The Bauer-Furuta Invariant And A Cohomotopy Refined Ruberman Invariant

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Abstract of the Dissertation

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Applying Bauer and Furuta's method to the 1-parameter Seiberg–Witten equation, a cohomotopy refinement of the Ruberman invariant can be defined. Using the refined Ruberman invariant, and the nontriviality of the Bauer-Furuta invariant, we will prove, there exist examples of 4-manifolds such that the set of unit-volume Riemannian metrics with scalar curvature bounded below by a fixed negative number is disconnected.

Contents

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Chapter 1

Introduction

Assume X is a closed smooth oriented 4-manifold. Let Γ_X be a *Spin*^{c} structure, and g a Riemannian metric on X. In [26], Witten defined the *Seiberg–Witten equation* for the *Spin*^{c} connection A and section ϕ of the positive spinor bundle,

$$\begin{aligned} D_A\phi &= 0, \\ F_A^+ &= \sigma(\phi) + i\eta, \end{aligned}$$

where $D_A : \Gamma(W^+) \to \Gamma(W^-)$ is the Dirac operator, F_A^+ is the self-dual part of the curvature F_A, η is a real self-dual 2-form, and $\sigma(\phi)$ is the trace-free endomorphism of W^+ defined by

$$\sigma(\phi) = \phi \otimes \phi^* - \frac{|\phi|^2}{2}\phi,$$

which will be identified as an imaginary-valued self-dual 2-form. The solution space of the Seiberg–Witten equation is equivariant under the action of the gauge group $\mathcal{M}ap(X, S^1)$. The *Seiberg–Witten moduli space* $\mathcal{M}(\Gamma; g, \eta)$ is the quotient space of the solution space by the gauge group action. Usually it is defined with suitable Soblev completions, but a standard argument shows the moduli space defined is not affected by the Soblev completions we choose. More importantly, $\mathcal{M}(\Gamma; g, \eta)$ is compact. If we choose a generic η and assume $b_+(X) > 1$, then it is either empty or a smooth compact orientable manifold with dimension

$$d(\Gamma_X) = \frac{c_1^2(\Gamma_X) - (2\chi(X) + 3\tau(X))}{4},$$

where $\chi(X)$ is the Euler number and $\tau(X)$ is the signature. The orientation of $\mathcal{M}(\Gamma_X; g, \eta)$ can be determined by a *homology orientation* of X, i.e. an orientation of the space $H^1(X;\mathbb{R}) \oplus H^+(X;\mathbb{R})$. When $\mathcal{M}(\Gamma_X; g, \eta)$ is 0-dimensional, the *Seiberg–Witten invariant* $SW(X, \Gamma_X)$ is the algebraic count of the points in $\mathcal{M}(\Gamma_X; g, \eta)$. When $\mathcal{M}(\Gamma_X; g, \eta)$ has a positive even dimension, $SW(X, \Gamma_X)$ is defined as the evaluation of the $d(\Gamma_X)/2$-th power of the first Chern class of the S^1-bundle

$$\widetilde{\mathcal{M}}(\Gamma_X; g, \eta) \to \mathcal{M}(\Gamma_X; g, \eta)$$

on the moduli space, where $\widetilde{\mathcal{M}}(\Gamma_X; g, \eta)$ is the quotient of the solution space of the Seiberg–Witten equation by the pointed gauge group $\mathcal{M}ap_0(X, S^1)$. If the moduli space is empty or it has an odd dimension, we just define $SW(X, \Gamma_X) = 0$. With the assumption $b_+(X) > 1$, the Seiberg–Witten invariant does not depend on the metric g or the self-dual 2-form η. (See [15], [17] or [22] for more details).

The Seiberg–Witten equation and the Seiberg–Witten invariant are widely used in the study of 4-manifolds' topology and geometry. One application is in the study of the scalar curvature of 4-manifolds. A direct argument using the Weitzenböck formula shows that no 4-manifold with $b_+ > 1$ and nontrivial

Seiberg–Witten invariant (for example Kähler manifold [4] or symplectic manifold [25]) has a positive scalar curvature metric. A more delicate discussion will give an L^2-estimate for the scalar curvature, which was shown to be exact in some interesting cases and used to study the Yamabe invariant by LeBrun in [10] and [12]. It can also be used to study other geometric invariants and to obtain an obstruction to the existence of the Einstein metrics [11].

The Seiberg–Witten equation is also used to study $\mathbf{PSC}(X)$, the space of positive scalar curvature metrics for the 4-manifold X. Ruberman proved

Theorem 1 *[20] There exist 4-manifolds of the form*

$$X = a\mathbb{CP}^2 \# b\overline{\mathbb{CP}}^2$$

such that $\mathbf{PSC}(X)$ *is disconnected.*

In [20] he studied the Seiberg–Witten equations for a path of metric g_t, $t \in [0,1]$. Then with a path of generic perturbations η_t, the 1-parameter moduli space, which is the union of the moduli spaces for each metric g_t, is a compact oriented smooth manifold with the virtual dimension

$$d' = \frac{c_1^2(\Gamma_X) - (2\chi(X) + 3\tau(X))}{4} + 1.$$

It will be closed if we assume the Seiberg–Witten moduli spaces for (g_t, η_t), $t = 0$ and 1, are empty. This 1-parameter moduli space gives some geometric information about the path of metrics we choose. Similar to the discussion for the Seiberg–Witten equation with a fixed metric, it gives an obstruction for the path of metrics to stay in $\mathbf{PSC}(X)$.

In [20] Ruberman only considered the case that the virtual dimension of the 1-parameter moduli space $d' = 0$. He defined the *Ruberman invariant* to be the algebraic count of the points in a generic 1-parameter moduli space. Similar to the Seiberg–Witten invariant, Ruberman invariant can measure the non-emptiness of the 1-parameter moduli space, but the definition has the restriction $d' = 0$. The natural question is how we can define a similar invariant with more generality.

A possible answer is given in this work. We will define a *cohomotopy refined* Ruberman invariant. It is the S^1-equivariant stable cohomotopy class of a 1-parameter family of Seiberg–Witten maps. We shall use the *finite dimensional approximation* by Bauer and Furuta, which was first used in Furuta's proof of his 10/8-theorem for *Spin* manifolds [5] and later generalized by Bauer and Furuta to non-*Spin* manifolds to define the cohomotopy refinement of Seiberg–Witten invariant [3], the *Bauer-Furuta invariant.* In the case the virtual dimension $d' = 0$, the cohomotopy refined Ruberman invariant can be reduced to the Ruberman invariant by a natural homomorphism. When $d' > 0$, the cohomotopy refined Ruberman invariant is also nontrivially well defined. We will see that sometimes the cohomotopy refined Ruberman invariant is a nonzero torsion class, which means that the cohomotopy refined Ruberman invariant has more generality than an integer invariant. This explains why we do not just make an analog of the definition of the Seiberg–Witten invariant in the general case to define our generalized Ruberman invariant, though this approach would also give a possible answer to our question.

As an application of the cohomotopy refined Ruberman invariant, we will consider the space of Riemannian metrics on the manifold

$$X = T^4 \# a\mathbb{CP}^2 \# b\overline{\mathbb{CP}}^2$$

and define a subset of metrics similar to $\mathbf{PSC}(X)$. From the work of Schoen and Yau [24], $\mathbf{PSC}(X)$ is empty. (There are even no zero scalar curvature metrics on X.) But as the Yamabe invariant of X is 0, for each $\varepsilon > 0$, there are unit-volume metrics with scalar curvature bounded below by $-\varepsilon$. So, if we denote the space of Riemannian metrics by $Met(X)$ and define the subset

$$Met_\varepsilon(X) \stackrel{def}{=} \{g \in Met(X) | s_g > -\varepsilon, vol_g(X) = 1\},$$

then $Met_\varepsilon(T^4 \# a\mathbb{CP}^2 \# b\overline{\mathbb{CP}}^2)$ is not empty. We will prove the following main result.

Theorem 2 *For any $\varepsilon > 0$, there exist 4-manifolds of the form*

$$X = T^4 \# a\mathbb{CP}^2 \# b\overline{\mathbb{CP}}^2$$

such that $Met_\varepsilon(X)$ is disconnected.

Theorem 2 is an analog of Theorem 1. In Ruberman's proof [20] of Theorem 1, there are three basic observations: the nontriviality of Seiberg–Witten invariant, the property of wall-crossing, and the gluing technique which gives a product formula for the 1-parameter moduli space. To prove Theorem 2, we need also to consider these three, but with the integer invariant replaced by its cohomotopy refined one. Thus we will consider the nontriviality of the Bauer–

Furuta invariant, the description of wall-crossing by a stable cohomotopy class, and the gluing technique which gives a product formula for the cohomotopy refined Ruberman invariant. The second of the three is quite obvious from the definition, while the other two deserve more careful discussions. Because the 4-manifold we consider has positive b_1, the Bauer–Furuta invariant is not as well understood as in the case $b_1 = 0$. Although in general we cannot establish the nontriviality of the Bauer–Furuta invariant when $b_1 > 0$, we can for some manifolds of the form $X = T^4 \# X'$, which is enough for proving Theorem 2.

Theorem 3 *The Bauer–Furuta invariant of $X = T^4 \# X'$ is non-trivial when $X' = T^4$ or X' is an almost complex 4-manifold such that $b_1(X') = 0$, $b_+(X') \equiv 3$ (mod 4) and the Seiberg–Witten invariant for the almost complex structure on X' is odd.*

For the third observation, we can use the gluing technique in [1] to prove a 1-parameter version of Bauer's product formula, which at the same time generalizes the one Ruberman proved in [20]. This product formula is the key to complete the proof of Theorem 2.

Besides its usage in the proof of Theorem 2, Theorem 3 can be applied to calculate the Yamabe invariant of some manifolds of the form $X = T^4 \# X'$ and so to give an alternative proof for the non-existence of positive scalar curvature metrics on those manifolds.

It needs to be pointed out that our cohomotopy refined Ruberman invariant, related to a self-diffeomorphism f of the 4-manifold, has not been defined in full generality. We have only discussed the case that f generates an infinite orbit for Γ_X, which is relative to our application. Defining a nontrivial cohomotopy refined Ruberman invariant related to other f and finding applications for it are fields which still need to be explored.

In Chapter 2, we will introduce the Bauer–Furuta invariant and two most important properties of it from [1] and [3]. From Chapter 3 to 5, we will study the nontriviality of the Bauer–Furuta invariant for some 4-manifolds with $b_1 > 0$. In Chapter 6, we will see the application of the nontriviality results we have proved. In Chapter 7, we will introduce the construction of Ruberman invariant and in Chapter 8 and 9, we will define a cohomotopy refined Ruberman invariant using Bauer and Furuta's construction. In Chapter 10 and 11 we state and prove the product formula for the cohomotopy refined Ruberman invariant. Finally in Chapter 12, we will finish the proof of Theorem 2.

Chapter 2

The Bauer–Furuta invariant

If not explicitly mentioned to the contrary, the 4-manifolds we will consider are closed smooth oriented and homology oriented (i.e. with a homology orientation).

For simplicity, we will first assume $b_1 = 0$. Let Γ_X be a $Spin^c$ structure on the 4-manifold X. Let $\widetilde{\mathcal{M}}(\Gamma; g, \eta)$ be the S^1-bundle on $\mathcal{M}(\Gamma; g, \eta)$ which is the quotient of the solution space of the Seiberg–Witten equation by the pointed gauge group. Fix a smooth $Spin^c$ connection A_0 and a Riemannian metric on X, then $\widetilde{\mathcal{M}}(\Gamma; g, \eta)$ can be identified with the preimage $sw^{-1}(0)$ for the following *Seiberg–Witten map*,

$$sw: \ \Gamma(W^+) \oplus i\Omega^1(X) \to \Gamma(W^-) \oplus i\Omega^0(X)/\mathbb{R} \oplus i\Omega^+(X),$$
$$(\phi, a) \mapsto (D_{A_0}\phi + a\phi, d^*a, d^+a + F^+_{A_0} - \sigma(\phi) - i\eta).$$

Fix $k > 3$. We will use the above notation to denote the induced map from the L^2_{k-1}-completion of $\Gamma(W^+) \oplus i\Omega^1(X)$ to the L^2_{k-2}-completion of

$$\Gamma(W^-) \oplus i\Omega^0(X)/\mathbb{R} \oplus i\Omega^+(X).$$

Let $l = (D_{A_0}, d^*, d^+)$ and $c = (a\phi, 0, F^+_{A_0} - \sigma(\phi) - i\eta)$. Then $sw = l + c$ is a decomposition of sw as the sum of a linear Fredholm operator and a compact operator between two Hilbert spaces. The compactness of the Seiberg–Witten moduli space implies the boundedness of the preimage $sw^{-1}(0)$. In fact a similar argument can prove the stronger *boundedness property* of sw, i.e. the preimage of any bounded set is bounded [3]. Then a stable cohomotopy class in $\pi^{st}_{\mathrm{ind}\, l}(S^0)$ can be defined for sw. For a suitable finite dimensional approximation c' for c, the map $l + c'$ induces maps between finite dimensional spheres. Changing the approximation c' or the dimensions of the spheres will only result a homotopic difference by suspensions, which means the stable cohomotopy class is well defined (see [3] or [23]).

To be precise, Bauer and Furuta introduced an equivalent way to define the stable cohomotopy class. They proved:

Lemma 4 *[3] Let $f = l + c : H' \to H$ be a bounded map between two Hilbert spaces and l is linear Fredholm and c is compact. Then there is a finite dimensional subspace $V \subset H$, so that the following holds:*

1. Together with the image $\mathrm{Im}\, l$ of the linear Fredholm map l, the subspace V spans the Hilbert space $H = \mathrm{Im}\, l + V$.

2. For any $W \supset V$ with $W = U \perp V$, $W' = l^{-1}(W)$ and $V' = l^{-1}(V)$, the restriction $f|_{W'^+} : W'^+ \to H^+$ of f to the one point compactification of W' misses the unit sphere $S(W^\perp)$ in the orthogonal complement of W.

3. Let $\rho_W : H^+ \backslash S(W^\perp) \to W^+$ be a naturally defined homotopy equivalence. The maps $\rho_W f|_{W'^+}$ and $id_{U^+} \wedge \rho_V f|_{V'^+}$ are homotopy equivalent as *pointed maps*

$$W'^{+} \cong U^{+} \wedge V' \longrightarrow U^{+} \wedge V^{+} = W^{+}.$$

Indeed, if H is separable, then the subspaces V satisfying these three conditions are cofinal in the direct system of finite dimensional subspaces in H.

In particular, the restrictions $f|_{l^{-1}(V)^{+}}$ with V satisfying the conditions in lemma 4 define an element in the colimit of pointed homotopy groups

$$\begin{aligned} [f] &= \overleftarrow{\lim_{V\subset H}}[f|_{l^{-1}(V)^{+}}] \in \overleftarrow{\lim_{V\subset H}}[l^{-1}(V)^{+}, H^{+}\backslash S(V^{\perp})] \\ &\cong \overleftarrow{\lim_{V\subset H}}[l^{-1}(V)^{+}, V^{+}] = \pi^{st}_{ind\, l}(S^{0}). \end{aligned}$$

In the case of the Seiberg–Witten map, the above invariant $[sw] \in \pi^{st}_{\mathrm{ind}\, l}(S^{0})$ can be further refined because the Seiberg–Witten map is S^1 -equivariant with respect to the trivial action of S^1 on forms and multiplication action on sections of spinors. The S^1-equivariant stable homotopy class for the Seiberg–Witten map will contain more precise information. Technically, all the key steps are still correct with the group action involved. So we can make the following definition.

Definition 5 *[3] If $b_1(X) = 0$, the S^1-equivariant map sw represents a stable S^1-cohomotopy class*

$$[sw] = \overleftarrow{\lim_{V\subset H}}[sw|_{l^{-1}(V)^{+}}]_{S^1} \in \overleftarrow{\lim_{V\subset H}}[l^{-1}(V)^{+}, V^{+}]_{S^1} = \pi^{b_+}_{S^1}(pt; \mathbb{C}^{d}),$$

where $H = L^2_{k-2}(\Gamma(W^{-}) \oplus i\Omega^{0}(X)/\mathbb{R} \oplus i\Omega^{+}(X))$ with natural S^1-action, V is any S^1-subspace of H, satisfying all the conditions in Lemma 4 equiv-

ariantly and d is the Atiyah-Singer index for the $Spin^c$ structure Γ_X. The S^1-equivariant stable cohomotopy class $[sw]$ will be called the Bauer–Furuta invariant and denoted by $BF(X, \Gamma_X)$.

The Bauer–Furuta invariant can also be defined in the case $b_1(X) > 0$. The Seiberg–Witten map can first be defined at the level

$$\begin{aligned} &\widetilde{sw} : \widetilde{\mathcal{A}} \longrightarrow \widetilde{\mathcal{C}}, \\ &(A, \phi, a) \longmapsto (A, D_A\phi + a\phi, d^*a, pr_{harm}a, d^+a + F_A^+ - \sigma(\phi) - i\eta), \end{aligned}$$

where

$$\begin{aligned} \widetilde{\mathcal{A}} &= (A_0 + i\ker d) \times L^2_{k-1}(\Gamma(W^+) \oplus i\Omega^1(X)), \\ \widetilde{\mathcal{C}} &= (A_0 + i\ker d) \times L^2_{k-2}(\Gamma(W^-) \oplus i\Omega^0(X)/\mathbb{R} \oplus iH^1(X;\mathbb{R}) \oplus i\Omega^+(X)). \end{aligned}$$

The map $\widetilde{sw}$ is equivariant under the action of $\mathcal{G} = L^2_k(\mathcal{M}ap(X, S^1))$ which acts naturally on connections, trivially on forms and by multiplication on the sections of spinors. Let $\mathcal{G}_0 \subset \mathcal{G}$ be the pointed gauge group. Then the Seiberg–Witten map

$$sw = \widetilde{sw}/\mathcal{G}_0 : \mathcal{A} = \widetilde{\mathcal{A}}/\mathcal{G}_0 \rightarrow \mathcal{C} = \widetilde{\mathcal{C}}/\mathcal{G}_0$$

is $S^1 = \mathcal{G}/\mathcal{G}_0$ equivariant, and $\mathcal{A}$ and $\mathcal{C}$ are Hilbert bundles over the torus $Pic^0(X) = (A_0 + i\ker d)/\mathcal{G}_0$. The Seiberg–Witten map can be decomposed as the sum of fiber-preserving S^1-maps $l + c$, where l is linear Fredholm and c is compact. As before, it satisfies the strong boundedness condition. Take a trivialization of the Hilbert bundle $\mathcal{C} = Pic^0(X) \times H$ and denote the projection to the fiber by pr . From the parametrized version of Lemma 4 (Lemma 2.5

of [3]), we can see the Seiberg–Witten map defines a stable cohomotopy class represented by

$$pr \circ sw|_{l^{-1}(V)^+} : TV' = l^{-1}(V)^+ \to H^+ \backslash S(V^\perp) \simeq V^+,$$

where V is any finite dimensional subspace of H satisfying the conditions similar to those listed in Lemma 4. Thus $V' = l^{-1}(V)$ is a finite dimensional subbundle of $\mathcal{C}$ over $Pic^0(X)$, the one-point compactification of which can be identified with its Thom space TV'. The stable cohomotopy class defined does not depend on the choice of l, but it depends on the K-group element, $\operatorname{ind} l = l^{-1}(V) - V = \operatorname{ind} D - \mathbb{R}^{b_+}$ on $Pic^0(X)$. Also we need to consider the S^1-equivariance of the Seiberg–Witten map, so we have:

Definition 6 *[3] The S^1-equivariant Seiberg–Witten map sw represents an S^1-equivariant stable cohomotopy class*

$$[sw] = \varprojlim_{V \subset H} [sw|_{l^{-1}(V)^+}]_{S^1} \in \varprojlim_{V \subset H} [TV', V^+]_{S^1} = \pi^{b_+}_{S^1}(Pic^0(X); \operatorname{ind} D),$$

where $\operatorname{ind} D$ *is the virtual index bundle for the Dirac operators parametrized on $Pic^0(X)$, $H = L^2_{k-2}(\Gamma(W^-) \oplus i\Omega^0(X)/\mathbb{R} \oplus iH^1(X) \oplus i\Omega^+(X))$ with the natural S^1-action and V is any finite dimensional S^1-subspace which satisfies Lemma 2.5 of [3]. We will call $[sw]$ the Bauer–Furuta invariant and denote it by $BF(X, \Gamma_X)$.*

Detailed discussions are contained in [1] and [3]. Here only two properties will be mentioned. The first indicates the relation between the Bauer–Furuta invariant and the Seiberg–Witten invariant.

Proposition 7 *[3] Let X be a closed 4-manifold with $b_+(X) > b_1(X)+1$. The choice of a homology orientation (i.e. an orientation of $H^1(X;\mathbb{R})\oplus H^+(X;\mathbb{R})$) then determines a homomorphism $t : \pi_{S^1}^{b_+}(Pic^0(X); \operatorname{ind} D) \to \mathbb{Z}$, which maps the Bauer–Furuta invariant to the Seiberg–Witten invariant.*

The second concerns the Bauer–Furuta invariant for manifolds of the form $X_1 \# X_2$. Bauer proved the following elegant product formula.

Theorem 8 *[1] For the connected sum $X = X_1 \# X_2$ of 4-manifolds, with Spin^c structure $\Gamma_X = \Gamma_{X_1} \# \Gamma_{X_2}$ induced by Γ_{X_1} and Γ_{X_2} on X_1 and X_2 respectively, then*

$$BF(X, \Gamma_X) = BF(X_1, \Gamma_{X_1}) \wedge BF(X_2, \Gamma_{X_2}).$$

Equivalently, if we have chosen S^1-maps $\mu_1 \in BF(X_1, \Gamma_{X_1})$ and $\mu_2 \in BF(X_2, \Gamma_{X_2})$, then $\mu_1 \times \mu_2$ with S^1 acting diagonally is in the class $BF(X, \Gamma_X)$.

Chapter 3

The nontriviality of the Bauer–Furuta invariant when $b_1 > 0$

The nontriviality of the Bauer–Furuta invariant implies the existence of solutions of Seiberg–Witten equations. So it is of great interest to us. When $b_1 = 0$, the Bauer–Furuta invariant can be identified with a non-equivariant stable cohomotopy class. In the more general case $b_+ > b_1 + 1$, Proposition 7 gives a possible way to determine the nontriviality of the Bauer–Furuta invariant as long as the corresponding Seiberg–Witten invariant is not zero. So two questions can be asked.

1. If $b_+ \leq b_1 + 1$, can we still get the nontriviality of the Bauer–Furuta invariant from that of the Seiberg–Witten invariant?

2. If $b_1 > 0$ and the Seiberg–Witten invariant is zero, can we still get a nontrivial Bauer–Furuta invariant?

We will give positive answers to these two questions for some 4-manifolds.

Denote the $Spin^c$ structure on X by Γ_X. Let V be any finite dimensional S^1-subspace satisfying the conditions in Definition 6 and TV' the Thom space for the bundle $V' = l^{-1}(V)$. We will denote by $\{,\}_{S^1}$ the equivariant stable homotopy group of S^1-maps. Then there is an exact sequence of equivariant stable cohomotopy groups [3]

$$\{\Sigma TV'^{S^1}, V^+\}_{S^1} \to \{(TV', TV'^{S^1}), (V^+, \emptyset^+)\}_{S^1}$$
$$\to \{TV', V^+\}_{S^1} \xrightarrow{res} \{TV'^{S^1}, V^+\}_{S^1},$$

in which TV'^{S^1} is the S^1-fixed subset. The Bauer–Furuta invariant $BF(X, \Gamma_X)$ is in $\pi^{b_+}_{S^1}(Pic^0(X); \operatorname{ind} D) = \{TV', V^+\}_{S^1}$. We will assume $b_+(X) > 0$. The restriction of the Seiberg–Witten map from the S^1-fixed subset TV'^{S^1} to the S^1-fixed subset V^{+S^1} is the following map:

$$(A_0 + i\ker d)/\mathcal{G}_0 \times L^2_{k-1}(i\Omega^1(X)) \to L^2_{k-2}(i\Omega^0(X)/\mathbb{R} \oplus iH^1(X;\mathbb{R}) \oplus i\Omega^+(X)),$$

$$(A, a) \mapsto (d^*a, pr_{harm}a, d^+a - i\eta),$$

which is not onto when $b_+(X) > 0$. So $res([BF(X, \Gamma_X)])$ will be zero in $\{TV'^{S^1}, V^+\}_{S^1}$. Thus $BF(X, \Gamma_X)$ is an element of

$$\overline{\pi}^{b_+}_{S^1}(Pic^0(X); \operatorname{ind} D) = \ker(\{TV', V^+\}_{S^1} \xrightarrow{res} \{TV'^{S^1}, V^+\}_{S^1}).$$

Clearly $BF(X, \Gamma_X)$ is nontrivial iff it is not zero in $\overline{\pi}^{b_+}_{S^1}(Pic^0(X); \operatorname{ind} D)$. We have an analog of Proposition 7.

Theorem 9 *Assume that X is a Spin 4-manifold with $b_+ > 1$, $b_1(X) - b_+(X) \le 2$ and $\tau(X) = 0$. Let Γ_X be a Spin structure on X. Then there* *is a homomorphism*

$$t : \overline{\pi}^{b_+}_{S^1}(Pic^0(X); \operatorname{ind} D) \longrightarrow \mathbb{Z},$$

such that the image of the Bauer–Furuta invariant $BF(X, \Gamma_X)$ *is the Seiberg–Witten invariant* $SW(X, \Gamma_X)$.

Proof. If $b_+(X) > b_1(X) + 1$, then it is correct by Proposition 7. So we can assume $0 \leq b_1 + 1 - b_+ \leq 3$. The virtual dimension $d(\Gamma_X)$ of the Seiberg–Witten moduli space for the *Spin* structure will satisfy

$$-2 \leq d(\Gamma_X) = b_1(X) - b_+(X) - 1 \leq 1.$$

If $d(\Gamma_X) \neq 0$, the Seiberg–Witten invariant is always zero. Then we can just choose t to be the zero homomorphism. So we need only to define the homomorphism t when $d(\Gamma_X) = 0$. By the exact sequence, all the elements in $\overline{\pi}^{b_+}_{S^1}(Pic^0; \operatorname{ind} D)$ can be lifted to the relative homotopy group

$$\{(TV', TV'^{S^1}), (V^+, \emptyset^+)\}_{S^1}.$$

There is a homomorphism

$$t : \{(TV', TV'^{S^1}), (V^+, \emptyset^+)\}_{S^1} \longrightarrow \mathbb{Z}$$

constructed in [3] for proving Proposition 7. When $d(\Gamma_X) = b_1(X) - b_+(X) - 1 = 0$, the homomorphism t maps each S^1-equivariant relative stable cohomotopy class to the algebraic count of the S^1-orbits in a generic preimage. We only need to show that this homomorphism t descends to $\pi_{S^1}^{b_+}(Pic^0(X); \operatorname{ind} D)$, i.e. different liftings of the Bauer-Furuta invariant $BF(X,\Gamma_X)$ have the same image under t. From [3], the lifting of $BF(X,\Gamma_X)$ given by a generic perturbation of the Seiberg-Witten equation is mapped to $SW(X,\Gamma_X)$ by t. Then the induced homomorphism t on $\pi_{S^1}^{b_+}(Pic^0(X); \operatorname{ind} D)$ will map $BF(X,\Gamma_X)$ to $SW(X,\Gamma_X)$. (In the proof of Proposition 7 in [3], this is trivial because the assumption $b_+(X) - b_1(X) - 1 > 0$ implies the uniqueness of the lifting. Though it is not necessary for the proof, the discussion below also shows the uniqueness of the lifting when Γ_X is a *Spin* structure, $\tau(X) = 0$ and $b_1 - b_+ \leq 2$.)

Let f_1 and f_2 be two maps representing two liftings of the same element in $\{TV', V^+\}_{S^1}$. Then we can choose a suitable V so that there is a S^1-homotopy $h : TV' \times I \longrightarrow V^+$ from f_1 to f_2. The homotopy h satisfies the property that for $t = 0$ or 1, $h(TV'^{S^1}, t) = \emptyset^+$, and for $t \in I$ and $x_0 = \emptyset^+ \in TV'$ (the base point of TV' at infinity), $h(x_0, t) = \emptyset^+$. It can be S^1-equivariantly perturbed so that $h|_{TV'^{S^1}}$ is smooth outside the preimage of the base point $h^{-1}(\emptyset^+)$. For example we can assume 0 is a regular value for $h|_{TV'^{S^1}}$, then $(h|_{TV'^{S^1}})^{-1}(0)$ is disconnected unions of oriented compact smooth manifolds with dimension $b_1(X) + 1 - b_+(X) \leq 3$. Denote a small tubular neighborhood of $(h|_{TV'^{S^1}})^{-1}(0)$ by U'. Let V be the direct sum of $V^{\mathbb{R}}$ and $V^{\mathbb{C}}$, the real and complex maximal subspaces of V (with trivial S^1-action and scalar multiplication action respectively). The bundle V' is also a complex bundle on V'^{S^1}. The virtual bundle on V'^{S^1} defined by $V' - V^{\mathbb{C}}$ is the lifting of the index bundle $\operatorname{ind} D$ from $Pic^0(X)$ to V'^{S^1}. The first chern class $c_1(\operatorname{ind} D)$ is 0 because Γ_X is a *Spin* structure. The same is true for the first chern class of the bundle V' on V'^{S^1} and its restriction to U'. Because U' can be contracted to an oriented manifold with dimension at most 3, the restriction of the complex bundle V' on V'^{S^1} to U' is trivial. So by the Atiyah–Singer index theorem and the fact $\tau(X) = 0$ there is a neighborhood U'' of $(h|_{TV'^{S^1}})^{-1}(0)$ in V' which is S^1-diffeomorphic to $U' \times \mathbb{C}^N$, where N is the complex dimension of $V^{\mathbf{C}}$. Define the S^1-map $h' : U' \times \mathbb{C}^N \to V = V^{\mathbf{R}} \oplus V^{\mathbb{C}} = V^{S^1} \oplus \mathbb{C}^N$, $h'(x, a) = (h(x), a)$ for $x \in U'$ and $a \in \mathbb{C}^N$. Therefore h' is also an S^1-equivariant map from U'' to V. Now choose an S^1-equivariant partition of $1 = \alpha + \beta$ on TV' such that the support of β contains a neighboorhood of $(h|_{TV'^{S^1}})^{-1}(0)$ and is contained in U''. Then $\alpha h + \beta h'$ is a S^1-homotopy between f_1 and f_2 such that

$$(\alpha h + \beta h')^{-1}(0) \cap TV'^{S^1} = (h|_{TV'^{S^1}})^{-1}(0)$$

is isolated from other componnents of $(\alpha h + \beta h')^{-1}(0)$. Therefore after further perturbation, $f_1^{-1}(0)$ and $f_2^{-1}(0)$ can be connected by a cobordism with free S^1-action. The homomorphism t defined by Bauer and Furuta is in fact an S^1-cobordism invariant. So it does not depend on the choice of the lifting. (In fact we can change h' such that

$$h'(x, a) = (h(x), \frac{a}{|a|^2})$$

for $x \in U'$ and $a \in \mathbb{C}^N$. Then $(\alpha h + \beta h')^{-1}(0) \cap TV'^{S^1}$ is empty and $\alpha h + \beta h'$ can be deformed to a homotopy from f_1 to f_2 in $(TV', TV'^{S^1}), (V^+, \emptyset^+)_{S^1}$. Thus we have proved the uniqueness of the lifting.) ∎

The following corollary gives a positive answer for the first question, with $X = T^4$.

Corollary 10 *The Bauer–Furuta invariant for T^4 with any Spin structure is nontrivial.*

Proof. Let Γ_{T^4} be a *Spin* structure on T^4. Then $SW(T^4, \Gamma_{T^4}) \equiv 1 \pmod 2$ by Ruberman and Strle' work [21]. The torus T^4 satisfies all the conditions in theorem 9. Since the image $t(BF(T^4, \Gamma_{T^4})) \neq 0$, $BF(T^4, \Gamma_{T^4})$ is nontrivial. ■

To answer the second question, we will consider the manifold $X = T^4 \# X'$, where X' is an almost complex manifold with $b_1(X') = 0$ and $b_+(X') \equiv 3$ (mod 4), and the Seiberg–Witten invariant for the almost complex structure is odd. Then from [3] and Corollary 10, each component in the connected sum has a nontrivial Bauer–Furuta invariant. The product formula in [1] (i.e. Theorem 8) implies the Bauer–Furuta invariant of X maybe nontrivial. We can also take X' with $b_1(X') > 0$, for example $X' = T^4$. Here we will list only the results, which is restatement of Theorem 3, and we postpone the proofs to the next two chapters.

Theorem 11 $BF(T^4 \# T^4, \Gamma) \neq 0$ *for any Spin structure on* $T^4 \# T^4$.

Theorem 12 *Let $\Gamma_{X'}$ be a Spin^c structure induced by a almost complex structure on 4-manifold X'. Assume $b_1(X') = 0$, $b_+(X') \equiv 3$ (mod 4), and $SW(X', \Gamma_{X'})$ is odd. Let Γ_{T^4} be a Spin structure on T^4. Then*

$$BF(T^4 \# X', \Gamma_{T^4} \# \Gamma_{X'}) \neq 0.$$

By [4], X' can be any complex surface with $b_1(X') = 0$ and $b_+(X') \equiv 3$ (mod 4). Another special case is,

Corollary 13 *Let X' be a homotopy K3, then $BF(T^4 \# X', \Gamma) \neq 0$ for any Spin structure Γ on $T^4 \# X'$.*

Proof. By Theorem 12 and [16]. ∎

For the connected sums in Theorem 11 and Theorem 12, it is a fundamental fact from [17] or [22] that their Seiberg–Witten invariants vanish. So those two theorems answered the second question.

Remark 14 *Bauer also generalized Proposition 7 to the case $b_+ - b_1 - 1 \leq 0$ [2]. His method involved a modification of the definition of the Bauer–Furuta invariant.*

Chapter 4

The δ-invariant and the nontriviality of the Bauer–Furuta invariant

No matter whether $X' = T^4$ or satisfies the conditions of Theorem 12, the corresponding Seiberg–Witten moduli spaces for X in those two theorems are 1-dimensional. The homomorphism t in theorem 9 is not useful for detecting the nontriviality of the Bauer–Furuta invariant. But it is known by [1] that the Bauer–Furuta invariant of $K3\#K3$, a simply connected analog, is a nontrivial torsion class in $\mathbb{Z}_2$. In [6], this $\mathbb{Z}_2$ torsion class has been described explicitly as the δ-invariant. The δ-invariant is convenient in the sense that though it is defined for S^1-maps between spheres, it can be generalized to S^1-maps from a Thom space to a sphere. In other words, we want to define the δ-invariant as a homomorphism

$$\delta : \pi^{b_+}_{S^1}(Pic^0(T^4\#X'); indD) \longrightarrow \mathbb{Z}_2$$

which will play the role of t. This turns out to be not quite accurate because unlike the case of $K3\#K3$ the first homology group of $X = T^4\#X'$ has an effect. We will define the homomorphism as

$$\overline{\pi}^{b_+}_{S^1}(Pic^0(X); \operatorname{ind} D) \longrightarrow \mathbb{Z}_2 \oplus H_{b_1}(Pic^0(X); \mathbb{Z}) = \mathbb{Z}_2 \oplus \mathbb{Z}^{b_1},$$

so that restricted to the subgroup which is mapped to 0 in the factor of $\mathbb{Z}^{b_1}$, it coincides with δ.

We first assume X is any *Spin* manifold and Γ_X is a *Spin* structure, with $\tau(X) = 0$, $b_+(X) > 1$, and $b_1 - b_+ = 2$. By the proof of Theorem 9, we need only to define the homomorphism for the relative stable S^1-cohomotopy group $\{(TV', TV'^{S^1}), (V^+, \emptyset^+)\}_{S^1}$ with V and V' defined in definition 6. Take any class in it with a representative $f : (TV', TV'^{S^1}) \to (V^+, \emptyset^+)$. Then it can be generically perturbed so that $f^{-1}(0)$ is an oriented smooth 2-dimensional submanifold in $V' \backslash V'^{S^1}$ with a free S^1-action. The quotient $f^{-1}(0)/S^1$ can be seen as the zero set of the section $\widetilde{f}$ of the bundle $\mathcal{E} = (V' \backslash V'^{S^1}) \times_{S^1} V$ induced by f. Assume the homology class of $f^{-1}(0)/S^1$ in $(V' \backslash V'^{S^1})/S^1$ is zero. Then there is an immersion $i_D : D \to (V' \backslash V'^{S^1})/S^1$ from a compact orientable surface D with boundary to $(V' \backslash V'^{S^1})/S^1$ so that the image of the boundary ∂D is $f^{-1}(0)/S^1$. Let $v(D)$ be the normal bundle of D. The restriction of $v(D)$ to ∂D has a canonical isomorphism $v(D) \oplus \mathbb{R} \cong v(\partial D)$ and the normal bundle $v(f^{-1}(0)/S^1)$ is isomorphic to the restriction of $\mathcal{E}$ to ∂D. So we get an isomorphism $\alpha_f : v(D)|_{\partial D} \oplus \mathbb{R} \to \mathcal{E}|_{\partial D}$. The triple $[v(D) \oplus \mathbb{R}, \mathcal{E}|_D, \alpha_f]$ gives an element in the relative KO-group $KO(D, \partial D)$. Define

$$\delta(f, D) = \Big\langle w_2([v(D) \oplus \mathbb{R}, \mathcal{E}|_D, \alpha_f]), [D, \partial D] \Big\rangle \in \mathbb{Z}_2,$$

where $\langle , \rangle$ denotes the obvious pairing.

Lemma 15 *$\delta(f, D)$ depends only on the homotopy class of f.*

Proof. First $\delta(f, D)$ is not changed by suspensions, i.e.

$$\delta(f, D) = \delta(f \wedge id, D),$$

where the right side is defined for $f \wedge id : TV' \wedge V_1^+ \to V^+ \wedge V_1^+$. The sphere $V_1^+ = (\mathbb{C}^{m_1} \oplus \mathbb{R}^{n_1})^+$ has the natural S^1-action. The normal bundle $v'(D)$ in $((TV' \wedge V_1^+) \backslash (TV'^{S^1} \wedge V_1^{+S^1}))/S^1$ can be naturally identified as the direct sum of the normal bundle $v(D)$ in $(TV' \backslash TV'^{S^1})/S^1$, the trivial real bundle $\mathbb{R}^{n_1}$ and the bundle $D' \times_{S^1} \mathbb{C}^{m_1}$, where D' is the preimage of D in $TV' \backslash TV'^{S^1}$, i.e. $D'/S^1 = D$. The corresponding $\mathcal{E}$ for D' has a similar decomposition. So $\alpha_{f \wedge id} = \alpha_f \oplus id$. Then w_2's for the two triples are the same and $\delta(f, D) = \delta(f \wedge id, D)$.

Let f_1 and f_2 be two maps in the same homotopy class in

$$\{(TV', TV'^{S^1}), (V^+, \emptyset^+)\}_{S^1}.$$

Assume they are perturbed generically so that 0 is a regular value for both f_1 and f_2. We can enlarge V by suspension so that there is an S^1-homotopy $h : (TV', TV'^{S^1}) \times I \to (V^+, \emptyset^+)$ between f_1 and f_2. We can also assume 0 is a regular value for h. Then $h^{-1}(0)/S^1 \in (TV' \backslash TV'^{S^1})/S^1 \times I$ is a cobordism between $f_1^{-1}(0)/S^1$ and $f_2^{-1}(0)/S^1$. Its projection to $Pic^0(X)$ gives a homology equivalence between the projections of $f_1^{-1}(0)/S^1$ and $f_2^{-1}(0)/S^1$. Because $(TV' \backslash TV'^{S^1})/S^1$ is a bundle over $Pic^0(X)$ with simply connected fibers, that the homology class of $f^{-1}(0)/S^1$ is 0 is equivalent to that the homology class of its projection to $Pic^0(X)$ is 0. Therefore if one of $\delta(D_1, f_1)$ and $\delta(D_2, f_2)$ is defined then so is the other. The union $D_1 \cup h^{-1}(0) \cup D_2$ gives an oriented surface with corners ∂D_1 and ∂D_2. Deform D_1 and D_2 to be immersed submanifolds in the thickening $(TV' \backslash TV'^{S^1})/S^1 \times [-1, 0]$ and $(TV' \backslash TV'^{S^1})/S^1 \times [1, 2]$ of $(TV' \backslash TV'^{S^1})/S^1 \times 0$ and $(TV' \backslash TV'^{S^1})/S^1 \times 0$ respectively, then we will make $N = D_1 \cup h^{-1}(0)/S^1 \cup D_2$ smooth. Compare δ_1 and δ_2,

$$\delta(f_1, D_1) - \delta(f_2, D_2) = \Big\langle w_2(v(N), [N] \Big\rangle - \Big\langle w_2((\mathcal{E} \times [-1, 2])|_N, [N] \Big\rangle.$$

Here on $h^{-1}(0)/S^1$ we have a canonical bundle isomorphism inducing α_{f_1} and α_{f_2} for $t = 0$ and 1,

$$\alpha_N : v(N) \to (\mathcal{E} \times I)|_N.$$

Denote the maximal real and complex subspaces of V by $V^{\mathbb{R}}$ and $V^{\mathbb{C}}$ as in the proof of Theorem 9. Similarly, denote the maximal real and complex subbundles of V' by $V'^{\mathbb{R}}$ and $V'^{\mathbb{C}}$. Let a be the first chern class of the complex line bundle $(V' \backslash V'^{S^1}) \times_{S^1} \mathbb{C}$. Then

$$\Big\langle w_2((\mathcal{E} \times [-1, 2])|_N, [N] \Big\rangle = \dim_{\mathbb{C}} V^{\mathbb{C}} \Big\langle a, [N] \Big\rangle \pmod 2.$$

Let N' be the 3-manifold so that $N'/S^1 = N$. Because Γ_X is a *Spin* structure, similar to the argument for proving Theorem 9, the restriction of the tangent bundle $T(V')|_{N'}$ can be S^1-equivariantly identified with $N' \times (U \oplus \mathbb{R}^{b_1})$, where U is a fiber of V'. The quotient $N' \times_{S^1} U$ is a bundle on N with $\Big\langle w_2(N' \times_{S^1} U), [N] \Big\rangle = \dim_{\mathbb{C}} U^{\mathbb{C}} \Big\langle a, [N] \Big\rangle \pmod 2$. The quotient $T(N')/S^1$ is a bundle isomorphic to $TN \oplus \mathbb{R}$ on N which satisfies $w_2 = 0$. So

$$\begin{aligned}\Big\langle w_2(v(N)), [N]\Big\rangle &= \Big\langle w_2(v(N')/S^1), [N]\Big\rangle \\ &= \Big\langle w_2(N' \times_{S^1} U, [N]\Big\rangle - \Big\langle w_2(T(N')/S^1, [N]\Big\rangle \\ &= \dim_{\mathbf{C}} U^{\mathbb{C}} \Big\langle a, [N]\Big\rangle \pmod 2.\end{aligned}$$

By the Atiyah–Singer index theorem and the condition $\tau(X) = 0$, $\dim_{\mathbf{C}} U^{\mathbb{C}} - \dim_{\mathbf{C}} V^{\mathbb{C}}$ is zero. So

$$\delta(f_1, D_1) - \delta(f_2, D_2) = 0.$$

∎

So we can use $\delta([f])$ to denote $\delta(f, D)$. Define the homomorphism

$$\begin{aligned}\mathrm{Pr} &: \overline{\pi}^{b_+}_{S^1}(Pic^0(X); \mathrm{ind}\, D) \longrightarrow H_1(Pic^0(X); \mathbb{Z}) = \mathbb{Z}^{b_1}, \\ \mathrm{Pr} &: [f] \longmapsto \text{the homology class of } f^{-1}(0).\end{aligned}$$

Then δ is defined on $\ker \mathrm{Pr}$. It is easy to see that δ satisfies the rule for a homomorphism. Thus $\ker \delta$ is a subgroup of $\ker \mathrm{Pr}$ and the quotient group $\overline{\pi}^{b_+}_{S^1}(Pic^0(X); indD)/\ker \delta$ is an extension of

$$\overline{\pi}^{b_+}_{S^1}(Pic^0(X); \mathrm{ind}\, D)/\ker \mathrm{Pr} \subset \mathbb{Z}^{b_1}$$

by $\ker \mathrm{Pr} / \ker \delta \subset \mathbb{Z}_2$. So $\overline{\pi}^{b_+}_{S^1}(Pic^0(X); \mathrm{ind}\, D)/\ker \delta$ is a subgroup of $\mathbb{Z}_2 \oplus \mathbb{Z}^{b_1}$, and the composition

$$\overline{\pi}^{b_+}_{S^1}(Pic^0(X); \mathrm{ind}\, D) \longrightarrow \overline{\pi}^{b_+}_{S^1}(Pic^0(X); \mathrm{ind}\, D)/\ker \delta \hookrightarrow \mathbb{Z}_2 \oplus \mathbb{Z}^{b_1}$$

is what we desire.

Proposition 16 *If a Spin manifold X satisfies $\tau(X) = 0$, $b_1(X) = b_+(X)+2$ and $b_+(X) > 1$, then we have a homomorphism*

$$\overline{\pi}^{b_+}_{S^1}(Pic^0(X); \operatorname{ind} D) \longrightarrow \mathbb{Z}_2 \oplus \mathbb{Z}^{b_1}$$

whose restriction to $\ker \Pr$ *coincides with δ.*

In fact the condition $\tau(X) = 0$ is not needed as long as the virtual dimension of the Seiberg–Witten moduli space is 1. So we have similarly,

Proposition 17 *If a Spin manifold X satisfies $b_+(X) > b_1(X) + 1$, and the Seiberg–Witten moduli space for the Spin structure has dimension 1, then we have a homomorphism*

$$\pi^{b_+}_{S^1}(Pic^0(X); \operatorname{ind} D) \longrightarrow \mathbb{Z}_2 \oplus \mathbb{Z}^{b_1}$$

whose restriction to $\ker \Pr$ *coincides with δ.*

If X is not *Spin*, the first chern class $c_1(\operatorname{ind} D)$ may not vanish and Atiyah–Singer index may not be even in general. Then δ may not be well defined if we change the representing map. But consider the case $X = T^4 \# X'$, where X' is an almost complex 4-manifold with $b_1(X') = 0$, $b_+ \equiv 3 \pmod 4$. Let Γ_{T^4} be a *Spin* structure on T^4, and $\Gamma_{X'}$ be induced from the almost complex structure. Then the Atiyah–Singer index is even from the index formula, $c_1(\operatorname{ind} D)$ vanishes because $b_1(X') = 0$, and the virtual dimension of the moduli space for $\Gamma_X = \Gamma_{T^4} \# \Gamma_{X'}$ is 1. This implies that δ is well-defined.

Proposition 18 *Let $X = T^4 \# X'$, where X' is an almost complex 4-manifold with $b_1 = 0$, $b_+ \equiv 3$ (mod 4). Let $\Gamma_X = \Gamma_{T^4} \# \Gamma_{X'}$. Then we have a homomorphism*

$$\pi_{S^1}^{b_+}(Pic^0(X); \operatorname{ind} D) \longrightarrow \mathbb{Z}_2 \oplus \mathbb{Z}^{b_1}$$

which restriction to $\ker \Pr$ *coincides with δ.*

In [6], it is shown δ maps the Bauer–Furuta invariant of a homotopy $K3 \# K3$ with the *Spin* structure to $1 \in \mathbb{Z}_2$. Here we see the homomorphism constructed above has a similar property. When there is no confusion, we will simply denote the homomorphisms constructed above by δ.

The next theorem will be proved in Chapter 5.

Theorem 19 *Let $X = T^4 \# X'$, where $X' = T^4$ or an almost complex manifold with $b_1 = 0$ and $b_+ \equiv 3$ (mod 4). Let Γ_{T^4} be a Spin structure on T^4, $\Gamma_{X'}$ be the $Spin^c$ structure on X' from the almost complex structure which has an odd Seiberg–Witten invariant. Then the Bauer–Furuta invariant for $(X, \Gamma_{T^4} \# \Gamma_{X'})$ will be mapped to a nontrivial torsion element by δ.*

Theorem 11 and 12 follow immediately from Theorem 19.

Chapter 5

The proof of Theorem 19

To use the product formula in [1] (i.e. Theorem 8) to prove Theorem 19, we need to find a good representing map for the Bauer–Furuta invariant of the manifold X' occuring in the connected sum.

If $X' = T^4$, we can use the special geometric properties of T^4 to prove the lemma below.

Lemma 20 *Let Γ_{T^4} be a Spin structure on T^4. The Bauer–Furuta invariant can be represented by*

$$\begin{aligned} \mu : \ \mathbb{H}' &\longrightarrow \mathbb{H} \oplus \mathbb{R}^3, \\ h &\mapsto (L(h), Q(h)), \end{aligned}$$

where $\mathbb{H}'$ is a $\mathbb{C}^2$ bundle over $Pic^0(T^4)$ with $c_1 = 0$ and $c_2 = \pm 1$, and $\mathbb{H}$ is a trivial quarternionic line bundle. The complex bundle homomorphism L is an isomorphism except at one fiber, where it is 0. We can choose a chart around the fiber of $\mathbb{H}'$ where L vanishes such that locally L can be represented as

$$L : \mathbb{R}^4 \oplus \mathbb{H} \to \mathbb{H}, \ L(a, h) = c(a)h,$$

where $\mathbb{H}$ corresponds to the fiber direction and $\mathbb{R}^4$ corresponds to the zero section, and $c : \mathbb{R}^4 \to \mathbb{H}$ is a linear isomorphism. The restriction of the quadratic map Q to that special fiber is nondegenergte.

Proof. Choose the standard flat metric on T^4. For any *Spin* structure Γ_{T^4}, there is a unique product $Spin^c$ connection A_0 which will be our base connection. Denote the decomposition of the unperturbed Seiberg–Witten map (i.e. $\eta = 0$) by $sw = l + c$, in which l is linear Fredholm bundle map and c contains all the quadratic terms. The Fredholm linear operator l can be decomposed as the direct sum of Dirac operators D_A parametrized by $Pic^0(T^4)$ and an injective map $d^* \oplus pr_{harm} \oplus d^+$ which maps $L^2_{k-1}(i\Omega^1(T^4;\mathbb{R}))$ onto $L^2_{k-2}(i\Omega^0(T^4/\mathbb{R};\mathbb{R}) \oplus iH^1(T^4;\mathbb{R}) \oplus iH^+(T^4;\mathbb{R})^\perp)$. Using the Weitzenböck formula, it is easy to see that D_A are all isomorphisms except at A_0 when both kernel and cokernel are the spaces of flat spinors, with complex dimension 2. Denote the Dirac operators parametrized by $Pic^0(X)$ by

$$D : \Gamma^+ \to \Gamma^-,$$

where

$$\Gamma^+ = (A_0 + \ker d) \times L^2_{k-1}(\Gamma(W^+))/\mathcal{G}_0$$

and

$$\Gamma^- = (A_0 + \ker d) \times L^2_{k-2}(\Gamma(W^-))/\mathcal{G}_0$$

are bundles over $Pic^0(T^4)$. We can find a complex 2-dimensional trivial subbundle $\mathbb{H}$ of Γ^- containing $\ker D^*_{A_0}$ at A_0. Then the preimage $D^{-1}(\mathbb{H})$ will be a complex 2-dimensional subbundle $\mathbb{H}'$, such that $\mathbb{H}' - \mathbb{H}$ gives the index bundle of D. From [21], we know $c_1(\mathbb{H}') = 0$ and $c_2(\mathbb{H}') = \pm 1$. We can choose the complements of $\mathbb{H}'$ and $\mathbb{H}$, so that D can be presented as a matrix

$$\begin{pmatrix} L & 0 \\ 0 & D' \end{pmatrix},$$

in which L is the restriction of D to $\mathbb{H}'$ and D' is an isomorphism. For any $\phi \in \ker D_{A_0}$, $\sigma(\phi)$ is a flat (therefore harmonic) self-dual 2-form. Let $Q : \mathbb{H}' \to H^+(T^4;\mathbb{R}) \cong \mathbb{R}^3$ be a fiberwise quadratic map which extends $\sigma(\cdot)$ on $\ker D_{A_0}$.

By the Weitzenböck formula, the moduli space for the unperturbed Seiberg–Witten equation is the 0-section of the bundle $\mathcal{A}$. So the Seiberg–Witten equation defines the same equivariant stable cohomotopy class as its restriction to the pair

$$(B, \partial B) \longrightarrow (H, H \backslash \{0\}),$$

where H is the fiber of $\mathcal{C}$, and B is any bounded disk bundle around the zero section in $\mathcal{A}$. The disk bundle B can be defined by

$$||x_1|| \leq r_1, ||x_2|| \leq r_2, \text{ for any } x = x_1 + x_2, \text{ with } x_1 \in \mathbb{H}', \text{ and } x_2 \in \mathbb{H}'^{\perp}.$$

Assume $r_2 << r_1$ and r_1 is very small. Consider the linear homotopy with $t \in I$

$$l + ((1-t) \cdot c + t \cdot Q).$$

Then for any $x \in \partial B$, the quadratic terms are dominated by either the linear terms or the nondegenerate quadratic term $\sigma(\cdot)$ on $\ker D_{A_0}$. So the homotopy above stays inside the space of the maps from $(B, \partial B)$ to $(H, H\backslash\{0\})$. Taking

$t = 1$ gives a representing map for $BF(T^4, \Gamma_{T^4})$, which is the direct sum of an isomorphism and $\mu = (L, Q) : \mathbb{H}' \longrightarrow \mathbb{H} \oplus \mathbb{R}^3$. So μ represents $BF(T^4, \Gamma_{T^4})$ for a suitable homology orientation.

We only need to find L that is of the required form around the fiber over A_0. We can modify our definition of the Seiberg–Witten map and use the harmonic connections $A_0 + iH^1(T^4; \mathbb{R})$ instead of $A_0 + \ker d$, with $\mathcal{G}_0$ replaced by its harmonic subgroup. We still get the same Seiberg–Witten map. Then $\ker D_{A_0}$ and $\ker D^*_{A_0}$ naturally define quarternionic line bundles around A_0. For any $A \in A_0 + iH^1(T^4; \mathbb{R})$, and $h \in \ker D_{A_0}$, $D_A h = ah \in \ker D^*_{A_0}$. We define the subbundles $\mathbb{H}$ and $\mathbb{H}'$ such that they extend the bundle $\ker D^*_{A_0}$ and $\ker D_{A_0}$ respectively. Then L will satisfy the required property around A_0. ■

We can also add a constant $C \in \mathbb{R}^3$ to μ without changing its cohomotopy class. Then the preimage $\mu^{-1}(0)$ will be a circle contained in the fiber where L vanishes. Also we can see from the proof, Q can be chosen such that in the local chart described in the lemma, Q is constant in the direction of $\mathbb{R}^4$.

Proof. (The proof of Theorem 19 when $X' = T^4$.) Now we can consider the Bauer–Furuta invariant of $T^4 \# T^4$. Assume the *Spin* structure Γ is induced by *Spin* structures Γ_1 and Γ_2 on each component. If f_1 and f_2 are maps representing the corresponding Bauer–Furuta invariants given by lemma 20, then $f_1 \times f_2$ represents $BF(T^4 \# T^4, \Gamma)$. The preimage $(f_1 \times f_2)^{-1}(0)$ is a 2-torus in one fiber, so its projection to $Pic^0(T^4)$ is a point. We have $\delta([f_1] \wedge [f_2]) \in \mathbb{Z}_2$. The quotient $(f_1 \times f_2)^{-1}(0)/S^1$ bounds a surface in the quotient of the punctured fiber. Using the local chart around that fiber, we can see $f_1 \times f_2$

factors as

$$\mathbb{R}^4 \oplus \mathbb{H} \oplus \mathbb{R}^4 \oplus \mathbb{H} \xrightarrow{F} \mathbb{H} \oplus \mathbb{H} \oplus \mathbb{H} \oplus \mathbb{H} \xrightarrow{G} \mathbb{H} \oplus \mathbb{R}^3 \oplus \mathbb{H} \oplus \mathbb{R}^3,$$

in which

$$F(a_1, h_1, a_2, h_2) = (c_1(a_1)h_1, h_1, c_2(a_2)h_2, h_2))$$

and

$$G(c_1(a_1)h_1, h_1, c_2(a_2)h_2, h_2)) = (c_1(a_1)h_1, Q_1(h_1) + C, c_2(a_2)h_2, Q_2(h_2) + C).$$

The map F maps a neighborhood of $0 \oplus \mathbb{H} \oplus 0 \oplus \mathbb{H}$ to a cone neighborhood of it and restriction of F to $0 \oplus \mathbb{H} \oplus 0 \oplus \mathbb{H}$ is identity. From the definition of δ, it is only relevant to an S^1-equivariant neighborhood of the 3-dimensional S^1-manifold $(f_1 \times f_2)^{-1}(0) \in 0 \oplus \mathbb{H} \oplus 0 \oplus \mathbb{H}$ bounds. So $\delta([f_1] \wedge [f_2]) = \delta([G])$. From [6], we know $\delta([G]) = 1 \in \mathbb{Z}_2$. ∎

If X' is the manifold in Theorem 12, then we can also find a good map to represent $BF(X', \Gamma_{X'})$ in $\overline{\pi}^{b_+}_{S^1}(Pic^0(T^4); \operatorname{ind} D) \otimes \mathbb{Z}_2$.

Lemma 21 *If X' satisfies $b_1 = 0$, $b_+ \equiv 3$ (mod 4), the Seiberg–Witten moduli space for the Spin^c structure $\Gamma_{X'}$ is 0-dimensional, and $SW(X', \Gamma_{X'})$ is odd, then in $\overline{\pi}^{b_+}_{S^1}(Pic^0(T^4); \operatorname{ind} D) \otimes \mathbb{Z}_2$, $BF(X', \Gamma_{X'})$ can be represented by the one point compactification of the composition*

$$\mathbb{C}^{2n+2} \longrightarrow \mathbb{C}^2 \oplus \mathbb{R}^{4n} \longrightarrow \mathbb{R}^{4n+3}$$

in which the second is the direct sum of a nondegenerate quadratic map Q from

$\mathbb{C}^2$ to $\mathbb{R}^3$ and the identity map on $\mathbb{R}^{4n}$, the first is identity map by forgetting the S^1-action on $\mathbb{C}^{2n}$. (We can also add a constant term C to Q so that the preimage of $0 \in \mathbb{R}^3$ will be a circle.)

Proof. From [1] or [3], $BF(X', \Gamma_{X'}) = SW(X', \Gamma_{X'})[\kappa]$, where $[\kappa]$ is the generator of the equivariant stable homotopy group. If and only if the coefficient n is odd, then the class $n[\kappa]$ can be nonequivariantly represented by the Hopf map. So the map constructed in Lemma 21 represents $n[\kappa]$ with an odd n, and in $\overline{\pi}^{b_+}_{S^1}(Pic^0(T^4); \operatorname{ind} D) \otimes \mathbb{Z}_2$ it belongs to the same class as $BF(X', \Gamma_{X'})$. ∎

Proof. (Completion of the proof of Theorem 19) If we choose generic maps f_1' and f_2' representing $BF(T^4, \Gamma_{T^4})$ and $BF(X', \Gamma_{X'})$ such that ${f'}_i^{-1}(0)$ is a finite set of S^1-orbits for each $i = 1$ and 2. Then the projection of $(f_1' \times f_2')^{-1}(0)$ to $Pic^0(T^4)$ is a finite set of points. By the product formula in [1], i.e. Theorem 8, and the definition of δ, the image

$$\delta(BF(T^4 \# X', \Gamma_{T^4} \# \Gamma_{X'})) = \delta([f_1' \times f_2'])$$

is a torsion class in $\mathbb{Z}^2$. To calculate $\delta(BF(T^4 \# X', \Gamma_{T^4} \# \Gamma_{X'}))$, we may assume f_1' and f_2' are maps constructed in Lemma 20 and Lemma 21 respectively. So we can repeat the same steps as in the case $X' = T^4$ to reduce the calculation of $\delta(BF(T^4 \# X', \Gamma_{T^4} \# \Gamma_{X'}))$ to the calculation of $\delta([f_1 \times f_2])$, where $f_1 : \mathbb{C}^2 \to \mathbb{R}^3$ is defined by $f_1(q) = Q(q) + C$, where Q is nondegenerate quadratic and C is non-zero constant, and $f_2 : \mathbb{C}^{2+2n} \to \mathbb{R}^{3+4n}$ as in Lemma 21,

$$f_2 : \mathbb{C}^2 \oplus \mathbb{C}^{2n} \xrightarrow{id} \mathbb{C}^2 \oplus \mathbb{R}^{4n} \xrightarrow{f_1 \oplus id} \mathbb{R}^{4n+3}.$$

So $f_1 \times f_2$ is S^1-equivariantly the product of identity map from $\mathbb{C}^{2n}$ to $\mathbb{R}^{4n}$ and $f_1 \times f_1$. We know from [6], $\delta([f_1 \times f_1]) = 1 \in \mathbb{Z}_2$. Let D be a surface in $(\mathbb{C}^4 \backslash \{0\})/S^1$, which boundary is $(f_1 \times f_1)^{-1}(0)$. Let $\overline{D}$ be the 3-dimensional S^1-manifold in $\mathbb{C}^4 \backslash \{0\}$ with quotient $\overline{D}/S^1 = D$. Then the normal bundle of $\overline{D}$ in $\mathbb{C}^4 \oplus \mathbb{C}^{2n}$ is S^1-isomorphic to the direct sum of the normal bundle of $\overline{D}$ in $\mathbb{C}^4$ and the product bundle $\overline{D} \times \mathbb{C}^{2n}$ with natural S^1-action on both factors. So the normal bundle of D in $((\mathbb{C}^4 \oplus \mathbb{C}^{2n}) \backslash \{0\})/S^1$ is isomorphic to the direct sum of the normal bundle of D in $(\mathbb{C}^4 \backslash \{0\})/S^1$ and $\overline{D} \times_{S^1} \mathbb{C}^{2n}$. Restricted to the boundary of D, the isomorphism $\alpha_{f_1 \times f_2}$ identifies the normal bundle of D in $(\mathbb{C}^4 \backslash \{0\})/S^1$ with $\mathcal{E} = \overline{D} \times_{S^1} \mathbb{R}^6$ and $\overline{D} \times_{S^1} \mathbb{C}^{2n}$ with $\overline{D} \times_{S^1} \mathbb{R}^{4n}$. So the relative KO-group element defined for $f_1 \times f_2$ in Chapter 4 is a sum of the relative KO-group element for $f_1 \times f_1$ and an even multiple of the relative KO-group element $[\overline{D} \times_{S^1} \mathbb{C}, \overline{D} \times_{S^1} \mathbb{R}^2]$ which w_2 is 0. So $\delta([f_1 \times f_2]) = \delta([f_1 \times f_1]) = 1 \in \mathbb{Z}_2$. We have finished the proof of Theorem 19. ■

Here we present a direct proof of the nontriviality of $\delta([f_1 \times f_2])$. In fact it also follows from the fact that $[f_1] \wedge [f_2]$ is nonequivariantly the square of the Hopf map so it is not trivial stable class, and δ is bijective from lemma 12 of [6].

Chapter 6

The application on calculating the Yamabe invariant

By Theorem 11 and 12, we can give an alternative proof that the 4-manifolds $X = T^4 \# X'$ which satisfy the conditions in Theorem 11 or 12 have no positive scalar curvature metrics.

Corollary 22 *Let $X' = T^4$, or a 4-manifold satisfying $b_1 = 0$, $b_+ \equiv 3 \pmod 4$, and with an almost complex structure for which the Seiberg–Witten invariant is odd. Let N be any 4-manifold with $b_+ = 0$. Then there are no positive scalar curvature metrics on $T^4 \# X' \# N$.*

Proof. The Bauer–Furuta invariant for $T^4 \# X'$ is not trivial from Theorem 11 and 12. Neither for $T^4 \# X' \# N$ from corollary 9 of [13]. So there are irreducible solutions for the Seiberg–Witten equation with a small generic perturbation. A standard argument with the Weitzenböck formula proves there are no positive scalar curvature metrics on those 4-manifolds. ∎

Similar to [13], we can calculate the Yamabe invariant of those 4-manifolds $X = T^4 \# X' \# N$. Recall the Yamabe invariant for a manifold X is defined as

$$Y(X) = \sup_{\gamma} \inf_{g \in \gamma} Vol_g^{(2-n)/n} \int s_g d\mu_g,$$

where the supremum is for all the conformal classes $\gamma = [g_0] = \{ug_0 \mid u \in C^\infty(X), u > 0\}$.

Theorem 23 *Let $X' = T^4$ or a minimal complex surface satisfying $b_1 = 0$, $b_+ \equiv 3$ (mod 4). Let N be any 4-manifold with $b_+ = 0$ whose Yamabe invariant $Y(N)$ is non-negative. Then*

$$Y(T^4 \# X' \# N) = -\sqrt{32\pi^2 c_1^2(X')}.$$

Proof. By Corollary 22, there is no positive scalar curvature on $T^4 \# X' \# N$, so

$$I_s(T^4 \# X' \# N) \stackrel{def}{=} \inf_g \int s_g^2 d\mu_g = Y(T^4 \# X' \# N)^2.$$

We need only to prove $I_s(T^4 \# X' \# N) = 32\pi^2 c_1^2(X')$.

First we can use the method in [10] and [13] to give the estimate $\int s_g^2 d\mu_g \geq 32\pi^2 c_1^2(X')$ for all metrics g. For simplicity, assume X' is the minimal complex surface with $b_1 = 0$ and $b_+ \equiv 3$ (mod 4). The case $X' = T^4$ can be proved similarly using Theorem 11. From corollary 8 in [13], Theorem 12, and [4], The Bauer–Furuta invariants of $T^4 \# X' \# N$ is not trivial. The $Spin^c$ structures for which the Bauer–Furuta invariant is not zero depend on $b_-(N)$. If $b_-(N) = 0$, for example $N = kS^1 \times S^3$, we need only the $Spin^c$ structure such that its restrictions on T^4 and N are the standard $Spin$ structures and its restriction on

X' is given by the complex structure. For any metrics g, the Seiberg–Witten equation with small generic perturbation will have solution. So we have [10]

$$\int s_g d\mu_g \geq 32\pi^2 c_1^2(X').$$

If $b_-(N) > 0$, then Γ_N, the restriction of the $Spin^c$ structure to N satisfies $c_1^2(\Gamma_N) = -b_-(N)$. So we have $2^{b_-(N)}$ ways to choose Γ_N to get a nontrivial Bauer–Furuta invariant on $T^4\#X'\#N$, i.e. Γ_N can be chosen from

$$\pm e_1 \pm e_2 \pm \cdots \pm e_k,$$

where $\{e_1, e_2,...,e_k\}$ is the basis so that the intersection form of N is given by $k(-1)$. We have the estimate

$$\int s_g d\mu_g \geq 32\pi^2 c_1^{+2}$$

where $c_1^+ + c_1^- = c_1$ is decomposition of the first chern class of the $Spin^c$ structure, according to the polarization by the metric. We have

$$\begin{aligned} c_1^{+2} &= (c_1^+(\Gamma_{T^4}) + c_1^+(X') + c_1^+(\Gamma_N))^2 \\ &= (c_1^+(X') + c_1^+(\Gamma_N))^2 \\ &\geq c_1^2(X') + 2c_1^+(X')c_1^+(\Gamma_N). \end{aligned}$$

Similar to [10] or [13], we can choose a Γ_N so that $c_1^+(X)c_1^+(\Gamma_N) \geq 0$. So we still have $\int s_g d\mu_g \geq 32\pi^2 c_1^2(X')$.

By [18]

$$I_s(T^4\#X'\#N) \leq I_s(T^4) + I_s(N) + I_s(X') = I_s(N) + I_s(X').$$

By the assumption $Y(N) \geq 0$, so $I_s(N) = 0$. From [12] or [11],

$$I_s(X') = 32\pi^2 c_1^2(X').$$

So $I_s(T^4\#X'\#N) = 32\pi^2 c_1^2(X')$ and $Y(T^4\#X'\#N) = -\sqrt{32\pi^2 c_1^2(X')}$. ■

There are some other applications of Theorem 11 and 12. For example, we may calculate the invariant I_r as [13], or get the nonexistence of the Einstein metrics on $T^4\#X'\#N$, provided with some condition on N.

Before the end of this chapter, we would like to mention, that a similar construction may be possible for manifolds $\#_{i=1}^3 X_i$ and $\#_{i=1}^4 X_i$ in which at least one of X_i is T^4 and the others satisfies the conditions for X' in Theorem 19. So we can conjecture

Conjecture 24 *Let $X = \#_{i=1}^3 X_i$ or $X = \#_{i=1}^4 X_i$, in which at least one of X_i is T^4 and the others satisfies the conditions for X' in Theorem 19, then the Bauer–Furuta invariant of X is not trivial.*

Chapter 7

The Ruberman invariant

In this chapter, we will introduce the work of Ruberman on the 1-parameter Seiberg–Witten equation and his invariant defined for a self-diffeomorphism of 4-manifold.

Let X be a 4-manifold, satisfying $b_+(X) > 2$. Let Γ_X be a $Spin^c$ structure on X. Define

$$\Pi \stackrel{def}{=} \{(g,\eta) \in Met(X) \times \Omega^2(Z) | *_g \eta = \eta\}.$$

Then for each $h \in \Pi$, the Seiberg–Witten equation defines a moduli space $\mathcal{M}(X,\Gamma;h)$. Similarly for a path $h : I \to \Pi$, we can define the *1-parameter Seiberg–Witten moduli space* to be the union

$$\mathcal{M}(X,\Gamma;h) = \cup_{0\leq t\leq 1}\mathcal{M}(X,\Gamma;h_t).$$

The virtual dimension of the 1-parameter Seiberg–Witten moduli space satisfies

$$\dim \mathcal{M}(X,\Gamma;h) = \dim \mathcal{M}(X,\Gamma;h_t) + 1$$

for any $t \in I$. If for each t, the virtual dimension $\dim \mathcal{M}(X,\Gamma;h_t) = -1$, we can choose a generic smooth path h, such that $\mathcal{M}(X,\Gamma;h_t) = \emptyset$, for $t = 0$ and 1, and $\mathcal{M}(X,\Gamma;h)$ is transversely cut by the Seiberg–Witten equation (which will be called the *1-parameter Seiberg–Witten equation* in the case we have the parameter t) for the path h and contains only finite points. Ruberman defined $SW(X,\Gamma;h)$ to be the algebraic count of the points in $\mathcal{M}(X,\Gamma;h)$. Then by [19] $SW(X,\Gamma;h)$ only depends on h_0 and $h_1 \in \Pi$. So $SW(X,\Gamma;h)$ can also be denoted by $SW(X,\Gamma;h_0,h_1)$. It is easy to see:

Lemma 25 *[20] Suppose $b_+ > 2$, and g_0, g_1 are generic Riemannian metrics in the same component of* **PSC(X)**. *Let $h_i = (g_i, \eta_i) \in \Pi$, $i = 0, 1$, be generic. Then $SW(X,\Gamma;h_0,h_1) = 0$ if η_i, $i = 0$ and 1, are sufficiently small.*

Let $f : X \to X$ be a diffeomorphism which keeps the homology orientation. Denote the orbit of Γ_X in the set of all *Spin*^{c} structures by the action of f^{*n}, $n \in \mathbb{Z}$, by $\mathcal{O}(f,\Gamma_X)$.

Definition 26 *[20] Suppose the Seiberg–Witten moduli space for the Spin^c structure Γ_X has virtual dimension -1. For an arbitrary generic point $h_0 \in \Pi$, define*

$$SW_{tot}(f,X,\Gamma_X) = \sum_{\Gamma' \in \mathcal{O}(f,\Gamma_X)} SW(X,\Gamma';h_0,f^*h_0).$$

The right side is in fact a finite sum and it does not depend on h_0 [20]. We will call $SW_{tot}(f,X,\Gamma_X)$ the *Ruberman invariant*. If we drop the condition that f keeps the homology orientation, $SW_{tot}(f,X,\Gamma_X)$ is still well defined in $\mathbb{Z}_2$.

One of the main results Ruberman proved in [20] is the nontriviality of this invariant for the diffeomorphism he constructed.

First consider $N = \mathbb{CP}^2 \# \overline{\mathbb{CP}}^2 \# \overline{\mathbb{CP}}^2$. Let S, E_1 and E_2 be the embedded 2-spheres with self square $1, -1, -1$ in each copy of $\mathbb{CP}^2$ or $\overline{\mathbb{CP}}^2$. Let $\Sigma_\pm = S \pm E_1 + E_2$ be spheres with squares -1 in the indicated classes. For $\Sigma = \Sigma_\pm$, let ρ^Σ be the diffeomorphism of N which induces the reflection $\rho^\Sigma_*(x) = x + 2(x \cdot \Sigma)\Sigma$ on $H_2(N)$. Let the diffeomorphism $\rho = \rho^{\Sigma_+} \circ \rho^{\Sigma_-}$. There is a $Spin^c$ structure Γ_N on N with $c_1(\Gamma_N) = s + e_1 + e_2$ where s, e_1 and e_2 are the Poincaré duals of S, E_1 and E_2.

Then take any almost complex manifold X with nontrivial Seiberg-Witten invariant for the almost complex structure. Assume $b_+(X) > 1$. Consider the manifold $Z = X \# N$ with $Spin^c$ structure $\Gamma_X \# \Gamma_N$, which restricts to Γ_X and Γ_N on X and N respectively. Consider the diffeomorphism $f = id \# \rho$ on Z. Ruberman proved,

Theorem 27 *[20]* $SW_{tot}(f, Z, \Gamma_X \# \Gamma_N) = SW(X, \Gamma_X)$.

We can take X to be a hypersurface in $\mathbb{CP}^3$ with a high degree. Then by [14] and [7] $Z = X \# N \cong m\mathbb{CP}^2 \# n\overline{\mathbb{CP}}^2$ has a positive scalar curvature metric. By Lemma 25, we get the disconnectedness of $\mathbf{PSC}(Z)$.

Chapter 8

The S^1-equivariant stable cohomotopy class of the 1-parameter Seiberg–Witten map

Now we will use Bauer–Furuta's method [3] to give a cohomotopy refinement of Ruberman work.

Let Γ_X be a fixed $Spin^c$ structure on X. For each smooth path h, with $h_t = (g_t, \eta_t)$, $t \in I$, in Π, we can define $\widetilde{sw}_t : \widetilde{\mathcal{A}}_t \to \widetilde{\mathcal{C}}_t$ and the Seiberg–Witten map $sw_t : \mathcal{A}_t \to \mathcal{C}_t$. We choose the same Soblev completions for all the $t \in I$. Then $\cup_{t\in I}\mathcal{A}_t$ and $\cup_{t\in I}\mathcal{C}_t$ are Hilbert bundles over $Pic^0(X) \times I$. We denote $\mathcal{A}_I = \cup_{t\in I}\mathcal{A}_t$, $\mathcal{C}_I = \cup_{t\in I}\mathcal{C}_t$, $\widetilde{\mathcal{A}}_I = \cup_{t\in I}\widetilde{\mathcal{A}}_t$, and $\widetilde{\mathcal{C}}_I = \cup_{t\in I}\widetilde{\mathcal{C}}_t$. Define

$$\widetilde{sw}_I : \widetilde{\mathcal{A}}_I \to \widetilde{\mathcal{C}}_I$$

to be the map whose restrictions to each $\widetilde{\mathcal{A}}_t$ is $\widetilde{sw}_t : \widetilde{\mathcal{A}}_t \to \widetilde{\mathcal{C}}_t$. Denote its quotient by the pointed gauge group $\mathcal{G}_0$ by

$$sw_I : \mathcal{A}_I \to \mathcal{C}_I.$$

It is easy to see sw_I is a bundle map which is equivariant under the action of $\mathcal{G}/\mathcal{G}_0 = S^1$.

The next lemma is proved below.

Lemma 28 *If $sw_0^{-1}(0) = sw_1^{-1}(0) = \emptyset$, then sw_I defines a stable homotopy class in $\pi_{S^1}^{b_+-1}(Pic^0(X); \operatorname{ind} D)$.*

Denote the fibers of $\mathcal{A}_I$ and $\mathcal{C}_I$ by H' and H. Then sw_I can be decomposed as $sw_I = l + c$, where l is a linear Fredholm bundle map and c is compact bundle map. We can trivialize the Hilbert bundle $\mathcal{C}_I = H \times (Pic^0(X) \times I)$. Denote the projection to the fiber H by $pr_H : \mathcal{C}_I \to H$. By lemma 2.5 of [3], there exists a finite dimensional S^1-subspace $V \subset H$ such that the following holds:

1. for each $y \in Pic^0(X) \times I$, the subspace V, together with the image of $pr_H \circ l_y : H'_y \to H$, spans the Hilbert space H, where l_y is the restriction of l at y. Let $F_1(V) = V \times (Pic^0(X) \times I) \subset \mathcal{C}_I$ and $F_0(V) = (pr_H \circ l)^{-1}(V)$. Then $F_0(V)$ is a bundle on $Pic^0(X) \times I$ and $\lambda = F_0(V) - F_1(V)$ is the virtual index bundle $\operatorname{ind} l$.

2. For any S^1-linear subspace $W = W' + V \subset H$ with $W' \subset V^\perp$, the restricted map $\mu_W^+ : (pr_H \circ sw_I)|_{F_0(W)}^+ : TF_0(W) \to H^+$ misses the unit sphere $S_1(W^\perp)$ centered at 0.

3. For any S^1-subspace W there is a canonical S^1-equivariant homotopy equivalence $\rho_W : H^+ \backslash S_1(W^\perp) \to W^+$. Then the maps $\rho_W \circ \mu_W^+$ and $id \wedge (\rho_V \circ \mu_V^+)$ are S^1-homotopic as pointed maps.

These properties set up the machine to produce an S^1-equivariant stable cohomotopy class associated to sw_I. But they are not complete. The extra property for $t = 0$ and 1 gives the following refinement,

Lemma 29 *There exist a finite dimensional S^1-subspace $V \subset H$ and constants r and r', $0 < r' < r$, such that the following hold:*

1. *For every $y \in Pic^0(X) \times I$, the subspace V, together with the image of $pr_H \circ l_y$, spans the Hilbert space H.*

2. *For any finite dimensional S^1-subspace $W = W' + V$ with $W' \subset V^\perp$, the one point compactification of the restricted map $\mu_W^+ = (pr_H \circ sw_I)|_{F_0(W)}^+ :$ $TF_0(W) \to H^+$ misses the radius r' sphere $S_{r'}(W^\perp)$ centered at 0.*

3. *There is a canonical S^1-homotopy equivalence $\rho_{W,r'} : H^+ \backslash S_{r'}(W^\perp) \to W^+$. The maps $\rho_{W,r'} \circ \mu_W^+$ and $id \wedge (\rho_{V,r'} \circ \mu_V^+)$ are S^1-homotopic as pointed maps.*

4. *For $t = 0$ and 1, the image of $pr_H \circ sw_t$ misses the radius r disk centered at 0. The homotopy constructed in 3 restricted to $t = 0$ and 1 also have images missing 0.*

Proof. (The proof of Lemma 28.) Assume Lemma 29. Then the 1-parameter Seiberg–Witten map sw_I defines an S^1-equivariant map

$$(TF_0(V), TF_0(V|_{t=0,1})) \longrightarrow (V^+, V^+\backslash\{0\}) \simeq (V^+, \emptyset^+).$$

The index bundle parametrized on $Pic^0(X) \times I$ is induced by $indl$ on $Pic^0(X)$. So we can trivialize $F_0(V)$ in the direction of I as $F_0(V) = V' \times I$. Hence the 1-parameter Seiberg–Witten map sw_I defines a S^1-equivariant map

$$(TV' \times I, TV' \times \{0,1\}) \to (TV' \wedge S^1, x_0) \to (V^+, \emptyset^+),$$

in which x_0 is the base point at infinity. Its stable cohomotopy class is a class in

$$\pi_{S^1}^{b_+-1}(Pic^0(X); \operatorname{ind} D),$$

which is independent of the choice of V ■

Proof. (The proof of Lemma 29.) By lemma 2.5 of [3], the first property can be satisfied.

It is also easy to prove the first statement in property 4. For example, restricted to $t = 0$, $pr_H \circ sw_0 : \mathcal{A}_0 \to H$ can be decomposed as $l_0 + c_0$ with l_0 linear Fredholm and c_0 compact. Assume there are a sequence of points $x_n \in \mathcal{A}_0$ such that $\lim_{n\to\infty} pr_H \circ sw_0(x_n) = 0$, then by the boundedness condition, x_n is a bounded sequence. Denote the orthogonal decomposition $x_n = x'_n + x''_n$ with $x_{n'} \in \ker l_0$ and $x''_n \in (\ker l_0)^\perp$. Passing to a subsequence of x_n, we can assume the projection of x_n to $Pic^0(X)$ is convergent, $c_0(x_n)$ is convergent and x'_n is convergent. Then x''_n converges because

$$l_0(x''_n) = (l_0 + c_0)(x_n) - c_0(x_n) = pr_H \circ sw_0(x_n) - c_0(x_n)$$

is convergent. Let $x = \lim x_n$ then $pr_H \circ sw_0(x) = 0$. This is a contradiction with the assumption.

Assume for $t = 0$ and 1 the images of $pr_H \circ sw_I$ will miss the radius R disk around the origin. The maps $\rho_{W;r'}$ can be defined as $r'\rho_W(r'^{-1})$ where $\rho_W : H^+ \backslash S_1(W^\perp) \to W^+$ is a homotopy equivariance. The map ρ_W can be the one constructed in [3]. But in fact we have more freedom on choosing it. For example, let $v = av_1 + v_2 \in H\backslash S_1(W^\perp)$, $v_1 \in S_1(W^\perp)$, $a \geq 0$ and $v_2 \in W$, $\rho_W(v) = \infty$ if $a \geq 1$ and $\rho_W(v) = v_2/(1-a)$ if $a < 1$. It is easy to see there is a constant $C > 4$, which is independent of W, so that for any $v \in H$ with

$||v|| > C$ then $||s\rho_W(v) + (1-s)v|| > 3$ for all $s \in I$. So for any v satisfying $||v|| > Cr'$, $||s\rho_{W;r'}(v) + (1-s)v|| > 3r'$ for all $s \in I$.

Take $r = R/C$ and $r' = r/2$ then property 2 and property 3 can be satisfied following a similar argument in the proofs of lemma 2.3 and lemma 2.5 in [3]. So we only need to prove that the second statement in property 4 can be satisfied by choosing a suitable V.

Let D' be a disk bundle in $\mathcal{A}_I$ containing the preimage $(pr_H \circ sw_I)^{-1}(B_R(0))$ and assume V is large enough so that the distance between $c(D')$ and V is less than $r'/4$. Similar to the proofs of lemma 2.3 and lemma 2.5 in [3], the homotopy is constructed first on $D' \times [0,3]$, i.e. $h : D' \times [0,3] \to H^+ \backslash S_{r'}(W^\perp)$,

$$h_s = l + ((1-s)id_H + s \cdot pr_V) \circ c, \text{ for } 0 \le s \le 1,$$

$$h_s = l + pr_V \circ c \circ ((2-s)id_{H'} + (s-1)pr_{F_0(V)}), \text{ for } 1 \le s \le 2,$$

and

$$h_s = pr_{V^\perp} \circ l + ((3-s)pr_V + (s-2)\rho_{V;r'}) \circ (l+c) \circ pr_{F_0(V)},$$

for $2 \le s \le 3$. Extend h to the complement of $D' \cap W'$ in W'^+, which will be mapped to $H^+ \backslash B_{r'}(W^\perp)$. Then $(\rho_{W;r'} \circ H)|^+_{F_0(W)}$ is the homotopy between $\rho_{W;r'} \circ \mu_W^+$ and $id \wedge (\rho_{V;r'} \circ \mu_V^+)$. By the contraction $\rho_{W;r'}$, $H^+ \backslash B_{r'}(W^\perp)$ will be mapped to $W^+ \backslash \{0\}$. So we need only to track the homotopy before the extension for the preimage of zero restricted to $t = 0$ and $t = 1$.

Restricted to $t = 0$ and $t = 1$, we will see h defined on $D' \times [0,3]$ misses

B_r in H. For $v \in D'$ and $s \in [0, 1]$,

$$||h_s(v)|| \geq ||sw_t(v)|| - dist(V, c(D')) > R - r'/4 > r.$$

For $s \in [2, 3]$,

$$\begin{aligned} ||h_s(v)|| &\geq \left|\left|((3-s)pr_V + (s-2)\rho_{V;r'}) \circ (l+c) \circ pr_{F_0(V)}(v)\right|\right| \\ &\geq \left|\left|((3-s)id + (s-2)\rho_{V;r'}) \circ (l+c) \circ pr_{F_0(V)}(v)\right|\right| \\ &\quad -(3-s)dist(V, c(D')). \end{aligned}$$

Because $(l+c) \circ pr_{F_0(V)}(v)$ has norm greater than $R > Cr'$, and $dist(V, c(D')) < r'/4$ so $||h_s(v)|| > 3r' - r'/4 = r$. For $s \in [1, 2]$, set $v = v_1 + v_2$, $v_1 \in F_0(V)$ and $v_2 \in F_0(V^\perp)$. If $||l(v_2)|| > r$ then $||h_s(v)|| \geq ||l(v_2)|| > r$. Otherwise

$$\begin{aligned} ||h_s(v)|| &\geq ||l(v) + c(v_1 + (2-s)v_2)|| - dist(V, c(D')) \\ &\geq ||(l+c)(v_1 + (2-s)v_2|| - ||l((s-1)v_2)|| - r'/4 \\ &> R - r - r'/4 > r. \end{aligned}$$

So the homotopy restricted to $t = 0$ and 1 will miss the radius r disk in H^+, and so it has image in $H^+ \backslash B_{r'}(W^\perp)$. ∎

Chapter 9

An cohomotopy refined Ruberman invariant

Let f be a diffeomorphism on the 4-manifold X. Assume the orbit $\mathcal{O}(f, \Gamma_X)$ of the $Spin^c$ structures Γ_X is infinite. Choose a Riemannian metric g on X, and a path of metrics g_t for $t \in I$, connecting g and f^*g, i.e. $g_0 = g$ and $g_1 = f^*g$. Then we have a path of metrics g_t for $t \in \mathbb{R}$ defined by $g_t = f^{*n} g_{t-n}$ for $t \in [n, n+1]$, $n \in \mathbb{Z}$. Denote by $sw_{\mathbb{R}} : \mathcal{A}_{\mathbb{R}} \to \mathcal{C}_{\mathbb{R}}$ the 1-parameter unperturbed Seiberg–Witten map (i.e. $\eta_t = 0$, $t \in \mathbb{R}$) for Γ_X and the path g_t, $t \in \mathbb{R}$. (If the perturbations $\eta_t = 0$, then g_t can also be seen as the path $(g_t, 0)$ in Π.)

Lemma 30 *The preimages $sw_t^{-1}(0)$ are empty when $t >> 0$ or $t << 0$.*

See [20] for the proof.

By Lemma 28 and Lemma 30, we can define a homotopy class $[sw_{\mathbb{R}}] \in \pi_{S^1}^{b_+-1}(Pic^0(X); \operatorname{ind} D)$, by restricting $t \in [t_0, t_1]$, where $t_0 << 0$ and $t_1 >> 0$. The homotopy class $[sw_{\mathbb{R}}]$ will not depend on the choice of t_0 and t_1. It is also independent of the metric g and the path g_t between g and f^*g. The proof is very similar to the one given in [20]. If g' is another metric and g'_t is a path between g' and f^*g', then there is a 2-parameter family of metrics $g_{t,s}$

connecting g_t and g_t' for $t, s \in I$, and we can use f to generate a 2-parameter family of metrics $g_{t,s}$ connecting g_t and g_t' for $s \in I$, $t \in \mathbb{R}$. We can prove 0 will not be mapped to by the 2-parameter Seiberg–Witten map for $g_{t,s}$ if $t >> 0$ or $t << 0$. So it defines an element in $\pi_{S^1}^{b_+-1}(Pic^0(X) \times I; \mathrm{ind}\, D)$. The restriction to either side gives the corresponding homotopy element $[sw_{\mathbb{R}}]$ and $[sw'_{\mathbb{R}}]$. This imply the identification between $[sw_{\mathbb{R}}]$ for different g and g_t.

In the case $b_+(X) > b_1(X)+2$ and the virtual dimension of the 1-parameter moduli space

$$d' = \frac{c_1^2(\Gamma_X) - \tau(X)}{4} - b_+(X) + b_1(X) = 0,$$

we may define a homomorphism

$$t : \pi_{S^1}^{b_+-1}(Pic^0(X); \mathrm{ind}\, D) \to \mathbb{Z}$$

by algebraically counting the S^1-orbits in the preimage of 0 for a generic map in the stable cohomotopy class. Then t will map $[sw_{\mathbb{R}}]$ to the algebraic count of the points in the 1-parameter Seiberg–Witten moduli space. This coincides with the Ruberman invariant, if f preserves the homology orientation.

So we have

Theorem 31 *Let X be a smooth closed oriented and homology-oriented 4-manifold with a $Spin^c$ structure Γ_X, and f be a diffeomorphism on X which has an infinite orbit $\mathcal{O}(f, \Gamma_X)$. Then we can define a cohomotopy refined invariant $[sw_{\mathbb{R}}] \in \pi_{S^1}^{b_+-1}(Pic^0(X); \mathrm{ind}\, D)$. If f preserves the homology orientation of X, $b_+(X) > b_1(X) + 2$, and the virtual dimension of the 1-*

parameter moduli space is 0, then the image of $[sw_{\mathbb{R}}]$ *under the homomorphism* $t : \pi^{b_+-1}_{S^1}(Pic^0(X); \operatorname{ind} D) \to \mathbb{Z}$ *is the Ruberman invariant.*

We will call $[sw_{\mathbb{R}}]$ the *cohomotopy refined Ruberman invariant*, and denote it by $BF_{tot}(X, f, \Gamma_X)$.

Chapter 10

The nontriviality of the refined Ruberman invariant

From Theorem 28, there are examples of manifold Z with $Spin^c$ structure Γ_Z and diffeomorphism f on Z such that $SW_{tot}(Z, f, \Gamma_Z) \neq 0$. If Z satisfies $b_+ > b_1 + 2$, then from Theorem 31, $BF_{tot}(Z, f, \Gamma_Z) \neq 0$.

To get more examples with nontrivial cohomotopy refined Ruberman invariants, we start by considering the manifold $N = \mathbb{CP}^2 \# 2\overline{\mathbb{CP}}^2$, with the $Spin^c$ structure and diffeomorphism ρ defined in Chapter 7. We have

Lemma 32 *Under a suitable homology orientation of N, $BF_{tot}(N, \rho, \Gamma_N) = [id] \in \pi^0_{S^1}(pt; \mathbb{C}^0)$.*

Proof. As $b_+(N) = 1$ and $b_1(N) = 0$, the homology orientation of N is an orientation of $H^+(N; \mathbb{R})$. So for each metric g_t in the path of metrics on N, it determines a unique harmonic self-dual 2-form ω_t. Define a function

$$f(t) = \int \omega_t \wedge c_1(\Gamma_N) = \int_N \omega_t \wedge (s + e_1 + e_2).$$

If g_t is generated by the diffeomorphism ρ, by the wall-crossing property proved by Ruberman in [20], we can choose the homology orientation so that $f(t) < 0$ when $t << 0$ and $f(t) > 0$ when $t >> 0$. The reason is the following. For any metric g_t, the cohomotopy class of ω_t is in the component $\mathcal{H}^+$ of

$$\mathcal{H} = \{\alpha \in H^2(N;\mathbb{R}) | \alpha^2 = 1\}.$$

We can identify $\mathcal{H}^+$ with the hyperbolic plane, then the action of f^* on $\mathcal{H}^+$ is a parabolic transformation. The set of $\alpha \in \mathcal{H}^+$ such that

$$\int_N \alpha \wedge (s + e_1 + e_2) = 0$$

is a geodesic passing through the only fixed point of f^* on the boundary of $\mathcal{H}^+$. So each orbit generated by f^{*n} will cross this geodesic exactly once. Therefore the family ω_t will have an odd intersection with it. If $f(t) < 0$ when $t << 0$ by choosing a suitable homology orientation, then $f(t) > 0$ when $t >> 0$.

The family of forms ω_t gives a trivialization of line bundle on $\mathbb{R}$ with fiber equal to the space of self-dual harmonic forms $iH^+(N;\mathbb{R})$. Restricted to the set of S^1-fixed points, the 1-parameter Seiberg–Witten map is the bundle map

$$\begin{aligned} \mathbb{R} \times L^2_k(i\Omega^1(N;\mathbb{R})) &\longrightarrow L^2_{k-1}(i\Omega^0(N;\mathbb{R})/i\mathbb{R} \oplus i\mathrm{Im}\, d^+ \oplus iH^+(N;\mathbb{R}) \\ (t,a) &\mapsto (d^*a, d^+a + pr_{\mathrm{Im}\, d^+} F_{A_0}, f(t)). \end{aligned}$$

It can be deformed to a linear diffeomorphism $(t,a) \mapsto (d^*a, d^+a, t)$ by a homotopy which satisfies the boundedness condition. So it is in the stable homotopy class of $[id]$. Using lemma 3.8 of [3] and Atiyah–Singer index theorem,

$$BF_{tot}(N, \rho, \Gamma_N) = [id] \in \pi^0_{S^1}(pt; \mathbb{C}^0).$$

■

Let X be any closed smooth oriented manifold with a given $Spin^c$ structure Γ_X, then on $Z = X\#N$, we have the diffeomorphism $id\#\rho$ and $Spin^c$ structure $\Gamma_Z = \Gamma_X\#\Gamma_N$. The homology orientation of Z is induced by those for X, and for N which is determined by Lemma 32. Then we will prove an analog for Ruberman's Theorem 27.

Theorem 33 $BF_{tot}(Z, id\#\rho, \Gamma_Z) = BF(X, \Gamma_X)$.

The proof will be postponed to the next chapter.

From Theorem 33, we can see there are more examples in which we can get a nontrivial cohomotopy refined Ruberman invariant. For example, take $X = X'\#X'$, where X' is a algebraic hypersurface in $\mathbb{CP}^3$ with $b_+ \equiv 3$ (mod 4). Let $\Gamma_{X'}$ be the $Spin^c$ structure induced by the complex structure. Then $SW(X', \Gamma_{X'}) = \pm 1$ by [4]. By the product formula of Bauer in [1] or theorem 5, $BF(X, \Gamma_X) \neq 0$ for $\Gamma_X = \Gamma_{X'}\#\Gamma_{X'}$. Then $BF_{tot}(Z, id\#\rho, \Gamma_Z) \neq 0$ for $Z = X\#N$ and $\Gamma_Z = \Gamma_X\#\Gamma_N$. The space of positive scalar curvature metrics on Z, $\mathbf{PSC}(Z)$, is not empty by [14]. So it follows easily from $BF_{tot}(Z, id\#\rho, \Gamma_Z) \neq 0$ that $\mathbf{PSC}(Z)$ is disconnected. Though $Z \cong m\mathbb{CP}^2\#n\overline{\mathbb{CP}}^2$ and the same statement was proved by Ruberman (see Chapter 7 or [20]), it is still worthwhile to point out that in Ruberman's work the manifold $Z \cong m\mathbb{CP}^2\#n\overline{\mathbb{CP}}^2$ with disconnected $\mathbf{PSC}(Z)$ will have even m. But here we see examples with odd m.

Chapter 11

The proof of Theorem 33

To prove Theorem 33, we need to use the gluing technique in [1].

We first assume the manifolds satisfy $b_1 = 0$ for simplicity.

Let X be the disjoint union of n closed connected Riemannian 4-manifolds X_i. On each component, there is a long neck $N(L)_i = [-L, L] \times S^3$ which divides X_i as $X_i = X_i^- \cup X_i^+$, where $X_i^\pm$ are closed submanifolds with common boundary $0 \times S^3$. The length $2L$ is a large number. For an even permutation $\epsilon \in A_n$, let X^ϵ be the manifold obtained from X by interchanging the positive parts of its components. It has components $X_i^\epsilon = X_i^- \cup X_{\epsilon(i)}^+$. The homology orientation of X, i.e. the orientation of $H^+(X;\mathbb{R})$ induces an orientation of $H^+(X^\epsilon;\mathbb{R})$, i.e. the homology orientation of X^ϵ. Assume f is a diffeomorphism which acts on the long necks as identity, then it induces a diffeomorphism f^ϵ on X^ϵ. A $Spin^c$ structure Γ on X also induces a $Spin^c$ structure Γ^ϵ on X^ϵ. Assume the orbit of $Spin^c$ structures $\mathcal{O}(f,\Gamma)$ is infinite, then the same is true for $\mathcal{O}(f^\epsilon,\Gamma^\epsilon)$. We want to prove the following product formula.

Theorem 34 *For any even permutation ϵ,*

$$BF_{tot}(X, f, \Gamma) = BF_{tot}(X^\epsilon, f^\epsilon, \Gamma^\epsilon).$$

Let g be a Riemannian metric on X whose restrictions to the long necks are the standard product metrics $dx^2 + g_{S^3}$ where g_{S^3} is the unit sphere metric on S^3. It induces a metric g^ϵ on X^ϵ. Let g_t, $t \in I$ be a path of metrics from g to f^*g which satisfy the same property as g on the long necks. Then g_t^ϵ, $t \in I$ will be a path from g^ϵ to $f^{\epsilon*}g^\epsilon$. Therefore by the actions of f and f^ϵ respectively, they generate the paths of metrics g_t and g_t^ϵ, $t \in \mathbb{R}$ respectively, which satisfy the same property as g on the long neck.

The Hilbert bundle $\mathcal{A}_\mathbb{R}$ and $\mathcal{C}_\mathbb{R}$ can be defined as the union of

$$\mathcal{A}_t = L^2_{k-1}(\Gamma(W^+) \oplus i\Omega^1(X))$$

and

$$\mathcal{C}_t = L^2_{k-2}(\Gamma(W^-) \oplus i\Omega^+(X) \oplus i\Omega^0(X)/\mathbb{R})$$

respectively, with $k > 3$ fixed. The Seiberg–Witten map for each t is

$$sw_t(\phi, a) = (D_{A_0+a}\phi, F^+_{A_0+a} - \sigma(\phi), d^*a).$$

For each t, the terms above are defined using the metric g_t. The fixed smooth base connection A_0 is assumed to be flat on the long neck.

There is a path $\psi^\epsilon : I \longrightarrow SO(n)$ such that $\psi^\epsilon(0) = id$ and $\psi^\epsilon(1) = \epsilon$, which is identified with the permutation matrix $(\delta_{i,\epsilon(j)}) \in SO(n)$. Choose a smooth function $\gamma : [-L, L] \times S^3 \longrightarrow [0,1]$, which is constant on each spherical slice, vanishes on $[-L,-1] \times S^3$ and is identical to 1 on $[1, L] \times S^3$.

For a section e of a bundle E over X, denote by e_i its restriction to the bundle $E_i|_{X_i}$. Assume the restrictions of E_i to the long necks are identified with a bundle F. Then on the long neck, e can be identified as the section $(e_1, e_2, ..., e_n)^T$ in $\oplus_{i=1}^n F$. Using this identification, $E|_{X_i^\pm}$ gives a bundle E^ϵ over X^ϵ. A smooth section e of E will induce smooth section e^ϵ of E^ϵ such that restricted to the long neck e^ϵ can be identified as $(\psi^\epsilon \circ \gamma) \cdot e$. Applying this gluing construction to forms a and spinors ϕ, and using the path of metric g_t, we get bundle isomorphisms $V : \mathcal{A}_\mathbb{R} \longrightarrow \mathcal{A}_\mathbb{R}^\epsilon$ and $V : \mathcal{C}_\mathbb{R} \longrightarrow \mathcal{C}_\mathbb{R}^\epsilon$, where $\mathcal{A}_\mathbb{R}^\epsilon$ and $\mathcal{C}_\mathbb{R}^\epsilon$ denote the corresponding $\mathcal{A}_\mathbb{R}$ and $\mathcal{C}_\mathbb{R}$ for X^ϵ. Denote the 1-parameter Seiberg-Witten map for X^ϵ by $sw_\mathbb{R}^\epsilon$, Then we have a diagram of bundle maps:

$$\begin{array}{ccc} \mathcal{A}_\mathbb{R} & \xrightarrow{sw_\mathbb{R}} & \mathcal{C}_\mathbb{R} \\ \downarrow & & \downarrow \\ \mathcal{A}_\mathbb{R}^\epsilon & \xrightarrow{sw_\mathbb{R}^\epsilon} & \mathcal{C}_\mathbb{R}^\epsilon \end{array}$$

in which the vertical maps are V. This diagram does not commute. As Bauer did in [1], we need to show it commutes up to homotopy within Fredholm maps, i.e. there is a S^1-homotopy H_s from $sw_\mathbb{R}$ to $V^{-1} \circ sw_\mathbb{R}^\epsilon \circ V$, such that $H_s^{-1}(0)$ is empty if $t >> 0$ or $t << 0$. As in [1] we cannot assume the homotopy H_s satisfies the strong boundedness condition, but if the long neck is stretched long enough, it satisfies a weaker one, i.e. there exist constants $U > 0$ and $S > 0$, such that if $(\phi, a) \in H_s^{-1}(0)$ for any s, $|\phi|_{C^0}^2 \leq 2S$ and $|a|_{C^0} < 2U$, then $|\phi|_{C^0}^2 \leq S$ and $|a|_{C^0} < U$. The homotopy H_s and the constants S and U are constructed by Bauer for each fixed metric. Naturally H_s can be defined for a path of metrics instead of a fixed one. We can see from [1] or the arguement below that S can be chosen independent of g_t, $t \in \mathbb{R}$. But the constant U depends on the C^0-norm of $F^+_{A_0}$, thus it depends on g_t (i.e. it depends on t). Similarly the minimal length of the long neck to be stretched may also depend on t. So it seems important that the boundary condition for the homotopy H_s related to $t >> 0$ and $t << 0$ is independent of the length of the long necks. To be precise, we will prove there is a constant $M > 0$ independent of the length of long neck such that if $|t| \geq M$ then $H_s^{-1}(0)$ is empty for all s. Therefore we need only to consider the homotopy restricted to $|t| \leq M$. Because the metrics change in a compact family, U can be chosen for all metrics g_t with $|t| \leq M$ and the long necks can be stretched uniformly. By Lemma 30, H_s with $|t| \leq M$ gives the homotopy from $sw_{\mathbb{R}}$ to $V^{-1} \circ sw^{\epsilon}_{\mathbb{R}} \circ V$ which satisfies the weaker boundedness condition.

Now we will describe the three step homotopy by Bauer for all g_t, $t \in \mathbb{R}$.

As the preparation for constructing the homotopy, we introduce a cutoff function ρ_R on X. It vanishes on the middle part $N(R-1)$ of the long neck. Outside $N(R)$ it is equal to 1. On the remaining part, it is constant on each sphere $\{r\} \times S^3$. Define the homotopy $\rho_{R,s} = (1-s)+s\rho_R$ between the constant map 1 and ρ_R. The function ρ_R and the homotopy $\rho_{R,s}$ naturally define their counterparts on X^{ϵ}.

Step 1. Consider a homotopy $\mu_s : \mathcal{A}_t \longrightarrow \mathcal{C}_t$ defined by

$$\mu_s(\phi, a) = (D_{A_0+a}\phi, F^+_{A_0+a} - \rho_{L,s}\sigma(\phi), d^*a),$$

with $s \in [0,1]$.

Lemma 35 *There exists a constant M which is independent of L, such that if $s \in [0,1]$ and $|t| \geq M$, $\mu_s^{-1}(0) = \emptyset$.*

Proof. To prove this lemma, we only need an estimate

$$\int_X |F^{+2}_{A_0+a}| < C$$

which is independent of t and L. Here $(\phi, a) \in \mu_s^{-1}(0)$ for any $s \in I$ and $t \in \mathbb{R}$. The reason is that we can use the actions of f^{*n}, $n \in \mathbb{Z}$ to make $t \in I$, at the same time change Γ to $f^{*n}\Gamma$. If the estimate is true, then for all n such that $\mu_s^{-1}(0)$ is not empty, $c_1^{+2}(f^{*n}\Gamma) < C/4\pi^2$ for some g_t, $t \in I$. Because $\mathcal{O}(f,\Gamma)$ is infinite and the metrics g_t change in a compact set, there are only a finite number of n's satisfying this estimate. Thus the constant $M > 0$ can be found such that $\mu_s^{-1}(0)$ is empty if $|t| > M$ and $s \in [0,1]$. The change of metric that corresponds to changing the length of the long neck can be realized by the action of a diffeomorphism which is isotopic to the identity map. Thus for each $t \in \mathbb{R}$, the subspace of $H^2(X;\mathbb{R})$ generated by all the harmonic self-dual 2-forms for the metric g_t is independent of L. For a similar reason, the actions of f^* on the cohomology groups are independent of L. If the constant C is independent of L (i.e. independent of the length of the long necks), then M, which only depends on f^*, C and g_t, $t \in I$, is independent of L.

We first look for a C^0 bound for $|\phi|^2$. By the Weitzenböck formula, at the maximum points for $|\phi^2|$

$$s_{g_t}|\phi|^2 + \rho_{L,s}|\phi|^4 \leq 0.$$

If $\rho_{L,s} < 1$ at a maximum point for $|\phi|^2$, then the maximum point is on the long neck, where the scalar curvature is positive. Then $|\phi^2|$ has to vanish at that point and therefore vanish everywhere. If $\rho_{L,s} = 1$ at a maximum point, we get the estimation

$$|\phi|^2 \leq S \overset{def}{=} \max(-s_{g_t}(x), 0).$$

On the complement of the long neck $N(L-1)$,

$$\int_{X \setminus N(L-1)} |F^{+2}_{A_0+a}| < C_1 |\phi|^4_{C^0} \leq C_1 S^2.$$

Here S is a constant satisfying $S \geq \max(-s_{g_t}, 0)$ for all the metrics g_t, $0 \leq t \leq 1$. We can use the action of f to change the $Spin^c$ structure and make $t \in I$. Then C_1 can be chosen independent of the metric for all $t \in I$ and the $Spin^c$ structure. Therefore C_1 is independent of $t \in \mathbb{R}$. On the long neck, $|\phi|^2$ satisfies the partial differential inequality,

$$\Delta|\phi|^2 + \frac{s'|\phi|^2}{2} \leq 0,$$

where $s' > 0$ is the scalar curvature of standard unit 3-sphere. Then $|\phi|^2$ is bounded above by a function which decays exponentially towards the middle of the long neck (for simplicity, we will just say $|\phi|^2$ exponentially decays towards the middle of the long neck.) Because $|\phi|^2 \leq S$ and $F^+_{A_0+a} = \rho_{L,s}\sigma(\phi)$, $|F^{+2}_{A_0+a}|$ also decays exponentially towards the middle of the long neck and its integration on $N(L)$ is bounded by $C_2 S^2$, where C_2 is a constant only dependent of s'. Combine the two estimates on the long neck and its complement, we get

$$\int_X F^{+2}_{A_0+a} < C = (C_1 + C_2)S^2,$$

which is independent of t and L. ∎

Step 2. For $s \in [0, 1]$, define

$$\mu_{s+1}(\phi, a) = (D_{A_0+\rho_{2,s}a}\phi, F^+_{A_0+a} - \rho_L\sigma(\phi), d^*a).$$

Similar to the Lemma 35, we have

Lemma 36 *There exists a constant $M > 0$ which is independent of L such that if $s \in [0, 1]$ and $|t| \geq M$,*

$$\mu^{-1}_{s+1}(0) \cap \{|\phi|^2_{C^0} \leq 2S\} = \emptyset.$$

Proof. Similar to the proof of the Lemma 35, we need an estimate

$$\int_X |F^{+2}_{A_0+a}| < C$$

for all $(\phi, a) \in \mu^{-1}_{s+1}(0) \cap \{|\phi|^2 \leq 2S\}$ which is independent of t and L. The estimate on the complement of the long neck $N(L-1)$ is the same as in the proof of Lemma 35. On the long neck $N(L-1)$, $F^+_{A_0+a} = 0$. ∎

Step 3. Denote μ_2 by P. Applying steps 1 and 2 to X^ϵ, we can define a homotopy with similar properties, which will be denoted by μ^ϵ_s, $s \in [0, 2]$. Denote μ^ϵ_2 by P^ϵ. The third homotopy is between P and $V^{-1}P^\epsilon V$. They differ only on the short neck $[-1,1]\times S^3$ by a multiplication operator,

$$V^{-1}P^{\epsilon}V = P + d\log V.$$

For $s\in[0,1]$, multiplication of spinors or forms on the long neck with the matrix valued function $\psi^{\epsilon}\circ s\gamma : [-L,L]\times S^3 \to SO(n)$ defines a map V_s on the long neck. The operator $V_s^{-1}P^{\epsilon}V_s = P + d\log V_s$ can be extended to the complement of the long neck to be the same as P. This family of extended operators gives the homotopy μ_{s+2} we need between P and $V^{-1}P^{\epsilon}V$.

Lemma 37 *There is a constant $M>0$ satisfying the following properties. For any $U>0$ there is a $L_0>0$ such that when $L>L_0$, $s\in[0,1]$ and $|t|\ge M$,*

$$\mu_{s+2}^{-1}(0)\cap\{|\phi|^2_{C^0}\le 2S, |a|_{C^0}\le 2U\} = \emptyset.$$

Proof. Let $(\phi,a)\in\mu_{s+2}^{-1}(0)$, with $s\in[0,1]$, such that $|\phi|^2\le 2S$ and $|a|\le 2U$. Then on the neck $N(L-1)$, the form a satisfies $V_s^{-1}a$ is harmonic. Let a_{sp} be the sphere direction component of a, then $|a_{sp}|=|(V_s^{-1}a)_{sp}|$. From [1], $|(V_s^{-1}a)_{sp}|$ decays exponentially towards the middle of neck. So there is a $L_0>0$ such that when $L>L_0$, the C^0-norm of restriction of a_{sp} on the middle short neck is smaller than 1. The constant L_0 only depends on U. On the middle short neck $N(1)$, $V_s^{-1}a$ is harmonic, then we know

$$d(V_s^{-1}a) = \frac{d}{dx}(V_s^{-1})dx\wedge a + V_s^{-1}da = 0.$$

Therefore

$$da = -V_s(\frac{d}{dx}V_s^{-1}dx\wedge a),$$

and

$$d^+a = -(V_s(\frac{d}{dx}V_s^{-1}dx \wedge a))^+.$$

Because $dx \wedge a = dx \wedge a_{sp}$, and we can assume the norm of dV_s^{-1}/dx is bounded by a constant $C_3 > 0$ which only depends on the definition of V_s, we get a C^0-bound for $F^+_{A_0+a} = d^+a$ on $N(1)$, which is independent of L and L_0. On $N(L-1)\backslash N(1)$, $F^+_{A_0+a}$ vanishes as in the step 2. and on the complement of $N(L-1)$, we can get an estimate similar to that in the proof of Lemma 35. Combine these together, we get the estimate $\int_X |F^{+2}_{A_0+a}| < C$ which is independent of t, L and L_0. ■

Proof. (Proof of Theorem 34 with $b_1 = 0$.) We can construct the S^1-homotopy H_s from $sw_{\mathbb{R}}$ to $V^{-1} \circ sw^\epsilon_{\mathbb{R}} \circ V$ by μ_s, $0 \leq s \leq 3$, from sw_I to $V^{-1}P^\epsilon V$, and then $V^{-1}\mu^\epsilon_{5-s}V$, $3 \leq s \leq 5$, from $V^{-1}P^\epsilon V$ to $V^{-1} \circ sw^\epsilon_{\mathbb{R}} \circ V$. We can choose a $M >> 0$ which satisfies Lemma 35, 36 and 37 respectively for all $s \in [0,5]$. Then the restriction $H_s|_{|t|\leq M}$ will satisfy the weaker boundedness condition, i.e. if L is big enough, then there are constants $U > 0$ and $S > 0$ such that if $(\phi, a) \in H_s|^{-1}_{|t|\leq M}(0)$, $s \in [0,5]$, $|\phi|^2 \leq 2S$ and $|a| \leq 2U$, then $|\phi|^2 \leq S$ and $|a| \leq U$. For the constant M we choose, H_s also satisfies the boundary condition, i.e.

$$H_s|^{-1}_{|t|=M}(0) \cap \{|\phi|^2_{C^0} \leq 2S, |a|_{C^0} \leq 2U\} = \emptyset,$$

for all $s \in [0,5]$ when $L >> 0$. So $H_s|_{|t|\leq M}$ defines a S^1-equivariant stable cohomotopy class which is identified to its restrictions to $s = 0$ and 5, $BF_{tot}(X, f, \Gamma_X)$ and $BF^\epsilon_{tot}(X^\epsilon, f, \Gamma_{X^\epsilon})$ respectively. ■

If the manifold X has a positive b_1, we have a family of base connections

$A_0 + \ker d$ together with a pointed gauge action by $\mathcal{G}_0$. But this is not compatible with the trivialization on the long neck. So we can choose a subspace $\mathcal{K}$ of $\ker d$, which is the subspace of the forms vanishing on the long necks. At the same time choose a subgroup $\mathcal{G}'$ of $\mathcal{G}_0$, which is the subgroup of all elements that takes value 1 on the long necks. We still have $(A_0 + \mathcal{K})/\mathcal{G}' \cong Pic^0(X)$ so we can use $\mathcal{K}$ and $\mathcal{G}'$ instead of $\ker d$ and $\mathcal{G}_0$ to define $\mathcal{A}_I$ and $\mathcal{C}_I$. Going through the steps above, the proof can be completed without any essential change.

Lemma 38 *Assume $X = X_1 \cup X_2$ as disconnected sum with a $Spin^c$ structure $\Gamma = \Gamma_1 \cup \Gamma_2$ which is induced by Γ_i on X_i. If f is a diffeomorphism of X_1, such that $\mathcal{O}(f, \Gamma_1)$ is infinite, then we have*

$$BF_{tot}(X, f \cup id, \Gamma) = BF_{tot}(X_1, f, \Gamma_1) \wedge BF(X_2, \Gamma_2).$$

Proof. Take a path of metrics g_t, $t \in \mathbb{R}$, on X_1, which repeats the part with $0 \le t \le 1$ between g_0 and $f^* g_0 = g_1$, by the action of powers of f^*. Take a constant path $g'_t = g'$ on X_2. Then the 1-parameter Seiberg–Witten map is defined as a product,

$$\mathcal{A}_I \times \mathcal{A}' \stackrel{sw_{\mathbb{R}} \times sw}{\longrightarrow} \mathcal{C}_I \times \mathcal{C}',$$

in which sw is the Seiberg–Witten map for X_2 with $Spin^c$ structure Γ_2, and it is independent of t and $sw_{\mathbb{R}}$ is the 1-parameter Seiberg–Witten map on X_1. The 1-parameter Seiberg–Witten map on X, followed by the projection to the fiber, naturally split as the direct sum of the $sw_{\mathbb{R}}$ on X_1 and the Seiberg–Witten map sw on X_2, followed by the projection to the fiber. Choose the finite dimensional S^1-subspace V as required by the definition which is compatible with this splitting, then the result follows the definition directly. ■

Proof. (for Theorem 33) Take $X_1 = X = X^- \cup D^4$, $X_2 = N = D^4 \cup N^+$, $X_3 = X_4 = S^4 = D^4 \cup D^4$. The permutation ϵ is given by exchanging D^4 in X_1 with N^+ and exchanging the the D^4's in X^3 and X^4. The path of metrics g_t is chosen such that each one is the standard product metrics along the long necks and coincides everywhere except on N^+. By the last lemma,

$$\begin{aligned} & BF_{tot}(\cup_{i=1}^4 X_i, id \cup \rho \cup id \cup id, \Gamma_X \cup \Gamma_N \cup \Gamma_{S^4} \cup \Gamma_{S^4}) \\ = \; & BF(X, \Gamma_X) \wedge BF_{tot}(N, \rho, \Gamma_N) \wedge BF(S^4, \Gamma_{S^4}) \wedge BF(S^4, \Gamma_{S^4}). \end{aligned}$$

By Lemma 38 and lemma 3.8 in [3], the last three terms in the last equality are $[id]$. So the left side is equal to $BF(X, \Gamma_X)$. On the other hand

$$\begin{aligned} & BF_{tot}((\cup_{i=1}^4 X_i)^\epsilon, (id \cup \rho \cup id \cup id)^\epsilon, (\Gamma_X \cup \Gamma_N \cup \Gamma_{S^4} \cup \Gamma_{S^4})^\epsilon) \\ = \; & BF_{tot}((\cup_{i=1}^4 X_i)^\epsilon, (\rho \# id) \cup id \cup id \cup id, (\Gamma_N \# \Gamma_X) \cup \Gamma_{S^4} \cup \Gamma_{S^4} \cup \Gamma_{S^4}) \\ = \; & BF_{tot}(N \# X, \rho \# id, \Gamma_N \# \Gamma_X) \wedge BF(S^4, \Gamma_{S^4})^{\wedge 3} \\ = \; & BF_{tot}(N \# X, \rho \# id, \Gamma_N \# \Gamma_X). \end{aligned}$$

By Theorem 34, $BF_{tot}(N \# X, \rho \# id, \Gamma_N \# \Gamma_X) = BF(X, \Gamma_X)$. ■

Chapter 12

The proof of Theorem 2

At the end we will prove Theorem 2. Let $X = T^4 \# X'$ where X' is an algebraic hypersurface of general type in $\mathbb{CP}^3$ with the $Spin^c$ structure $\Gamma_{X'}$ induced by the complex structure. Assume the degree d of X' can be divided by 4, i.e.

$$b_+(X') = \frac{d^3 - 6d^2 + 11d}{3} - 1 \equiv 3 \text{ (mod 4)}.$$

Let Γ_{T^4} be the $Spin$ structure on T^4. Then by theorem 12, for $\Gamma_X = \Gamma_{T^4} \# \Gamma_{X'}$, $BF(X, \Gamma_X) \neq 0$. By Theorem 33, $BF_{tot}(X \# N, id \# \rho, \Gamma_X \# \Gamma_N) = BF(X, \Gamma_X)$.

Fix constant $\varepsilon > 0$. Then the set $Met_\varepsilon(X \# N)$ is not empty because by [14] $X \# N \cong T^4 \# m\mathbb{CP}^2 \# n\overline{\mathbb{CP}}^2$ has nonnegative Yamabe invariant. So we can find a metric g with unit volume and scalar curvature $s_g > -\varepsilon$. Using the diffeomorphism $id \# \rho$, we can construct a path of metrics g_t, $t \in \mathbb{R}$. If $Met_\varepsilon(X \# N)$ is connected, then we can choose the path g_t inside $Met_\varepsilon(X \# N)$. The 1-parameter Seiberg–Witten moduli space is nonempty because $BF_{tot}(X \# N, id \# \rho, \Gamma_X \# \Gamma_N)$ is not zero. (This can be proved applying the argument at the beginning of the proof of Lemma 29.) Assume for g_{t_0}, the Seiberg–Witten moduli space is not empty. Then we have the estimate for the scalar curvature $s = s_{g_{t_0}}$,

$$\int s_-^2 d\mu_{g_{t_0}} \geq 32\pi^2 c_1^2(\Gamma_X \# \Gamma_N) = 32\pi^2 (c_1^2(\Gamma_{X'}) - 1),$$

where $s_- = \min(s, 0)$. Because $g_{t_0} \in Met_\varepsilon(X \# N)$,

$$\varepsilon^2 \geq \int s_-^2 d\mu_{g_{t_0}} \geq 32\pi^2 c_1^2(\Gamma_X \# \Gamma_N) = 32\pi^2 (c_1^2(\Gamma_{X'}) - 1),$$

and then

$$c_1^2(\Gamma_{X'}) \leq \frac{1}{32\pi^2}\varepsilon^2 + 1.$$

It is always possible to find general type hypersurface in $\mathbb{CP}^3$ with degree $d >> 0$ such that $c_1^2 = (4-d)^2 d > 1 + \varepsilon^2/32\pi^2$. This means for those X', the corresponding $T^4 \# X' \# N \cong T^4 \# m\mathbb{CP}^2 \# n\overline{\mathbb{CP}}^2$ will have disconnected $Met_\varepsilon(X \# N)$.

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