QUANTUM GAUGE THEORY

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<u>Principal Fiber Bundles</u> Let

$$\begin{array}{ccc} G & \longrightarrow & P \\ & & \Big\downarrow \pi \\ & & M \end{array}$$

be a principal bundle with structure group G, which we shall take to be a Lie group. Therefore P is a free right G-space:

$$\begin{cases} P \times G \longrightarrow P \\ (p, \sigma) \longrightarrow p \cdot \sigma = R_\sigma(p) \end{cases}$$

with

$$M \approx P/G.$$

Moreover, π is a submersion and $\pi(p_1) = \pi(p_2)$ iff $\exists\, \sigma \in G : p_1 \cdot \sigma = p_2$. Finally, there is an open cover $\{U_i\}$ of M such that $\forall i$, $P|U_i$ is equivariantly diffeomorphic to $U_i \times G$ over U_i:

$$\begin{tikzcd} P|U_i \arrow[rr, "\Phi_i"] \arrow[dr, "\pi"'] & & U_i \times G \arrow[dl, "pr_1"] \\ & U_i & \end{tikzcd}$$

$$\begin{cases} \Phi_i(p) = (\pi(p), \phi_i(p)) \\ \qquad \phi_i(p \cdot \sigma) = \phi_i(p) \cdot \sigma \quad . \end{cases}$$

Definition: A <u>local trivialization</u> is an open set $U \subset M$ and a diffeomorphism

$$\begin{tikzcd} P|U \arrow[rr, "\Phi"] \arrow[dr, "\pi"'] & & U \times G \arrow[dl, "pr_1"] \\ & U & \end{tikzcd}$$

$$\begin{cases} \Phi(p) = (\pi(p), \phi(p)) \\ \qquad \phi(p \cdot \sigma) = \phi(p) \cdot \sigma \ . \end{cases}$$

Observation: Fix U -- then there is a one-to-one correspondence between the Φ and the sections s over U.

[Given Φ, define s by $s(x) = \Phi^{-1}(x,e)$. Given s, define Φ by $\Phi(s(x) \cdot \sigma) = (x, \sigma)$.]

LEMMA A principal G-bundle is trivial iff it admits a global section.

Rappel: There is an injective morphism of Lie algebras

$$\begin{cases} \underline{g} \longrightarrow \mathcal{D}^1(P) \\ X \longrightarrow \overline{X} \end{cases}$$

with the property that

$$(R_\sigma)_* \overline{X} = \overline{\mathrm{Ad}(\sigma^{-1}) X}.$$

Given $p \in P$, denote by $T_p^v(P)$ the <u>vertical subspace</u> of $T_p(P)$:

$$T_p^v(P) = \{ T \in T_p(P) : d\pi_p(T) = 0 \}.$$

<u>FACT</u> $\forall\ X \in \underline{g}$, $\overline{X}_p \in T_p^v(P)$ and the arrow

$$\begin{cases} \underline{g} \longrightarrow T_p^v(P) \\ X \longrightarrow \overline{X}_p \end{cases}$$

is a linear isomorphism.

Suppose that F is a left G-space -- then the prescription

$$(p,x)\cdot\sigma = (p\cdot\sigma, \sigma^{-1}\cdot x)$$

defines a right action of G on $P \times F$. Put

$$P \times_G F = (P \times F)/G.$$

Then there is a commutative diagram

$$\begin{array}{ccc} P \times F & \xrightarrow{pr_1} & P \\ {\scriptstyle pro}\downarrow & & \downarrow{\scriptstyle \pi} \\ P \times_G F & \xrightarrow[\pi_P]{} & M. \end{array}$$

Here

$$\pi_P([p,x]) = \pi(p)\ ([p,x] = pro(p,x)).$$

[Note: $\forall\ p \in P$, the map

$$\zeta_p : F \longrightarrow (P \times_G F)_{\pi(p)}$$

defined by

$$x \longrightarrow [p,x]$$

is a diffeomorphism with the property that

$$\zeta_{p\cdot\sigma}(x) = \zeta_p(\sigma\cdot x).]$$

Definition: $(P \times_G F,\ M,\ \pi_P, F)$ is the fiber bundle associated with

$$\begin{tikzcd} G \arrow[r] & P \arrow[d, "\pi"] \\ & M. \end{tikzcd}$$

Let

$$\mathrm{map}_G(P,F)$$

be the set of G-equivariant maps

$$f: P \longrightarrow F,$$

so $\forall \sigma \in G$,

$$f(p \cdot \sigma) = \sigma^{-1} \cdot f(p).$$

LEMMA There is a one-to-one correspondence

$$\mathrm{map}_G(P,F) \longrightarrow \sec(P \times_G F).$$

[Assign to $f \in \mathrm{map}_G(P,F)$ the section s_f of $P \times_G F$ defined by

$$s_f(x) = [p, f(p)] \quad (p \in \pi^{-1}(x)).$$

In the other direction, assign to $s \in \sec(P \times_G F)$ the map $f_s: P \longrightarrow F$ defined by

$$f_s(p) = \xi_p^{-1}(s(\pi(p))),$$

the claim being that

$$f_s(p \cdot \sigma) = \sigma^{-1} \cdot f_s(p) \qquad \forall \sigma \in G.$$

First, $\forall x \in F$,

$$x = \zeta_{p\cdot\sigma}^{-1}\ (\zeta_{p\cdot\sigma}\ (x)) = \zeta_{p\cdot\sigma}^{-1}\ (\zeta_p(\sigma\cdot x))$$

$$\Longrightarrow$$

$$\sigma^{-1}\cdot x \ = \zeta_{p\cdot\sigma}^{-1}\ (\zeta_p(x)).$$

Now specialize and take $x = f_s(p)$ -- then

$$\begin{aligned} f_s(p\cdot\sigma) &= \zeta_{p\cdot\sigma}^{-1}\ (s(\pi(p\cdot\sigma))) \\ &= \zeta_{p\cdot\sigma}^{-1}\ (s(\pi(p))) \\ &= \zeta_{p\cdot\sigma}^{-1}\ (\zeta_p(x)) \\ &= \sigma^{-1}\cdot x \\ &= \sigma^{-1}\cdot\ \zeta_p^{-1}(s(\pi(p))) \\ &= \sigma^{-1}\cdot f_s(p).] \end{aligned}$$

Example: Take F=G and let the action be Int -- then

$$G^P = P \times_G G$$

is the bundle of Lie groups associated with P.

Example: Take $F=\underline{g}$ and let the action be Ad -- then

$$\underline{g}^P = P \times_G \underline{g}$$

is the bundle of Lie algebras associated with P.

Suppose that $E \longrightarrow M$ is a vector bundle -- then the sections of

$$E \otimes \Lambda^k T^*M$$

are the k-forms on M with values in E.

Notation: Put

$$\Lambda^k(M;E) = \sec(E \otimes \Lambda^k T^*M).$$

[Note: Conventionally,

$$\Lambda^0(M;E) = \sec(E).]$$

So, for $k \geq 1$, a given $\omega \in \Lambda^k(M;E)$ can be viewed at each $x \in M$ as a multilinear antisymmetric map $\omega_x : T_x(M) \times \cdots \times T_x(M) \longrightarrow E_x$.

Structurally,

$$\Lambda^k(M;E) \approx \Lambda^0(M;E) \otimes_{C^\infty(M)} \Lambda^k(M),$$

where

$$(s \otimes \omega)_x (X_1, \ldots, X_k) = \omega_x(X_1, \ldots, X_k)s(x).$$

Remark: If E is a trivial vector bundle with fiber V, then $\Lambda^k(M;E)$ is the space of k-forms on M with values in V and is denoted by $\Lambda^k(M;V)$.

Let ρ be a representation of G on a finite dimensional vector space V -- then a k-form

$$\omega \in \Lambda^k(P;V)$$

is said to be of <u>type ρ</u> if

$$(R_\sigma)^* \omega = \rho(\sigma^{-1})\omega \quad \forall\, \sigma \in G$$

and

$$\omega(T_1,\ldots,T_k) = 0$$

whenever one of the T_i is vertical.

Notation: Write

$$\Lambda^k_\rho(P;V)$$

for the space of k-forms of type ρ and let E be the vector bundle

$$P \times_G V.$$

LEMMA There is a one-to-one correspondence

$$\Lambda^k_\rho(P;V) \longrightarrow \Lambda^k(M;E).$$

[The element $s_\omega \in \Lambda^k(M;E)$ corresponding to $\omega \in \Lambda^k_\rho(P;V)$ is defined by the prescription

$$s_\omega \Big|_x (X_1,\ldots,X_k) = \zeta_p(\omega\big|_p(T_1,\ldots T_k)) \quad (p \in \pi^{-1}(x)),$$

where the $T_i \in T_p(P)$ are such that $d\pi_p(T_i) = X_i$ $(1 \leq i \leq k)$.]

<u>Classification</u>   Suppose given

$$\begin{array}{ccc} G & \longrightarrow & P \\ & & \big\downarrow \pi \\ & & M\,. \end{array}$$

THEOREM Assume that M is contractible -- then P is trivial:

$$P \approx M \times G.$$

In particular: Principal G-bundles over $\mathbf{R}^n$, $[0,1]^n$, $\mathbf{B}^n$ and $\mathbf{D}^n$ are trivial.

THEOREM Take $M=\mathbf{S}^n$ and G path connected -- then the set of isomorphism classes of principal G-bundles over M is in a one-to-one correspondence with the elements of $\pi_{n-1}(G)$.

In particular: If G is path connected, then every principal G-bundle over $\mathbf{S}^1$ is trivial.

THEOREM Suppose that G and M are path connected. Assume:

$$\pi_q(G) = 0 \quad (q < \dim M).$$

Then every principal G-bundle over M is trivial.

[Note: This is ordinarily proved in a more general context, viz. when M is a CW complex. In our situation, M is a C^∞ manifold, thus M can be triangulated, hence carries a CW structure.]

Example: Let $G=\underset{\sim}{SU}(2)$ and assume that M is path connected with dim M=3 -- then every principal G-bundle over M is trivial.

[This is because $\pi_q(\underset{\sim}{SU}(2)) = 0$ $(q=0, 1, 2)$.]

<u>Connections</u> Suppose given

$$\begin{array}{ccc} G & \longrightarrow & P \\ & & \big\downarrow \pi \\ & & M. \end{array}$$

Then a <u>connection</u> Γ is a G-invariant distribution on P which projects isomorphically onto TM. In other words, Γ consists in the smooth assignment

$$p \longrightarrow T_p^h(P) \subset T_p(P)$$

of subspaces, said to be <u>horizontal</u>, satisfying:

(1) $T_p(P) = T_p^v(P) \oplus T_p^h(P)$;

(2) $dR_\sigma(T_p^h(P)) = T_{p\cdot\sigma}^h(P)$.

Remark: There is a short exact sequence

$$0 \longrightarrow T_p^v(P) \longrightarrow T_p(P) \longrightarrow T_{\pi(p)}(M) \longrightarrow 0,$$

hence

$$T_p^h(P) \approx T_{\pi(p)}(M).$$

<u>LEMMA</u> If $X \in \underline{g}$ and $T \in \mathcal{D}^1(P)$ is horizontal, then

$$[\tilde{X}, T]$$

is horizontal.

A connection Γ gives rise to a 1-form

$$\omega_\Gamma \in \Lambda^1(P;\underline{g}),$$

viz.:

$$\overline{X}_p \oplus Y \in T_p^v(P) \oplus T_p^h(P)$$

$$\longrightarrow \quad X \in \underline{g}.$$

Therefore $\omega_\Gamma(T)=0$ iff T is horizontal. And:

(1) $\omega_\Gamma(\overline{X})=X$;

(2) $(R_\sigma)^*\omega_\Gamma = \mathrm{Ad}(\sigma^{-1})\omega_\Gamma$.

[Note: Conversely, if

$$\omega : \mathcal{D}^1(P) \longrightarrow C^\infty(P;\underline{g})$$

satisfies these two conditions, then $\exists !\ \Gamma$ such that

$$\omega = \omega_\Gamma .$$

Indeed, the assignment

$$p \longrightarrow T_p^h(P) = \{ T \in T_p(P) : \omega_p(T)=0 \}$$

defines the connection Γ .]

<u>FACT</u> Every $X \in \mathcal{D}^1(M)$ admits a unique lifting X^h to a horizontal vector field on P such that $\pi_* X^h = X$.

[Note: $X^h \in \mathcal{D}^1(P)$ is invariant under the action of G and every horizontal vector field on P with this property is the lift of some vector field on M.]

Remark: Let ρ be a representation of G on a finite dimensional vector space V -- then in the presence of the connection Γ, the correspondence

$$\Lambda_\rho^k(P;V) \longrightarrow \Lambda^k(M;E)$$

which sends ω to s_ω is defined by the prescription

$$s_\omega\Big|_x(X_1,\ldots,X_k) = [p,\omega_p(X_1^h,\ldots,X_k^h)] \qquad (p\in\pi^{-1}(x)).$$

If Γ_1, Γ_2 are connections, then

$$\omega_{\Gamma_1} - \omega_{\Gamma_2} \in \Lambda_{\mathrm{Ad}}^1(P;\underline{g}).$$

Conversely, if Γ is a connection and if $\omega\in\Lambda_{\mathrm{Ad}}^1(P;\underline{g})$, then

$$\omega_\Gamma + \omega$$

determines a connection.

Notation: $\mathfrak{A}(P)$ is the set of connections.

Agreeing to identify Γ with ω_Γ, it follows that $\mathfrak{A}(P)$ is an affine space with translation group $\Lambda_{\mathrm{Ad}}^1(P;\underline{g})$. Indeed, the action

$$\omega_\Gamma\cdot\omega = \omega_\Gamma + \omega \qquad (\omega\in\Lambda_{\mathrm{Ad}}^1(P;\underline{g}))$$

is free and transitive. Since

$$\Lambda_{\mathrm{Ad}}^1(P;\underline{g}) \approx \Lambda^1(M;\underline{g}^P),$$

one can also say that $\mathfrak{A}(P)$ is an affine space with translation group $\Lambda^1(M;\underline{g}^P)$.

Example: Consider $P = M\times G$ -- then the assignment

$$(x,\sigma)\longrightarrow T^h_{(x,\sigma)}(M\times G) = T_x(M)$$

is a connection Γ. Let Θ be the canonical 1-form on G, i.e., Θ is the left invariant $\underline{g}$-valued 1-form on G characterized by the condition

$$\Theta_\sigma(X) = (dL_{\sigma^{-1}})_\sigma\ (X).$$

Then

$$\omega_\Gamma = pr_2^*(\Theta),$$

where

$$pr_2 : M \times G \longrightarrow G.$$

[Note: This particular connection on $M \times G$ is called the <u>standard</u> connection. If P is arbitrary and if $\Gamma \in \mathfrak{A}(P)$, then Γ is said to be <u>flat</u> if every $x \in M$ admits a trivializing neighborhood U such that $\Phi : \pi^{-1}(U) \longrightarrow U \times G$ sends the induced connection on $\pi^{-1}(U)$ to the standard connection on $U \times G$.]

insert 4.5

LOCAL CRITERION Let $\{U_i\}$ be a trivializing open cover of M. Suppose that $\forall j$, $\mathfrak{A}_j$ is a $\underline{g}$-valued 1-form on U_j such that whenever $U_j \cap U_i \neq \emptyset$,

$$\mathfrak{A}_j = \mathrm{Ad}(g_{ij}^{-1}) \circ \mathfrak{A}_i + \Theta_{ij}$$

on $U_j \cap U_i$, where $g_{ij} : U_j \cap U_i \longrightarrow G$ is the transition function and $\Theta_{ij} = g_{ij}^* \Theta$ -- then $\exists$ a unique connection Γ such that $\forall j$,

$$\mathfrak{A}_j = s_j^* \omega_\Gamma,$$

$s_j : U_j \longrightarrow \pi^{-1}(U_j)$ the section associated with the trivialization (U_j, Φ_j).

Let $\{U_i\}$ be a trivializing open cover of M -- then $\forall$ i, we have

$$\begin{tikzcd} P|U_i \arrow[rr, "\Phi_i"] \arrow[dr, "\pi"'] & & U_i \times G \arrow[dl, "pr_1"] \\ & U_i & \end{tikzcd}$$

$$\begin{cases} \Phi_i(p) = (\pi(p), \phi_i(p)) \\ \phi_i(p\cdot\sigma) = \phi_i(p)\cdot\sigma \end{cases}$$

and

$$\begin{cases} s_i : U_i \longrightarrow P|U_i \\ s_i(x) = \Phi_i^{-1}(x,e). \end{cases}$$

Suppose that $U_i \cap U_j \neq \emptyset$ -- then the function

$$g_{ji} : U_i \cap U_j \longrightarrow G$$

defined by the rule

$$g_{ji}(x) = \phi_j(p)(\phi_i(p))^{-1} \quad (p \in \pi^{-1}(x))$$

is called a <u>transition function</u>.

[Note: It follows from the definitions that

$$s_i(x) = s_j(x)\cdot g_{ji}(x).]$$

Properties:

$$g_{ii} = e,\ g_{ij} = (g_{ji})^{-1},\ g_{kj}g_{ji} = g_{ki}.$$

[By definition,

$$\Phi_j : \pi^{-1}(U_j) \longrightarrow U_j \times G.$$

Put

$$\omega_j = (\mathrm{pr}_1 \circ \Phi_j)^* \mathfrak{A}_j + (\mathrm{pr}_2 \circ \Phi_j)^* \Theta.$$

Then the element $\omega \in \Lambda^1(P;\underline{g})$ for which

$$\omega \mid \pi^{-1}(U_j) = \omega_j$$

determines the connection Γ.]

Application: Take $P = M \times G$ -- then for every $\underline{g}$-valued 1-form $\mathfrak{A}$ on M, there is a unique connection Γ such that

$$\mathfrak{A} = s^* \omega_\Gamma,$$

where $s(x) = (x,e) \quad (x \in M)$.

<u>Exterior Differentiation</u> Suppose given

$$\begin{array}{ccc} G & \longrightarrow & P \\ & & \downarrow \pi \\ & & M \end{array}$$

and let ρ be a representation of G on a finite dimensional vector space V. Fix an element $\Gamma \in \mathfrak{A}(P)$.

Definition: Put

$$d^{\Gamma}\omega = d\omega \circ h \quad (\omega \in \wedge^*(P;V)).$$

It is easy to show that

$$\omega \in \wedge^k_\rho(P;V) \Rightarrow d^{\Gamma}\omega \in \wedge^{k+1}_\rho(P;V).$$

Define now a bilinear map

$$\begin{cases} \underline{g} \times V \longrightarrow V \\ (A,v) \longrightarrow A\cdot v, \end{cases}$$

where

$$A\cdot v = \frac{d}{dt}\left(\rho(\exp(tA))(v)\right)\Big|_{t=0}.$$

Given

$$\begin{cases} \alpha \in \wedge^k(P;\underline{g}) \\ \beta \in \wedge^\ell(P;V), \end{cases}$$

let

$$\alpha \wedge_\rho \beta \in \wedge^{k+\ell}(P;V)$$

be defined at each point of P by

$$(\alpha \wedge_\rho \beta)(T_1, \ldots, T_{k+\ell})$$

$$= \frac{1}{k!\,\ell!} \sum_{\sigma \in S_{k+\ell}} (\operatorname{sgn} \sigma)\, \alpha(T_{\sigma(1)}, \ldots, T_{\sigma(k)})$$

$$\cdot\ \beta(T_{\sigma(k+1)}, \ldots, T_{\sigma(\ell)}).$$

E.g.: Take $V = \underline{g}$, $\rho = \mathrm{Ad}$ -- then

$$(\alpha \wedge_{\mathrm{Ad}} \beta)(T_1, \ldots, T_{k+\ell})$$

$$= \frac{1}{k!\,\ell!} \sum_{\sigma \in S_{k+\ell}} (\operatorname{sgn} \sigma)\, [\alpha(T_{\sigma(1)}, \ldots, T_{\sigma(k)}),$$

$$\beta(T_{\sigma(k+1)}, \ldots, T_{\sigma(k+\ell)})].$$

Specialized to the case when $\alpha = \beta = \omega$ and $k = \ell = 1$, we get

$$(\omega \wedge_{\mathrm{Ad}} \omega)(X,Y) = [\omega(X), \omega(Y)] - [\omega(Y), \omega(X)]$$

$$= 2\, [\omega(X), \omega(Y)].$$

Rappel: A graded Lie algebra over a commutative ring R with unit is a graded R-module $L = \bigoplus_{n \geq 0} L_n$ together with bilinear pairings $[\ ,\ ]: L_n \times L_m \longrightarrow L_{n+m}$ such that

$$[x,y] = (-1)^{|x||y|+1} [y,x]$$

and

$$(-1)^{|x||z|} [[x,y],z] + (-1)^{|y||x|} [[y,z],x] + (-1)^{|z||y|} [[z,x],y] = 0.$$

E.g.: Let $L = \Lambda^*(P;\underline{g})$ and $[\ ,\ ] = \Lambda_{Ad}$ -- then L is a graded Lie algebra.

<u>FACT</u> If $\alpha \in \Lambda^k(P;\underline{g})$, $\beta \in \Lambda^\ell(P;\underline{g})$, then

$$d(\alpha \wedge_{Ad} \beta) = d\alpha \wedge_{Ad} \beta + (-1)^k \alpha \wedge_{Ad} d\beta .$$

Returning to the general case, one has the following fundamental result.

<u>THEOREM</u> Let $\omega \in \Lambda^k_\rho(P;V)$ -- then

$$d^\Gamma \omega = d\omega + \omega_\Gamma \wedge_\rho \omega .$$

[Note: Written out, this says that for each $p \in P$ and all $T_1, \ldots, T_{k+1} \in T_p(P)$,

$$(d\omega)_p (hT_1, \ldots, hT_{k+1}) = (d\omega)_p (T_1, \ldots, T_{k+1})$$

$$+ \frac{1}{k!} \sum_{\sigma \in S_{k+1}} (\operatorname{sgn} \sigma)(\omega_\Gamma)_p(T_{\sigma(1)}) \cdot \omega_p(T_{\sigma(2)}, \ldots, T_{\sigma(k+1)}).]$$

Definition: A <u>matter field</u> is an equivariant map $\phi: P \longrightarrow V$.

[Note: This means that

$$\phi(p \cdot \sigma) = \rho(\sigma^{-1}) \phi(p) \equiv \sigma^{-1} \cdot \phi(p).]$$

E.g.: When $V = \underline{g}$ and $\rho = Ad$, ϕ is called a <u>Higgs field</u>.

Remark: Since

$$\operatorname{map}_G(P,V) \longleftrightarrow \sec(P \times_G V),$$

a matter field can also be viewed as a global section of the vector bundle $P \times_G V$.

Let $\phi : P \longrightarrow V$ be a matter field -- then $\phi \in \Lambda^0_\rho(P;V)$, hence by the theorem,

$$d^\Gamma \phi = d\phi + \omega_\Gamma \wedge_\rho \phi .$$

$$\equiv d\phi + \omega_\Gamma \cdot \phi .$$

Here

$$(\omega_\Gamma \wedge_\rho \phi)_p(T) = \omega_\Gamma(T) \cdot \phi(p).$$

Suppose that $s : U \longrightarrow \pi^{-1}(U)$ is a section -- then it is clear that

$$s^*(d^\Gamma \phi) = d(\phi \circ s) + s^* \omega_\Gamma \cdot (\phi \circ s).$$

Suppose in addition that $\varphi : U \longrightarrow \underset{\sim}{R}^n$ is a chart with coordinates $x^1, \ldots, x^n$ -- then still

$$(s \circ \varphi^{-1})^*(d^\Gamma \phi) = d(\phi \circ s \circ \varphi^{-1}) + (s \circ \varphi^{-1})^* \omega_\Gamma \cdot (\phi \circ s \circ \varphi^{-1}).$$

To simplify, write $\phi(x^1, \ldots, x^n)$ in place of $\phi \circ s \circ \varphi^{-1}(x^1, \ldots, x^n)$. Put $\mathfrak{A} = s^* \omega_\Gamma$ and let

$$(\varphi^{-1})^* \mathfrak{A} = \sum_\mu \mathfrak{A}_\mu \, dx^\mu ,$$

where each $\mathfrak{A}_\mu$ is $\underline{g}$-valued.

Specialize now to the case when $G = \underset{\sim}{SU}(2)$ and take for ρ the fundamental representation of $\underset{\sim}{SU}(2)$ on $\underset{\sim}{C}^2$:

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \longrightarrow \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} .$$

Let $\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$ be an equivariant $\underset{\sim}{C}^2$-valued map on P. Working in local coordinates, the exterior derivative is computed componentwise, i.e.,

$$d\phi = \begin{pmatrix} d\phi_1 \\ d\phi_2 \end{pmatrix} = \begin{bmatrix} \sum_\mu \frac{\partial \phi_1}{\partial x^\mu} dx^\mu \\ \\ \sum_\mu \frac{\partial \phi_2}{\partial x^\mu} dx^\mu \end{bmatrix}$$

$$= \begin{bmatrix} \sum_\mu \frac{\partial \phi_1}{\partial x^\mu} \\ \\ \sum_\mu \frac{\partial \phi_2}{\partial x^\mu} \end{bmatrix} dx^\mu = \sum_\mu \frac{\partial}{\partial x^\mu} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} dx^\mu .$$

Since

$$\mathfrak{A}_\mu \cdot \phi = \mathfrak{A}_\mu \phi \text{ (matrix multiplication)},$$

it follows that the local expression for $d^\Gamma \phi$ is

$$\sum_\mu \left(\frac{\partial}{\partial x^\mu} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} + \mathfrak{A}_\mu \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \right) dx^\mu .$$

If ρ is the trivial representation ($\rho(\sigma)v=v \;\; \forall \sigma \in G$), then $P \times_G V \approx M \times V$ and the elements of $\Lambda^k_\rho (P;V)$ project uniquely to the elements of $\Lambda^k(M;V)$, i.e., $\forall \omega \in \Lambda_\rho^k(P;V), \exists ! \, \bar{\omega} \in \Lambda^k(M;V)$: $\pi^*\bar{\omega} = \omega$. Here

$$\bar{\omega}_x(X_1,\ldots,X_k) = \omega_p(T_1,\ldots,T_k) \quad (p \in \pi^{-1}(x),\ d\pi_p(T_i) = X_i),$$

a definition which does not depend on the choices. So, if $s: U \rightarrow \pi^{-1}(U)$ is a section, then one can take $T_i = s_{*x}(X_i)$ (since $X_i = (\underline{id}_U)_{*x}(X_i) = (\pi \circ s)_{*x}(X_i) = \pi_{*s(x)}(s_{*x}(X_i))$). Therefore

$$\bar{\omega}_x(X_1,\ldots,X_k) = \omega_{s(x)}(s_{*x}(X_1),\ldots,s_{*x}(X_k))$$

$$= (s^*\omega)_x(X_1,\ldots,X_k)$$

$$\Rightarrow$$

$$\bar{\omega} = s^*\omega$$

on U.

LEMMA We have

$$d^\Gamma\omega = d\omega.$$

[In fact,

$$(d\omega)_p(T_1,\ldots,T_{k+1}) = (d(\pi^*\bar{\omega}))_p(T_1,\ldots,T_{k+1})$$

$$= (\pi^*(d\bar{\omega}))_p(T_1,\ldots,T_{k+1})$$

$$= (d\bar{\omega})_{\pi(p)}(\pi_{*p}(T_1),\ldots,\pi_{*p}(T_{k+1}))$$

$$= (d\bar{\omega})_{\pi(p)}(\pi_{*p}(hT_1),\ldots,\pi_{*p}(hT_{k+1}))$$

$$= (\pi^*(d\bar{\omega}))_p \ (hT_1,\ldots,hT_{k+1})$$

$$= (d(\pi^*\bar{\omega}))_p \ (hT_1,\ldots,hT_{k+1})$$

$$= (d\,\omega)_p \ (hT_1,\ldots,hT_{k+1})$$

$$= (d^{\Gamma}\omega)_p \ (T_1,\ldots,T_{k+1}).]$$

Curvature Suppose given

$$\begin{array}{ccc} G & \longrightarrow & P \\ & & \downarrow \pi \\ & & M \end{array}$$

and let Γ be a connection.

Definition: The curvature form Ω_Γ is

$$d^\Gamma \omega_\Gamma \quad (= d\,\omega_\Gamma \circ h).$$

I.e.:

$$\Omega_\Gamma(X,Y) = d\,\omega_\Gamma\,(hX,hY).$$

STRUCTURAL EQUATION We have

$$\Omega_\Gamma(X,Y) = d\,\omega_\Gamma\,(X,Y) + [\,\omega_\Gamma(X),\ \omega_\Gamma(Y)].$$

[Note: The theorem in the previous section does not apply (since $\omega_\Gamma \not\in \Lambda^1_{Ad}\,(P;\underline{g})$). It would lead in any event to an incorrect result as we'd be off by a factor of 1/2:

$$[\ \omega_\Gamma\,,\ \omega_\Gamma\ ] = \frac{1}{2}\ \omega_\Gamma \wedge_{Ad} \omega_\Gamma\ .]$$

FACT Γ is flat iff $\Omega_\Gamma = 0$.

Example: The standard connection on $M \times G$ is flat.

[In this situation, $\omega_\Gamma = pr_2^*(\Theta)$. And:

$$d\Theta + [\,\Theta\,,\Theta\,] = 0 \quad \text{(Maurer-Cartan)}$$

$\Longrightarrow$

$$\begin{aligned}
d\omega_\Gamma &= d\,pr_2^*\,(\Theta) \\
&= pr_2^*\,(d\Theta) \\
&= pr_2^*\,(-\,[\Theta,\Theta]) \\
&= -\,[pr_2^*\,(\Theta),\ pr_2^*\,(\Theta)] \\
&= -\,[\,\omega_\Gamma,\ \omega_\Gamma\,].
\end{aligned}$$

On the other hand,

$$\Omega_\Gamma = d\omega_\Gamma + [\,\omega_\Gamma,\omega_\Gamma\,],$$

hence $\Omega_\Gamma = 0$.]

Example: Take $M = \underset{\sim}{R}$ -- then every $\Gamma \in \mathfrak{C}(P)$ is flat.

[The horizontal subspaces are one dimensional, hence

$$\begin{aligned}
\Omega_\Gamma(X,Y) &= d\,\omega_\Gamma(hX,hY) \\
&= d\,\omega_\Gamma(\lambda 1,\ \mu 1) \\
&= \lambda\mu\, d\,\omega_\Gamma(1,1) \\
&= 0.]
\end{aligned}$$

LEMMA Let $\omega \in \Lambda_\rho^k(P;V)$ -- then

$$d^\Gamma d^\Gamma \omega = \Omega_\Gamma \wedge_\rho \omega.$$

[We have

$$\begin{aligned}
d^\Gamma d^\Gamma \omega &= dd^\Gamma \omega + \omega_\Gamma \wedge_\rho d^\Gamma \omega \\
&= d(d\omega + \omega_\Gamma \wedge_\rho \omega) + \omega_\Gamma \wedge_\rho (d\omega + \omega_\Gamma \wedge_\rho \omega) \\
&= d\omega_\Gamma \wedge_\rho \omega - \omega_\Gamma \wedge_\rho d\omega + \omega_\Gamma \wedge_\rho d\omega + \omega_\Gamma \wedge_\rho (\omega_\Gamma \wedge_\rho \omega) \\
&= d\omega_\Gamma \wedge_\rho \omega + \frac{1}{2}(\omega_\Gamma \wedge_{\mathrm{Ad}} \omega_\Gamma) \wedge_\rho \omega \\
&= (d\omega_\Gamma + [\omega_\Gamma, \omega_\Gamma]) \wedge_\rho \omega \\
&= \Omega_\Gamma \wedge_\rho \omega .]
\end{aligned}$$

So, if Γ is flat, then

$$(d^\Gamma)^2 \equiv 0.$$

Observation: $\Omega_\Gamma \in \Lambda^2_{Ad}(P;\underline{g})$.

[In fact,

$$(R_\sigma)^* \Omega_\Gamma = (R_\sigma)^*(d\omega_\Gamma + [\omega_\Gamma, \omega_\Gamma])$$

$$= d((R_\sigma)^* \omega_\Gamma) + [(R_\sigma)^*\omega_\Gamma, (R_\sigma)^*\omega_\Gamma]$$

$$= d(Ad(\sigma^{-1})\omega_\Gamma) + [Ad(\sigma^{-1})\omega_\Gamma, Ad(\sigma^{-1})\omega_\Gamma]$$

$$= Ad(\sigma^{-1})(d\omega_\Gamma + [\omega_\Gamma, \omega_\Gamma])$$

$$= Ad(\sigma^{-1})\Omega_\Gamma \;.]$$

Therefore

$$d^\Gamma\Omega_\Gamma = d\Omega_\Gamma + \omega_\Gamma \wedge_{Ad} \Omega_\Gamma .$$

Claim (<u>Bianchi Identity</u>): We have

$$d^\Gamma\Omega_\Gamma = 0.$$

[To begin with,

$$d\Omega_\Gamma + \omega_\Gamma \wedge_{Ad} \Omega_\Gamma$$

$$= d(d\omega_\Gamma + \tfrac{1}{2}\omega_\Gamma \wedge_{Ad} \omega_\Gamma)$$

$$+ \omega_\Gamma \wedge_{Ad}(d\omega_\Gamma + \tfrac{1}{2}\omega_\Gamma \wedge_{Ad} \omega_\Gamma)$$

$$= \tfrac{1}{2} d(\omega_\Gamma \wedge_{Ad} \omega_\Gamma)$$

$$+ \omega_\Gamma \wedge_{Ad} d\omega_\Gamma + \tfrac{1}{2}\omega_\Gamma \wedge_{Ad}(\omega_\Gamma \wedge_{Ad} \omega_\Gamma).$$

But

$$\begin{cases} d(\omega_\Gamma \wedge_{Ad} \omega_\Gamma) = d\omega_\Gamma \wedge_{Ad} \omega_\Gamma - \omega_\Gamma \wedge_{Ad} d\omega_\Gamma \\ \qquad\qquad \omega_\Gamma \wedge_{Ad} d\omega_\Gamma = (-1)^{1\cdot 2+1} d\omega_\Gamma \wedge_{Ad} \omega_\Gamma . \end{cases}$$

Therefore

$$d^{\Gamma}\Omega_{\Gamma} = d\,\omega_{\Gamma} \wedge_{\mathrm{Ad}} \omega_{\Gamma}$$

$$+ \omega_{\Gamma} \wedge_{\mathrm{Ad}} d\omega_{\Gamma} + \frac{1}{2}\, \omega_{\Gamma} \wedge_{\mathrm{Ad}} (\omega_{\Gamma} \wedge_{\mathrm{Ad}} \omega_{\Gamma})$$

$$= \frac{1}{2}\, \omega_{\Gamma} \wedge_{\mathrm{Ad}} (\omega_{\Gamma} \wedge_{\mathrm{Ad}} \omega_{\Gamma}).$$

And, thanks to the graded Jacobi identity,

$$\omega_{\Gamma} \wedge_{\mathrm{Ad}} (\omega_{\Gamma} \wedge_{\mathrm{Ad}} \omega_{\Gamma}) = 0,$$

from which the claim.]

Definition: The <u>field strength</u> $\mathfrak{F}_{\Gamma}$ is that element of $\Lambda^2(M;\underline{g}^P)$ which corresponds to Ω_{Γ} under the identification

$$\Lambda^2_{\mathrm{Ad}}(P;\underline{g}) \longleftrightarrow \Lambda^2(M;\underline{g}^P).$$

Given a section $s:U \longrightarrow \pi^{-1}(U)$, write

$$\mathfrak{A} = s^*\omega_{\Gamma} \quad \text{(the } \underline{\text{local gauge potential}}\text{)}$$

and

$$\mathfrak{F} = s^*\Omega_{\Gamma} \quad \underline{\text{(the local field strength)}}.$$

Then

$$\mathfrak{F} = d\mathfrak{A} + [\mathfrak{A}, \mathfrak{A}].$$

Assuming that U is a chart with coordinates $x^1,\ldots,x^n$, we have

$$\mathfrak{A} = \sum_{\mu} \mathfrak{A}_{\mu}\, dx^{\mu} \text{ and } \mathfrak{F} = \frac{1}{2} \sum_{\mu,\nu} \mathfrak{F}_{\mu\nu}\, dx^{\mu} \wedge dx^{\nu},$$

where the $\mathfrak{A}_{\mu}$ and the $\mathfrak{F}_{\mu\nu}$ are $\underline{g}$-valued functions on U. Consequently,

$$\mathfrak{F}_{\mu\nu} = \partial_{\mu}\mathfrak{A}_{\nu} - \partial_{\nu}\mathfrak{A}_{\mu} + [\mathfrak{A}_{\mu}, \mathfrak{A}_{\nu}],$$

the derivatives being computed componentwise in $\underline{g}$.

Remark: If $s_i : U_i \longrightarrow \pi^{-1}(U_i)$ and $s_j : U_j \longrightarrow \pi^{-1}(U_j)$ are sections and if $g_{ij} : U_j \cap U_i \longrightarrow G$ is the associated transition function, then on $U_j \cap U_i$,

$$\mathfrak{A}_j = \mathrm{Ad}(g_{ij}^{-1}) \circ \mathfrak{A}_i + \Theta_{ij}$$

and

$$\mathfrak{F}_j = \mathrm{Ad}(g_{ij}^{-1}) \circ \mathfrak{F}_i .$$

So, when $\underline{g}$ is abelian, $\mathfrak{F}_j = \mathfrak{F}_i$ on $U_j \cap U_i$, and the local field strengths can be pieced together to give a globally defined $\underline{g}$-valued 2-form $\mathfrak{F}$ on M, namely $\mathfrak{F} = \mathfrak{F}_\Gamma$.

Gauge Transformations Suppose given

$$\begin{array}{ccc} G & \longrightarrow & P \\ & & \downarrow \pi \\ & & M. \end{array}$$

Then by a gauge transformation we understand an equivariant diffeomorphism

$$P \xrightarrow{f} P$$

over M. So:

(1) $f(p\cdot\sigma) = f(p)\cdot\sigma$;

(2) $\pi \circ f = \pi$.

Notation: $\mathcal{G}(P)$ is the group of gauge transformations.

Let

$$\mathrm{Int}(P,G)$$

be the set of C^{∞} functions $\mu : P \longrightarrow G$ such that

$$\mu(p\cdot\sigma) = \sigma^{-1}\mu(p)\sigma .$$

Then on general grounds,

$$\mathrm{Int}(P,G) \longleftrightarrow \sec(G^P).$$

LEMMA There is a one-to-one correspondence

$$\mathcal{G}(P) \longrightarrow \mathrm{Int}(P,G).$$

[Assign to $f \in \mathcal{G}(P)$ the element $\mu_f \in \mathrm{Int}(P,G)$ defined as follows: $\mu_f(p)$ is the unique element of G such that $f(p) = p\cdot\mu_f(p)$.]

[Note: In the special case when $P = M \times G$, we have

$$\mu_f(x,\sigma) = \mu_f((x,e)\cdot\sigma)$$
$$= \sigma^{-1}\mu_f(x,e)\sigma,$$

thus μ_f is completely determined by

$$\begin{cases} M \longrightarrow G \\ x \longrightarrow \mu_f(x,e). \end{cases}$$

Conversely, if $g:M \longrightarrow G$, then the prescription

$$g(x,\sigma) = \sigma^{-1}g(x)\sigma$$

extends g to an element of Int(P,G).]

Remark: The preceding identifications respect the underlying group structures.

insert 2.5 Suppose that $\Gamma \in \mathfrak{A}(P)$ -- then $\Gamma \longleftrightarrow \omega_\Gamma$ and $\forall\, f \in \mathcal{G}(P)$, $\Gamma\cdot f \longleftrightarrow f^*\omega_\Gamma$. Here

$$f^*\omega_\Gamma = \mathrm{Ad}(\mu_f^{-1})\,\omega_\Gamma + \mu_f^*\Theta .$$

The prescription

$$\Gamma\cdot f \longleftrightarrow f^*\omega_\Gamma$$

defines a right action of $\mathcal{G}(P)$ on $\mathfrak{A}(P)$:

$$\mathfrak{A}(P) \times \mathcal{G}(P) \longrightarrow \mathfrak{A}(P).$$

Definition: Two connections $\Gamma_1, \Gamma_2 \in \mathfrak{A}(P)$ are said to be <u>gauge equivalent</u> if $\exists\, f \in \mathcal{G}(P)$:

$$\omega_{\Gamma_2} = f^*\omega_{\Gamma_1} .$$

Informally, Int(P,G) is a Lie group with Lie algebra $\Lambda^0_{Ad}(P;\underline{g})$.

[Note: The exponential map

$$\exp: \Lambda^0_{Ad}(P;\underline{g}) \longrightarrow \mathrm{Int}(P,G)$$

is defined by

$$(\exp \alpha)(p) = \exp(\alpha(p)).]$$

Therefore each $\alpha \in \Lambda^0_{Ad}(P;\underline{g})$ induces a one parameter family of gauge transformations:

$$f_{\alpha,\lambda} \in \mathcal{G}(P),$$

where

$$f_{\alpha,\lambda}(p) = p \cdot \exp(\lambda\alpha(p)) \quad (\lambda \in \underset{\sim}{R}).$$

And

$$\frac{d}{d\lambda} f^*_{\alpha,\lambda}\,\omega_\Gamma \Big|_{\lambda=0} = d\alpha + \omega_\Gamma \wedge_{Ad} \alpha = d^\Gamma \alpha .$$

The orbit space

$$\mathfrak{A}(P)/\mathcal{G}(P)$$

is the <u>configuration space</u> of the theory.

Definition: An automorphism of $(P,M;G)$ is a pair (f,f_M), where $f:P \to P$ is an equivariant diffeomorphism, $f_M:M \to M$ is a diffeomorphism, and the diagram

$$\begin{array}{ccc} P & \xrightarrow{f} & P \\ \pi\downarrow & & \downarrow\pi \\ M & \xrightarrow[f_M]{} & M \end{array}$$

commutes.

[Note: If $f:P \to P$ is equivariant, $f_M:M \to M$ is a diffeomorphism, and $f_M \circ \pi = \pi \circ f$, then f is necessarily a diffeomorphism.]

There is an evident exact sequence

$$1 \to \mathcal{G}(P) \to \text{Aut } P \to \text{Diff } M,$$

but the map on the right need not be onto. For example, consider the Hopf bundle

$$\begin{array}{ccc} \underset{\sim}{S}^1 & \to & \underset{\sim}{S}^3 \\ & & \downarrow \\ & & \underset{\sim}{S}^2 \end{array}.$$

Then the antipodal map $\underset{\sim}{S}^2 \to \underset{\sim}{S}^2$ does not lift to an automorphism of $(\underset{\sim}{S}^3,\underset{\sim}{S}^2;\underset{\sim}{S}^1)$. However, when $P=M\times G$, the arrow $\text{Aut } P \to \text{Diff } M$ is obviously surjective.

Let $M \xrightarrow{f_M} M \xleftarrow{\pi} P$ be a 2-sink, where $f_M \in \text{Diff } M$, and form the pullback square

$$\begin{tikzcd} f_M^*P \arrow[r, "\eta"] \arrow[d, "\xi"'] & P \arrow[d, "\pi"] \\ M \arrow[r, "f_M"'] & M . \end{tikzcd}$$

Let $M \xleftarrow{\pi} P \xrightarrow{f} P$ be a 2-source with $f_M \circ \pi = \pi \circ f$ -- then there is an arrow $P \xrightarrow{\phi} f_M^*P$ and a commutative diagram

$$\begin{tikzcd} P \arrow[r, "\phi"] \arrow[dr, "\pi"'] & f_M^*P \arrow[r, "\eta"] \arrow[d, "\xi"] & P \arrow[d, "\pi"] \\ & M \arrow[r, "f_M"'] & M \end{tikzcd} \qquad (f=\eta\circ\phi).$$

Rewriting the triangle as a commutative square

$$\begin{tikzcd} P \arrow[r, "\phi"] \arrow[d, "\pi"'] & f_M^*P \arrow[d, "\xi"] \\ M \arrow[r, "id_M"'] & M , \end{tikzcd}$$

it follows that ϕ is an equivariant diffeomorphism. Conversely, if we are given a commutative diagram

$$\begin{tikzcd} P \arrow[r, "\phi"] \arrow[d, "\pi"'] & f_M^*P \arrow[d, "\xi"] \\ M \arrow[r, "id_M"'] & M , \end{tikzcd}$$

where ϕ is an equivariant diffeomorphism, then it is clear that the lifting problem admits a solution.

Rappel: If $f_M : M \longrightarrow M$ is smoothly homotopic to id_M, then there is an equivariant diffeomorphism $\phi : P \longrightarrow f_M^* P$ and a commutative diagram

$$\begin{array}{ccc} P & \xrightarrow{\phi} & f_M^* P \\ \pi \downarrow & & \downarrow \xi \\ M & \xrightarrow[id_M]{} & M \end{array}.$$

So, when $f \simeq id_M$, the lifting problem admits a solution.

Notation: $\mathrm{Diff}_0 M$ is the subgroup of Diff M consisting of those diffeomorphisms f_M which are diffeotopic to the identity, i.e., for which $\exists$ a smooth one parameter family $H_t \in$ Diff M: $H_0 = id_M$, $H_1 = f_M$.

LEMMA $\forall f_M \in \mathrm{Diff}_0 M$, $\exists$ an equivariant diffeomorphism $f : P \longrightarrow P$ such that $f_M \circ \pi = \pi \circ f$.

Let ρ be a representation of G on a finite dimensional vector space V -- then $\forall f \in \mathcal{G}(P)$, f^* defines an isomorphism

$$f^* : \Lambda_\rho^k (P;V) \longrightarrow \Lambda_\rho^k (P;V)$$

and

$$f^* \alpha = \mu_f^{-1} \cdot \alpha \quad (\alpha \in \Lambda_\rho^k (P;V)).$$

Example: $\forall \ \Gamma \in \mathcal{A}(P)$, we have

$$\Omega_{\Gamma \cdot f} = f^* \Omega_\Gamma = \mathrm{Ad}(\mu_f^{-1}) \, \Omega_\Gamma .$$

To have a concrete illustration of the foregoing, consider

$$\begin{array}{ccc} G & \longrightarrow & P \\ & & \downarrow \pi \\ & & \underset{\sim}{S}^1, \end{array}$$

where G is path connected -- then P is trivial: $P \approx \underset{\sim}{S}^1 \times G$.

Agreeing to work with $\underset{\sim}{S}^1 \times G$, specialize and assume that G is a compact connected semisimple matrix Lie group. Write $\mathfrak{A}$ in place of $\mathfrak{A}(P)$ and $\mathfrak{G}$ in place of $\mathfrak{G}(P)$.

Convention: View the circle $\underset{\sim}{S}^1$ as the unit interval [0,1] with boundary points identified, parameterized by $\tau \in [0,1]$.

<u>Ad $\mathfrak{A}$:</u> We have

$$\mathfrak{A} \longleftrightarrow C^\infty(\underset{\sim}{S}^1;\underline{g}).$$

<u>Ad $\mathfrak{G}$:</u> We have

$$\mathfrak{G} \longleftrightarrow C^\infty(\underset{\sim}{S}^1;G).$$

The right action of $\mathfrak{G}$ on $\mathfrak{A}$ is given by the prescription

$$A \longrightarrow A^g = g^{-1}Ag + g^{-1}g'.$$

[Note: The precise meaning of A^g is this:

$$A^g(\tau) = g^{-1}(\tau)A(\tau)g(\tau) + g(\tau)^{-1}g'(\tau).$$

Here

$$L_{g(\tau)^{-1}}(g(\tau)) = e$$

$$\Longrightarrow$$

$$dL_{g(\tau)^{-1}} : G_{g(\tau)} \longrightarrow G_e = \underline{g}.$$

But

$$g'(\tau) \in G_{g(\tau)}$$

$$\Longrightarrow$$

$$dL_{g(\tau)^{-1}} (g'(\tau)) \in \underline{g}.$$

I.e.:

$$\frac{d}{d\lambda} [g(\tau)^{-1} g(\tau + \lambda)] \Big|_{\lambda = 0}$$

$$= g(\tau)^{-1} \frac{d}{d\lambda} g(\tau + \lambda) \Big|_{\lambda = 0}$$

$$= g(\tau)^{-1} g'(\tau) \in \underline{g}.]$$

Put

$$\mathcal{G}_e = \left\{ g \in \mathcal{G} : g(0) = g(1) = e \right\}.$$

Then $\mathcal{G}_e$ is a normal subgroup of $\mathcal{G}$.

Observation: The map

$$\begin{cases} \mathcal{G} \longrightarrow \mathcal{G}_e \times G \\ g \longrightarrow (g\, g(0)^{-1}, g(0)) \end{cases}$$

is bijective.

Given $\sigma \in G$ and $g_e \in \mathcal{G}_e$, let

$$\sigma \cdot g_e = \sigma g_e \sigma^{-1}.$$

Then the multiplication per the semidirect product $\mathcal{G}_e \rtimes G$ is given by the rule

$$(g_e, \sigma)(g'_e, \sigma') = (g_e(\sigma \cdot g'_e), \ \sigma\sigma').$$

Claim: The canonical bijection

$$\mathcal{G} \longrightarrow \mathcal{G}_e \times G$$

is an isomorphism of groups.

[In fact,

$$(g\ g(0)^{-1},\ g(0))\ (h\ h(0)^{-1},\ h(0))$$

$$= (g\ g(0)^{-1}\ (g(0) \cdot h\ h(0)^{-1}),\ g(0)h(0))$$

$$= (g\ g(0)^{-1}\ g(0)h\ h(0)^{-1}\ g(0)^{-1},\ g(0)h(0))$$

$$= (gh\ h(0)^{-1}\ g(0)^{-1},\ g(0)h(0)).]$$

LEMMA We have

$$\mathfrak{A} / \mathcal{G}_e \approx G$$

and

$$\mathfrak{A} / \mathcal{G} \approx G/\mathrm{Int},$$

the set of conjugacy classes in G.

[This is a simple application of holonomy theory.]

Remark: Let T be a maximal torus in G, $W = N(T)/T$ the associated Weyl group -- then

$$G/\mathrm{Int}\, G \approx T/W.$$

Parallel Transport Suppose given

$$\begin{array}{ccc} G & \longrightarrow & P \\ & & \downarrow \pi \\ & & M, \end{array}$$

where G and M are path connected, and let Γ be a connection.

Convention: Curves are continuous and piecewise smooth.

THEOREM Let $\gamma : [0,1] \longrightarrow M$ be a curve. Fix a point $p_0 \in \pi^{-1}(\gamma(0))$ -- then there is a unique curve $\gamma^{\uparrow} : [0,1] \longrightarrow P$ such that (i) $\gamma^{\uparrow}(0) = p_0$, (ii) $\pi \circ \gamma^{\uparrow} = \gamma$, (iii) $\dot{\gamma}^{\uparrow}(t) \in T^h_{\gamma^{\uparrow}(t)}(P)\ (0 \leq t \leq 1)$.

Application: There is a diffeomorphism

$$T_\gamma : \pi^{-1}(\gamma(0)) \longrightarrow \pi^{-1}(\gamma(1))$$

called parallel transport from $\gamma(0)$ to $\gamma(1)$ along γ satisfying the condition

$$T_\gamma \circ R_\sigma = R_\sigma \circ T_\gamma \quad \forall \sigma \in G.$$

[In fact,

$$T_\gamma(p_0) = \gamma^{\uparrow}(1).]$$

[Note: If $\phi : [0,1] \longrightarrow [a,b]$ is a homeomorphism with $\phi(0) = a$ & $\phi(1) = b$ such that ϕ and ϕ^{-1} are C^∞ except at a finite number of points, then the parallel transport per γ is the same as the parallel transport per $\gamma \circ \phi^{-1}$.]

Remark: The parallel transport along γ^{-1} is the inverse of the parallel transport along γ.

[Note: As usual,

$$\gamma^{-1}(t) = \gamma(1-t).]$$

If $\mu:[0,1] \longrightarrow M$ is a curve from x to y and $\nu:[0,1] \longrightarrow M$ is a curve from y to z, then the composite

$$\nu \circ \mu(t) = \begin{cases} \mu(2t) & (0 \leq t \leq 1/2) \\ \nu(2t-1) & (1/2 \leq t \leq 1) \end{cases}$$

is a curve from x to z and

$$T_{\nu \circ \mu} = T_{\nu} \circ T_{\mu} .$$

Let $f:P \longrightarrow P$ be a gauge transformation. Put $\Gamma' = \Gamma \cdot f$ -- then

$$T'_{\gamma} = f^{-1} \circ T_{\gamma} \circ f.$$

<u>Holonomy</u> Suppose given

$$\begin{array}{ccc} G & \longrightarrow & P \\ & & \big\downarrow \pi \\ & & M, \end{array}$$

where G and M are path connected, and let Γ be a connection.

Notation: $\forall\ x \in M$, $\Omega(x)$ is the loop space at x, i.e., the set of all closed curves starting and ending at x.

For each $\gamma \in \Omega(x)$,

$$T_\gamma : \pi^{-1}(x) \longrightarrow \pi^{-1}(x)$$

is a diffeomorphism, the set of all such being the <u>holonomy group</u> of Γ at x:

$$\mathrm{Hol}(\Gamma, x).$$

The subgroup of $\mathrm{Hol}(\Gamma,x)$ consisting of those T_γ for which γ is nullhomotopic is the <u>restricted holonomy group</u> of Γ at x:

$$\mathrm{Hol}^0(\Gamma, x).$$

Let $p \in \pi^{-1}(x)$ -- then $\forall\, \gamma \in \Omega(x)$, $\exists\, g_\gamma \in G$:

$$T_\gamma(p) = p \cdot g_\gamma .$$

Observation:

$$\begin{aligned} p \cdot g_{\nu \circ \mu} &= T_{\nu \circ \mu}(p) \\ &= T_\nu(T_\mu(p)) \\ &= T_\nu(p \cdot g_\mu) \\ &= T_\nu \circ R_{g_\mu}(p) \\ &= R_{g_\mu} \circ T_\nu(p) \end{aligned}$$

$$= R_{g_\mu}(p\cdot g_\nu)$$

$$= p\cdot g_\nu \cdot g_\mu$$

$$= p\cdot (g_\nu g_\mu)$$

$$\Longrightarrow$$

$$g_{\nu\circ\mu} = g_\nu g_\mu .$$

Observation:

$$T_\gamma \circ T_{\gamma^{-1}}(p) = p = p\cdot e$$

$$\Longrightarrow$$

$$p\cdot (g_\gamma g_{\gamma^{-1}}) = p\cdot e$$

$$\Longrightarrow g_\gamma g_{\gamma^{-1}} = e$$

$$\Longrightarrow g_\gamma^{-1} = g_{\gamma^{-1}} .$$

Put

$$\mathrm{Hol}(\Gamma,p) = \{ g_\gamma : \gamma \in \Omega(x)\} .$$

Then $\mathrm{Hol}(\Gamma,p)$ is a subgroup of G and $\forall\, \sigma \in G$,

$$\mathrm{Hol}(\Gamma,p\cdot\sigma) = \sigma^{-1}\mathrm{Hol}(\Gamma,p)\sigma .$$

[Note: $\mathrm{Hol}^0(\Gamma,p)$ is defined analogously.]

LEMMA The arrow

$$T_\gamma \longrightarrow g_\gamma$$

is an isomorphism

$$\mathrm{Hol}(\Gamma,x) \longrightarrow \mathrm{Hol}(\Gamma,p)$$

of groups.

[One has only to check injectivity. Suppose therefore that

$$g_{\gamma_1} = g_{\gamma_2} .$$

Then

$$T_{\gamma_1}(p) = T_{\gamma_2}(p).$$

So, $\forall \sigma \in G$,

$$\begin{aligned} T_{\gamma_1}(p\cdot\sigma) &= T_{\gamma_1} \circ R_\sigma(p) \\ &= R_\sigma \circ T_{\gamma_1}(p) \\ &= R_\sigma \circ T_{\gamma_2}(p) \\ &= T_{\gamma_2} \circ R_\sigma(p) \\ &= T_{\gamma_2}(p\cdot\sigma) \end{aligned}$$

$\Rightarrow$

$$T_{\gamma_1} = T_{\gamma_2}.]$$

Rappel: $\mathrm{Hol}^0(\Gamma,p)$ is the identity component of $\mathrm{Hol}(\Gamma,p)$ and is a connected Lie subgroup of G. There is a surjective homomorphism

$$\pi_1(M,x) \longrightarrow \mathrm{Hol}(\Gamma,p)/\mathrm{Hol}^0(\Gamma,p)$$

of groups, hence

$$\mathrm{Hol}(\Gamma,p) = \mathrm{Hol}^0(\Gamma,p)$$

when M is simply connected.

~~(Note: The Hol(Γ,p) are conjugate to one another in G.)~~

AMBROSE-SINGER THEOREM Fix a point $p_0 \in P$ -- then the Lie algebra of $\mathrm{Hol}(\Gamma, p_0)$ is spanned by the $\Omega_p(X,Y)$ $(X,Y \in T_p^h(P))$, where p ranges over the points in P which can be joined to p_0 by a horizontal curve.

Remark: Let

$$\mathcal{G}(P)_\Gamma = \{ f \in \mathcal{G}(P) : \Gamma \cdot f = \Gamma \} .$$

Then the image of the arrow

$$\begin{cases} \mathcal{G}(P)_\Gamma \longrightarrow G \\ \qquad f \longrightarrow \mu_f(p) \end{cases}$$

is the centralizer of $\mathrm{Hol}(\Gamma, p)$.

Write

$$h(\Gamma, p; \gamma) = g_\gamma \quad .$$

Then

$$\begin{aligned} T_\gamma(p \cdot \sigma) &= T_\gamma \circ R_\sigma(p) \\ &= R_\sigma \circ T_\gamma(p) \\ &= p \cdot g_\gamma \sigma \\ &= (p \cdot \sigma) \cdot (\sigma^{-1} g_\gamma \sigma) \end{aligned}$$

$\Longrightarrow$

$$h(\Gamma, p \cdot \sigma ; \gamma) = \sigma^{-1} h(\Gamma, p; \gamma) \sigma .$$

Let $f:P \longrightarrow P$ be a gauge transformation. Put $\Gamma' = \Gamma\cdot f$ -- then

$$h(\Gamma',p;\gamma) = g'_\gamma .$$

But

$$T'_\gamma = f^{-1} \circ T_\gamma \circ f.$$

And

$$\begin{aligned} f^{-1}&(T_\gamma(f(p))) \\ &= f^{-1}(T_\gamma(p\cdot\mu_f(p))) \\ &= f^{-1}(T_\gamma \circ R_{\mu_f(p)}(p)) \\ &= f^{-1}(R_{\mu_f(p)} \circ T_\gamma(p)) \\ &= f^{-1}(p\cdot g_\gamma\cdot\mu_f(p)) \\ &= f^{-1}(p)\cdot g_\gamma\cdot\mu_f(p) \\ &= p\cdot\mu_{f^{-1}}(p)\cdot g_\gamma\cdot\mu_f(p) \\ &= p\cdot\mu_f(p)^{-1}\cdot g_\gamma\cdot\mu_f(p) \\ &= p\cdot g'_\gamma \end{aligned}$$

$\Longrightarrow$

$$h(\Gamma',p;\gamma) = \mu_f(p)^{-1}\, h(\Gamma,p;\gamma)\mu_f(p).$$

Example: Suppose that G is compact. Let ρ be a representation of G on a finite dimensional vector space V. Define a function

$$W_\rho : \mathfrak{M}(P) \times \Omega(x) \longrightarrow \underset{\sim}{C}$$

by

$$W_\rho(\Gamma,\gamma) = \mathrm{tr}(\rho(h(\Gamma,p;\gamma))).$$

Then W_ρ does not depend on the choice of $p \in \pi^{-1}(x)$. Furthermore, W_ρ is gauge invariant, i.e., $\forall\ f \in \mathfrak{A}(P)$,

$$W_\rho(\Gamma\cdot f,\gamma) = W_\rho(\Gamma,f).$$

Therefore $W_\rho(\text{—},\gamma)$ defines a function on $\mathfrak{A}(P)/\mathcal{G}(P)$.

[Note: Per ρ , $W_\rho(\text{—},\gamma)$ is the <u>Wilson loop</u> associated with γ.]

LEMMA Let Γ_1, Γ_2 be connections. Suppose that $\forall\ \gamma \in \Omega(x)$,

$$h(\Gamma_1,p;\gamma) = h(\Gamma_2,p;\gamma).$$

Then Γ_1, Γ_2 are gauge equivalent, hence

$$[\Gamma_1] = [\Gamma_2]$$

in $\mathfrak{A}(P)/\mathcal{G}(P)$.

[To define $f \in \mathcal{G}(P)$ such that $\Gamma_1 \underset{f}{\approx} \Gamma_2$, take any point $p_0 \in P$, let γ be a curve joining $\pi(p_0)$ to $\pi(p)$, and put

$$f(p_0) = T^2_{\gamma^{-1}} \circ T^1_\gamma \ (p_0),$$

where T^1 is the parallel transport per Γ_1 and T^2 is the parallel transport per Γ_2. This makes sense. Thus let γ_1, γ_2 be two curves joining $\pi(p_0)$ to $\pi(p)$ -- then

$$T^1_{\gamma_2 \circ \gamma_1^{-1}} = T^2_{\gamma_2 \circ \gamma_1^{-1}} \qquad \text{(by hypothesis)}$$

$\Rightarrow$

$$T^1_{\gamma_2} \circ T^1_{\gamma_1^{-1}} = T^2_{\gamma_2} \circ T^2_{\gamma_1^{-1}}$$

$\Rightarrow$

$$T^1_{\gamma_2} = T^2_{\gamma_2} \circ T^2_{\gamma_1^{-1}} \circ T^1_{\gamma_1}$$

$\Rightarrow$

$$T^2_{\gamma_2^{-1}} \circ T^1_{\gamma_2} = T^2_{\gamma_1^{-1}} \circ T^1_{\gamma_1} \quad .]$$

[Note: By construction, f is the identity in the fiber over x.]

If $\exists\, \sigma \in G: \forall \gamma \in \Omega(x)$,

$$h(\Gamma_1, p; \gamma) = \sigma^{-1} h(\Gamma_2, p; \gamma)\sigma,$$

then it is still the case that

$$[\Gamma_1] = [\Gamma_2].$$

In fact,

$$\sigma^{-1} h(\Gamma_2, p; \gamma)\sigma = h(\Gamma_2, p\cdot\sigma; \gamma).$$

Choose a gauge transformation $f: P \rightarrow P$ such that $f(p) = p\cdot\sigma$ ($\Rightarrow \sigma = \mu_f(p)$) -- then

$$h(\Gamma_2, p\cdot\sigma; \gamma) = h(\Gamma_2\cdot f, p; \gamma).$$

So, $\forall \gamma \in \Omega(x)$,

$$h(\Gamma_1, p; \gamma) = h(\Gamma_2\cdot f, p; \gamma).$$

The lemma thus implies that

$$[\Gamma_1] = [\Gamma_2 \cdot f] = [\Gamma_2].$$

Remark: If instead one assumes that $\forall\, \gamma \in \Omega(x)$, $\exists\, \sigma_\gamma \in G$:

$$h(\Gamma_1, p; \gamma) = \sigma_\gamma^{-1}\, h(\Gamma_2, p; \gamma)\, \sigma_\gamma \quad ,$$

then it need not be true that

$$[\Gamma_1] = [\Gamma_2].$$

The preceding considerations can be generalized. Suppose given

$$\begin{array}{ccc} G & \longrightarrow & P \\ & & \big\downarrow \pi \\ & & M \end{array} \qquad \text{and} \qquad \begin{array}{ccc} G & \longrightarrow & \widetilde{P} \\ & & \big\downarrow \widetilde{\pi} \\ & & M \end{array} .$$

Let $\Gamma \in \mathfrak{A}(P)$, $\widetilde{\Gamma} \in \mathfrak{A}(\widetilde{P})$ and assume that $\forall\, \gamma \in \Omega(x)$,

$$h(\Gamma, p; \gamma) = h(\widetilde{\Gamma}, \widetilde{p}; \gamma),$$

for some

$$p \in \pi^{-1}(x), \widetilde{p} \in \widetilde{\pi}^{-1}(x).$$

Claim: $\exists$ an equivariant diffeomorphism

$$\begin{tikzcd} P \arrow[rr, "f"] \arrow[dr, "\pi"'] & & \widetilde{P} \arrow[dl, "\widetilde{\pi}"] \\ & M & \end{tikzcd}$$

over M such that $f_*\Gamma = \widetilde{\Gamma}$.

To see this, let

$$\begin{cases} \varsigma_p : G \longrightarrow P_x \\ \varsigma_{\widetilde{p}} : G \longrightarrow \widetilde{P}_x \end{cases}$$

be defined by

$$\begin{cases} \sigma \longrightarrow p\cdot\sigma \\ \sigma \longrightarrow \widetilde{p}\cdot\sigma \end{cases} \qquad (\sigma \in G).$$

Put

$$\zeta = \zeta_{\widetilde{p}} \circ \zeta_p^{-1} : P_x \longrightarrow \widetilde{P}_x .$$

Then

$$\begin{aligned} \zeta(p) &= \zeta_{\widetilde{p}} \circ \zeta_p^{-1}(p) \\ &= \zeta_{\widetilde{p}}(e) = \widetilde{p}. \end{aligned}$$

Furthermore,

$$\zeta \circ R_\sigma = \widetilde{R}_\sigma \circ \zeta .$$

Now define $f : P \longrightarrow \widetilde{P}$ fiberwise by the rule

$$f \mid \pi^{-1}(y) = \widetilde{T}_{\gamma^{-1}} \circ \zeta \circ T_\gamma .$$

Here y is any point in M and γ is any curve joining y to x. This makes sense. Thus let γ, δ be two curves joining y to x -- then we have to show that

$$\widetilde{T}_{\gamma^{-1}} \circ \zeta \circ T_\gamma = \widetilde{T}_{\delta^{-1}} \circ \zeta \circ T_\delta$$

or still, that

$$\zeta \circ T_{\gamma \circ \delta^{-1}} = \widetilde{T}_{\gamma \circ \delta^{-1}} \circ \zeta .$$

By hypothesis,

$$g_{\gamma \circ \delta^{-1}} = h(\Gamma, p; \gamma \circ \delta^{-1}) = h(\widetilde{\Gamma}, \widetilde{p}; \gamma \circ \delta^{-1}) = \widetilde{g}_{\gamma \circ \delta^{-1}}$$

and

$$\begin{cases} T_{\gamma \circ \delta^{-1}}(p) = p \cdot g_{\gamma \circ \delta^{-1}} \\ \widetilde{T}_{\gamma \circ \delta^{-1}}(\widetilde{p}) = \widetilde{p} \cdot \widetilde{g}_{\gamma \circ \delta^{-1}} \end{cases} .$$

Therefore

$$\begin{aligned}
&\zeta \circ T_{\gamma \circ \delta^{-1}}(p \cdot \sigma) \\
&= \zeta \circ T_{\gamma \circ \delta^{-1}} \circ R_{\sigma}(p) \\
&= \zeta \circ R_{\sigma} \circ T_{\gamma \circ \delta^{-1}}(p) \\
&= \zeta(p \cdot g_{\gamma \circ \delta^{-1}} \sigma) \\
&= \zeta \circ R_{g_{\gamma \circ \delta^{-1}} \sigma}(p) \\
&= \widetilde{R}_{g_{\gamma \circ \delta^{-1}} \sigma}\, \zeta(p) \\
&= \widetilde{R}_{\widetilde{g}_{\gamma \circ \delta^{-1}} \sigma}\, \zeta(p) \\
&= \widetilde{R}_{\widetilde{g}_{\gamma \circ \delta^{-1}} \sigma}(\widetilde{p}) \\
&= \widetilde{p} \cdot \widetilde{g}_{\gamma \circ \delta^{-1}}\, \sigma .
\end{aligned}$$

But

$$\begin{aligned}
&\widetilde{T}_{\gamma \circ \delta^{-1}} \circ \zeta(p \cdot \sigma) \\
&\qquad = \widetilde{T}_{\gamma \circ \delta^{-1}} \circ \zeta \circ R_{\sigma}(p) \\
&\qquad = \widetilde{T}_{\gamma \circ \delta^{-1}} \circ \widetilde{R}_{\sigma} \circ \zeta(p) \\
&\qquad = \widetilde{R}_{\sigma} \circ \widetilde{T}_{\gamma \circ \delta^{-1}}(\widetilde{p})
\end{aligned}$$

$$= \widetilde{R}_{\sigma}(\widetilde{p}\cdot\widetilde{g}_{\gamma\circ\delta^{-1}})$$

$$= \widetilde{p}\cdot\widetilde{g}_{\gamma\circ\delta^{-1}}\,\sigma\,.$$

Specialize and assume that

$$G = G_1^{k_1} \times G_2^{k_2} \times G_3^{k_3} \times G_4^{k_4},$$

where $G_1 = \underset{\sim}{U}(n)$, $G_2 = \underset{\sim}{SU}(n)$, $G_3 = \underset{\sim}{O}(n)$, $G_4 = \underset{\sim}{SO}(2n+1)$.

[Note: This covers the case of $\underset{\sim}{U}(1) \times \underset{\sim}{SU}(2) \times \underset{\sim}{SU}(3)$, which is the group involved in the standard model. One can also include $G_5^{k_5}$ with $G_5 = \underset{\sim}{SO}(2)$ or $\underset{\sim}{SO}(4)$.]

LEMMA Suppose that $\{\sigma_i : i \in I\}$ and $\{\tau_i : i \in I\}$ are collections of elements of G such that $\forall\, i_1, \ldots, i_k \in I$, $\sigma_{i_1} \cdots \sigma_{i_k}$ is conjugate to $\tau_{i_1} \cdots \tau_{i_k}$ -- then $\exists\, g \in G$:

$$\sigma_i = g^{-1} \tau_i\, g \quad \forall\, i \in I.$$

Let $\Gamma_1, \Gamma_2 \in \mathfrak{A}(P)$. Assume: $\forall$ irreducible character χ of

$$\chi(h(\Gamma_1, p; \gamma)) = \chi(h(\Gamma_2, p; \gamma)) \qquad \forall\, \gamma \in \Omega(x).$$

Then

$$[\Gamma_1] = [\Gamma_2].$$

In fact, since the χ separate conjugacy classes, $\forall\, \gamma \in \Omega(x)$, $h(\Gamma_1, p; \gamma)$ is conjugate to $h(\Gamma_2, p; \gamma)$. But this persists to products, so it follows from the lemma that $\exists\, \sigma \in G$:

$$h(\Gamma_1, p; \gamma) = \sigma^{-1} h(\Gamma_2, p; \gamma)\sigma \qquad \forall \gamma \in \Omega(x),$$

which, as has been seen earlier, implies that

$$[\Gamma_1] = [\Gamma_2].$$

Remark: Let

$$W_\chi([\Gamma], \gamma) = \chi(h(\Gamma, p; \gamma)).$$

Then

$$W_\chi(\text{—}, \gamma) : \mathfrak{A}(P)/\mathfrak{G}(P) \longrightarrow \underset{\sim}{C}$$

and the above discussion shows that the $W_\chi(\text{—}, \gamma)$ separate the points of $\mathfrak{A}(P)/\mathfrak{G}(P)$, i.e.,

$$W_\chi([\Gamma_1], \gamma) = W_\chi([\Gamma_2], \gamma) \quad \forall \chi \,\&\, \forall \gamma$$

$$\Longrightarrow$$

$$[\Gamma_1] = [\Gamma_2].$$

Let K be a positive integer -- then G operates on G^K:

$$(\sigma_1, \ldots, \sigma_K) \cdot \sigma = (\sigma^{-1}\sigma_1\sigma, \ldots, \sigma^{-1}\sigma_K\sigma)$$

and the functions

$$(\sigma_1, \ldots, \sigma_K) \rightarrow \chi(\sigma_{i_1} \cdots \sigma_{i_k}) \quad (i_1, \ldots, i_k \in \{1, \ldots, K\})$$

associated with the irreducible characters χ of G are invariant under the action of G.

Claim: The algebra A generated by these functions is dense in $C(G^K/G)$.

[Since A is closed under conjugation and contains the constants, it need only be shown that A separates points. Assume therefore that

$$\left\{\begin{array}{l} (\sigma_1, \ldots, \sigma_K) \\ \\ (\tau_1, \ldots, \tau_K) \end{array}\right. \in G^K$$

have the property that $\forall \chi$ and all $i_1, \ldots, i_k \in \{1, \ldots, K\}$,

$$\chi(\sigma_{i_1} \cdots \sigma_{i_k}) = \chi(\tau_{i_1} \cdots \tau_{i_k}).$$

An application of the lemma then gives a $g \in G$:

$$\sigma_i = g^{-1} \tau_i \, g \qquad (i=1, \ldots, K),$$

from which the claim.]

<u>Reconstruction Theory</u> Let P run through a set of representatives for the isomorphism classes of principal G-bundles over M.

Problem: Identify

$$\coprod_{P} \mathfrak{A}(P)/\mathcal{G}(P)$$

in terms of M and G alone.

Fix a point $* \in M$ -- then a <u>smooth family of loops</u> is a map $\psi : U \subset \mathbf{R}^n \longrightarrow \Omega(*)$, where U is open, such that the function $\Psi : U \times [0,1] \longrightarrow M$ defined by the rule

$$\Psi(x,t) = \psi(x)(t)$$

is smooth.

<u>LEMMA</u> For every smooth family of loops ψ, the function

$$\begin{cases} U \longrightarrow G \\ x \longrightarrow h(\Gamma, p; \psi(x)) \end{cases}$$

is smooth.

~~Assume now that M is path connected and fix a point $* \in M$~~

Definition: Two loops $\gamma, \delta \in \Omega(*)$ are said to be <u>thinly homotopic</u> if they are homotopic via a homotopy $H : [0,1] \times [0,1] \longrightarrow M$ such that

$$H([0,1] \times [0,1]) \subset \gamma([0,1]) \cup \delta([0,1]),$$

where H is piecewise smooth for some paving of $[0,1] \times [0,1]$ by polygons.

[Note: Accordingly,

$$\begin{cases} H(t,0) = \gamma(t) \\ \qquad\qquad\qquad\qquad (0 \leq t \leq 1) \\ H(t,1) = \delta(t) \end{cases}$$

and, since H is rel $\partial[0,1]$,

$$\begin{cases} H(0,t) = * \\ \qquad\qquad\qquad\qquad (0 \leq t \leq 1).] \\ H(1,t) = * \end{cases}$$

Remark: The image of a smooth curve cannot fill a two dimensional submanifold.

Two loops $\gamma, \delta \in \Omega(*)$ are thinly equivalent, written $\gamma \underset{t}{\simeq} \delta$, if $\exists$ a finite sequence $\eta_1, \ldots, \eta_n \in \Omega(*)$ such that $\eta_1 = \gamma, \ldots, \eta_n = \delta$ with η_i thinly homotopic to η_{i+1}.

FACT Composition and inversion of loops gives rise to a group structure on

$$\Omega(*)/ \underset{t}{\simeq} \; \equiv \pi_1^t(M),$$

the thin fundamental group of M.

[Note: The homotopies used in the proof that $\Omega(*)/\simeq = \pi_1(M)$ is a group are thin (after smoothing at a finite number of non-differentiable points).]

Remark: There is a canonical surjection

$$\pi_1^t(M) \longrightarrow \pi_1(M)$$

which is an injection when dim M = 1.

LEMMA Suppose that $\gamma \underset{t}{\sim} \delta$ -- then $\forall P$ & $\forall \Gamma \in \mathfrak{A}(P)$,

$$h(\Gamma, p; \gamma) = h(\Gamma, p; \delta) \quad (p \in \pi^{-1}(*)).$$

[Note: In general, if $\gamma \simeq \delta$, then

$$h(\Gamma, p; \gamma) \neq h(\Gamma, p; \delta).]$$

Therefore $h(\Gamma, p; —)$ gives rise to a homomorphism

$$\pi_1^t(M) \longrightarrow G$$

which is smooth in the following sense.

Definition: A homomorphism

$$h: \pi_1^t(M) \longrightarrow G$$

is <u>smooth</u> if for every smooth family of loops $\Psi: U \to \Omega(*)$ the composition

$$U \xrightarrow{\Psi} \Omega(*) \xrightarrow{\text{pro}} \pi_1^t(M) \xrightarrow{h} G$$

is smooth.

Notation: $\mathrm{Hom}^{\infty}(\pi_1^t(M), G)$ is the set of smooth homomorphisms $h: \pi_1^t(M) \longrightarrow G$.

The group G operates to the right on $\mathrm{Hom}(\pi_1^t(M), G)$, viz:

$$h \cdot \sigma = \sigma^{-1} h \sigma .$$

Denote by

$$\mathrm{Hom}(\pi_1^t(M), G)/G$$

the associated set $\{[h]\}$ of equivalence classes.

Observation: $\mathrm{Hom}^{\infty}(\pi_1^t(M),G)$ is a G-stable subset of $\mathrm{Hom}(\pi_1^t(M),G)$.

If $p_1,p_2 \in \pi^{-1}(*)$, then

$$[h(\Gamma,p_1;—)] = [h(\Gamma,p_2;—)],$$

hence the class of

$$h(\Gamma,p;—)$$

in

$$\mathrm{Hom}^{\infty}(\pi_1^t(M),G)/G$$

is independent of the choice of $p \in \pi^{-1}(*)$. On the other hand,

$$[\Gamma_1] = [\Gamma_2] \Rightarrow [h(\Gamma_1,p;—)] = [h(\Gamma_2,p;—)].$$

THEOREM The arrow

$$[\Gamma] \longrightarrow [h(\Gamma,p;—)]$$

implements a bijection

$$\coprod_P \mathfrak{A}(P)/\mathcal{G}(P) \longrightarrow \mathrm{Hom}^{\infty}(\pi_1^t(M),G)/G.$$

Remark: Let $\mathfrak{A}_F(P)$ be the subset of $\mathfrak{A}(P)$ consisting of the flat connections -- then it follows from the Ambrose-Singer theorem that $\forall\, \Gamma \in \mathfrak{A}_F(P)$, $\mathrm{Hol}^0(\Gamma,p) = \{e\}$, so

$$\gamma \simeq \delta \Rightarrow h(\Gamma,p;\gamma) = h(\Gamma,p;\delta),$$

thus the map

$$\gamma \longrightarrow h(\Gamma,p;\gamma)$$

passes to the quotient and induces an arrow

$$\pi_1(M) \longrightarrow G$$

which is a homomorphism of groups. If $h: \pi_1(M) \longrightarrow G$ is a homomorphism, then the composition

$$\pi_1^t(M) \longrightarrow \pi_1(M) \xrightarrow{h} G$$

is necessarily smooth. It is wellknown that

$$\coprod_P \mathfrak{A}_F(P)/\mathcal{G}(P) \longleftrightarrow \mathrm{Hom}(\pi_1(M),G)/G.$$

[Note: Let $\tilde{M}$ be the universal covering space of M -- then $\tilde{M} \longrightarrow M$ is a principal $\pi_1(M)$-bundle. Each $h \in \mathrm{Hom}(\pi_1(M),G)$ determines a left action of $\pi_1(M)$ on G. The associated fiber bundle $\tilde{M} \times_{\pi_1(M)} G$ is a principal G-bundle which admits a natural flat connection.]

Example: Suppose that dim M = 1.

Case 1: $M = \mathbf{R}$ -- then every principal G-bundle is trivial: $P \approx \mathbf{R} \times G$. Here

$$\pi_1^t(\mathbf{R}) = \pi_1(\mathbf{R})$$

and

$$\begin{aligned} &\mathrm{Hom}(\pi_1(\mathbf{R}),G)/G \\ &= \mathrm{Hom}(*,G)/G \\ &= \{*\}, \end{aligned}$$

thus $\mathfrak{A}_F(P)/\mathcal{G}(P)$ is a singleton.

Case 2: $M = \mathbf{S}^1$. Here

$$\pi_1^t(\mathbf{S}^1) = \pi_1(\mathbf{S}^1)$$

and

$$\mathrm{Hom}(\pi_1(\underset{\sim}{S}^1),G)/G$$

$$= \mathrm{Hom}(\underset{\sim}{Z},G)/G$$

$$= G/\mathrm{Int},$$

the set of conjugacy classes in G.

[Note: In both cases, $\forall$ P, $\mathfrak{A}(P) = \mathfrak{A}_F(P)$. This is obvious when $M=\underset{\sim}{R}$: All connections are flat and, up to gauge equivalence, there is only one, namely the standard connection. When $M=\underset{\sim}{S}^1$, in a local trivialization consisting of a coordinate neighborhood U diffeomorphic to $\underset{\sim}{R}$, we have $\pi^{-1}(U) \approx U \times G \approx \underset{\sim}{R} \times G$, thus $\forall \Gamma \in \mathfrak{A}(P)$, the induced connection on $\pi^{-1}(U)$ "is" the standard connection, i.e., Γ is flat.]

Two loops $\gamma, \delta \in \Omega(*)$ are said to be <u>holonomically equivalent</u> if $\forall$ P & $\forall \Gamma \in \mathfrak{A}(P)$,

$$h(\Gamma,p;\gamma) = h(\Gamma,p;\delta) \quad (p \in \pi^{-1}(*)).$$

Accordingly,

γ, δ thinly equivalent $\Rightarrow$ γ, δ holonomically equivalent.

Notation: $\mathcal{HG}_G$ is $\Omega(*)$ modulo the holonomy relation.

<u>FACT</u> With the obvious operations, $\mathcal{HG}_G$ is a group, the <u>G-hoop group</u> of M.

Remark: There is a canonical surjection

$$\pi_1^t(M) \longrightarrow \mathcal{HG}_G.$$

The preceding theory can be written in terms of $\mathcal{HG}_G$ as opposed to $\pi_1^t(M)$, the upshot being the following conclusion.

THEOREM The arrow

$$[\Gamma] \longrightarrow [h(\Gamma, p; -)]$$

implements a bijection

$$\coprod_{P} \mathfrak{A}(P)/\mathcal{G}(P) \longrightarrow \mathrm{Hom}^{\infty}(\mathcal{H}\mathcal{G}_G, G)/G.$$

Definition: A connection $\Gamma \in \mathfrak{A}(P)$ is irreducible if

$$\mathrm{Hol}(\Gamma, p) = G.$$

FACT Suppose that

$$\begin{cases} \Gamma_1 \in \mathfrak{A}(P_1) \\ \Gamma_2 \in \mathfrak{A}(P_2) \end{cases}$$

are irreducible with

$$\ker h(\Gamma_1, p; -) = \ker h(\Gamma_2, p; -),$$

where

$$\begin{cases} h(\Gamma_1, p; -) \\ h(\Gamma_2, p; -) \end{cases}$$

are viewed as homomorphisms $\mathcal{H}\mathcal{G}_G \longrightarrow G$ -- then $\exists$ an equivariant diffeomorphism $f: P_1 \longrightarrow P_2$ over M such that $f_* \Gamma_1 = \Gamma_2$.

Rappel: A groupoid G is a small category in which every morphism is invertible. So, $\forall$ $X \in \mathrm{Ob}\, G$, $\mathrm{Mor}(X,X)$ is a group under composition.

[Note: A group G is a groupoid with one object e:Mor(e,e) = G.]

The notion of "holonomically equivalent" for loops can be generalized to arbitrary curves.

Definition: Let γ, δ be curves such that $x = \gamma(0) = \delta(0)$ & $y = \gamma(1) = \delta(1)$ -- then γ, δ are <u>holonomically equivalent</u> if $\forall$ P & $\forall\ \Gamma \in \mathfrak{O}(P)$,

$$T_\gamma = T_\delta \quad .$$

$\mathcal{O}\mathcal{G}_G$ is the groupoid whose objects are the points of M and whose morphisms are the equivalence classes of curves from x to y per the holonomy relation.

[Note: Therefore

$$\mathrm{Mor}(*,*) = \mathcal{H}\mathcal{G}_G .]$$

By a <u>point structure</u> on P, we understand the specification of a point in each fiber of π.

Claim: Fix a point structure on P -- then every $\Gamma \in \mathfrak{O}(P)$ determines a functor

$$h_\Gamma : \mathcal{O}\mathcal{G}_G \longrightarrow G.$$

[Send the objects of $\mathcal{O}\mathcal{G}_G$ to the identity of G. As for the morphisms, take $[\gamma] \in \mathrm{Mor}(x,y)$ and, relative to the given point structure on P, let $p \in \pi^{-1}(x)$, $q \in \pi^{-1}(y)$. Define $g_\gamma \in G$ by the relation

$$T_\gamma(p) = q \cdot g_\gamma \quad .$$

Then g_γ depends only on the holonomy class of γ. Putting

$$h_\Gamma[\gamma] = g_\gamma \quad ,$$

one checks without difficulty that h_Γ respects composition, hence is indeed a functor.]

Rappel: Let $\begin{cases} F:\underset{\sim}{C} \longrightarrow \underset{\sim}{D} \\ G:\underset{\sim}{C} \longrightarrow \underset{\sim}{D} \end{cases}$ be functors -- then a natural transformation Ξ from F to G is a function that assigns to each $X \in \mathrm{Ob}\,\underset{\sim}{C}$ an element $\Xi_X \in \mathrm{Mor}(FX,GX)$ such that for every $f \in \mathrm{Mor}(X,Y)$ the square

$$\begin{array}{ccc} FX & \xrightarrow{\Xi_X} & GX \\ {\scriptstyle Ff}\downarrow & & \downarrow{\scriptstyle Gf} \\ FY & \xrightarrow[\Xi_Y]{} & GY \end{array}$$

commutes, Ξ being termed a natural isomorphism if all the Ξ_X are isomorphisms, in which case F and G are naturally isomorphic.

[Note: If $\underset{\sim}{C},\underset{\sim}{D}$ are groupoids, then it is automatic that the Ξ_X are isomorphisms.]

Example: Let G,K be groups, thought of as groupoids (thus functors $G \longrightarrow K$ are homomorphisms). If $f,g \in \mathrm{Hom}(G,K)$, then a natural transformation $\Xi : f \longrightarrow g$ consists in the specification of an element $\varkappa \in K$ such that $\forall\, \sigma \in G$, there is a commutative diagram

$$\begin{array}{ccc} e & \xrightarrow{\varkappa} & e \\ {\scriptstyle f(\sigma)}\downarrow & & \downarrow{\scriptstyle g(\sigma)} \\ e & \xrightarrow[\varkappa]{} & e \end{array} .$$

Of course, Ξ is necessarily a natural isomorphism. Therefore, to say that f and g are naturally isomorphic amounts to saying that

$$[f] = [g] \text{ in } \mathrm{Hom}(G,K)/K.$$

Example: Let $\underset{\sim}{G}$ be a groupoid, K a group. Let $\Phi, \Psi : \underset{\sim}{G} \to K$ be functors -- then a natural transformation $\Xi : \Phi \to \Psi$ is a function $X \to \Xi_X$ from Ob $\underset{\sim}{G}$ to $K(= \mathrm{Mor}(e,e))$ such that $\forall \phi \in \mathrm{Mor}(X,Y)$, there is a commutative diagram

$$\begin{array}{ccc} e & \xrightarrow{\Xi_X} & e \\ {\scriptstyle \Phi\phi}\downarrow & & \downarrow{\scriptstyle \Psi\phi} \\ e & \xrightarrow[\Xi_Y]{} & e \end{array} .$$

The construction of

$$h_\Gamma : \mathcal{O}\mathcal{B}_G \longrightarrow G$$

hinges on a choice of the point structure for P. If this is changed, say

$$\begin{cases} p \longrightarrow p' = p \cdot \sigma \\ q \longrightarrow q' = q \cdot \tau \end{cases} ,$$

then g_γ is replaced by $g'_\gamma = \tau^{-1} g_\gamma \sigma$. Proof:

$$\begin{aligned} T_\gamma (p \cdot \sigma) &= T_\gamma \circ R_\sigma (p) \\ &= R_\sigma \circ T_\gamma (p) \\ &= R_\sigma (q \cdot g_\gamma) \\ &= q \cdot g_\gamma \sigma \\ &= (q \cdot \tau) \cdot (\tau^{-1} g_\gamma \sigma). \end{aligned}$$

From this, it follows that there is a natural isomorphism

$$\Xi : h'_\Gamma \longrightarrow h_\Gamma .$$

In fact, assign to each $x \in M$ the element

$$\Xi_x \in \mathrm{Mor}(h'_\Gamma x, h_\Gamma x)$$

corresponding to σ. Let $[\gamma] \in \mathrm{Mor}(x,y)$ -- then by the above, the diagram

$$\begin{tikzcd} e \arrow[r, "\sigma"] \arrow[d, "g'_\gamma"'] & e \arrow[d, "g_\gamma"] \\ e \arrow[r, "\tau"'] & e \end{tikzcd}$$

commutes, i.e., the diagram

$$\begin{tikzcd} h'_\Gamma x \arrow[r, "\Xi_x"] \arrow[d, "h'_\Gamma[\gamma]"'] & h_\Gamma x \arrow[d, "h_\Gamma[\gamma]"] \\ h'_\Gamma y \arrow[r, "\Xi_y"'] & h'_\Gamma y \end{tikzcd}$$

commutes.

Notation: $\mathrm{Hom}(\mathcal{P}\mathcal{G}_G, G)$ is the set of functors $h: \mathcal{P}\mathcal{G}_G \longrightarrow G$.

So, each $\Gamma \in \mathfrak{A}(P)$ determines an element

$$h_\Gamma \in \mathrm{Hom}(\mathcal{P}\mathcal{G}_G, G).$$

~~which does not depend on the choice of a point structure for P~~

Moreover, it can be shown that the arrow

$$\Gamma \longrightarrow h_\Gamma$$

implements a bijection

$$\coprod_P \mathfrak{A}(P) \longrightarrow \mathrm{Hom}^\infty(\mathcal{P}\mathcal{G}_G, G).$$

There is one final point in this circle of ideas.

Fix a point structure on P -- then every $f \in \mathcal{G}(P)$ determines a function $F_f : M \longrightarrow G$, namely

$$x \longrightarrow \mu_f(p).$$

Obviously, $F_f = F_g \Rightarrow f=g$, so we have an injection

$$\mathcal{G}(P) \longrightarrow \mathrm{Map}^{\infty}(M,G) \subset \mathrm{Map}(M,G).$$

[Note: If

$$p \longrightarrow p' = p\cdot\sigma ,$$

then

$$\begin{aligned} f(p') &= f(p)\cdot\sigma \\ &= p\cdot \mu_f(p)\sigma \\ &= p'\cdot\sigma^{-1}\mu_f(p)\sigma \end{aligned}$$

$$\Rightarrow$$

$$F_f' = \sigma^{-1}F_f\,\sigma.]$$

<u>The Analytic Setting</u> In what follows, we shall take G compact and assume that the base of our principal G-bundle is analytic rather than smooth.

[Note: Every paracompact C^{∞} manifold admits an analytic structure which is unique up to a C^{∞} diffeomorphism.]

Suppose therefore that M is analytic and path connected with $\dim M \geq 2$ -- then in this context, a curve is a continuous piecewise analytic map $\gamma : [0,1] \longrightarrow M$ which is a piecewise embedding, thus

$$\gamma : [0,t_1] \cup \cdots \cup [t_{n-1},1] \longrightarrow M$$

and $\gamma'(t) \neq 0$ on $[t_i,t_{i+1}]$ unless $\gamma[t_i,t_{i+1}] = \{x\}$ for some $x \in M$.

[Note: In the analytic category, two curves can intersect in an infinite set only if they overlap on some closed interval. This is false in the smooth category.]

An <u>edge</u> is a curve $e:[0,1] \longrightarrow M$ whose restriction to $]0,1[$ is an embedding.

[Note: We shall not distinguish between edges which differ by a reparametrization, i.e., by an analytic orientation preserving diffeomorphism of $[0,1]$.]

<u>FACT</u> Given a finite set of curves $\gamma_k (k \in K)$, $\exists$ a finite set of edges e_{ℓ} $(\ell \in L)$ such that

(a) $\forall$ $k \in K$, $\exists$ $L_k \subset L$ such that

$$\gamma_k = \prod_{L_k} e_{\ell}^{\pm} ;$$

(b) $\forall$ $\ell_1 \neq \ell_2$, $e_{\ell_1}(t_1) = e_{\ell_2}(t_2) \Rightarrow t_1, t_2 \in \{0,1\}$.

Remark: An <u>embedded graph</u> is a nonempty subset $\Lambda \subset M$ for which there exists a finite set of edges e_ℓ ($\ell \in L$) such that

$$\Lambda = \bigcup_\ell e_\ell$$

and

$$\forall\ \ell_1 \neq \ell_2,\ e_{\ell_1}(t_1) = e_{\ell_2}(t_2) \Rightarrow t_1, t_2 \in \{0,1\}.$$

The preceding result thus says that given a finite set of curves, $\exists$ an embedded graph with the property that each curve admits a representation as a product of certain edges of the graph (and their inverses).

[Note: Λ is a finite one dimensional CW-complex. As such, there is no unique choice of the e_ℓ satisfying the stated conditions.]

Example: If $\gamma : \underset{\sim}{S}^1 \rightarrow M$ is a loop, then the range of γ is an embedded graph.

Consider now the definition of "holonomically equivalent". Ostensibly, this definition depends on the choice of G. However, since we are working in the analytic category, this dependence can be partially eliminated.

Definition: Let γ_1, γ_2 be curves -- then γ_2 is said to arise from γ_1 by <u>inserting a retracing</u> if there is a $\tau \in [0,1]$ and a curve η such that

$$\gamma_2(t) = \begin{cases} \gamma_1(2t) & (0 \leq t \leq \frac{\tau}{2}) \\ \eta(4(t - \frac{\tau}{2})) & (\frac{\tau}{2} \leq t \leq \frac{\tau}{2} + \frac{1}{4}) \\ \eta(4(\frac{\tau}{2} + \frac{1}{2} - t)) & (\frac{\tau}{2} + \frac{1}{4} \leq t \leq \frac{\tau}{2} + \frac{1}{2}) \\ \gamma_1(2t-1) & (\frac{\tau}{2} + \frac{1}{2} \leq t \leq 1). \end{cases}$$

THEOREM Suppose that G is a compact connected nonabelian Lie group -- then two curves γ, δ are holonomically equivalent iff $\exists$ a finite sequence $\eta_1, \ldots, \eta_n$ of curves η_i such that $\eta_1 = \gamma, \ldots, \eta_n = \delta$, where η_i and η_{i+1} differ by a reparametrization or η_{i+1} (η_i) arises from η_i (η_{i+1}) by inserting a retracing.

[Note: This description is completely internal to M.]

Under the foregoing circumstances, we shall write $\mathcal{PG}$, $\mathcal{HG}$ in place of $\mathcal{PG}_G$, $\mathcal{HG}_G$.

Remark: It is clear from the theorem that if two loops $\gamma, \delta \in \Omega(*)$ are holonomically equivalent, then they are thinly equivalent. Consequently, the canonical surjection

$$\pi_1^t(M) \longrightarrow \mathcal{HG}$$

is an isomorphism.

[Note: The fundamental groupoid ΠM is a quotient of the holonomy groupoid $\mathcal{PG}$.]

The relation figuring in the statement of the theorem is an equivalence relation of general applicability, call it $\sim$.

FACT Composition and inversion of loops gives rise to a group structure on

$$\Omega(*)/\sim \; \equiv \mathcal{L}(*).$$

So, in the nonabelian case, $\mathcal{L}(*)$ is the hoop group. On the other hand, if G is a compact connected abelian Lie group, then

$$\mathcal{HG}_G = \mathcal{L}(*)/[\mathcal{L}(*), \mathcal{L}(*)],$$

a description which is again completely internal to M.

Remark: Suppose that G is a compact connected Lie group (e.g., $\mathbf{U}(1) \times \mathbf{SU}(2) \times \mathbf{SU}(3)$) -- then there are just two possibilities for $\mathcal{H}\mathcal{G}_G$:

$$
\begin{cases}
\mathcal{L}(*)/[\mathcal{L}(*), \mathcal{L}(*)] & \text{(G abelian)} \\
\mathcal{L}(*) & \text{(G nonabelian).}
\end{cases}
$$

[Note: G is necessarily reductive.]

INTERPOLATION PRINCIPLE Suppose given

$$
\begin{tikzcd}
G \arrow[r] & P \arrow[d, "\pi"] \\
 & M,
\end{tikzcd}
$$

where G is a compact connected Lie group. Let $[\gamma_1], \ldots, [\gamma_n] \in \mathcal{H}\mathcal{G}_G$ -- then $\forall\, h \in \mathrm{Hom}(\mathcal{H}\mathcal{G}_G, G)$, $\exists\ \Gamma \in \mathfrak{A}(P)$:

$$
h[\gamma_k] = h(\Gamma, p; \gamma_k) \quad (k=1,\ldots,n) \ .
$$

Ashtekar Space Suppose that M is analytic and path connected with dim $M \geq 2$ -- then by the term "graph" we shall mean a connected embedded graph, Gra M standing for the set of graphs in M.

Notation: Given a graph Λ, denote by $E(\Lambda)$ its set of edges and $V(\Lambda)$ its set of vertices.

If Λ_1, Λ_2 are graphs, then $\Lambda_1 \leq \Lambda_2$ if each edge of Λ_1 is a product $(^{\pm})$ of edges of Λ_2 and $V(\Lambda_1) \subset V(\Lambda_2)$.

FACT Gra M is directed by $\leq$.

One may attach to each $\Lambda \in$ Gra M the groupoid $\mathcal{PG}_\Lambda$ which is freely generated by the edges of Λ. Thus the objects of $\mathcal{PG}_\Lambda$ are the vertices of Λ and the morphisms are all possible compositions of the edges and their inverses.

Assume now that G is a compact connected nonabelian Lie group -- then the notion of "holonomically equivalent" does not depend on G and the groupoid $\mathcal{PG}$ is generated by edges (but it is not freely generated by edges). In fact,

$$\mathcal{PG} = \operatorname*{colim}_{\Lambda} \mathcal{PG}_\Lambda .$$

Let

$$\begin{cases} \overline{\mathfrak{A}}_\Lambda = \operatorname{Hom}(\mathcal{PG}_\Lambda, G) \\ \overline{\mathcal{G}}_\Lambda = \operatorname{Map}(V(\Lambda), G). \end{cases}$$

Since a functor $h: \mathcal{PG}_\Lambda \longrightarrow G$ is determined by the images of the $e \in E(\Lambda)$, it is clear that

$$\overline{\mathfrak{A}}_\Lambda \approx G^{\#E(\Lambda)}.$$

Analogously,

$$\overline{\mathcal{G}}_\Lambda \approx G^{\#V(\Lambda)}.$$

Therefore $\overline{\mathfrak{A}}_\Lambda$ and $\overline{\mathcal{G}}_\Lambda$ are compact Hausdorff spaces.

Observation: There is a right action of $\overline{\mathcal{G}}_\Lambda$ on $\overline{\mathfrak{A}}_\Lambda$, viz.

$$\begin{cases} \overline{\mathfrak{A}}_\Lambda \times \overline{\mathcal{G}}_\Lambda \longrightarrow \overline{\mathfrak{A}}_\Lambda \\ (h, \phi) \longrightarrow h \cdot \phi, \end{cases}$$

where

$$h \cdot \phi(e) = \phi(e(1))^{-1} h(e) \phi(e(0)).$$

<u>FACT</u> $\overline{\mathfrak{A}}_\Lambda / \overline{\mathcal{G}}_\Lambda$ is a compact Hausdorff space.

Let $\pi_1(\Lambda)$ be the fundamental group of Λ (based at a vertex) -- then $\pi_1(\Lambda)$ is free on $1-\chi(\Lambda)$ generators $(\chi(\Lambda) = \#V(\Lambda) - \#E(\Lambda))$. The hoop group of Λ "is" the fundamental group of Λ and

$$\overline{\mathfrak{A}}_\Lambda / \overline{\mathcal{G}}_\Lambda \approx \mathrm{Hom}(\pi_1(\Lambda), G)/G.$$

Suppose that $\Lambda_1 \leq \Lambda_2$ -- then there are arrows of restriction

$$\begin{cases} \overline{\mathfrak{A}}_{\Lambda_2} \longrightarrow \overline{\mathfrak{A}}_{\Lambda_1} \\ \overline{\mathcal{G}}_{\Lambda_2} \longrightarrow \overline{\mathcal{G}}_{\Lambda_1} \\ \overline{\mathfrak{A}}_{\Lambda_2} / \overline{\mathcal{G}}_{\Lambda_2} \longrightarrow \overline{\mathfrak{A}}_{\Lambda_1} / \overline{\mathcal{G}}_{\Lambda_1}, \end{cases}$$

denoted in all three cases by π_1^2.

<u>FACT</u> These maps are continuous, open and surjective.

One can check that

$$\Lambda_1 \leq \Lambda_2 \leq \Lambda_3 \Rightarrow \pi_1^2 \circ \pi_2^3 = \pi_1^3,$$

which sets the stage for passage to the limit.

Definition: Put

$$\begin{cases} \overline{\mathfrak{A}} = \lim \overline{\mathfrak{A}}_\Lambda \left(\subset \prod_\Lambda \overline{\mathfrak{A}}_\Lambda\right) \\ \overline{\mathcal{G}} = \lim \overline{\mathcal{G}}_\Lambda \left(\subset \prod_\Lambda \overline{\mathcal{G}}_\Lambda\right) \\ \overline{\mathfrak{A}/\mathcal{G}} = \lim \overline{\mathfrak{A}}_\Lambda/\overline{\mathcal{G}}_\Lambda \left(\subset \prod_\Lambda \overline{\mathfrak{A}}_\Lambda/\overline{\mathcal{G}}_\Lambda\right). \end{cases}$$

Obviously, $\overline{\mathfrak{A}}$, $\overline{\mathcal{G}}$, and $\overline{\mathfrak{A}/\mathcal{G}}$ are compact Hausdorff spaces.

There are projections

$$\begin{cases} \overline{\mathfrak{A}} \longrightarrow \overline{\mathfrak{A}}_\Lambda \\ \overline{\mathcal{G}} \longrightarrow \overline{\mathcal{G}}_\Lambda \\ \overline{\mathfrak{A}/\mathcal{G}} \longrightarrow \overline{\mathfrak{A}}_\Lambda/\overline{\mathcal{G}}_\Lambda, \end{cases}$$

denoted in all three cases by π_Λ.

<u>FACT</u> These maps are continuous, open and surjective.

<u>THEOREM</u> We have

$$\begin{cases} \overline{\mathfrak{A}} \approx \mathrm{Hom}(\mathcal{P}\mathcal{G}, G) \\ \overline{\mathcal{G}} \approx \mathrm{Map}(M, G). \end{cases}$$

Observation: There is a right action of $\overline{\mathcal{G}}$ on $\overline{\mathfrak{A}}$, viz.

$$\begin{cases} \overline{\mathfrak{A}} \times \overline{\mathcal{G}} \longrightarrow \overline{\mathfrak{A}} \\ (\{h_\Lambda\}, \{\phi_\Lambda\}) \longrightarrow \{h_\Lambda \cdot \phi_\Lambda\}. \end{cases}$$

THEOREM We have

$$\overline{\mathfrak{A}/\mathcal{G}} \approx \overline{\mathfrak{A}}/\overline{\mathcal{G}} \approx \mathrm{Hom}(\mathcal{H}\mathcal{G}, G)/G.$$

Remark: A choice of a point structure on P leads to embeddings

$$\begin{cases} \mathfrak{A}(P) \longrightarrow \overline{\mathfrak{A}} \\ \mathcal{G}(P) \longrightarrow \overline{\mathcal{G}} \end{cases}, \quad \mathfrak{A}(P)/\mathcal{G}(P) \longrightarrow \overline{\mathfrak{A}}/\overline{\mathcal{G}}.$$

Each has a dense image.

Let $\overline{\mathcal{G}}_*$ be the subgroup of $\overline{\mathcal{G}}$ consisting of those strings $\sigma_x (x \in M)$ such that $\sigma_* = e$ -- then it is clear that

$$\overline{\mathcal{G}} \approx G \times \overline{\mathcal{G}}_* .$$

Claim: We have

$$\mathrm{Hom}(\mathcal{P}\mathcal{G}, G) \approx \mathrm{Hom}(\mathcal{H}\mathcal{G}, G) \times \overline{\mathcal{G}}_* .$$

[Let $E = \{e_x : x \in M\}$, where e_* is the trivial loop and $\forall x \neq *$, $e_x \in \mathrm{Mor}(*, x)$ is an edge. Define

$$\Theta_E: \mathrm{Hom}(\mathcal{P}\mathcal{G}, G) \longrightarrow \mathrm{Hom}(\mathcal{H}\mathcal{G}, G) \times \overline{\mathcal{G}}_*$$

by the prescription

$$\Theta_E h = (H, \phi_0): \begin{cases} H[\gamma] = h[\gamma] \\ \phi_0(x) = he_x. \end{cases}$$

Then it can be shown that Θ_E is a homeomorphism.]

[Note: $\mathrm{Hom}(\mathcal{P}\mathcal{G}, G)$ is a right $\overline{\mathcal{G}}$-space:

$$h\cdot\phi([\gamma]) = \phi(\gamma(1))^{-1}(h[\gamma])\phi(\gamma(0)).$$

The same is true of $\operatorname{Hom}(\mathcal{H}\mathcal{G}, G) \times \overline{\mathcal{G}}_*$. Indeed,

$$(H, \phi_0)\cdot\phi = (H\cdot\phi\,,\ \phi_0\cdot\phi)\ :\ \begin{cases} H\cdot\phi\,([\gamma]) = \phi(*)^{-1}(H[\gamma])\,\phi(*) \\ \\ (\phi_0\cdot\phi)(x) = \phi(x)^{-1}\phi_0(x)\,\phi(*). \end{cases}$$

Working through the definitions, one finds that Θ_E is actually $\overline{\mathcal{G}}$-equivariant.]

Application: We have

$$\begin{aligned} &\operatorname{Hom}(\mathcal{O}\mathcal{G}, G)/\overline{\mathcal{G}} \\ &\approx (\operatorname{Hom}(\mathcal{H}\mathcal{G}, G)\times\overline{\mathcal{G}}_*)/(G\times\overline{\mathcal{G}}_*) \\ &\approx \operatorname{Hom}(\mathcal{H}\mathcal{G}, G)/G \times \overline{\mathcal{G}}_*/\overline{\mathcal{G}}_* \\ &\approx \operatorname{Hom}(\mathcal{H}\mathcal{G}, G)/G. \end{aligned}$$

I.e.:

$$\overline{\mathfrak{A}}/\overline{\mathcal{G}} \approx \operatorname{Hom}(\mathcal{H}\mathcal{G}, G)/G.$$

SLICE THEOREM For any $h \in \overline{\mathfrak{A}}$, $\exists$ a subset $\overline{\mathcal{S}} \subset \overline{\mathfrak{A}}$ such that

(1) $\overline{\mathcal{S}}\cdot\overline{\mathcal{G}}$ is a neighborhood of $h\cdot\overline{\mathcal{G}}$ with $h \in \overline{\mathcal{S}}$;

(2) $\exists$ an equivariant retraction $r: \overline{\mathcal{S}}\cdot\overline{\mathcal{G}} \longrightarrow h\cdot\overline{\mathcal{G}}$ with $r^{-1}(\{h\}) = \overline{\mathcal{S}}$.

~~Suppose that M is analytic and path connected with dim M ≥ 2~~

<u>Types</u> Suppose that M is analytic and path connected with $\dim M \geq 2$ and G is a compact connected nonabelian Lie group.

~~XNotexxxThesexarexourxstandingxassumptionsxinxthexsequelxx~~

Let $h \in \overline{\mathfrak{A}} = \mathrm{Hom}(\mathcal{P}\mathcal{G}, G)$ be given -- then

$$H_h = h(\mathcal{H}\mathcal{G}) \subset G$$

is the <u>holonomy group</u> of h and

$$Z_h = \mathrm{Cen}_G\, H_h$$

is the <u>holonomy centralizer</u> of h.

<u>FACT</u> Let H be any subgroup of G -- then $\exists\, h \in \overline{\mathfrak{A}} : H_h = H$.

[Note: There are no topological requirements on H.]

Remark: Fix a point structure on P -- then each $\Gamma \in \mathfrak{A}(P)$ determines a functor $h_\Gamma : \mathcal{P}\mathcal{G} \longrightarrow G$, viz.

$$h_\Gamma[\gamma] = g_\gamma \quad (T_\gamma(p) = q \cdot g_\gamma).$$

So, working at the base point $*$ and taking $\gamma \in \Omega(*)$, we have by definition

$$\begin{aligned} \mathrm{Hol}(\Gamma, p) &= \{g_\gamma : \gamma \in \Omega(*)\} \\ &= h_\Gamma(\mathcal{H}\mathcal{G}). \end{aligned}$$

<u>LEMMA</u> $\forall\, \phi \in \overline{\mathcal{G}}$,

$$\begin{cases} H_{h\cdot\phi} = \phi(*)^{-1}\, H_h\, \phi(*) \\ Z_{h\cdot\phi} = \phi(*)^{-1}\, Z_h\, \phi(*). \end{cases}$$

The orbit $h \cdot \overline{\mathcal{G}}$ is a compact Hausdorff space:

$$h \cdot \overline{\mathcal{G}} \approx \overline{\mathcal{G}}_h \backslash \overline{\mathcal{G}}.$$

<u>FACT</u> We have

$$\overline{\mathcal{G}}_h \backslash \overline{\mathcal{G}} \approx (Z_h \backslash G) \times \overline{\mathcal{G}}_*.$$

So, as a corollary, if Z_{h_1} and Z_{h_2} are conjugate in G, then the orbits $h_1 \cdot \overline{\mathcal{G}}$ and $h_2 \cdot \overline{\mathcal{G}}$ are homeomorphic.

Definition: The <u>type</u> typ(h) of an orbit $h \cdot \overline{\mathcal{G}}$ is the conjugacy class in G of Z_h.

[Note: This definition depends only on $[h] \in \overline{\mathfrak{A}} / \overline{\mathcal{G}}$. In fact, if $h' = h \cdot \phi$, then $Z_{h'} = \phi(*)^{-1} Z_h \phi(*)$.]

From the above, therefore, if two orbits have the same type, then they are homeomorphic.

Rappel: A subgroup H of G is said to be a <u>Howe subgroup</u> if there is a set $S \subset G$ such that $H = \text{Cen}_G\, S$.

Example: Take $G = \underset{\sim}{SU}(2)$ -- then the maximal tori are Howe subgroups.

Notation: $\mathfrak{T}$ is the set of conjugacy classes of Howe subgroups of G.

[Note: Since G is compact, $\mathfrak{T}$ is at most countable.]

Given $t_1, t_2 \in \mathfrak{T}$, write $t_1 \leq t_2$ if $\exists\, H_1 \in t_1,\ H_2 \in t_2$ such that $H_1 \supset H_2$.

Example: The maximal element in $\mathfrak{T}$ is the class t_{max} of the center Z_G of G.

Example: The minimal element in $\mathfrak{T}$ is the class t_{min} of G itself.

Notation: Given $t \in \mathfrak{T}$, let

$$\begin{cases} \bar{\mathfrak{A}}_{\geq t} = \{h \in \bar{\mathfrak{A}} : \mathrm{typ}(h) \geq t\} \\ \bar{\mathfrak{A}}_{=t} = \{h \in \bar{\mathfrak{A}} : \mathrm{typ}(h) = t\} \\ \bar{\mathfrak{A}}_{\leq t} = \{h \in \bar{\mathfrak{A}} : \mathrm{typ}(h) \leq t\}. \end{cases}$$

Properties:

(1) $\bar{\mathfrak{A}}_{\geq t}$ is open;

(2) $\bar{\mathfrak{A}}_{\leq t}$ is compact;

(3) $\bar{\mathfrak{A}}_{=t}$ is open in $\bar{\mathfrak{A}}_{\leq t}$;

(4) $\bar{\mathfrak{A}}_{=t}$ is dense in $\bar{\mathfrak{A}}_{\leq t}$.

THEOREM $\forall\ t \geq \mathrm{typ}(h)$, $\exists\ h_t \in \bar{\mathfrak{A}}$:

$$\mathrm{typ}(h_t) = t.$$

Let h be the trivial element of $\mathrm{Hom}(\mathcal{P}\mathcal{Y}, G)$, i.e., $h[\gamma] = e$ $\forall\ \gamma$ -- then $H_h = \{e\} \Rightarrow Z_h = G$, hence $\mathrm{typ}(h) = t_{\min}$. It therefore follows that $\forall\ t \in \mathcal{T}$, $\exists\ h_t \in \bar{\mathfrak{A}}$:

$$\mathrm{typ}(h_t) = t.$$

In other words, the set of orbit types exhausts the set of conjugacy classes of Howe subgroups of G.

Rappel: Let X be a topological space -- then a collection $\mathcal{S} = \{S\}$ of nonempty subsets of X is said to be a <u>stratification</u> of X (the S being <u>strata</u>) if $X = \coprod_{\mathcal{S}} S$ and

$$\bar{S} \cap S' \neq \emptyset \Rightarrow \begin{cases} \bar{S} \supset S' \\ \bar{S}' \cap (S \cup S') = S'. \end{cases}$$

[Note: Write $S \prec S'$ if $\overline{S} \cap S' \neq \emptyset$ -- then it is easy to prove that

$$\overline{S} = \bigcup_{S \prec S'} S'.]$$

Example: Take for X the unit cube in $\underset{\sim}{R}^3$. Let $\mathcal{S}$ consist of the interior of the cube, the relative interiors of the six faces, the relative interiors of the twelve bounding segments, and the eight corners -- then $\mathcal{S}$ is a stratification of X.

THEOREM The collection $\{\overline{\mathfrak{A}}_{=t} : t \in \mathcal{T}\}$ is a stratification of $\overline{\mathfrak{A}}$.

An element $h \in \overline{\mathfrak{A}}$ is said to be <u>generic</u> if

$$\text{typ}(h) = t_{\max}.$$

[Note: Therefore, when h is generic, $Z_h = Z_G$.]

Let

$$\overline{\mathfrak{A}}_{\text{gen}} = \overline{\mathfrak{A}}_{=t_{\max}}.$$

Then

$$\overline{\mathfrak{A}}_{\text{gen}} = \overline{\mathfrak{A}}_{\geq t_{\max}},$$

so $\overline{\mathfrak{A}}_{\text{gen}}$ is an open subset of $\overline{\mathfrak{A}}$. On the other hand,

$$\overline{\mathfrak{A}} = \overline{\mathfrak{A}}_{\leq t_{\max}},$$

hence $\overline{\mathfrak{A}}_{\text{gen}}$ is a dense subset of $\overline{\mathfrak{A}}$.

[Note: It is clear that $\overline{\mathfrak{A}}_{\text{gen}}$ is $\overline{\mathcal{G}}$-invariant.]

An element $h \in \overline{\mathfrak{A}}$ is <u>irreducible</u> if $H_h = G$.

Obviously, h irreducible $\Rightarrow$ h generic. The converse is false.
Proof: Fix a proper subgroup $H \subset G$: $\text{Cen}_G\, H = Z_G$ -- then $\exists\, h \in \bar{\mathfrak{A}}$: $H_h = H$, thus h is generic but not irreducible.

[Note: One can take for H the subgroup generated by a countable dense set (H is countable, hence is a proper subgroup of G).]

<u>The Holonomy Algebra</u> By way of motivation, we shall first look at a special case. So suppose that M is analytic and path connected with $\dim M = 3$.

Consider

$$\begin{array}{ccc} \mathbf{SU}(2) & \longrightarrow & P \\ & & \big\downarrow \pi \\ & & M. \end{array}$$

Then P is trivial: $P \approx M \times \mathbf{SU}(2)$.

[Note: The canonical section $s : M \longrightarrow M \times \mathbf{SU}(2)$ is given by

$$s(x) = (x,e).$$

If $g : M \longrightarrow \mathbf{SU}(2)$ is smooth, then

$$s^g(x) = s(x) \cdot g(x) = (x, g(x))$$

is another section and all such have this form.]

Agreeing to work only with $M \times \mathbf{SU}(2)$, write $\mathfrak{A}$ in place of $\mathfrak{A}(P)$ and $\mathcal{G}$ in place of $\mathcal{G}(P)$.

Given a connection Γ on $M \times \mathbf{SU}(2)$, let

$$W(\Gamma, \gamma) = \frac{1}{2}\,\mathrm{tr}(h(\Gamma, p; \gamma)).$$

Then

$W(\Gamma, \gamma)$ depends only on $[\Gamma] \in \mathfrak{A}/\mathcal{G}$ and $[\gamma] \in \mathcal{HG}$, thus $W(—, [\gamma])$ is a real valued function on $\mathfrak{A}/\mathcal{G}$.

[Note: Recall too that

$$h(\Gamma, p; —) : \mathcal{HG} \longrightarrow \mathbf{SU}(2)$$

is a homomorphism.]

Since

$$\forall \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \in \mathbf{SU}(2), \quad |a|^2 + |b|^2 = 1$$

$$\Rightarrow$$

$$0 \leq |a| \leq 1 \Rightarrow -2 \leq a + \bar{a} \leq 2,$$

it follows that

$$|W(\text{—},[\gamma])| \leq 1.$$

Claim: The complex vector space spanned by the $W(\text{—},[\gamma])$ is closed under the formation of products.

[$\forall\, g,h \in \underset{\sim}{SU}(2)$, we have

$$\operatorname{tr}(g)\operatorname{tr}(h) = \operatorname{tr}(gh) + \operatorname{tr}(gh^{-1}).$$

Therefore

$$W(\text{—},[\gamma_1])W(\text{—},[\gamma_2])$$

$$= \frac{1}{2}\,(W(\text{—},[\gamma_1][\gamma_2]) + W(\text{—},[\gamma_1][\gamma_2]^{-1})).]$$

Denote this algebra by $\mathcal{H}\mathfrak{A}$ and call it the <u>holonomy algebra</u>.

Claim: $\mathcal{H}\mathfrak{A}$ is a unital commutative $*$-algebra.

[The $*$-operation is

$$\Big(\sum_i c_i W(\text{—},[\gamma_i])\Big)^* = \sum_i \bar{c}_i W(\text{—},[\gamma_i]).]$$

Equip $\mathcal{H}\mathfrak{A}$ with the sup norm -- then its completion $\overline{\mathcal{H}\mathfrak{A}}$ is a unital commutative C*-algebra, thus $\overline{\mathcal{H}\mathfrak{A}} \approx C(\operatorname{Spec}\overline{\mathcal{H}\mathfrak{A}})$. And (see below),

$$\overline{\mathfrak{A}/\mathfrak{G}} \approx \operatorname{Spec}\overline{\mathcal{H}\mathfrak{A}}.$$

[Note: The identification

$$\begin{cases} \operatorname{Hom}(\mathcal{H}\mathfrak{G}, G)/G \approx \operatorname{Spec}\overline{\mathcal{H}\mathfrak{A}} \\ \qquad\qquad h \longleftrightarrow \varphi_h \end{cases} \qquad (G = \underset{\sim}{SU}(2))$$

is characterized by the relation

$$\varphi_h(W(\text{—},[\gamma])) = \frac{1}{2}\operatorname{tr}(h[\gamma]).]$$

To generalize these conclusions requires some preparation.

Assume that M is analytic and path connected with $\dim M \geq 2$ and let G be a compact connected nonabelian normal subgroup of $\underset{\sim}{U}(N)$ $(N \geq 2)$.

Notation: Per $M \times G$, $\mathfrak{A}$ is the set of connections and $\mathcal{G}$ is the set of gauge transformations.

Rappel: Let H be a topological group -- then $\exists$ a compact topological group $\overline{H}$ and a continuous homomorphism $\alpha : H \longrightarrow \overline{H}$ with the following property: $\forall$ compact topological group $\overline{H}'$ and $\forall$ continuous homomorphism $\alpha' : H \longrightarrow \overline{H}'$, $\exists$ a unique continuous homomorphism $\beta : \overline{H} \longrightarrow \overline{H}'$ such that $\alpha' = \beta \circ \alpha$:

$$\begin{tikzcd} H \arrow[r, "\alpha"] \arrow[dr, "\alpha'"'] & \overline{H} \arrow[d, dashed, "\beta"] \\ & \overline{H}' \end{tikzcd}$$

[Note: $\alpha(H)$ is dense in $\overline{H}$ and $\overline{H}$ is unique up to isomorphism.]

Definition: Let H be a topological group -- then H is said to be <u>injectable</u> if $\operatorname{Ker} \alpha = \{e\}$.

[Note: In general, the kernel of α is equal to the intersection of the kernels of the continuous homomorphisms of H into all compact groups or, equivalently, is equal to the intersection of the kernels of the finite dimensional irreducible unitary representations of H.]

Example: Equip $\mathcal{H}\mathcal{G}$ with the discrete topology -- then $\mathcal{H}\mathcal{G}$ is injectable.

[In fact,

$$\bigcap_{\Gamma \in \mathfrak{A}} \ker h(\Gamma, p; \text{—}) = \{\mathrm{id}_{\mathcal{H}\mathcal{G}}\}\ .]$$

Definition: Let H be a topological group -- then a bounded continuous function $f: H \longrightarrow \underset{\sim}{C}$ is said to be <u>almost periodic</u> if f is the uniform limit of finite linear combinations of matrix coefficients of the finite dimensional irreducible unitary representations of H.

<u>FACT</u> Let $f \in C_b(H)$ -- then f is almost periodic iff $\exists$ a continuous function $\overline{f}: \overline{H} \longrightarrow \underset{\sim}{C}$ such that $f = \overline{f} \circ \alpha$.

Example: Equip $\mathcal{HG}$ with the discrete topology -- then

$$\mathrm{Hom}(\mathcal{HG}, G) \approx \mathrm{Hom}_c(\overline{\mathcal{HG}}, G),$$

the subscript standing for continuous.

Denote by AP(H) the set of almost periodic functions on H -- then AP(H) is a closed subspace of $C_b(H)$, hence is a unital commutative C*-algebra. And: $AP(H) \approx C(\overline{H})$ via $f \longrightarrow \overline{f}$.

Pass now to $M \times G$ and define

$$W: \mathfrak{A}/\mathcal{G} \times \mathcal{HG} \longrightarrow \underset{\sim}{C}$$

by

$$W([\Gamma], [\gamma]) = \frac{1}{N} \operatorname{tr}(h(\Gamma, p; \gamma)).$$

Definition: The <u>holonomy algebra</u> $\mathcal{H}\mathfrak{A}$ is the algebra over $\underset{\sim}{C}$ generated by the $W(—, [\gamma])$ $([\gamma] \in \mathcal{HG})$.

[Note: The elements of $\mathcal{H}\mathfrak{A}$ thus have the form

$$\sum_{i=1}^{n} c_i \prod_{j_i=1}^{n_i} W(—, [\gamma_{j_i}]).]$$

Since

$$\overline{W(—, [\gamma])} = W(—, [\gamma^{-1}]),$$

it follows that $\mathcal{H}\mathfrak{A}$ is an involutive subalgebra of $B(\mathfrak{A}/\mathfrak{G})$, the C*-algebra of bounded complex valued functions on $\mathfrak{A}/\mathfrak{G}$.

LEMMA There is a canonical map

$$h \longrightarrow \varphi_h$$

from Hom($\mathcal{H}\mathfrak{G}$,G) to the continuous multiplicative linear functionals on $\mathcal{H}\mathfrak{A}$.

[Given h, define

$$\varphi_h : \mathcal{H}\mathfrak{A} \longrightarrow \mathbb{C}$$

by

$$\varphi_h(W(\text{---},[\gamma])) = \frac{1}{N}\,\mathrm{tr}(h[\gamma]).$$

That φ_h actually does extend to a multiplicative linear functional on $\mathcal{H}\mathfrak{A}$ is implied by:

1. If

$$\left\{\begin{matrix} [\gamma_1],\ldots,[\gamma_n] \\ \\ [\delta_1],\ldots,[\delta_m] \end{matrix}\right. \in \mathcal{H}\mathfrak{G}$$

and if

$$W(\text{---},[\gamma_1])\cdots W(\text{---},[\gamma_n]) = W(\text{---},[\delta_1])\cdots W(\text{---},[\delta_m]),$$

then

$$\prod_{k=1}^{n} \varphi_h(W(\text{---},[\gamma_k])) = \prod_{\ell=1}^{m} \varphi_h(W(\text{---},[\delta_\ell])).$$

2. If

$$\sum_{i=1}^{n} c_i \prod_{j_i=1}^{n_i} W(—,[\gamma_{j_i}]) = 0,$$

then

$$\sum_{i=1}^{n} c_i \prod_{j_i=1}^{n_i} \varphi_h(W(—,[\gamma_{j_i}])) = 0.$$

Ad 1: Owing to the interpolation principle, $\exists\ \Gamma \in \mathfrak{A}$:

$$\begin{cases} h[\gamma_k] = h(\Gamma,p;\gamma_k) & (k=1,\ldots,n) \\ h[\delta_\ell] = h(\Gamma,p;\delta_\ell) & (\ell=1,\ldots,m). \end{cases}$$

Therefore

$$\prod_{k=1}^{n} \varphi_h(W(—,[\gamma_k]))$$

$$= \prod_{k=1}^{n} \frac{1}{N}\,\mathrm{tr}(h[\gamma_k])$$

$$= \prod_{k=1}^{n} \frac{1}{N}\,\mathrm{tr}(h(\Gamma,p;\gamma_k))$$

$$= \prod_{k=1}^{n} W(\Gamma,[\gamma_k])$$

$$= \prod_{\ell=1}^{m} W(\Gamma, [\delta_\ell])$$

$$= \prod_{\ell=1}^{m} \frac{1}{N} \operatorname{tr}(h(\Gamma, p; \delta_\ell))$$

$$= \prod_{\ell=1}^{m} \frac{1}{N} \operatorname{tr}(h[\delta_\ell])$$

$$= \prod_{\ell=1}^{m} \varphi_h(W(-, [\delta_\ell])).$$

<u>Ad 2:</u> Owing to the interpolation principle, $\exists\, \Gamma_0 \in \mathfrak{A}$:

$$h[\gamma_{j_i}] = h(\Gamma_0, p; \gamma_{j_i}) \quad (1 \leq i \leq n,\ 1 \leq j_i \leq n_i).$$

Therefore

$$\sum_{i=1}^{n} c_i \prod_{j_i=1}^{n_i} \varphi_h(W(-, [\gamma_{j_i}]))$$

$$= \sum_{i=1}^{n} c_i \prod_{j_i=1}^{n_i} \frac{1}{N} \operatorname{tr}(h[\gamma_{j_i}])$$

$$= \sum_{i=1}^{n} c_i \prod_{j_i=1}^{n_i} \frac{1}{N} \operatorname{tr}(h(\Gamma_0, p; \gamma_{j_i}))$$

$$= \sum_{i=1}^{n} c_i \prod_{j_i=1}^{n_i} W(\Gamma_0,[\gamma_{j_i}])$$

$$= 0.$$

To establish the continuity of φ_h, choose Γ_0 as above and then note that

$$\left|\varphi_h\left(\sum_{i=1}^{n} c_i \prod_{j_i=1}^{n_i} W(\text{—},[\gamma_{j_i}])\right)\right|$$

$$= \left|\sum_{i=1}^{n} c_i \prod_{j_i=1}^{n_i} W(\Gamma_0,[\gamma_{j_i}])\right|$$

$$\leq \sup_{\Gamma\in\mathfrak{A}/\mathfrak{G}} \left|\sum_{i=1}^{n} c_i \prod_{j_i=1}^{n_i} W(\Gamma,[\gamma_{j_i}])\right|$$

$$= \left\|\sum_{i=1}^{n} c_i \prod_{j_i=1}^{n_i} W(\text{—},[\gamma_{j_i}])\right\|_{\infty}.]$$

Denote by $\overline{\mathcal{H}\mathfrak{A}}$ the closure of $\mathcal{H}\mathfrak{A}$ in $B(\mathfrak{A}/\mathfrak{G})$ -- then $\overline{\mathcal{H}\mathfrak{A}}$ is a unital commutative C*-algebra. And, thanks to the lemma, there is an arrow

$$\mathrm{Hom}(\mathcal{H}\mathfrak{G},G) \longrightarrow \mathrm{Spec}\,\overline{\mathcal{H}\mathfrak{A}},$$

viz. $h \longrightarrow \varphi_h$.

[Note: Tacitly, φ_h has been extended by continuity to $\overline{\mathcal{HA}}$, which is permissible (φ_h is a continuous linear functional, hence, of necessity, is uniformly continuous).]

Obviously, $\forall\ \sigma \in G$,

$$\varphi_{h\cdot\sigma} \ (= \varphi_{\sigma^{-1}h\sigma}) = \varphi_h.$$

This said, suppose that $\varphi_{h_1} = \varphi_{h_2}$ -- then $\forall\ [\gamma] \in \mathcal{HB}$,

$$\mathrm{tr}(h_1[\gamma]) = \mathrm{tr}(h_2[\gamma]).$$

But

$$\mathrm{Hom}(\mathcal{HB}, G) \approx \mathrm{Hom}_c(\overline{\mathcal{HB}}, G).$$

So, if

$$\begin{cases} \bar{h}_1 : \overline{\mathcal{HB}} \longrightarrow G \\ \\ \bar{h}_2 : \mathcal{HB} \longrightarrow G \end{cases}$$

correspond to

$$\begin{cases} h_1 : \mathcal{HB} \longrightarrow G \\ \\ h_2 : \mathcal{HB} \longrightarrow G, \end{cases}$$

then there is an equality of characters

$$\chi_{\bar{h}_1} = \chi_{\bar{h}_2}.$$

Since $\overline{\mathcal{HB}}$ is compact, it follows that $\exists\ U \in \underset{\sim}{U}(N)$:

$$\bar{h}_2 = U^{-1}\bar{h}_1 U.$$

Therefore

$$h_2 = U^{-1} h_1 U$$

$$\Longrightarrow$$

$$\varphi_{h_2}(W(—,[\gamma]))$$

$$= \frac{1}{N}\,\mathrm{tr}(h_2[\gamma])$$

$$= \frac{1}{N}\,\mathrm{tr}(U^{-1}h_1[\gamma]U)$$

$$= \frac{1}{N}\,\mathrm{tr}(h_1[\gamma])$$

$$= \varphi_{h_1}(W(—,[\gamma])).$$

LEMMA The conjugacy classes in G per $\underset{\sim}{U}(N)$ and G are one and the same.

Consequently,

$$\mathrm{Hom}(\partial\mathcal{H}\mathcal{G}, G)/\underset{\sim}{U}(N) = \mathrm{Hom}(\partial\mathcal{H}\mathcal{G}, G)/G.$$

Summary: The arrow $h \longrightarrow \varphi_h$ passes to the quotient and induces an injection

$$\iota : \mathrm{Hom}(\partial\mathcal{H}\mathcal{G}, G)/G \longrightarrow \mathrm{Spec}\ \overline{\partial\mathcal{H}\mathfrak{A}}\ .$$

[This is the upshot of the foregoing discussion.]

View $\mathrm{Hom}(\partial\mathcal{H}\mathcal{G}, G)$ as a subset of $G^{\partial\mathcal{H}\mathcal{G}}$ and take $G^{\partial\mathcal{H}\mathcal{G}}$ in the product topology -- then $\text{Hom}(\mathcal{H}\mathcal{G},G)$ is closed, hence is a compact Hausdorff space. Next, give $\text{Hom}(\mathcal{H}\mathcal{G},G)/G$ the quotient topology -- then it too is a compact Hausdorff space. Finally, equip Spec $\overline{\mathcal{H}\mathfrak{A}}$ with the Gelfand topology.

Observation: Since the $W(—,[\gamma])$ generate $\mathcal{H}\mathfrak{A}$, the Gelfand topology on Spec $\overline{\mathcal{H}\mathfrak{A}}$ is the initial topology determined by the

$$\hat{W}(—,[\gamma])\colon \text{Spec}\ \overline{\mathcal{H}\mathfrak{A}} \longrightarrow \mathbf{C},$$

i.e., is the coarsest topology for which the functions

$$\omega \longrightarrow \omega(W(—,[\gamma])) \quad (\omega \in \text{Spec}\ \overline{\mathcal{H}\mathfrak{A}})$$

are continuous.

<u>LEMMA</u> The injection

$$\iota : \text{Hom}(\mathcal{H}\mathcal{G},G)/G \longrightarrow \text{Spec}\ \overline{\mathcal{H}\mathfrak{A}}$$

is continuous.

[Bearing in mind that

$$\text{Hom}(\mathcal{H}\mathcal{G},G)/G$$

carries the quotient topology, one has only to check that the composite

$$\text{Hom}(\mathcal{H}\mathcal{G},G) \longrightarrow \text{Spec}\ \overline{\mathcal{H}\mathfrak{A}}$$

is continuous. In turn, this will be the case iff $\forall\ [\gamma]$, the function

$$\text{Hom}(\mathcal{H}\mathcal{G},G) \longrightarrow \text{Spec}\ \overline{\mathcal{H}\mathfrak{A}} \xrightarrow{\hat{W}(—,[\gamma])} \mathbf{C}$$

is continuous. But, from the definitions,

$$\hat{W}(—,[\gamma])\ (\varphi_h) = \varphi_h(W(—,[\gamma]))$$

$$= \frac{1}{N}\ \mathrm{tr}(h[\gamma]),$$

and $\forall\,[\gamma]$, the function

$$\begin{cases} \mathrm{Hom}(\mathcal{H}\mathcal{G}, G) \longrightarrow \underset{\sim}{C} \\ \quad h \longrightarrow \frac{1}{N}\ \mathrm{tr}(h[\gamma]) \end{cases}$$

is certainly continuous.]

We have

$$\mathfrak{A}/\mathcal{G} \hookrightarrow \mathrm{Hom}(\mathcal{H}\mathcal{G}, G)/G \xrightarrow{\iota} \mathrm{Spec}\ \overline{\mathcal{H}\mathfrak{A}}\ ,$$

where

$$\iota\ [\Gamma]\ (W(—,[\gamma]))$$

$$= W([\Gamma],\ [\gamma])$$

$$= \frac{1}{N}\ \mathrm{tr}(h(\Gamma, p; \gamma)).$$

Claim: The image $\iota(\mathfrak{A}/\mathcal{G})$ is dense in Spec $\overline{\mathcal{H}\mathfrak{A}}$.

[Suppose that $f:\mathrm{Spec}\ \overline{\mathcal{H}\mathfrak{A}} \longrightarrow \underset{\sim}{C}$ is a continuous function which vanishes on $\iota(\mathfrak{A}/\mathcal{G})$. Choose $\phi \in \overline{\mathcal{H}\mathfrak{A}}$: $f = \hat{\phi}$ -- then $\forall\,[\Gamma]$,

$$0 = f(\iota[\Gamma]) = \hat{\phi}(\iota[\Gamma]) = \phi([\Gamma])$$

$$\Longrightarrow$$

$$\phi \equiv 0 \Longrightarrow f \equiv 0.]$$

Therefore

$$\iota : \mathrm{Hom}(\mathcal{H}\mathfrak{G}, G)/G \longrightarrow \mathrm{Spec}\, \overline{\mathcal{H}\mathfrak{A}}$$

is a homeomorphism, hence

$$\mathrm{Hom}(\mathcal{H}\mathfrak{G}, G)/G \approx \mathrm{Spec}\, \overline{\mathcal{H}\mathfrak{A}}$$

or still,

$$\mathrm{Hom}_C(\overline{\mathcal{H}\mathfrak{G}}, G)/G \approx \mathrm{Spec}\, \overline{\mathcal{H}\mathfrak{A}}\ .$$

Remark: Introduce the constructs $\overline{\mathfrak{A}}$, $\overline{\mathfrak{G}}$, $\overline{\mathfrak{A}/\mathfrak{G}}$ -- then, as we have seen earlier,

$$\overline{\mathfrak{A}/\mathfrak{G}} \approx \overline{\mathfrak{A}}/\overline{\mathfrak{G}} \approx \mathrm{Hom}(\mathcal{H}\mathfrak{G}, G)/G.$$

On the other hand, $\iota(\mathfrak{A}/\mathfrak{G})$ is dense in Spec $\overline{\mathcal{H}\mathfrak{A}}$, thus the notation is consistent.

<u>Abelian Theory</u> Maintaining the assumption that M is analytic and path connected with dim $M \geq 2$, let us turn now to the case when $G = \underset{\sim}{U}(1)$ -- then again

$$\begin{cases} \operatorname{Hom}(\mathcal{H}\mathfrak{G}, \underset{\sim}{U}(1)) \approx \operatorname{Spec} \overline{\mathcal{H}\mathfrak{A}} \\ \qquad\qquad h \longleftrightarrow \varphi_h . \end{cases}$$

Observation: $\operatorname{Hom}(\mathcal{H}\mathfrak{G}, \underset{\sim}{U}(1))$ is a compact abelian topological group, call it $\overline{\mathfrak{A}/\mathfrak{G}}$.

Let μ_0 be normalized Haar measure on $\overline{\mathfrak{A}/\mathfrak{G}}$ -- then $\overline{\mathcal{H}\mathfrak{A}}$ can be represented on $L^2(\overline{\mathfrak{A}/\mathfrak{G}}\,;\mu_0)$:

$$(\Pi_0(\phi)f)(\varphi_h) = \hat{\phi}(\varphi_h)f(\varphi_h) \quad (f \in L^2(\overline{\mathfrak{A}/\mathfrak{G}}\,;\mu_0)).$$

[Note: In particular,

$$(\Pi_0(W(\text{—},[\gamma]))f)(\varphi_h) = \hat{W}(\text{—},[\gamma])(\varphi_h)f(\varphi_h)$$

$$= \varphi_h(W(\text{—},[\gamma]))f(\varphi_h)$$

$$= h[\gamma]f(\varphi_h).]$$

On the other hand, there is also the regular representation of $\overline{\mathfrak{A}/\mathfrak{G}}$ on $L^2(\overline{\mathfrak{A}/\mathfrak{G}}\,;\mu_0)$:

$$\rho(\varphi_h)f(\varphi_{h'}) = f(\varphi_{h'} \cdot \varphi_h)$$

$$= f(\varphi_{h'h}).$$

Given $[\gamma] \in \partial\mathcal{H}\mathcal{G}$, define

$$\chi_{[\gamma]} : \overline{\mathcal{O}\!\mathcal{L}/\mathcal{G}} \longrightarrow \underset{\sim}{U}(1)$$

by

$$\chi_{[\gamma]}(\varphi_h) = h[\gamma].$$

Then $\chi_{[\gamma]}$ is a character of $\overline{\mathcal{O}\!\mathcal{L}/\mathcal{G}}$.

<u>FACT</u> $\partial\mathcal{H}\mathcal{G}$ (discrete topology) is isomorphic to the character group of $\overline{\mathcal{O}\!\mathcal{L}/\mathcal{G}}$ via the map

$$\begin{cases} \partial\mathcal{H}\mathcal{G} \longrightarrow \widehat{\overline{\mathcal{O}\!\mathcal{L}/\mathcal{G}}} \\ [\gamma] \longrightarrow \chi_{[\gamma]}. \end{cases}$$

Remark: Given $f \in L^1(\overline{\mathcal{O}\!\mathcal{L}/\mathcal{G}})$, its Fourier transform $\hat{f} : \widehat{\overline{\mathcal{O}\!\mathcal{L}/\mathcal{G}}} \longrightarrow \underset{\sim}{C}$ is

$$\hat{f}[\gamma] = \int_{\overline{\mathcal{O}\!\mathcal{L}/\mathcal{G}}} f\,\overline{\chi}_{[\gamma]}\, d\mu_0.$$

Let ω_0 be the state on $\overline{\partial\mathcal{H}\mathcal{O}\!\mathcal{L}}$ determined by μ_0:

$$\omega_0(\phi) = \int_{\overline{\mathcal{O}\!\mathcal{L}/\mathcal{G}}} \hat{\phi}\, d\mu_0.$$

Then

$$\omega_0(W(-,[\gamma])) = \int_{\overline{\mathcal{O}\!\mathcal{L}/\mathcal{G}}} \hat{W}(-,[\gamma])\, d\mu_0$$

$$= \int_{\overline{\mathfrak{A}/\mathcal{G}}} \chi_{[\gamma]} \, d\mu_0$$

$$= \begin{cases} 1 \text{ if } [\gamma] = \mathrm{id}_{\partial \ell \mathcal{G}} \\ 0 \text{ if } [\gamma] \neq \mathrm{id}_{\partial \ell \mathcal{G}} \end{cases} .$$

To illustrate the preceding generalities, take $M = \mathbb{R}^3$ and denote by

$$\begin{cases} \mathfrak{A} \text{ -- the set of connections} \\ \mathcal{G} \text{ -- the set of gauge transformations} \end{cases}$$

per

$$\begin{array}{ccc} \mathrm{U}(1) \longrightarrow & \mathbb{R}^3 \times \mathrm{U}(1) & \\ & \downarrow & \\ & \mathbb{R}^3 . & \end{array}$$

<u>Ad $\mathfrak{A}$:</u> There are identifications

$$\mathfrak{A} \longleftrightarrow \Lambda^1(\mathbb{R}^3; \sqrt{-1}\,\mathbb{R}) \longleftrightarrow C^\infty(\mathbb{R}^3; \mathbb{R}^3),$$

viz.

$$\Gamma \longleftrightarrow \omega_\Gamma \longleftrightarrow A_\Gamma ,$$

where

$$\begin{cases} \omega_\Gamma = -\sqrt{-1}\, A_a dx^a \\ A_\Gamma = (A_1, A_2, A_3). \end{cases}$$

<u>Ad $\mathcal{G}$:</u> The elements of $\phi \in C^\infty(\mathbb{R}^3)$ operate on $C^\infty(\mathbb{R}^3; \mathbb{R}^3)$ via

$$A \longrightarrow A + \nabla \phi .$$

Holonomy We have

$$h(\Gamma;\gamma) = \exp(-\int_\gamma \omega_\Gamma)$$
$$= \exp(\sqrt{-1}\int_\gamma A_a dx^a).$$

[Note: Since $\underset{\sim}{U}(1)$ is abelian, it is permissible to write $h(\Gamma;\gamma)$ in place of $h(\Gamma,p;\gamma)$.]

The assignment

$$A \longrightarrow \int_\gamma A_a dx^a$$

is a compactly supported distribution with components $(X_\gamma^1, X_\gamma^2, X_\gamma^3)$. So, symbolically:

$$\int_\gamma A_a dx^a = \int_{\underset{\sim}{R}^3} A_a X_\gamma^a \, dx.$$

Observation: We have

$$\frac{\partial X_\gamma^1}{\partial x^1} + \frac{\partial X_\gamma^2}{\partial x^2} + \frac{\partial X_\gamma^3}{\partial x^3} = 0.$$

[In fact,

$$\sum_{a=1}^{3} \left\langle \varphi, \frac{\partial X_\gamma^a}{\partial x^a} \right\rangle = - \sum_{a=1}^{3} \left\langle \frac{\partial \varphi}{\partial x^a}, X_\gamma^a \right\rangle$$

$$= - \int_\gamma \frac{\partial \varphi}{\partial x^a} dx^a$$

$$= - \int_\gamma \nabla\varphi = 0.]$$

Remark: There is a unitary representation U of $\mathcal{S}(\underset{\sim}{R}^3;\underset{\sim}{R}^3)^T$ on $L^2(\overline{\mathfrak{A}/\mathcal{G}}\ ;\mu_0)$, namely

$$U(F)f(\varphi_h) = f(\varphi_h\varphi_F),$$

where

$$\varphi_F[\gamma] = \exp(\sqrt{-1}\int_\gamma F),$$

and a unitary representation V of $\mathcal{H}\mathcal{G}$ on $L^2(\overline{\mathfrak{A}/\mathcal{G}}\ ;\mu_0)$, namely

$$V([\gamma])f(\varphi_h) = \chi_{[\gamma]}(\varphi_h)f(\varphi_h).$$

[Note: From the definitions,

$$U(F)V([\gamma]) = \exp(\sqrt{-1}\int_\gamma F)V([\gamma])U(F),$$

which are the analogs of the canonical commutation relations in this setup.]

Given $t > 0$, let

$$f_t(x) = \frac{1}{(2\pi t)^{3/2}}\exp(-\frac{x^2}{2t}).$$

Then, in the sense of distributions.

$$\lim_{t\downarrow 0} f_t = \delta .$$

Definition: The <u>form factor</u> attached to t,γ is that element $X_{t,\gamma} \in \mathcal{S}(\underset{\sim}{R}^3;\underset{\sim}{R}^3)$ defined by the convolution

$$X^a_{t,\gamma}(x) = \int_{\underset{\sim}{R}^3} f_t(x-y)X^a_\gamma(y)dy.$$

Observation: We have

$$X_{t,\gamma} \in \mathcal{S}(\underset{\sim}{R}^3;\underset{\sim}{R}^3)^{T}.$$

[In fact,

$$\operatorname{div} X_{t,\gamma} = \frac{\partial X^a_{t,\gamma}}{\partial x^a}$$

$$= - \int_{\underset{\sim}{R}^3} \frac{\partial}{\partial y^a} f_t(x-y) X^a_{\gamma}(y)\,dy$$

$$= - \int_{\gamma} \nabla f_t(x -)$$

$$= 0.]$$

It is clear that holonomically equivalent loops have the same form factor, hence $\forall\, t > 0$, there is a map

$$\begin{cases} \partial\mathcal{H}\mathcal{L} \longrightarrow \mathcal{S}(\underset{\sim}{R}^3;\underset{\sim}{R}^3)^{T} \\ [\gamma] \longrightarrow X_{t,\gamma} \end{cases}$$

which respects composition, i.e.,

$$X_{t,\gamma\circ\delta} = X_{t,\gamma} + X_{t,\delta} .$$

Recalling that $\mathcal{T} = \mathcal{S}(\underset{\sim}{R}^3;\underset{\sim}{R}^3)^{T}$, define

$$\Theta_t : \mathcal{T}^* \longrightarrow \overline{\mathcal{O}\mathcal{V}\mathcal{L}}$$

by

$$(\Theta_t \lambda)[\gamma] = \exp(\sqrt{-1}\,\lambda(X_{t,\gamma})).$$

LEMMA $\forall\, t$, Θ_t is measurable.

[The relevant σ -algebra on $\mathcal{J}^*$ is

$$\underline{\mathrm{Cyl}}\,\mathcal{J}^* = \underline{\mathrm{Bor}}\,\mathcal{J}^*_s .$$

On the other hand, the relevant σ -algebra on $\overline{\mathcal{O}\!\mathcal{L}/\mathcal{G}}$ is the σ -algebra generated by the $\chi_{[\gamma]}$ ($[\gamma] \in \mathcal{H}\mathcal{G}$), i.e., the Baire σ -algebra. Therefore Θ_t is measurable iff $\forall\, [\gamma] \in \mathcal{H}\mathcal{G}$, $\chi_{[\gamma]} \circ \Theta_t$ is Borel measurable (as a function from $\mathcal{J}^*$ to $\underline{U}(1)$). But

$$(\chi_{[\gamma]} \circ \Theta_t)(\lambda)$$

$$= \chi_{[\gamma]}(\Theta_t \lambda)$$

$$= (\Theta_t \lambda)[\gamma] = \exp(\sqrt{-1}\,\lambda(X_{t,\gamma}))$$

and the composition

$$\lambda \longrightarrow \lambda(X_{t,\gamma}) \longrightarrow \exp(\sqrt{-1}\,\lambda(X_{t,\gamma}))$$

is obviously Borel.]

Let μ be a Borel measure on $\mathcal{J}^*$ -- then $(\Theta_t)_* \mu$ is a Baire measure on $\overline{\mathcal{O}\!\mathcal{L}/\mathcal{G}}$, hence admits a unique extension to a Radon measure on $\overline{\mathcal{O}\!\mathcal{L}/\mathcal{G}}$.

Remark: The topology on $\overline{\mathcal{O}\!\mathcal{L}/\mathcal{G}}$ is the initial topology determined by the $\chi_{[\gamma]}$ ($[\gamma] \in \mathcal{H}\mathcal{G}$) and $\forall\, [\gamma]$, $\chi_{[\gamma]} \circ \Theta_t$ is weakly continuous, hence strongly continuous. Therefore

$$\Theta_t : \mathcal{T}_s^* \to \overline{\mathfrak{A}/\mathfrak{B}}$$

is continuous. But $\mathcal{T}_s^*$ is a Souslin space, thus so is its image $\Theta_t(\mathcal{T}_s^*)$, which, while not necessarily Borel, is at least measurable w.r.t. $(\Theta_t)_* \mu$ (i.e., is in the domain of the completion of $(\Theta_t)_* \mu$).

[Note: A compact Hausdorff space is Souslin iff it is second countable, a property that in all likelihood $\overline{\mathfrak{A}/\mathfrak{B}}$ does not have.]

Take now for μ the unique gaussian measure γ_r on $\mathcal{T}^*$ with Fourier transform $e^{-Q_r/2}$, where

$$Q_r(F) = \langle F, (-\Delta)^r F \rangle_{L^2(\underset{\sim}{R}^3; \underset{\sim}{R}^3)} \quad (F \in \mathcal{T}).$$

To be in agreement with the physics literature, choose $r = -\frac{1}{2}$ -- then

$$\hat{\gamma}_{-1/2}(F) = \int_{\mathcal{T}^*} e^{\sqrt{-1}\,\lambda(F)}\, d\gamma_{-\frac{1}{2}}(\lambda)$$

$$= \exp\left(-\frac{1}{2} \langle F, (-\Delta)^{-1/2} F \rangle_{L^2(\underset{\sim}{R}^3; \underset{\sim}{R}^3)} \right)$$

$$= \exp\left(-\frac{1}{2} \int_{\underset{\sim}{R}^3} \frac{\hat{F} \cdot \hat{F}}{\|\xi\|}\, d\xi \right).$$

Put

$$\mu_t = (\Theta_t)_* \gamma_{-1/2} .$$

Then μ_t determines a state ω_t on $\overline{\mathcal{H}\mathcal{P}\mathfrak{A}}$:

$$\omega_t(\phi) = \int_{\overline{\mathfrak{A}/\mathcal{G}}} \hat{\phi}\, d\mu_t .$$

Therefore

$$\begin{aligned}
&\omega_t(W(—, [\gamma])) \\
&= \int_{\overline{\mathfrak{A}/\mathcal{G}}} \hat{W}(—, [\gamma])\, d\mu_t \\
&= \int_{\overline{\mathfrak{A}/\mathcal{G}}} \chi_{[\gamma]}\, d\mu_t \\
&= \int_{\mathcal{T}^*} \chi_{[\gamma]}(\Theta_t \lambda)\, d\gamma_{-1/2}(\lambda) \\
&= \int_{\mathcal{T}^*} (\Theta_t \lambda)[\gamma]\, d\gamma_{-1/2}(\lambda) \\
&= \int_{\mathcal{T}^*} \exp(\sqrt{-1}\, \lambda(X_{t,\gamma}))\, d\gamma_{-1/2}(\lambda) \\
&= \exp\left(-\frac{1}{2} \int_{\underset{\sim}{R}^3} \frac{\hat{X}_{t,\gamma} \cdot \hat{X}_{t,\gamma}}{\|\xi\|}\, d\xi\right).
\end{aligned}$$

<u>FACT</u> The measures in the set $\{\mu_t : t > 0\} \cup \{\mu_0\}$ are singular w.r.t. one another.

Since $\overline{\mathfrak{A}/\mathfrak{B}}$ is a compact abelian topological group, it follows that none of the μ_t is quasi-invariant under the action of $\overline{\mathfrak{A}/\mathfrak{B}}$ on itself by multiplication.

[Note: Every quasi-invariant measure on $\overline{\mathfrak{A}/\mathfrak{B}}$ is equivalent to the Haar measure μ_0.]

Thanks to the Hahn-Banach theorem, the arrow of restriction

$$\begin{cases} \mathcal{S}(\underset{\sim}{R}^3;\underset{\sim}{R}^3)^* \longrightarrow \mathcal{T}^* \\ \quad \lambda \longrightarrow \lambda|\mathcal{T} \end{cases}$$

is surjective. This said, suppose that $\nabla \times \lambda = 0$ -- then $\lambda | \mathcal{T} = 0$. Thus let $F \in \mathcal{T}$:

$$F = -\nabla \times (G * \text{curl } F)$$

$$\Longrightarrow$$

$$\begin{aligned}\langle F, \lambda \rangle &= -\langle \nabla \times (G * \text{curl } F), \lambda \rangle \\ &= -\langle G * \text{curl } F, \nabla \times \lambda \rangle \\ &= 0.\end{aligned}$$

[Note: This argument is suggestive but formal, there being no assurance that $G * \text{curl } F$ is rapidly decreasing so, strictly speaking, integration by parts is not permissible. The way out is to appeal to the homology theory of currents which implies that an element $\lambda \in \mathcal{S}(R^3;R^3)^*$ admits a potential $\phi \in \mathcal{S}(R^3)^*$ iff $\nabla \times \lambda = 0$. But then

$$\begin{aligned}\langle F, T \rangle &= \langle F, \nabla \phi \rangle \\ &= -\langle \text{div } F, \phi \rangle = 0.]\end{aligned}$$

LEMMA $\forall$ t, the $X_{t,\gamma}$ separate the points of $\mathcal{T}^*$.

[Let $\lambda_1 \neq \lambda_2$ be distinct elements of $\mathcal{T}^*$ -- then the claim is that $\exists\, \gamma : \lambda_1(X_{t,\gamma}) \neq \lambda_2(X_{t,\gamma})$ or, rephrased:

$$\lambda(X_{t,\gamma}) = 0 \quad \forall \gamma \Rightarrow \lambda = 0.$$

But $\forall \gamma$,

$$0 = \lambda(X_{t,\gamma}) = \langle f_t * X_\gamma , \lambda \rangle$$

$$= \langle f_t * \lambda , X_\gamma \rangle$$

$$= \int_\gamma f_t * \lambda$$

$\Rightarrow$

$$0 = \operatorname{curl}(f_t * \lambda)$$

$$= f_t * \operatorname{curl} \lambda$$

$\Rightarrow$

$$0 = \hat{f}_t \, (\nabla \times \lambda)^\wedge$$

$\Rightarrow$

$$(\nabla \times \lambda)^\wedge = 0$$

$\Rightarrow$

$$\nabla \times \lambda = 0.]$$

Rappel: Let $\mathcal{H}$, $\mathcal{K}$ be Hilbert spaces. Suppose that $T: \mathcal{H} \longrightarrow \mathcal{K}$ is an isometry such that $\overline{\text{ran } T} = \mathcal{K}$ -- then T is surjective.

[Given $y \in \mathcal{K}$, $\exists$ a sequence $\{x_n\} \subset \mathcal{H}$: $Tx_n \longrightarrow y$. But $\|Tx_n\| = \|x_n\|$, hence $\{x_n\}$ is Cauchy, so $x_n \longrightarrow x \Longrightarrow y = \lim Tx_n = Tx$, i.e., ran $T = \mathcal{K}$.]

The map

$$f \longrightarrow f \circ \Theta_t$$

induces an isometry

$$T: L^2(\overline{\mathfrak{a}/\mathfrak{g}}; \mu_t) \longrightarrow L^2(\mathfrak{T}^*; \gamma_{-1/2})$$

via the change of variable formula

$$\int_{\overline{\mathfrak{a}/\mathfrak{g}}} |f|^2 \, d\mu_t = \int_{\mathfrak{T}^*} |f \circ \Theta_t|^2 \, d\gamma_{-1/2}.$$

Since the $X_{t,\gamma}$ separate the points of $\mathfrak{T}^*$, standard generalities then imply that the functions

$$\lambda \longrightarrow e^{\sqrt{-1}\,\lambda(X_{t,\gamma})}$$

constitute a total subset of $L^2(\mathfrak{T}^*; \gamma_{-1/2})$. But

$$(\chi_{[\gamma]} \circ \Theta_t)(\lambda) = e^{\sqrt{-1}\,\lambda(X_{t,\gamma})}.$$

Therefore T is surjective.

Fix $t > 0$ and consider the restriction of Θ_t to $\mathcal{T} \subset \mathcal{T}^*$:

$$\Theta_{t,F} \equiv \Theta_t F$$

$$\Longrightarrow$$

$$\Theta_{t,F}[\gamma] = \exp(\sqrt{-1} \int_{\underset{\sim}{R}^3} F_a X^a_{t,\gamma} \, dx).$$

Since

$$\Theta_{t,F_1+F_2} = \Theta_{t,F_1} \cdot \Theta_{t,F_2},$$

there is an action Ξ_t of $\mathcal{T}$ on $\overline{\mathfrak{A}/\mathfrak{G}}$:

$$\begin{cases} \overline{\mathfrak{A}/\mathfrak{G}} \times \mathcal{T} \longrightarrow \overline{\mathfrak{A}/\mathfrak{G}} \\ (\varphi_h, F) \longrightarrow \varphi_h \cdot \Theta_{t,F} . \end{cases}$$

<u>FACT</u> μ_t is quasi-invariant w.r.t. Ξ_t.

Let

$$\Xi_{t,F} : \overline{\mathfrak{A}/\mathfrak{G}} \longrightarrow \overline{\mathfrak{A}/\mathfrak{G}}$$

be the map

$$\varphi_h \longrightarrow \varphi_h \cdot \Theta_{t,F} \quad (= \Xi_t(\varphi_h, F))$$

and put

$$\mu_{t,F} = (\Xi_{t,F})_* \mu_t \quad (= (\Theta_t)_* \gamma_{-1/2,F}),$$

so that

$$\int_{\overline{\mathfrak{A}/\mathfrak{G}}} f\, d\mu_{t,F} = \int_{\overline{\mathfrak{A}/\mathfrak{G}}} f \circ \Xi_{t,F}\, d\mu_t .$$

Then the prescription

$$U_t(F)f(\varphi_h) = f(\varphi_h \cdot \Theta_{t,F}) \left[\frac{d\mu_{t,-F}}{d\mu_t}(\varphi_h) \right]^{1/2}$$

defines a unitary representation of $\mathcal{T}$ on $L^2(\overline{\mathfrak{A}/\mathfrak{G}}\,;\mu_t)$. On the other hand, the prescription

$$U(F)f(\lambda) = f(\lambda + F) \left[\frac{d\gamma_{-1/2,-F}}{d\gamma_{-1/2}}(\lambda) \right]^{1/2}$$

defines a unitary representation of $\mathcal{T}$ on $L^2(\mathcal{T}^*;\gamma_{-1/2})$.

LEMMA The diagram

$$\begin{tikzcd}
L^2(\overline{\mathfrak{A}/\mathfrak{G}}\,;\mu_t) \arrow[r, "T"] \arrow[d, "U_t(F)"'] & L^2(\mathcal{T}^*;\gamma_{-1/2}) \arrow[d, "U(F)"] \\
L^2(\overline{\mathfrak{A}/\mathfrak{G}}\,;\mu_t) \arrow[r, "T"'] & L^2(\mathcal{T}^*;\gamma_{-1/2})
\end{tikzcd}$$

commutes.

[As a function of λ,

$$U(F)Tf\,\Big|_{\lambda}$$

$$= Tf(\lambda + F) \left[\frac{d\gamma_{-1/2,-F}}{d\gamma_{-1/2}} (\lambda) \right]^{1/2}$$

$$= f(\Theta_t(\lambda + F)) \left[\frac{d\gamma_{-1/2,-F}}{d\gamma_{-1/2}} (\lambda) \right]^{1/2}$$

$$= f(\Theta_t \lambda \cdot \Theta_{t,F}) \left[\frac{d\gamma_{-1/2,-F}}{d\gamma_{-1/2}} (\lambda) \right]^{1/2},$$

while

$$TU_t(F) f \Big|_{\lambda}$$

$$= U_t(F) f(\Theta_t \lambda)$$

$$= f(\Theta_t \lambda \cdot \Theta_{t,F}) \left[\frac{d\mu_{t,-F}}{d\mu_t} (\Theta_t \lambda) \right]^{1/2}.$$

Since

$$\left[\frac{d\gamma_{-1/2,-F}}{d\gamma_{-1/2}} \right]^{1/2} \in L^2(\mathcal{T}^*; \gamma_{-1/2}),$$

there exists a nonnegative function

$$f_F \in L^2(\overline{\mathfrak{A}/\mathcal{B}}; \mu_t)$$

such that

$$\left[\frac{d\gamma_{-1/2,-F}}{d\gamma_{-1/2}} \right]^{1/2} = f_F \circ \Theta_t .$$

But for every Borel set $B \subset \overline{\mathfrak{A}/\mathfrak{B}}$,

$$\mu_{t,-F}(B) = (\Theta_t)_* \gamma_{-1/2,-F}(B)$$

$$= \gamma_{-1/2,-F}(\Theta_t^{-1} B)$$

$$= \int_{\Theta_t^{-1} B} \left[\frac{d\gamma_{-1/2,-F}}{d\gamma_{-1/2}} (\lambda) \right] d\gamma_{-1/2}(\lambda)$$

$$= \int_{\Theta_t^{-1} B} (f_F \circ \Theta_t)^2 (\lambda)\, d\gamma_{-1/2}(\lambda)$$

$$= \int_{\Theta_t^{-1} B} f_F(\Theta_t \lambda)^2\, d\gamma_{-1/2}(\lambda)$$

$$= \int_B f_F^2\, d\mu_t$$

$$= \int_B \left[\frac{d\mu_{t,-F}}{d\mu_t} \right] d\mu_t$$

$\Longrightarrow$

$$f_F = \left[\frac{d\mu_{t,-F}}{d\mu_t} \right]^{1/2}$$

$$\Longrightarrow \left[\frac{d\gamma_{-1/2,-F}}{d\gamma_{-1/2}}(\lambda)\right]^{1/2} = \left[\frac{d\mu_{t,-F}}{d\mu_t}(\Theta_t\lambda)\right]^{1/2}.$$

Therefore

$$U(F)\circ T = T\circ U_t(F).]$$

There is another unitary representation of $\mathcal{T}$ on $L^2(\mathcal{T}^*;\gamma_{-1/2})$, viz.

$$V(F)f = \chi_F f,$$

where we have written

$$\chi_F(\lambda) = e^{\sqrt{-1}\,\lambda(F)}.$$

However, the analog of this in the $\overline{\mathfrak{A}/\mathfrak{G}}$ -picture is not so transparent: Put

$$V_t(F) = T^{-1}\circ V(F)\circ T$$

and define

$$m_F\in L^2(\overline{\mathfrak{A}/\mathfrak{G}};\mu_t)$$

by

$$\chi_F = m_F\circ\Theta_t.$$

Then

$$V_t(F)f = T^{-1}(\chi_F(f\circ\Theta_t))$$

$$= T^{-1}((m_F\circ\Theta_t)(f\circ\Theta_t))$$

$$= m_F f.$$

Example: We have

$$\chi_{X_{t,\gamma}} = \chi_{[\gamma]} \circ \Theta_t$$

$$\Longrightarrow$$

$$m_{X_{t,\gamma}} = \chi_{[\gamma]}.$$

<u>Cylinder Functions</u> Suppose that M is analytic and path connected with dim $M \geq 2$ and G is a compact connected nonabelian Lie group -- then

$$\overline{\mathfrak{A}} = \lim \overline{\mathfrak{A}}_{\Lambda} = \mathrm{Hom}(\mathcal{P}\mathcal{G}, G).$$

Definition: A function $f \in C(\overline{\mathfrak{A}})$ is said to be a <u>cylinder function</u> if $\exists\ \Lambda \in \text{Gra } M$ and $f_\Lambda \in C(\overline{\mathfrak{A}}_\Lambda)$ such that the triangle

$$\begin{tikzcd} \overline{\mathfrak{A}} \arrow[r, "\pi_\Lambda"] \arrow[dr, "f"'] & \overline{\mathfrak{A}}_\Lambda \arrow[d, "f_\Lambda"] \\ & \mathbf{C} \end{tikzcd}$$

commutes.

Write $\text{Cyl}(\overline{\mathfrak{A}})$ for the set of cylinder functions on $\overline{\mathfrak{A}}$.

<u>LEMMA</u> $\text{Cyl}(\overline{\mathfrak{A}})$ is a $*$-subalgebra of $C(\overline{\mathfrak{A}})$.

[It is obvious that $\text{Cyl}(\overline{\mathfrak{A}})$ is closed under conjugation and scalar multiplication. Therefore the issue is closure under sums and products. This hinges on the fact that Gra M is directed. Consider, e.g., sums. Let $f_1, f_2 \in \text{Cyl}(\overline{\mathfrak{A}})$: $f_1 = f_{\Lambda_1} \circ \pi_{\Lambda_1}$, $f_2 = f_{\Lambda_2} \circ \pi_{\Lambda_2}$. Choose $\Lambda_3 : \Lambda_3 \geq \Lambda_1, \Lambda_2$ -- then

$$\begin{aligned} f_1 + f_2 &= f_{\Lambda_1} \circ \pi_{\Lambda_1} + f_{\Lambda_2} \circ \pi_{\Lambda_2} \\ &= f_{\Lambda_1} \circ \pi_1^3 \circ \pi_{\Lambda_3} + f_{\Lambda_2} \circ \pi_2^3 \circ \pi_{\Lambda_3} \\ &= (f_{\Lambda_1} \circ \pi_1^3 + f_{\Lambda_2} \circ \pi_2^3) \circ \pi_{\Lambda_3} \\ &\in \text{Cyl}(\overline{\mathfrak{A}}).] \end{aligned}$$

Rappel: Let X be a compact Hausdorff space. Let $A \subset C(X)$ be a *-subalgebra of $C(X)$ which separates the points of X ($x_1 \neq x_2 \Rightarrow \exists\, f \in A$: $f(x_1) \neq f(x_2)$) -- then the uniform closure of A is all of $C(X)$.

LEMMA $\mathrm{Cyl}(\overline{\mathfrak{A}})$ is dense in $C(\overline{\mathfrak{A}})$.

[Take $h_1 \neq h_2$ in $\overline{\mathfrak{A}}$ and choose $\Lambda \in$ Gra M: $\pi_\Lambda(h_1) \neq \pi_\Lambda(h_2)$ -- then $\exists\, f_\Lambda \in C(\overline{\mathfrak{A}}_\Lambda)$:

$$f_\Lambda(\pi_\Lambda(h_1)) \neq f_\Lambda(\pi_\Lambda(h_2)).$$

Since $f_\Lambda \circ \pi_\Lambda \in \mathrm{Cyl}(\overline{\mathfrak{A}})$, it follows that $\mathrm{Cyl}(\overline{\mathfrak{A}})$ separates the points of $\overline{\mathfrak{A}}$.]

<u>The Ashtekar-Lewandowski Measure</u> Let X be a compact Hausdorff space -- then a linear functional $I:C(X) \longrightarrow \underset{\sim}{C}$ is said to be <u>positive</u> if $f \geq 0 \Rightarrow I(f) \geq 0$. This said, the Riesz representation theorem provides a one-to-one correspondence between the positive linear functionals I on C(X) and the Radon measures μ on $X: I \longleftrightarrow \mu$, where

$$I(f) = \int_X f d\mu .$$

[Note: C(X) is a unital commutative C*-algebra and its states are the normalized positive linear functionals, hence are parameterized by the Radon probability measures on X.]

Specialize now to the case when $X = \overline{\mathfrak{A}}$ -- then given any state ω on $C(\overline{\mathfrak{A}})$, there exists a unique Radon probability measure μ_ω on $\overline{\mathfrak{A}}$ such that

$$\omega(f) = \int_{\overline{\mathfrak{A}}} f d\mu_\omega .$$

Moreover, the assignment

$$\begin{cases} f \longrightarrow \Pi_\omega(f) \\ \Pi_\omega(f)\phi = f\phi \end{cases}$$

defines a cyclic representation of $C(\overline{\mathfrak{A}})$ on $L^2(\overline{\mathfrak{A}}; \mu_\omega)$. Here, of course,

$$\omega(f) = \langle 1, \Pi_\omega(f)1 \rangle .$$

On the other hand, every cyclic representation of $C(\overline{\mathfrak{A}})$ is unitarily equivalent to a representation of this type.

Suppose given a collection $\{\mu_\Lambda\}$, where $\forall\Lambda$, μ_Λ is a Radon probability measure on $\overline{\mathfrak{A}}_\Lambda$ -- then $\{\mu_\Lambda\}$ is said to be <u>consistent</u> if

$$\Lambda_2 \geq \Lambda_1$$
$$\Longrightarrow \int_{\overline{\mathfrak{A}}_{\Lambda_2}} f \circ \pi_1^2 \, d\mu_{\Lambda_2} = \int_{\overline{\mathfrak{A}}_{\Lambda_1}} f d\mu_{\Lambda_1} \quad (f \in C(\overline{\mathfrak{A}}_{\Lambda_1})).$$

Example: Every Radon probability measure on $\overline{\mathfrak{A}}$ gives rise to a consistent collection $\{\mu_\Lambda\}$ of Radon probability measures on the $\overline{\mathfrak{A}}_\Lambda$.

[A state ω on $C(\overline{\mathfrak{A}})$ defines a state ω_Λ on $C(\overline{\mathfrak{A}}_\Lambda)$ via the prescription

$$\omega_\Lambda(f) = \omega(f \circ \pi_\Lambda).]$$

The converse is also true: Every consistent collection $\{\mu_\Lambda\}$ of Radon probability measures on the $\overline{\mathfrak{A}}_\Lambda$ gives rise to a Radon probability measure on $\overline{\mathfrak{A}}$. To see this, let ω_Λ be the state on $C(\overline{\mathfrak{A}}_\Lambda)$ determined by μ_Λ. Given $f \in \mathrm{Cyl}(\overline{\mathfrak{A}})$, write $f = f_\Lambda \circ \pi_\Lambda$ and put

$$\omega(f) = \omega_\Lambda(f_\Lambda).$$

Then

$$\omega : \mathrm{Cyl}(\overline{\mathfrak{A}}) \longrightarrow \underset{\sim}{C}$$

is a welldefined normalized positive linear functional.

Observation: A positive linear functional I on Cyl($\overline{\mathfrak{A}}$) is necessarily continuous.

[Take f real -- then

$$|f| \leq \|f\| \Rightarrow \|f\| \pm f \geq 0$$

$$\Rightarrow I(1) \cdot \|f\| \pm I(f) \geq 0$$

$$\Rightarrow |I(f)| \leq I(1) \cdot \|f\| .]$$

Since Cyl($\overline{\mathfrak{A}}$) is dense in C($\overline{\mathfrak{A}}$), it follows that ω admits a unique extension to a state on C($\overline{\mathfrak{A}}$), which in turn determines a Radon probability measure on $\overline{\mathfrak{A}}$.

Let μ_Λ be the normalized Haar measure on

$$\overline{\mathfrak{A}}_\Lambda \approx G^{\#E(\Lambda)} .$$

Then the collection $\{\mu_\Lambda\}$ is consistent. Granted this, the Radon probability measure on $\overline{\mathfrak{A}}$ thereby produced is denoted by μ_{AL} and is called the <u>Ashtekar-Lewandowski measure</u>.

[Note: Write ω_{AL} for the state on C($\overline{\mathfrak{A}}$) corresponding to μ_{AL}.]

Suppose that f is cylindrical w.r.t. Λ_1 and Λ_2, i.e., $f = f_{\Lambda_1} \circ \pi_{\Lambda_1}$ and $f = f_{\Lambda_2} \circ \pi_{\Lambda_2}$ -- then f is cylindrical w.r.t. any $\Lambda_3 \geq \Lambda_1, \Lambda_2$. Accordingly, it will be enough to prove that

$$\int_{\overline{\mathfrak{A}}_\Lambda} f_\Lambda \, d\mu_\Lambda = \int_{\overline{\mathfrak{A}}_{\Lambda'}} f_{\Lambda'} \, d\mu_{\Lambda'},$$

where $\Lambda \leq \Lambda'$.

Let $e_1, \ldots, e_n$ be the edges of Λ ($\Rightarrow n = \#E(\Lambda)$); let $e'_1, \ldots, e'_{n'}$ be the edges of Λ' ($\Rightarrow n' = \#E(\Lambda')$) -- then, since $\Lambda \leq \Lambda'$, each edge e_i admits a decomposition in terms of the edges $e'_{i'}$:

$$e_i = \prod_{i'} (e'_{r(i,i')})^{\varepsilon(i,i')},$$

where $r(i,i') \in \{1, \ldots, n'\}$, $\varepsilon(i,i') \in \{-1,+1\}$. Furthermore, for each $i \in \{1, \ldots, n\}$, $\exists\, k(i) \in \{1, \ldots, n'\}$ with the following properties:

(a) $i \neq j \Rightarrow k(i) \neq k(j)$;

(b) $e'_{k(i)}$ is not used in the decomposition of the e_j $(j < i)$;

(c) $(e'_{k(i)})^{\pm}$ is used in the decomposition of e_i exactly once.

Observation: The arrow of restriction $\pi_{\Lambda}^{\Lambda'} : \overline{\mathfrak{A}}_{\Lambda'} \longrightarrow \overline{\mathfrak{A}}_\Lambda$ can be identified with the map $\pi_n^{n'} : G^{n'} \longrightarrow G^n$ that sends

$$(\sigma_1, \ldots, \sigma_{n'})$$

to

$$\Big(\prod_{i'} (\sigma_{r(1,i')})^{\varepsilon(1,i')}, \ldots, \prod_{i'} (\sigma_{r(n,i')})^{\varepsilon(n,i')} \Big).$$

With this preparation, we can now prove that

$$\int_{\overline{\mathfrak{A}}_\Lambda} f_\Lambda \, d\mu_\Lambda = \int_{\overline{\mathfrak{A}}_{\Lambda'}} f_{\Lambda'} \, d\mu_{\Lambda'} ,$$

where $\Lambda \leq \Lambda'$. Thus

$$\int_{\overline{\mathfrak{A}}_{\Lambda'}} f_{\Lambda'} \, d\mu_{\Lambda'}$$

$$= \int_{G^{n'}} \prod_{i'=1}^{n'} d\mu_{i'} \, f_{\Lambda'}(\sigma_1, \ldots, \sigma_{n'})$$

$$= \int_{G^{n'}} \prod_{i'=1}^{n'} d\mu_{i'} \, f_\Lambda \circ \pi_n^{n'}(\sigma_1, \ldots, \sigma_{n'})$$

$$= \int_{G^{n'}} \prod_{i'=1}^{n'} d\mu_{i'} \, f_\Lambda \Big(\prod_{i'} (\sigma_{r(1,i')})^{\varepsilon(1,i')}, \ldots, \prod_{i'} (\sigma_{r(n,i')})^{\varepsilon(n,i')} \Big)$$

$$= \int_{G^{n'-n}} \prod_{\substack{i'=1 \\ i' \notin k[1,n]}}^{n'} d\mu_{i'} \int_G d\mu_{k(1)} \cdots \int_G d\mu_{k(n)}$$

$$f_\Lambda(\cdots \sigma_{k(1)} \cdots, \ldots, \cdots \sigma_{k(n)} \cdots)$$

$$= \int_{G^{n'-n}} \prod_{\substack{i'=1 \\ i' \notin k[1,n]}}^{n'} d\mu_{i'} \int_G d\mu_{k(1)} \cdots \int_G d\mu_{k(n)} \, f_\Lambda(\sigma_{k(1)}, \ldots, \sigma_{k(n)})$$

$$= \int_{G^n} \prod_{i=1}^{n} d\mu_i \, f_\Lambda(\sigma_1, \ldots, \sigma_n)$$

$$= \int_{\overline{\mathfrak{A}}_\Lambda} f_\Lambda \, d\mu_\Lambda \, .$$

Example: Consider the simplest case, viz. when $n'=2$, $n=1$, $\varepsilon=+1$ -- then

$$\int_{G^2} d\mu(\sigma_1) d\mu(\sigma_2) f_{\Lambda'}(\sigma_1, \sigma_2)$$

$$= \int_{G^2} d\mu(\sigma_1) d\mu(\sigma_2) f_\Lambda(\sigma_1\sigma_2)$$

$$= \int_G d\mu(\sigma_1) \int_G d\mu(\sigma_2) \, f_\Lambda(\sigma_1\sigma_2)$$

$$= \int_G d\mu(\sigma) f_\Lambda(\sigma) .$$

Example: Consider an analytic circle Λ_u with a single vertex u and a single edge e_u:

u  e_u.

Fix a point $v \neq u$ and thereby determine a second analytic circle Λ_v with a single vertex v and a single edge e_v:

e_v  v

Then

e_1

u  v

e_2

is an element $\Lambda_{u,v}$ of Gra M refining Λ_u and Λ_v:

$$\begin{cases} e_u = e_2 e_1 \\ e_v = e_1 e_2. \end{cases}$$

Suppose that

$$\begin{cases} f = f_u \circ \pi_{\Lambda_u} & (f_u \in C(\overline{\mathfrak{A}}_{\Lambda_u})) \\ g = g_v \circ \pi_{\Lambda_v} & (g_v \in C(\overline{\mathfrak{A}}_{\Lambda_v})). \end{cases}$$

Then

$$\begin{cases} f = f_u \circ \pi_u \circ \pi_{u,v} \\ g = g_v \circ \pi_v \circ \pi_{u,v}, \end{cases}$$

where

$$\begin{cases} \pi_u : \overline{\mathfrak{A}}_{\Lambda_{u,v}} \longrightarrow \overline{\mathfrak{A}}_{\Lambda_u} \\ \\ \pi_v : \overline{\mathfrak{A}}_{\Lambda_{u,v}} \longrightarrow \overline{\mathfrak{A}}_{\Lambda_v} \end{cases} \quad \& \quad \pi_{u,v} : \overline{\mathfrak{A}} \longrightarrow \overline{\mathfrak{A}}_{\Lambda_{u,v}}.$$

So, from the definitions,

$$\begin{aligned} &\int_{\overline{\mathfrak{A}}} fg \, d\mu_{AL} \\ &= \int_{\overline{\mathfrak{A}}_{\Lambda_{u,v}}} (f_u \circ \pi_u)(g_v \circ \pi_v) d\mu_{\Lambda_{u,v}} \\ &= \int_{G \times G} d\sigma \, d\tau \; f_u \circ \pi_u(\sigma, \tau) \; g_v \circ \pi_v(\sigma, \tau) \\ &= \int_{G \times G} d\sigma d\tau \; f_u(\tau\sigma) \; g_v(\sigma\tau) \\ &= \int_G d\tau \; \left(\int_G d\sigma \; f_u(\sigma) \; g_v(\tau\sigma\tau^{-1}) \right). \end{aligned}$$

Therefore

$$\int_{\overline{\mathfrak{A}}} fg \, d\mu_{AL} = \left(\int_G f_u \right) \left(\int_G g_v \right)$$

provided that g_v is G-central.

Properties of μ_{AL}:

(1) $\mu_{AL}(\overline{\mathfrak{A}}) = 1$;

(2) $f \in C(\overline{\mathfrak{A}})$, $f \neq 0 \Rightarrow \omega_{AL}(\bar{f}f) > 0$;

(3) $\mu_{AL}(\overline{\mathfrak{A}}_{gen}) = 1.$

[The first property is obvious (look at the construction of μ_{AL}). Turning to the second property, note first that if $U \subset G$ is open and nonempty, then $\mu(U) > 0$. In fact, $\{ U\sigma : \sigma \in G \}$ is an open covering of G, so $\exists\ \sigma_1, \ldots, \sigma_n \in G : G = \bigcup_{i=1}^{n} U\sigma_i$

$$\Rightarrow \quad \mu(U) = \frac{1}{n} \sum_i \mu(U\sigma_i) \geq \frac{1}{n}\ \mu(G) = \frac{1}{n}.$$

Now put $F = \bar{f}f$ and let $U = F^{-1}(]\ \|F\|_\infty/2, +\infty[)$. Since $\overline{\mathfrak{A}} = \lim \overline{\mathfrak{A}}_\Lambda$, $\exists\ \Lambda : \pi_\Lambda^{-1}(U_\Lambda) \subset U$ ($U_\Lambda \subset \overline{\mathfrak{A}}_\Lambda$ open and nonempty)(true by the definition of the topology on $\overline{\mathfrak{A}}$). We then have

$$\begin{aligned}
\omega_{AL}(F) &= \int_{\overline{\mathfrak{A}}} F\, d\mu_{AL} \\
&\geq \int_U F\, d\mu_{AL} \\
&\geq \int_U \|F\|_\infty/2\ d\mu_{AL} \\
&\geq \|F\|_\infty/2 \int_U 1\ d\mu_{AL} \\
&\geq \|F\|_\infty/2 \int_{\pi_\Lambda^{-1}(U_\Lambda)} 1\ d\mu_{AL} \\
&\geq \|F\|_\infty/2 \int_{U_\Lambda} 1\ d\mu_\Lambda \\
&= (\|F\|_\infty/2) \cdot \mu_\Lambda(U_\Lambda) > 0.
\end{aligned}$$

The verification of the third property is more difficult and will be omitted (recall that $\overline{\mathfrak{A}}_{gen}$ is open, hence is measurable).]

Fix an edge $e:[0,1] \longrightarrow M$ and for $s \in [0,1]$, put

$$e_s(t) = e(st) \quad (0 \leq t \leq 1).$$

Define a map

$$\begin{cases} \overline{\mathfrak{A}} \xrightarrow{\ v\ } G^{[0,1]} \\ h \longrightarrow v_h \end{cases}$$

by

$$v_h(s) = h(e_s).$$

Let $\bar{\mu}_{AL}$ be the completion of μ_{AL} -- then it can be shown that the $\bar{\mu}_{AL}$-measure of

$$\{ h \in \overline{\mathfrak{A}} : \exists \ s : v_h \text{ is continuous at } s \}$$

is 0. Therefore the $\bar{\mu}_{AL}$-measure of

$$\{ h \in \overline{\mathfrak{A}} : \forall \ s : v_h \text{ is discontinuous at } s \}$$

is 1.

[Note: $\forall$ P, $\exists$ an embedding $\mathfrak{A}(P) \longrightarrow \overline{\mathfrak{A}}$ with a dense image. The v_h associated with the $\Gamma \in \mathfrak{A}(P)$ are certainly continuous, so $\bar{\mu}_{AL}(\mathfrak{A}(P))=0$.]

Rappel: There is a right action of $\overline{\mathcal{G}}_\wedge$ on $\overline{\mathfrak{A}}_\wedge$, viz.

$$\begin{cases} \overline{\mathfrak{A}}_\wedge \times \overline{\mathcal{G}}_\wedge \longrightarrow \overline{\mathfrak{A}}_\wedge \\ (h, \phi) \longrightarrow h \cdot \phi , \end{cases}$$

where

$$h\cdot\phi(e) = \phi(e(1))^{-1}\, h(e)\phi(e(0)).$$

Let $n = \#E(\Lambda)$. Fix elements $\sigma_i^1, \sigma_i^2 \in G$ $(i=1,\ldots,n)$ -- then $\forall\, F \in C(G^n)$,

$$\int_{G^n} \prod_{i=1}^{n} d\mu_i\, F(\sigma_1^1\sigma_1\sigma_1^2, \ldots, \sigma_n^1\sigma_n\sigma_n^2)$$

$$= \int_{G^n} \prod_{i=1}^{n} d\mu_i\, F(\sigma_1, \ldots, \sigma_n).$$

Therefore μ_Λ is $\bar{\mathcal{G}}_\Lambda$-invariant.

Rappel: There is a right action of $\bar{\mathcal{G}}$ on $\bar{\mathfrak{A}}$, viz.

$$\begin{cases} \bar{\mathfrak{A}} \times \bar{\mathcal{G}} \longrightarrow \bar{\mathfrak{A}} \\ (\{h_\Lambda\}, \{\phi_\Lambda\}) \longrightarrow \{h_\Lambda \cdot \phi_\Lambda\} \end{cases}.$$

LEMMA μ_{AL} is $\bar{\mathcal{G}}$-invariant.

[It suffices to check invariance on the cylinder functions. But, on the basis of what has been said above, this is immediate.]

It follows that $L^2(\bar{\mathfrak{A}}; \mu_{AL})$ supports a unitary representation of $\bar{\mathcal{G}}$, viz.

$$f \longrightarrow \phi\cdot f \qquad (\phi \in \bar{\mathcal{G}}),$$

where

$$\phi\cdot f \Big|_h = f(h\cdot\phi) \qquad (h \in \overline{\mathfrak{A}}).$$

Indeed,

$$\int_{\overline{\mathfrak{A}}} |\phi\cdot f|^2 \, d\mu_{AL} = \int_{\overline{\mathfrak{A}}} |f|^2 \, d\mu_{AL},$$

μ_{AL} being $\overline{\mathcal{G}}$-invariant.

Remark: Let $\pi : \overline{\mathfrak{A}} \longrightarrow \overline{\mathfrak{A}}/\overline{\mathcal{G}}$ be the projection and put $\mu_0 = \pi_*(\mu_{AL})$ -- then it is clear that

$$L^2_{inv}(\overline{\mathfrak{A}}; \mu_{AL}) \approx L^2(\overline{\mathfrak{A}}/\overline{\mathcal{G}}; \mu_0).$$

Let $\text{Diff}^{\omega} M$ denote the analytic diffeomorphism group of M -- then $\text{Diff}^{\omega} M$ operates on Gra M in the obvious way:

$$\Lambda \longrightarrow \varphi\Lambda.$$

Moreover, this action preserves the partial order on Gra M:

$$\Lambda_1 \leq \Lambda_2 \Rightarrow \varphi\Lambda_1 \leq \varphi\Lambda_2.$$

LEMMA μ_{AL} is Diff^{ω} M-invariant.

[Let $\varphi \in \text{Diff}^{\omega} M$ -- then $\forall \Lambda \in$ Gra M, there is an isomorphism of groupoids $\mathcal{P}\mathcal{G}_{\Lambda} \rightarrow \mathcal{P}\mathcal{G}_{\varphi\Lambda}$, hence, by contravariance, a homeomorphism

$$\text{Hom}(\mathcal{P}\mathcal{G}_{\varphi\Lambda}, G) \longrightarrow \text{Hom}(\mathcal{P}\mathcal{G}_{\Lambda}, G)$$

or still, a homeomorphism

$$\overline{\varphi}_\Lambda : \overline{\mathfrak{A}}_{\varphi\Lambda} \longrightarrow \overline{\mathfrak{A}}_\Lambda .$$

When combined, the $\overline{\varphi}_\Lambda$ lead to a homeomorphism $\overline{\varphi}: \overline{\mathfrak{A}} \longrightarrow \overline{\mathfrak{A}}$ rendering the diagram

$$\begin{array}{ccc} \overline{\mathfrak{A}} & \xrightarrow{\overline{\varphi}} & \overline{\mathfrak{A}} \\ {\scriptstyle \pi_{\varphi\Lambda}} \downarrow & & \downarrow {\scriptstyle \pi_\Lambda} \\ \overline{\mathfrak{A}}_{\varphi\Lambda} & \xrightarrow[\overline{\varphi}_\Lambda]{} & \overline{\mathfrak{A}}_\Lambda \end{array}$$

commutative. Suppose now that $f \in \mathrm{Cyl}(\overline{\mathfrak{A}})$, say $f = f_\Lambda \circ \pi_\Lambda$ -- then

$$\int_{\overline{\mathfrak{A}}} f \, d(\overline{\varphi})_* \mu_{AL} = \int_{\overline{\mathfrak{A}}} f \circ \overline{\varphi} \, d\mu_{AL}$$

$$= \int_{\overline{\mathfrak{A}}} f_\Lambda \circ \pi_\Lambda \circ \overline{\varphi} \, d\mu_{AL}$$

$$= \int_{\overline{\mathfrak{A}}} f_\Lambda \circ \overline{\varphi}_\Lambda \circ \pi_{\varphi\Lambda} \, d\mu_{AL}$$

$$= \int_{\overline{\mathfrak{A}}_{\varphi\Lambda}} f_\Lambda \circ \overline{\varphi}_\Lambda \, d\mu_{\varphi\Lambda} .$$

But

$$
\begin{tikzcd}
\overline{\mathfrak{A}}_{\varphi\Lambda} \arrow[r, "\approx"] \arrow[d, "\overline{\varphi}_{\Lambda}"'] & G^{\#E(\varphi\Lambda)\,=\,n} \arrow[d, dashed, "\overline{\varphi}_{\Lambda}"] \\
\overline{\mathfrak{A}}_{\Lambda} \arrow[r, "\approx"] & G^{\#E(\Lambda)\,=\,n}
\end{tikzcd}
.
$$

Here, the arrow on the right is a topological automorphism of G^n, hence has modular function $\equiv 1$ (G^n being compact). Therefore

$$\int_{\overline{\mathfrak{A}}_{\varphi\Lambda}} f_{\Lambda} \circ \overline{\varphi}_{\Lambda} \;\; d\mu_{\varphi\Lambda}$$

$$= \int_{G^n} \prod_{i=1}^{n} d\mu_i \;\; f_{\Lambda} \circ \overline{\varphi}_{\Lambda} \;\; (\sigma_1, \ldots, \sigma_n)$$

$$= \int_{G^n} \prod_{i=1}^{n} d\mu_i \;\; f_{\Lambda} \; (\sigma_1, \ldots, \sigma_n)$$

$$= \int_{\overline{\mathfrak{A}}_{\Lambda}} f_{\Lambda} \; d\mu_{\Lambda}$$

$$= \int_{\overline{\mathfrak{A}}} f_{\Lambda} \circ \pi_{\Lambda} \;\; d\mu_{AL} = \int_{\overline{\mathfrak{A}}} f \; d\mu_{AL}.]$$

The lemma implies that there is a natural unitary representation of $\mathrm{Diff}^{\omega} M$ on $L^2(\overline{\mathfrak{A}};\mu_{AL})$. However, it turns out that the only invariant elements in $L^2(\overline{\mathfrak{A}};\mu_{AL})$ are the constants, hence that the action of $\mathrm{Diff}^{\omega} M$ is ergodic.

A curve $\gamma:[0,1] \longrightarrow M$ is said to have no <u>self-intersections</u> if $\gamma(t') = \gamma(t'') \Rightarrow t' = t''$ or $t' = 0$, $t'' = 1$ or $t' = 1$, $t'' = 0$.

Definition: A <u>piecewise analytic edge</u> is a curve which has no self-intersections.

Every edge is, of course, a piecewise analytic edge.

Suppose that $\varphi: M \longrightarrow M$ is a homeomorphism -- then φ is said to be a <u>C^0 diffeomorphism</u> if $\forall$ piecewise analytic edge e, the composite $\varphi \circ e$ is again a piecewise analytic edge.

Remark: Using piecewise analytic edges, one can form piecewise analytic graphs. Therefore a homeomorphism $\varphi: M \longrightarrow M$ is a C^0 diffeomorphism iff φ sends piecewise analytic graphs to piecewise analytic graphs.

Notation: $\mathrm{Diff}^0 M$ is the group of C^0 diffeomorphisms.

<u>FACT</u> μ_{AL} is Diff^0 M-invariant.

<u>Two Dimensional Yang-Mills Theory</u> Recall our standing assumptions: M is analytic and path connected with dim $M \geq 2$ and G is a compact connected nonabelian Lie group.

Suppose given

$$\begin{array}{ccc} G & \longrightarrow & P \\ & & \downarrow \pi \\ & & M, \end{array}$$

where M is orientable and semiriemannian.

Rappel: The Yang-Mills lagrangian $\mathcal{L}_{YM}$ is the functional with domain $\mathfrak{A}(P)$ defined by the prescription

$$\mathcal{L}_{YM}(\Gamma) = \int_M \langle \mathfrak{F}_\Gamma , \mathfrak{F}_\Gamma \rangle \text{ vol}.$$

It is invariant under $\mathfrak{G}(P)$, hence passes to the quotient and defines a functional on the configuration space $\mathfrak{A}(P)/\mathfrak{G}(P)$.

Now let P run through a set of representatives for the isomorphism classes of principal G-bundles over M -- then $\mathcal{L}_{YM}$ is defined on

$$\coprod_P \mathfrak{A}(P)/\mathfrak{G}(P)$$

or still, $\mathcal{L}_{YM}$ is defined on

$$\mathrm{Hom}^\infty(\mathfrak{HG}, G)/G.$$

Problem: Extend the definition of $\mathcal{L}_{YM}$ to

$$\mathrm{Hom}(\mathfrak{HG}, G)/G,$$

i.e., to

$$\overline{\mathfrak{A}/\mathfrak{G}} \approx \overline{\mathfrak{A}}/\overline{\mathfrak{G}}.$$

[Note: The motivation is the quantization of Yang-Mills theories.

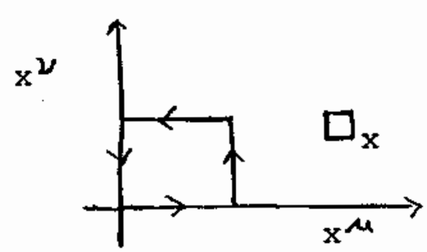
Since $\overline{\mathfrak{A}/\mathfrak{G}}$ is the quantum configuration space, heuristics arising from constructive field theory and functional integration suggest that one should consider

$$d\mu_{YM} = e^{-\mathcal{L}_{YM}} d\mu_{AL}.$$

Here μ_{AL} is the Ashtekar-Lewandowski measure on $\overline{\mathfrak{A}/\mathfrak{G}}$ but the exact meaning of $e^{-\mathcal{L}_{YM}}$ remains problematic.]

In what follows, we shall consider a particular case, viz. when $G = \underset{\sim}{SU}(N)$ $(N \geq 2)$ and M is the plane. However, even in this situation, the analysis is by no means simple.

First we need to deal with a generality (valid for arbitrary M). Fix $\Gamma \in \mathfrak{A}(P)$ and let $U \subset M$ be a local trivialization of P with coordinates $x^1, \ldots, x^n$. Consider a small square of side length ε in the $x^\mu - x^\nu$ plane:



Here $\square_x$ is the plaquette loop traced in the counterclockwise direction defined by its four corners $(x, x+\varepsilon\vec{e}_\mu, x+\varepsilon\vec{e}_\mu + \varepsilon\vec{e}_\nu, x+\varepsilon\vec{e}_\nu)$, where $x \in U$ and $\vec{e}_\mu, \vec{e}_\nu$ are unit vectors. Let $\{T_a\}$ be a basis for $\underline{su}(N)$ and write

$$\mathfrak{F}_{\mu\nu} = \sum_a \mathfrak{F}^a_{\mu\nu} T_a.$$

[Note: By definition, $\mathfrak{F} = s^*\Omega_\Gamma$ is the local field strength:

$$\mathfrak{F} = \frac{1}{2} \sum_{\mu,\nu} \mathfrak{F}_{\mu\nu}\, dx^{\mu} \wedge dx^{\nu},$$

where the $\mathfrak{F}_{\mu\nu}$ are $\underline{\mathrm{su}}(N)$-valued functions on U.]

APPROXIMATION LEMMA We have

$$\frac{1}{N}\operatorname{tr}(h(\Gamma, p;\ \square_x))$$

$$= 1 + \frac{\varepsilon^2}{N} \sum_a \mathfrak{F}^a_{\mu\nu}(x)\operatorname{tr}(T_a)$$

$$+ \frac{\varepsilon^4}{N} \sum_{a,b} \mathfrak{F}^a_{\mu\nu}(x)\,\mathfrak{F}^b_{\mu\nu}(x)\operatorname{tr}(T_a T_b) + O(\varepsilon^6).$$

Remark: Since it is a question of $\underline{\mathrm{su}}(N)$, $\operatorname{tr}(T_a) = 0$. Therefore the expansion reduces to

$$\frac{1}{N}\operatorname{tr}(h(\Gamma, p;\ \square_x)) = 1 + \frac{\varepsilon^4}{N} \sum_{a,b} \mathfrak{F}^a_{\mu\nu}(x)\,\mathfrak{F}^b_{\mu\nu}(x)\operatorname{tr}(T_a T_b) + O(\varepsilon^6).$$

Now take $M = \underline{R}^2$ (base point the origin) -- then P is trivial. And:

$$\mathfrak{F}_\Gamma = \frac{1}{2}\mathfrak{F}_{12}dx^1 \wedge dx^2 + \frac{1}{2}\mathfrak{F}_{21}dx^2 \wedge dx^1$$

$$= \mathfrak{F}_{12}dx^1 \wedge dx^2.$$

Write

$$\mathfrak{F}_{12} = \sum_a \mathfrak{F}^a_{12}\, T_a.$$

Then

$$\mathfrak{F}_\Gamma = \sum_a (\mathfrak{F}^a_{12}\, T_a)dx^1 \wedge dx^2$$

$\Rightarrow$

$$\langle \mathfrak{F}_\Gamma , \mathfrak{F}_\Gamma \rangle (x)$$

$$= \sum_{a,b} g(\mathfrak{F}^a_{12}(x)dx^1 \wedge dx^2, \mathfrak{F}^b_{12}(x)dx^1 \wedge dx^2)(\mathrm{IP}(T_a, T_b))$$

$$= -\frac{1}{N} \sum_{a,b} \mathfrak{F}^a_{12}(x) \mathfrak{F}^b_{12}(x) \ \mathrm{tr}(T_a T_b).$$

Therefore

$$1 - \frac{1}{N} \mathrm{Re}(\mathrm{tr}(h(\Gamma, p; \ \Box_x)))$$

$$\sim \ \varepsilon^4 \langle \mathfrak{F}_\Gamma, \ \mathfrak{F}_\Gamma \rangle (x).$$

[Note: The Killing form $\underset{\sim}{K}(X,Y)$ of $\underline{su}(N)$ is $2N\mathrm{tr}(X,Y)$.]

Let Λ be a finite square lattice in $\underset{\sim}{R}^2$ with spacing ε and length $\ell\varepsilon$ ($\ell \in \underset{\sim}{N}$) having the origin as a vertex, thus Λ contains ℓ^2 plaquette loops $\Box$.

Notation: Given $\Box$, choose a path $\rho_\Box$ in Λ from the base point to the lower left hand corner of $\Box$ and then put

$$\gamma_\Box = \rho_\Box^{-1} \circ \Box \circ \rho_\Box \quad .$$

[Note: The $\gamma_\Box$ thus generate $\pi_1(\Lambda)$.]

Let $\mathfrak{A} = \mathfrak{A}(P)$ ($P = \underset{\sim}{R}^2 \times \underset{\sim}{SU}(N)$) -- then the <u>Wilson lagrangian</u> $\mathcal{L}^\Lambda_W$ is the functional with domain $\mathfrak{A}$ defined by the prescription

$$\mathcal{L}^\Lambda_W(\Gamma) = \frac{1}{\varepsilon^2} \sum_\Box \left(1 - \frac{1}{N} \mathrm{Re}(\mathrm{tr}(h(\Gamma \ ; \Box)))\right),$$

or still,

$$\mathcal{L}^\Lambda_W(\Gamma) = \frac{1}{\varepsilon^2} \sum_\Box \left(1 - \frac{1}{N} \mathrm{Re}(\mathrm{tr}(h(\Gamma \ ; \gamma_\Box)))\right).$$

It is clear that $\mathcal{L}^{\wedge}_{W}$ is gauge invariant: $\forall\ f \in \mathcal{G}\ (= \mathcal{G}(P))$,

$$\mathcal{L}^{\wedge}_{W}(\Gamma \cdot f) = \mathcal{L}^{\wedge}_{W}(\Gamma),$$

so $\mathcal{L}^{\wedge}_{W}$ lives on $\mathfrak{A}/\mathcal{G}$. More is true: $\mathcal{L}^{\wedge}_{W}$ extends to $\overline{\mathfrak{A}/\mathcal{G}}$. Indeed, for any $h \in \mathrm{Hom}(\partial\ell\mathcal{G}, \underset{\sim}{SU}(N))/\underset{\sim}{SU}(N)$,

$$\mathcal{L}^{\wedge}_{W}(h) = \frac{1}{\epsilon^2} \sum_{\square} \left(1 - \frac{1}{N} \mathrm{Re}(\mathrm{tr}(h(\gamma_{\square})))\right).$$

Notation: $\wedge \longrightarrow \underset{\sim}{R}^2$ means that $\ell \longrightarrow \infty$, $\epsilon \longrightarrow 0$.

HEURISTIC PRINCIPLE $\forall\ \Gamma \in \mathfrak{A}$,

$$\lim_{\wedge \to \underset{\sim}{R}^2} \mathcal{L}^{\wedge}_{W}(\Gamma) = \mathcal{L}_{YM}(\Gamma).$$

[In fact,

$$\begin{aligned} \mathcal{L}^{\wedge}_{W}(\Gamma) &\sim \sum_{\square} \langle \mathcal{F}_{\Gamma}, \mathcal{F}_{\Gamma} \rangle \epsilon^2 \\ &\xrightarrow[\wedge \to \underset{\sim}{R}^2]{} \int_M \langle \mathcal{F}_{\Gamma}, \mathcal{F}_{\Gamma} \rangle \,\mathrm{vol} \\ &= \mathcal{L}_{YM}(\Gamma).] \end{aligned}$$

To simplify, henceforth write G for $\underset{\sim}{SU}(N)$ and $\underline{g}$ for $\underline{su}(N)$, where $N \geq 2$.

Let $p_{\wedge} : \overline{\mathfrak{A}/\mathcal{G}} \longrightarrow G^{\ell^2}$ be the projection

$$h \longrightarrow (h[\gamma_{\square}]).$$

Given a continuous function $\phi_{\wedge} : G^{\ell^2} \longrightarrow \underset{\sim}{C}$, we have

$$\int_{\overline{\mathfrak{A}/\mathcal{G}}} \phi_\Lambda \circ p_\Lambda \, d\mu_{AL} = \int_{G^{\ell^2}} \phi_\Lambda \, d\mu_\Lambda ,$$

μ_Λ being the normalized Haar measure on G^{ℓ^2}.

Example: $\mathcal{L}_W^\Lambda = \phi_\Lambda \circ p_\Lambda$, where

$$\phi_\Lambda(\sigma_\square) = \frac{1}{\varepsilon^2} \sum_\square (1-\frac{1}{N}\operatorname{Re}(\operatorname{tr}(\sigma_\square))).$$

Put

$$Z(\Lambda) = \int_{\overline{\mathfrak{A}/\mathcal{G}}} \exp(-\mathcal{L}_W^\Lambda(h))\, d\mu_{AL}(h).$$

Since the function $h \longrightarrow \exp(-\mathcal{L}_W^\Lambda(h))$ is continuous and $\overline{\mathfrak{A}/\mathcal{G}}$ is compact, it is clear that the integral defining $Z(\Lambda)$ is finite. In fact,

$$Z(\Lambda) = \left(\int_G \exp(-\frac{1}{\varepsilon^2}(1 - \frac{1}{N}\operatorname{Re}(\operatorname{tr}(\sigma))))\, d\mu(\sigma) \right)^{\ell^2}.$$

Given a loop γ in Λ , let

$$\chi(\gamma;\Lambda) = \frac{1}{Z(\Lambda)} \int_{\overline{\mathfrak{A}/\mathcal{G}}} \exp(-\mathcal{L}_W^\Lambda(h))\operatorname{tr}(h[\gamma])\, d\mu_{AL}(h).$$

Write

$$\gamma = \gamma_{\square_{i_1}}^{\varepsilon_1} \cdots \gamma_{\square_{i_k}}^{\varepsilon_k} \qquad (\varepsilon_j = \pm 1).$$

Then

$$h[\gamma] = h[\gamma_{\square_{i_1}}^{\varepsilon_1}] \cdots h[\gamma_{\square_{i_k}}^{\varepsilon_k}]$$

$\Longrightarrow$

$$\chi(\gamma;\Lambda) = \frac{1}{Z(\Lambda)} \int_{G^{\ell^2}} \exp(-\mathcal{L}_W^\Lambda(\sigma_\square))$$

$$\times \operatorname{tr}\left(\prod_{j=1}^{k} \sigma_{\square_{i_j}}^{\varepsilon_j}\right) d\mu_\Lambda(\sigma_\square).$$

The next step is to calculate $\chi(\gamma;\Lambda)$ in closed form. This is a difficult undertaking and the final result, while explicit, is somewhat complicated to state, thus I will omit the details. From here, one then proceeds to the main conclusion, which simply says:

$$\lim_{\Lambda \to \underset{\sim}{R}^2} \chi(\gamma;\Lambda)$$

exists.

Example: If γ is simple in the sense that $[\gamma]$ contains a loop with no self-intersections, then

$$\lim_{\Lambda \to \underset{\sim}{R}^2} \chi(\gamma;\Lambda) = e^{-cA(\gamma)},$$

where c is a certain positive constant and $A(\gamma)$ is the area enclosed by γ.

<u>THEOREM</u> There is a Radon measure μ_{YM} on $\overline{\mathcal{O}\!\mathcal{L}/\mathcal{G}}$ of total mass 1 such that

$$\lim_{\Lambda \to \underset{\sim}{R}^2} \chi(\gamma;\Lambda) = \int_{\overline{\mathcal{O}\!\mathcal{L}/\mathcal{G}}} \operatorname{tr}(h[\gamma])\, d\mu_{YM}(h).$$

[Note: μ_{YM} is called the <u>Yang-Mills</u> measure.]

Remark: Proceeding formally, it is tempting to write

$$\lim_{\Lambda \to \underset{\sim}{R}^2} \chi(\gamma;\Lambda)$$

$$= \frac{1}{\lim_{\Lambda \to \mathbb{R}^2} Z(\Lambda)} \cdot \int_{\overline{\mathcal{A}/\mathcal{G}}} \lim_{\Lambda \to \mathbb{R}^2} \exp(-\mathcal{L}_W^{\Lambda}(h))\,\mathrm{tr}(h[\gamma])\,d\mu_{AL}(h)$$

$$= \frac{1}{Z} \cdot \int_{\overline{\mathcal{A}/\mathcal{G}}} \exp(-\mathcal{L}_{YM}(h))\,\mathrm{tr}(h[\gamma])\,d\mu_{AL}(h).$$

However (see below), such a procedure is necessarily doomed to fail.

Properties of μ_{YM}:

(1) μ_{AL} and μ_{YM} are mutually singular, i.e., $\exists$ a measurable set $W \subset \overline{\mathcal{A}/\mathcal{G}}$ with

$$\mu_{AL}(W) = 0 \;\&\; \mu_{YM}(W) = 1;$$

(2) $\mu_{YM}(\overline{\mathcal{A}}_{gen}/\overline{\mathcal{G}}) = 1$;

(3) $\mu_{YM}(\mathcal{A}(P)/\mathcal{G}(P)) = 0$.

Properties (2) and (3) are the analogs of what we know to be true of μ_{AL}; on the other hand, (1) implies that $\nexists$ a measurable function S_{YM} on $\overline{\mathcal{A}/\mathcal{G}}$ such that

$$\mu_{YM} = e^{-S_{YM}} \mu_{AL}.$$

Decomposition Theory   Suppose that M is analytic and path connected with $\dim M \geq 2$ and G is a compact connected nonabelian Lie group.

Let Π be a set of representatives for the unitary equivalence classes of irreducible unitary representations of G. Given $\pi \in \Pi$, denote by d_π its dimension and write $[\pi(\sigma)]_{ij}$ $(1 \leq i \leq d_\pi,\ 1 \leq j \leq d_\pi)$ for the matrix elements of $\pi(\sigma)$ $(\sigma \in G)$.

Rappel: The functions

$$\sqrt{d_\pi}\,[\pi(\cdot)]_{ij}$$

are a complete orthonormal system in $L^2(G)$ and their linear span is dense in $C(G)$.

[Note: If $L^{2,*}(G)$ is the closed linear span of the $\sqrt{d_\pi}\,[\pi(\cdot)]_{ij}$, where $\pi \neq \pi_t$ (the trivial one dimensional representation of G), then

$$L^2(G) = \underset{\sim}{C}1 \oplus L^{2,*}(G).]$$

Remark: Up to unitary equivalence, every irreducible unitary representation of G^n $(n > 1)$ has the form $\bigotimes_{k=1}^{n} \pi_k$ $(\pi_k \in \Pi)$.

Therefore the functions

$$\prod_{k=1}^{n} \sqrt{d_{\pi_k}}\,[\pi_k(\cdot)]_{i_k j_k}$$

are a complete orthonormal system in $L^2(G^n)$ and their linear span is dense in $C(G^n)$.

Suppose now that $\Lambda \in$ Gra M -- then

$$\overline{\mathfrak{A}} \xrightarrow{\pi_\Lambda} \overline{\mathfrak{A}}_\Lambda \approx G^{\#E(\Lambda)}$$

and

$$L^2(\overline{\mathfrak{A}}_\Lambda) \approx \bigotimes_{e \in E(\Lambda)} L^2(G).$$

Furthermore, there is an arrow of insertion

$$L^2(\overline{\mathfrak{A}}_\Lambda) \longrightarrow L^2(\overline{\mathfrak{A}};\mu_{AL})$$

and the union

$$\bigcup_\Lambda L^2(\overline{\mathfrak{A}}_\Lambda)$$

is dense in $L^2(\overline{\mathfrak{A}};\mu_{AL})$.

[Note: The subspace corresponding to the empty graph is $\underset{\sim}{C}1$.]

Let $\Pi(\Lambda)$ stand for the set of all functions $\underline{\pi}:E(\Lambda) \longrightarrow \Pi$. Determine i_e and j_e per π_e ($\equiv \underline{\pi}(e)$) and put

$$\begin{cases} \underline{i} = \{i_e\} & (1 \leq i_e \leq d_{\pi_e}) \\ \underline{j} = \{j_e\} & (1 \leq j_e \leq d_{\pi_e}). \end{cases}$$

Definition: The <u>edge network</u>

$$T_{\Lambda;\underline{\pi},\underline{i},\underline{j}}$$

is the cylinder function $\overline{\mathfrak{A}} \longrightarrow \underset{\sim}{C}$ defined by

$$h \longrightarrow \prod_{e\in E(\Lambda)} \sqrt{d_{\pi_e}}\ [\pi_e(\pi_\Lambda(h)\big|_e)]_{i_e j_e}.$$

[Note: If the orientation of an edge is reversed and the corresponding representation is dualized, then the edge network is unchanged. This type of overcompleteness will be ignored in the sequel.]

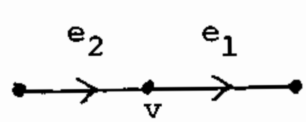
LEMMA The span of the

$$T_{\Lambda;\underline{\pi},\underline{i},\underline{j}} \qquad (\Lambda \in \text{Gra } M)$$

is dense in $C(\overline{\mathfrak{A}})$.

It follows from this that the set of edge networks is total in $L^2(\overline{\mathfrak{A}};\mu_{AL})$.

Example (Fleischhack): Contrary to what might be expected, the set of edge networks is not orthonormal. To see this, take for Λ an edge e and then decompose e into the product e_1e_2 of two edges by placing a vertex in the interior of e:



.

Denote by Λ' the graph thus obtained, so that $\Lambda < \Lambda'$. Fix $\pi \in \Pi : d_\pi > 1$ and fix indices

$$\begin{cases} 1 \leq i \leq d_{\pi} \\ 1 \leq j \leq d_{\pi} . \end{cases}$$

Put

$$T = \sqrt{d_{\pi}}\,[\pi(\Pi_{\Lambda}(\quad)|_e)]_{ij} ,$$

an edge network per Λ. Define $\underline{\pi} \in \Pi(\Lambda')$ by

$$\begin{cases} \pi_{e_1} = \pi \\ \pi_{e_2} = \pi. \end{cases}$$

Let

$$\begin{cases} \underline{i}_m = \left\{ i(= i_{e_1}),\ m(= i_{e_2}) \right\} \\ \underline{j}_m = \left\{ m(= j_{e_1}),\ j(= j_{e_2}) \right\} \end{cases} \quad (1 \leq m \leq d_{\pi}).$$

Then the

$$T_m = \sqrt{d_{\pi}}\,[\pi(\Pi_{\Lambda'}(\quad)|_{e_1})]_{im} \cdot \sqrt{d_{\pi}}\,[\pi(\Pi_{\Lambda'}(\quad)|_{e_2})]_{mj}$$

are edge networks per Λ'. From the definitions, $\forall\, h \in \overline{\mathfrak{O}\mathfrak{l}}$,

$$\Pi_{\Lambda'}(h)|_{e_1}\ \Pi_{\Lambda'}(h)|_{e_2}$$

$$= h(e_1)h(e_2)$$

$$= h(e_1e_2)$$

$$= h(e) = \Pi_{\Lambda}(h)|_e,$$

hence

$$\sqrt{d_\pi}\,T = \sum_m T_m.$$

But

$$\langle T_m, T_n \rangle = \int_{\overline{\mathfrak{A}}} \overline{T}_m T_n \, d\mu_{AL}$$

$$= (d_\pi)^2 \int_{\overline{\mathfrak{A}}} \overline{[\pi(\pi_{\Lambda'}(h)|_{e_1})]_{im}} \cdot [\pi(\pi_{\Lambda'}(h)|_{e_1})]_{in}$$

$$\times \overline{[\pi(\pi_{\Lambda'}(h)|_{e_2})]_{mj}} \cdot [\pi(\pi_{\Lambda'}(h)|_{e_2})]_{nj} \, d\mu_{AL}(h)$$

$$= (d_\pi)^2 \int_G \overline{\pi(\sigma)}_{im} \cdot \pi(\sigma)_{in} \, d\mu(\sigma)$$

$$\times \int_G \overline{\pi(\sigma)}_{mj} \cdot \pi(\sigma)_{nj} \, d\mu(\sigma)$$

$$= (d_\pi)^2 \cdot \frac{\delta_{mn}}{d_\pi} \cdot \frac{\delta_{mn}}{d_\pi}$$

$$= \delta_{mn}$$

$\Longrightarrow$

$$\langle T, T_n \rangle = \frac{1}{\sqrt{d_\pi}} \left\langle \sum_m T_m, T_n \right\rangle$$

$$= \frac{1}{\sqrt{d_\pi}} \sum_m \langle T_m, T_n \rangle$$

$$= \frac{1}{\sqrt{d_\pi}} .$$

Since $d_\pi > 1$, it is clear that $T \neq T_n$, yet, by the above, T and T_n are not orthogonal.

Consider $L^2(G)$ -- then $G \times G$ operates to the right on G, viz.

$$\sigma \cdot (\sigma_1, \sigma_2) = \sigma_1^{-1} \sigma \sigma_2,$$

the corresponding unitary representation on $L^2(G)$ being the assignment

$$f(\sigma) \longrightarrow f(\sigma_1^{-1} \sigma \sigma_2),$$

which decomposes as

$$\bigoplus_{\pi \in \hat{G}} \bar{\pi} \otimes \pi \quad .$$

[Note: For us, the inner product is conjugate linear in the first slot, hence it is a question of $\bar{\pi} \otimes \pi$, not $\pi \otimes \bar{\pi}$.]

Therefore

$$L^2(\overline{\mathfrak{A}}_\Lambda) \approx \bigotimes_{e \in E(\Lambda)} L^2(G)$$

$$\approx \bigotimes_{e \in E(\Lambda)} \bigoplus_{\pi \in \hat{G}} \bar{\pi} \otimes \pi .$$

Here $\phi \in \bar{\mathcal{G}}_\Lambda$ operates as

$$\bigotimes_{e \in E(\Lambda)} \bigoplus_{\pi \in \hat{G}} \bar{\pi}(\phi(e(1))) \otimes \pi(\phi(e(0))).$$

[Note: Recall that $\forall\, h \in \overline{\mathfrak{A}}_\Lambda$,

$$h \cdot \phi(e) = \phi(e(1))^{-1} h(e) \phi(e(0))$$

$$\equiv h(e) \cdot (\phi(e(1)), \phi(e(0))).]$$

Taking into account the associativity of tensor products and direct sums then gives

$$L^2(\overline{\mathfrak{A}}_\Lambda) \approx \bigoplus_{\underline{\Pi} \in \Pi(\Lambda)} \bigotimes_{e \in E(\Lambda)} \overline{\Pi}_e \otimes \Pi_{e'}$$

the action of $\phi \in \overline{\mathcal{G}}_\Lambda$ becoming

$$\bigoplus_{\underline{\Pi} \in \Pi(\Lambda)} \bigotimes_{e \in E(\Lambda)} \overline{\Pi}_e(\phi(e(1))) \otimes \Pi_e(\phi(e(0))).$$

Put

$$L^2(\overline{\mathfrak{A}}_\Lambda ; \underline{\Pi}) = \bigotimes_{e \in E(\Lambda)} \overline{\Pi}_e \otimes \Pi_e .$$

Then $L^2(\overline{\mathfrak{A}}_\Lambda ; \underline{\Pi})$ is a finite dimensional $\overline{\mathcal{G}}_\Lambda$ -invariant subspace of $L^2(\overline{\mathfrak{A}}_\Lambda)$. Since $\overline{\mathcal{G}}_\Lambda \approx G^{\#V(\Lambda)}$, its irreducible unitary representations are in a one-to-one correspondence with the functions

$$\underline{\rho} : V(\Lambda) \rightarrow \Pi .$$

So, denoting by

$$L^2(\overline{\mathfrak{A}}_\Lambda ; \underline{\Pi} ; \underline{\rho})$$

the isotypic $\overline{\mathcal{G}}_\Lambda$ -subspace of $L^2(\overline{\mathfrak{A}}_\Lambda ; \underline{\Pi})$ of type $\underline{\rho}$, we have

$$L^2(\overline{\mathfrak{A}}_\Lambda ; \underline{\Pi}) = \bigoplus_{\underline{\rho}} L^2(\overline{\mathfrak{A}}_\Lambda ; \underline{\Pi} ; \underline{\rho}).$$

[Note: There are, of course, but finitely many $\underline{\rho}$ for which

$$L^2(\overline{\mathfrak{A}}_\Lambda ; \underline{\Pi} ; \underline{\rho})$$

is nonzero.]

<u>Spin Networks</u> Maintaining the assumptions of the preceding section, suppose that $\Lambda \in \mathrm{Gra}\ M$.

Notation: Given $v \in V(\Lambda)$, let

$$\begin{cases} S(v) = \{e \in E(\Lambda) : e(0) = v\} \\ T(v) = \{e \in E(\Lambda) : e(1) = v\}. \end{cases}$$

With the understanding that an empty tensor product of representations is the trivial one dimensional representation π_t of G, we can then write

$$L^2(\bar{\mathfrak{A}}_\Lambda) \approx \bigoplus_{\underline{\pi} \in \Pi(\Lambda)} \ \bigotimes_{v \in V(\Lambda)} \ \Big(\bigotimes_{e \in T(v)} \bar{\pi}_e \otimes \bigotimes_{e \in S(v)} \pi_e\Big).$$

In this description, the action of $\phi \in \bar{\mathscr{G}}_\Lambda$ is

$$\bigoplus_{\underline{\pi} \in \Pi(\Lambda)} \ \bigotimes_{v \in V(\Lambda)} \ \Big(\bigotimes_{e \in T(v)} \bar{\pi}_e(\phi(v)) \otimes \bigotimes_{e \in S(v)} \pi_e(\phi(v))\Big).$$

Notation: Given $\underline{\pi} \in \Pi(\Lambda)$ and $v \in V(\Lambda)$, let

$$\mathrm{Inv}(\underline{\pi}, v)$$

be the G-invariants in

$$\bigotimes_{e \in T(v)} \bar{\pi}_e \otimes \bigotimes_{e \in S(v)} \pi_e.$$

[Note: Since

$$\bigotimes_{e \in T(v)} \bar{\pi}_e \otimes \bigotimes_{e \in S(v)} \pi_e$$

is a unitary representation of G, it can be decomposed into irreducibles. Assuming that π_t actually appears, $\mathrm{Inv}(\underline{\pi}, v)$ is simply a direct sum of a certain number of copies of $\underset{\sim}{C}$ on which G acts trivially.]

The space $L^2(\bar{\mathfrak{a}}_\Lambda / \bar{\mathfrak{g}}_\Lambda)$ can be viewed as the subspace of $\bar{\mathfrak{g}}_\Lambda$-invariant elements in $L^2(\bar{\mathfrak{a}}_\Lambda)$. Therefore

$$L^2(\bar{\mathfrak{a}}_\Lambda / \bar{\mathfrak{g}}_\Lambda) \approx \bigoplus_{\underline{\Pi} \in \Pi(\Lambda)} \bigotimes_{v \in V(\Lambda)} \mathrm{Inv}(\underline{\Pi}, v).$$

Rappel: If π_1 and π_2 are finite dimensional unitary representations of G, then the subspace of G-invariant vectors in $\bar{\pi}_1 \otimes \pi_2$ is isomorphic to the space $\mathrm{Hom}(\pi_1, \pi_2)$ of intertwining operators from π_1 to π_2.

[Note: Let H_i be the representation space of π_i $(i=1,2)$ -- then $\bar{H}_1 \otimes H_2$ can be identified with the set of linear transformations $T: H_1 \longrightarrow H_2$, the inner product being

$$\langle T, S \rangle = \mathrm{tr}(TS^*).$$

Here,

$$T_{x_1, x_2} \equiv x_1 \otimes x_2$$

sends y_1 to $\langle x_1, y_1 \rangle x_2$, thus $T^*_{x_1, x_2}$ sends y_2 to $\langle x_2, y_2 \rangle x_1$. To run a reality check, fix an orthonormal basis $\{e_i\}$ in H_2 -- then

$$\langle x_1 \otimes x_2, x_1' \otimes x_2' \rangle$$

$$= \langle T_{x_1, x_2}, T_{x_1', x_2'} \rangle$$

$$= \mathrm{tr}(T_{x_1, x_2} \cdot T^*_{x_1', x_2'})$$

$$= \sum_i \langle T_{x_1,x_2} T^*_{x_1',x_2'} e_i, e_i \rangle$$

$$= \sum_i \langle T^*_{x_1',x_2'} e_i, T^*_{x_1,x_2} e_i \rangle$$

$$= \sum_i \langle \langle x_2', e_i \rangle x_1', \langle x_2, e_i \rangle x_1 \rangle$$

$$= \langle x_1', x_1 \rangle \sum_i \overline{\langle x_2', e_i \rangle} \langle x_2, e_i \rangle$$

$$= \langle x_1', x_1 \rangle \sum_i \overline{\langle e_i, x_2 \rangle} \langle e_i, x_2' \rangle$$

$$= \langle x_1', x_1 \rangle \langle x_2, x_2' \rangle$$

$$= \langle x_1, x_1' \rangle_{\overline{H}_1} \langle x_2, x_2' \rangle_{H_2} .$$

Next, if

$$\begin{cases} A: H_1 \longrightarrow H_1 \\ B: H_2 \longrightarrow H_2 \end{cases}$$

are linear transformations, then

$$A \otimes B: \overline{H}_1 \otimes H_2 \longrightarrow \overline{H}_1 \otimes H_2$$

is defined by

$$(A \otimes B) T = BTA^*$$

and we have

$$(A \otimes B)(x_1 \otimes x_2) = Ax_1 \otimes Bx_2.$$

In fact,

$$(A\otimes B)(x_1\otimes x_2)\Big|_{y_1} = BT_{x_1,x_2}\,A^*y_1$$

$$= B(\langle x_1,A^*y_1\rangle x_2)$$

$$= \langle Ax_1,y_1\rangle Bx_2,$$

while

$$Ax_1\otimes Bx_2\Big|_{y_1} = T_{Ax_1,Bx_2}\,y_1$$

$$= \langle Ax_1,y_1\rangle Bx_2.$$

This said, $T\in \mathrm{Hom}(\pi_1,\pi_2)$ iff $\forall\,\sigma\in G$,

$$T\pi_1(\sigma) = \pi_2(\sigma)T$$

$$\Longleftrightarrow$$

$$T = \pi_2(\sigma)T\,\pi_1(\sigma^{-1})$$

$$\Longleftrightarrow$$

$$T = \pi_2(\sigma)T\,\pi_1(\sigma)^*$$

$$\Longleftrightarrow$$

$$T = (\pi_1(\sigma)\otimes\pi_2(\sigma))T,$$

the condition that T be G-invariant.]

Consequently

$$\mathrm{Inv}(\underline{\pi},v) \approx \mathrm{Hom}\Big(\bigotimes_{e\in T(v)}\pi_e,\ \bigotimes_{e\in S(v)}\pi_e\Big),$$

so

$$L^2(\overline{\mathfrak{A}}_\Lambda/\overline{\mathfrak{B}}_\Lambda) \approx \bigoplus_{\underline{\Pi} \in \Pi(\Lambda)} \bigotimes_{v \in V(\Lambda)} \mathrm{Hom}\Big(\bigotimes_{e \in T(v)} \Pi_e, \bigotimes_{e \in S(v)} \Pi_e\Big).$$

It remains to make this explicit.

Let

$$\begin{cases} \{\Pi_{a_1}, \ldots, \Pi_{a_K}\} \\ \{\Pi_{b_1}, \ldots, \Pi_{b_L}\} \end{cases}$$

be two finite subsets of Π. Fix an orthonormal basis

$$\begin{cases} \{e_{k;i_{a_k}} \quad (1 \leq i_{a_k} \leq d_{\Pi_{a_k}})\} \\ \{e_{\ell;j_{b_\ell}} \quad (1 \leq j_{b_\ell} \leq d_{\Pi_{b_\ell}})\} \end{cases}$$

in the representation space of

$$\begin{cases} \Pi_{a_k} & (k=1,\ldots,K) \\ \Pi_{b_\ell} & (\ell=1,\ldots,L). \end{cases}$$

Suppose that

$$I \in \mathrm{Hom}\Big(\bigotimes_{k=1}^{K} \Pi_{a_k}, \bigotimes_{\ell=1}^{L} \Pi_{b_\ell}\Big)$$

is an intertwining operator -- then

$$I(e_{1;i_{a_1}} \otimes \cdots \otimes e_{K;i_{a_K}})$$

$$= I^{j_{b_1}\cdots j_{b_L}}_{i_{a_1}\cdots i_{a_K}} \; e_{1;j_{b_1}} \otimes \cdots \otimes e_{L;j_{b_L}} .$$

But $\forall \sigma \in G$,

$$I(\pi_{a_1}(\sigma)\otimes\cdots\otimes\pi_{a_K}(\sigma))$$

$$= (\pi_{b_1}(\sigma)\otimes\cdots\otimes\pi_{b_L}(\sigma))I$$

or still,

$$I = (\pi_{b_1}(\sigma)\otimes\cdots\otimes\pi_{b_L}(\sigma))I(\pi_{a_1}(\sigma^{-1})\otimes\cdots\otimes\pi_{a_K}(\sigma^{-1})).$$

Therefore

$$I^{j_{b_1}\cdots j_{b_L}}_{i_{a_1}\cdots i_{a_K}}$$

$$= [\pi_{b_1}(\sigma)]^{j_{b_1}}_{n_{b_1}} \cdots [\pi_{b_L}(\sigma)]^{j_{b_L}}_{n_{b_L}}$$

$$\times \; I^{n_{b_1}\cdots n_{b_L}}_{m_{a_1}\cdots m_{a_K}}$$

$$\times \; [\pi_{a_1}(\sigma^{-1})]^{m_{a_1}}_{i_{a_1}} \cdots [\pi_{a_K}(\sigma^{-1})]^{m_{a_K}}_{i_{a_K}} .$$

So, e.g., these observations apply to the

$$I_v \in \mathrm{Hom}\Big(\bigotimes_{e\in T(v)} \pi_e, \; \bigotimes_{e\in S(v)} \pi_e\Big).$$

Note the index pattern

$$\begin{cases} \text{subscripts} \longleftrightarrow T(v) \\ \text{superscripts} \longleftrightarrow S(v). \end{cases}$$

Definition: A <u>spin network</u> is a triple $(\Lambda;\underline{\Pi},\underline{I})$ consisting of a graph $\Lambda \in \text{Gra } M$, an element $\underline{\Pi} \in \Pi(\Lambda)$, and a set $\underline{I} = \{I_v : v \in V(\Lambda)\}$, where

$$I_v \in \text{Hom}\left(\bigotimes_{e\in T(v)} \pi_e, \bigotimes_{e\in S(v)} \pi_e\right).$$

Every spin network determines a function $\Psi_{\Lambda;\underline{\Pi},\underline{I}}$ on $\overline{\mathfrak{A}}_\Lambda$ via the following procedure: Assign to a given $h \in \overline{\mathfrak{A}}_\Lambda$ the number

$$\left[\bigotimes_{e\in E(\Lambda)} \pi_e(h(e))\right] \bullet \left[\bigotimes_{v\in V(\Lambda)} I_v\right],$$

where the bullet $\bullet$ stands for contracting at each $v \in V(\Lambda)$ the upper indices of the matrices corresponding to the incoming edges, the lower indices of the matrices corresponding to the outgoing edges, and the corresponding indices of I_v.

[Note: The function $\Psi_{\Lambda;\underline{\Pi},\underline{I}}$ is called a <u>spin network state</u>.]

(Loops) Take for Λ the graph

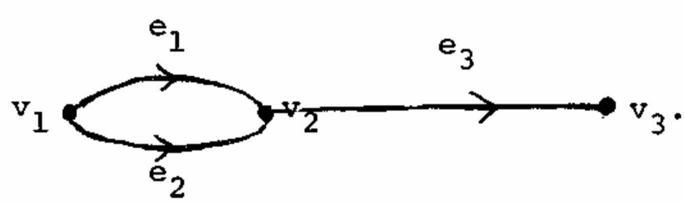
v  e_v.

Let $\pi \in \Pi$ and let I_v be the identity intertwining operator -- then

$$\Psi_{\Lambda;\underline{\Pi},\underline{I}}(h) = \pi(h(e_v))^i_j \, \delta^j_i$$

$$= \mathrm{tr}(\pi(h(e_v))).$$

Example: Consider the graph



Here

$$\begin{cases} S(v_1) = \{e_1, e_2\}, & T(v_1) = \emptyset \\ S(v_2) = \{e_3\}, & T(v_2) = \{e_1, e_2\} \\ S(v_3) = \emptyset, & T(v_3) = \{e_3\}. \end{cases}$$

We have

$$\begin{cases} I_{v_1} : \pi_t \longrightarrow \pi_{e_1} \otimes \pi_{e_2} \\ I_{v_2} : \pi_{e_1} \otimes \pi_{e_2} \longrightarrow \pi_{e_3} \\ I_{v_3} : \pi_{e_3} \longrightarrow \pi_t. \end{cases}$$

Therefore

$$\Psi_{\Lambda;\underline{\pi},\underline{I}}(h) =$$

$$[\pi_{e_1}(h(e_1))]^{i_1}_{j_1} \; [\pi_{e_2}(h(e_2))]^{i_2}_{j_2} \; [\pi_{e_3}(h(e_3))]^{i_3}_{j_3}$$

$$\times \ (I_{v_1})^{j_1 j_2} \ (I_{v_2})^{j_3}_{i_1 i_2} \ (I_{v_3})_{i_3} .$$

LEMMA The functions

$$\Psi_{\Lambda;\underline{\Pi},\underline{I}}$$

are $\overline{\mathcal{G}}_\Lambda$ -invariant.

[This is a bit of a mess to write out in general but the idea can be illustrated with the preceding example. Thus let $\phi \in \overline{\mathcal{G}}_\Lambda$ -- then the claim is that $\forall\, h \in \overline{\mathfrak{A}}_\Lambda$,

$$\Psi_{\Lambda;\underline{\Pi},\underline{I}}(h\cdot\phi) = \Psi_{\Lambda;\underline{\Pi};\underline{I}}(h),$$

where

$$h\cdot\phi\,(e_a) = \phi(e_a(1))^{-1}\, h(e_a)\,\phi\,(e_a(0)) \quad (1 \leq a \leq 3).$$

Consider

$$[\Pi_{e_1}(h\cdot\phi(e_1))]^{i_1}_{j_1} \ [\Pi_{e_2}(h\cdot\phi(e_2))]^{i_2}_{j_2} \ [\Pi_{e_3}(h\cdot\phi(e_3))]^{i_3}_{j_3}$$

$$\times \ (I_{v_1})^{j_1 j_2} \ (I_{v_2})^{j_3}_{i_1 i_2} \ (I_{v_3})_{i_3}$$

$$= [\Pi_{e_1}(\phi(e_1(1))^{-1}\, h(e_1)\,\phi\,(e_1(0)))]^{i_1}_{j_1}$$

$$\cdot\ [\Pi_{e_2}(\phi(e_2(1))^{-1}\, h(e_2)\,\phi\,(e_2(0)))]^{i_2}_{j_2}$$

$$\cdot\ [\pi_{e_3}(\phi(e_3(1))^{-1}\, h(e_3)\,\phi(e_3(0)))]^{i_3}_{j_3}$$

$$\times\ (I_{v_1})^{j_1 j_2}\ (I_{v_2})^{j_3}_{i_1 i_2}\ \ (I_{v_3})_{i_3}$$

$$= [\pi_{e_1}(\phi(v_2)^{-1})]^{i_1}_{k_1}\ [\pi_{e_1}(h(e_1))]^{k_1}_{\ell_1}\ [\pi_{e_1}(\phi(v_1))]^{\ell_1}_{j_1}$$

$$\cdot\ [\pi_{e_2}(\phi(v_2)^{-1})]^{i_2}_{k_2}\ [\pi_{e_2}(h(e_2))]^{k_2}_{\ell_2}\ [\pi_{e_2}(\phi(v_1))]^{\ell_2}_{j_2}$$

$$\cdot\ [\pi_{e_3}(\phi(v_3)^{-1})]^{i_3}_{k_3}\ [\pi_{e_3}(h(e_3))]^{k_3}_{\ell_3}\ [\pi_{e_3}(\phi(v_2))]^{\ell_3}_{j_3}$$

$$\times\ (I_{v_1})^{j_1 j_2}\ (I_{v_2})^{j_3}_{i_1 i_2}\ \ (I_{v_3})_{i_3}$$

$$= [\pi_{e_1}(h(e_1))]^{k_1}_{\ell_1}\ \ [\pi_{e_2}(h(e_2))]^{k_2}_{\ell_2}\ \ [\pi_{e_3}(h(e_3))]^{k_3}_{\ell_3}$$

$$\times\ [\pi_{e_1}(\phi(v_1))]^{\ell_1}_{j_1}\ [\pi_{e_2}(\phi(v_1))]^{\ell_2}_{j_2}\ (I_{v_1})^{j_1 j_2}$$

$$\times\ [\pi_{e_3}(\phi(v_2))]^{\ell_3}_{j_3}\ (I_{v_2})^{j_3}_{i_1 i_2}\ [\pi_{e_1}(\phi(v_2)^{-1})]^{i_1}_{k_1}\ [\pi_{e_2}(\phi(v_2)^{-1})]^{i_2}_{k_2}$$

$$\times\ (I_{v_3})_{i_3}\ [\pi_{e_3}\ (\phi(v_3)^{-1})]^{i_3}_{k_3}$$

$$= [\pi_{e_1}(h(e_1))]^{k_1}_{\ell_1} \; [\pi_{e_2}(h(e_2))]^{k_2}_{\ell_2} \; [\pi_{e_3}(h(e_3))]^{k_3}_{\ell_3}$$

$$\times \; (I_{v_1})^{\ell_1 \ell_2} \; (I_{v_2})^{\ell_3}_{k_1 k_2} \; (I_{v_3})_{k_3} \,,$$

which is precisely

$$\Psi_{\Lambda;\underline{\pi},\underline{I}}(h).]$$

Consider now the inner product

$$\langle \Psi_{\Lambda;\underline{\pi},\underline{I}} \,,\; \Psi_{\Lambda;\pi',\underline{I}'} \rangle \;.$$

Omitting for the moment the terms involving the intertwining operators (which, being constant, can be taken outside the integral sign), consider

$$\prod_{e \in E(\Lambda)} \int_G \overline{[\pi_e(\sigma)]^{i_e}_{j_e}} \; [\pi'_e(\sigma)]^{i'_e}_{j'_e} \, d\sigma$$

or still,

$$\prod_{e \in E(\Lambda)} \frac{\delta_{\pi_e,\pi'_e}}{(d_{\pi_e} d_{\pi'_e})^{1/2}} \; \delta_{i_e,i'_e} \; \delta_{j_e,j'_e} \;.$$

Thus there is no contribution unless $\pi_e = \pi'_e$, leaving

$$\prod_{e \in E(\Lambda)} \frac{1}{d_{\pi_e}} \, \delta_{i_e,i'_e} \; \delta_{j_e,j'_e} \;.$$

Restoring the terms involving the intertwining operators then gives

$$\langle \Psi_{\Lambda;\underline{\Pi},\underline{I}}, \Psi_{\Lambda;\underline{\Pi},\underline{I}'} \rangle$$

$$= \prod_{v \in V(\Lambda)} D(v)\overline{\langle I_v, I_v' \rangle},$$

where

$$D(v) = \prod_{e \in S(v)} \frac{1}{d_{\Pi_e}}.$$

[Note: The fact that $\overline{\langle I_v, I_v' \rangle}$ appears as opposed to $\langle I_v, I_v' \rangle$ is a consequence of our definition of the inner product on

$$\mathrm{Hom}(\bigotimes_{e \in T(v)} \Pi_e, \bigotimes_{e \in S(v)} \Pi_e),$$

viz.

$$\langle I_v, I_v' \rangle = \mathrm{tr}(I_v I_v'^{*}),$$

which is conjugate linear in the second slot rather than the first slot.]

Fix $\underline{\Pi} \in \Pi(\Lambda)$ and adjust the definitions in the obvious way -- then the foregoing discussion implies that the arrow

$$\underline{I} \longrightarrow \Psi_{\Lambda;\underline{\Pi},\underline{I}}$$

injects

$$\bigotimes_{v \in V(\Lambda)} \mathrm{Hom}(\bigotimes_{e \in T(v)} \Pi_e, \bigotimes_{e \in S(v)} \Pi_e)$$

isometrically into $L^2(\overline{\mathfrak{A}}_\Lambda / \overline{\mathfrak{G}}_\Lambda)$.

[Note: The map

$$\prod_{v \in V(\Lambda)} \mathrm{Hom}\Big(\bigotimes_{e \in T(v)} \pi_e, \bigotimes_{e \in S(v)} \pi_e\Big) \longrightarrow L^2(\bar{\mathfrak{A}}_\Lambda / \bar{\mathcal{G}}_\Lambda)$$

that sends $\{I_v\}$ to $\Psi_{\Lambda;\underline{\pi},\underline{I}}$ is multilinear, hence gives rise to a map

$$\bigotimes_{v \in V(\Lambda)} \mathrm{Hom}\Big(\bigotimes_{e \in T(v)} \pi_e, \bigotimes_{e \in S(v)} \pi_e\Big) \longrightarrow L^2(\bar{\mathfrak{A}}_\Lambda / \bar{\mathcal{G}}_\Lambda)$$

that sends $\otimes I_v$ to $\Psi_{\Lambda;\underline{\pi},\underline{I}}$.]

Remark: Choose an orthonormal basis $\{I_v(b) : b \in B_v\}$ for

$$\mathrm{Hom}\Big(\bigotimes_{e \in T(v)} \pi_e, \bigotimes_{e \in S(v)} \pi_e\Big)$$

and let $\underline{I} = \{I_v(b) : v \in V(\Lambda)\}$ run through all possible combinations thereof -- then the spin network states

$$\Psi_{\Lambda;\underline{\pi},\underline{I}} \quad (\underline{\pi} \in \Pi(\Lambda))$$

constitute an orthonormal basis for $L^2(\bar{\mathfrak{A}}_\Lambda / \bar{\mathcal{G}}_\Lambda)$.

Suppose that $\Lambda \leq \Lambda'$ -- then there is an isometric injection

$$L^2(\bar{\mathfrak{A}}_\Lambda / \bar{\mathcal{G}}_\Lambda) \longrightarrow L^2(\bar{\mathfrak{A}}_{\Lambda'} / \bar{\mathcal{G}}_{\Lambda'})$$

which takes spin network states to spin network states.

<u>LEMMA</u> The spin network states

$$\Psi_{\Lambda;\underline{\pi},\underline{I}} \quad (\Lambda \in \mathrm{Gra}\ M)$$

span $L^2(\bar{\mathfrak{A}} / \bar{\mathcal{G}} ; \mu_0)$.

[Note: Recall that the union

$$\bigcup_{\Lambda} L^2(\overline{\mathfrak{A}}_\Lambda / \overline{\mathfrak{G}}_\Lambda)$$

is dense in $L^2(\overline{\mathfrak{A}} / \overline{\mathfrak{G}} ; \mu_0)$.]

There are certain redundancies in the description of spin network states.

Example: Suppose that Λ' arises from Λ by subdividing an edge of Λ into two edges labeled with the same representation by inserting a vertex to which one has attached the identity intertwining operator -- then, as functions on $\overline{\mathfrak{A}}$,

$$\Psi_{\Lambda ; \underline{\Pi}, \underline{I}} = \Psi_{\Lambda', \underline{\Pi}', \underline{I}'} .$$

[Consider

$$\Lambda : v_1 \xrightarrow{\ e\ } v_2$$

and

$$\Lambda' : v_1 \xrightarrow{\ e_1\ } \underset{v_3}{\bullet} \xrightarrow{\ e_2\ } v_2 .$$

Then $\forall\, h \in \overline{\mathfrak{A}}_\Lambda$,

$$\Psi_{\Lambda ; , \underline{\Pi}, \underline{I}} (h) = [\Pi(h(e))]^i_j \, (I_{v_1})^j \, (I_{v_2})_i$$

and $\forall\, h' \in \overline{\mathfrak{A}}_{\Lambda'}$,

$$\Psi_{\Lambda', \underline{\Pi}', \underline{I}'} (h')$$

$$= [\Pi(h'(e_1))]^{i_1}_{j_1} [\Pi(h'(e_2))]^{i_2}_{j_2} (I_{v_1})^{j_1} (I_{v_2})_{i_2} (I_{v_3})^{j_2}_{i_1}$$

$$= [\Pi(h'(e_1))]^{i_1}_{j_1} [\Pi(h'(e_2))]^{i_2}_{j_2} (I_{v_1})^{j_1} (I_{v_2})_{i_2} \delta^{j_2}_{i_1}$$

$$= [\Pi(h'(e_1))]^{k}_{j_1} [\Pi(h'(e_2))]^{i_2}_{k} (I_{v_1})^{j_1} (I_{v_2})_{i_2} .$$

Let

$$\pi^{\Lambda'}_{\Lambda} : \bar{\mathfrak{A}}_{\Lambda'} \longrightarrow \bar{\mathfrak{A}}_{\Lambda}$$

be the canonical projection and take

$$h = \pi^{\Lambda'}_{\Lambda} (h').$$

Then

$$h(e) = h'(e_2e_1) = h'(e_2)h'(e_1)$$

$\Longrightarrow$

$$\Pi(h(e)) = \Pi(h'(e_2))\Pi(h'(e_1))$$

$\Longrightarrow$

$$[\Pi(h(e))]^{i_2}_{j_1} = [\Pi(h'(e_2))]^{i_2}_{k} [\Pi(h'(e_1))]^{k}_{j_1}$$

$\Longrightarrow$

$$\Psi_{\Lambda';\underline{\Pi}',\underline{I}'}(h') = [\Pi(h(e))]^{i_2}_{j_1} (I_{v_1})^{j_1} (I_{v_2})_{i_2}$$

$$= [\Pi(h(e))]^{i}_{j} (I_{v_1})^{j} (I_{v_2})_{i}$$

$$= \Psi_{\Lambda;\underline{\Pi},\underline{I}}(h).]$$

Another type of redundancy involves π_t.

[Note: Let us agree that the spin network state attached to the empty graph is the function $\equiv 1$ -- then for any $\Lambda \neq \emptyset$, the spin network state $\Psi_{\Lambda;\underline{\Pi},\underline{I}}$, where $\forall e$, $\pi_e = \pi_t$ and $\forall v$, $I_v = \mathrm{id}_{\pi_t}$, is also $\equiv 1$.

Example: Consider

$$\Lambda : v_1 \xrightarrow{e_1} \underset{v_3}{\bullet} \xrightarrow{e_2} v_2.$$

Let $\pi_{e_1} = \pi$, $\pi_{e_2} = \pi_t$, so

$$\begin{cases} I_{v_1} : \pi_t \longrightarrow \pi_{e_1} \\ I_{v_3} : \pi_{e_1} \longrightarrow \pi_{e_2} \quad (= \pi_t) \\ I_{v_2} : \pi_{e_2} \ (= \pi_t) \longrightarrow \pi_t. \end{cases}$$

Then $\forall\, h \in \overline{\mathfrak{A}}_{\Lambda}$,

$$\Psi_{;\underline{\Pi},\underline{I}}(h)$$

$$= [\pi(h(e_1))]^{i_1}_{j_1}\, [\pi_t(h(e_2))]^{i_2}_{j_2}\, (I_{v_1})^{j_1}\, (I_{v_2})_{i_2}\, (I_{v_3})^{j_2}_{i_1}.$$

But $i_2 = 1$, $j_2 = 1$, hence

$$\Psi_{\Lambda;\underline{\Pi},\underline{I}}(h)$$

$$= [\Pi(h(e_1))]^{i_1}_{j_1} (I_{v_1})^{j_1} (I_{v_3})_{i_1} ,$$

where, to normalize the situation, we have taken $I_{v_2} = 1$. Let

$$\Lambda_1 : v_1 \xrightarrow{e_1} v_3$$

and take $\Pi_{e_1} = \Pi$. Suppose now that $h \in \overline{\mathfrak{A}}$ -- then, in view of the commutative triangle,

$$\begin{tikzcd} \overline{\mathfrak{A}} \arrow[r, "\pi_\Lambda"] \arrow[dr, "\pi_{\Lambda_1}"'] & \overline{\mathfrak{A}}_\Lambda \arrow[d, "\pi^{\Lambda}_{\Lambda_1}"] \\ & \overline{\mathfrak{A}}_{\Lambda_1} \end{tikzcd} ,$$

we have

$$\pi_{\Lambda_1}(h)\Big|_{e_1} = \pi^{\Lambda}_{\Lambda_1}(\pi_\Lambda(h))\Big|_{e_1}$$

$$= \pi_\Lambda(h)\Big|_{e_1} .$$

Therefore

$$\Psi_{\Lambda_1;\underline{\Pi},\underline{I}}(\pi_{\Lambda_1}(h))$$

$$= [\Pi(\pi_{\Lambda_1}(h)\Big|_{e_1})]^{i_1}_{j_1} (I_{v_1})^{j_1} (I_{v_3})_{i_1}$$

$$= [\pi(\pi_\Lambda(h)\,|\,e_1)]^{i_1}{}_{j_1}\,(I_{v_1})^{j_1}\,(I_{v_3})_{i_1}$$

$$= \Psi_{\Lambda;\underline{\pi},\underline{I}}\,(\pi_\Lambda(h))$$

$\Longrightarrow$

$$\Psi_{\Lambda_1;\underline{\pi},\underline{I}} \circ \pi_{\Lambda_1} = \Psi_{\Lambda;\underline{\pi},\underline{I}} \circ \pi_\Lambda .$$

Remark: There are two other ways to modify a spin network without changing the state it defines, viz. reparametrization and orientation reversal.

THEOREM There exists a subset $\mathrm{Gra}_0\ M$ of Gra M such that the spin network states

$$\Psi_{\Lambda;\underline{\pi},\underline{I}} \qquad (\Lambda \in \mathrm{Gra}_0\ M)$$

constitute an orthonormal basis for $L^2(\overline{\mathfrak{A}}/\overline{\mathfrak{G}};\mu_0)$.

[Note: An element $\Lambda \in \mathrm{Gra}_0\ M$ is minimal in the sense that it cannot be obtained from another graph Λ' by subdividing edges of Λ' (but to prevent overdetermination, not all minimal graphs are allowed...). Moreover, $\forall\ e \in E(\Lambda)$, $\pi_e \neq \pi_t$ and each $\underline{I} = \{ I_v(b) : v \in V(\Lambda)\}$ is as before. Bear in mind that the empty graph determines the constants.]

Remark: It follows that $L^2(\overline{\mathfrak{A}}/\overline{\mathfrak{G}};\mu_0)$ is not separable.

Let μ be a Radon measure on $\overline{\mathfrak{A}}/\overline{\mathfrak{G}}$. Given an element $\Psi_{\Lambda;\underline{\Pi},\underline{I}}$ of the orthonormal basis for $L^2(\overline{\mathfrak{A}}/\overline{\mathfrak{G}};\mu_0)$ per the theorem, put

$$\langle \Psi_{\Lambda;\underline{\Pi},\underline{I}} \rangle_\mu = \int_{\overline{\mathfrak{A}}/\overline{\mathfrak{G}}} \Psi_{\Lambda;\underline{\Pi},\underline{I}}\, d\mu.$$

LEMMA Assume that the set

$$\{ (\Lambda;\underline{\Pi},\underline{I}) : \langle \Psi_{\Lambda;\underline{\Pi},\underline{I}} \rangle_\mu \neq 0 \}$$

is uncountable -- then $\nexists\, f \in L^1(\overline{\mathfrak{A}}/\overline{\mathfrak{G}};\mu_0)$ such that $d\mu = f d\mu_0$.

[Special Case: There is no square integrable f such that $d\mu = f d\mu_0$. In fact, if this were true, then

$$f = \sum \langle \Psi_{\Lambda;\underline{\Pi},\underline{I}}, f \rangle f$$

$$= \sum \left(\int_{\overline{\mathfrak{A}}/\overline{\mathfrak{G}}} \overline{\Psi}_{\Lambda;\underline{\Pi},\underline{I}}\, f d\mu_0 \right) f$$

$$= \sum \left(\int_{\overline{\mathfrak{A}}/\overline{\mathfrak{G}}} \overline{\Psi}_{\Lambda;\underline{\Pi},\underline{I}}\, d\mu \right) f$$

$$= \sum \overline{\langle \Psi_{\Lambda;\underline{\Pi},\underline{I}} \rangle}_\mu\, f.$$

But the set of nonzero Fourier coefficients of f is at most countable, so we have a contradiction.

General Case: There is no integrable f such that $d\mu = f d\mu_0$.

Supposing the opposite, choose a sequence $f_n \in \mathrm{Cyl}(\overline{\mathfrak{A}}/\overline{\mathfrak{G}})$: $f_n \longrightarrow f$ in $L^1(\overline{\mathfrak{A}}/\overline{\mathfrak{G}};\mu_0)$ (with f_n real valued) -- then

$$\overline{\langle \Psi_{\Lambda;\underline{\Pi},\underline{I}}, f_n \rangle}$$

$$= \int_{\overline{\mathfrak{A}}/\overline{\mathfrak{G}}} \Psi_{\Lambda;\underline{\Pi},\underline{I}} f_n d\mu_0$$

$$\longrightarrow \int_{\overline{\mathfrak{A}}/\overline{\mathfrak{G}}} \Psi_{\Lambda;\underline{\Pi},\underline{I}} f d\mu_0$$

$$= \int_{\overline{\mathfrak{A}}/\overline{\mathfrak{G}}} \Psi_{\Lambda;\underline{\Pi},\underline{I}} d\mu$$

$$= \langle \Psi_{\Lambda;\underline{\Pi},\underline{I}} \rangle_\mu .$$

Since the set

$$\bigcup_{n=1}^{\infty} \{ (\Lambda;\underline{\Pi},\underline{I}) : \langle \Psi_{\Lambda;\underline{\Pi},\underline{I}}, f_n \rangle \neq 0 \}$$

is at most countable, we once again have a contradiction.]

Example: Consider two dimensional Yang-Mills theory (thus $M=\underset{\sim}{R}^2$, $G=\underset{\sim}{SU}(N), N \geq 2$). Take γ simple -- then

$$\int_{\overline{\mathfrak{A}}/\overline{\mathfrak{G}}} \mathrm{tr}(h[\gamma]) d\mu_{YM}(h)$$

$$= e^{-cA(\gamma)},$$

which is nonzero for uncountably many γ. Therefore μ_{YM} cannot be absolutely continuous w.r.t. μ_0.

[Note: The standard representation of $\underset{\sim}{SU}(N)$ on $\underset{\sim}{C}^N$ is irreducible and relative to it, the function $h \longrightarrow tr(h[\gamma])$ is a spin network state.]

The Weyl Algebra Suppose that M is analytic and path connected with dim $M \geq 2$ and G is a compact connected nonabelian Lie group.

Let S be a nonempty subset of M.

Definition: A curve $\gamma : [0,1] \longrightarrow M$ is

$$\begin{cases} \text{S-external if } \operatorname{int}\gamma \cap S = \emptyset \\ \text{S-internal if } \operatorname{int}\gamma \subset S. \end{cases}$$

Let $\gamma : [0,1] \longrightarrow M$ be a curve -- then curves $\gamma_1, \ldots, \gamma_n$ are said to be an S-admissible decomposition of γ if $\gamma = \gamma_n \cdots \gamma_1$ and $\forall i$, γ_i is either S-external or S-internal. An S-admissible decomposition $\gamma = \gamma_n \cdots \gamma_1$ is termed minimal if for any other S-admissible decomposition $\gamma = \gamma'_{n'} \cdots \gamma'_1$ there are indices

$$1 = j_0 < j_1 < j_2 < \cdots < j_{n-1} < j_n = n'$$

such that

$$\gamma_1 = \gamma'_{j_1} \cdots \gamma'_{j_0}$$

$$\gamma_2 = \gamma'_{j_2} \cdots \gamma'_{j_1+1}$$

$$\vdots$$

$$\gamma_n = \gamma'_{j_n} \cdots \gamma'_{j_{n-1}+1}.$$

LEMMA If a curve γ has an S-admissible decomposition, then it has a minimal S-admissible decomposition.

[Let $\gamma = \gamma'_{n'} \cdots \gamma'_1$ be an S-admissible decomposition of γ -- then there is a partition $\bigcup_{j=1}^{n'} I'_j$ of [0,1] into closed subintervals $I'_j = [t_{j-1}, t_j]$ $(t_0 = 0,\ t_{n'} = 1)$ with $\gamma \mid I'_j \longleftrightarrow \gamma'_j$. Cancel from the set $T' = \{t_0, \ldots, t_j, \ldots, t_{n'}\}$ those $t_j \neq 0,1$ such that

$$\mathrm{int}\ \gamma \mid [t_{j-1}, t_{j+1}] \cap S = \emptyset$$

or

$$\mathrm{int}\ \gamma \mid [t_{j-1}, t_{j+1}] \subset S.$$

Let $T = \{t_0, \ldots, t_i, \ldots t_n\}$ be the resulting subset of T', thus $[0,1] = \bigcup_{i=1}^{n} I_i$, where $I_i = [t_{i-1}, t_i]$ $(t_0 = 0,\ t_n = 1)$. So, if $\gamma \mid I_i \longleftrightarrow \gamma_i$, then the decomposition $\gamma = \gamma_n \cdots \gamma_1$ is S-admissible and we claim that it is in fact minimal. To see this, let $\gamma = \delta_m \cdots \delta_1$ be an arbitrary S-admissible decomposition of γ.

Here $\gamma \mid J_k \longleftrightarrow \delta_k$ and $[0,1] = \bigcup_{k=1}^{m} J_k$. Can J_k overlap I_i and I_{i+1}? I.e., does $\exists\ \varepsilon > 0 : [t_i - \varepsilon,\ t_i + \varepsilon] \subset J_k$? This would mean that $\gamma(t_i) \in \mathrm{int}\ \delta_k$ and there are then two possibilities: $\gamma(t_i) \in S$ or $\gamma(t_i) \notin S$. Consider the first:

$$\gamma(t_i) \in S \implies \text{int}\, \delta_k = \text{int}\, \gamma | J_k \subset S$$

$$\implies$$

$$\text{int}\, \gamma | I_i \subset S \ \& \ \text{int}\, \gamma | I_{i+1} \subset S$$

$$\implies$$

$$\text{int}\, \gamma | (I_i \cup I_{i+1})$$

$$= \text{int}\, \gamma | I_i \cup \{\gamma(t_i)\} \cup \text{int}\, \gamma | I_{i+1}$$

$$\subset S$$

$$\implies$$

$$t_i \notin T,$$

a contradiction. Ditto for the second. Therefore the S-admissible decomposition $\gamma = \gamma_n \cdots \gamma_1$ is minimal.]

[Note: A minimal S-admissible decomposition is unique (up to parametrization of its components).]

Definition: S is called a pseudosurface if every curve γ has an S-admissible decomposition.

LEMMA The embedded analytic submanifolds of M are pseudosurfaces.

Example: Let S be the open subset of $M = \mathbf{R}^2$ lying above $y = x \sin(1/x)$ and bounded by $x = 0$, $x = 1$ -- then the straight line

γ between (0,0) and (1,0) leaves and returns to S infinitely often, hence does not have an S-admissible decomposition. Therefore S is not a pseudosurface.

Let $\gamma, \gamma' : [0,1] \longrightarrow M$ be curves -- then γ, γ' have the same initial (final) segment, written $\gamma \uparrow\uparrow \gamma'$ ($\gamma \downarrow\downarrow \gamma'$), if

$\exists\ 0 < \varepsilon < 1 : \gamma \mid [0,\varepsilon] = \gamma' \mid [0,\varepsilon]\ (\gamma \mid [1-\varepsilon,1] = \gamma' \mid [1-\varepsilon,1])$.

Definition: Suppose that S is a pseudosurface. Let σ_S^-, σ_S^+ be $\underset{\sim}{Z}$-valued functions defined on curves.

σ_S^- is called an <u>outgoing intersection function</u> for S if

1. $\gamma(0) \notin S \Rightarrow \sigma_S^-(\gamma) = 0$;
2. $\gamma \uparrow\uparrow \gamma' \Rightarrow \sigma_S^-(\gamma) = \sigma_S^-(\gamma')$.

σ_S^+ is called an <u>incoming intersection function</u> for S if

1. $\gamma(1) \notin S \Rightarrow \sigma_S^+(\gamma) = 0$;
2. $\gamma \downarrow\downarrow \gamma' \Rightarrow \sigma_S^+(\gamma) = \sigma_S^+(\gamma')$.

Let σ_S^- be an outgoing intersection function for S and let σ_S^+ be an incoming intersection function for S -- then the pair

$$\sigma_S = (\sigma_S^-, \sigma_S^+)$$

is said to be an <u>intersection function</u> for S if $\forall\ \gamma$,

$$\sigma_S^-(\gamma) + \sigma_S^+(\gamma) = 0.$$

Example: Let S be an oriented embedded analytic hypersurface in M -- then S carries two natural intersection functions.

<u>Type I</u>: Put $\sigma_S^-(\gamma) = 0$ if $\gamma(0) \notin S$ or $\dot{\gamma}(0)$ is tangent to S and put $\sigma_S^-(\gamma) = 1$ (-1) if $\gamma(0) \in S$ and $\dot{\gamma}(0)$ is not tangent to S but some initial segment of γ lies above (below) S (except $\gamma(0)$).

[Note: The definition of σ_S^+ is dual.]

<u>Type II</u>: Put $\sigma_S^-(\gamma) = 0$ if $\gamma(0) \notin S$ or some initial segment of γ is contained in S and put $\sigma_S^-(\gamma) = 1$ (-1) if $\gamma(0) \in S$ and no initial segment of γ is contained in S but some initial segment of γ lies above (below) S (except $\gamma(0)$).

[Note: The definition of σ_S^+ is dual.]

In both cases, the terms "above" and "below" refer to the orientation of S. There are, of course, two choices for the orientation and the associated intersection functions differ by a sign.

Rappel: We have

$$\begin{cases} \bar{\mathfrak{A}} \approx \mathrm{Hom}(\mathcal{P}\mathfrak{G}, G) \\ \bar{\mathfrak{G}} \approx \mathrm{Map}(M,G) \end{cases}$$

and there is a right action of $\bar{\mathfrak{G}}$ on $\bar{\mathfrak{A}}$, viz.

$$\begin{cases} \bar{\mathfrak{A}} \times \bar{\mathfrak{G}} \longrightarrow \bar{\mathfrak{A}} \\ h \cdot \phi([\gamma]) = \phi(\gamma(1))^{-1}(h[\gamma])\phi(\gamma(0)). \end{cases}$$

Fix a pseudosurface S and an intersection function $\sigma_S = (\sigma_S^-, \sigma_S^+)$ for S.

Given $h \in \overline{\mathfrak{N}}$, $\phi \in \overline{\mathfrak{Y}}$, define a G-valued function $K_{h,\phi}$ on curves as follows. If γ is S-external, put

$$K_{h,\phi}(\gamma) = \phi(\gamma(1))^{\sigma_S^-(\gamma)} h(\gamma)\, \phi(\gamma(0))^{\sigma_S^+(\gamma)}$$

and if γ is S-internal, put

$$K_{h,\phi}(\gamma) = h(\gamma).$$

If γ is arbitrary, let $\gamma = \gamma_n \cdots \gamma_1$ be its minimal S-admissible decomposition and put

$$K_{h,\phi}(\gamma) = K_{h,\phi}(\gamma_n) \cdots K_{h,\phi}(\gamma_1).$$

Example: $\forall\, \gamma$, we have

$$K_{h,\phi}(\gamma^{-1}) = K_{h,\phi}(\gamma)^{-1}.$$

[Take γ S-external -- then

$$K_{h,\phi}(\gamma^{-1}) = \phi(\gamma^{-1}(1))^{\sigma_S^-(\gamma^{-1})} h(\gamma^{-1})\, \phi(\gamma^{-1}(0))^{\sigma_S^+(\gamma^{-1})}$$

$$= \phi(\gamma(0))^{-\sigma_S^+(\gamma)} h(\gamma)^{-1}\, \phi(\gamma(1))^{-\sigma_S^-(\gamma)}$$

$$= K_{h,\phi}(\gamma)^{-1}.]$$

LEMMA $K_{h,\phi}$ passes to the quotient and defines a map from

$\mathcal{P}\mathcal{Y}$ to G. As such, $K_{h,\phi}$ is a functor, i.e., $K_{h,\phi} \in \overline{\mathfrak{A}}$.

Fix $\phi \in \overline{\mathcal{Y}}$ and put

$$K_{\phi}(h) = K_{h,\phi} .$$

Then

$$K_{\phi} : \overline{\mathfrak{A}} \longrightarrow \overline{\mathfrak{A}}$$

is a homeomorphism.

LEMMA μ_{AL} is K_{ϕ}-invariant.

[Let $\Lambda \in \text{Gra } M$ -- then $\exists$ a graph $\Lambda' \geq \Lambda$ such that every edge e' of Λ' is S-external or S-internal. This said, define

$$K_{e'}:G \longrightarrow G$$

by

$$K_{e'}(\sigma) = \phi(e'(1))^{\sigma_S^-(e')} \sigma \phi(e'(0))^{\sigma_S^+(e')}$$

if e' is S-external and

$$K_{e'}(\sigma) = \sigma$$

if e' is S-internal. Now enumerate the edges of Λ' : $e_1',\ldots,e_{n'}'$, -- then $\forall\ h \in \overline{\mathfrak{A}}$,

$$\pi_{\Lambda'}(K_{\phi}(h)) \in \overline{\mathfrak{A}}_{\Lambda'} \longleftrightarrow G^{n'} \quad (n' = \#E(\Lambda'))$$

and indeed

$$\pi_{\Lambda'}(K_{\phi}(h)) = (K_{e_1'} \times \cdots \times K_{e'_{n'}})\ \pi_{\Lambda'}(h).$$

Therefore

$$(\pi_\Lambda)_* (K_\phi)_* \mu_{AL}$$

$$= (\pi^{\Lambda'}_{\Lambda})_* (\pi_{\Lambda'} \circ K_\phi)_* \mu_{AL}$$

$$= (\pi^{\Lambda'}_{\Lambda})_* (K_{e'_1} \times \cdots \times K_{e'_{n'}})_* (\pi_{\Lambda'})_* \mu_{AL}$$

$$= (\pi^{\Lambda'}_{\Lambda})_* (K_{e'_1} \times \cdots \times K_{e'_{n'}})_* \mu_{\Lambda'}$$

$$= (\pi^{\Lambda'}_{\Lambda})_* \mu_{\Lambda'}$$

$$= (\pi^{\Lambda'}_{\Lambda})_* (\pi_{\Lambda'})_* \mu_{AL}$$

$$= (\pi_\Lambda)_* \mu_{AL}$$

$$= \mu_\Lambda$$

$\Longrightarrow$

$$(K_\phi)_* \mu_{AL} = \mu_{AL}.]$$

Definition: The <u>Weyl operator</u> attached to $\phi \in \overline{\mathcal{G}}$ is the unitary operator

$$\begin{cases} W_\phi : L^2(\overline{\mathfrak{A}}; \mu_{AL}) \longrightarrow L^2(\overline{\mathfrak{A}}; \mu_{AL}) \\ \qquad\qquad W_\phi f = f \circ K_\phi \quad . \end{cases}$$

[Note: Since μ_{AL} is K_ϕ-invariant, we have

$$\int_{\overline{\mathfrak{A}}} | W_\phi f |^2 \, d\mu_{AL} = \int_{\overline{\mathfrak{A}}} | f \circ K_\phi |^2 \, d\mu_{AL}$$

$$= \int_{\overline{\mathfrak{A}}} | f |^2 \, d(K_\phi)_* \mu_{AL}$$

$$= \int_{\overline{\mathfrak{A}}} | f |^2 \, d\mu_{AL}.]$$

A Weyl operator depends on S and σ_S:

$$W_\phi = W_\phi^{S,\sigma_S} .$$

Therefore

$$(W_\phi^{S,\sigma_S})^* = (W_\phi^{S,\sigma_S})^{-1}$$

$$= W_{\phi^{-1}}^{S,\sigma_S}$$

$$= W_\phi^{S,-\sigma_S} .$$

LEMMA Let S_1 and S_2 be disjoint pseudosurfaces with respective intersection functions σ_{S_1} and σ_{S_2} -- then $\forall\ \phi_1, \phi_2 \in \overline{\mathcal{G}}$, we have

$$W_{\phi_1}^{S_1,\sigma_{S_1}} \circ W_{\phi_2}^{S_2,\sigma_{S_2}} = W_{\phi_2}^{S_2,\sigma_{S_2}} \circ W_{\phi_1}^{S_1,\sigma_{S_1}} .$$

Let $\zeta \in \text{Map}(M,\underline{g})$ and define

$$E_{\zeta} : \underset{\sim}{R} \longrightarrow \bar{\mathcal{G}} \quad (= \text{Map}(M,G))$$

by

$$E_{\zeta}(t)\Big|_x = \exp(t\,\zeta(x)) \qquad (x \in M).$$

<u>FACT</u> The map

$$t \longrightarrow W_{E_{\zeta}(t)}^{S,\sigma_S}$$

is a one parameter unitary group.

[Note: In particular, this entails continuity in the strong operator topology.]

By <u>structure data</u> for the theory we shall understand a nonempty subset $\mathcal{S}$ of the set of pseudosurfaces in M plus:

- $\forall S \in \mathcal{S}$, a nonempty subset $\Sigma(S)$ of the set of intersection functions for S;

- $\forall S \in \mathcal{S}$, a nonempty subset $\Phi(S)$ of the set of functions from M to G.

Per some choice of structure data, put

$$W = \bigcup_{S \in \mathcal{S}} \quad \bigcup_{\sigma_S \in \Sigma(S)} \quad \bigcup_{\phi \in \Phi(S)} \{W_{\phi}^{S,\sigma_S}\}.$$

Then the <u>Weyl algebra (of quantum geometry)</u> is the C*-subalgebra $\mathcal{W}$ of $\mathcal{B}(L^2(\bar{\mathcal{A}};\mu_{AL}))$ generated by $C(\bar{\mathcal{A}})$ and W.

[Note: Here, the elements of $C(\bar{\mathcal{A}})$ are to be regarded as multiplication operators on $L^2(\bar{\mathcal{A}};\mu_{AL})$. Accordingly, $\mathcal{W}$ admits

<u>Irreducibility</u> By its very construction, $\mathfrak{W}$ depends on the choice of structure data and the problem now is to find conditions on the structure data which serve to ensure that $\mathfrak{W}$ operates irreducibly on $L^2(\overline{\mathfrak{A}};\mu_{AL})$.

[Note: Since $\mathfrak{W} \supset C(\overline{\mathfrak{A}})$, the constant function 1 is cyclic.]

To this end, we shall impose the following assumptions:

<u>$\mathcal{S}$</u> : $\mathcal{S}$ consists of the oriented embedded analytic hypersurfaces in M.

• $\forall\ S \in \mathcal{S}$, $\Sigma(S)$ is the Type I intersection function carried by S;

• $\forall\ S \in \mathcal{S}$, $\Phi(S)$ is the set of G-valued constant functions on M.

<u>THEOREM</u> $\mathfrak{W}$ operates irreducibly on $L^2(\overline{\mathfrak{A}};\mu_{AL})$.

The proof requires some preparation.

Definition: Let $S \in \mathcal{S}$, let Λ be a graph, and let γ be an edge -- then a point $x \in S$ is called a <u>puncture</u> of S and (Λ, γ) if $S \cap \Lambda = \emptyset$ and $S \cap \operatorname{int} \gamma = \{x\}$, where $\dot{\gamma}(x)$ is not tangent to S and

$$\begin{cases} \sigma_S^-(\gamma \mid [t,1]) = 1 \\ \sigma_S^+(\gamma \mid [0,t]) = 1. \end{cases} \qquad (x = \gamma(t),\ 0 < t < 1)$$

[Note: The empty graph is allowed.]

<u>FACT</u> Let γ be an edge and Λ a graph. Assume: γ and the edges

of Λ intersect at most at their endpoints -- then $\forall\ x \in \operatorname{int} \gamma$, $\exists\ S \in \mathcal{S}$ such that $x \in S$ is a puncture of S and (Λ, γ).

Notation: Given $S \in \mathcal{S}$, write W_{σ}^{S} in place of $W_{\Phi_{\sigma}}^{S,\sigma_S}$, where $\Phi_{\sigma}(M) = \{\sigma\}$ $(\sigma \in G)$, and given $\pi \in \Pi$, denote by χ_{π} its character.

LEMMA Suppose that γ and the edges of Λ intersect at most at their endpoints. Put

$$T = T_{\gamma;\pi,m,n} \cdot T_{\Lambda;\underline{\Pi},\underline{i},\underline{j}} .$$

Let S_1, S_2 be elements of $\mathcal{S}$ such that the punctures x_1, x_2 are distinct -- then

$$\langle W_{\sigma_1}^{S_1}(T), W_{\sigma_2}^{S_2}(T) \rangle = \frac{\overline{\chi_{\pi}(\sigma_1^2)} \cdot \chi_{\pi}(\sigma_2^2)}{d_{\pi}^2} .$$

[Determine points $t_1, t_2 \in]0,1[$ $(t_1 \neq t_2)$:

$$\begin{cases} x_1 = \gamma(t_1) \\ x_2 = \gamma(t_2) . \end{cases}$$

Take $t_1 < t_2$ and let

$$\begin{cases} \gamma_2 \longleftrightarrow \gamma \mid [0, t_1] \\ \gamma_0 \longleftrightarrow \gamma \mid [t_1, t_2] \\ \gamma_1 \longleftrightarrow \gamma \mid [t_2, 1]. \end{cases}$$

Then

$$ {}^{T}\gamma_{;\pi,m,n} $$

$$ = \frac{1}{d_\pi} \sum_{p,q} {}^{T}\gamma_{1;\pi,m,p} \cdot {}^{T}\gamma_{0;\pi,p,q} \cdot {}^{T}\gamma_{2;\pi,q,n} . $$

So, from the definitions,

$$ W_{\sigma_1}^{S_1} ({}^{T}\gamma_{;\pi,m,n}) $$

$$ = \frac{1}{d_\pi} \sum_{k_1,\ell_1} \sum_{p,q} {}^{T}\gamma_{1;\pi,m,k_1} \pi(\sigma_1)_{k_1 p} \pi(\sigma_1)_{p \ell_1} {}^{T}\gamma_{0;\pi,\ell_1,q} \cdot {}^{T}\gamma_{2;\pi,q,n} $$

$$ = \frac{1}{d_\pi} \sum_{k_1,\ell_1} \pi(\sigma_1^2)_{k_1 \ell_1} \sum_{q} {}^{T}\gamma_{1;\pi,m,k_1} \cdot {}^{T}\gamma_{0;\pi,\ell_1,q} \cdot {}^{T}\gamma_{2;\pi,q,n} $$

and

$$ W_{\sigma_2}^{S_2} ({}^{T}\gamma_{;\pi,m,n}) $$

$$ = \frac{1}{d_\pi} \sum_{k_2,\ell_2} \sum_{p,q} {}^{T}\gamma_{1;\pi,m,p} \cdot {}^{T}\gamma_{0;\pi,p,k_2} \pi(\sigma_2)_{k_2 q} \pi(\sigma_2)_{q \ell_2} {}^{T}\gamma_{2;\pi,\ell_2,n} $$

$$ = \frac{1}{d_\pi} \sum_{k_2,\ell_2} \pi(\sigma_2^2)_{k_2 \ell_2} \sum_{p} {}^{T}\gamma_{1;\pi,m,p} \cdot {}^{T}\gamma_{0;\pi,p,k_2} \cdot {}^{T}\gamma_{2;\pi,\ell_2,n} . $$

Since

$$ \begin{cases} W_1^{S_1}(T) = W_{\sigma_1}^{S_1} ({}^{T}\gamma_{;\pi,m,n}) \; {}^{T}\Lambda_{;\underline{\pi},\underline{i},\underline{j}} \\[2ex] W_{\sigma_2}^{S_2}(T) = W_{\sigma_2}^{S_2} ({}^{T}\gamma_{;\pi,m,n}) \; {}^{T}\Lambda_{;\underline{\pi},\underline{i},\underline{j}} , \end{cases} $$

it follows that

$$\left\langle W^{S_1}_{\sigma_1}(T),\ W^{S_2}_{\sigma_2}(T)\right\rangle$$

$$= \left\langle W^{S_1}_{\sigma_1}(T_{\gamma;\pi,m,n}),\ W^{S_2}_{\sigma_2}(T_{\gamma;\pi,m,n})\right\rangle \cdot \left\langle T_{\Lambda;\underline{\pi},\underline{i},\underline{j}},\ T_{\Lambda;\underline{\pi},\underline{i},\underline{j}}\right\rangle$$

$$= \frac{1}{d^2_\pi} \sum_{k_1,\ell_1} \sum_{k_2,\ell_2} \sum_p \sum_q$$

$$\overline{\pi(\sigma^2_1)}_{k_1\ell_1} \times \pi(\sigma^2_2)_{k_2\ell_2}$$

$$\times \left\langle T_{\gamma_1;\pi,m,k_1},\ T_{\gamma_1;\pi,m,p}\right\rangle$$

$$\times \left\langle T_{\gamma_0;\pi,\ell_1,q},\ T_{\gamma_0;\pi,p,k_2}\right\rangle$$

$$\times \left\langle T_{\gamma_2;\pi,q,n},\ T_{\gamma_2;\pi,\ell_2,n}\right\rangle$$

$$= \frac{1}{d^2_\pi} \sum_{k_1,\ell_1} \sum_{k_2,\ell_2} \sum_p \sum_q$$

$$\overline{\pi(\sigma^2_1)}_{k_1\ell_1} \times \pi(\sigma^2_2)_{k_2\ell_2}$$

$$\times\ \delta_{k_1 p}\ \delta_{\ell_1 p}\ \delta_{q k_2}\ \delta_{q \ell_2}$$

$$= \frac{1}{d^2_\pi} \sum_{k_1,\ell_1} \sum_{k_2,\ell_2} \delta_{k_1\ell_1}\ \delta_{k_2\ell_2}\ \overline{\pi(\sigma^2_1)}_{k_1\ell_1}\ \pi(\sigma^2_2)_{k_2\ell_2}$$

$$= \frac{\overline{\mathrm{tr}(\Pi(\sigma_1^2))}\cdot \mathrm{tr}(\Pi(\sigma_2^2))}{d_\Pi^2}$$

$$= \frac{\overline{\chi_\Pi(\sigma_1^2)}\cdot \chi_\Pi(\sigma_2^2)}{d_\Pi^2}\ .]$$

Remark: If Π is abelian (hence χ_Π is multiplicative and $d_\Pi = 1$), then

$$W_\sigma^S(T) = \chi_\Pi(\sigma^2)T \quad (S \in \mathcal{S}).$$

To prove that $\mathcal{W}$ is irreducible, it suffices to prove that $\mathcal{W}'$ consists of scalars only. On general grounds,

$$C(\overline{\mathfrak{A}}) \subset \mathcal{W} \Rightarrow \mathcal{W}' \subset C(\overline{\mathfrak{A}})' = L^\infty(\overline{\mathfrak{A}};\mu_{AL}).$$

Let $f \in \mathcal{W}'$ -- then $\forall\, w \in \langle W \rangle$,

$$f \circ w = w \circ f.$$

But

$$w \circ f = w(f) \circ w.$$

Therefore

$$w(f) = f$$

in $L^\infty(\overline{\mathfrak{A}};\mu_{AL})$.

Consider now a nonconstant edge network T -- then we claim that

$$\langle T, f \rangle = 0.$$

Because T is arbitrary, this implies that $f \in \underset{\sim}{\mathbb{C}}1$, as desired.

Bearing in mind that $w \in \langle W \rangle \Rightarrow w^* \in \langle W \rangle$, we have

$$\langle T, f \rangle = \langle T, w^*(f) \rangle$$
$$= \langle w(T), f \rangle .$$

Therefore

$$\langle w_1(T), f \rangle = \langle T, f \rangle = \langle w_2(T), f \rangle$$

for all $w_1, w_2 \in \langle W \rangle$.

Write

$$T = T_{\gamma ; \pi, m, n} \cdot T_{\Lambda ; \underline{\pi}, \underline{i}, \underline{j}} .$$

Here γ and the edges of Λ intersect at most at their endpoints and $\pi \neq \pi_t$ (however, $T_{\Lambda ; \underline{\pi}, \underline{i}, \underline{j}}$ might be trivial).

Case 1: π is abelian -- then

$$W_\sigma^S (T) = \chi_\pi(\sigma^2) T$$
$$\Longrightarrow$$
$$\langle T, f \rangle = \langle W_\sigma^S (T), f \rangle$$
$$= \overline{\chi_\pi(\sigma^2)} \langle T, f \rangle .$$

Since $\pi \neq \pi_t$, $\exists \ \sigma \in G$: $\chi_\pi(\sigma^2) \neq 1$, thus $\langle T, f \rangle = 0$.

Case 2: π is nonabelian -- then $\exists \ \tau \in G$: $\chi_\pi(\tau) = 0$ (due, in essence, to the Weyl character formula). But $\exists \ \sigma \in G$: $\sigma^2 = \tau$

$$\Longrightarrow$$
$$\chi_\pi(\sigma^2) = 0.$$

So, in view of the lemma,

$$\left\langle W_{\sigma}^{S_1}(T),\ W_{\sigma}^{S_2}(T)\right\rangle = \frac{\overline{\chi_{\pi}(\sigma^2)}\ \chi_{\pi}(\sigma^2)}{d_{\pi}^2}$$

$$= 0.$$

Choose an infinite subset $\mathcal{S}_{(\Lambda,\gamma)} \subset \mathcal{S}$:

$$S_1, S_2 \in \mathcal{S}_{(\Lambda,\gamma)} \Longrightarrow x_1 \neq x_2.$$

Then the collection

$$\left\{ W_{\sigma}^{S}(T)\ :\ S \in \mathcal{S}_{(\Lambda,\gamma)} \right\}$$

is an infinite orthonormal set in $L^2(\overline{\mathfrak{O}\!\mathfrak{l}}\,;\mu_{AL})$. Call P the orthogonal projection onto its span:

$$Pf = \sum_{S \in \mathcal{S}_{(\Lambda,\gamma)}} \left\langle W_{\sigma}^{S}(T),\ Pf \right\rangle W_{\sigma}^{S}(T)$$

$$= \sum_{S \in \mathcal{S}_{(\Lambda,\gamma)}} \left\langle W_{\sigma}^{S}(T),\ f \right\rangle W_{\sigma}^{S}(T).$$

By the above, all the Fourier coefficients are equal. Since

$$\sum_{S \in \mathcal{S}_{(\Lambda,\gamma)}} \left| \left\langle W_{\sigma}^{S}(T),\ f \right\rangle \right|^2 < \infty,$$

the conclusion is that $\forall S \in \mathcal{S}_{(\Lambda,\gamma)}$,

$$\left\langle W_{\sigma}^{S}(T),\ f \right\rangle = 0.$$

Therefore

$$\left\langle T, f \right\rangle = 0.$$