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The linear combination of two triangular numbers is a perfect square

Junyao Peng[1,2]

[1] Chongqing Fuling No.15 Middle School
Chongqing, 400000, China
[2] School of Mathematics and Statistics, Changsha University of Science and Technology
Hunan Provincial Key Laboratory of Mathematical Modeling and Analysis in Engineering
Changsha, 410114, China
e-mail: junyaopeng906@163.com

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Abstract: By the basic properties of Pell equation and the theory of congruence, we investigate the problem about the linear combination of two triangular numbers is a perfect square. First, we show that if $2n$ is not a perfect square, the Diophantine equation

$$1 + n\binom{y}{2} = z^2$$

has infinitely many positive integer solutions (y, z). Second, we prove that if m, n are some special values, the Diophantine equation

$$m\binom{x}{2} + n\binom{y}{2} = z^2$$

has infinitely many positive integer solutions (x, y, z). At last, we raise some related questions.
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1 Introduction and main results

A triangular number is a positive integer of the form

$$t_{x-1} = \binom{x}{2} = \frac{x(x-1)}{2}, x \geq 2, x \in \mathbb{Z}.$$

Research on triangular numbers can be traced back to Pythagoras (570–501 B. C.). Many remarkable properties of triangular numbers have been discovered by Fermat, Euler, Legendre, Gauss and other great mathematicians [4]: Legendre proved that no triangular number, except 1, is a cube or fourth power; Gauss showed that every natural number is a sum of at most three triangular numbers; Euler determined infinitely many triangular numbers which are perfect squares.

In 2005, Bencze [1] raised a problem: find out all positive integers n which make the form $1+\frac{9}{2}n(n+1)$ to be a perfect square. In 2007, Le [9] showed that all positive integers n of the form $1+\frac{9}{2}n(n+1)$ that are perfect squares are given by

$$n=\frac{1}{2}\big(\frac{1}{6}(a^{2k+1}+b^{2k+1})-1\big),$$

where $a=3+\sqrt{8}, b=3-\sqrt{8}$, and $k\in\mathbb{Z}^+$. In 2011, Guan [6] proved that all positive integers n which are of the form $1+\frac{4n(n+1)s^2}{s^2-1}$ that are perfect squares are given by

$$n=\frac{1}{2}\big(\frac{1}{2s}(a^{2k+1}+b^{2k+1})-1\big),$$

where s is a positive odd integer with $s>1$, and $k\in\mathbb{Z}^+$. In 2013, Hu [7] considered the positive integer solutions of the Diophantine equation

$$1+n\binom{y}{2}=z^2, \tag{1.1}$$

where

$$n=\begin{cases}\dfrac{t^2\pm 1}{2}, t\equiv 1 \pmod 2, t\geq 3,\\ \dfrac{t^2\pm 2}{2}, t\equiv 0 \pmod 2, t\geq 2,\\ \dfrac{t(t-1)}{2}=\dbinom{t}{2}, t\geq 2.\end{cases}$$

There are more related results in [2, 10, 12].

In this paper, by the theory of Pell equation, we have the following results.

Theorem 1.1. *If $2n$ is not a perfect square, then Eq. (1.1) has infinitely many positive integer solutions.*

In order to illustrate Theorem 1.2, we give an integer solution of the Pell equation $x^2-d(t)y^2=1$ $(t\equiv 0 \pmod 2, t\geq 2)$ in Table 1.

$d(t)$	$x(t), y(t)$
$t(s^2t\pm 1), s\in\mathbb{Z}^+$	$(2s^2t\pm 1, 2s)$
$t(s^2t\pm 2), s\in\mathbb{Z}^+$	$(s^2t\pm 1, s)$
$9t^2\pm 8t+2$	$((9t\pm 4)^2+1, 3(9t\pm 4))$
$49t^2\pm 20t+2$	$((49t\pm 10)^2-1, 7(49t\pm 10))$
$t(t^3\pm 2)$	$(t^3\pm 1, t)$

Table 1. An integer solution of the Pell equation $x^2-d(t)y^2=1$

Theorem 1.2. *When $n = \frac{d(t)}{2}$, then Eq. (1.1) has infinitely many positive integer solutions.*

Notice that $1 = \binom{2}{2}$, then $1 + n\binom{y}{2} = z^2$ can be written as $\binom{2}{2} + n\binom{y}{2} = z^2$. So we consider the Diophantine equation

$$m\binom{x}{2} + n\binom{y}{2} = z^2, \tag{1.2}$$

where $m, n \in \mathbb{Z}^+$. More remarkable, in 2009, Sun [11] investigated the number of representations of n by

$$a\frac{x(x-1)}{2} + b\frac{y(y-1)}{2},$$

where $a \geq 1, b \geq 1$.

When $m = n = 1$, let $y = x + 1$, Eq. (1.2) becomes

$$\binom{x}{2} + \binom{x+1}{2} = x^2,$$

i.e., $z = x$. In 1897, Fauquembergue [4] noticed that $\binom{x}{2} + \binom{y}{2} = z^2$ equals

$$(2x-1)^2 + (2y-1)^2 = (2z+1)^2 + (2z-1)^2,$$

by Euler's formula, if there exist $a, b, c, d \in \mathbb{Z}^+$ such that $bc + ad = ac - bd + 2$, then all solutions of the above equation are given by

$$\begin{cases} 2x - 1 = ac + bd, \\ 2y - 1 = bc - ad, \\ 2z + 1 = bc + ad. \end{cases}$$

According to the condition of the quadratic equation with multiple roots, we have the following theorem.

Theorem 1.3. *When $\frac{m(m+1)}{2} = u^2, n = 1$, there exist infinitely many pairs (a, b) of positive integer numbers such that Eq. (1.2) has integer parametric solutions $(t, at + b, u(ct + d))$, where t is a positive integer greater than 1.*

Moreover, we get

Theorem 1.4. *If $2(m + n)$ is not a perfect square, $r \in \mathbb{Z}$, and the Pell equation*

$$X^2 - 2(m+n)Z^2 = \left(\frac{m+n}{2}\right)^2 - r^2mn$$

has a positive integer solution satisfying

$$X_0 - rn + \frac{m+n}{2} \equiv 0 \pmod{m+n},$$

then Eq. (1.2) has infinitely many positive integer solutions.

In particular,

Theorem 1.5. *Let u, v be integers with $u > \sqrt{2}v$, and u being a positive even integer. When $m = (u^2 - 2v^2)^2, n = 8u^2v^2$, then the Eq. (1.2) has infinitely many positive integer solutions.*

2 Preliminaries

In order to prove the above results, we need the following lemmas.

Lemma 2.1. *[8] Let D be a positive integer which is not a perfect square, then the Pell equation $x^2 - Dy^2 = 1$ has infinitely many positive integer solutions. If (U, V) is the least positive integer solution of the Pell equation $x^2 - Dy^2 = 1$, then all positive solutions are given by*

$$x_k + y_k\sqrt{D} = (U + V\sqrt{D})^k,$$

where k is an arbitrary integer.

Lemma 2.2. *[8] Let D be a positive integer which is not a perfect square, N be a nonzero integer, and (U, V) is the least positive integer solution of $x^2 - Dy^2 = 1$. If (x_0, y_0) is a positive integer solution of $x^2 - Dy^2 = N$, all positive solutions are given by*

$$x_k + y_k\sqrt{D} = (x_0 + y_0\sqrt{D})(U + V\sqrt{D})^k,$$

where k is an arbitrary integer.

Lemma 2.3. *[5] Let D be a positive integer which is not a perfect square, m be a positive integer, and N be a nonzero integer. If the Pell equation $x^2 - Dy^2 = N$ has a positive integer solution satisfying*

$$(u_0, v_0) \equiv (a, b) \pmod{m},$$

then it has infinitely many positive integer solutions satisfying

$$(u, v) \equiv (a, b) \pmod{m}.$$

3 Proofs of the theorems

Proof of Theorem 1.1. Multiply Eq. (1.1) by 8, we have

$$(n(2y-1))^2 - 2n(2z)^2 = n(n-8).$$

Set $Y = n(2y-1), Z = 2z$, we get the Pell equation

$$Y^2 - 2nZ^2 = n(n-8). \tag{3.1}$$

By Lemma 2.1, if $2n$ is not a perfect square, the Pell equation $Y^2 - 2nZ^2 = 1$ always has an infinite number of integer solutions. And suppose (u, v) is the least positive integer solution of $Y^2 - 2nZ^2 = 1$. It is easy to note that $(Y_0, Z_0) = (n, 2)$ is an integer solution of Eq. (3.1). By Lemma 2.2, an infinity of positive integer solutions of Eq. (3.1) are given by

$$Y_k + Z_k\sqrt{2n} = \left(n + 2\sqrt{2n}\right)\left(u + v\sqrt{2n}\right)^k, k \geq 0.$$

Clearly, we have a solution $(Y_0, Z_0) = (n, 2)$ of Eq. (3.1) satisfying

$$Y_0 + n \equiv 0 \pmod{2n},\ Z_0 \equiv 0 \pmod{2}.$$

Lemma 2.3 guarantees that Eq. (3.1) has infinitely many positive integer solutions (Y, Z) with the above condition. Then there are infinitely many

$$y = \frac{1}{2}\left(\frac{Y}{n} + 1\right) \in \mathbb{Z}^+,\ z = \frac{Z}{2} \in \mathbb{Z}^+.$$

Thus, if $2n$ is not a perfect square, Eq. (1.1) has infinitely many positive integer solutions (y, z). □

Example 3.1. *When $n = 10$, $2n = 20$ is not a perfect square, we have the Pell equation*

$$Y^2 - 20Z^2 = 20.$$

It is easy to see that $(u, v) = (9, 2)$ is the least positive integer solution of $Y^2 - 20Z^2 = 1$, and $(Y_0, Z_0) = (10, 2)$ is a positive integer solution of $Y^2 - 20Z^2 = 20$. Then an infinity of positive integer solutions of $Y^2 - 20Z^2 = 20$ are given by

$$Y_k + Z_k\sqrt{20} = \left(10 + 2\sqrt{20}\right)\left(9 + 2\sqrt{20}\right)^k, k \geq 0.$$

Thus

$$\begin{cases} Y_k = 18Y_{k-1} - Y_{k-2},\ Y_0 = 10,\ Y_1 = 170, \\ Z_k = 18Z_{k-1} - Z_{k-2},\ Z_0 = 2,\ Z_1 = 38. \end{cases}$$

From the above recurrence relations, we have

$$Y_k + 10 \equiv 0 \pmod{20},\ Z_k \equiv 0 \pmod{2},$$

then

$$y_k = \frac{1}{2}\left(\frac{Y_k}{10} + 1\right) \in \mathbb{Z}^+,\ z_k = \frac{Z_k}{2} \in \mathbb{Z}^+.$$

Therefore, when $n = 10$, Eq. (1.1) has infinitely many positive integer solutions (y_k, z_k).

Proof of Theorem 1.2. We consider only the case $n = \frac{d(t)}{2} = \frac{t(s^2t\pm1)}{2}$, where $s = 2, t \equiv 0 \pmod{2}$ and $t \geq 2$. The other cases are dealt with similarly. Then Eq. (3.1) becomes

$$Y^2 - t(4t \pm 1)Z^2 = \frac{t(4t \pm 1)}{2}\left(\frac{t(4t \pm 1)}{2} - 8\right). \tag{3.2}$$

Let us note that the pair $(Y_0, Z_0) = \left(\frac{t(4t\pm1)}{2}, 2\right)$ is a solution of Eq. (3.2). Moreover, the pair $(u, v) = (8t \pm 1, 4)$ solves the equation $Y^2 - t(4t \pm 1)Z^2 = 1$. Then an infinity of positive integer solutions of Eq. (3.2) are given by

$$Y_k + Z_k\sqrt{t(4t \pm 1)} = \left(\frac{t(4t \pm 1)}{2} + 2\sqrt{t(4t \pm 1)}\right)\left(8t \pm 1 + 4\sqrt{t(4t \pm 1)}\right)^k, k \geq 0.$$

Thus

$$\begin{cases} Y_k = 2(8t \pm 1)Y_{k-1} - Y_{k-2},\ Y_0 = \frac{t(4t\pm1)}{2},\ Y_1 = \frac{t(4t\pm1)\left((8t\pm1)+16\right)}{2}, \\ Z_k = 2(8t \pm 1)Z_{k-1} - Z_{k-2},\ Z_0 = 2,\ Z_1 = 2(8t \pm 1) + 2t(4t \pm 1). \end{cases}$$

Then

$$\begin{cases} y_k = 2(8t \pm 1)y_{k-1} - y_{k-2} - (8t \pm 1) + 1, \ y_0 = 1, \ y_1 = \frac{(8t\pm1)+17}{2}, \\ z_k = 2(8t \pm 1)z_{k-1} - z_{k-2}, \ z_0 = 1, \ z_1 = (8t \pm 1) + t(4t \pm 1). \end{cases}$$

From the above recurrence relations, we have $y_k, z_k \in \mathbb{Z}^+$. Thus, when $n = \frac{d(t)}{2} = \frac{t(s^2t\pm1)}{2}$, where $s = 2, t \equiv 0 \pmod 2$ and $t \geq 2$, Eq. (1.1) has infinitely many positive integer solutions (y_k, z_k). □

Example 3.2. *When* $n = \frac{d(t)}{2} = \frac{t(s^2t\pm1)}{2}$, *where* $s = 2, t \equiv 0 \pmod 2$ *and* $t \geq 2$, *we take* $t = 2$, *then* $n = 7$ *or* $n = 9$.

When $t = 2, n = 7$, *Eq. (1.1) has infinitely many positive integer solutions*

$$\begin{cases} y_k = 30y_{k-1} - y_{k-2} - 14, \ y_0 = 1, \ y_1 = 16, \\ z_k = 30z_{k-1} - z_{k-2}, \ z_0 = 1, \ z_1 = 29. \end{cases}$$

When $t = 2, n = 9$, *Eq. (1.1) has infinitely many positive integer solutions*

$$\begin{cases} y_k = 34y_{k-1} - y_{k-2} - 16, \ y_0 = 1, \ y_1 = 17, \\ z_k = 34z_{k-1} - z_{k-2}, \ z_0 = 1, \ z_1 = 35. \end{cases}$$

Proof of Theorem 1.3. When $\frac{m(m+1)}{2} = u^2, n = 1$, let

$$x = t, \ y = at + b,$$

then Eq. (1.2) equals to

$$\frac{a^2 + m}{2}t^2 + \frac{2ab - a - m}{2}t + \frac{b^2 - b}{2} = z^2. \tag{3.3}$$

Consider

$$g(t) = \frac{a^2 + m}{2}t^2 + \frac{2ab - a - m}{2}t + \frac{b^2 - b}{2}$$

as a quadratic polynomial of t, if $g(t) = 0$ has multiple roots, the discriminant of $g(t)$ is zero, i.e.,

$$-abm - b^2m + \frac{1}{4}a^2 + \frac{1}{2}am + bm + \frac{1}{4}m^2 = 0.$$

It implies

$$a = 2bm - m \pm 2\sqrt{m(m+1)b(b-1)}.$$

To find $a \in \mathbb{Z}^+$, we take $m(m+1)b(b-1) = v^2$, then

$$(2v)^2 - m(m+1)(2b-1)^2 = -m(m+1).$$

Let $X = 2v, Y = 2b - 1$, we obtain the Pell equation

$$X^2 - m(m+1)Y^2 = -m(m+1). \tag{3.4}$$

It easy to see that the pair $(X_0, Y_0) = (2m(m+1), 2m+1)$ is a solution of Eq. (3.4), and the pair $(U, V) = (2m+1, 2)$ solves the equation $X^2 - m(m+1)Y^2 = 1$. So an infinity of positive integer solutions of Eq. (3.4) are given by

$$X_k+Y_k\sqrt{m(m+1)} = \left(2m(m+1)+(2m+1)\sqrt{m(m+1)}\right)\left(2m+1+2\sqrt{m(m+1)}\right)^k, k \geq 0.$$

Thus

$$\begin{cases} X_k = 2(2m+1)X_{k-1} - X_{k-2},\ X_0 = 2m(m+1),\ X_1 = 4m(m+1)(2m+1), \\ Y_k = 2(2m+1)Y_{k-1} - Y_{k-2},\ Y_0 = 2m+1,\ Y_1 = 8m^2+8m+1. \end{cases}$$

According the above recurrence relations, we have

$$\begin{aligned} b_k =& \frac{Y_k+1}{2} \in \mathbb{Z}^+, \\ a_k =& 2m\frac{Y_k+1}{2} - m \pm 2\frac{X_k}{2} = 2mb_k - m \pm 2v_k \in \mathbb{Z}^+. \end{aligned}$$

From $X = 2v$ and $X_k = 2(2m+1)X_{k-1} - X_{k-2}$, we have

$$m(m+1) \mid v.$$

Let $v = m(m+1)w$, then

$$\frac{b(b-1)}{2} = \frac{m(m+1)}{2}w^2.$$

Because of $g(t) = 0$ has multiple roots, then Eq. (3.3) can be reformulated in the form

$$\frac{a^2+m}{2}t^2 + \frac{2ab-a-m}{2}t + \frac{b^2-b}{2} = \frac{m(m+1)}{2}(ct+d)^2.$$

Thus there exist infinitely many positive pairs (a_k, b_k) such that $m\binom{x}{2} + \binom{y}{2} = z^2$ has positive integer parametric solutions $\big(t, a_kt+b_k, u(c_kt+d_k)\big)$, where $t > 1, t \in \mathbb{Z}$. □

Example 3.3. *When* $m = 1, \frac{m(m+1)}{2} = 1^2, n = 1$, *we have* $a_0 = 7, b_0 = 2$ *i.e.,* $y = 7t+2$, *then Eq. (1.2) has positive integer parametric solutions* $(t, 7t+2, 5t+1)$, *where* $t > 1, t \in \mathbb{Z}$.

When $m = 8, \frac{m(m+1)}{2} = 6^2, n = 1$, *we take* $a_0 = 280, b_0 = 9$, *i.e.,* $y = 280t+9$, *then Eq. (1.2) has positive integer parametric solutions* $(t, 280t+9, 6(33t+1))$, *where* $t > 1, t \in \mathbb{Z}$.

Proof of Theorem 1.4. Let $y = x+r, r \in \mathbb{Z}$, Eq. (1.2) equals

$$\left((m+n)x + rn - \frac{m+n}{2}\right)^2 - 2(m+n)z^2 = \left(\frac{m+n}{2}\right)^2 - r^2mn.$$

Take $X = (m+n)x + rn - \frac{m+n}{2}, Z = z$, we get

$$X^2 - 2(m+n)Z^2 = \left(\frac{m+n}{2}\right)^2 - r^2mn. \tag{3.5}$$

By Lemma 2.1, if $2(m+n)$ is not a perfect square, the Pell equation

$$X^2 - 2(m+n)Z^2 = 1$$

has infinitely many positive integer solutions. By Lemma 2.2, if Eq. (3.5) has a positive integer solution, it has infinitely many positive integer solutions. Assume that Eq. (3.5) has a positive integer solution (X_0, Z_0) satisfying

$$X_0 - rn + \frac{m+n}{2} \equiv 0 \pmod{m+n},$$

By Lemma 2.3, Eq. (3.5) has infinitely many positive integer solutions (X, Z) satisfying the above condition, which leads to infinitely many $x, z \in \mathbb{Z}^+$. Then there are infinitely many $y = x + r \in \mathbb{Z}^+$. Hence, Eq. (1.2) has infinitely many positive integer solutions (x, y, z). □

Example 3.4. *When $m = 1, r = 2$, i.e., $y = x + 2$, suppose that $2(n+1)$ is not a perfect square, where n is odd. We get*

$$X^2 - 2(n+1)Z^2 = \frac{1}{4}n^2 - \frac{7}{2}n + \frac{1}{4}.$$

And the above Pell equation has a solution $(X_0, Z_0) = \left(\frac{n-3}{2}, 1\right)$ satisfying

$$X_0 - 2n + \frac{n+1}{2} \equiv 0 \pmod{n+1},$$

where $\frac{n-3}{2} \in \mathbb{Z}$.

When $n = 5$, we have

$$X^2 - 12Z^2 = -11.$$

An infinity of positive integer solutions are given by

$$X_k + Z_k\sqrt{12} = \left(1 + \sqrt{12}\right)\left(7 + 2\sqrt{12}\right)^k, k \geq 0.$$

Thus

$$\begin{cases} X_k = 14X_{k-1} - X_{k-2},\ X_0 = 1,\ X_1 = 31, \\ Z_k = 14Z_{k-1} - Z_{k-2},\ Z_0 = 1,\ Z_1 = 9. \end{cases}$$

It is easy to prove that

$$X_k - 7 \equiv 0 \pmod{6},$$

then $x_k, z_k \in \mathbb{Z}^+$. Hence, we have

$$\begin{cases} x_k = 14x_{k-1} - x_{k-2} + 14,\ x_1 = 4,\ x_2 = 63, \\ y_k = x_k + 2,\ y_1 = 6,\ y_2 = 65, \\ z_k = 14z_{k-1} - z_{k-2},\ z_1 = 9,\ z_2 = 125. \end{cases}$$

Therefore, when $m = 1, n = 5$, Eq. (1.2) has infinitely many positive integer solutions (x_k, y_k, z_k).

Proof of Theorem 1.5. By Theorem 1.4, we need to find a positive integer solution (X_0, Z_0) satisfying

$$X_0 - rn + \frac{m+n}{2} \equiv 0 \pmod{m+n}.$$

Suppose that $X_0 = \frac{m+n}{2}$, then Z_0 satisfying

$$2(m+n)Z_0^2 = r^2 mn. \tag{3.6}$$

From $X_0 = (m+n)x_0 + rn - \frac{m+n}{2}$, we have $x_0 = 1 - \frac{rn}{m+n}$, then it is sufficient to show that

$$\frac{rn}{m+n} \in \mathbb{Z}.$$

Take $r = -t(m+n)$, where $t > 0$, the condition (3.6) is equivalent to

$$r^2 = t^2(m+n)^2 = \frac{2(m+n)}{mn} Z_0^2.$$

Thus

$$Z_0^2 = \frac{t^2 mn(m+n)}{2}.$$

In view of Z_0 is an integer, then $\frac{mn(m+n)}{2}$ is a perfect square. Let

$$m = \alpha^2, \frac{n}{2} = \beta^2, m+n = \gamma^2,$$

where $\alpha, \beta, \gamma \in \mathbb{Z}^+$. Then we get a quadratic equation

$$\alpha^2 + 2\beta^2 = \gamma^2,$$

which has a solution

$$\alpha = u^2 - 2v^2, \beta = 2uv, \gamma = u^2 + 2v^2,$$

where $u, v \in \mathbb{Z}^+, u > \sqrt{2}v$, and u is a positive even integer. Hence,

$$m = (u^2 - 2v^2)^2, n = 8u^2v^2, Z_0 = \alpha\beta\gamma t = 2uv(u^4 - 4v^4)t,$$

and

$$X_0 = \frac{m+n}{2} = \frac{(u^2+2v^2)^2}{2} \in \mathbb{Z}^+.$$

Notice that $2(m+n) = 2\gamma^2$ is not a perfect square, by Lemma 2.1, the Pell equation $X^2 - 2(u^2+2v^2)^2Z^2 = 1$ has infinitely many positive integer solutions. Let (U_0, V_0) be the least positive integer solution of $X^2 - 2(u^2+2v^2)^2Z^2 = 1$, where U_0 is odd. And let the Pell equation

$$X^2 - 2(u^2+2v^2)^2Z^2 = \left(\frac{(u^2+2v^2)^2}{2}\right)^2 - 8u^2v^2(u^4-4v^4)^2(u^2+2v^2)^2t^2 \tag{3.7}$$

have a positive integer solution $(X_0, Z_0) = \left(\frac{(u^2+2v^2)^2}{2}, 2uv(u^4-4v^4)t\right)$. It is easy to prove that

$$X_0 - 8u^2v^2(u^2+2v^2)^2t + \frac{(u^2+2v^2)^2}{2} \equiv 0 \pmod{(u^2+2v^2)^2}.$$

By Lemma 2.2, an infinity of positive integer solutions are given by

$$X_k+Z_k\sqrt{2(u^2+2v^2)^2}=\left(\frac{(u^2+2v^2)^2}{2}+2uv(u^4-4v^4)t\sqrt{2(u^2+2v^2)^2}\right)\left(U_0+V_0\sqrt{2(u^2+2v^2)^2}\right)^k, k\geq 0.$$

Thus

$$\begin{cases} X_k=2U_0X_{k-1}-X_{k-2}, & X_0=\dfrac{(u^2+2v^2)^2}{2},\\ & X_1=4V_0uvt(u^2+2v^2)^2(u^4-4v^4)+U_0\dfrac{(u^2+2v^2)^2}{2},\\ Z_k=2U_0Z_{k-1}-Z_{k-2}, & Z_0=2uvt(u^4-4v^4),\\ & Z_1=2U_0uvt(u^4-4v^4)+V_0\dfrac{(u^2+2v^2)^2}{2}.\end{cases}$$

Then

$$\begin{cases} x_k=2U_0x_{k-1}-x_{k-2}-(U_0-1)(16tu^2v^2+1),\\ y_k=x_k-t(u^2+2v^2)^2,\\ z_k=2U_0z_{k-1}-z_{k-2},\end{cases}$$

where

$$\begin{aligned} x_0=&1+8tu^2v^2,\ x_1=4V_0uvt(u^4-4v^4)+8tu^2v^2+\frac{1}{2}(U_0+1),\\ y_0=&1-(u^2-2v^2)^2t,\ y_1=(u^2-2v^2)t\big(4V_0uv(u^2+2v^2)-(u^2-2v^2)\big)+\frac{1}{2}(U_0+1),\\ z_0=&2uvt(u^4-4v^4),\ z_1=2U_0uvt(u^4-4v^4)+V_0\frac{(u^2+2v^2)^2}{2}.\end{aligned}$$

When $u,v\in\mathbb{Z}^+, u>\sqrt{2}v$, and u is a positive even integer, for any $k\geq 1$, we deduce that x_k,y_k,z_k are positive integers greater than 1. Thus Eq. (1.2) has infinitely many positive integer solutions (x_k,y_k,z_k). □

Example 3.5. *When $u=2,v=1$, then $m=4,n=32,r=-36t$, and $2(m+n)=72$ is not a perfect square. We have*

$$X^2-72Z^2=18^2-72(48t)^2.$$

An infinity of positive integer solutions are given by

$$X_k+Z_k\sqrt{72}=(18+48t\sqrt{72})(17+2\sqrt{72})^k, k\geq 0.$$

Thus

$$\begin{cases} X_k=34X_{k-1}-X_{k-2},\ X_0=18,\ X_1=6912t+306,\\ Z_k=34Z_{k-1}-Z_{k-2},\ Z_0=48t,\ Z_1=816t+36.\end{cases}$$

It is easy to prove that

$$X_k+1152t+18\equiv 0 \pmod{36},$$

then $x_k, z_k \in \mathbb{Z}^+$. Hence, we have

$$\begin{cases} x_k = 34x_{k-1} - x_{k-2} - 1024t - 16,\ x_1 = 224t + 9,\ x_2 = 6560t + 289, \\ y_k = x_k - 36t,\ y_1 = 188t + 9,\ y_2 = 6524t + 289, \\ z_k = 34z_{k-1} - z_{k-2},\ z_1 = 816t + 36,\ z_2 = 27696t + 1224. \end{cases}$$

Therefore, when $m = 4, n = 32$, Eq. (1.2) has infinitely many positive integer solutions (x_k, y_k, z_k).

4 Some related questions

In this paper, we have investigated the problem about the linear combination of two triangular numbers is a perfect square. Similarly, we can ask

Question 4.1. *Are there positive integer solutions of the Diophantine equation*

$$mP_k(x) + nP_k(y) = z^2,$$

where $P_k(x) = \frac{x\big((x-1)(k-2)+2\big)}{2}$ is a polygonal number, and $k > 4$? If it has, are they infinitely many?

When $k = 4$, $P_4(x) = x^2$ is a square number. H. Cohen [3, Corollary 6.3.6.] gave the general solutions of the Diophantine equation

$$Ax^2 + By^2 = Cz^2,$$

i.e., "Assume that $ABC \neq 0$, let (x_0, y_0, z_0) be a particular nontrivial solution of $Ax^2 + By^2 = Cz^2$, and assume that $z_0 \neq 0$. The general solution in rational numbers to the equation is given by

$$\begin{cases} x = d\big(x_0(As^2 - Bt^2) + 2y_0Bst\big), \\ y = d\big(2x_0Ast - y_0(As^2 - Bt^2)\big), \\ z = dz_0(As^2 + Bt^2), \end{cases}$$

where $d \in \mathbb{Q}, s, t \in \mathbb{Z}$, and $gcd(s,t) = 1$."

Question 4.2. *Are there positive integer solutions of the Diophantine equation*

$$m\binom{x}{k} + n\binom{y}{k} = z^2,$$

where $\binom{x}{k} = \frac{x!}{(x-k)!k!}$ is a binomial coefficient, where $k \geq 3$? If it has, are they infinitely many?

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