n\le x}v(n)^s=\sum_{n\le x}\frac{n^s}{u(n)^s}\le x^s\sum_{n\le x}\frac{1}{u(n)^s}$$

□

Remark 2.9. *The function $\sum_{n\le x}\frac{1}{u(n)}$ (the case $s=1$ in this article) has been very studied. In 1962, N. G. de Bruijn obtained the asymptotic formula $\log\left(\sum_{n\le x}\frac{1}{u(n)}\right)=(1+o(1))\sqrt{\frac{8\log x}{\log\log x}}$, see [2]. An immediate consequence of this formula are limits (6) and (7) ($s=1$). In 1965, W. Schwarz obtained a function $F(x)$ such that $\sum_{n\le x}\frac{1}{u(n)}\sim F(x)$, see [2].*

Now, we establish a general theorem.

Theorem 2.10. *Let us consider the inequality*

$$r_1x_1+\cdots+r_nx_n\le x\qquad (x\ge 0),$$

where r_i ($i=1,\ldots,n$) are fixed positive real numbers. The number of solutions $(x_1,\ldots,x_n)$ to this inequality, where x_i ($i=1,\ldots,n$) are positive integers, will be denoted by $S_n(x)$.

The following inequalities hold

$$S_n(x)\ge\frac{1}{n!}\frac{x^n}{r_1\cdots r_n}-\frac{1}{(n-1)!}\frac{r_1+\cdots+r_n}{r_1\cdots r_n}x^{n-1}\qquad (x\ge n\,(r_1+\cdots+r_n))$$

$$S_n(x)\ge 0\qquad (0\le x<n\,(r_1+\cdots+r_n))$$

Proof: First, we shall prove that the inequality

$$S_n(x)\ge\frac{1}{n!}\frac{x^n}{r_1\cdots r_n}-\frac{1}{(n-1)!}\frac{r_1+\cdots+r_n}{r_1\cdots r_n}x^{n-1}\tag{26}$$

holds for $x\ge 0$.

If $x\ge r_1$, then the solutions to the inequality $r_1x_1\le x$ are $x_1=1,\ldots,\left\lfloor\frac{x}{r_1}\right\rfloor$ and, consequently, $S_1(x)=\left\lfloor\frac{x}{r_1}\right\rfloor\ge\frac{x}{r_1}-1$. On the other hand, if $0\le x<r_1$, we have $S_1(x)=0$ and, consequently, also $S_1(x)\ge\frac{x}{r_1}-1$. Therefore, inequality (26) is true for $n=1$. Suppose that

inequality (26) is true for $n \geq 1$, we shall prove that inequality (26) is also true for $n+1$. Suppose that $x \geq r_{n+1}$, then

$$
\begin{aligned}
S_{n+1}(x) = \sum_{x_{n+1}=1}^{\left\lfloor \frac{x}{r_{n+1}} \right\rfloor} S_n\left(x - r_{n+1}x_{n+1}\right) \geq \frac{1}{n!}\frac{1}{r_1\cdots r_n}\sum_{x_{n+1}=1}^{\left\lfloor \frac{x}{r_{n+1}} \right\rfloor}\left(x - r_{n+1}x_{n+1}\right)^n \\
- \quad \frac{1}{(n-1)!}\frac{r_1+\cdots+r_n}{r_1\cdots r_n}\sum_{x_{n+1}=1}^{\left\lfloor \frac{x}{r_{n+1}} \right\rfloor}\left(x - r_{n+1}x_{n+1}\right)^{n-1} \\
\geq \quad \frac{1}{n!}\frac{1}{r_1\cdots r_n}\int_0^{\frac{x}{r_{n+1}}}\left(x - r_{n+1}x_{n+1}\right)^n dx_{n+1} - \frac{1}{n!}\frac{1}{r_1\cdots r_n}x^n \\
- \quad \frac{1}{(n-1)!}\frac{r_1+\cdots+r_n}{r_1\cdots r_n}\int_0^{\frac{x}{r_{n+1}}}\left(x - r_{n+1}x_{n+1}\right)^{n-1} dx_{n+1} \\
= \quad \frac{1}{(n+1)!}\frac{x^{n+1}}{r_1\cdots r_{n+1}} - \frac{1}{n!}\frac{r_1+\cdots+r_{n+1}}{r_1\cdots r_{n+1}}x^n.
\end{aligned}
$$

On the other hand, if $0 < x < r_{n+1}$, then $S_{n+1}(x) = 0$ and, consequently, the inequality also holds, since

$$
\begin{aligned}
\frac{1}{(n+1)!}\frac{x^{n+1}}{r_1\cdots r_{n+1}} - \frac{1}{n!}\frac{r_1+\cdots+r_{n+1}}{r_1\cdots r_{n+1}}x^n = \frac{1}{(n+1)!}\frac{x^{n+1}}{r_1\cdots r_{n+1}} \\
- \quad \frac{1}{n!}\frac{1}{r_1\cdots r_n}x^n - \frac{1}{n!}\frac{r_1+\cdots+r_n}{r_1\cdots r_{n+1}}x^n = \frac{1}{n!}\frac{1}{r_1\cdots r_n}x^n\left(\frac{x}{(n+1)r_{n+1}} - 1\right) \\
- \quad \frac{1}{n!}\frac{r_1+\cdots+r_n}{r_1\cdots r_{n+1}}x^n < 0.
\end{aligned}
$$

Therefore, inequality (26) is proved for $x \geq 0$. Now, inequality (26) can be written in the form

$$
\begin{aligned}
S_n(x) \geq \frac{1}{n!}\frac{x^n}{r_1\cdots r_n} \\
- \quad \frac{1}{(n-1)!}\frac{r_1+\cdots+r_n}{r_1\cdots r_n}x^{n-1} = \frac{1}{n!}\frac{x^n}{r_1\cdots r_n}\left(1 - \frac{n(r_1+\cdots+r_n)}{x}\right),
\end{aligned}
\tag{27}
$$

which completes the proof. □

We have the following immediate corollary (see Theorem 2.1).

Corollary 2.11. *Let k and s be arbitrary but fixed positive integers. If $x \geq (p_1\cdots p_k)^k$, then the following inequalities hold*

$$
\sum_{a\leq x} 1 \geq \frac{1}{k!}\frac{1}{\log p_1\cdots\log p_k}\log^k x - \frac{1}{(k-1)!}\frac{\log p_1+\cdots+\log p_k}{\log p_1\cdots\log p_k}\log^{k-1} x
$$

$$
\begin{aligned}
\sum_{a\leq x}\frac{1}{u(a)^s} \geq \frac{1}{k!}\frac{1}{(p_1\cdots p_k)^s\log p_1\cdots\log p_k}\log^k x \\
- \quad \frac{1}{(k-1)!}\frac{\log p_1+\cdots+\log p_k}{(p_1\cdots p_k)^s\log p_1\cdots\log p_k}\log^{k-1} x.
\end{aligned}
$$

Theorem 2.12. *Let s be an arbitrary but fixed positive integer. The following inequality holds*

$$\sum_{n\le x}\frac{1}{u(n)^s}\ge F_{5,s}(x)\qquad (x\ge 1),$$

where

$$F_{5,s}(x)=\sum_{\substack{p_1\cdots p_k\le x\\ (p_1\cdots p_k)^k\le x}}\left(\frac{1}{k!}\frac{1}{(p_1\cdots p_k)^s\log p_1\cdots\log p_k}\log^k x\right.$$
$$\left.-\quad\frac{1}{(k-1)!}\frac{\log p_1+\cdots+\log p_k}{(p_1\cdots p_k)^s\log p_1\cdots\log p_k}\log^{k-1}x\right).$$

Besides, for all $\beta>0$ and all $\alpha>0$ the following limits hold

$$\lim_{x\to\infty}\frac{F_{5,s}(x)}{\log^\beta x}=\infty,$$

$$\lim_{x\to\infty}\frac{F_{5,s}(x)}{x^\alpha}=0.$$

Proof: It is an immediate consequence of Corollary 2.11 and the definition of the function $F_{5,s}(x)$. □

Let q_n be the n-th prime. Note that in the series (12) the greatest term is

$$\frac{1}{(q_1\cdots q_k)^s\log q_1\cdots\log q_k}.$$

In the following theorem, we compare the sum $b_{k,s}$ of the series (12) with its greatest term. Before that, we need the following well-known lemma.

Lemma 2.13. *Let us consider the two series of positive terms $\sum_{i=1}^\infty a_i$ and $\sum_{i=1}^\infty b_i$. If $a_i\sim b_i(i\to\infty)$ and the series $\sum_{i=1}^\infty b_i$ diverges, then $\sum_{i=1}^n a_i\sim\sum_{i=1}^n b_i(n\to\infty)$*

Proof: See, for example, [7]. □

Theorem 2.14. *Let s be an arbitrary but fixed positive integer. The following limit holds*

$$\lim_{k\to\infty}\frac{b_{k,s}}{\frac{1}{(q_1\cdots q_k)^s\log q_1\cdots\log q_k}}=\infty,$$

where q_n denotes the n-th prime.

Proof: Note that (due to the Prime Number Theorem and Lemma 2.13) we have

$$\sum_{i=1}^k q_i^s\log q_i\sim\sum_{i=1}^k i^s\log^{s+1} i.$$

Now,

$$\sum_{i=1}^{k} i^s \log^{s+1} i = \int_1^k x^s \log^{s+1} x \, dx + O\left(k^s \log^{s+1} k\right) \sim \frac{k^{s+1}}{s+1} \log^{s+1} k \sim \frac{q_k^{s+1}}{s+1}$$

(since L'Hospital's rule)

$$\lim_{x\to\infty} \frac{\int_1^x t^s \log^{s+1} t \, dt}{\frac{x^{s+1}}{s+1} \log^{s+1} x} = 1.$$

Therefore,

$$\begin{aligned} \frac{b_{k,s}}{\frac{1}{(q_1\cdots q_k)^s \log q_1 \cdots \log q_k}} &= \sum_{p_1\cdots p_k} \frac{(q_1\cdots q_k)^s \log q_1 \cdots \log q_k}{(p_1\cdots p_k)^s \log p_1 \cdots \log p_k} \\ \geq & \sum_{i=1}^{k} \frac{(q_1\cdots q_k)^s \log q_1 \cdots \log q_k (q_i^s \log q_i)}{(q_1\cdots q_k)^s \log q_1 \cdots \log q_k \left(q_{k+1}^s \log q_{k+1}\right)} = \frac{1}{q_{k+1}^s \log q_{k+1}} \sum_{i=1}^{k} q_i^s \log q_i \\ \sim & \frac{k}{s+1}. \end{aligned}$$

This completes the proof. □

Theorem 2.15. *Let s be an arbitrary but fixed positive integer. The following asymptotic formulas hold*

$$\log b_{k,s} \sim -sk \log k \sim -sq_k, \tag{28}$$

$$b_{k,s} = e^{-s(1+o(1))k\log k} = e^{-s(1+o(1))q_k}, \tag{29}$$

where q_n denotes the n-th prime.

Proof: The following equation is well-known (see [4])

$$\log(q_1\cdots q_k) = k\log k + k\log\log k - k + o(k). \tag{30}$$

From the Stirling's formula $k! \sim \frac{\sqrt{2\pi}k^k\sqrt{k}}{e^k}$, we obtain

$$\log(k!) = k\log k - k + o(k), \tag{31}$$

and besides we have

$$\begin{aligned} \sum_{i=2}^{k} \log\log i &= \log\log 2 + \int_e^k \log\log x \, dx + O(\log\log k) = \log\log 2 \\ + & \ k\log\log k - \int_e^k \frac{1}{\log x}\, dx + O(\log\log k) = k\log\log k + o(k). \end{aligned}$$

Therefore, we have

$$\log\log q_1 + \cdots + \log\log q_k = \log\log q_1 + \log\log q_2 + \sum_{i=3}^{k} \log\log i + \sum_{i=3}^{k} o(1)$$
$$= \quad k \log\log k + o(k), \tag{32}$$

since the Prime Number Theorem $q_n \sim n \log n$ implies $\log\log q_n = \log\log n + o(1)$.

Now, we have the inequality (see Theorem 2.5 and the definition of $b_{k,s}$)

$$\frac{1}{(q_1 \cdots q_k)^s \log q_1 \cdots \log q_k} \leq b_{k,s} \leq \frac{(c_s^s)^k}{(k!)^s}. \tag{33}$$

Therefore,

$$-sk \log k \sim \log\left(\frac{1}{(q_1 \cdots q_k)^s \log q_1 \cdots \log q_k}\right) \leq \log b_{k,s} \leq \log\left(\frac{(c_s^s)^k}{(k!)^s}\right)$$
$$\sim \quad -sk \log k. \tag{34}$$

This completes the proof. □

Theorem 2.16. *Let s be an arbitrary but fixed positive integer. The following limit holds*

$$\lim_{k\to\infty} \frac{b_{k,s}}{\frac{1}{(q_1\cdots q_k)^{s-1} \log q_1 \cdots \log q_k}} = 0,$$

where q_n denotes the n-th prime.

Proof: We have (see Theorem 2.15)

$$\log\left(\frac{b_{k,s}}{\frac{1}{(q_1\cdots q_k)^{s-1} \log q_1 \cdots \log q_k}}\right) = \log b_{k,s} + (s-1)\log(q_1 \cdots q_k)$$
$$+ \quad (\log\log q_1 + \cdots + \log\log q_k) \sim -k \log k.$$

This completes the proof. □

Theorem 2.17. *Let s be an arbitrary but fixed positive integer. There exists k_0, such that if $k \geq k_0$, then we have*

$$b_{k,s} = \frac{1}{(q_1 \cdots q_k)^{\beta_k} \log q_1 \cdots \log q_k} \tag{35}$$

where $s-1 < \beta_k < s$, $\lim_{k\to\infty} \beta_k = s$ and q_n denotes the n-th prime.

Besides, the following asymptotic formulas hold

$$\log b_{k,s} = -\beta_k k \log k + (1-\beta_k) k \log\log k + \beta_k k + o(k) \tag{36}$$

$$b_{k,s} = \exp\left(-\beta_k k \log k + (1-\beta_k) k \log\log k + \beta_k k + o(k)\right) \tag{37}$$

$$(b_{k,s})^{1/k} \sim \exp\left(-\beta_k \log k + (1-\beta_k) \log\log k + \beta_k\right) \tag{38}$$

Proof: From the definition of $b_{k,s}$ and Theorem 2.16, there exists k_0, such that if $k \geq k_0$, the following inequality holds

$$\frac{1}{(q_1 \cdots q_k)^s \log q_1 \cdots \log q_k} < b_{k,s} < \frac{1}{(q_1 \cdots q_k)^{s-1} \log q_1 \cdots \log q_k}, \tag{39}$$

therefore, there exists a unique β_k, such that $s - 1 < \beta_k < s$ and

$$b_{k,s} = \frac{1}{(q_1 \cdots q_k)^{\beta_k} \log q_1 \cdots \log q_k}. \tag{40}$$

From equation (40) and using the formulas proved in Theorem 2.16, we obtain (36). Equation (36) can be written in the form

$$\log b_{k,s} = -\beta_k k \log k + o(k \log k). \tag{41}$$

On the other hand, we have by Theorem 2.16

$$\log b_{k,s} = -sk \log k + o(k \log k). \tag{42}$$

Equations (41) and (42) give $\lim_{k\to\infty} \beta_k = s$. □

Theorem 2.18. *Let s be an arbitrary but fixed positive integer. There exists k_0, such that if $k \geq k_0$, the following inequality holds*

$$b_{k,s} < b_{k-1,s}. \tag{43}$$

Besides, the following limit holds

$$\lim_{k\to\infty} \frac{b_{k,s}}{b_{k-1,s}} = 0. \tag{44}$$

Proof: We have, $p_1 < \cdots < p_k$,

$$\begin{aligned}
b_{k,s} &= \sum_{p_1 \cdots p_k} \frac{1}{(p_1 \cdots p_k)^s \log p_1 \cdots \log p_k} \\
&= \sum_{p_1 \cdots p_{k-1}} \left(\frac{1}{(p_1 \cdots p_{k-1})^s \log p_1 \cdots \log p_{k-1}} \left(\sum_{p \geq p_k} \frac{1}{p^s \log p} \right) \right) \\
&\leq \sum_{p_1 \cdots p_{k-1}} \left(\frac{1}{(p_1 \cdots p_{k-1})^s \log p_1 \cdots \log p_{k-1}} \left(\sum_{p \geq q_k} \frac{1}{p^s \log p} \right) \right) \\
&= b_{k-1,s} \left(\sum_{p \geq q_k} \frac{1}{p^s \log p} \right)
\end{aligned}$$

Now, $\lim_{k\to\infty} \sum_{p \geq q_k} \frac{1}{p^s \log p} = 0$. Therefore, (44) is proved.
Let us choose k_0, such that $\sum_{p \geq q_{k_0}} \frac{1}{p \log p} < 1$. Hence, if $k \geq k_0$, we have

$$\sum_{p \geq q_k} \frac{1}{p^s \log p} \leq \sum_{p \geq q_{k_0}} \frac{1}{p^s \log p} \leq \sum_{p \geq q_{k_0}} \frac{1}{p \log p} < 1.$$

Therefore, (43) is proved. □

Theorem 2.19. *Let s be an arbitrary but fixed positive integer. We have*

$$\frac{b_{k,s}}{k!}(\log x)^k = e^{E(k)}, \tag{45}$$

where

$$E(k) = \\ \left(-\left(\beta_k+1\right)\log k + \left(1-\beta_k\right)\log\log k + \beta_k + 1 + \log\log x + o(1)\right)k \tag{46}$$

and $k \geq 2$, $\beta_k \to s$.

Let $\epsilon > 0$. There exists k_0, such that if $k \geq k_0$, the following inequalities hold

$$E(k) < \left(-(s+1-\epsilon)\log k + (1-s+\epsilon)\log\log k + s + 1 + \epsilon + \log\log x\right)k, \tag{47}$$

$$E(k) > \left(-(s+1)\log k + (1-s)\log\log k + s + 1 - 2\epsilon + \log\log x\right)k. \tag{48}$$

Proof: Equation (46) is an immediate consequence of equation (36). Let $\epsilon > 0$. There exist k_0 such that if $k \geq k_0$ the following inequalities hold

$$-\epsilon < o(1) < \epsilon, \qquad s - \epsilon < \beta_k < s, \tag{49}$$

and, consequently, the following inequalities hold

$$-(s+1) < -(\beta_k+1) < -(s+1-\epsilon), \qquad 1-s < 1-\beta_k < 1-s+\epsilon, \tag{50}$$

$$s + 1 - \epsilon + \log\log x < \beta_k + 1 + \log\log x < s + 1 + \log\log x. \tag{51}$$

Inequalities (49), (50), (51) and equation (46) give inequalities (47) and (48). $\square$

Theorem 2.20. *Let s be an arbitrary but fixed positive integer. The following asymptotic formula holds*

$$F_{3,s}(x) = \sum_{k=1}^{\infty}\frac{b_{k,s}}{k!}(\log x)^k = \exp\left((\log x)^{\alpha_s(x)}\right), \tag{52}$$

where $\lim_{x\to\infty}\alpha_s(x) = \frac{1}{s+1}$.

Proof: Let $\epsilon > 0$. Let us consider the positive integer depending on x

$$k' = \left\lfloor (\log x)^{\frac{1}{s+1+\epsilon}} \right\rfloor = f(x)(\log x)^{\frac{1}{s+1+\epsilon}} = f(x)(\log x)^{\frac{1}{s+1}-\epsilon'}, \tag{53}$$

where $\lim_{x\to\infty} f(x) = 1$. We have

$$\log k' = g(x)\frac{1}{s+1+\epsilon}\log\log x, \qquad \log\log k' = h(x)\log\log\log x, \tag{54}$$

where $\lim_{x\to\infty} g(x) = 1$ and $\lim_{x\to\infty} h(x) = 1$. Substituting (53) into (48), we obtain

$$\lim_{x\to\infty} \frac{E(k')}{k'} = \infty.$$

Therefore, from a certain value of x we have

$$k' < E(k'), \tag{55}$$

and, consequently, we have (see equations (55), (45), (18) and (21))

$$e^{k'} < e^{E(k')} = \frac{b_{k',s}}{k'!}(\log x)^{k'} < F_{3,s}(x) \le e^{C\,\sqrt[s+1]{\log x}}, \tag{56}$$

where $C = (s+1)c_s^{s/s+1}$. Hence, (see (56) and (53)) if we put $\log F_{3,s}(x) = (\log x)^{\alpha_s(x)}$, then

$$k' = f(x)(\log x)^{\frac{1}{s+1}-\epsilon'} \le (\log x)^{\alpha_s(x)} \le C(\log x)^{\frac{1}{s+1}}$$

From here we obtain $\lim_{x\to\infty} \alpha_s(x) = \frac{1}{s+1}$, since ϵ and, consequently, ϵ' can be arbitrarily small. □

Theorem 2.21. *Let $s \ge 2$ an arbitrary but fixed positive integer. The following asymptotic formulas hold (see Theorems 2.8 and 2.12)*

$$\sum_{n\le x} \frac{1}{u(n)^s} \sim F_{i,s}(x) \qquad (i = 1,2,3,5). \tag{57}$$

Besides

$$\sum_{n\le x} \frac{1}{u(n)^s} = \exp\left((\log x)^{\beta_s(x)}\right), \tag{58}$$

where $\lim_{x\to\infty} \beta_s(x) = \frac{1}{s+1}$.

Proof: We have (see Theorem 2.12)

$$F_{5,s}(x) = \sum_{\substack{p_1\cdots p_k \le x\\ (p_1\cdots p_k)^k \le x}} \left(\frac{1}{k!}\frac{1}{(p_1\cdots p_k)^s \log p_1 \cdots \log p_k} \log^k x\right.$$
$$- \left.\frac{1}{(k-1)!}\frac{\log p_1 + \cdots + \log p_k}{(p_1\cdots p_k)^s \log p_1\cdots \log p_k} \log^{k-1} x\right) = G_{1,s}(x) - G_{2,s}(x). \tag{59}$$

If we put

$$a_{k,s}(x) = \sum_{p_1\cdots p_k \le x^{1/k}} \frac{1}{(p_1\cdots p_k)^s \log p_1 \cdots \log p_k}, \tag{60}$$

$$b_{k,s}(x) = \sum_{p_1\cdots p_k > x^{1/k}} \frac{1}{(p_1\cdots p_k)^s \log p_1 \cdots \log p_k}, \tag{61}$$

$$c_{k,s}(x) = \sum_{p_1\cdots p_k \le x^{1/k}} \frac{\log p_1 + \cdots + \log p_k}{(p_1\cdots p_k)^s \log p_1 \cdots \log p_k}, \tag{62}$$

then

$$\begin{aligned} G_{1,s}(x) &= \sum_{\substack{p_1\cdots p_k \le x \\ (p_1\cdots p_k)^k \le x}} \frac{1}{k!} \frac{1}{(p_1\cdots p_k)^s \log p_1 \cdots \log p_k} \log^k x \\ &= \sum_{k=1}^{h'} a_{k,s}(x) \frac{(\log x)^k}{k!}, \end{aligned} \tag{63}$$

$$\begin{aligned} G_{2,s}(x) &= \sum_{\substack{p_1\cdots p_k \le x \\ (p_1\cdots p_k)^k \le x}} \frac{1}{(k-1)!} \frac{\log p_1 + \cdots + \log p_k}{(p_1\cdots p_k)^s \log p_1 \cdots \log p_k} \log^{k-1} x \\ &= \sum_{k=1}^{h'} c_{k,s}(x) \frac{(\log x)^{k-1}}{(k-1)!}, \end{aligned} \tag{64}$$

where h' is the greatest k such that there exist $p_1 \cdots p_{h'}$, such that $(p_1 \cdots p_{h'})^{h'} \le x$. Note that (see (60), (61) and (12))

$$b_{k,s} = a_{k,s}(x) + b_{k,s}(x). \tag{65}$$

Let q_n be the n-th prime. Let us consider the inequality $(q_1 \cdots q_k)^k \le x$, that is $k \log(q_1 \cdots q_k) \le \log x$, that is (see Theorem 2.16) $f(k)k^2 \log k \le \log x$, where $f(k) \to 1$. From here we obtain

$$\left\lfloor (\log x)^{5/12} \right\rfloor \le h' \le \left\lfloor (\log x)^{1/2} \right\rfloor. \tag{66}$$

Note that $1/3 < 5/12 < 1/2$.

Let $k_0 = \lfloor \log x \rfloor$. We have (see (43) and (29))

$$\begin{aligned} 0 &\le \sum_{k=k_0}^{\infty} \frac{b_{k,s}}{k!} (\log x)^k \le b_{k_0,s} \sum_{k=k_0}^{\infty} \frac{(\log x)^k}{k!} \le b_{k_0,s} e^{\log x} \\ &= \exp\left(-s(1+o(1)) \lfloor \log x \rfloor \log \lfloor \log x \rfloor + \log x\right) \\ &= \exp\left(-s(1+o(1)) \log x \log\log x + \log x\right) = o(1). \end{aligned}$$

That is

$$\sum_{k=\lfloor \log x \rfloor}^{\infty} \frac{b_{k,s}}{k!} (\log x)^k = o(1). \tag{67}$$

Let us put

$$k' = \left\lfloor (\log x)^{\frac{1}{s+1-2\epsilon}} \right\rfloor = \left\lfloor (\log x)^{\frac{1}{s+1}+\epsilon'} \right\rfloor. \tag{68}$$

Equation (47) gives

$$\lim_{x\to\infty}\frac{E(k')}{k'}=-\infty.$$

Therefore, from a certain value of x

$$E(k')<0$$

and also (see (47)) if $k\geq k'$,

$$E(k)<0.$$

Therefore, if $k\geq k'$,

$$\frac{b_{k,s}}{k!}(\log x)^k=e^{E(k)}<1. \tag{69}$$

On the other hand, we have

$$\begin{aligned}\sum_{p_1\cdots p_k>x}\frac{1}{(p_1\cdots p_k)^s\log p_1\cdots\log p_k}&\leq\sum_{p_1\cdots p_k>x}\frac{1}{\log 2}\frac{1}{(p_1\cdots p_k)^s}\leq\frac{1}{\log 2}\sum_{n>x}\frac{1}{n^s}\\ \leq\ \frac{1}{\log 2}\int_{\lfloor x\rfloor}^{\infty}t^{-s}\,dt&=\frac{1}{\log 2}\frac{1}{(s-1)\left(\lfloor x\rfloor\right)^{s-1}}=C\frac{1}{\left(\lfloor x\rfloor\right)^{s-1}}. \end{aligned}\tag{70}$$

Consequently, if $k=1,2,\ldots,k'=\left\lfloor(\log x)^{\frac{1}{s+1}+\epsilon'}\right\rfloor$, then (see (61) and (70))

$$b_{k,s}(x)\leq C\frac{1}{\left(\lfloor x^{1/k}\rfloor\right)^{s-1}}\leq C\frac{1}{\left(\lfloor x^{1/k'}\rfloor\right)^{s-1}}. \tag{71}$$

Therefore, (see (71)) we have

$$\begin{aligned}0\leq\sum_{k=1}^{k'}b_{k,s}(x)\frac{(\log x)^k}{k!}&\leq\frac{Ck'}{\left(\lfloor x^{1/k'}\rfloor\right)^{s-1}}\frac{(\log x)^{k'}}{1!}=\frac{Ck'}{f(x)x^{(s-1)/k'}}(\log x)^{k'}\\ =\ \exp\left(-(s-1)(1+o(1))(\log x)^{\frac{s}{s+1}-\epsilon'}\right)&=o(1).\end{aligned}$$

That is,

$$\sum_{k=1}^{k'}b_{k,s}(x)\frac{(\log x)^k}{k!}=o(1). \tag{72}$$

Now, (see equations (21), (60), (61), (63), (65), (66), (68), (69), (72) and (67)), we have

$$\begin{aligned}F_{3,s}(x)&=\sum_{k=1}^{\infty}b_{k,s}\frac{(\log x)^k}{k!}=\sum_{k=1}^{h'}(a_{k,s}(x)+b_{k,s}(x))\frac{(\log x)^k}{k!}\\ &+\sum_{k=h'+1}^{\infty}b_{k,s}\frac{(\log x)^k}{k!}=G_{1,s}(x)+\sum_{k=1}^{h'}b_{k,s}(x)\frac{(\log x)^k}{k!}\\ &+\sum_{k=h'+1}^{\infty}b_{k,s}\frac{(\log x)^k}{k!}=G_{1,s}(x)+\sum_{k=1}^{k'}b_{k,s}(x)\frac{(\log x)^k}{k!}\\ &+\sum_{k=k'+1}^{h'}b_{k,s}(x)\frac{(\log x)^k}{k!}+\sum_{k=h'+1}^{\lfloor\log x\rfloor-1}b_{k,s}\frac{(\log x)^k}{k!}+\sum_{k=\lfloor\log x\rfloor}^{\infty}b_{k,s}\frac{(\log x)^k}{k!}\\ &=G_{1,s}(x)+o(1)+O\left(\log x\right)+o(1)=G_{1,s}(x)+O\left(\log x\right)\\ &=G_{1,s}(x)+o\left(F_{3,s}(x)\right).\end{aligned}\tag{73}$$

Besides,

$$\sum_{p_1\cdots p_k}\frac{\log p_1+\cdots+\log p_k}{(p_1\cdots p_k)^s\log p_1\cdots\log p_k}=\sum_{p_1\cdots p_k}\frac{\log(p_1\cdots p_k)}{(p_1\cdots p_k)^s\log p_1\cdots\log p_k}$$
$$\le \frac{1}{\log 2}\sum_{n=1}^{\infty}\frac{\log n}{n^s},$$

that is, the series converges. Hence, we have

$$\sum_{p_1\cdots p_k}\frac{\log p_1+\cdots+\log p_k}{(p_1\cdots p_k)^s\log p_1\cdots\log p_k}=c_{k,s}. \tag{74}$$

If $p_1<\cdots<p_k$, we have

$$\begin{aligned}
c_{k,s} &= \sum_{p_1\cdots p_k}\frac{\log p_1+\cdots+\log p_k}{(p_1\cdots p_k)^s\log p_1\cdots\log p_k}\\
&= \sum_{i=k}^{\infty}\left(\sum_{p_1\cdots p_{k-1}q_i}\frac{\log p_1+\cdots+\log p_{k-1}+\log q_i}{(p_1\cdots p_{k-1}q_i)^s\log p_1\cdots\log p_{k-1}\log q_i}\right)\\
&\le \sum_{i=k}^{\infty}\left(\sum_{p_1\cdots p_{k-1}q_i}\frac{k\log q_i}{(p_1\cdots p_{k-1}q_i)^s\log p_1\cdots\log p_{k-1}\log q_i}\right)\\
&\le kb_{k-1,s}\sum_{i=k}^{\infty}\frac{1}{q_i^s}\le kb_{k-1,s}\sum_{n\ge q_k}^{\infty}\frac{1}{n^s}\le b_{k-1,s}\frac{1}{s-1}\frac{k}{(q_k-1)^{s-1}}
\end{aligned}$$

Therefore,

$$\lim_{k\to\infty}\frac{c_{k,s}}{b_{k-1,s}}=0 \tag{75}$$

Consequently, if $\alpha>0$ there exists k_0 such that if $k\ge k_0$, then $\frac{c_{k,s}}{b_{k-1,s}}<\alpha$. Hence, (see (64))

$$\begin{aligned}
0 &\le G_{2,s}(x)=\sum_{k=1}^{h'}c_{k,s}(x)\frac{(\log x)^{k-1}}{(k-1)!}\le\sum_{k=1}^{\infty}c_{k,s}\frac{(\log x)^{k-1}}{(k-1)!}\\
&= \sum_{k=1}^{k_0-1}a_{k,s}\frac{(\log x)^{k-1}}{(k-1)!}+\alpha\sum_{k=k_0}^{\infty}b_{k-1,s}\frac{(\log x)^{k-1}}{(k-1)!}=\sum_{k=1}^{k_0-1}a_{k,s}\frac{(\log x)^{k-1}}{(k-1)!}\\
&+ \alpha\sum_{k=k_0-1}^{\infty}b_{k,s}\frac{(\log x)^{k}}{(k)!}.
\end{aligned}$$

That is, from a certain value of x we have (see (21) and (22))

$$0\le\frac{G_{2,s}(x)}{F_{3,s}(x)}\le o(1)+\alpha\le 2\alpha$$

and since α can be arbitrarily small, we find that

$$G_{2,s}(x)=o\left(F_{3,s}(x)\right). \tag{76}$$

Equations (59), (76) and (73) give $F_{3,s}(x) \sim F_{5,s}(x)$ and, consequently, (see (59), (18) and Theorem 2.12) equation (57) is proved. Finally, equations (52) and (57) give (58). □

To finish, we establish the following question.

Question: Does equation (57) hold when $s = 1$?

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References

[1] Beukers, F. (1975). The lattice points of n-dimensional tetrahedra, *Indag. Math.*, 37, 365–372.

[2] De Koninck, J., Diouf, I., & Doyon, N. (2012). On the truncated kernel function, *J. Integer Seq.*, 15, Article 12.3.2.

[3] Hardy, G. H., & Wright, E. M. (1960). *An Introduction to the Theory of Numbers,* Oxford.

[4] Jakimczuk, R. (2008). An observation on the Cipolla's expansion, *Mathematical Sciences. Quarterly Journal*, 2, 219–222.

[5] Jakimczuk, R. (2007). Integers of the form $p_1^{s_1} p_2^{s_2} \cdots p_k^{s_k}$, where $p_1, p_2, \ldots, p_k$ are primes fixed, *International Journal of Contemporary Mathematical Sciences*, 2, 1327–1333.

[6] Jakimczuk, R. (2017). On the kernel function, *International Mathematical Forum*, 12, 693–703.

[7] Rey Pastor, J., Pi Calleja, P., & Trejo, C. A. (1969). *Análisis Matemático, Volume 1,* Kapelusz.