Semidefinite Relaxations of OPF

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Outline

Mathematical preliminaries

Bus injection model

- OPF formulation
- 3 convex relaxations & relationship

Branch flow model

- OPF formulation
- SOCP relaxation & equivalence

Exact relaxation

- Radial networks
- Mesh networks

Multiphase unbalanced networks

Outline

Largely following a 2-part tutorial

SL, Convex relaxation of OPF, 2014
http://netlab.caltech.edu

Mathematical preliminaries

- Semidefinite programs
- QCQP and semidefinite relaxations
- Partial matrices and completions
- Chordal relaxation

2^{nd} order cone program (SOCP)

$$\begin{array}{lll} \min & c_0^H x & \\ \text{s.t.} & \left\| C_k x + b_k \right\| \le c_k^H x + d_k & k \ge 1 \end{array}$$

- $C_k \in \mathbf{R}^{(n_k - 1) \times n}, \; b_k \in \mathbf{R}^{n_k - 1}, \; c_k \in \mathbf{C}^n, \; d_k \in \mathbf{R}$

- || || : Euclidean norm
- Feasible set is 2^{nd} order cone and convex
- Includes LP, convex QP as special cases
- Special case of SDP, but much simpler computationally

SOCP in rotated form

$$
\begin{aligned}
\min \quad & c_0^H x \\
\text{s.t.} \quad & \left\| C_k x + b_k \right\|^2 \leq \left(c_k^H x + d_k \right)\left(\hat{c}_k^H x + \hat{d}_k \right)
\end{aligned}
$$

- Useful for OPF:

$$
\begin{aligned}
\min \quad & c_0^H x \\
\text{s.t.} \quad & C_k x = b_k \qquad k \geq 1 \\
& \left\| w_m \right\|^2 \leq y_m z_m \qquad m \geq 1
\end{aligned}
$$

- Transformation:

$$
\left\| w \right\|^2 \leq yz,\ y \geq 0,\ z \geq 0 \ \Leftrightarrow\ \left\| \begin{bmatrix} 2w \\ y - z \end{bmatrix} \right\| \leq y + z
$$

Semidefinite program (SDP)

Primal: $\min_{x \in \mathbf{R}^n} \sum_{i=1}^{n} c_i x_i \quad \text{s. t.} \quad A_0 + \sum_{i=1}^{n} x_i A_i \preceq 0$

Lagrangian: for $\Lambda \succeq 0$

$$L(x;\Lambda) := \sum_{i=1}^{n} c_i x_i + \operatorname{tr} \Lambda \left(A_0 + \sum_{i=1}^{n} x_i A_i \right)$$

$$= \operatorname{tr}\left(A_0 \Lambda\right) + \sum_{i=1}^{n} \left(\operatorname{tr}\left(A_i \Lambda\right) + c_i\right) x_i$$

$$D(\Lambda) = \begin{cases} \operatorname{tr}\left(A_0 \Lambda\right) & \text{if } \operatorname{tr}\left(A_i \Lambda\right) + c_i = 0 \ \ \forall i \\ -\infty & \text{else} \end{cases}$$

Semidefinite program (SDP)

Primal: $$\min_{x \in \mathbf{R}^n} \sum_{i=1}^{n} c_i x_i \quad \text{s. t.} \quad A_0 + \sum_{i=1}^{n} x_i A_i \le 0$$

Dual: $$\max_{\Lambda \ge 0} \operatorname{tr}(A_0 \Lambda) \quad \text{s.t.} \quad \operatorname{tr}(A_i \Lambda) + c_i = 0 \quad \forall\, i$$

We will later use an inequality form:

$$\max_{\Lambda \ge 0} \operatorname{tr}(A_0 \Lambda)$$

$$\text{s.t.} \quad \operatorname{tr}(A_i \Lambda) \le c_i \quad \forall\, i$$

equivalent to equality form through slack variables

PSD cones are convex

- Hermitian matrices

$$\mathbf{S}^n := \left\{ A \in \mathbf{C}^{n \times n} \,\middle|\, A = A^H \right\}$$

- Positive semidefinite (psd) matrices

$$\mathbf{S}_+^n := \left\{ A \in \mathbf{S}^n \,\middle|\, x^T A x \geq 0 \ \text{ for all } x \in \mathbf{C}^n \right\}$$

- Positive definite (pd) matrices

$$\mathbf{S}_{++}^n := \left\{ A \in \mathbf{S}^n \,\middle|\, x^T A x > 0 \ \text{ for all } x \in \mathbf{C}^n \right\}$$

Semidefinite program (SDP)

Primal: $$\min_{x \in \mathbf{R}^n} \sum_{i=1}^{n} c_i x_i \quad \text{s. t.} \quad A_0 + \sum_{i=1}^{n} x_i A_i \le 0$$

Dual: $$\max_{\Lambda \ge 0} \operatorname{tr}\left(A_0 \Lambda\right) \quad \text{s.t.} \quad \operatorname{tr}\left(A_i \Lambda\right) + c_i = 0 \quad \forall\, i$$

Theorem: strong duality

primal optimal value = dual optimal value

Semidefinite program (SDP)

Theorem: The following are equivalent

- (x^*, Λ^*) is primal-dual optimal
- (x^*, Λ^*) is a saddle pt of Lagrangian

$$L(x^*, \Lambda) \le L(x^*, \Lambda^*) \le L(x, \Lambda^*) \quad \forall \text{ feasible } x, \Lambda$$

- KKT:

$$A_0 + \sum_{i=1}^{n} x_i^* A_i \le 0,$$

$$\Lambda^* \ge 0, \quad \mathrm{tr}\,(A_i \Lambda^*) + c_i = 0 \quad \forall\ i$$

$$\mathrm{tr}\,\Lambda^* \left(A_0 + \sum_{i=1}^{n} x_i^* A_i \right) = 0$$

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QCQP

$$\begin{aligned} &\min && x^H C_0 x \\ &\text{over} && x \in \mathbf{C}^n \\ &\text{s.t.} && x^H C_k x \le b_k \qquad k \ge 1 \end{aligned}$$

- $C_k, k \ge 0$, Hermitian $\Rightarrow$ $x^H C_k x$ is real
  $b_k \in \mathbf{R}^n$
- Convex problem if all C_k are psd
  Nonconvex otherwise

QCQP

$$\begin{array}{lll} \text{min} & x^H C_0 x & \\ \text{over} & x \in \mathbf{C}^n & \\ \text{s.t.} & x^H C_k x \le b_k & k \ge 1 \end{array}$$

- $x^H C_k x = \text{tr}\, x^H C_k x = \text{tr}\, C_k \left(x x^H \right)$

QCQP

$$\begin{array}{lll} \min & \operatorname{tr} C_0\left(xx^H\right) & \\ \text{over} & x \in \mathbf{C}^n & \\ \text{s.t.} & \operatorname{tr} C_k\left(xx^H\right) \le b_k & k \ge 1 \end{array}$$

- $x^H C_k x \;=\; \operatorname{tr} x^H C_k x \;=\; \operatorname{tr} C_k\left(xx^H\right)$

QCQP

$$
\begin{array}{lll}
\text{min} & \operatorname{tr} C_0\left(xx^H\right) & \\
\text{over} & x \in \mathbf{C}^n & \\
\text{s.t.} & \operatorname{tr} C_k \underbrace{\left(xx^H\right) \le}_{X \in \mathbf{S}_+^n} b_k & k \ge 1
\end{array}
$$

- $x^H C_k x = \operatorname{tr} x^H C_k x = \operatorname{tr} C_k\left(xx^H\right)$

QCQP

$$\min \quad \operatorname{tr} C_0 X$$

$$\text{over} \quad X \in \mathbf{S}_+^n$$

$$\text{s.t.} \quad \operatorname{tr} C_k X \le b_k \qquad k \ge 1$$

~~$\operatorname{rank} X = 1$~~ ← only nonconvexity

- Any solution X yields a unique x through

$$X = xx^H$$

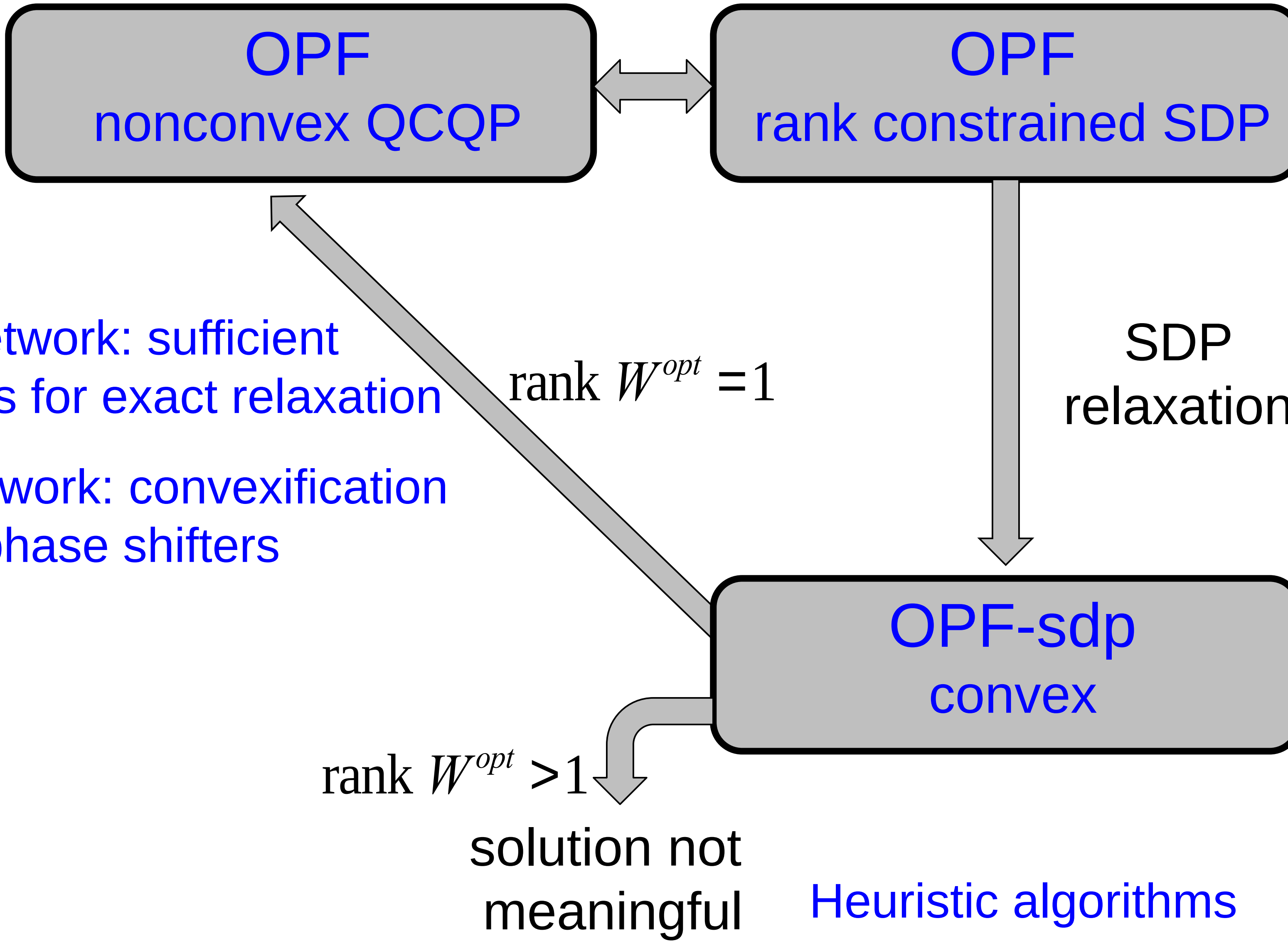
- Feasible sets are *equivalent*

Semidefinite program (SDP)

$$\begin{aligned} \min \quad & \operatorname{tr} C_0 X \\ \text{s.t.} \quad & \operatorname{tr} C_k X \le b_k \qquad k \ge 1 \\ & X \ge 0 \end{aligned}$$

- Feasible set of QCQP is an *effective subset* of feasible set of SDP
- SDP is a *relaxation* of QCQP

Preview: solution strategy



Radial network: sufficient conditions for exact relaxation

Mesh network: convexification through phase shifters

Heuristic algorithms

SOCP in rotated form

$$
\begin{aligned}
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\text{s.t.} \quad & \left\| C_k x + b_k \right\|^2 \le \left(c_k^H x + d_k \right)\left(\hat{c}_k^H x + \hat{d}_k \right)
\end{aligned}
$$

- Useful for OPF:

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\end{aligned}
$$

- Transformation:

$$
\|w\|^2 \le yz,\; y \ge 0,\; z \ge 0 \iff \left\| \begin{bmatrix} 2w \\ y - z \end{bmatrix} \right\| \le y + z
$$

Recap: QCQP, SDP, SOCP

QCQP

$$
\begin{aligned}
\min \quad & x^H C_0 x \\
\text{s.t.} \quad & x^H C_k x \le b_k \qquad k \ge 1
\end{aligned}
$$

SDP

$$
\begin{aligned}
\min \quad & \operatorname{tr} C_0 X \\
\text{s.t.} \quad & \operatorname{tr} C_k X \le b_k \qquad k \ge 1 \\
& X \ge 0
\end{aligned}
$$

SOCP

$$
\begin{aligned}
\min \quad & c_0^H x \\
\text{s.t.} \quad & C_k x = b_k \qquad k \ge 1 \\
& \|w_m\|^2 \le y_m z_m \qquad m \ge 1
\end{aligned}
$$

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Graphs

Graph $G = (V, E)$

Complete graph: all node pairs adjacent

Clique: complete subgraph of G

- An edge is a clique
- *Maximal* clique: a clique that is not a subgraph of another clique

Chordal graph: all minimal cycles have length 3

- *Minimal* cycle: cycle without chord

Chordal ext: chordal graph containing G

- Every graph has a chordal extension
- Chordal extensions are not unique

Partial matrices

Fix an undirected graph $G = (V, E)$

Partial matrix X_G :

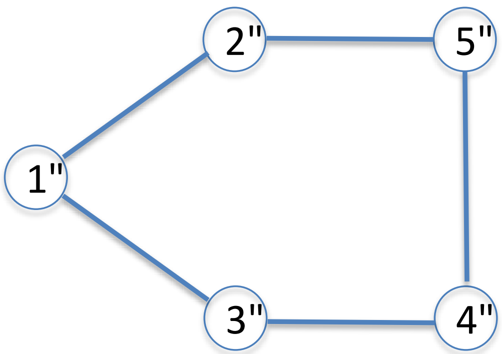
$$X_G := \left([X_G]_{jj}, j \in V, \ [X_G]_{jk}, (j,k) \in E \right)$$

Completion X of a partial matrix X_G :

$$X = X_G \text{ on } G$$

Example

partial matrix X_G := { complex numbers on G }



G

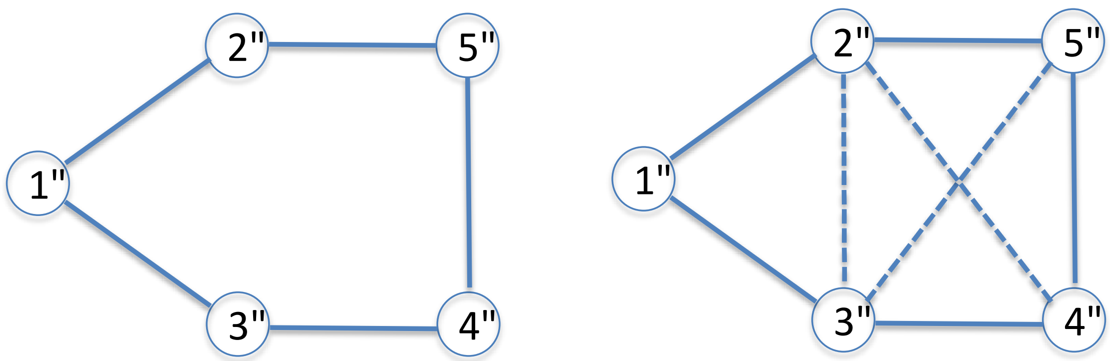
n-vertex complete graph

$$X_G = \begin{bmatrix} x_{11} & x_{12} & x_{13} & & \\ x_{21} & x_{22} & & & x_{25} \\ x_{31} & & x_{33} & x_{34} & \\ & & x_{43} & x_{44} & x_{45} \\ & x_{52} & & x_{54} & x_{55} \end{bmatrix}$$

completion: full matrix X that agrees with X_G on G

Example

$$\text{chordal ext } X_{c(G)} := \{ \text{ complex numbers on } c(G) \ \}$$

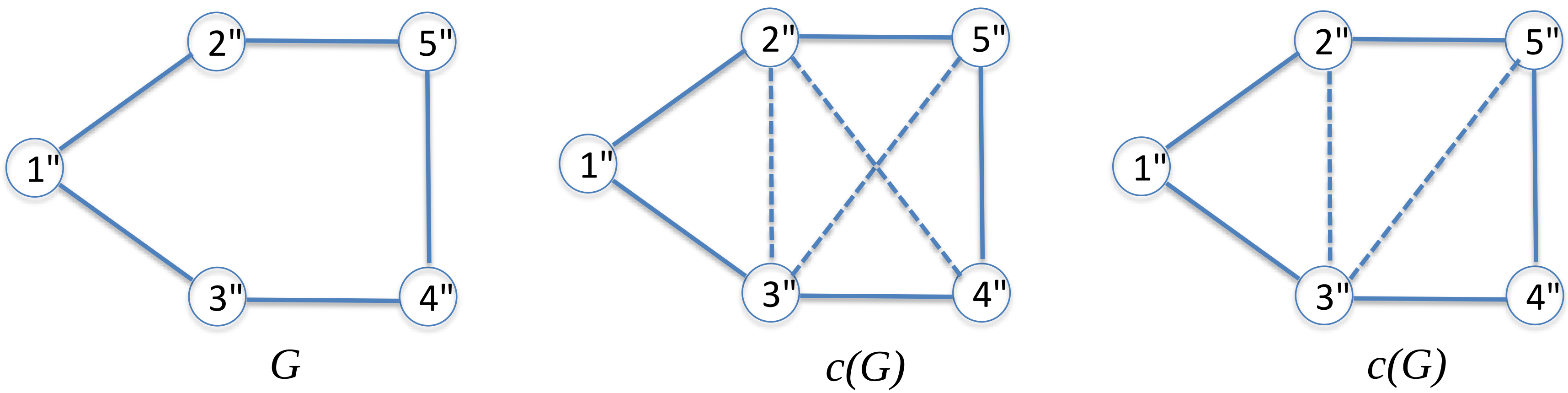


G $\qquad$ $c(G)$

$$X_G = \begin{bmatrix} x_{11} & x_{12} & x_{13} & & \\ x_{21} & x_{22} & & & x_{25} \\ x_{31} & & x_{33} & x_{34} & \\ & & x_{43} & x_{44} & x_{45} \\ & x_{52} & & x_{54} & x_{55} \end{bmatrix} \qquad X_{c(G)} = \begin{bmatrix} x_{11} & x_{12} & x_{13} & & \\ x_{21} & x_{22} & x_{23} & x_{24} & x_{25} \\ x_{31} & x_{32} & x_{33} & x_{34} & x_{35} \\ & x_{42} & x_{43} & x_{44} & x_{45} \\ & x_{52} & x_{53} & x_{54} & x_{55} \end{bmatrix}$$

Example

$$\text{chordal ext } X_{c(G)} := \{ \text{ complex numbers on } c(G) \ \}$$



$$X_G = \begin{bmatrix} x_{11} & x_{12} & x_{13} & & \\ x_{21} & x_{22} & & & x_{25} \\ x_{31} & & x_{33} & x_{34} & \\ & & x_{43} & x_{44} & x_{45} \\ & x_{52} & & x_{54} & x_{55} \end{bmatrix} \quad X_{c(G)} = \begin{bmatrix} x_{11} & x_{12} & x_{13} & & \\ x_{21} & x_{22} & x_{23} & x_{24} & x_{25} \\ x_{31} & x_{32} & x_{33} & x_{34} & x_{35} \\ & x_{42} & x_{43} & x_{44} & x_{45} \\ & x_{52} & x_{53} & x_{54} & x_{55} \end{bmatrix} \quad X_{c(G)} = \begin{bmatrix} x_{11} & x_{12} & x_{13} & & \\ x_{21} & x_{22} & x_{23} & & x_{25} \\ x_{31} & x_{32} & x_{33} & x_{34} & x_{35} \\ & & x_{43} & x_{44} & x_{45} \\ & x_{52} & x_{53} & x_{54} & x_{55} \end{bmatrix}$$

Partial matrices

Fix an undirected graph $G = (V, E)$

A partial matrix X_G is *psd* if

$$X_G(q) \succeq 0 \quad \text{for all maximal cliques } q$$

A partial matrix X_G is *rank-1* if

$$\operatorname{rank} X_G(q) = 1 \quad \text{for all maximal cliques } q$$

Matrix completion

Theorem [Grone et al 1984]

Every psd partial matrix X_G has a psd completion if and only if G is chordal

- Motivates chordal relaxation

Feasible set

Consider

dec
vars

- full matrix W
- partial matrix $W_{c(G)}$ defined on a chordal ext of G
- partial matrix W_G defined on G

C1: $$W \succeq 0, \;\; \text{rank}\; W = 1$$

Feasible set

Consider

dec
vars

- full matrix W
- partial matrix $W_{c(G)}$ defined on a chordal ext of G
- partial matrix W_G defined on G

C1: $$W \succeq 0, \ \text{rank } W = 1$$

C2: $$W_{c(G)} \succeq 0, \ \text{rank } W_{c(G)} = 1$$

Feasible set

Consider

dec
\# vars

- full matrix W
- partial matrix $W_{c(G)}$ defined on a chordal ext of G
- partial matrix W_G defined on G

C1: $$W \succeq 0, \ \ \text{rank } W = 1$$

C2: $$W_{c(G)} \succeq 0, \ \ \text{rank } W_{c(G)} = 1$$

C3: $$\begin{cases} W_G(j,k) \succeq 0, \ \ \text{rank } W_G(j,k) = 1, \quad (j,k) \in E \\ \sum_{(j,k)\in c} \text{Ð}\left[W_G\right]_{jk} = 0 \quad \mod 2\pi \end{cases}$$

Feasible set

Theorem

C1 = C2 = C3

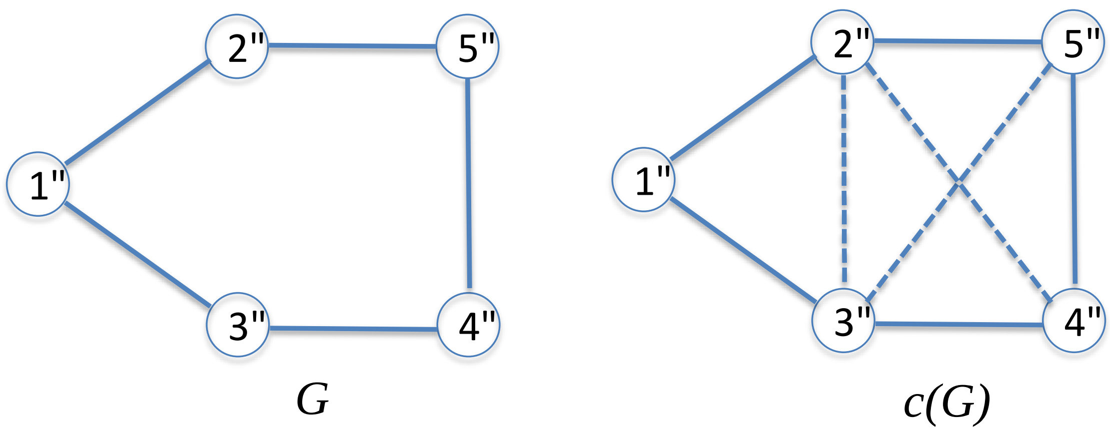
C1: $$W \succeq 0, \ \text{rank}\ W = 1$$

C2: $$W_{c(G)} \succeq 0, \ \text{rank}\ W_{c(G)} = 1$$

C3: $$\begin{cases} W_G(j,k) \succeq 0, \ \text{rank}\ W_G(j,k) = 1, \quad (j,k) \in E \\ \sum_{(j,k)\in c} \angle [W_G]_{jk} = 0 \mod 2\pi \end{cases}$$

Bose, Low, Chandy Allerton 2012
Bose, Low, Teeraratkul, Hassibi TAC2014

Example



C1 = C2 means:

W is psd rank-1
iff
$W_{c(G)}$ is psd rank-1

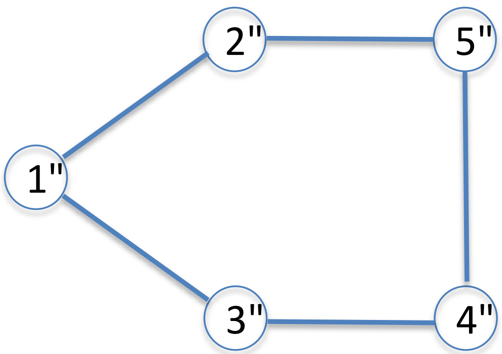
$$W_G = \begin{bmatrix} W_{11} & W_{12} & W_{13} & & \\ W_{21} & W_{22} & & & W_{25} \\ W_{31} & & W_{33} & W_{34} & \\ & & W_{43} & W_{44} & W_{45} \\ & W_{52} & & W_{54} & W_{55} \end{bmatrix}$$

$$W_{c(G)} = \begin{bmatrix} W_{11} & W_{12} & W_{13} & & \\ W_{21} & W_{22} & W_{23} & W_{24} & W_{25} \\ W_{31} & W_{32} & W_{33} & W_{34} & W_{35} \\ & W_{42} & W_{43} & W_{44} & W_{45} \\ & W_{52} & W_{53} & W_{54} & W_{55} \end{bmatrix}$$

iff
these 2 submatrices are psd rank-1

much smaller for large sparse network

Example



G

$$W_G = \begin{bmatrix} W_{11} & W_{12} & W_{13} & & \\ W_{21} & W_{22} & & & W_{25} \\ W_{31} & & W_{33} & W_{34} & \\ & & W_{43} & W_{44} & W_{45} \\ & W_{52} & & W_{54} & W_{55} \end{bmatrix}$$

C1 = C3 means:

W is psd rank-1 iff
W_G is psd rank-1 and satisfies cycle cond

iff
5 2x2 submatrices are psd rank-1 and satisfies cycle cond

much much smaller for large sparse network

Feasible set

Theorem

C1 = C2 = C3

Moreover, given W_G that satisfies C3, there is a unique completion W that satisfies C1

C1: $$W \succeq 0, \ \text{rank } W = 1$$

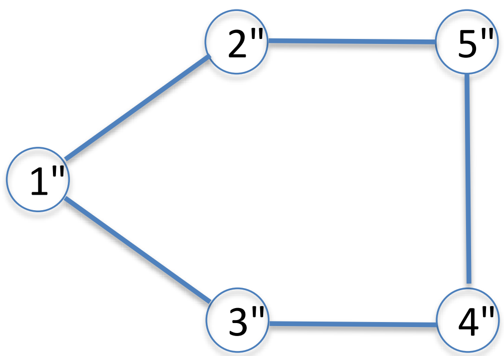
C2: $$W_{c(G)} \succeq 0, \ \text{rank } W_{c(G)} = 1$$

C3: $$\begin{cases} W_G(j,k) \succeq 0, \ \text{rank } W_G(j,k) = 1, \quad (j,k) \in E \\ \sum_{(j,k)\in c} \angle [W_G]_{jk} = 0 \mod 2\pi \end{cases}$$

Bose, Low, Chandy Allerton 2012

Bose, Low, Teeraratkul, Hassibi TAC2014

Example



G

$$W_G = \begin{bmatrix} W_{11} & W_{12} & W_{13} & & \\ W_{21} & W_{22} & & & W_{25} \\ W_{31} & & W_{33} & W_{34} & \\ & & W_{43} & W_{44} & W_{45} \\ & W_{52} & & W_{54} & W_{55} \end{bmatrix}$$

Given W_G that satisfies C3, there is only one way to fill in missing entries to get an W from which an V can be recovered

Chordal relaxation

QCQP

$$\min \quad x^H C_0 x$$

$$\text{s.t.} \quad x^H C_k x \leq b_k \qquad k \geq 1$$

SDP

$$\min \quad \operatorname{tr} C_0 X$$

$$\text{s.t.} \quad \operatorname{tr} C_k X \leq b_k \qquad k \geq 1$$

$$X \geq 0$$

Chordal

$$\min_{X_{c(G)}} \quad \operatorname{tr} C_0 X_G$$

$$\text{s.t.} \quad \operatorname{tr} C_k X_G \leq b_k \qquad k \geq 1$$

$$X_{c(G)} \geq 0$$

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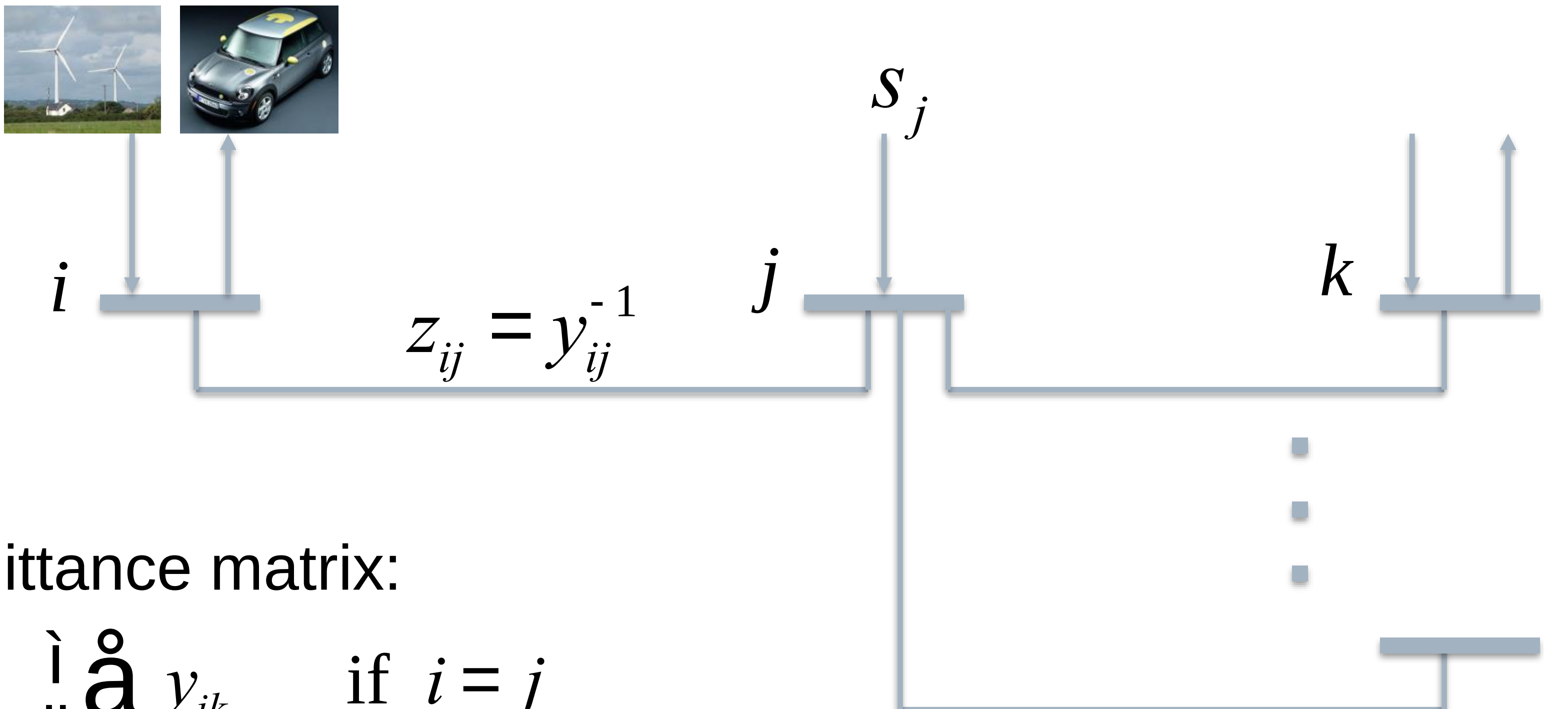
Bus injection model



admittance matrix:

$$Y_{ij} := \begin{cases} \sum_{k \sim i} y_{ik} & \text{if } i = j \\ -y_{ij} & \text{if } i \sim j \\ 0 & \text{else} \end{cases}$$

graph G: undirected

Y specifies topology of G and impedances z on lines

Bus injection model

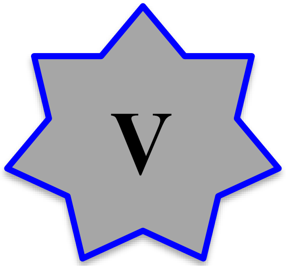
In terms of V:

$$s_j = \mathrm{tr}\left(Y_j^H V V^H\right) \qquad \text{for all } j$$

$$Y_j = Y^H e_j e_j^T$$

Power flow problem:

Given (Y, s) find V



isolated solutions

OPF: bus injection model

min $\quad \mathrm{tr}\left(CVV^H\right)$

gen cost,
power loss

over $\quad \left(V, s\right)$

subject to $\quad \underline{s}_j \;\text{£}\; s_j \;\text{£}\; \overline{s}_j \qquad \underline{V}_j \;\text{£}\; |V_j| \;\text{£}\; \overline{V}_j$

OPF: bus injection model

min	$\mathrm{tr}\left(CVV^H\right)$	gen cost, power loss
over	$\left(V, s\right)$	
subject to	$\underline{s}_j$ £ s_j £ $\overline{s}_j$	$\underline{V}_j$ £ $\lvert V_j \rvert$ £ $\overline{V}_j$
	$s_j = \mathrm{tr}\left(Y_j^H VV^H\right)$	power flow equation

Summary: OPF (bus injection model)

$$\min \quad \mathrm{tr}\, C V V^H$$

$$\text{subject to} \quad \underline{s}_j \; £ \; \mathrm{tr}\left(Y_j V V^H \right) \; £ \; \bar{s}_j \qquad \underline{v}_j \; £ \; |V_j|^2 \; £ \; \bar{v}_j$$

nonconvex QCQP

(quad constrained quad program)

Other features

Security constraint OPF

- Solve for operating points after each single contingency (N-1 security)
- N sets of variables and constraints, one for each contingency

Unit commitment

- Discrete variables

Stochastic OPF

- Chance constraints Pr(bad event) < e

Other constraints

- Line flow, line loss, stability limit, …

… OPF in practice is a lot harder

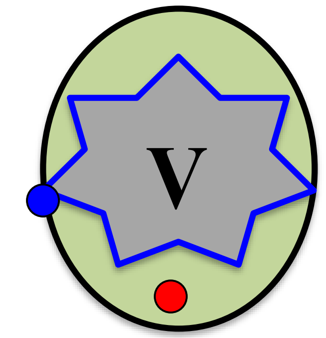
Literature

Convex relaxations of OPF

relaxation	model	first proposed	first analyzed
SOCP	BIM	Jabr 2006 TPS	
SDP	BIM	Bai et al 2008 EPES	Lavaei, Low 2012 TPS
Chordal	BIM	Bai, Wei 2011 EPES Jabr 2012 TPS	Molzahn et al 2013 TPS Bose et al 2014 TAC

Low. Convex relaxation of OPF (I, II), IEEE Trans Control of Network Systems, 2014

Basic idea



$$\min \quad \text{tr}\, C V V^H$$

$$\text{subject to} \quad \underbrace{\underline{s}_j \;\text{£}\; \text{tr}\left(Y_j V V^H\right) \;\text{£}\; \bar{s}_j \qquad \underline{v}_j \;\text{£}\; |V_j|^2 \;\text{£}\; \bar{v}_j}_{\mathbf{V}}$$

Approach

1. Three equivalent characterizations of $\mathbf{V}$
2. Each suggests a lift and relaxation

- What is the relation among different relaxations ?
- When will a relaxation be <u>exact</u> ?

Feasible set & SDP

$$\begin{aligned} \min \quad & \operatorname{tr} CVV^H \\ \text{subject to} \quad & \underline{s}_j \le \operatorname{tr}\left(Y_j VV^H\right) \le \bar{s}_j \qquad \underline{v}_j \le |V_j|^2 \le \bar{v}_j \end{aligned}$$

quadratic in V

linear in W

Equivalent problem:

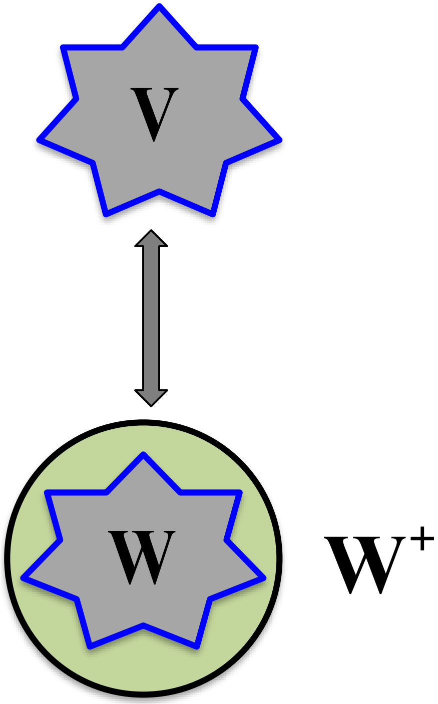
$$\begin{aligned} \min \quad & \operatorname{tr} CW \\ \text{subject to} \quad & \underline{s}_j \le \operatorname{tr}\left(Y_j W\right) \le \bar{s}_j \qquad \underline{v}_i \le W_{ii} \le \bar{v}_i \\ & W \ge 0, \quad \operatorname{rank} W = 1 \end{aligned}$$

convex in W

except this constraint

Equivalent feasible sets

V:= $\{V$: quadratic constraints $\}$



QCQP: n variables

SDP: n^2 vars !

W:= $\{W$: linear constraints $\} \cap \{W \geq 0$ ~~rank-1~~$\}$

idea: $W = VV^H$

Feasible set

only $n+2m$ vars !

linear in $\left(W_{jj}, W_{jk}\right)$ ⟸ W_{jj} W_{jk}

$$\sum_{k:k\sim j} y_{jk}^H \left(|V_j|^2 - V_j V_k^H \right) : \textbf{ only } |V_j|^2 \text{ and } V_j V_k^H$$

corresponding to edges (j,k) in G!

$$\begin{aligned} &\min && \mathrm{tr}\, C V V^H \\ &\text{subject to} && \underline{s}_j \le \mathrm{tr}\left(Y_j V V^H\right) \le \bar{s}_j \qquad \underline{v}_j \le |V_j|^2 \le \bar{v}_j \end{aligned}$$

$\mathbf{V}$

Feasible set

only $n+2m$ vars !

linear in (W_{jj}, W_{jk}) ⟵ W_{jj} W_{jk}

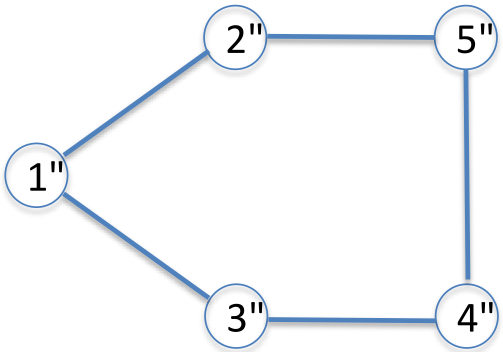
$$\sum_{k:k\sim j} y_{jk}^H \left(|V_j|^2 - V_j V_k^H \right) : \textbf{ only } |V_j|^2 \text{ and } V_j V_k^H$$

$$\text{partial matrix } W_G \text{ defined on } G$$

$$W_G := \left\{ [W_G]_{jj}, [W_G]_{jk} \,\middle|\, j, jk \in G \right\}$$

Kirchhoff's laws depend <u>directly</u> only on W_G

Example



G

$$W_G = \begin{bmatrix} W_{11} & W_{12} & W_{13} & & \\ W_{21} & W_{22} & & & W_{25} \\ W_{31} & & W_{33} & W_{34} & \\ & & W_{43} & W_{44} & W_{45} \\ & W_{52} & & W_{54} & W_{55} \end{bmatrix}$$

Want to solve for W_G

$n+2m$ variables

$$W = \begin{bmatrix} W_{11} & W_{12} & W_{13} & W_{14} & W_{15} \\ W_{21} & W_{22} & W_{23} & W_{24} & W_{25} \\ W_{31} & W_{32} & W_{33} & W_{34} & W_{35} \\ W_{41} & W_{42} & W_{43} & W_{44} & W_{45} \\ W_{51} & W_{52} & W_{53} & W_{54} & W_{55} \end{bmatrix}$$

SDP solves for $W \in \mathbf{C}^{n^2}$

n^2 variables

Feasible sets

OPF

$$\mathbf{V} := \left\{ V \,\middle|\, \underline{s}_j \le \operatorname{tr}\left(Y_j V V^H\right) \le \bar{s}_j, \quad \underline{v}_j \le |V_j|^2 \le \bar{v}_j \right\}$$

nonconvexity

SDP

$$\mathbf{W} := \left\{ W \,\middle|\, \underline{s}_j \le \operatorname{tr}\left(Y_j W\right) \le \bar{s}_j, \ \underline{v}_j \le W_{jj} \le \bar{v}_j \right\} \cap \left\{ W \succeq 0, \text{rank-}1 \right\}$$

depend only on W_G

depend on all entries of W

Feasible sets

OPF $\qquad \mathbf{V} := \left\{ V \,\middle|\, \underline{s}_j \le \mathrm{tr}\left(Y_j V V^H\right) \le \bar{s}_j, \quad \underline{v}_j \le |V_j|^2 \le \bar{v}_j \right\}$

SDP

$$\mathbf{W} := \left\{ W \,\middle|\, \underline{s}_j \le \mathrm{tr}\left(Y_j W\right) \le \bar{s}_j, \ \underline{v}_j \le W_{jj} \le \bar{v}_j \right\} \cap \left\{ W \succeq 0, \mathrm{rank}\text{-}1 \right\}$$

first idea:

$$\mathbf{W}_G := \left\{ W_G \,\middle|\, \underline{s}_j \le \mathrm{tr}\left(Y_j W_G\right) \le \bar{s}_j, \ \underline{v}_j \le [W_G]_{jj} \le \bar{v}_j \right\} \cap \left\{ W_G \succeq 0, \mathrm{rank}\text{-}1 \right\}$$

W_G is equivalent to V when G is **chordal**

Not equivalent otherwise …

Equivalent feasible sets

$$\mathbf{W}_{c(G)} := \left\{ W_{c(G)} : \ \underline{\text{linear}} \text{ constraints } \right\}$$

$$\text{idea: } \ W_{c(G)} = \left(VV^H \text{ on } c(G) \right)$$

$$\mathbf{W} := \left\{ W : \ \underline{\text{linear}} \text{ constraints } \right\} \cap \left\{ W \geq 0 \text{ rank-1} \right\}$$

$$\text{idea: } \ W = VV^H$$

Equivalent feasible sets

$$\mathbf{W}_{c(G)} := \left\{ W_{c(G)} : \ \underline{\text{linear}} \text{ constraints } \right\} \cap \left\{ W_{c(G)} \geq 0 \text{ rank-1} \right\}$$

$$\text{idea: } W_{c(G)} = \left(VV^H \text{ on } c(G) \right)$$

$$\mathbf{W} := \left\{ W : \ \underline{\text{linear}} \text{ constraints } \right\} \cap \left\{ W \geq 0 \text{ rank-1} \right\}$$

$$\text{idea: } W = VV^H$$

Equivalent feasible sets

$$\mathbf{W}_G := \{W_G : \underline{\text{linear}} \text{ constraints } \}$$

idea: $W_G = \left(VV^H \text{ only on } G\right)$

$$\mathbf{W}_{c(G)} := \{W_{c(G)} : \underline{\text{linear}} \text{ constraints } \} \cap \{W_{c(G)} \geq 0 \text{ rank-1}\}$$

idea: $W_{c(G)} = \left(VV^H \text{ on } c(G)\right)$

$$\mathbf{W} := \{W : \underline{\text{linear}} \text{ constraints } \} \cap \{W \geq 0 \text{ rank-1}\}$$

idea: $W = VV^H$

Equivalent feasible sets

$$\mathbf{W}_G := \left\{ W_G : \underline{\text{linear}} \text{ constraints } \right\} \cap \left\{ \begin{matrix} W(j,k) \geq 0 \text{ rank-1}, \\ \text{cycle cond on } \angle W_{jk} \end{matrix} \right\}$$

idea: $W_G = \left(VV^H \text{ only on } G \right)$

$$\mathbf{W}_{c(G)} := \left\{ W_{c(G)} : \underline{\text{linear}} \text{ constraints } \right\} \cap \left\{ W_{c(G)} \geq 0 \text{ rank-1} \right\}$$

idea: $W_{c(G)} = \left(VV^H \text{ on } c(G) \right)$

$$\mathbf{W} := \left\{ W : \underline{\text{linear}} \text{ constraints } \right\} \cap \left\{ W \geq 0 \text{ rank-1} \right\}$$

idea: $W = VV^H$

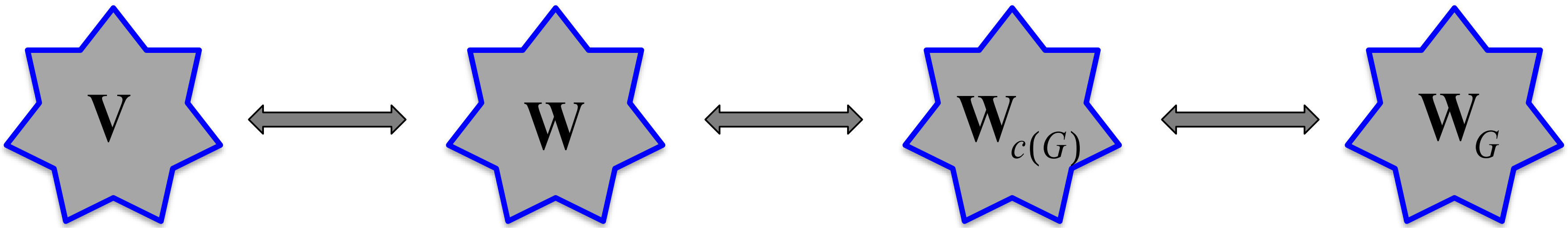
Cycle condition

local $$W_G(j,k) \succeq 0, \ \ \text{rank } W_G(j,k) = 1, \quad (j,k) \in E$$

global $$\sum_{(j,k)\in c} \text{Đ}\left[W_G\right]_{jk} = 0 \qquad \text{mod } 2\pi$$

← cycle cond

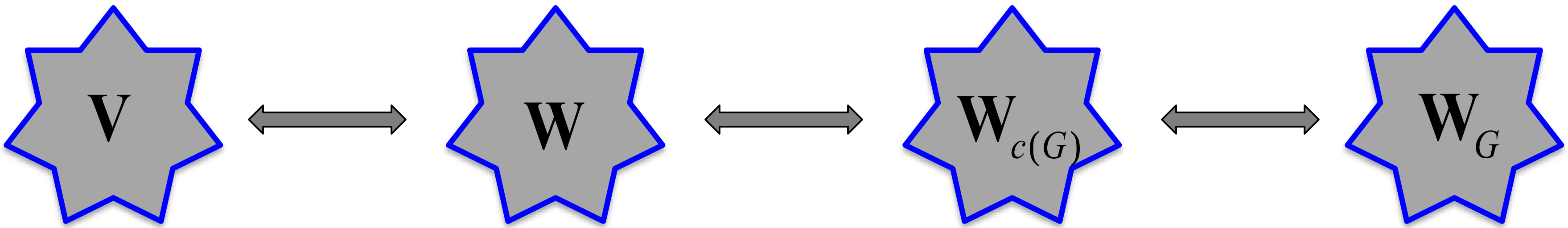
Equivalent feasible sets



<u>Theorem:</u> $\mathbf{V} \;^{o}\; \mathbf{W} \;^{o}\; \mathbf{W}_{c(G)} \;^{o}\; \mathbf{W}_{G}$

Bose, Low, Chandy Allerton 2012
Bose, Low, Teeraratkul, Hassibi TAC2014

Equivalent feasible sets



<u>Theorem:</u> $\mathbf{V} \equiv \mathbf{W} \equiv \mathbf{W}_{c(G)} \equiv \mathbf{W}_G$

Given $W_G \in \mathbf{W}_G$ or $W_{c(G)} \in \mathbf{W}_{c(G)}$ there is unique completion $W \in \mathbf{W}$ and unique $V \in \mathbf{V}$

Can minimize cost over **any** of these sets, but …

Equivalent feasible sets

$$\mathbf{W}_G := \{W_G : \underline{\text{linear}} \text{ constraints }\} \cap \left\{ \begin{matrix} W(j,k) \geq 0 \ \text{~~rank-1~~}, \\ \text{~~cycle cond on } \angle W_{jk}\text{~~} \end{matrix} \right\}$$

$$\text{idea: } W_G = \left(VV^H \text{ only on } G\right)$$

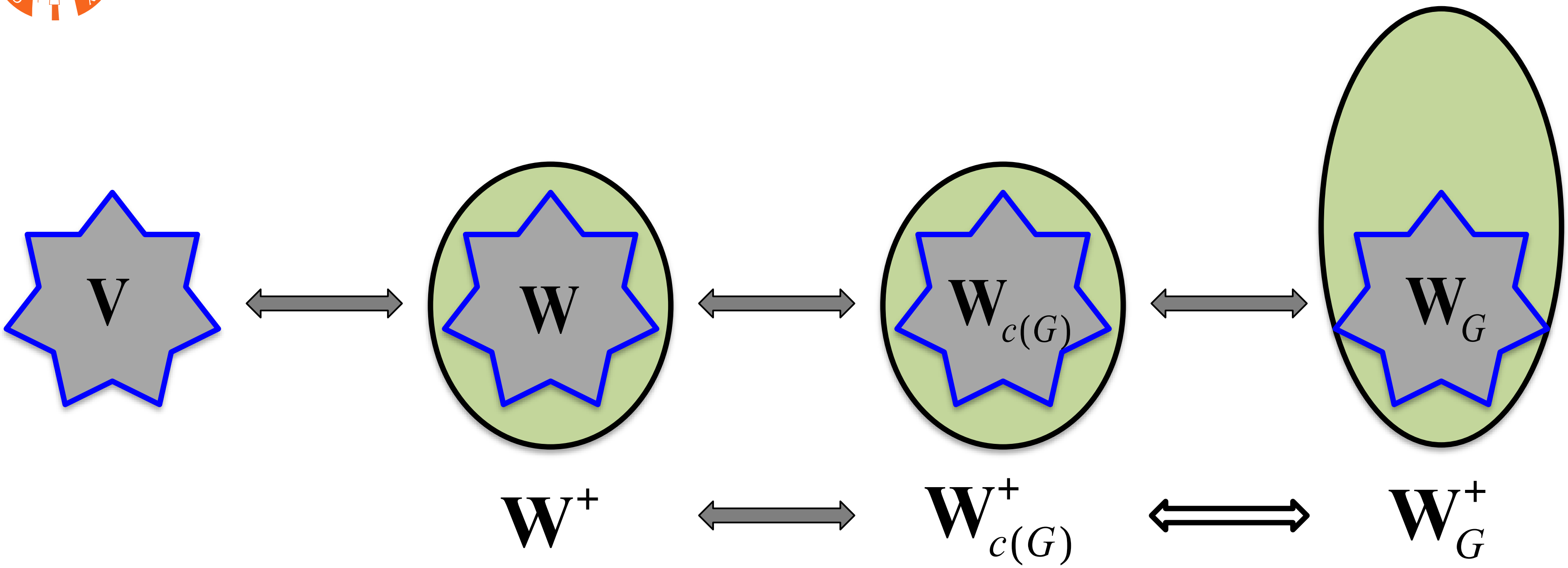
$$\mathbf{W}_{c(G)} := \{W_{c(G)} : \underline{\text{linear}} \text{ constraints }\} \cap \{W_{c(G)} \geq 0 \ \text{~~rank-1~~}\}$$

$$\text{idea: } W_{c(G)} = \left(VV^H \text{ on } c(G)\right)$$

$$\mathbf{W} := \{W : \underline{\text{linear}} \text{ constraints }\} \cap \{W \geq 0 \ \text{~~rank-1~~}\}$$

$$\text{idea: } W = VV^H$$

Relaxations

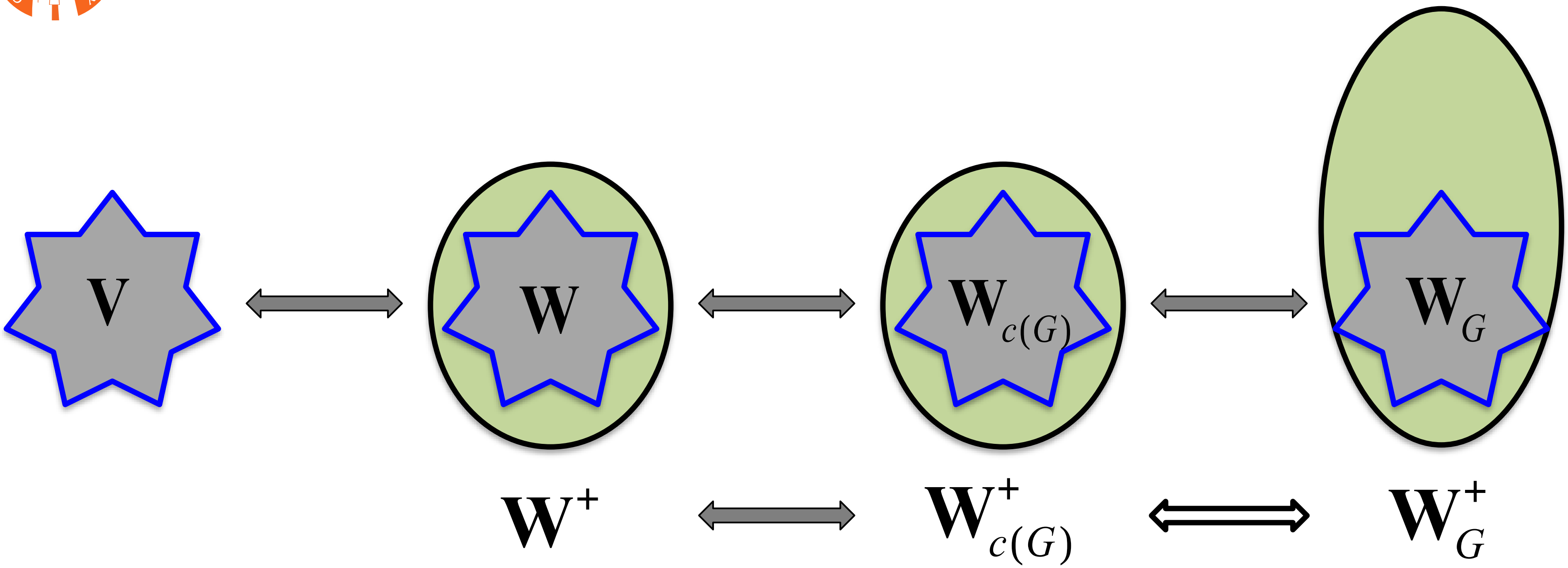


<u>Theorem</u>

- Radial G : $\mathbf{V} \subseteq \mathbf{W}^+ @ \mathbf{W}^+_{c(G)} @ \mathbf{W}^+_G$
- Mesh G : $\mathbf{V} \subseteq \mathbf{W}^+ @ \mathbf{W}^+_{c(G)} \subseteq \mathbf{W}^+_G$

Bose, Low, Chandy Allerton 2012
Bose, Low, Teeraratkul, Hassibi TAC2014

Relaxations



<u>Theorem</u>

- Radial G : $\mathbf{V} \subseteq \mathbf{W}^+ @ \mathbf{W}^+_{c(G)} @ \mathbf{W}^+_G$
- Mesh G : $\mathbf{V} \subseteq \mathbf{W}^+ @ \mathbf{W}^+_{c(G)} \subseteq \mathbf{W}^+_G$

For radial networks: always solve SOCP !

Convex relaxations

OPF

$$\min_{V} \; C(V) \quad \text{subject to} \quad V \in \mathbf{V}$$

OPF-sdp:

$$\min_{W} \; C(W_G) \quad \text{subject to} \quad W \in \mathbb{W}^{+}$$

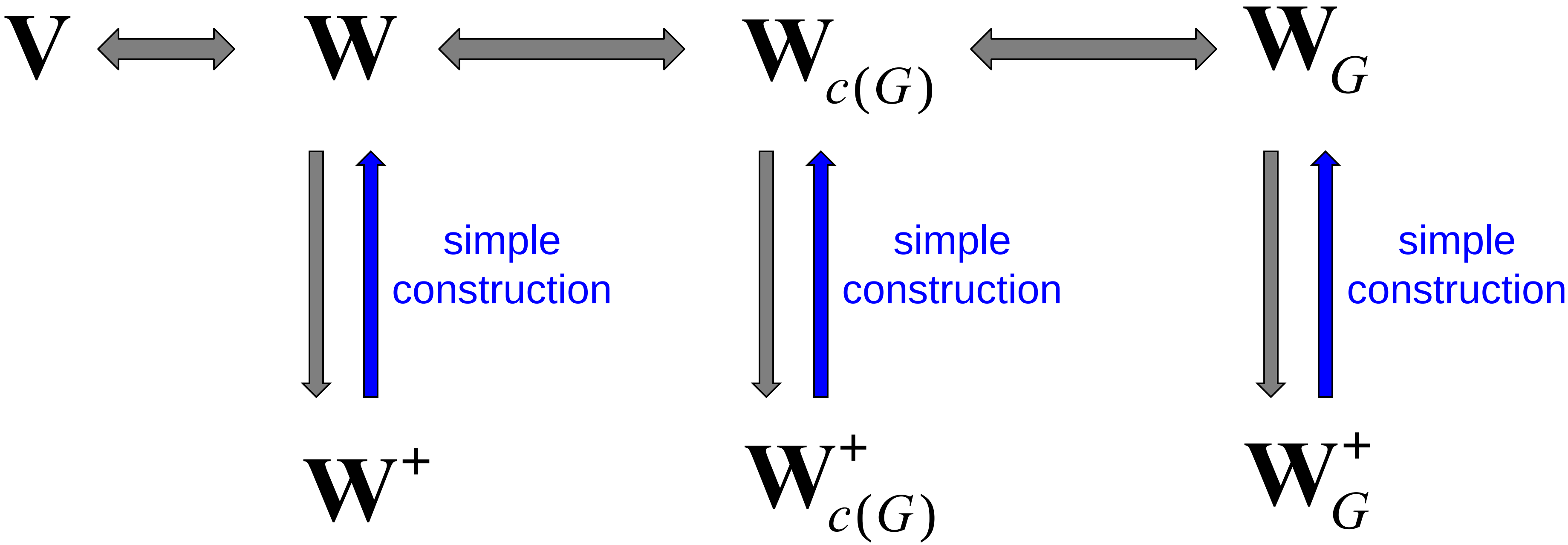
OPF-ch:

$$\min_{W_{c(G)}} \; C(W_G) \quad \text{subject to} \quad W_{c(G)} \in \mathbb{W}^{+}_{c(G)}$$

OPF-socp:

$$\min_{W_G} \; C(W_G) \quad \text{subject to} \quad W_G \in \mathbb{W}^{+}_{G}$$

Recap: convex relaxations

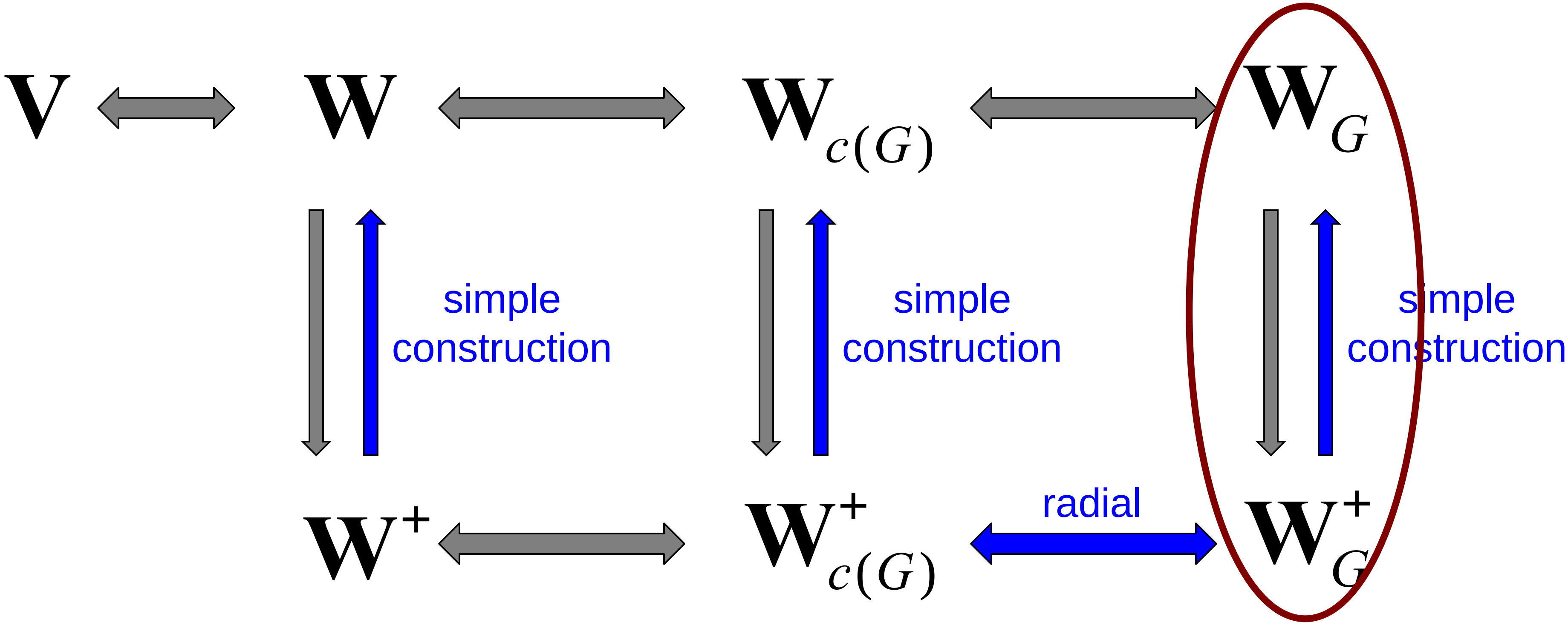


SDP relaxation
- tightest superset
- max # variables
- slowest

Chordal relaxation
- equivalent superset
- much faster for sparse networks

SOCP relaxation
- coarsest superset
- min # variables
- fastest

Recap: convex relaxations



<u>SDP relaxation</u>

- tightest superset
- max # variables
- slowest

<u>Chordal relaxation</u>

- equivalent superset
- much faster for sparse networks

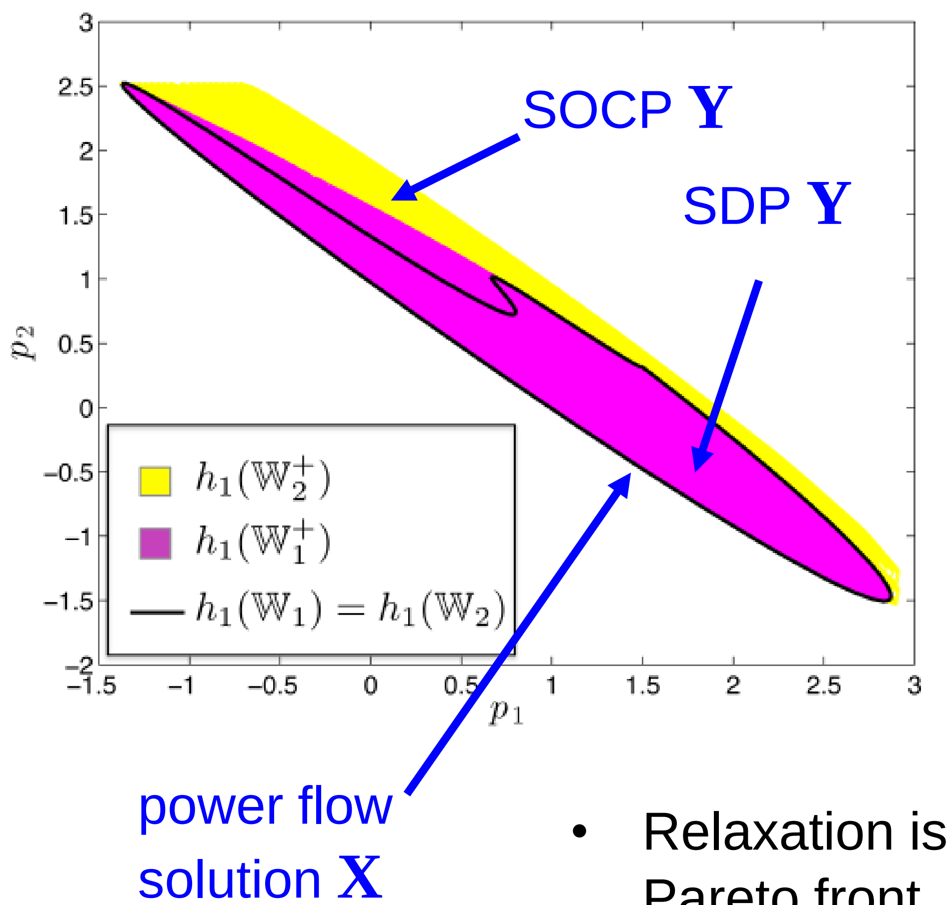
<u>SOCP relaxation</u>

- coarsest superset
- min # variables
- fastest

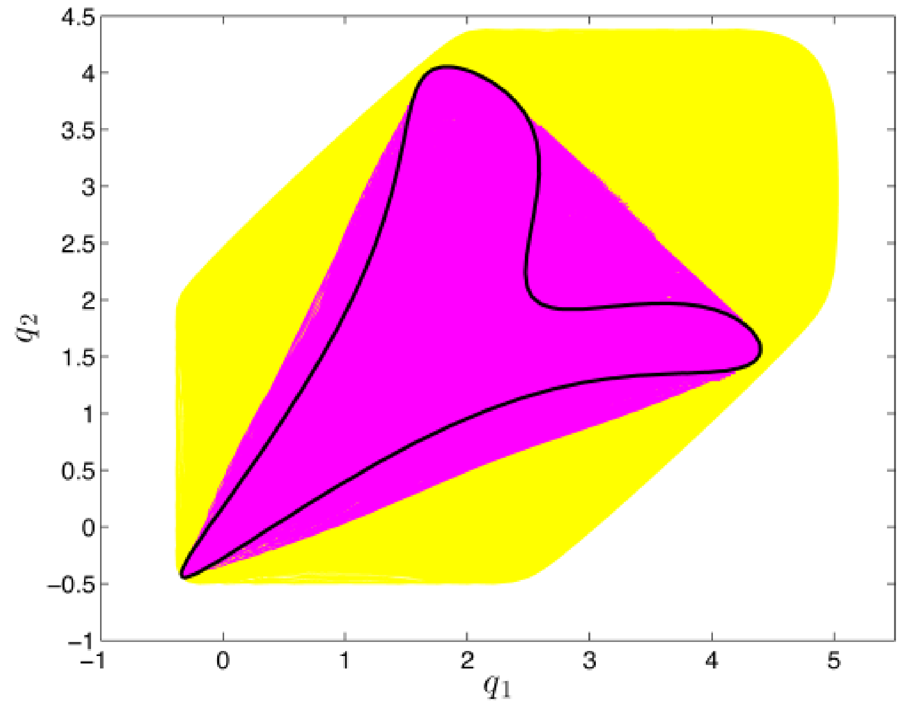
For radial network: always solve SOCP !

Examples

Real Power



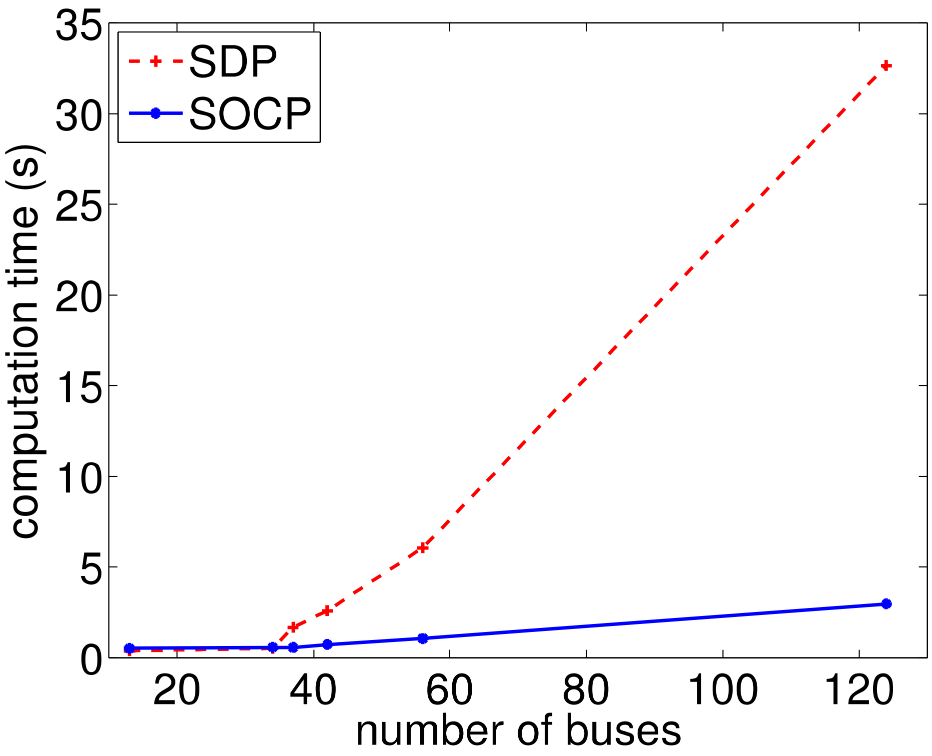
Reactive Power

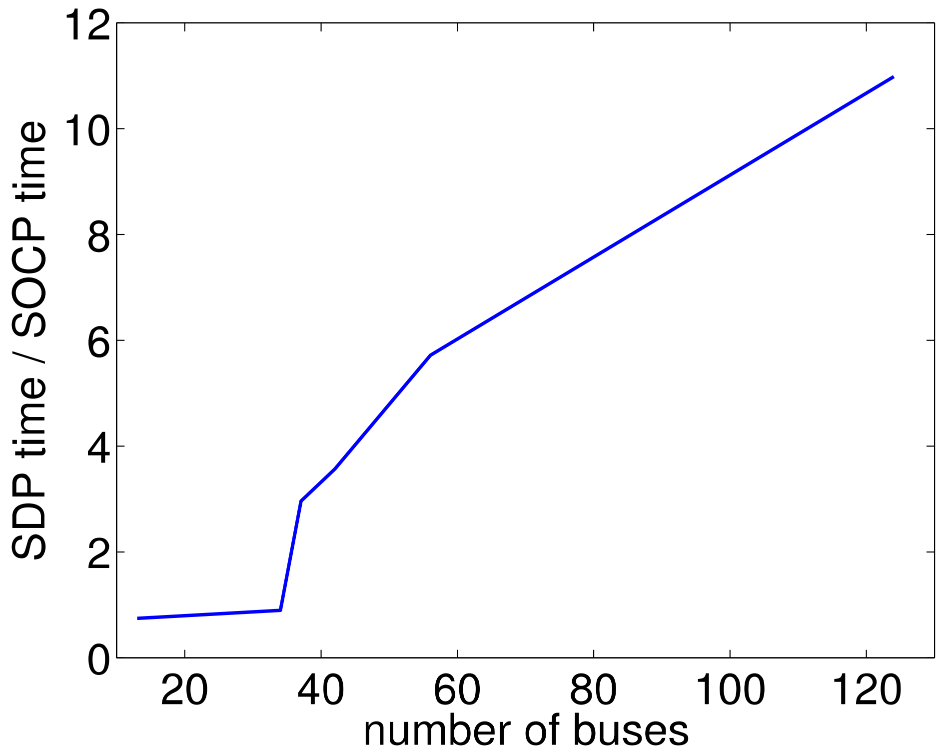


- Relaxation is exact if $\mathbf{X}$ and $\mathbf{Y}$ have same Pareto front
- SOCP is faster but coarser than SDP

Bose, Low, Teeraratkul, Hassibi TAC 2014

SOCP more efficient than SDP





Relaxations are exact in all cases

- IEEE networks: IEEE 13, 34, 37, 123 buses (0% DG)
- SCE networks 47 buses (57% PV), 56 buses (130% PV)
- Single phase; SOCP using BFM
- Matlab 7.9.0.529 (64-bit) with CVX 1.21 on Mac OS X 10.7.5 with 2.66GHz Intel Core 2 Due CPU and 4GB 1067MHz DDR3 memory

Outline

Mathematical preliminaries

Bus injection model

- OPF formulation
- 3 convex relaxations & relationship

Branch flow model

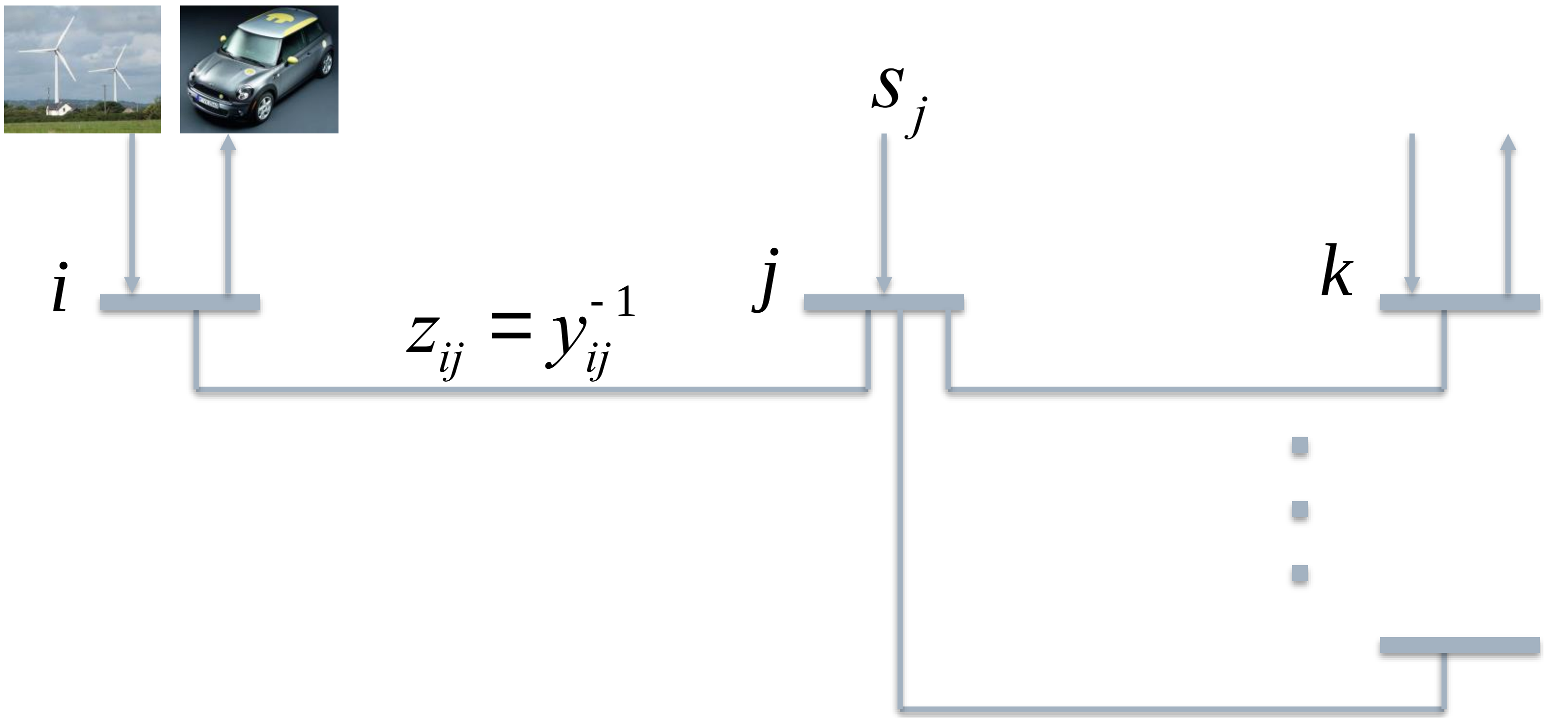
- OPF formulation
- SOCP relaxation & equivalence

Exact relaxation

- Radial networks
- Mesh networks

Multiphase unbalanced networks

Branch flow model



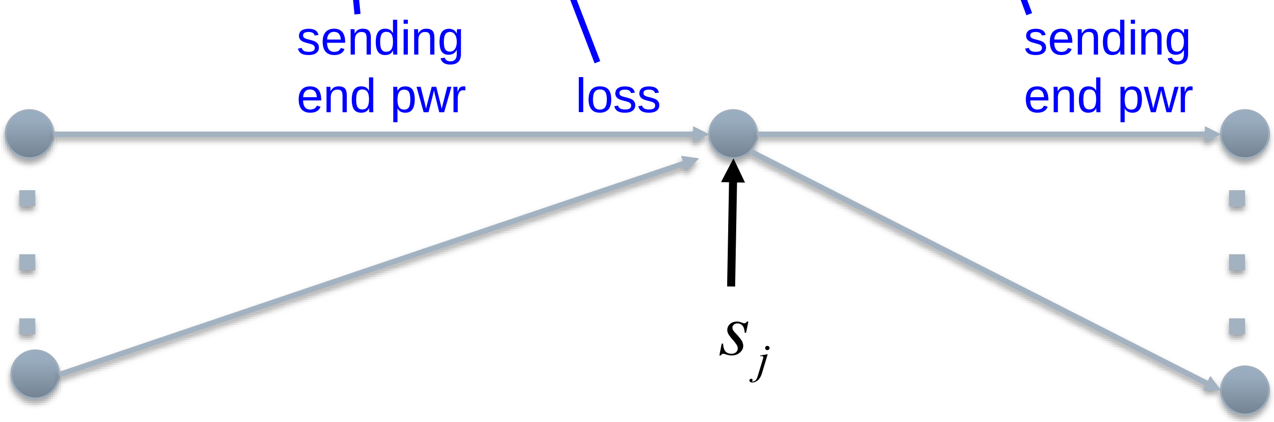
graph model G: directed

Branch flow model

$$V_i - V_j = z_{ij} I_{ij} \qquad \text{for all } i \to j \qquad \text{Kirchhoff law}$$

$$S_{ij} = V_i I_{ij}^H \qquad \text{for all } i \to j \qquad \text{power definition}$$

$$\sum_{i \to j} \left(S_{ij} - z_{ij} \left| I_{ij} \right|^2 \right) + s_j = \sum_{j \to k} S_{jk} \qquad \text{for all } j \qquad \text{power balance}$$



S_{ij} : branch power

I_{ij} : branch current

V_j : voltage

Branch flow model

$$V_i - V_j = z_{ij} I_{ij} \quad \text{for all } i \to j$$

Kirchhoff law

$$S_{ij} = V_i I_{ij}^H \quad \text{for all } i \to j$$

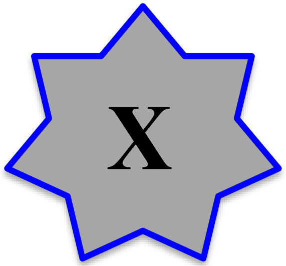
power definition

$$\sum_{i \to j} \left(S_{ij} - z_{ij} \left| I_{ij} \right|^2 \right) + s_j = \sum_{j \to k} S_{jk} \quad \text{for all } j$$

power balance

Power flow problem:

Given (z, s) find (S, I, V)



isolated sols

Recap

Bus injection model

$$s_j = \mathrm{tr}\left(Y_j V V^H\right)$$

$$(V, s) \in \mathbf{C}^{2(n+1)}$$

solution set $\mathbf{V}$

Branch flow model

$$V_i - V_j = z_{ij} I_{ij}$$

$$S_{ij} = V_i I_{ij}^H$$

$$\sum_{j \to k} S_{jk} = \sum_{i \to j} \left(S_{ij} - z_{ij} \left| I_{ij} \right|^2 \right) + s_j$$

$$(S, I, V, s) \in \mathbf{C}^{2(m+n+1)}$$

$\mathbf{X}$

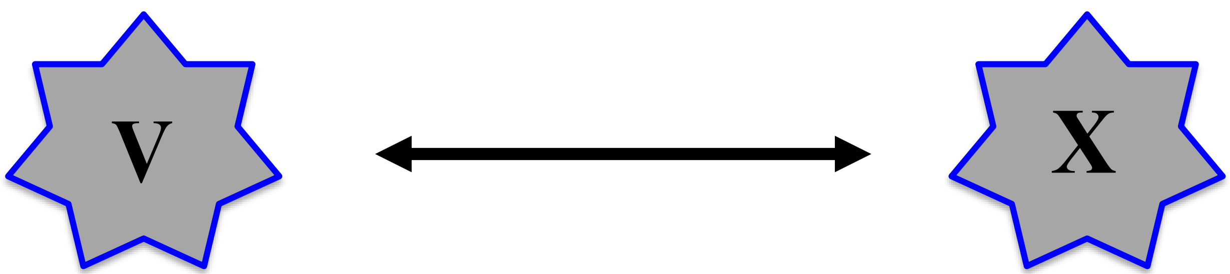
Equivalence

<u>Theorem:</u> $\mathbf{V}$ º $\mathbf{X}$

- BIM and BFM are equivalent in this sense
- Any result in one model is in principle provable in the other,
- … but some results are easier to formulate or prove in one than the other
- BFM seems to be much more numerically stable (radial networks)

(V, s) Î $\mathbf{C}^{2(n+1)}$ (S, I, V, s) Î $\mathbf{C}^{2(m+n+1)}$

solution set



OPF: branch flow model

$$\min \quad f(x)$$

$$\text{over} \quad x := (S, I, V, s)$$

s. t.

OPF: branch flow model

$$\min \quad f(x)$$

$$\text{over} \quad x := (S, I, V, s)$$

$$\text{s. t.} \quad \underline{s}_j \text{ £ } s_j \text{ £ } \overline{s}_j \qquad \underline{v}_j \text{ £ } v_j \text{ £ } \overline{v}_j$$

Summary: OPF (branch flow model)

$$\min \quad f(x)$$

$$\text{over} \quad x := (S, I, V, s)$$

$$\text{s. t.} \quad \underline{s}_j \le s_j \le \overline{s}_j \qquad \underline{v}_j \le v_j \le \overline{v}_j$$

branch flow model

$$\sum_{i \to j} \left(S_{ij} - z_{ij} \left| I_{ij} \right|^2 \right) - \sum_{j \to k} S_{jk} = s_j$$

$$V_j = V_i - z_{ij} I_{ij} \qquad S_{ij} = V_i I_{ij}^H$$

nonconvex (quadratic)

Literature

Convex relaxations of OPF

relaxation	model	first proposed	first analyzed
SOCP	BIM	Jabr 2006 TPS	
SDP	BIM	Bai et al 2008 EPES	Lavaei, Low 2012 TPS
Chordal	BIM	Bai, Wei 2011 EPES Jabr 2012 TPS	Molzahn et al 2013 TPS Bose et al 2014 TAC
SOCP	BFM	Farivar et al 2011 SGC Farivar, Low 2013 TPS	Farivar et al 2011 SGC Farivar, Low 2013 TPS

Low. Convex relaxation of OPF (I, II), IEEE Trans Control of Network Systems, 2014

Branch flow model

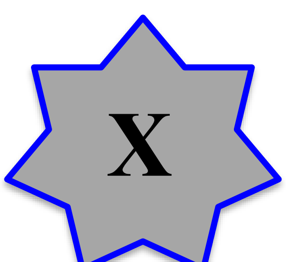
Branch flow model

$$\sum_{j\to k} S_{jk} = \sum_{i\to j}\left(S_{ij} - z_{ij}\left|I_{ij}\right|^2\right) + s_j$$

$$V_i - V_j = z_{ij} I_{ij}$$

$$V_i I_{ij}^H = S_{ij}$$

$$(S, I, V, s) \in \mathbf{C}^{2(m+n+1)}$$



SOCP relaxation

$$\sum_{j\to k} P_{jk} = \sum_{i\to j}\left(P_{ij} - r_{ij}\left|I_{ij}\right|^2\right) + p_j$$

$$\sum_{j\to k} Q_{jk} = \sum_{i\to j}\left(Q_{ij} - x_{ij}\left|I_{ij}\right|^2\right) + q_j$$

Branch flow model

$$\ell_{ij} := |I_{ij}|^2$$
$$v_i := |V_i|^2$$

Branch flow model

$$\sum_{j \to k} S_{jk} = \sum_{i \to j} \left(S_{ij} - z_{ij} |I_{ij}|^2 \right) + s_j$$

$$V_i - V_j = z_{ij} I_{ij}$$

$$V_i I_{ij}^H = S_{ij}$$

$$(S, I, V, s) \in \mathbf{C}^{2(m+n+1)}$$

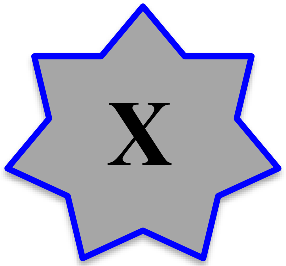
SOCP relaxation

$$\sum_{j \to k} S_{jk} = \sum_{i \to j} \left(S_{ij} - z_{ij} \ell_{ij} \right) + s_j$$

$$v_i - v_j = 2\ \mathrm{Re}\left(z_{ij}^H S_{ij} \right) - |z_{ij}|^2 \ell_{ij}$$

$$v_i \ell_{ij} = |S_{ij}|^2$$

$$(S, \ell, v, s) \in \mathbf{R}^{3(m+n+1)}$$



DistFlow model for **radial** networks
Baran and Wu 1989

Branch flow model

$$\ell_{ij} := |I_{ij}|^2$$
$$v_i := |V_i|^2$$

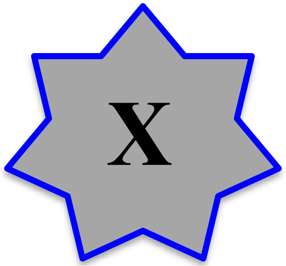
Branch flow model

$$\sum_{j \to k} S_{jk} = \sum_{i \to j} \left(S_{ij} - z_{ij} |I_{ij}|^2 \right) + s_j$$

$$V_i - V_j = z_{ij} I_{ij}$$

$$V_i I_{ij}^H = S_{ij}$$

$$(S, I, V, s) \,\hat{\text{I}}\, \mathbf{C}^{2(m+n+1)}$$



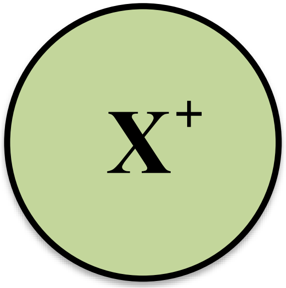
SOCP relaxation

$$\sum_{j \to k} S_{jk} = \sum_{i \to j} \left(S_{ij} - z_{ij} \ell_{ij} \right) + s_j$$

$$v_i - v_j = 2 \operatorname{Re}\left(z_{ij}^H S_{ij} \right) - |z_{ij}|^2 \ell_{ij}$$

$$v_i \ell_{ij} \;\; 3 \;\; |S_{ij}|^2$$

$$(S, \ell, v, s) \,\hat{\text{I}}\, \mathbf{R}^{3(m+n+1)}$$



Branch flow model

$$\mathbb{X}^+ := \{ x : \underline{\text{linear}} \text{ constraints} \} \cap \left\{ \ell_{jk} v_j \geq |S|^2 \right\}$$

SOC

$$\mathbb{P} := \left\{ \begin{array}{l} x : \ \ell_{jk} v_j = |S|^2 \\ \quad \text{cycle cond on } x \end{array} \right\}$$

<u>Theorem</u> $\mathbb{X} \equiv \mathbb{X}^+ \cap \mathbb{P}$

Cycle condition

A solution x satisfies the cycle condition if

$$\$q \quad \text{s.t.} \quad Bq = b(x) \qquad \text{mod } 2p$$

incidence matrix;
depends on topology

$$x := (S, \ell, v, s)$$

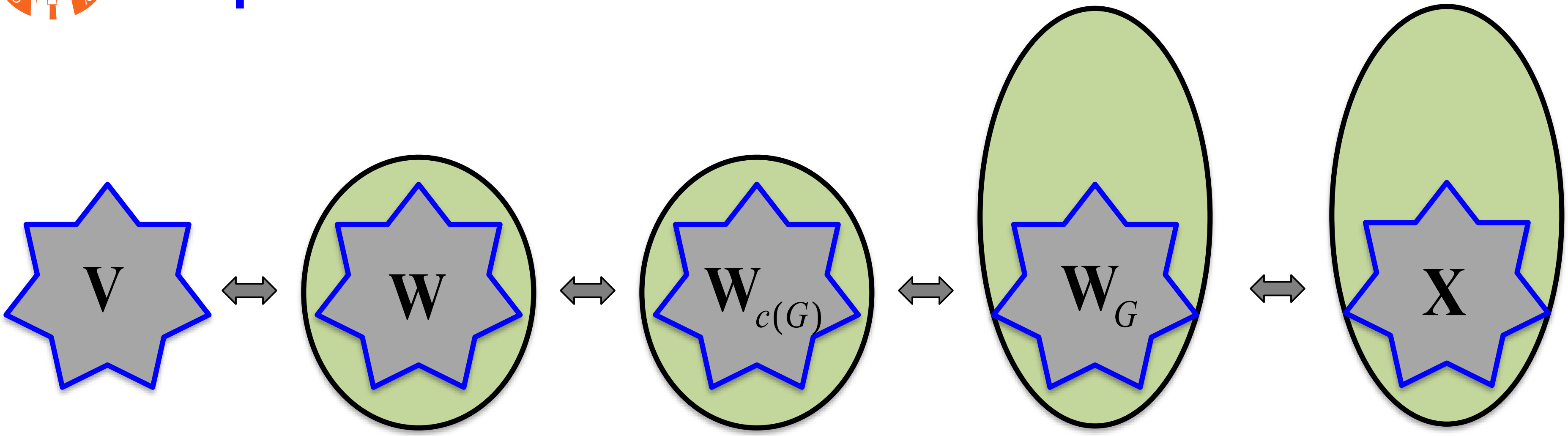
$$b_{jk}(x) := \text{Đ}\left(v_j \text{ - } z_{jk}^H S_{jk}\right)$$

BFM: SOCP relaxation of OPF

OPF: $$\min_{x \in \mathbf{X}} f(x)$$

SOCP: $$\min_{x \in \mathbf{X}^+} f(x)$$

Equivalence



$$\mathbf{W}^+ \Leftrightarrow \mathbf{W}^+_{c(G)} \Leftrightarrow \mathbf{W}^+_G \Leftrightarrow \mathbf{X}^+$$

<u>Theorem</u>

$\mathbf{W}_G \circ \mathbf{X}$ and $\mathbf{W}^+_G \circ \mathbf{X}^+$

BFM for radial networks

Table 5.3: Objective values and CPU times of CVX and IPM

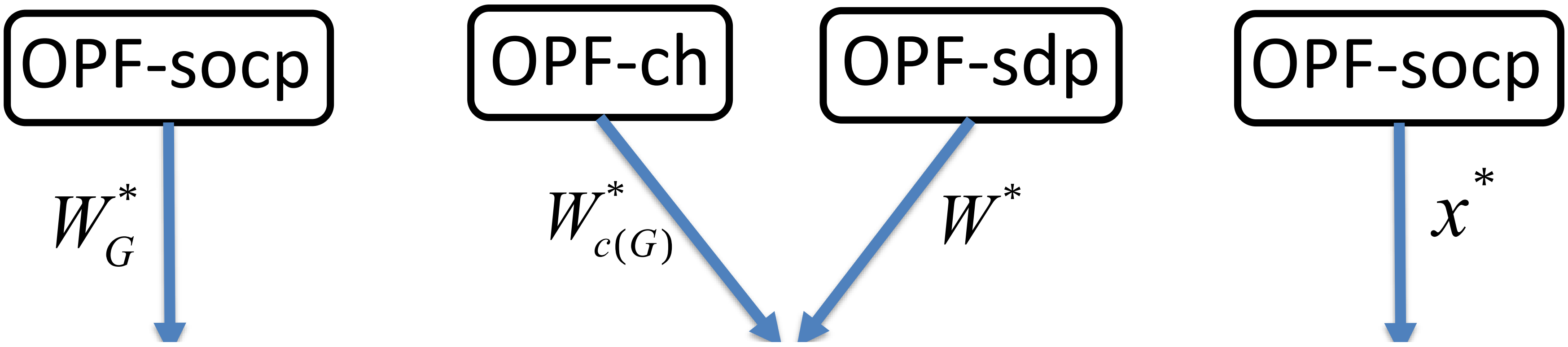
# bus	CVX		IPM		error	speedup
	obj	time(s)	obj	time(s)		
42	10.4585	6.5267	10.4585	0.2679	-0.0e-7	24.36
56	34.8989	7.1077	34.8989	0.3924	+0.2e-7	18.11
111	0.0751	11.3793	0.0751	0.8529	+5.4e-6	13.34
190	0.1394	20.2745	0.1394	1.9968	+3.3e-6	10.15
290	0.2817	23.8817	0.2817	4.3564	+1.1e-7	5.48
390	0.4292	29.8620	0.4292	2.9405	+5.4e-7	10.16
490	0.5526	36.3591	0.5526	3.0072	+2.9e-7	12.09
590	0.7035	43.6932	0.7035	4.4655	+2.4e-7	9.78
690	0.8546	51.9830	0.8546	3.2247	+0.7e-7	16.12
790	0.9975	62.3654	0.9975	2.6228	+0.7e-7	23.78
890	1.1685	67.7256	1.1685	2.0507	+0.8e-7	33.03
990	1.3930	74.8522	1.3930	2.7747	+1.0e-7	26.98
1091	1.5869	83.2236	1.5869	1.0869	+1.2e-7	76.57
1190	1.8123	92.4484	1.8123	1.2121	+1.4e-7	76.27
1290	2.0134	101.0380	2.0134	1.3525	+1.6e-7	74.70
1390	2.2007	111.0839	2.2007	1.4883	+1.7e-7	74.64
1490	2.4523	122.1819	2.4523	1.6372	+1.9e-7	74.83
1590	2.6477	157.8238	2.6477	1.8021	+2.0e-7	87.58
1690	2.8441	147.6862	2.8441	1.9166	+2.1e-7	77.06
1790	3.0495	152.6081	3.0495	2.0603	+2.1e-7	74.07
1890	3.8555	160.4689	3.8555	2.1963	+1.9e-7	73.06
1990	4.1424	171.8137	4.1424	2.3586	+1.9e-7	72.84

Recursive structure

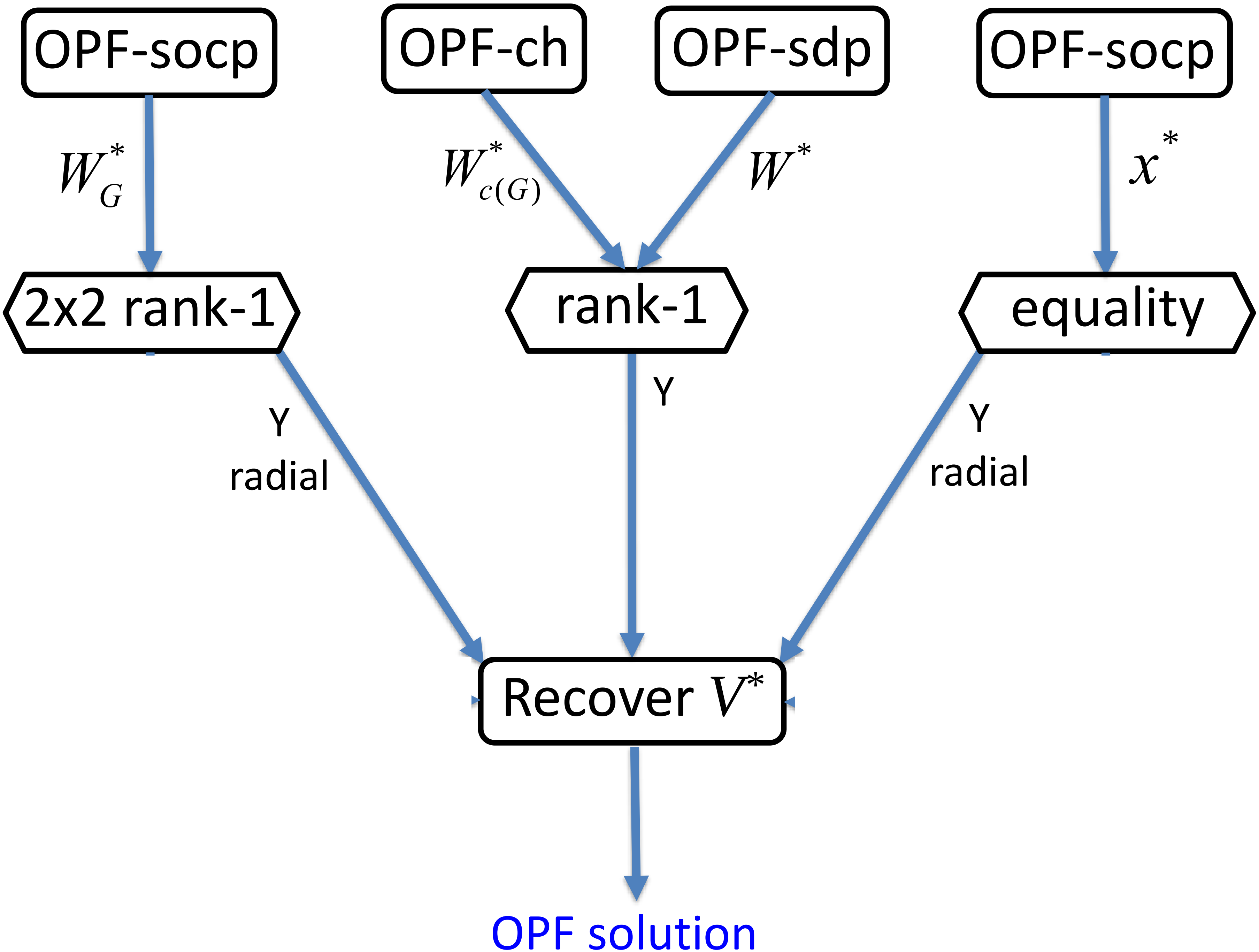
- backward-forward sweep for PF solution

Advantages over BIM

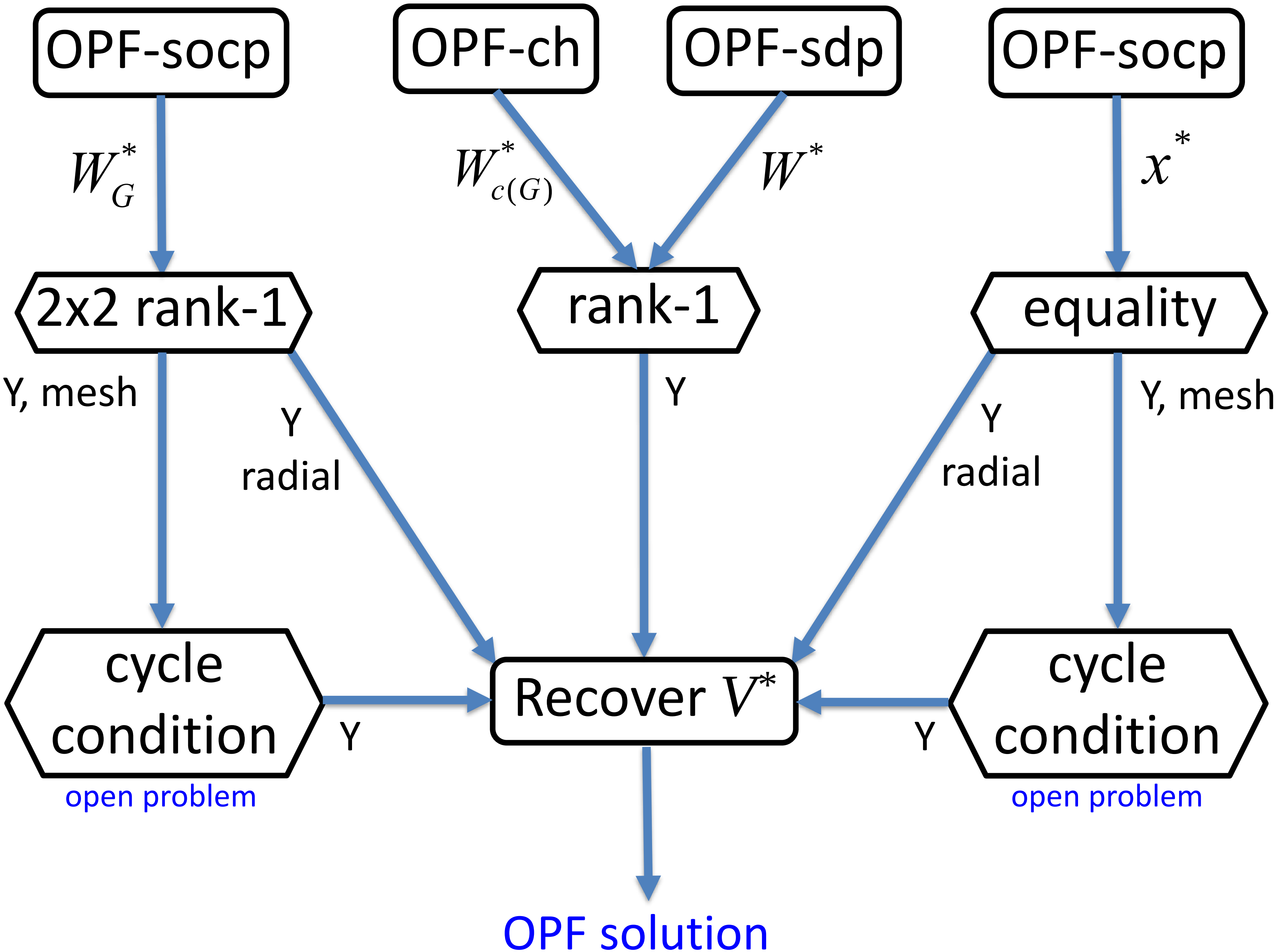
- much faster
- much more stable numerically



OPF-socp
OPF-ch
OPF-sdp
OPF-socp
W_G^*
$W_{c(G)}^*$
W^*
x^*

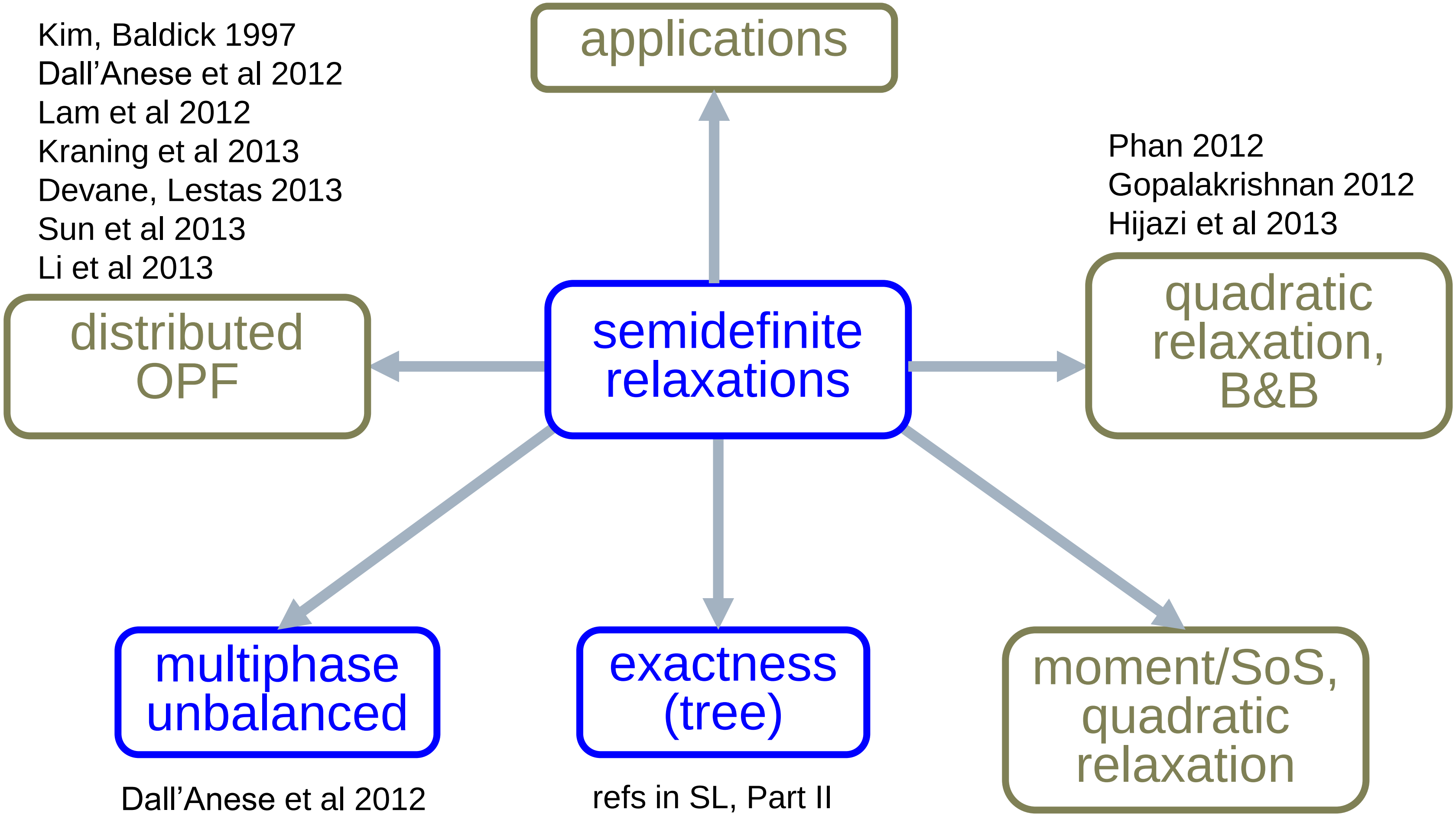


OPF-socp
OPF-ch
OPF-sdp
OPF-socp
W_G^*
$W_{c(G)}^*$
W^*
x^*
2x2 rank-1
rank-1
equality
Y
radial
Y
Y
radial
Recover V^*
OPF solution



OPF-socp
OPF-ch
OPF-sdp
OPF-socp
W_G^*
$W_{c(G)}^*$
W^*
x^*
2x2 rank-1
rank-1
equality
Y, mesh
Y
radial
Y
Y
radial
Y, mesh
cycle condition
Recover V^*
cycle condition
Y
Y
open problem
open problem
OPF solution

OPF: extensions



Kim, Baldick 1997
Dall'Anese et al 2012
Lam et al 2012
Kraning et al 2013
Devane, Lestas 2013
Sun et al 2013
Li et al 2013

Phan 2012
Gopalakrishnan 2012
Hijazi et al 2013

Dall'Anese et al 2012
Gan, Low 2014

refs in SL, Part II
TCNS 2014

Molzahn, Hiskens 2014
Josz et al 2014
Ghaddar et al 2014

Digression:

Branch flow model for radial networks

BFM for radial networks

$$\sum_{j \to k} S_{jk} = S_{ij} - z_{ij}\ell_{ij} + s_j$$

$$v_i - v_j = 2\,\mathrm{Re}\left(z_{ij}^H S_{ij}\right) - \left|z_{ij}\right|^2 \ell_{ij}$$

$$\ell_{ij} v_i = \left|S_{ij}\right|^2$$

DistFlow model
Baran and Wu 1989

$$\ell_{ij} := \left|I_{ij}\right|^2$$
$$v_i := \left|V_i\right|^2$$

Advantages

- PF: recursive structure ➔ backward/forward sweep
- OPF: more numerically stable SOCP
- Linear approx. suitable for **radial** networks (unlike DC)
- Variables represent physical quantities

Lin DistFlow for radial networks

$$\sum_{j \to k} S_{jk}^{\text{lin}} = S_{ij}^{\text{lin}} + s_j$$

$$v_i^{\text{lin}} - v_j^{\text{lin}} = 2 \, \text{Re}\left(z_{ij}^H S_{ij}^{\text{lin}}\right)$$

Linear DistFlow
Baran and Wu 1989

Advantages over DC power flow

- Includes voltages and reactive power as vars
- Allows nonzero resistance
- Accurate when line loss is small compared with with branch power flow
- … more …

Lin DistFlow for radial networks

$$\sum_{j\to k} S_{jk}^{\text{lin}} = S_{ij}^{\text{lin}} + s_j$$

$$v_i^{\text{lin}} - v_j^{\text{lin}} = 2\,\text{Re}\left(z_{ij}^H S_{ij}^{\text{lin}}\right)$$

Linear DistFlow

Baran and Wu 1989

- Explicit solution:

$$S_{ij}^{\text{lin}} = -\sum_{k\in \mathbf{T}_j} s_k$$

$$v_j^{\text{lin}} = v_0 - \sum_{(i,k)\in \mathbf{P}_j} 2\,\text{Re}\left(z_{ik}^H S_{ik}^{\text{lin}}\right)$$

- Bounding true solution: $v_j \le v_j^{\text{lin}} \qquad S_{ij} \ge S_{ij}^{\text{lin}}$

Outline

Mathematical preliminaries

Bus injection model

- OPF formulation
- 3 convex relaxations & relationship

Branch flow model

- OPF formulation
- SOCP relaxation & equivalence

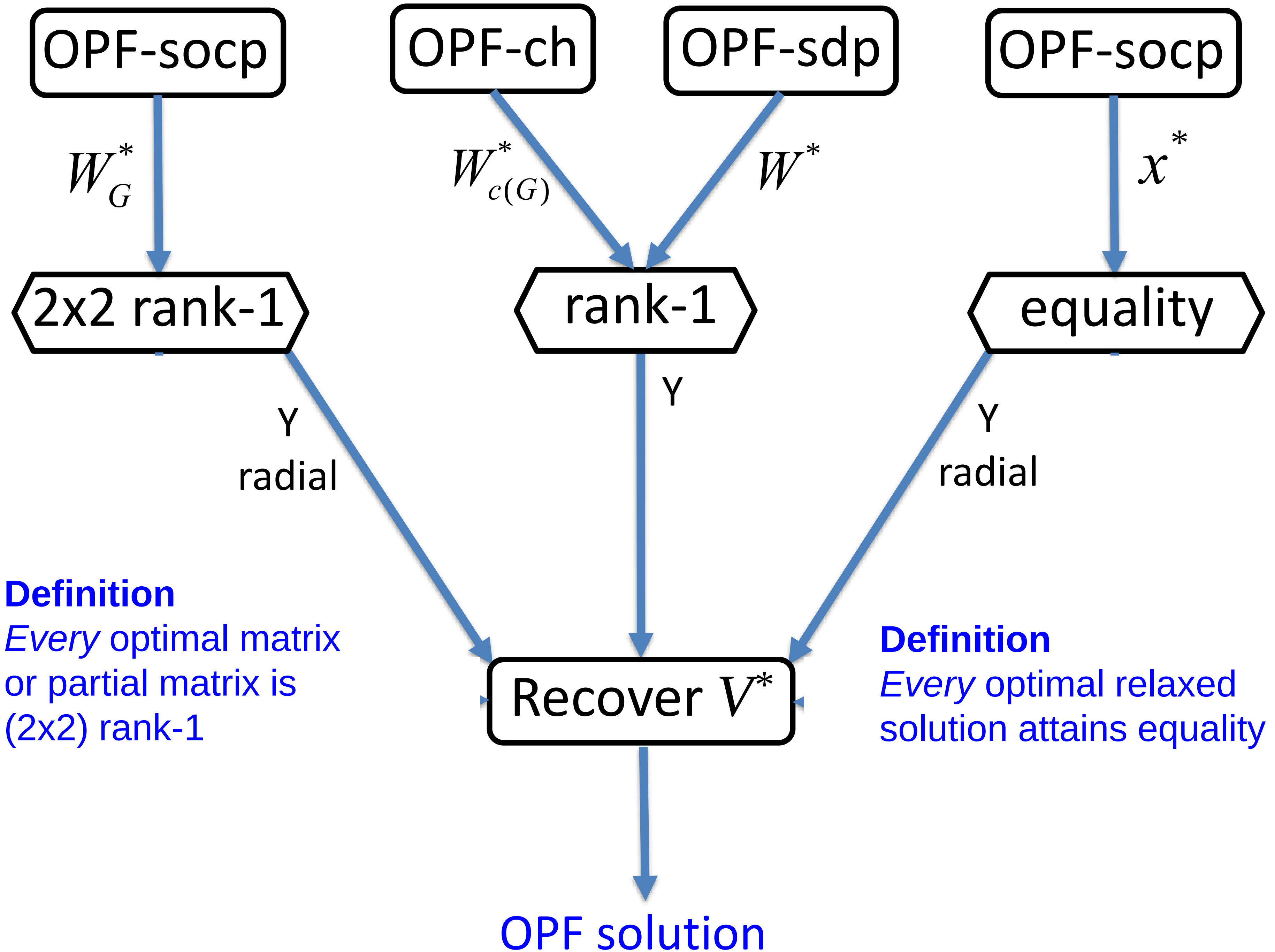
Exact relaxation

- Radial networks
- Mesh networks

Multiphase unbalanced networks

Exact relaxation

A relaxation is exact if an optimal solution of the original OPF can be recovered from *every* optimal solution of the relaxation


OPF-socp
OPF-ch
OPF-sdp
OPF-socp
W_G^*
$W_{c(G)}^*$
W^*
x^*
2x2 rank-1
rank-1
equality
Y
radial
Y
Y
radial
Definition
Every optimal matrix or partial matrix is (2x2) rank-1
Recover V^*
Definition
Every optimal relaxed solution attains equality
OPF solution

Summary of sufficient conds

type	condition	model	reference	remark
A	power injections	BIM, BFM	[25], [26], [27], [28], [29] [30], [16], [17]	
B	voltage magnitudes	BFM	[31], [32], [33], [34]	allows general injection region
C	voltage angles	BIM	[35], [36]	makes use of branch power flows

TABLE I: Sufficient conditions for radial (tree) networks.

network	condition	reference	remark
with phase shifters	type A, B, C	[17, Part II], [37]	equivalent to radial networks
direct current	type A	[17, Part I], [19], [38]	assumes nonnegative voltages
	type B	[39], [40]	assumes nonnegative voltages

TABLE II: Sufficient conditions for mesh networks

1. QCQP over tree

QCQP (C, C_k)

$$\begin{aligned} &\min && x^* C x \\ &\text{over} && x \in \mathbf{C}^n \\ &\text{s.t.} && x^* C_k x \leq b_k \qquad k \in K \end{aligned}$$

graph of QCQP

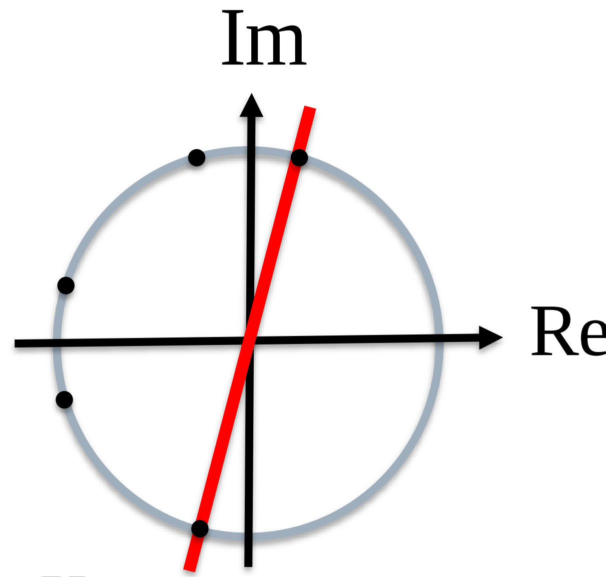
$G(C, C_k)$ has edge (i, j) $\Leftrightarrow$

$C_{ij} \neq 0$ or $[C_k]_{ij} \neq 0$ for some k

QCQP over tree

$G(C, C_k)$ is a tree

1. Linear separability



QCQP (C, C_k)

$$\begin{aligned} &\min && x^* C x \\ &\text{over} && x \in \mathbf{C}^n \\ &\text{s.t.} && x^* C_k x \le b_k \qquad k \in K \end{aligned}$$

Key condition

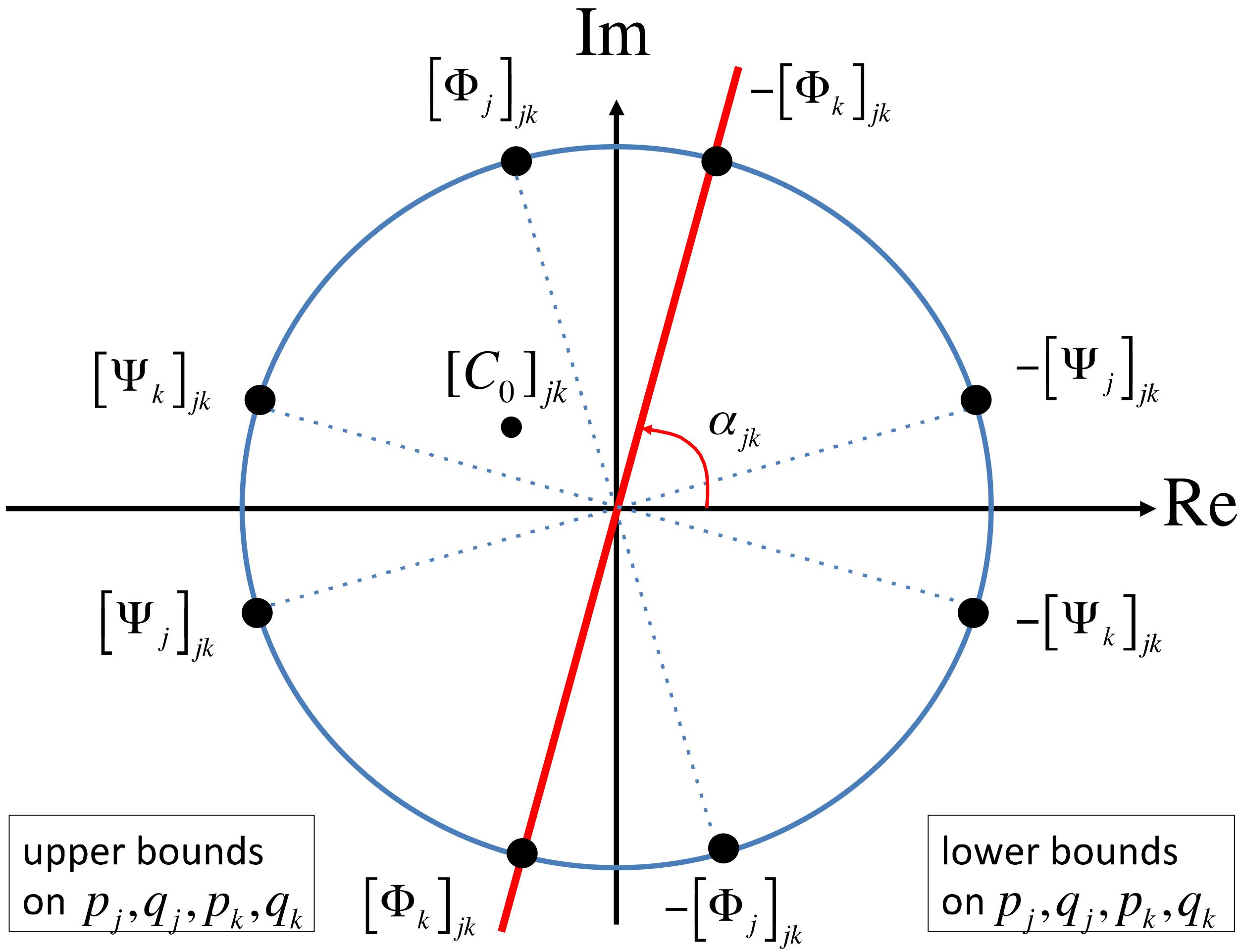
$i \sim j: \ \left(C_{ij}, [C_k]_{ij}, \ \forall k\right)$ lie on half-plane through 0

<u>**Theorem**</u>

SOCP relaxation is exact for QCQP over tree

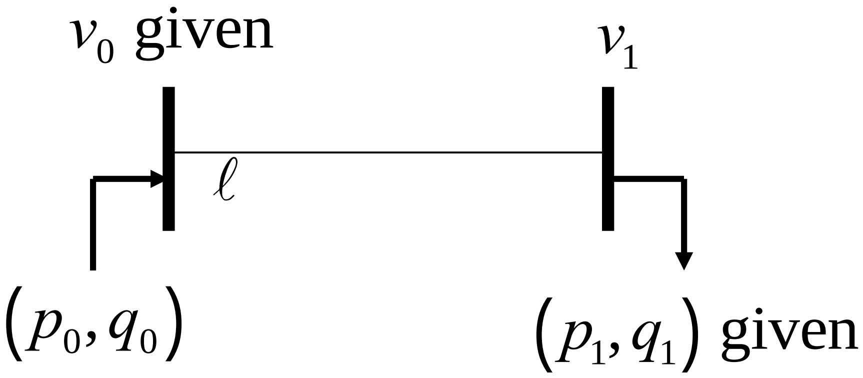
Bose et al 2012, 2014
Sojoudi, Lavaei 2013

Implication on OPF



Not both lower & upper bounds on real & reactive powers at both ends of a line can be finite

2. Voltage upper bounds



geometric insight

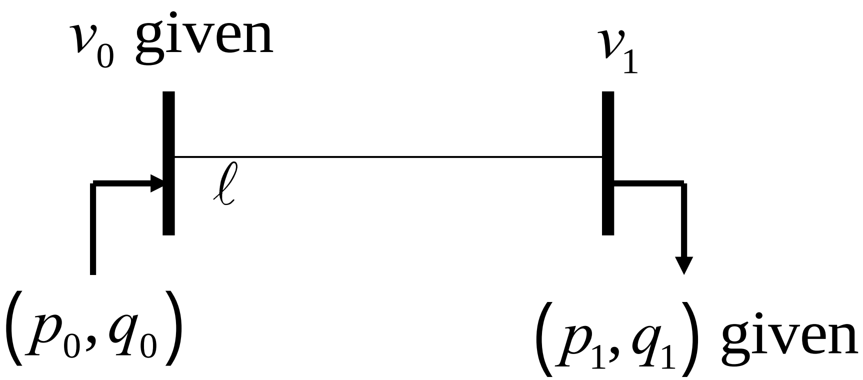
vars are: $(p_0, q_0), \ell, v_1$

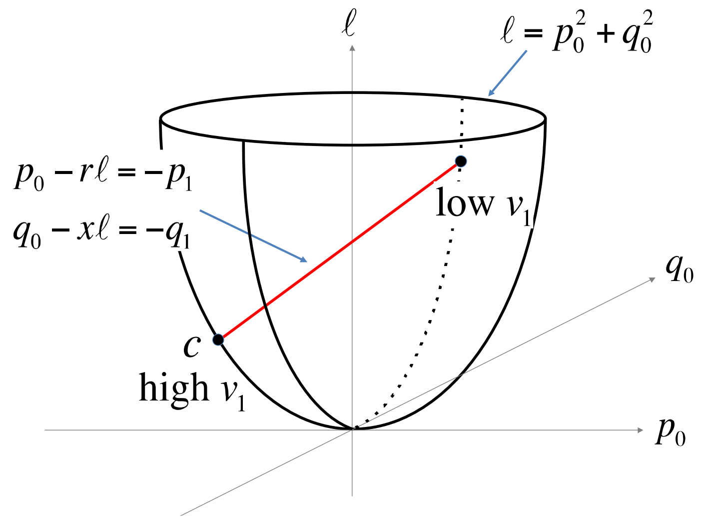
$$p_0^2 + q_0^2 \;=\; \ell$$

$$p_0 - r\ell \;=\; -p_1, \qquad q_0 - x\ell \;=\; -q_1$$

$$v_1 - v_0 \;=\; 2(rp_0 + xq_0) - |z|^2 \ell$$

2. Voltage upper bounds



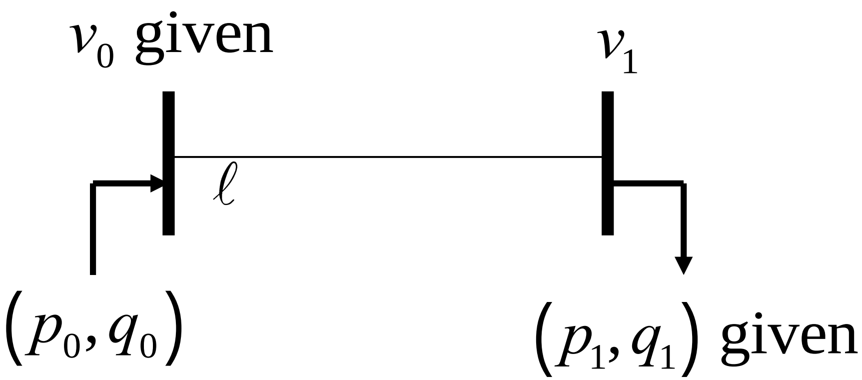


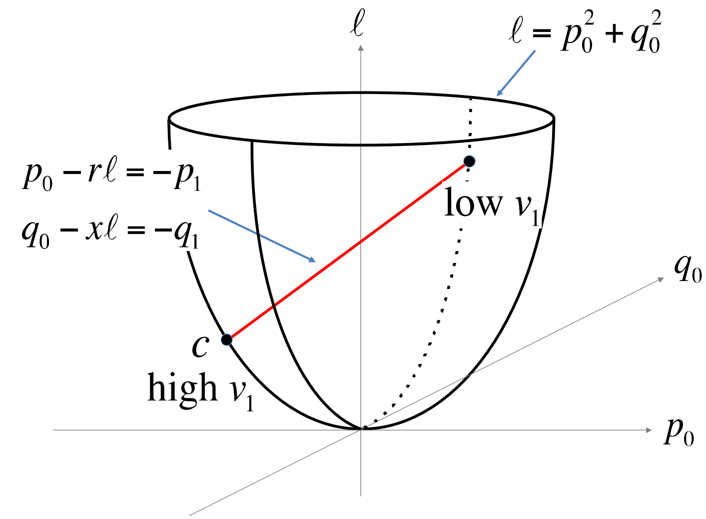
when there is no voltage constraint

- feasible set : 2 intersection pts
- relaxation: line segment
- exact relaxation: c is optimal

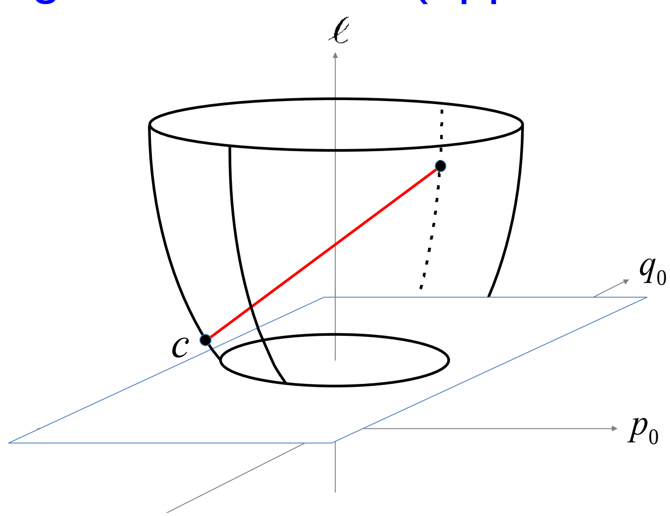
  … as long as cost increasing in ℓ, p_0, q_0

2. Voltage upper bounds

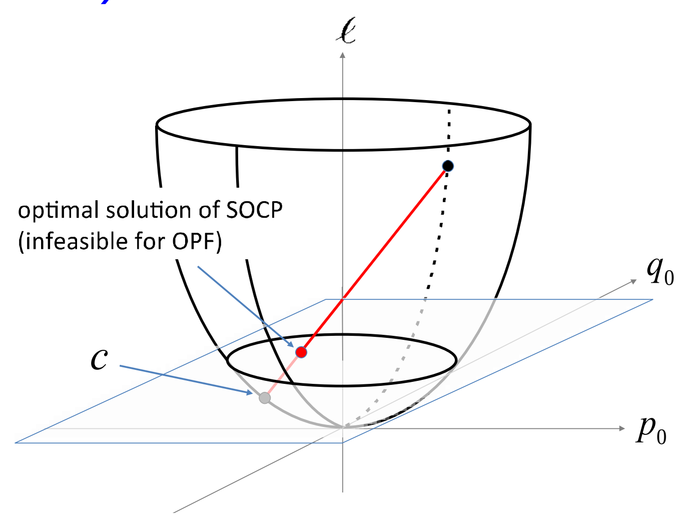




voltage lower bound (upper bound on l) does not affect relaxation



(a) Voltage constraint not binding



(b) Voltage constraint binding

2. Voltage upper bounds

OPF: $\min_{x \in \mathbf{X}} f(x) \quad \text{s.t.} \quad \underline{v} \le v \le \overline{v}, \quad s \in \mathcal{S}$

SOCP: $\min_{x \in \mathbf{X}^{+}} f(x) \quad \text{s.t.} \quad \underline{v} \le v \le \overline{v}, \quad s \in \mathcal{S}$

Key conditions:

- $v^{\text{lin}}(s) \le \overline{v}$ — voltages if network were lossless
- Jacobian condition — if upward current were reduced then all subsequent powers dec

  $\underline{A}_{i_t} \cdots \underline{A}_{i_{t'}} z_{i_{t'+1}} > 0$ for all $1 \le t \le t' < k$

Theorem

SOCP relaxation is exact for radial networks

Gan, Li, Topcu, Low TAC2014

2. Voltage upper bounds

OPF: $\min_{x \in \mathbf{X}} f(x) \quad \text{s.t.} \quad \underline{v} \le v \le \overline{v}, \quad s \in S$

SOCP: $\min_{x \in \mathbf{X}^+} f(x) \quad \text{s.t.} \quad \underline{v} \le v \le \overline{v}, \quad s \in S$

Key conditions:

- $v^{\text{lin}}(s) \le \overline{v}$
- Jacobian condition

  $\underline{A}_{i_t} \cdots \underline{A}_{i_{t'}} z_{i_{t'+1}} > 0 \text{ for all } 1 \le t \le t' < k$

satisfied with large margin in IEEE circuits and SCE circuits

Theorem

SOCP relaxation is exact for radial networks

Gan, Li, Topcu, Low TAC2014

3. Voltage angles

$$\min_{p,P,V} \quad C(p)$$

$$\text{s.t.} \quad \underline{p}_j \le p_j \le \overline{p}_j$$

$$\underline{\theta}_{jk} \le \theta_{jk} \le \overline{\theta}_{jk}$$

$$p_j = \sum_{k:k \leftarrow j} P_{jk}$$

$$P_{jk} = |V_j|^2 g_{jk} - |V_j||V_k| g_{jk} \cos\theta_{jk} + |V_j||V_k| b_{jk} \sin\theta_{jk}$$

Can represent constraints on

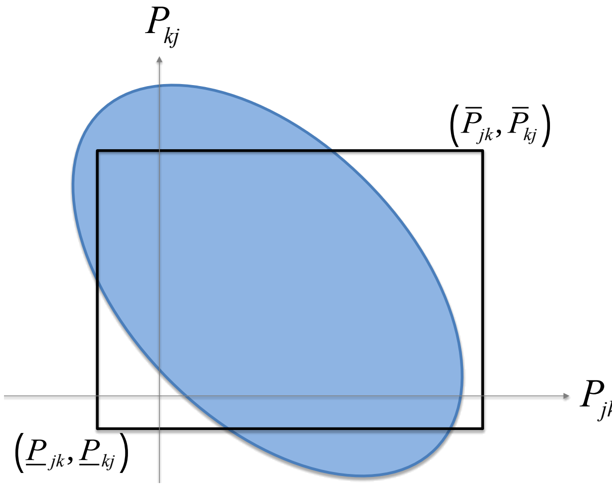
- Line flows
- Line loss
- Stability

assumptions:

- fixed voltage magnitudes
- real power only

Zhang & Tse, TPS 2013
Lavaei, Zhang, Tse, 2012

3. Voltage angles



OPF:

$$\min_{p,\theta} \quad C(p)$$

$$\text{s.t.} \quad \underline{p}_j \le p_j \le \overline{p}_j$$

$$\underline{\theta}_{jk} \le \theta_{jk} \le \overline{\theta}_{jk}$$

$$p_j = \sum_{k:k\sim j} g_{jk} - g_{jk}\cos\theta_{jk} + b_{jk}\sin\theta_{jk}$$

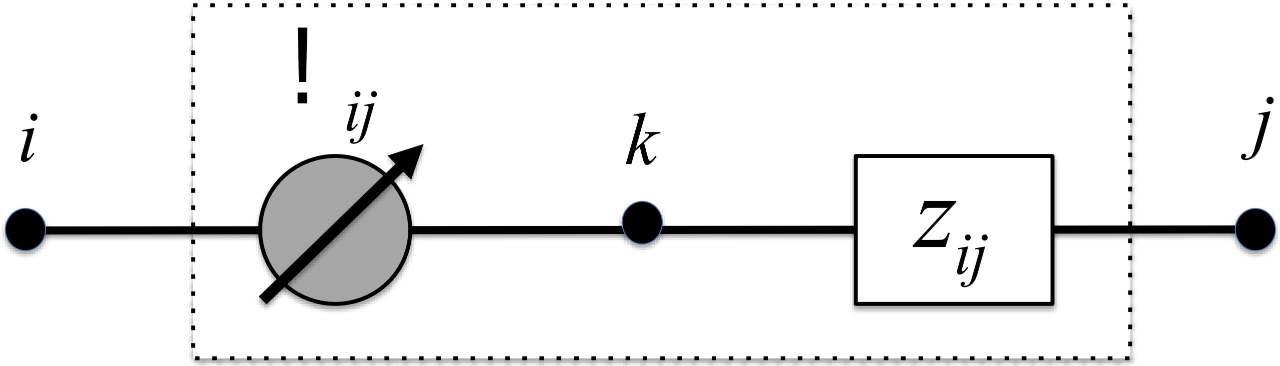
Key condition: $-\tan^{-1}\left(\frac{x_{jk}}{r_{jk}}\right) < \underline{\theta}_{jk} \le \overline{\theta}_{jk} < \tan^{-1}\left(\frac{x_{jk}}{r_{jk}}\right)$

Theorem

SOCP relaxation is exact for radial networks $\left(|V_j| \text{ constant}\right)$

Lavaei, Tse, Zhang 2012

Mesh networks with phase shifter



ideal phase shifter

Mesh networks with phase shifter

<u>BFM without phase shifters:</u>

$$I_{ij} = y_{ij}\left(V_i - V_j\right)$$

$$S_{ij} = V_i I_{ij}^*$$

$$s_j = \sum_{k: j \to k} S_{jk} - \sum_{i: i \to j} \left(S_{ij} - z_{ij}|I_{ij}|^2\right) + y_j^* |V_j|^2$$

<u>BFM with phase shifters:</u>

$$I_{ij} = y_{ij}\left(V_i - V_j\, e^{-i\varphi_{ij}}\right)$$

$$S_{ij} = V_i I_{ij}^*$$

$$s_j = \sum_{k: j \to k} S_{jk} - \sum_{i: i \to j} \left(S_{ij} - z_{ij}|I_{ij}|^2\right) + y_j^* |V_j|^2$$

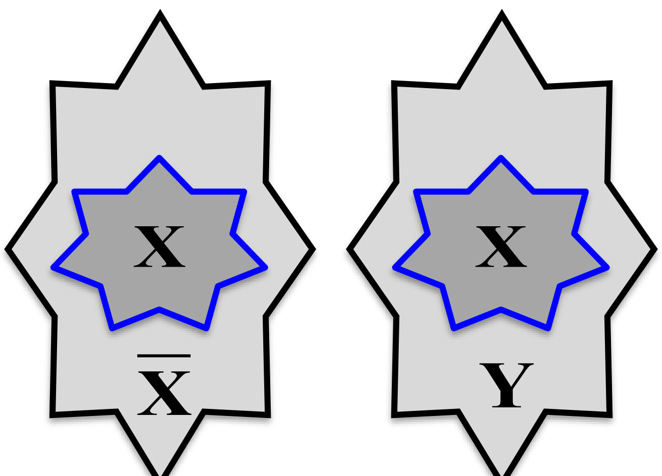
Convexification of mesh networks

OPF $\quad \min_{x} \; f\big(h(x)\big) \quad \text{s.t.} \quad x \in \mathbf{X}$

OPF-ar $\quad \min_{x} \; f\big(h(x)\big) \quad \text{s.t.} \quad x \in \mathbf{Y}$

OPF-ps $\quad \min_{x,\phi} \; f\big(h(x)\big) \quad \text{s.t.} \quad x \in \overline{\mathbf{X}}$

optimize over phase shifters as well



<u>Theorem</u>

- $\overline{\mathbf{X}} = \mathbf{Y}$
- Need phase shifters only outside spanning tree

Cycle condition

A solution x satisfies the cycle condition if

- without PS:

$$\$q \quad \text{s.t.} \quad Bq = b(x) \qquad \text{mod } 2p$$

$$x := (S, \ell, v, s)$$

$$b_{jk}(x) := Đ\left(v_j - z_{jk}^H S_{jk}\right)$$

- without PS:

$$\$q, f \quad \text{s.t.} \quad Bq = b(x) - f \qquad \text{mod } 2p$$

can always satisfy with PS at strategic locations

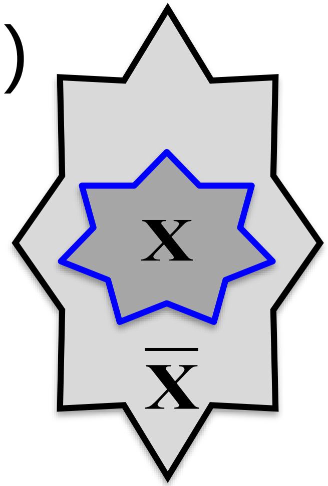
Convexification of mesh networks

OPF-ps $\min_{x,f} \; f\left(h(x)\right) \quad \text{s.t.} \quad x \in \overline{\mathbf{X}}$

optimize over phase shifters as well

<u>Optimization of ϕ</u>

- Min # phase shifters (#lines - #buses + 1)
- Min $\|f\|_2$: NP hard (good heuristics)
- Given existing network of PS, min # or angles of additional PS



Farivar & Low TPS 2013

Examples

Test cases	# links (m)	No PS Min loss (OPF, MW)	With PS Min loss (OPF-cr, MW)
IEEE 14-Bus	20	0.546	0.545
IEEE 30-Bus	41	1.372	1.239
IEEE 57-Bus	80	11.302	10.910
IEEE 118-Bus	186	9.232	8.728
IEEE 300-Bus	411	211.871	197.387
New England 39-Bus	46	29.915	28.901
Polish (case2383wp)	2,896	433.019	385.894
Polish (case2737sop)	3,506	130.145	109.905

Examples

Test cases	# links (m)	# active PS $\lvert\phi_i\rvert > 0.1^\circ$	Min #PS ($^\circ$) $[\phi_{\min}, \phi_{\max}]$
IEEE 14-Bus	20	2 (10%)	$[-2.09,\ 0.58]$
IEEE 30-Bus	41	3 (7%)	$[-0.20,\ 4.47]$
IEEE 57-Bus	80	19 (24%)	$[-3.47,\ 3.15]$
IEEE 118-Bus	186	36 (19%)	$[-1.95,\ 2.03]$
IEEE 300-Bus	411	101 (25%)	$[-13.3,\ 9.40]$
New England 39-Bus	46	7 (15%)	$[-0.26,\ 1.83]$
Polish (case2383wp)	2,896	373 (13%)	$[-19.9,\ 16.8]$
Polish (case2737sop)	3,506	395 (11%)	$[-10.9,\ 11.9]$

Examples

Test cases	# links (m)	Min #PS (°) $[\phi_{\min}, \phi_{\max}]$	Min $\|\phi\|^2$ (°) $[\phi_{\min}, \phi_{\max}]$
IEEE 14-Bus	20	$[-2.09,\ 0.58]$	[-0.63, 0.12]
IEEE 30-Bus	41	$[-0.20,\ 4.47]$	[-0.95, 0.65]
IEEE 57-Bus	80	$[-3.47,\ 3.15]$	[-0.99, 0.99]
IEEE 118-Bus	186	$[-1.95,\ 2.03]$	[-0.81, 0.31]
IEEE 300-Bus	411	$[-13.3,\ 9.40]$	[-3.96, 2.85]
New England 39-Bus	46	$[-0.26,\ 1.83]$	[-0.33, 0.33]
Polish (case2383wp)	2,896	$[-19.9,\ 16.8]$	[-3.07, 3.23]
Polish (case2737sop)	3,506	$[-10.9,\ 11.9]$	[-1.23, 2.36]

Outline

Mathematical preliminaries

Bus injection model

- OPF formulation
- 3 convex relaxations & relationship

Branch flow model

- OPF formulation
- SOCP relaxation & equivalence

Exact relaxation

- Sufficient conditions

Multiphase unbalanced networks

Distribution systems

Mostly radial networks

Multiphase unbalanced

- Lines may not be transposed
- Loads may not be balanced

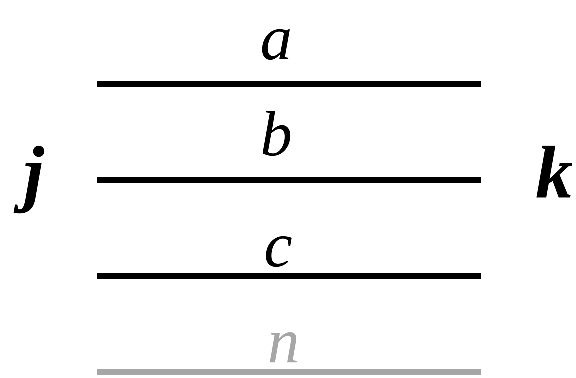
Some references

- Kersting (2002)
- Shirmohammadi, et al (1988), Chen et al (1991)
- Lo and Zhang (1993), Arboleya et al (2014)
- Dall'Anese, Zhu and Giannakis (2012)
- ……

Bus injection model (phase frame)

3-phase balanced (positive sequence)

$$\begin{bmatrix} I_{jk}^{a} \\ I_{jk}^{b} \\ I_{jk}^{c} \end{bmatrix} = \begin{bmatrix} y_{jk}^{aa} & 0 & 0 \\ 0 & y_{jk}^{bb} & 0 \\ 0 & 0 & y_{jk}^{cc} \end{bmatrix} \left(\begin{bmatrix} V_{j}^{a} \\ V_{j}^{b} \\ V_{j}^{c} \end{bmatrix} - \begin{bmatrix} V_{k}^{a} \\ V_{k}^{b} \\ V_{k}^{c} \end{bmatrix} \right)$$



3-phase unbalanced

$$\begin{bmatrix} I_{jk}^{a} \\ I_{jk}^{b} \\ I_{jk}^{c} \end{bmatrix} = \begin{bmatrix} y_{jk}^{aa} & y_{jk}^{ab} & y_{jk}^{ac} \\ y_{jk}^{ba} & y_{jk}^{bb} & y_{jk}^{bc} \\ y_{jk}^{ca} & y_{jk}^{cb} & y_{jk}^{cc} \end{bmatrix} \left(\begin{bmatrix} V_{j}^{a} \\ V_{j}^{b} \\ V_{j}^{c} \end{bmatrix} - \begin{bmatrix} V_{k}^{a} \\ V_{k}^{b} \\ V_{k}^{c} \end{bmatrix} \right)$$

Assume 3 phases everywhere. See paper for general multiphase

Bus injection model (phase frame)

3-phase balanced (positive sequence)

$$\begin{bmatrix} I_{jk}^a \\ I_{jk}^b \\ I_{jk}^c \end{bmatrix} = \begin{bmatrix} y_{jk}^{aa} & 0 & 0 \\ 0 & y_{jk}^{bb} & 0 \\ 0 & 0 & y_{jk}^{cc} \end{bmatrix} \left(\begin{bmatrix} V_j^a \\ V_j^b \\ V_j^c \end{bmatrix} - \begin{bmatrix} V_k^a \\ V_k^b \\ V_k^c \end{bmatrix} \right)$$

per-phase analysis

$$I_{jk}^a = y_{jk}^{aa} \left(V_j^a - V_k^a \right)$$

3-phase unbalanced

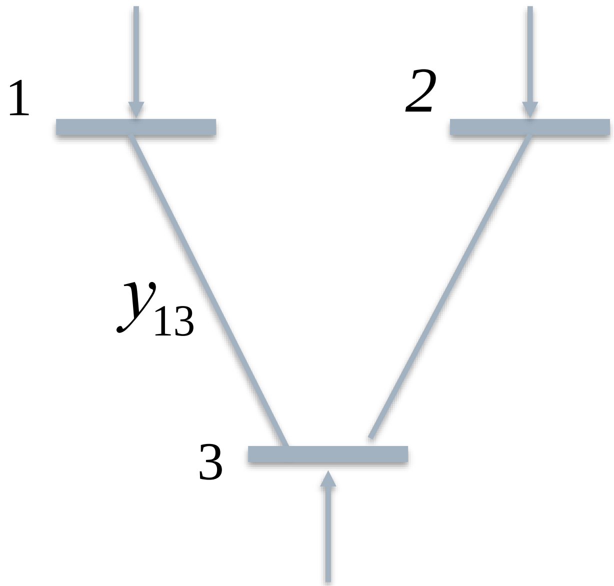
$$\begin{bmatrix} I_{jk}^a \\ I_{jk}^b \\ I_{jk}^c \end{bmatrix} = \begin{bmatrix} y_{jk}^{aa} & y_{jk}^{ab} & y_{jk}^{ac} \\ y_{jk}^{ba} & y_{jk}^{bb} & y_{jk}^{bc} \\ y_{jk}^{ca} & y_{jk}^{cb} & y_{jk}^{cc} \end{bmatrix} \left(\begin{bmatrix} V_j^a \\ V_j^b \\ V_j^c \end{bmatrix} - \begin{bmatrix} V_k^a \\ V_k^b \\ V_k^c \end{bmatrix} \right)$$

3-phase analysis

$$I_{jk} = y_{jk} \left(V_j - V_k \right)$$

3x3 matrix

Admittance matrix



per-phase:

$$Y = \begin{bmatrix} y_{13} & 0 & -y_{13} \\ 0 & y_{23} & -y_{23} \\ -y_{13} & -y_{23} & y_{13}+y_{23} \end{bmatrix}$$

$$I = YV$$

N x N matrix

Admittance matrix (phase frame)

3-phase:

$$
Y = \begin{bmatrix}
\begin{bmatrix} y_{13}^{aa} & y_{13}^{ab} & y_{13}^{ac} \\ y_{13}^{ba} & y_{13}^{bb} & y_{13}^{bc} \\ y_{13}^{ca} & y_{13}^{cb} & y_{13}^{cc} \end{bmatrix} & 0 & -\begin{bmatrix} y_{13}^{aa} & y_{13}^{ab} & y_{13}^{ac} \\ y_{13}^{ba} & y_{13}^{bb} & y_{13}^{bc} \\ y_{13}^{ca} & y_{13}^{cb} & y_{13}^{cc} \end{bmatrix} \\
0 & [y_{23}] & -[y_{23}] \\
-[y_{13}] & -[y_{23}] & [y_{13}] + [y_{23}]
\end{bmatrix}
$$

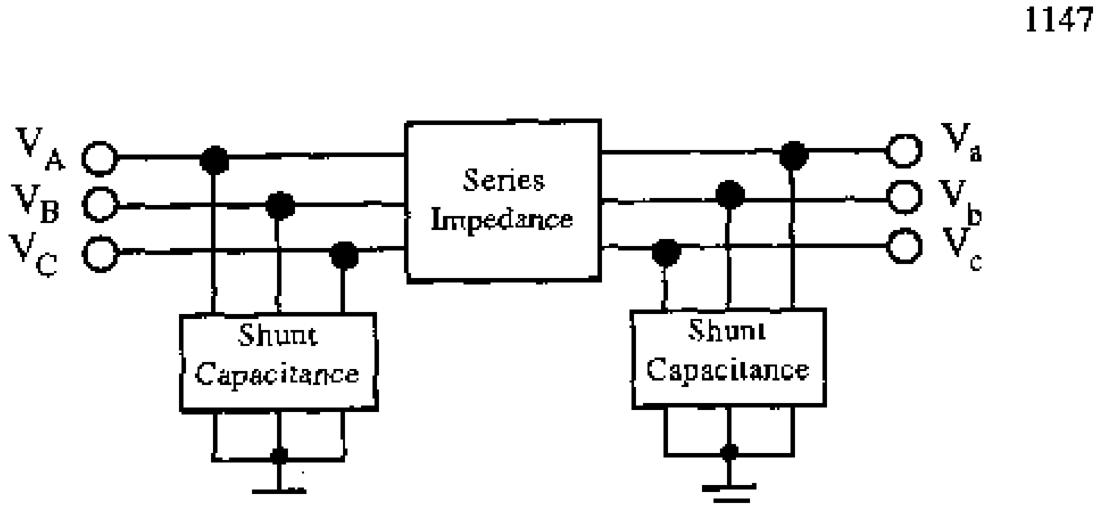
$$
I = YV
$$

$3N$ x $3N$ matrix

Single-phase equivalent

Single-phase equivalent is a chordal graph for radial networks !

- with a maximal clique for each line (j,k)



T. H. Chen et al (1991)

BIM: OPF and relaxations

OPF: reduced to single-phase case

- Each node is indexed by (bus, phase)

Standard SDP relaxation applies

- Dall'Anese, Zhu and Giannakis (TSG 2012)
- Distribute OPF into areas (maximal cliques) in chordal extension

Chordal relaxation applies

- Simpler for large sparse networks
- Gan and L (PSCC 2014)

BFM for radial: advantages

SOCP relaxation

- Much more scalable than SDP

Linearized model

- Baran and Wu (TPD 1989)
- More suitable for distribution systems
  - nonzero R, variable V, includes Q (unlike DC)
  - explicit solution given power injections

Much more stable numerically than BIM

ALL extend to multiphase unbalanced case !

Gan and Low, PSCC 2014

BFM for radial

scalar

Single phase

$$V_i - V_j = z_{ij} I_{ij}$$

$$S_{ij} = V_i I_{ij}^*$$

$$\sum_{j \to k} S_{jk} = \sum_{i \to j} \left(S_{ij} - z_{ij} \left| I_{ij} \right|^2 \right) + s_j$$

3x3 matrix

Multiphase

vector →

$$V_i - V_j = z_{ij} I_{ij}$$

$$S_{ij} = V_i I_{ij}^*$$

$$\sum_{j \to k} \mathrm{diag}\left(S_{jk} \right) = \sum_{i \to j} \mathrm{diag}\left(S_{ij} - z_{ij} I_{ij} I_{ij}^* \right) + s_j$$

SOCP relaxation: single phase

power flow solutions: $x := (S, \ell, v, s)$ satisfy

$$\sum_{k:j\to k} S_{jk} = \sum_{i:i\to j} \left(S_{ij} - z_{ij}\ell_{ij}\right) + s_j$$

$$v_i - v_j = 2\ \mathrm{Re}\left(z_{ij}^* S_{ij}\right) - \left|z_{ij}\right|^2 \ell_{ij}$$

linear

$$\ell_{ij} v_i = \left|S_{ij}\right|^2$$

nonconvexity

$$\ell_{ij} := \left|I_{ij}\right|^2$$

$$v_i := \left|V_i\right|^2$$

Baran and Wu 1989
for **radial** networks

SOCP relaxation: single phase

power flow solutions: $x := (S, \ell, v, s)$ satisfy

$$\sum_{k: j\to k} S_{jk} = \sum_{i: i\to j} \left(S_{ij} - z_{ij}\ell_{ij}\right) + s_j$$

$$v_i - v_j = 2\,\mathrm{Re}\left(z_{ij}^* S_{ij}\right) - |z_{ij}|^2 \ell_{ij}$$

linear

$$\ell_{ij} v_i \geq |S_{ij}|^2$$

second-order cone

$$\ell_{ij} := |I_{ij}|^2$$

$$v_i := |V_i|^2$$

Farivar et al 2011

SOCP relaxation: multiphase

Single phase

$$\sum_{k:j\to k} S_{jk} = \sum_{i:i\to j} \left(S_{ij} - z_{ij}\ell_{ij}\right) + s_j$$

$$v_i - v_j = \left(S_{ij}z_{ij}^* + z_{ij}S_{ij}^*\right) - |z_{ij}|^2 \ell_{ij}$$

Multiphase

$$\sum_{j\to k} \mathrm{diag}\left(S_{jk}\right) = \sum_{i\to j} \mathrm{diag}\left(S_{ij} - z_{ij}\ell_{ij}\right) + s_j$$

3x3 matrix → $$v_i - v_j = \left(S_{ij}z_{ij}^* + z_{ij}S_{ij}^*\right) - z_{ij}\ell_{ij}z_{ij}^*$$

SOCP relaxation: multiphase

Single phase	Multiphase
$\ell_{ij} v_i \geq \lvert S_{ij} \rvert^2$	$\begin{bmatrix} v_i & S_{ij} \\ S_{ij}^* & \ell_{ij} \end{bmatrix} \succeq 0$
exact: $\ell_{ij} v_i = \lvert S_{ij} \rvert^2$	$\text{rank} \begin{bmatrix} v_i & S_{ij} \\ S_{ij}^* & \ell_{ij} \end{bmatrix} = 1$

recovery:

$$\begin{bmatrix} v_i & S_{ij} \\ S_{ij}^H & \ell_{ij} \end{bmatrix} = \begin{bmatrix} V_i \\ I_{ij} \end{bmatrix} \begin{bmatrix} V_i^H & I_{ij}^H \end{bmatrix}$$

Equivalence: multiphase

Theorem

- BFM and BIM are equivalent
- Linear bijection between solution/feasible sets

Theorem

- Relaxation is exact for BFM iff it is for BIM

Gan and Low, PSCC 2014

Simulation results: multiphase

network	BIM-SDP			BFM-SDP		
	value	time	ratio	value	time	ratio
IEEE 13-bus	152.7	1.05	8.2e-9	152.7	0.74	2.8e-10
IEEE 34-bus	~~-100.0~~	~~2.22~~	~~1.0~~	279.0	1.64	3.3e-11
IEEE 37-bus	212.3	2.66	1.5e-8	212.2	1.95	1.3e-10
IEEE 123-bus	~~-8917~~	~~7.21~~	~~3.2e-2~~	229.8	8.86	0.6e-11
Rossi 2065-bus	~~-100.0~~	~~115.50~~	~~1.0~~	19.15	96.98	4.3e-8

numerically unstable

numerically stable

BFM is much more numerically stable

Linear approximation in BFM

Single phase

- Simple DistFlow equations
- Baran and Wu (1989)

Multiphase

- Extension to multiphase unbalanced networks
- Closed-form solution given power injections

Gan and Low, PSCC 2014

Summary

Bus injection model

- OPF formulation
- 3 convex relaxations & relationship

Branch flow model

- OPF formulation
- SOCP relaxation & equivalence

Exact relaxation

- Radial networks
- Mesh networks

Multiphase unbalanced networks