QUANTUM FIELD THEORY SEMINAR

(School of Haag-Kastler et al.)

Garth Warner

Department of Mathematics

University of Washington

QUANTUM FIELD THEORY SEMINAR

TOPICS

1. Causality
2. The Axioms of Local Quantum Physics
3. CCR
4. Example: The Free Particle
5. CAR
6. Example: The Dirac Field
7. States
8. W*-Algebras
9. Particle Theories
10. Edge of the Wedge
11. Reeh-Schlieder
12. The Intersection Property
13. Wightman's Inequality
14. A Theorem of Borchers
15. The Schlieder Property
16. The Borchers Property
17. Simplicity of the Quasilocal Algebra
18. The Totality Lemma
19. Uniqueness of the Translation Representation
20. Inner Symmetries
21. Tools from Harmonic Analysis
22. Projections and Classification
23. Type III
24. The Split Property
25. The Modular Theory

Causality Let $M=\mathbb{R}^{1,d}$ $(d \geq 1)$ be the Minkowski space-time of dimension d+1.

Definition: Let $S \subset M$ -- then the causal complement $S^{\perp}$ of S is the set of points in M which lie spacelike to all points of S.

[Note: Subsets $S, T \subset M$ are said to be causally disjoint, written $S \perp T$, if $S \subset T^{\perp}$.]

Properties:

(1) $S \subset S^{\perp\perp}$;

(2) $S \cap S^{\perp} = \emptyset$;

(3) $S^{\perp\perp\perp} = S^{\perp}$;

(4) $S \subset T \Rightarrow T^{\perp} \subset S^{\perp}$;

(5) $(\bigcup_i S_i)^{\perp} = \bigcap_i S_i^{\perp}$.

Definition: The set $S^{\perp\perp}$ is called the causal closure of S. If $S=S^{\perp\perp}$, then S is said to be causally closed.

LEMMA The set $\mathcal{L}_M$ of causally closed subsets of M is a lattice.

[The operations are

$$\begin{cases} S \wedge T = S \cap T \\ S \vee T = (S \cup T)^{\perp\perp} \end{cases}.]$$

Let $f_M : M \times M \to \mathbb{R}$ be the function defined by the formula

$$f_M(x,y) = \langle x-y, x-y \rangle .$$

Put

$$C_x = \{ y \in M : f_M(x,y) < 0 \} .$$

Then

$$\{x\}^{\perp} = C_x .$$

LEMMA Let S be a nonempty subset of M -- then

$$S^{\perp} = \bigcap_{x \in S} C_x.$$

Let $x,y \in M$ -- then the double cone $D_{x,y}$ generated by x,y is

$$D_{x,y} = (\{x\} \cup \{y\})^{\perp\perp} \quad (= (C_x \cap C_y)^{\perp}).$$

[Note: Therefore double cones are the causal closures of two-point sets.]

Notation: Given $x \in M$, let

$$\begin{cases} V_+(x) = \{y \in M : f_M(x,y) > 0,\ y_0 - x_0 > 0\} \\ V_-(x) = \{y \in M : f_M(x,y) > 0,\ y_0 - x_0 < 0\}. \end{cases}$$

LEMMA Suppose that $y \in V_+(x)$ -- then the interior of $D_{x,y}$ is

$$V_+(x) \cap V_-(y).$$

It is also necessary to relate causality to the topology on M.

LEMMA Let $K \subset M$ be compact -- then $K^{\perp} \neq \emptyset$ is open.

[The point is that the function $y \rightarrow \sup_{x \in K} f_M(x,y)$ is continuous and takes on negative values.]

Application: Let O be a bounded open subset of M -- then $\text{int}\ O^{\perp} \neq \emptyset$.

[For $O \subset \bar{O} \Rightarrow \bar{O}^{\perp} \subset O^{\perp}$ and by the lemma, $\bar{O}^{\perp}$ is open and nonempty.]

[Note: It is also true that $O^{\perp}$ is closed, hence $O^{\perp} \supset \overline{\underline{\text{int}}\ O^{\perp}}$.]

A bounded open O is said to be connected with $O^{\perp}$ if

$$\bar{O} \cap \overline{(\bar{O})^{\perp}} \neq \emptyset.$$

[Note: Since

$$\overline{(\bar{O})^{\perp}} = \underline{\text{fr}}\ (\bar{O})^{\perp} \cup \underline{\text{int}}\ (\bar{O})^{\perp}$$

and $\bar{O} \cap (\bar{O})^{\perp} = \emptyset$, we have

$$\bar{O} \cap \underline{\text{fr}}\ (\bar{O})^{\perp} \neq \emptyset,$$

thus $\exists\, x_0 \in \bar{O}$ and $\{x_n\} \subset (\bar{O})^{\perp}$ such that $x_n \to x_0$. But

$$\underline{\text{fr}}\ \bar{O} = \bar{\bar{O}} - \underline{\text{int}}\ \bar{O} = \bar{O} - \underline{\text{int}}\ \bar{O}.$$

Therefore $x_0 \notin \underline{\text{int}}\ \bar{O}$ (otherwise $x_n \in \bar{O}$ $(n \gg 0) \Rightarrow \bar{O} \cap (\bar{O})^{\perp} \neq \emptyset$), which means that

$$\underline{\text{fr}}\ \bar{O} \cap \underline{\text{fr}} (\bar{O})^{\perp} \neq \emptyset.]$$

In practice, it is often convenient to replace the collection $\mathcal{O} = \{O\}$ of all bounded open subsets of M by a smaller subcollection $\mathcal{O}_0 = \{O_0\}$.

Definition: A final subcollection $\mathcal{O}_0 \subset \mathcal{O}$ which is a basis for the topology on M is said to be causal if

(i) $\forall O_0,\ O_0 = O_0^{\perp\perp}$;

(ii) $\forall O_0,\ O_0 = \underline{\text{int}}\ \bar{O}_0$;

(iii) $\forall O_0,\ O_0^{\perp} = \overline{\underline{\text{int}}\ O_0^{\perp}}$;

(iv) $\forall O_0,\ \bar{O}_0 \cap \overline{(\bar{O}_0)^{\perp}} \neq \emptyset.$

LEMMA The collection

$$V_+(x) \cap V_-(y) \quad (x, y \in M; y \in V_+(x))$$

is a causal subcollection of $\mathcal{O}$.

Denote by $\mathcal{D}$ the collection figuring in the lemma -- then the elements of $\mathcal{D}$ are called <u>double cones</u> and are denoted generically by $D(x,y)$ (so $D(x,y) = \underline{\text{int}}\, D_{x,y}$).

[Note: $\mathcal{D}$ is invariant under the operations of $\mathcal{P}_+^\uparrow$.]

Now put

$$W_R = \{ x \in M : |x_0| < x_1 \}$$

and let

$$\mathcal{W} = \{ (\Lambda,a)\cdot W_R : (\Lambda,a) \in \mathcal{P}_+^\uparrow \},$$

the <u>wedges</u> in M.

Fact: $\forall\, W \in \mathcal{W},\ (\overline{W})^\perp = \underline{\text{int}}\, W^\perp$.

<u>LEMMA</u> $\mathcal{W}$ generates $\mathcal{D}$ by intersection, i.e., $\forall\, D(x,y)$,

$$D(x,y) = \underline{\text{int}} \bigcap W,$$

where $\bigcap$ is taken over all W containing $D(x,y)$.

<u>LEMMA</u> $\mathcal{W}$ separates $\mathcal{D}$, i.e., if $D(x_1,y_1),\ D(x_2,y_2) \in \mathcal{D}$ and $D(x_1,y_1) \perp D(x_2,y_2)$, then $\exists\, W_1, W_2 \in \mathcal{W}$:

$$\begin{cases} D(x_1,y_1) \subset W_1 \\ D(x_2,y_2) \subset W_2 \end{cases} \ \& \ W_1 \perp W_2.$$

<u>The Axioms of Local Quantum Physics</u> Results from the theory of C*-algebras will be recalled as needed.

Definition: Let $\mathfrak{A}$ be a Banach algebra, $*: \mathfrak{A} \to \mathfrak{A}$ an involution -- then the pair $(\mathfrak{A}, *)$ is said to be a <u>C*-algebra</u> if $\forall\ A \in \mathfrak{A}$,

(i) $\|A^*\| = \|A\|$ & (ii) $\|A^*A\| = \|A\|^2$.

[Note: A morphism of C*-algebras is a linear map $\Phi: \mathfrak{A} \to \mathfrak{B}$ such that

$$\Phi(A_1 A_2) = \Phi(A_1)\,\Phi(A_2) \ \ \& \ \ \Phi(A^*) = \Phi(A)^*.$$

Every morphism is automatically continuous: $\|\Phi(A)\| \leq \|A\| \ \forall\ A \in \mathfrak{A}$. Furthermore, the kernel of Φ is a closed ideal in $\mathfrak{A}$ and the image of Φ is a C*-subalgebra of $\mathfrak{B}$. Finally, Φ injective $\Rightarrow$ Φ isometric: $\|\Phi(A)\| = \|A\| \ \forall\ A \in \mathfrak{A}$.]

Example: Let X be a LCH space, $C_\infty(X)$ the algebra of continuous functions on X that vanish at infinity. Equip $C_\infty(X)$ with the sup norm and let the involution * be complex conjugation -- then the pair $(C_\infty(X), *)$ is a commutative C*-algebra.

[Note: If $\mathfrak{A}$ is an arbitrary commutative C*-algebra, then $\exists$ a LCH space X and an isomorphism $\mathfrak{A} \to C_\infty(X)$. Such an X is unique up to homeomorphism and is compact when $\mathfrak{A}$ is unital.]

Example: Let $\mathcal{H}$ be a Hilbert space, $\mathfrak{B}(\mathcal{H})$ the algebra of bounded linear operators on $\mathcal{H}$. Equip $\mathfrak{B}(\mathcal{H})$ with the operator norm and let the involution * be the adjunction -- then the pair $(\mathfrak{B}(\mathcal{H}), *)$ is a C*-algebra.

[Note: A norm closed *-algebra $\mathfrak{A} \subset \mathfrak{B}(\mathcal{H})$ is a C*-algebra. Conversely, every C*-algebra is isomorphic to a norm closed *-algebra in $\mathfrak{B}(\mathcal{H})$ for some $\mathcal{H}$.]

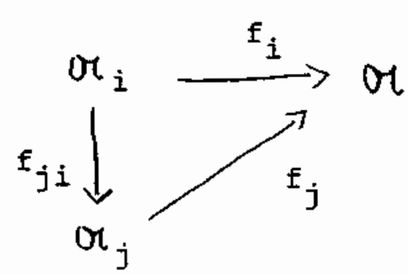
Philosophy: The idea is to produce a mathematical construct which reflects the claim that everything that can be known about a physical system is contained in the assignment between regions of space-time and their observables.

Let $M=\mathbb{R}^{1,d}$ $(d \geq 1)$ be the Minkowski space-time of dimension d+1.

<u>Axiom I</u>: To every bounded open subset $O \subset M$ there is associated a unital C*-algebra $\mathfrak{A}(O)$ such that

$$O_1 \subset O_2 \Rightarrow \mathfrak{A}(O_1) \subset \mathfrak{A}(O_2).$$

Rappel: Let $\{\mathfrak{A}_i : i \in I\}$ be a collection of unital C*-algebras indexed by a directed set I. Assume: $\exists$ injective morphisms $f_{ji}: \mathfrak{A}_i \rightarrow \mathfrak{A}_j$ $(i \leq j)$ with $f_{ki}=f_{kj} \circ f_{ji}$ and $f_{ii}=\underline{id}$ -- then $\exists$ a unital C*-algebra $\mathfrak{A} = C^*(\bigcup_I f_i(\mathfrak{A}_i))$ and injective morphisms $f_i: \mathfrak{A}_i \rightarrow \mathfrak{A}$ such that $\bigcup_I f_i(\mathfrak{A}_i)$ is norm dense in $\mathfrak{A}$ and $\forall i \leq j$, the triangle



commutes.

In the context of Axiom I, the $\mathfrak{A}(O)$ are called the <u>local algebras</u> of the theory, $\bigcup_O \mathfrak{A}(O)$ is called the <u>local algebra</u> of the theory, and $\mathfrak{A} = C^*(\bigcup_O \mathfrak{A}(O))$ is called the <u>quasilocal algebra</u> of the theory.

[Note: The selfadjoint elements of $\mathfrak{A}(O)$ are the observables which can be measured in O.]

Remark: If G is an unbounded open subset of $\mathbb{R}^{1,d}$, then by definition

$$\mathfrak{A}(G) = C^*(\bigcup_{O \subset G} \mathfrak{A}(O)),$$

thus

$$\mathfrak{A}(\underset{\sim}{R}^{1,d}) = \mathfrak{A}.$$

<u>Axiom II</u>: If O_1 and O_2 are spacelike separated (i.e., if $O_1 \perp O_2$), then the elements of $\mathfrak{A}(O_1)$ commute with the elements of $\mathfrak{A}(O_2)$.

Rappel: Let $\mathfrak{A}$ be a C*-algebra, G a topological group -- then a representation of G on $\mathfrak{A}$ is a homomorphism $\alpha : G \to \underline{\text{Aut}}\, \mathfrak{A}$.

<u>Axiom III</u>: There is a representation α of $\tilde{\mathcal{P}}_+^\uparrow$ on $\mathfrak{A}$ such that $\forall\, O \in \mathcal{O}$,

$$\alpha(\tilde{\Lambda}, a) \cdot \mathfrak{A}(O) = \mathfrak{A}((\tilde{\Lambda}, a) \cdot O).$$

[Note: Sometimes this assumption is made only for the translation subgroup of $\tilde{\mathcal{P}}_+^\uparrow$ ("Axiom III_T").]

Rappel: Let $\mathfrak{A}$ be a C*-algebra, G a topological group. Let $\alpha : G \to \underline{\text{Aut}}\, \mathfrak{A}$ be a representation of G on $\mathfrak{A}$ -- then a triple $\{\mathcal{H}, \pi, U\}$, where $\mathcal{H}$ is a Hilbert space and

$$\begin{cases} \pi \text{ is a nondegenerate representation of } \mathfrak{A} \text{ on } \mathcal{H} \\ U \text{ is a unitary representation of } G \text{ on } \mathcal{H}, \end{cases}$$

is said to <u>implement</u> α if

$$U(\sigma)\pi(A)U(\sigma)^{-1} = \pi(\alpha(\sigma)A) \quad (A \in \mathfrak{A},\ \sigma \in G).$$

[Note: Recall that a representation π of $\mathfrak{A}$ on $\mathcal{H}$ is a morphism $\pi : \mathfrak{A} \to \mathcal{B}(\mathcal{H})$ (thus π is automatically continuous: $\|\pi(A)\| \leq \|A\|$ $\forall\, A \in \mathfrak{A}$ (which sharpens to $\|\pi(A)\| = \|A\|$ $\forall\, A \in \mathfrak{A}$ if π is faithful)). There is an orthogonal decomposition

$$\mathcal{H} = \mathcal{H}_{\pi} \oplus \mathcal{H}_{d}$$

into invariant subspaces $\mathcal{H}_{\pi}$ and $\mathcal{H}_{d}$, where $\mathcal{H}_{\pi}$ is the closure of the linear span of the $\pi(A)x$ $(A \in \mathfrak{A}, x \in \mathcal{H})$ and $\mathcal{H}_{d}$ is the set of $x \in \mathcal{H} : \pi(A)x = 0 \ \forall A \in \mathfrak{A}$. One calls π nondegenerate if $\mathcal{H}_{d} = \{0\}$. Every nondegenerate representation is a direct sum of cyclic representations.]

Let U be a unitary representation of $\underset{\sim}{R}^{1,d}$ on $\mathcal{H}$ -- then

$$U(a) = \int_{\underset{\sim}{R}^{1,d}} e^{\sqrt{-1}\,\langle a,p \rangle} \, dE_p,$$

the support of E being the spectrum of U.

Axiom IV: There is a faithful representation π of $\mathfrak{A}$ on $\mathcal{H}$ and a unitary representation U of $\tilde{\mathcal{P}}_{+}^{\uparrow}$ on $\mathcal{H}$ such that the triple $\{\mathcal{H}, \pi, U\}$ implements α, where the spectrum of $U|\underset{\sim}{R}^{1,d}$ is contained in $\overline{V}_{+}$.

[Note: Sometimes this assumption is made only for the translation subgroup ("Axiom IV_T").]

Definition: A theory satisfying Axioms I-IV is called a theory of local observables: $\{\mathfrak{A}, \alpha, \mathcal{H}, \pi, U\}$.

CCR Let $E \neq 0$ be a real linear space equipped with a nondegenerate alternating bilinear form σ (so either $\underline{\dim}\, E = +\infty$ or $\underline{\dim}\, E = 2n$ $(n = 1, 2, \ldots)$).

Example: Take for E a complex pre-Hilbert space, view E as a real linear space via restriction of scalars, and let

$$\sigma(x,y) = \underline{\operatorname{Im}} \langle x, y \rangle .$$

Definition: A CCR realization of (E, σ) is a C*-algebra $\operatorname{CCR}(E, \sigma)$ which is generated by nonzero elements $W(f)$ $(f \in E)$ subject to

$$W(f)^* = W(-f) \quad (f \in E)$$

and

$$W(f)W(g) = \underline{\exp}\left(- \frac{\sqrt{-1}}{2}\, \sigma(f,g) \right) W(f+g) \quad (f, g \in E).$$

Example: Let $\mathcal{H}$ be a Hilbert space. Consider $\mathfrak{F}_s(\mathcal{H})$ -- then one can attach to each $f \in \mathcal{H}$ the Segal field operator

$$\Phi_S(f) = \frac{1}{\sqrt{2}} (\underset{\sim}{a}(f) + \underset{\sim}{c}(f)).$$

As we know, $\Phi_S(f)$ is essentially selfadjoint. This said, put

$$W(f) = \underline{\exp}\, (\sqrt{-1}\, \overline{\Phi_S(f)}),$$

a unitary operator on $\mathfrak{F}_s(\mathcal{H})$. One can show that

$$W(f)^* = W(-f)$$

and

$$W(f)W(g) = \underline{\exp}\left(- \frac{\sqrt{-1}}{2}\, \underline{\operatorname{Im}} \langle f, g \rangle \right) W(f+g).$$

Therefore the C*-subalgebra of $\mathcal{B}(\mathfrak{F}_s(\mathcal{H}))$ generated by the $W(f)$ is a CCR realization of $(\mathcal{H}, \underline{\operatorname{Im}} \langle \, , \rangle)$.

THEOREM OF EXISTENCE The pair (E,σ) always admits a CCR realization.

[Fix an infinite dimensional Hilbert space $\mathcal{L}$. Put $\mathcal{L}_E = \bigoplus_{f\in E} \mathcal{L}_f$

$(\mathcal{L}_f = \mathcal{L} \ \forall f \in E)$ and define $W(f) \in \mathcal{B}(\mathcal{L}_E)$ by the rule

$$(W(f)\Lambda)(x) = \exp\left(\frac{\sqrt{-1}}{2}\sigma(x,f)\right)\Lambda(x+f) \ (x,f\in E;\ \Lambda \in \mathcal{L}_E).]$$

Remark: Let $\mathfrak{A}_1(E,\sigma)$, $\mathfrak{A}_2(E,\sigma)$ be two C*-algebras per the theorem of existence -- then there is one and only one isomorphism $\phi : \mathfrak{A}_1(E,\sigma) \rightarrow \mathfrak{A}_2(E,\sigma)$ such that

$$\phi(W_1(f)) = W_2(f) \quad \forall f \in E.$$

[Note: Uniqueness is, of course, trivial.]

Properties of $CCR(E,\sigma)$:

(1) $W(0) = I$, $W(f)^* = W(f)^{-1}$;

(2) $CCR(E,\sigma)$ is not separable;

(3) $CCR(E,\sigma)$ is simple.

[Note: Another point is this. Let M be a subspace of E -- then the C*-subalgebra of $CCR(E,\sigma)$ generated by $\{W(f) : f \in M\}$ is equal to $CCR(E,\sigma)$ iff $M=E$.]

A linear bijection $T:E \rightarrow E$ is said to be symplectic if

$$\sigma(Tf,Tg) = \sigma(f,g) \quad \forall f,g \in E.$$

LEMMA Given a symplectic map $T:E \rightarrow E$, $\exists$ an automorphism α_T of $CCR(E,\sigma)$ such that

$$\alpha_T(W(f)) = W(Tf) \quad (f \in E).$$

[The W(Tf) satisfy the same general conditions as the W(f) and both generate CCR(E,σ). Now apply the preceding remark.]

[Note: Needless to say, the condition

$$\alpha_T(W(f)) = W(Tf) \quad (f \in E)$$

determines α_T uniquely.]

The α_T are called the <u>Bogolubov automorphisms</u>. They form a subgroup of <u>Aut</u> CCR(E,σ) and the arrow

$$T \longrightarrow \alpha_T$$

is a representation of the symplectic group of (E,σ) on CCR(E,σ).

Rappel: Let $E \neq 0$ be a real linear space -- then a complex structure on E is a linear map $J:E \longrightarrow E$ such that $J^2 = -I$.

[Note: E becomes a complex linear space if we write $\sqrt{-1}\, f = Jf$.]

Example: Let $\mathcal{H}$ be a real Hilbert space, σ a nondegenerate alternating continuous bilinear form on $\mathcal{H}$ -- then $\exists$ a complex structure J on $\mathcal{H}$ such that $J^* = -J = -J^{-1}$ with J symplectic:

$$\sigma(Jx,Jy) = \sigma(x,y).$$

Now view $\mathcal{H}$ as a complex linear space via $-J$ and put

$$\langle x,y \rangle_\sigma = -\sigma(x,Jy) + \sqrt{-1}\ \sigma(x,y).$$

Then the pair $(\mathcal{H}, \langle\ ,\ \rangle_\sigma)$ is a pre-Hilbert space.

[Note: Implicitly, matters have been arranged so as to ensure that $-\sigma(x,Jy)$ is an inner product on $\mathcal{H}$. In addition, it is necessary to work with $-J$ to get the correct signs. E.g.:

$$\langle x, \sqrt{-1}\, y \rangle_\sigma = -\sigma(x, J(-Jy)) + \sqrt{-1}\ \sigma(x, -Jy)$$

$$= -\sigma(x,y) - \sqrt{-1}\ \sigma(x, Jy)$$

$$= \sqrt{-1}\ (\sqrt{-1}\ \sigma(x,y) - \sigma(x,Jy))$$

$$= \sqrt{-1}\,\langle x,y \rangle_\sigma\ .]$$

<u>Example</u> The free relativistic particle of spin zero and mass $m > 0$ admits a theory of local observables. Thus take for E the real linear space $C_c^\infty(\underset{\sim}{R}^{1,3};\underset{\sim}{R})$ and let

$$\sigma(f,g) = \underline{\operatorname{Im}} \langle f,g \rangle ,$$

where

$$\langle f,g \rangle = \int_{X_m} \overline{\hat{f}(p)}\hat{g}(p)d\mu_m(p).$$

But there is a technical problem: Our standing hypothesis of non-degeneracy is not met by σ . To remedy this, put

$$N = \left\{ f \in E:\hat{f}\,|X_m = 0 \right\} .$$

Then the arrow

$$\begin{cases} [f] \longrightarrow \hat{f}\,|X_m \\ E/N \longrightarrow L^2(X_m,\mu_m) \end{cases}$$

is one-to-one and has a dense image. Therefore the prescription

$$\langle [f],[g] \rangle = \int_{X_m} \overline{\hat{f}(p)}\hat{g}(p)d\mu_m(p)$$

equips E/N with the structure of a pre-Hilbert space, so

$$\sigma([f],[g]) = \underline{\operatorname{Im}} \langle [f],[g] \rangle$$

is nondegenerate.

Remark: The elements of N are those elements in E of the form

$$(\square^2 + m^2)f.$$

Definition: The quasilocal algebra of the theory is

$$\mathfrak{A} = CCR(E/N, \sigma) \hookrightarrow CCR(L^2(X_m, \mu_m), \sigma).$$

Notation: Write

$$W(f) \equiv W(\hat{f}|X_m),$$

a unitary operator on $\mathcal{F}_s(L^2(X_m, \mu_m))$ which depends only on the equivalence class of f, hence

$$W([f]) = W(f).$$

We have now to deal with Axioms I-IV.

<u>Ad I</u>: Let

$$\mathfrak{A}(O) = C^*\{W(\hat{f}|X_m) : \underline{spt}\ f \subset O\}.$$

Obviously,

$$O_1 \subset O_2 \Rightarrow \mathfrak{A}(O_1) \subset \mathfrak{A}(O_2).$$

And:

$$\mathfrak{A} = C^*(\bigcup_O \mathfrak{A}(O)).$$

<u>Ad II</u>: Suppose that $O_1 \perp O_2$ -- then

$$\left\{ \begin{array}{l} f_1 \in E : \underline{spt}\ f_1 \subset O_1 \\ \\ f_2 \in E : \underline{spt}\ f_2 \subset O_2 \end{array} \right. \Rightarrow \sigma(f_1, f_2) = 0.$$

To see this, recall first that

$$\Delta_m(x) = \Delta_+(x;m^2) - \overline{\Delta_+(x;m^2)}.$$

On the other hand,

$$\begin{aligned} x \text{ spacelike} &\Rightarrow \Delta_m(x) = 0 \\ &\Rightarrow \underline{\text{Im}}\ \Delta_+(x;m^2) = 0. \end{aligned}$$

Therefore

$$\sigma(f_1,f_2) = \int_{O_1} \int_{O_2} f_1(x_1)\ \underline{\mathrm{Im}}\ \Delta_+(x_1-x_2;m^2) f_2(x_2)dx_1dx_2$$

$$= 0.$$

From this, it follows that

$$W(f_1)W(f_2) = \underline{\exp}\left(-\frac{\sqrt{-1}}{2}\ \sigma(f_1,f_2)\right) W(f_1+f_2)$$

$$= W(f_1+f_2)$$

$$= W(f_2+f_1)$$

$$= \underline{\exp}\left(-\frac{\sqrt{-1}}{2}\ \sigma(f_2,f_1)\right) W(f_2+f_1)$$

$$= W(f_2)W(f_1),$$

which proves that the elements of $\mathfrak{A}(O_1)$ commute with the elements of $\mathfrak{A}(O_2)$.

<u>Ad III</u>: The Poincaré group $\mathcal{P}_+^\uparrow$ operates on E:

$$(\Lambda,a)\cdot f = f_{\Lambda,a}\ ,$$

where

$$f_{\Lambda,a}(x) = f(\Lambda^{-1}(x-a)).$$

Since

$$\hat{f}_{\Lambda,a}(p) = e^{\sqrt{-1}\ \langle a,p\rangle}\ \hat{f}(\Lambda^{-1}p),$$

this action passes to the quotient:

$$(\Lambda,a)\cdot[f] = [f_{\Lambda,a}].$$

Bearing in mind that μ_m is invariant, we have

$$\sigma((\Lambda,a)\cdot[f],\ (\Lambda,a)\cdot[g]) = \sigma([f],\ [g]).$$

So, by Bogolubov, $\exists$ a unique automorphism $\alpha_{(\Lambda,a)}$ of $\mathfrak{A}$ such that

$$\alpha_{(\Lambda,a)}(W(f)) = W(f_{\Lambda,a}).$$

But this implies that

$$\alpha_{(\Lambda,a)}\cdot \mathfrak{A}(O) = \mathfrak{A}((\Lambda,a)\cdot O).$$

Finally, the arrows

$$\begin{cases} \mathcal{P}_+^\uparrow \longrightarrow \underline{\text{Sym}}(E/N,\sigma) \\ \\ \underline{\text{Sym}}(E/N,\sigma) \longrightarrow \underline{\text{Aut}}\ \text{CCR}(E/N,\sigma) \end{cases}$$

are homomorphisms, thus the composite

$$\alpha : \mathcal{P}_+^\uparrow \longrightarrow \underline{\text{Aut}}\ \text{CCR}(E/N,\sigma)$$

is a representation of $\mathcal{P}_+^\uparrow$ on $\mathfrak{A}$.

<u>Ad IV</u>: By construction,

$$\mathfrak{A} = \text{CCR}(E/N,\sigma) \hookrightarrow \text{CCR}(L^2(X_m,\mu_m),\sigma)$$

and

$$\text{CCR}(L^2(X_m,\mu_m),\sigma)$$

is realized as a C*-subalgebra of $\mathcal{B}(\mathcal{F}_s(L^2(X_m,\mu_m)))$. Take any $\phi \in L^2(X_m,\mu_m)$ -- then

$$U^{(m)}(\Lambda,a)W(\phi)U^{(m)}(\Lambda,a)^{-1} = W(U^{(m)}(\Lambda,a)\phi).$$

So here Π is simply the inclusion

$$\mathfrak{A} \rightarrow \mathfrak{B}(\mathcal{F}_s(L^2(X_m,\mu_m)))$$

and the implementation of α is immediate. Indeed, $\forall$ $f \in E$,

$$W(U^{(m)}(\Lambda,a)f) \equiv W(U^{(m)}(\Lambda,a)\hat{f}|X_m)$$

and

$$U^{(m)}(\Lambda,a)\hat{f}|X_m = \hat{f}_{\Lambda,a}|X_m$$

$$\Longrightarrow$$

$$(W(U^{(m)}(\Lambda,a)f) = W(f_{\Lambda,a})$$

$$= \alpha_{(\Lambda,a)}(W(f)).$$

That the spectrum of $U^{(m)}|\underset{\sim}{R}^{1,3}$ is contained in $\overline{V}_+$ has been seen before.

Remark: The foregoing analysis can be extended to cover the case of the free relativistic particle of spin $s > 0$ and mass $m > 0$.

[Note: The massless case can also be incorporated.]

<u>CAR</u> Let E be an infinite dimensional complex pre-Hilbert space equipped with an antiunitary involution $\Gamma : E \to E$, so

$$\Gamma^2 = I, \quad \langle \Gamma x, \Gamma y \rangle = \langle y, x \rangle \quad \forall\, x,y \in E.$$

Definition: A <u>CAR realization</u> of (E,Γ) is a C*-algebra $CAR(E,\Gamma)$ which is generated by nonzero elements $A(f)$ $(f \in E)$ subject to

$$A(f)^* = A(\Gamma f) \quad (f \in E)$$

and

$$A(f)A(g) + A(g)A(f) = \langle \Gamma f, g \rangle I \quad (f,g \in E).$$

[Note: Here it is assumed that the map $A: E \to CAR(E,\Gamma)$ is linear.]

Parallel to the CCR situation, there is a theorem of existence and essential uniqueness. Moreover, if M is a Γ-stable subspace of E, then

$$CAR(M,\Gamma) \hookrightarrow CAR(E,\Gamma).$$

Remark: It is easy to see that

$$\frac{\|f\|}{\sqrt{2}} \leq \|Af\| \leq \|f\| .$$

One can in fact be precise:

$$\|Af\| = \frac{1}{\sqrt{2}}(\|f\|^2 + (\|f\|^4 - |\langle \Gamma f, f \rangle|^2)^{1/2})^{1/2}.$$

Therefore the map

$$A: E \to CAR(E,\Gamma)$$

can be extended by continuity to a map

$$\bar{A}:\bar{E} \to CAR(E,\Gamma).$$

On the other hand, it can be shown that the arrow

$$CAR(E,\Gamma) \to CAR(\bar{E},\bar{\Gamma})$$

is an isomorphism.

Denote now by $\mathcal{U}(E,\Gamma)$ the subgroup of the unitary group of E consisting of those U which commute with Γ.

LEMMA To each $U \in \mathcal{U}(E,\Gamma)$ there corresponds an automorphism α_U of $CAR(E,\Gamma)$ such that

$$\alpha_U(A(f)) = A(Uf) \quad (f \in E).$$

The α_U are called the Bogolubov automorphisms. They form a subgroup of Aut $CAR(E,\Gamma)$ and the arrow

$$U \to \alpha_U$$

is a representation of $\mathcal{U}(E,\Gamma)$ on $CAR(E,\Gamma)$.

Put

$$\tilde{E} = E \oplus E, \quad \tilde{\Gamma} = \begin{pmatrix} 0 & \Gamma \\ \Gamma & 0 \end{pmatrix} .$$

Then $\tilde{\Gamma}:\tilde{E} \to \tilde{E}$ is an antiunitary involution, thus it makes sense to form $CAR(\tilde{E},\tilde{\Gamma})$.

[Note: View the elements of $\tilde{E}$ as column vectors, so

$$\tilde{\Gamma}\,\tilde{f} = \begin{pmatrix} 0 & \Gamma \\ \Gamma & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \Gamma y \\ \Gamma x \end{pmatrix} \quad (x,y \in E).]$$

This said, define a unitary operator $U_\theta : \widetilde{E} \to \widetilde{E}$ $(0 \leq \theta < 2\pi)$ by

$$U_\theta \widetilde{f} = \begin{pmatrix} e^{\sqrt{-1}\theta} x \\ e^{-\sqrt{-1}\theta} y \end{pmatrix} .$$

Claim: $\forall \theta$,

$$U_\theta \widetilde{\Gamma} = \widetilde{\Gamma} U_\theta .$$

In fact,

$$U_\theta \widetilde{\Gamma} \widetilde{f} = \begin{pmatrix} e^{\sqrt{-1}\theta} \Gamma y \\ e^{-\sqrt{-1}\theta} \Gamma x \end{pmatrix}$$

while

$$\widetilde{\Gamma} U_\theta \widetilde{f} = \begin{pmatrix} \Gamma e^{-\sqrt{-1}\theta} y \\ \Gamma e^{\sqrt{-1}\theta} x \end{pmatrix} = \begin{pmatrix} e^{\sqrt{-1}\theta} \Gamma y \\ e^{-\sqrt{-1}\theta} \Gamma x \end{pmatrix} .$$

Accordingly, $\forall \theta$, there corresponds an automorphism α_θ of $\mathrm{CAR}(\widetilde{E}, \widetilde{\Gamma})$ such that

$$\alpha_\theta(A(\widetilde{f})) = A(U_\theta \widetilde{f}) \quad (\widetilde{f} \in \widetilde{E}).$$

Moreover, the assignment

$$\theta \longrightarrow \alpha_\theta$$

defines a representation of $\underset{\sim}{T}$ on $\mathrm{CAR}(\widetilde{E}, \widetilde{\Gamma})$. Since $\hat{\underset{\sim}{T}} = \underset{\sim}{Z}$, there is a decomposition

$$CAR(\widetilde{E},\widetilde{\Gamma}) = \bigoplus_{-\infty}^{+\infty} CAR_n\ (\widetilde{E},\widetilde{\Gamma})$$

into norm closed linear subspaces

$$CAR_n\ (\widetilde{E},\widetilde{\Gamma}) = \left\{ A:\ \alpha_\theta A = e^{\sqrt{-1}\,n\theta} A \right\} .$$

Obviously,

$$CAR_n\ (\widetilde{E},\widetilde{\Gamma})\ CAR_m\ (\widetilde{E},\widetilde{\Gamma}) \subset CAR_{n+m}\ (\widetilde{E},\widetilde{\Gamma})$$

and

$$CAR_n\ (\widetilde{E},\widetilde{\Gamma})^* = CAR_{-n}\ (\widetilde{E},\widetilde{\Gamma}).$$

Therefore $CAR(\widetilde{E},\widetilde{\Gamma})$ is a $\underset{\sim}{Z}$-graded C*-algebra.

Example The Dirac field admits a theory of local observables. Thus take for E the complex linear space $C_c^\infty(\mathbb{R}^{1,3};\mathbb{C}^4)$ with inner product

$$\langle f,g\rangle = \int_{\mathbb{R}^4}\int_{\mathbb{R}^4} \langle f(x), \gamma^0 D_m(x-y)g(y)\rangle \, dxdy$$

$$= \int_{\mathbb{R}^4}\int_{\mathbb{R}^4} \sum_{\mu,\nu} \overline{f_{\mu\nu}(x)}(\gamma^0 D_m(x-y))_{\mu,\nu}\, g_\nu(y)\,dxdy,$$

where

$$D_m(x) = \frac{1}{2m}\left(m + \frac{1}{\sqrt{-1}}\gamma^j \partial_j\right)\Delta_m(x).$$

Take for $\Gamma: E \to E$ the complex conjugation and pass from (E,Γ) to $(\tilde{E},\tilde{\Gamma})$. Here,

$$\tilde{E} = C_c^\infty(\mathbb{R}^{1,3};\mathbb{C}^4 \oplus \mathbb{C}^4).$$

Definition: The field algebra of the theory is

$$\mathfrak{F} = \mathrm{CAR}(\tilde{E},\tilde{\Gamma}).$$

[Note: It is clear that $\mathbb{R}^{1,3}$ operates on $\tilde{E}$:

$$\tilde{f} \to \tilde{f}_a \quad (\tilde{f}_a(x) = \tilde{f}(x-a))$$

from which an automorphism $\alpha_a: \mathfrak{F} \to \mathfrak{F}$ characterized by the relation

$$\alpha_a(A(\tilde{f})) = A(\tilde{f}_a).]$$

Notation: Given $\tilde{f} = \begin{pmatrix} f \\ 0 \end{pmatrix}$, write $\underset{\sim}{a}(f)$ in place of $A(\tilde{f})$, and given $\tilde{g} = \begin{pmatrix} 0 \\ g \end{pmatrix}$, write $\underset{\sim}{c}(g)$ in place of $A(\tilde{g})$.

[Note: Since A is linear, we have

$$\begin{aligned} A(\tilde{f}) = A\begin{pmatrix} \phi \\ \psi \end{pmatrix} &= A\left(\begin{pmatrix} \phi \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \psi \end{pmatrix} \right) \\ &= A\begin{pmatrix} \phi \\ 0 \end{pmatrix} + A\begin{pmatrix} 0 \\ \psi \end{pmatrix} \\ &= \underset{\sim}{a}(\phi) + \underset{\sim}{c}(\psi). \end{aligned}$$

In addition,

$$\begin{aligned} \underset{\sim}{a}(f)^* &= \left(A\begin{pmatrix} f \\ 0 \end{pmatrix} \right)^* \\ &= A\left(\begin{pmatrix} 0 \\ \Gamma f \end{pmatrix} \right) = \underset{\sim}{c}(\bar{f}).] \end{aligned}$$

It is easy to check that

$$\begin{cases} \underset{\sim}{a}(f)\underset{\sim}{a}(g) + \underset{\sim}{a}(g)\underset{\sim}{a}(f) = 0 \\ \underset{\sim}{c}(f)\underset{\sim}{c}(g) + \underset{\sim}{c}(g)\underset{\sim}{c}(f) = 0 \end{cases}$$

and

$$\underset{\sim}{a}(f)\underset{\sim}{c}(g) + \underset{\sim}{c}(g)\underset{\sim}{a}(f) = \langle \Gamma f, g \rangle \, I.$$

Notation: Given a monomial M in the $\underset{\sim}{a}(f)$ and $\underset{\sim}{c}(g)$, let $n(M)$ be the number of $\underset{\sim}{a}(f)$ and $n^*(M)$ the number of $\underset{\sim}{c}(g)$, so $n(M) + n^*(M)$ is the degree of M.

[Note: Monomials are total in $\mathfrak{F}$.]

Let

$$\mathfrak{F}(\mathcal{O}) = C^*\{ A(\tilde{f}) : \underline{\text{spt}}\ \tilde{f} \subset \mathcal{O} \}.$$

Then

$$\mathfrak{F} = C^*\Big(\bigcup_{O} \mathfrak{F}(O)\Big) .$$

LEMMA $\mathfrak{F}(O)$ is a $\underset{\sim}{Z}$-graded C*-algebra:

$$\mathfrak{F}(O) = \bigoplus_{-\infty}^{+\infty} \mathfrak{F}_k(O).$$

And a monomial $M \in \mathfrak{F}(O)$ belongs to $\mathfrak{F}_k(O)$ iff

$$n(M) - n^*(M) = k.$$

[From the definitions, $\alpha_\theta(\underset{\sim}{a}(f)) = e^{\sqrt{-1}\,\theta}\,\underset{\sim}{a}(f)$ and $\alpha_\theta(\underset{\sim}{c}(g)) = e^{-\sqrt{-1}\,\theta}\,\underset{\sim}{c}(g)$. Since α_θ is an automorphism, it follows that

$$\alpha_\theta M = \underline{\exp}(\sqrt{-1}\,(n(M) - n^*(M))\theta)M,$$

from which the assertion.]

LEMMA Suppose that $O_1 \perp O_2$ -- then

$$\begin{cases} A_1 \in \mathfrak{F}_k(O_1) \\ A_2 \in \mathfrak{F}_\ell(O_2) \end{cases} \Rightarrow A_1A_2 = (-1)^{k\ell} A_2A_1.$$

[If

$$\begin{cases} \underline{\text{spt}}\ f_1 \subset O_1 \\ \underline{\text{spt}}\ f_2 \subset O_2 \end{cases},$$

then

$$\underset{\sim}{a}(f_1)\underset{\sim}{c}(f_2) + \underset{\sim}{c}(f_2)\underset{\sim}{a}(f_1)$$

$$= \langle \Gamma f_1, f_2 \rangle I = \langle \bar{f}_1, f_2 \rangle I$$

and

$$\langle \bar{f}_1, f_2 \rangle = \int_{O_1} \int_{O_2} \sum_{\mu,\nu} f_{1\mu}(x)\,(\gamma^0 D_m(x-y))_{\mu,\nu}\, f_{2\nu}(y)\,dx\,dy$$

$$= 0.$$

So, for monomials

$$\begin{cases} M_1 \in \mathfrak{F}_k(O_1) \\ M_2 \in \mathfrak{F}_\ell(O_2), \end{cases}$$

we have

$$M_1 M_2 = (-1)^{(n(M_1) + n^*(M_1))(n(M_2) + n^*(M_2))} M_2 M_1$$

$$= (-1)^{(n(M_1) - n^*(M_1))(n(M_2) - n^*(M_2))} M_2 M_1$$

$$= (-1)^{k\ell} M_2 M_1$$

and the result follows by density.]

Now put

$$\begin{cases} \mathfrak{F}^n(O) = \bigoplus_{-\infty}^{+\infty} \mathfrak{F}_{kn}(O) \qquad (n \neq 0) \\ \mathfrak{F}^0(O) = \mathfrak{F}_0(O) \end{cases}$$

and let

$$\begin{cases} \mathfrak{F}^n = C^*(\bigcup_O \mathfrak{F}^n(O)) \\ \mathfrak{F}^0 = C^*(\bigcup_O \mathfrak{F}_0(O)). \end{cases}$$

Notation: $\underset{\sim}{Z}/n\underset{\sim}{Z}$ is the subgroup of $\underset{\sim}{T}$ generated by $\alpha_{\frac{2\pi}{n}}$ $(n\neq 0)$.

LEMMA An element $A \in \mathfrak{F}$ belongs to $\mathfrak{F}^n$ $(n\neq 0)$ iff A is $\underset{\sim}{Z}/n\underset{\sim}{Z}$-invariant and an element $A \in \mathfrak{F}$ belongs to $\mathfrak{F}^0$ iff A is $\underset{\sim}{T}$-invariant.

[It suffices to consider the case when $n\neq 0$. Trivially, $\forall\, A \in \mathfrak{F}^n$, $\theta \in \underset{\sim}{Z}/n\underset{\sim}{Z} \Rightarrow \alpha_\theta A=A$. On the other hand, if $A \in \mathfrak{F}$ and $\alpha_{\frac{2\pi m}{n}} A=A$, then with $A = \sum_{-\infty}^{+\infty} A_j$, one has

$$\alpha_{\frac{2\pi m}{n}} A = \sum_j \underline{\exp}\left(\sqrt{-1}\, j\, \frac{2\pi m}{n}\right) A_j$$

$$= \sum_{k=0}^{n-1} \underline{\exp}\left(\sqrt{-1}\, \frac{2\pi km}{n}\right) \sum_{\ell=-\infty}^{+\infty} A_{\ell n+k}$$

$$= A$$

$\Rightarrow$

$$\sum_{\ell=-\infty}^{+\infty} A_{\ell n+k} = 0 \quad (k = 1,\ldots,n-1)$$

$\Rightarrow$

$$A = \sum_{-\infty}^{+\infty} A_{\ell n} \in \mathfrak{F}^n.]$$

From this data, for each n=0,1,... one can generate a theory of local observables. Here the local algebras are

$$\mathcal{O} \longrightarrow \mathfrak{F}^{2n}(\mathcal{O}),$$

the associated quasilocal algebra being $\mathfrak{F}^{2n}$. Axioms I, II, and III_T are then immediate. In this connection, observe that if $O_1 \perp O_2$, then

$$\left\{\begin{array}{l} A_1 \in \mathfrak{F}_{2kn}\ (O_1) \\ \\ A_2 \in \mathfrak{F}_{2\ell n}\ (O_2) \end{array}\right. \Rightarrow A_1A_2 = (-1)^{4k\ell n^2} A_2A_1 = A_2A_1 \ .$$

Remark: It is possible to incorporate a representation α of $\widetilde{\mathcal{P}}_+^\uparrow$ on $\mathfrak{F}^{2n}$ so as to strengthen III_T to III. To get a true theory of local observables, it is also necessary to verify IV but I shall omit the details.

States Let $\mathfrak{A}$ be a unital C*-algebra.

Notation:

(1) $\mathfrak{A}_{\underset{\sim}{R}}$ is the collection of all selfadjoint elements in $\mathfrak{A}$, i.e.,

$$\mathfrak{A}_{\underset{\sim}{R}} = \{A \in \mathfrak{A} : A^* = A\}.$$

(2) $\mathfrak{A}^+$ is the collection of all positive elements in $\mathfrak{A}$, i.e.,

$$\begin{aligned}\mathfrak{A}^+ &= \{A^2 : A \in \mathfrak{A}_{\underset{\sim}{R}}\} \\ &= \{A^*A : A \in \mathfrak{A}\}.\end{aligned}$$

Definition: A state on $\mathfrak{A}$ is a linear functional $\omega : \mathfrak{A} \to \underset{\sim}{C}$ such that

$$\begin{cases} \omega(A) \geq 0 & \forall\, A \in \mathfrak{A}^+ \\ \omega(I) = 1. \end{cases}$$

[Note: A state ω is necessarily hermitian: $\omega(A^*) = \overline{\omega(A)}$ $\forall\, A \in \mathfrak{A}$.]

Let $\mathcal{S}(\mathfrak{A})$ be the state space of $\mathfrak{A}$ -- then $\mathcal{S}(\mathfrak{A})$ is a convex set and its elements are necessarily continuous of norm 1, thus $\mathcal{S}(\mathfrak{A})$ is contained in the unit ball of the dual of $\mathfrak{A}$. It is easy to verify that $\mathcal{S}(\mathfrak{A})$ is closed in the weak* topology, so $\mathcal{S}(\mathfrak{A})$ is compact (Alaoglu).

Fact: $\forall\, A \in \mathfrak{A}_{\underset{\sim}{R}}$,

$$\|A\| = \underline{\sup}\,\{|\omega(A)| : \omega \in \mathcal{S}(\mathfrak{A})\}.$$

[Note: The supremum is achieved, i.e., $\exists\, \omega : \|A\| = |\omega(A)|$.]

If π is a cyclic representation of $\mathfrak{A}$ on $\mathcal{H}$, then one can attach to any cyclic unit vector $\Omega \in \mathcal{H}$ a state

$$\omega(A) = \langle \Omega , \pi(A)\Omega \rangle .$$

Conversely, given a state ω, the GNS construction produces a cyclic representation π_ω of $\mathfrak{A}$ on a Hilbert space $\mathcal{H}_\omega$ with cyclic unit vector Ω_ω such that

$$\omega(A) = \langle \Omega_\omega, \pi_\omega(A)\Omega_\omega \rangle .$$

[Note: Suppose that π is a cyclic representation of $\mathfrak{A}$. Take any cyclic unit vector Ω and perform the GNS construction on

$$\omega(A) = \langle \Omega , \pi(A)\Omega \rangle .$$

Then π_ω is unitarily equivalent to π .]

Definition: A state ω is <u>faithful</u> if π_ω is faithful.

[Note: On general grounds, π_ω is faithful iff $A > 0 \Rightarrow \pi_\omega(A) > 0$ or still, $A > 0 \Rightarrow \omega(A) > 0$.]

Example: Recall that a <u>density operator</u> is a bounded linear operator W on $\mathcal{H}$ with the following properties:

(1) W is nonnegative;

(2) W is selfadjoint;

(3) W is trace class and $\underline{\text{tr}}(W)=1$.

This said, take $\mathfrak{A} = \mathcal{B}(\mathcal{H})$ and fix W -- then the state

$$A \longrightarrow \underline{\text{tr}}(AW)$$

is faithful iff W is invertible.

Remark: Let π be a representation of $\mathfrak{A}$ on $\mathcal{H}$ -- then each density operator W determines a state ω_W, viz. $A \longrightarrow \underline{\text{tr}}(\pi(A)W)$. Denote by $\mathcal{S}_\pi(\mathfrak{A})$ the set of such -- then π is faithful iff $\mathcal{S}_\pi(\mathfrak{A})$

is weak* dense in $\mathcal{S}(\pi)$.

Definition: The <u>universal representation</u> π_U of $\mathfrak{A}$ is the direct sum of all its GNS representations π_ω ($\omega \in \mathcal{S}(\mathfrak{A})$), thus

$$\mathcal{H}_U = \bigoplus_{\omega \in \mathcal{S}(\mathfrak{A})} \mathcal{H}_\omega .$$

Remark: π_U is faithful. In fact,

$$\begin{aligned} \pi_U(A) &= 0 \\ &\Rightarrow \pi_\omega(A)\Omega_\omega = 0 \quad \forall \omega \\ &\Rightarrow \| \pi_\omega(A)\Omega_\omega \|^2 = 0 \quad \forall \omega \\ &\Rightarrow \omega(A^*A) = 0 \quad \forall \omega \\ &\Rightarrow A^*A = 0 \\ &\Rightarrow \| A^*A \| = \| A \|^2 = 0 \\ &\Rightarrow A = 0. \end{aligned}$$

Rappel: Let π be a representation of $\mathfrak{A}$ on a Hilbert space $\mathcal{H}$ -- then the following conditions are equivalent:

(1) π is irreducible;

(2) $\pi(\mathfrak{A})' = \underset{\sim}{C}I$;

(3) $\pi(\mathfrak{A})'' = \mathcal{B}(\mathcal{H})$;

(4) $\overline{\pi(\mathfrak{A})\Omega} = \mathcal{H} \ \forall \ \Omega \neq 0$ in $\mathcal{H}$.

Definition: Let $\omega \in \mathcal{S}(\mathfrak{A})$ -- then ω is <u>pure</u> iff it is an extreme point of $\mathcal{S}(\mathfrak{A})$.

Fact: The GNS representation π_ω associated with a state ω is irreducible iff ω is pure.

Remark: Let π run through the unitary equivalence classes of irreducible representations of $\mathfrak{A}$ -- then

$$\forall\, A \in \mathfrak{A}, \ \sup_{\pi} \ \|\pi(A)\| = \|A\| .$$

Let $\alpha: \mathfrak{A} \rightarrow \mathfrak{A}$ be an automorphism of $\mathfrak{A}$ -- then a state $\omega \in \mathcal{S}(\mathfrak{A})$ is said to be invariant w.r.t. α if

$$\alpha^{T}(\omega) \ (= \omega \circ \alpha) = \omega .$$

Consider the GNS construction per ω -- then there exists one and only one unitary operator $U_{\omega} : \mathcal{H}_{\omega} \rightarrow \mathcal{H}_{\omega}$ such that $U_{\omega} \Omega_{\omega} = \Omega_{\omega}$ and

$$U_{\omega} \pi_{\omega}(A) U_{\omega}^{-1} = \pi_{\omega}(\alpha A) \quad \forall A \in \mathfrak{A} .$$

[Note: The definition of U_{ω} is the obvious one:

$$U_{\omega} \pi_{\omega}(A) \Omega_{\omega} = \pi_{\omega}(\alpha A) \Omega_{\omega} .$$

Therefore U_{ω} has a dense domain and a dense range. That U_{ω} is unitary follows from the α-invariance of ω:

$$\begin{aligned} &\| U_{\omega} \pi_{\omega}(A) \Omega_{\omega} \|^{2} \\ &\quad = \langle \pi_{\omega}(\alpha A) \Omega_{\omega}, \pi_{\omega}(\alpha A) \Omega_{\omega} \rangle \\ &\quad = \langle \Omega_{\omega}, \pi_{\omega}(\alpha A)^{*} \pi_{\omega}(\alpha A) \Omega_{\omega} \rangle \\ &\quad = \omega(\alpha(A^{*}A)) \\ &\quad = \omega(A^{*}A) \\ &\quad = \langle \Omega_{\omega}, \pi_{\omega}(A^{*}A) \Omega_{\omega} \rangle \\ &\quad = \| \pi_{\omega}(A) \Omega_{\omega} \|^{2} .] \end{aligned}$$

Let $\alpha : G \rightarrow \underline{\text{Aut}}\, \mathfrak{A}$ be a representation of G on $\mathfrak{A}$. Suppose that ω is a G-invariant state:

$$\alpha(\sigma)^{T} \omega = \omega \quad \forall\, \sigma \in G.$$

Consider the GNS construction per ω -- then $\exists$ a unitary representation U_ω of G on $\mathcal{H}_\omega$ such that $\forall\, \sigma \in G$, $U_\omega(\sigma)\Omega_\omega = \Omega_\omega$ and

$$U_\omega(\sigma)\, \pi_\omega(A) U_\omega(\sigma)^{-1} = \pi_\omega(\alpha(\sigma)A) \quad \forall\, A \in \mathfrak{A}\,.$$

Therefore the triple $\{\mathcal{H}_\omega, \pi_\omega, U_\omega\}$ implements α.

[Note: Cyclic representations are necessarily nondegenerate.]

THEOREM (Markoff-Kakutani) Let K be a convex compact subset of a linear topological space X. Let $\mathcal{J}$ be a commuting family of continuous linear maps $X \longrightarrow X$ which leave K invariant -- then $\exists\, p \in K$: $Tp=p \;\forall\, T \in \mathcal{J}$.

We shall now apply the preceding machinery to

$$\mathfrak{A} = C^*(\bigcup_{O} \mathfrak{A}(O)).$$

Suppose that Axiom III_T is in force -- then $\forall\, a \in \underset{\sim}{R}^{1,d}$, $\alpha(a)$ is an automorphism of $\mathfrak{A}$ and it is clear that $\alpha(a)^T$ leaves $\mathcal{S}(\mathfrak{A})$ invariant. On the other hand, the $\alpha(a)$ commute among themselves, hence so do the $\alpha(a)^T$. In Markoff-Kakutani, take for X the dual of $\mathfrak{A}$ equipped with the weak* topology and let $K = \mathcal{S}(\mathfrak{A})$ -- then the conclusion is that $\exists$ a translation invariant state ω_T, thus the triple $\{\mathcal{H}_{\omega_T}, \pi_{\omega_T}, U_{\omega_T}\}$ implements α.

Remark: Part of Axiom IV_T is therefore automatic. However these generalities tell us nothing about the validity of the spectral condition.

<u>W*-Algebras</u> Let $\mathcal{H}$ be a Hilbert space, $\mathfrak{M}$ a nonempty subset of $\mathcal{B}(\mathcal{H})$ -- then the <u>commutant</u> $\mathfrak{M}'$ of $\mathfrak{M}$ is the set of all $T \in \mathcal{B}(\mathcal{H})$ that commute with the elements of $\mathfrak{M}$. The bicommutant $\mathfrak{M}''$ is the commutant of $\mathfrak{M}'$. It is easy to check that $\mathfrak{M}' = \mathfrak{M}'''$.

Suppose that $\mathfrak{M}$ is a *-subalgebra of $\mathcal{B}(\mathcal{H})$ containing I -- then the following are equivalent:

(1) $\mathfrak{M}'' = \mathfrak{M}$;

(2) $\mathfrak{M}$ is closed in the weak operator topology;

(3) $\mathfrak{M}$ is closed in the strong operator topology.

[Note: In general, $\overline{\mathfrak{M}}$ (weak or strong closure) is $\mathfrak{M}''$.
Proof: $\mathfrak{M} \subset \overline{\mathfrak{M}} \Rightarrow \overline{\mathfrak{M}}' \subset \mathfrak{M}' \Rightarrow \mathfrak{M}'' \subset \overline{\mathfrak{M}}'' = \overline{\mathfrak{M}}$. But $\mathfrak{M}''$ is weakly closed and contains $\mathfrak{M}$, hence $\mathfrak{M}'' = \overline{\mathfrak{M}}$.]

Definition: A *-subalgebra $\mathfrak{M}$ of $\mathcal{B}(\mathcal{H})$ containing I is a <u>W*-algebra</u> if it satisfies one (hence all) of the preceding conditions.

Example: For any nonempty subset $\mathfrak{M}$ of $\mathcal{B}(\mathcal{H})$, $(\mathfrak{M} \cup \mathfrak{M}^*)'$ is a W*-algebra.

Example: Let π be a representation of a unital C*-algebra on a Hilbert space $\mathcal{H}$ -- then $\pi(\mathfrak{A})'$ and $\pi(\mathfrak{A})''$ are W*-algebras.

<u>LEMMA</u> Suppose that $\mathfrak{A}$ is a unital C*-subalgebra of $\mathcal{B}(\mathcal{H})$. Let $\mathfrak{M}$ be the strong closure of $\mathfrak{A}$ -- then the unit ball of $\mathfrak{A}$ is strongly dense in the unit ball of $\mathfrak{M}$.

Given a W*-algebra $\mathfrak{M}$, let $W_{\mathfrak{M}}$ be the set of all linear functionals on $\mathfrak{M}$ continuous in the weak operator topology -- then

$W_{\mathfrak{M}}$ is a normed linear subspace of $\mathfrak{M}^*$ (which, however, need not be closed) and the arrow

$$\begin{cases} \mathfrak{M} \rightarrow W^*_{\mathfrak{M}} \\ M \rightarrow \lambda^*_M \end{cases} \quad \langle \lambda, \lambda^*_M \rangle = \lambda(M)$$

is an isometric isomorphism.

Example: Take $\mathcal{H} = L^2[0,1]$, $\mathfrak{M} = L^\infty[0,1]$ (realized as multiplication operators) -- then here $W_{\mathfrak{M}} = L^1[0,1]$ and $\mathfrak{M} = W^*_{\mathfrak{M}}$.

[Note: In this situation, $W_{\mathfrak{M}}$ is a Banach space, which is generally not the case.]

LEMMA Every W*-algebra $\mathfrak{M}$ is isometrically isomorphic to the dual of a Banach space.

[Let $\overline{W}_{\mathfrak{M}}$ be the completion of $W_{\mathfrak{M}}$ -- then the arrow of restriction $\overline{W}^*_{\mathfrak{M}} \rightarrow W^*_{\mathfrak{M}}$ is an isometric isomorphism, so $\mathfrak{M} \approx \overline{W}^*_{\mathfrak{M}}$.]

Remark: A W*-algebra $\mathfrak{M}$ is closed in the norm topology, hence is a C*-algebra. On the other hand, not every C*-algebra is isomorphic to a W*-algebra. For example, it is wellknown that C[0,1] is not the dual of any Banach space.

Rappel: Suppose that $\{A_i\} \subset \mathfrak{B}(\mathcal{H})$ is an increasing net of selfadjoint operators: $i < j \Rightarrow A_i \leq A_j$. Assume: $\exists\ C > 0$: $\|A_i\| \leq C \ \forall i$ -- then $\exists$ a bounded selfadjoint operator A of norm $\leq C$ such that $A_i \rightarrow A$ in the strong operator topology.

[Note: It is customary to write $A = \underline{\lim}\, A_i$.]

Let $\lambda : \mathfrak{M} \to \underset{\sim}{C}$ be a continuous linear functional -- then λ is said to be <u>normal</u> if for each bounded, increasing net $\{M_i\}$ in $\mathfrak{M}_{\underset{\sim}{R}}$, we have

$$\underline{\lim}\ \lambda(M_i) = \lambda(\underline{\lim}\ M_i).$$

LEMMA Let $\lambda \in \mathfrak{M}^*$ -- then λ is normal iff $\exists$ a trace class operator T on $\mathcal{H}$ such that

$$\lambda(M) = \underline{tr}(MT) \quad \forall\, M \in \mathfrak{M}.$$

[Note: Accordingly, a state $\omega \in \mathcal{S}(\mathfrak{M})$ is normal iff $\omega(M) = \underline{tr}(MW)$ $(M \in \mathfrak{M})$ for some density operator W, so, e.g., if $\Omega \in \mathcal{H}$ is a unit vector and if P_Ω is the orthogonal projection of $\mathcal{H}$ onto $\underset{\sim}{C}\Omega$, then $M \to \langle \Omega, M\Omega \rangle = \underline{tr}(MP_\Omega)$ is a normal state.]

Notation: $\mathfrak{M}_*$ is the set of normal elements of $\mathfrak{M}^*$.

Fact: $\mathfrak{M}_*$ is a norm closed subspace of $\mathfrak{M}^*$.

[Note: Therefore $\mathfrak{M}_*$ is a Banach space, the <u>predual</u> of $\mathfrak{M}$.]

Remark: One can be more precise: $\mathfrak{M}_*$ is the norm closure in $\mathfrak{M}^*$ of $W_{\mathfrak{M}}$, i.e., $\mathfrak{M}_* = \overline{W}_{\mathfrak{M}}$, hence $\mathfrak{M} \approx (\mathfrak{M}_*)^*$.

To see the point, take $\mathfrak{M} = \mathcal{B}(\mathcal{H})$ -- then $\mathfrak{M}_* \approx \mathcal{B}_1(\mathcal{H})$ (the trace class operators on $\mathcal{H}$) and, as is wellknown, $\mathcal{B}(\mathcal{H})$ can be identified with $\mathcal{B}_1(\mathcal{H})^*$. The associated weak* topology on $\mathcal{B}(\mathcal{H})$ is called the <u>σ-weak topology</u>. It is generated by the seminorms

$$A \to |\underline{tr}(AT)| \quad (T \in \mathcal{B}_1(\mathcal{H})).$$

It is clear that the σ-weak topology contains the weak operator topology and is contained in the weak topology (the smallest topology on $\mathcal{B}(\mathcal{H})$ for which all norm continuous linear functionals on $\mathcal{B}(\mathcal{H})$

are continuous).

<u>LEMMA</u> Let $\lambda \in \mathfrak{M}^*$ -- then λ is normal iff λ is σ-weakly continuous.

Example: Suppose that $\mathcal{H}$ is separable. Fix an orthonormal basis $\{e_n\}$ and let $\mathfrak{M} = \mathcal{B}(\mathcal{H})$ -- then the assignment

$$A \longrightarrow \sum_n \frac{1}{2^n} \langle e_n, Ae_n \rangle$$

is a faithful normal state, thus is in the predual $\mathfrak{M}_*$. But it is not continuous in the weak operator topology, thus is not in $W_{\mathfrak{M}}$.

<u>LEMMA</u> Suppose that ω is a normal state on $\mathfrak{M}$. Let $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$ be the associated GNS data -- then $\pi_\omega(\mathfrak{M}) \subset \mathcal{B}(\mathcal{H}_\omega)$ is a W*-algebra.

Definition: Let $\mathfrak{M}$ be a W*-algebra -- then $\Omega \in \mathcal{H}$ is <u>separating</u> for $\mathfrak{M}$ if $M\Omega = 0$ $(M \in \mathfrak{M}) \Rightarrow M=0$.

[Note: If $\mathfrak{M}$ admits a separating vector Ω, then $\forall$ normal state ω, $\exists\, x \in \mathcal{H} : \omega(M) = \langle x, Mx \rangle$ $(M \in \mathfrak{M})$ (x can be chosen cyclic provided ω is also faithful).]

<u>LEMMA</u> Ω is cyclic for $\mathfrak{M}$ iff Ω is separating for $\mathfrak{M}'$.

Fact: $\mathfrak{M}$ admits a faithful normal state iff $\mathfrak{M}$ is isomorphic to a W*-algebra $\pi(\mathfrak{M})$ which has a cyclic and separating vector.

A W*-algebra $\mathfrak{M}$ is <u>σ-finite</u> if every collection of mutually orthogonal projections in $\mathfrak{M}$ is at most countable.

If $\mathcal{H}$ is separable (as we suppose in the applications), then every W*-algebra on $\mathcal{H}$ is necessarily σ-finite. On the other hand, it is not difficult to prove that every σ-finite $\mathfrak{M}$ possesses a faithful normal state. Consequently, in the separable case, there is no essential loss of generality in assuming that $\mathfrak{M}$ has a cyclic and separating vector.

The center of a W*-algebra $\mathfrak{M}$ is $Z_{\mathfrak{M}} = \mathfrak{M} \cap \mathfrak{M}'$. One says that $\mathfrak{M}$ is a <u>factor</u> if $Z_{\mathfrak{M}} = \underset{\sim}{C}I$.

[Note: In some sense, the study of W*-algebras can be reduced to the study of factors (decomposition theory) but I'll omit the specifics as they are not particularly enlightening.]

<u>Particle Theories</u> Let $\{\mathfrak{A}, \alpha, \mathcal{H}, \pi, U\}$ be a theory of local observables -- then π is faithful, so $\mathfrak{A}$ "is" $\pi(\mathfrak{A})$. Accordingly, we shall agree henceforth to identify $\mathfrak{A}$ with $\pi(\mathfrak{A})$ and work entirely in $\mathcal{H}$. One can then embed $\mathfrak{A}(O)$ in $\mathfrak{A}(O)''$ and this is how W*-algebras make their appearance.

Definition: A <u>particle theory</u> in a Hilbert space $\mathcal{H}$ is the assignment of a W*-algebra $\mathfrak{M}(O)$ to each bounded open subset $O \subset M$ subject to the following assumptions:

<u>PT</u>$_1$: $O_1 \subset O_2 \Rightarrow \mathfrak{M}(O_1) \subset \mathfrak{M}(O_2)$;

<u>PT</u>$_2$: $O_1 \perp O_2 \Rightarrow \mathfrak{M}(O_1) \subset \mathfrak{M}(O_2)'$;

<u>PT</u>$_3$: $\exists$ a unitary representation U of $\tilde{\mathcal{P}}_+^\uparrow$ on $\mathcal{H}$ such that

$$U(\tilde{\Lambda}, a)\,\mathfrak{M}(O)\,U(\tilde{\Lambda}, a)^{-1} = \mathfrak{M}((\tilde{\Lambda}, a)\cdot O),$$

where

$$\underline{\text{spec}}(U|\underset{\sim}{R}^{1,d}) \subset \bar{V}_+.$$

The <u>quasilocal algebra</u> $\mathfrak{A}$ of a particle theory is the norm closure of $\bigcup_O \mathfrak{M}(O)$ and the <u>global algebra</u> $\mathfrak{M}$ of a particle theory is the weak closure of $\bigcup_O \mathfrak{M}(O)$.

[Note: $\mathfrak{A}$ is a C*-algebra, $\mathfrak{M}$ is a W*-algebra, and $\mathfrak{M} = \mathfrak{A}''$.]

Remark: If G is an unbounded open subset of $\underset{\sim}{R}^{1,d}$, then by definition

$$\mathfrak{M}(G) = \Big(\bigcup_{O \subset G} \mathfrak{M}(O)\Big)'',$$

thus

$$\mathfrak{M}(\underset{\sim}{R}^{1,d}) = \mathfrak{M}.$$

Observation: From the definitions, $\tilde{\mathcal{P}}_+^\uparrow$ is represented on $\mathfrak{M}$ by conjugation. This carries over to $\mathfrak{M}'$: $\forall$ $M' \in \mathfrak{M}'$, $U(\tilde{\Lambda},a)M'U(\tilde{\Lambda},a)^{-1} \in \mathfrak{M}'$. Thus take any $M \in \mathfrak{M}$ -- then

$$MU(\tilde{\Lambda},a)M'U(\tilde{\Lambda},a)^{-1}$$

$$= U(\tilde{\Lambda},a)U(\tilde{\Lambda},a)^{-1}\ MU(\tilde{\Lambda},a)M'U(\tilde{\Lambda},a)^{-1}$$

$$= U(\tilde{\Lambda},a)M'U(\tilde{\Lambda},a)^{-1}\ MU(\tilde{\Lambda},a)U(\tilde{\Lambda},a)^{-1}$$

$$= U(\tilde{\Lambda},a)M'U(\tilde{\Lambda},a)^{-1}M.$$

A particle theory admits a vacuum if $\exists$ a $\tilde{\mathcal{P}}_+^\uparrow$-invariant unit vector Ω_0 such that $\left(\bigcup_O \mathfrak{M}(O)\right)\Omega_0$ is dense in $\mathcal{H}$.

[Note: Therefore $\mathfrak{A}\Omega_0$ and $\mathfrak{M}\Omega_0$ are dense in $\mathcal{H}$.]

Notation: A PTV is a particle theory that admits a vacuum.

Example: Take $\mathcal{H} = \underset{\sim}{C}$, so $\mathcal{B}(\mathcal{H}) = \underset{\sim}{C}$ and assign to each O the W*-algebra $\mathfrak{M}(O) = \underset{\sim}{C}$. Take $\Omega_0 = 1$ and let $U(\tilde{\Lambda},a) = I$ $\forall(\tilde{\Lambda},a)$ -- then all the requirements for a PTV are met.

[Note: To eliminate this triviality, in the sequel, we shall assume that $\underline{\dim}\,\mathcal{H} > 1$.]

THEOREM Suppose given a PTV -- then

$$U(\underset{\sim}{R}^{1,d}) \subset \mathfrak{M}.$$

[Fix $M \in \mathfrak{M}$, $M' \in \mathfrak{M}'$ and put $U(a) = U(I,a)$ -- then, on the one hand,

$$\langle \Omega_0, U(a)M'U(a)^{-1}M\,\Omega_0\rangle$$

$$= \langle M^*\Omega_0, U(a)M'\Omega_0\rangle,$$

while on the other,

$$\langle \Omega_0, U(a)M'U(a)^{-1}M\Omega_0 \rangle$$

$$= \langle \Omega_0, M'U(-a)M\,\Omega_0 \rangle$$

$$= \langle M'^{*}\Omega_0, U(-a)M\Omega_0 \rangle .$$

The Fourier transform of

$$a \to \langle M^{*}\,\Omega_0, U(a)M'\,\Omega_0 \rangle$$

has its support in $\overline{V}_+$ and the Fourier transform of

$$a \longrightarrow \langle M'^{*}\,\Omega_0, U(-a)M\,\Omega_0 \rangle$$

has its support in $\overline{V}_-$. Therefore the support of the Fourier transform of

$$a \to \langle \Omega_0, U(a)M'U(a)^{-1}M\,\Omega_0 \rangle$$

is the origin, so, by the usual argument,

$$\langle \Omega_0, U(a)M'U(a)^{-1}M\,\Omega_0 \rangle$$

is a constant function of a, thus $\forall$ a,

$$\langle M^{*}\,\Omega_0, U(a)M'U(a)^{-1}\,\Omega_0 \rangle$$

$$= \langle M^{*}\,\Omega_0, M'\,\Omega_0 \rangle .$$

Since $\mathcal{M}\Omega_0$ is dense in $\mathcal{H}$, this implies that $\forall$ a,

$$U(a)M'U(a)^{-1}\,\Omega_0 = M'\Omega_0$$

or still,

$$(U(a)M'U(a)^{-1} - M')\Omega_0 = 0$$

$$\Rightarrow$$

$$U(a)M'U(a)^{-1} = M',$$

Ω_0 being separating for $\mathfrak{M}'$. It then follows that

$$U(a) \in \mathfrak{M}'' = \mathfrak{M},$$

as contended.]

[Note: PT_3 provides a representation of $\tilde{\mathcal{P}}_+^\uparrow$ on $\mathfrak{M}$, i.e., a homomorphism $\tilde{\mathcal{P}}_+^\uparrow \rightarrow \underline{\text{Aut}}\,\mathfrak{M}$. This representation restricts to a representation of $\mathbf{R}^{1,d}$ on $\mathfrak{M}$ and the theorem says that the action is via inner automorphisms.]

The vacuum of a PTV is said to be unique if

$$\underline{\dim}\,\{x \in \mathcal{H} : U(I,a)x = x \quad \forall\, a\} = 1.$$

[Note: This condition implies that the space of $\tilde{\mathcal{P}}_+^\uparrow$-invariants is one dimensional ($\tilde{\mathcal{L}}_+^\uparrow$ is semisimple).]

Remark: If the vacuum of a PTV is unique, then $\forall$ nonzero $a \in \mathbf{R}^{1,d}$,

$$\underline{\dim}\,\{x \in \mathcal{H} : U(I,ta)x = x \quad \forall\, t \in \mathbf{R}\} = 1.$$

[Here is a sketch of the proof. Let E_0 be the orthogonal projection of $\mathcal{H}$ onto $\mathbf{C}\,\Omega_0$ -- then $\forall\, x \in E_0^\perp \mathcal{H}$, $d\langle x, E_a x\rangle$ is absolutely continuous w.r.t. Lebesgue measure, which implies that

$$\lim_{t \rightarrow +\infty} U(I,ta) = E_0$$

in the weak operator topology. If now $x \perp \mathbf{C}\,\Omega_0$ and $U(I,ta)x = x$ $\forall\, t \in \mathbf{R}$,

then

$$\|x\|^2 = \langle x, U(I,ta)x \rangle$$

$$\longrightarrow \langle x, E_0 x \rangle = 0 \quad (t \longrightarrow +\infty),$$

so x=0.]

THEOREM Suppose given a PTV with a unique vacuum -- then $\mathfrak{M} = \mathcal{B}(\mathcal{H})$.

[From the previous theorem,

$$U(I,a) \in \mathfrak{M} \quad \forall a.$$

So, $\forall M' \in \mathfrak{M}'$,

$$U(I,a)M'U(I,a)^{-1} = M'$$

$$\Longrightarrow$$

$$U(I,a)M'\Omega_0 = M'\Omega_0$$

$$\Longrightarrow \quad M'\Omega_0 = c\,\Omega_0 \quad (\exists\, c)$$

$$\Longrightarrow \quad M' = cI$$

$$\Longrightarrow \quad \mathfrak{M}' = \underset{\sim}{C} I$$

$$\Longrightarrow \quad \mathfrak{M} = \mathfrak{M}'' = \mathcal{B}(\mathcal{H}).]$$

LEMMA Suppose given a PTV. Assume: $\mathfrak{M}$ is a factor. Let $a \in \underset{\sim}{R}^{1,d}$ be spacelike -- then $\forall\, M \in \bigcup_{O} \mathfrak{M}(O)$,

$$\lim_{\lambda \to +\infty} U(I, \lambda a) M U(I, \lambda a)^{-1} = \langle \Omega_0, M\Omega_0 \rangle I$$

in the weak operator topology.

[Since $\mathfrak{M}$ is the dual of $\mathfrak{M}_*$, $\forall r > 0$, the ball $\{M \in \mathfrak{M} : \|M\| \leq r\}$ is σ-weakly compact (Alaoglu). Let $M_0 \in \mathfrak{M}(O_0)$, where O_0 is arbitrary but fixed. Consider the net

$$\{U(I, \lambda a) M_0 \; U(I, \lambda a)^{-1} : \lambda \geq 0\} .$$

To prove that it is convergent to $\langle \Omega_0, M_0 \Omega_0 \rangle I$ in the weak operator topology, it suffices to prove that every subnet has a subnet convergent in the weak operator topology to $\langle \Omega_0, M_0 \Omega_0 \rangle I$. Since

$$\| U(I, \lambda a) M_0 \; U(I, \lambda a)^{-1} \| \leq \|M_0\| ,$$

every subnet has a subnet convergent in the σ-weak topology to some point in the ball of radius $\|M_0\|$, say

$$\lim_i \; U(I, \lambda_i a) M_0 \; U(I, \lambda_i a) = A_0 .$$

Because a is spacelike ($\langle a,a \rangle = a_0^2 - |\underset{\sim}{a}|^2 < 0$), $\forall O$, $\exists \lambda_0 : \lambda \geq \lambda_0 \Rightarrow (O_0 + \lambda a) \perp O$, thus $\exists i_0 : i \geq i_0$

$\Rightarrow$

$$(O_0 + \lambda_i a) \perp O$$

$\Rightarrow$

$$U(I, \lambda_i a) M_0 \; U(I, \lambda_i a)^{-1} \in \mathfrak{M}(O_0 + \lambda_i a) \subset \mathfrak{M}(O)',$$

so $\forall M \in \mathfrak{M}(O)$,

$$U(I, \lambda_i a) M_0 \; U(I, \lambda_i a)^{-1} \cdot M$$

$$= M \cdot U(I, \lambda_i a) M_0 \; U(I, \lambda_i a)^{-1} .$$

Passing to the σ-weak limit, we get

$$A_0 M = M A_0 \Rightarrow A_0 \in \mathfrak{M}(O)' .$$

But this holds $\forall\, O$, hence $A_0 \in \mathfrak{M}'$. On the other hand, $A_0 \in \mathfrak{M}$ (convergence in the σ-weak topology implies convergence in the weak operator topology), thus $A_0 \in \mathfrak{M} \cap \mathfrak{M}' = \underset{\sim}{C}I$, $\mathfrak{M}$ being by hypothesis a factor. Therefore

$$A_0 = C(A_0)I \quad (C(A_0) \in \underset{\sim}{C}).$$

To calculate $C(A_0)$, note that

$$\begin{aligned}
&\underset{i}{\underline{\lim}} \langle \Omega_0, U(I, \lambda_i a) M_0\, U(I, \lambda_i a)^{-1} \Omega_0 \rangle \\
&\quad = \langle \Omega_0, C(A_0) I\, \Omega_0 \rangle \\
&\quad = C(A_0) \langle \Omega_0, \Omega_0 \rangle \\
&\quad = C(A_0).
\end{aligned}$$

And: $\forall\, i$,

$$\begin{aligned}
&\langle \Omega_0, U(I, \lambda_i a) M_0\, U(I, \lambda_i a)^{-1} \Omega_0 \rangle \\
&\quad = \langle \Omega_0, M_0 \Omega_0 \rangle .
\end{aligned}$$

Combining these two facts gives

$$C(A_0) = \langle \Omega_0, M_0\, \Omega_0 \rangle,$$

from which the lemma.]

THEOREM Suppose given a PTV. Assume: $\mathfrak{M}$ is a factor -- then the vacuum is unique.

[Let $x \in \mathcal{H}$:

$$U(I,a)x = x \quad \forall\, a.$$

Take x orthogonal to Ω_0. Fix a spacelike a -- then $\forall\, M \in \bigcup_{O} \mathfrak{M}(O)$,

$$\lim_{\lambda \to +\infty} \langle x, U(I, \lambda a) M\, U(I, \lambda a)^{-1} \Omega_0 \rangle$$

$$= \langle \Omega_0, M \Omega_0 \rangle \langle x, \Omega_0 \rangle$$

$$= 0.$$

But

$$\langle x, U(I, \lambda a) M\, U(I, \lambda a)^{-1} \Omega_0 \rangle$$

$$= \langle U(I, -\lambda a) x, M\, U(I, -\lambda a) \Omega_0 \rangle$$

$$= \langle x, M \Omega_0 \rangle$$

$\Longrightarrow$

$$x \perp \Big(\bigcup_{O} \mathfrak{M}(O) \Big) \Omega_0$$

$$\Longrightarrow x = 0.$$

Therefore the vacuum is unique.]

Remark: Suppose given a PTV for which $\mathfrak{M}$ is a factor -- then $\mathfrak{M}$ must be $\mathcal{B}(\mathcal{H})$.

A PTV is said to be <u>additive</u> if

$$O = \bigcup_j O_j \Rightarrow \mathfrak{M}(O) = \Big(\bigcup_j \mathfrak{M}(O_j) \Big)''.$$

A PTV is said to be <u>weakly additive</u> if $\forall$ O,

$$\Big(\bigcup_a \mathfrak{M}(O+a) \Big)'' = \mathfrak{M}.$$

LEMMA An additive PTV is weakly additive.

[Fix a double cone $D \in \mathcal{O}$ and choose a point $x_0 \in O$ -- then

$$D \subset \bigcup_{a \in I_D} (O+a),$$

where a runs through all $d-x_0$ $(d=x_0+(d-x_0))$. This guarantees that

$$\bigcup_{a\in I_D} (O+a)$$

is bounded. In fact, $\forall\, x \in O$ & $\forall\, d \in D$,

$$\begin{aligned}\| x + a \| &= \| x + (d-x_0) \| \\ &\leq \|x\| + \|d\| + \|x_0\|\end{aligned}$$

which is uniformly bounded in x and d. Therefore

$$\begin{aligned}\mathfrak{M}(D) &\subset \mathfrak{M}\Big(\bigcup_{a\in I_D} (O+a)\Big) \\ &= \Big(\bigcup_{a\in I_D} \mathfrak{M}(O+a)\Big)''.\end{aligned}$$

But $\mathcal{D}$ is final in $\mathcal{O}$, hence

$$\begin{aligned}\mathfrak{M} &= \Big(\bigcup_{D} \mathfrak{M}(D)\Big)'' \\ &\subset \Big(\bigcup_{D} \big(\bigcup_{a\in I_D} \mathfrak{M}(O+a)\big)''\Big)'' \\ &\subset \Big(\Big(\bigcup_{a} \mathfrak{M}(O+a)\Big)''\Big)'' \\ &= \Big(\bigcup_{a} \mathfrak{M}(O+a)\Big)'' \\ &\subset \mathfrak{M}\end{aligned}$$

$$\Rightarrow \qquad \Big(\bigcup_{a} \mathfrak{M}(O+a)\Big)'' = \mathfrak{M}\ .]$$

Remark: Suppose given a weakly additive PTV with a unique vacuum -- then we shall prove later that $\forall$ a,

$$\bigcap_{O \ni a} \mathfrak{M}(O) = \underset{\sim}{C}I.$$

[Note: The interpretation of this fact is that there are no nontrivial observables at a point.]

Edge of the Wedge We shall need the generalization to several complex variables of the following standard statement from one complex variable.

Rappel: Let

$$\begin{cases} D^+ = \{z:\ |z| < 1\ \&\ \underline{\mathrm{Im}}\ z > 0\} \\ D^- = \{z:\ |z| < 1\ \&\ \underline{\mathrm{Im}}\ z < 0\}. \end{cases}$$

Suppose given two functions $\begin{cases} f^+ \\ f^- \end{cases}$ holomorphic in $\begin{cases} D^+ \\ D^- \end{cases}$ having continuous boundary values at real points $|x| < 1$ and that these boundary values coincide -- then $\exists$ a function f holomorphic in $\{z:\ |z| < 1\}$ such that

$$\begin{cases} f \,|\, D^+ = f^+ \\ f \,|\, D^- = f^-. \end{cases}$$

Extensions of this result to several complex variables are called edge of the wedge theorems.

Notation: Let $C \subset \underline{R}^n$ be a proper convex open cone with apex 0,

$$T(C) = \underline{R}^n + \sqrt{-1}\, C$$

$$(= \{z \in \underline{C}^n:\ \underline{\mathrm{Im}}\ z \in C\})$$

the tube based at C.

THEOREM Let

$$B = \{z \in \underline{C}^n: \|z\| < 1\}$$

and put

$$\begin{cases} B_C^+ = B \cap T(C) \\ B_C^- = B \cap T(-C). \end{cases}$$

Suppose given two functions $\begin{cases} f^+ \\ f^- \end{cases}$ holomorphic in $\begin{cases} B_C^+ \\ B_C^- \end{cases}$ having continuous boundary values at real points $\|x\| < 1$ and that these boundary values coincide -- then $\exists$ a complex neighborhood η of $\|x\| < 1$ and a function f holomorphic in $\eta \cup B_C^+ \cup B_C^-$ such that

$$\begin{cases} f \mid B_C^+ = f^+ \\ f \mid B_C^- = f^- . \end{cases}$$

Remark: In one dimension, take

$$C = \{y : y > 0\} .$$

Then

$$\begin{cases} B_C^+ = D^+ \\ B_C^- = D^- \end{cases}$$

and the role of η is played by D itself.

Application: Suppose that F is holomorphic in $B \cap T(C)$. Suppose further that

$$\lim_{\substack{y \to 0 \\ y \in C}} F(x + \sqrt{-1}\, y) = 0 \quad (\|x\| < 1).$$

Then F=0 in $B \cap T(C)$.

[Define a holomorphic function G in $B \cap T(-C)$ by

$$G(x + \sqrt{-1}\, y) = \overline{F(x - \sqrt{-1}\, y)}.$$

Obviously,

$$\lim_{\substack{y \to 0 \\ y \in -C}} G(x + \sqrt{-1}\, y) = 0 \quad (\|x\| < 1).$$

So, thanks to the edge of the wedge theorem, there exists a function Φ holomorphic in a complex neighborhood $\mathcal{N}$ of $\|x\| < 1$ which is an analytic continuation of F. But

$$\|x\| < 1 \Rightarrow \Phi(x) = \lim_{\substack{y \to 0 \\ y \in C}} F(x + \sqrt{-1}\,y) = 0.$$

Therefore $\Phi \equiv 0 \Rightarrow F \equiv 0$.]

[Note: The last step uses the identity principle from several complex variables: If f is a holomorphic function in a domain D which, together with all its derivatives $\partial^\alpha f$, vanishes at some point $p_0 \in D$, then $f \equiv 0$ in D. Corollary: A holomorphic function that vanishes in a real or complex neighborhood of a point of a domain must vanish identically in that domain.]

Here is an example. Let U be a unitary representation of $\underset{\sim}{R}^{1,d}$ on a Hilbert space $\mathcal{H}$ with the property that the spectrum of U is contained in $\overline{V}_+$:

$$U(a) = \int_{\overline{V}_+} e^{\sqrt{-1}\langle a,p\rangle}\, dE_p.$$

Put

$$U(z) = \int_{\overline{V}_+} e^{\sqrt{-1}\langle z,p\rangle}\, dE_p,$$

where

$$z = a + \sqrt{-1}\, b \in \underset{\sim}{R}^{1,d} + \sqrt{-1}\, V_+.$$

Since

$$e^{\sqrt{-1}\langle z,p\rangle} = e^{\sqrt{-1}\langle a,p\rangle}\, e^{-\langle b,p\rangle}$$

and

$$\begin{cases} b \in V_+ \\ p \in \overline{V}_+ \end{cases} \Rightarrow \langle b,p \rangle \geq 0,$$

the integral exists. This said, $\forall\ \psi \in \mathcal{H}$,

$$\| U(z)\psi \|^2 = \int_{\overline{V}_+} | e^{\sqrt{-1}\langle z,p \rangle} |^2 \, d\langle \psi, E_p \psi \rangle$$

$$\leq \int_{\overline{V}_+} e^{-2\langle b,p \rangle} d\langle \psi, E_p \psi \rangle$$

$$\leq \int_{\overline{V}_+} d\langle \psi, E_p \psi \rangle = \|\psi\|^2,$$

thus $\|U(z)\| \leq 1$, so $U(z)$ is a contraction.

LEMMA We have

$$\lim_{b \to 0} U(a + \sqrt{-1}\, b) = U(a)$$

in the strong operator topology.

[For any $\psi \in \mathcal{H}$,

$$\| U(a + \sqrt{-1}\, b)\psi - U(a)\psi \|^2$$

$$= \int_{\overline{V}_+} | e^{\sqrt{-1}\langle a + \sqrt{-1}\, b,p \rangle} - e^{\sqrt{-1}\langle a,p \rangle} |^2 d\langle \psi, E_p \psi \rangle$$

$$= \int_{\overline{V}_+} | e^{\sqrt{-1}\langle a,p \rangle} (e^{-\langle b,p \rangle} - 1) |^2 \, d\langle \psi, E_p \psi \rangle$$

$$= \int_{\overline{V}_+} |1 - e^{-\langle b,p \rangle}|^2 \, d\langle \psi, E_p \psi \rangle \; .$$

But

$$|1 - e^{-\langle b,p \rangle}|^2 \leq 4$$

and for fixed p,

$$\lim_{b \to 0} |1 - e^{-\langle b,p \rangle}|^2 = 0.$$

Therefore, by dominated convergence,

$$\lim_{b \to 0} \int_{\overline{V}_+} |1 - e^{-\langle b,p \rangle}|^2 \, d\langle \psi, E_p \psi \rangle = 0,$$

from which the assertion.]

Now fix $\psi, \Omega_0 \in \mathcal{H}$ and $A \in \mathcal{B}(\mathcal{H})$. Let

$$f_{\psi,A}(z) = \langle \psi, U(z) A \Omega_0 \rangle \; .$$

Then $f_{\psi,A}$ is holomorphic in $\underset{\sim}{R}^{1,d} + \sqrt{-1}\, V_+$ and continuous on $\underset{\sim}{R}^{1,d} + \sqrt{-1}\, (V_+ \cup \{0\})$:

$$f_{\psi,A}(a) = \lim_{b \to 0} f_{\psi,A}(a + \sqrt{-1}\, b).$$

Suppose that $\exists \; \varepsilon > 0$:

$$\|a\| < \varepsilon \Rightarrow f_{\psi,A}(a) = 0.$$

Then

$$f_{\psi,A}(z) = 0 \quad \forall \; z \in \underset{\sim}{R}^{1,d} + \sqrt{-1}\, V_+$$

$$\Rightarrow$$

$$f_{\psi,A}(a) = 0 \quad \forall \; a \in \underset{\sim}{R}^{1,d} \; .$$

<u>Reeh-Schlieder</u> Suppose given a weakly additive PTV -- then $\forall$ O, $\mathfrak{M}(O)\Omega_0$ is dense in $\mathcal{H}$.

To prove this, fix O_0 and $\varepsilon > 0$:

$$O_0 + a \subset O \quad \forall a: \|a\| < \varepsilon .$$

Consider any $\Psi \in \mathcal{H}$: $\Psi \perp \mathfrak{M}(O)\Omega_0$. Given $M_0 \in \mathfrak{M}(O_0)$, form

$$f_{\Psi,M_0}(z) = \langle \Psi, U(I,z)M_0\Omega_0 \rangle .$$

Since

$$\begin{aligned} & U(I,a)\mathfrak{M}(O_0)U(I,a)^{-1} \\ & \quad = \mathfrak{M}(O_0+a) \\ & \quad \subset \mathfrak{M}(O), \end{aligned}$$

$\forall a: \|a\| < \varepsilon$,

$$\begin{aligned} f_{\Psi,M_0}(a) &= \langle \Psi, U(I,a)M_0\, U(I,a)^{-1}\Omega_0 \rangle \\ &= 0 \end{aligned}$$

$\Rightarrow$

$$f_{\Psi,M_0}(a) = 0 \quad \forall a.$$

I.e.: $\forall$ a & $\forall$ $M \in \mathfrak{M}(O_0+a)$,

$$\langle \Psi, M\Omega_0 \rangle = 0.$$

But, by weak additivity,

$$\Big(\bigcup_a \mathfrak{M}(O_0+a)\Big)'' = \mathfrak{M},$$

so $\forall\, M \in \mathfrak{M}$,

$$\langle \Psi, M\Omega_0 \rangle = 0$$

$$\Longrightarrow \quad \Psi = 0,$$

$\mathfrak{M}\Omega_0$ being dense in $\mathcal{H}$.

Remark: Ω_0 is separating for $\mathfrak{M}(O)$. Thus choose $P: O \subset P^{\perp} \Rightarrow O \perp P \Rightarrow \mathfrak{M}(O) \subset \mathfrak{M}(P)'$ (cf. PT_2). By the above, Ω_0 is cyclic for $\mathfrak{M}(P)$, hence separating for $\mathfrak{M}(P)'$, hence separating for $\mathfrak{M}(O)$.

Example: Let E_0 be the orthogonal projection of $\mathcal{H}$ onto $\underset{\sim}{C}\,\Omega_0$ -- then $E_0 \notin \mathfrak{M}(O)$.

[In fact, $E_0 \in \mathfrak{M}(O) \Rightarrow I - E_0 \in \mathfrak{M}(O)$ & $(I-E_0)\Omega_0 = 0 \Rightarrow I-E_0 = 0 \Rightarrow I = E_0$.]

The restriction of U to $\underset{\sim}{R}^1$ ($\underset{\sim}{R}^{1,d} = \underset{\sim}{R}^1 \times \underset{\sim}{R}^d$) is the one parameter group of time translations, hence by Stone, $U(t) = e^{\sqrt{-1}\,tH}$, where H is positive and selfadjoint, the <u>energy operator</u>.

Definition: $\Psi_0 \in \mathcal{H}$ is <u>analytic for the energy</u> if Ψ_0 is an analytic vector for H.

[Note: Therefore $\Psi_0 \in \underline{\mathrm{Dom}}_{H^n}$ $\forall$ n and

$$\sum_{n=0}^{\infty} \frac{\| H^n \Psi_0 \|}{n!} \, t^n < +\infty$$

for some $t > 0$.]

Example: Ω_0 is analytic for the energy (in fact, $H\Omega_0 = 0$).

The foregoing can now be extended: If $\Psi_0 \neq 0$ is analytic for the energy, then $\forall$ O, $\Psi \perp \mathfrak{M}(O)\Psi_0 \Rightarrow \Psi \perp \mathfrak{M}\Psi_0$, hence $\forall$ O, $\mathfrak{M}(O)\Psi_0$ is dense in $\mathcal{H}$ provided that the vacuum is unique (so $\mathfrak{M} = \mathcal{B}(\mathcal{H})$).

Remark: Suppose that $f \in L^1(\underset{\sim}{R})$ and <u>spt</u> $\hat{f}$ is compact -- then

$$\hat{f}(H) = \int_0^{\infty} \hat{f}(\lambda) dE_{\lambda}$$

$$= \int_0^{\infty} \left(\int_{-\infty}^{\infty} e^{\sqrt{-1}\, t \lambda} f(t) dt \right) dE_{\lambda}$$

$$= \int_{-\infty}^{\infty} f(t) \left(\int_0^{\infty} e^{\sqrt{-1}\, t \lambda} dE_{\lambda} \right) dt$$

$$= \int_{-\infty}^{\infty} f(t) U(t) dt.$$

And: $\forall \ \Psi \in \mathcal{H}$, $\hat{f}(H)\Psi$ is analytic for the energy.

LEMMA In the weak operator topology,

$$\underline{\lim}_{t \to +\infty} e^{-tH} = E_0.$$

[First, $\forall \ \lambda \geq 0$,

$$e^{-t\lambda} \to \begin{cases} 1 & (\lambda = 0) \\ 0 & (\lambda > 0) \end{cases} .$$

Therefore, $\forall \ x, y \in \mathcal{H}$,

$$\langle x, e^{-tH} y \rangle = \int_0^{\infty} e^{-t\lambda} d\langle x, E_{\lambda} y \rangle$$

$$\to \int_0^{\infty} \chi_{\{0\}} d\langle x, E_{\lambda} y \rangle$$

$$= \langle x, E_0 y \rangle .]$$

<u>The Intersection Property</u> Suppose given a weakly additive PTV with a unique vacuum -- then $\forall$ a,

$$\bigcap_{O \ni a} \mathfrak{M}(O) = \underset{\sim}{C} I.$$

To prove this, fix a and put

$$\begin{cases} S_a = \{ b \in \underset{\sim}{R}^{1,d} : a + b \in \{a\}^{\perp} \} \\ S_O = \{ b \in \underset{\sim}{R}^{1,d} : O + b \subset \underline{\text{int}}\ O^{\perp} \}. \end{cases}$$

Then

$$S_a = \bigcup_{O \ni a} S_O.$$

Take now an $M \in \bigcap_{O \ni a} \mathfrak{M}(O)$ -- then

$$M = \frac{M + M^*}{2} + \sqrt{-1}\ \frac{M - M^*}{2\sqrt{-1}},$$

so we can suppose that M is selfadjoint. Let

$$F_M(b) = \langle \Omega_0, M U(I,b) M \Omega_0 \rangle,$$

where $b \in S_a$. Since $M \in \mathfrak{M}(O)$ $(O \ni a)$, $\forall$ $b \in S_O$:

$$\begin{aligned} U(I,b) M U(I,b)^{-1} &\in \mathfrak{M}(O+b) \\ &\subset \mathfrak{M}(\underline{\text{int}}\ O^{\perp}) \end{aligned}$$

$\Longrightarrow$

$$\begin{aligned} &M \cdot U(I,b) M U(I,b)^{-1} \\ &\quad = U(I,b) M U(I,b)^{-1} \cdot M \end{aligned}$$

$\Longrightarrow$

$$F_M(b) = \langle \Omega_0, M U(I,b) M \Omega_0 \rangle$$

$$= \langle \Omega_0, MU(I,b)MU(I,b)^{-1}\Omega_0 \rangle$$

$$= \langle \Omega_0, U(I,b)MU(I,b)^{-1}M\Omega_0 \rangle$$

$$= \langle \Omega_0, MU(I,-b)M\,\Omega_0 \rangle$$

$$= F_M(-b).$$

But $S_a = \bigcup_{O \ni a} S_O$, hence $F_M(b) = F_M(-b)\ \forall\, b \in S_a$, a relation which obviously persists to $\overline{S}_a$. Let

$$W = \{ w \in \underset{\sim}{R}^{1,d} : |w_0| < |\underset{\sim}{w}| \}.$$

Then

$$W \subset S_a \Rightarrow \overline{W} \subset \overline{S}_a.$$

Fix $\overline{w} \in \overline{W} : \overline{w}_0^2 = |\underset{\sim}{\overline{w}}|^2$ $(\overline{w}_0 \neq 0)$ and write

$$\phi(t) = F_M(t\overline{w}) \quad (t \in \underset{\sim}{R}).$$

On the basis of the definitions, it is easy to see that the Fourier transform of $t \to \phi(t)$ has its support in $\overline{V}_+$ and the Fourier transform of $t \to \phi(-t)$ has its support in $\overline{V}_-$. Since $\phi(t) = \phi(-t)$, the usual argument implies that $\phi(t)$ is a constant. Therefore

$$\langle \Omega_0, MU(I,t\overline{w})M\,\Omega_0 \rangle$$

$$= \langle \Omega_0, M^2\Omega_0 \rangle$$

$$\Longrightarrow$$

$$\langle M\Omega_0, U(I,t\overline{w})M\,\Omega_0 \rangle = \langle \Omega_0, M^2\Omega_0 \rangle$$

$$= \langle M\Omega_0, M\Omega_0 \rangle$$

$$= \|M\Omega_0\| \cdot \|M\Omega_0\|$$

$$= \| M\Omega_0 \| \cdot \| U(I,t\bar{w})M\Omega_0 \|$$

$\Rightarrow$

$$U(I,t\bar{w})M\Omega_0 = c_t \, M\Omega_0 \quad (\exists\, c_t)$$

$\Rightarrow$

$$\langle M\Omega_0, U(I,t\bar{w})M\Omega_0 \rangle$$

$$= \langle M\Omega_0, c_t \, M\Omega_0 \rangle$$

$$= c_t \langle M\Omega_0, M\Omega_0 \rangle$$

$$= \langle M\Omega_0, M\Omega_0 \rangle$$

$\Rightarrow$

$$c_t = 1 \quad \forall\, t$$

$\Rightarrow$

$$U(I,t\bar{w})M\Omega_0 = M\Omega_0 \quad \forall\, t.$$

Finally, for an arbitrary $x \in \mathcal{H}$,

$$\lim_{t \to +\infty} \langle x, U(I,t\bar{w})M\Omega_0 \rangle = \langle x, E_0 M\Omega_0 \rangle \ .$$

I.e.:

$$\langle x, M\Omega_0 \rangle = \langle x, E_0 \, M\Omega_0 \rangle$$

$$\Rightarrow \quad M\Omega_0 = E_0 \, M\Omega_0$$

$$\Rightarrow \quad M\Omega_0 = \langle \Omega_0, M\Omega_0 \rangle \Omega_0$$

$$\Rightarrow \quad M = \langle \Omega_0, M\Omega_0 \rangle \, I,$$

Ω_0 being separating for $\mathfrak{M}(0)$.

Remark: The fact that $\phi(t)$ is a constant can be established by using complex variables. To see this, note that

$$\phi(t) = \int_{\overline{V}_+} e^{\sqrt{-1}\, t \langle \overline{w},p \rangle}\, d\langle M\Omega_0, E_p\, M\Omega_0 \rangle .$$

Since $\langle \overline{w},p \rangle \geq 0 \;\; \forall\, p \in \overline{V}_+$,

$$\overline{\phi(t)} = \int_{\overline{V}_+} e^{-\sqrt{-1}\, t \langle \overline{w},p \rangle}\, d\langle M\Omega_0, E_p\, M\Omega_0 \rangle$$

$$= \phi(-t) = \phi(t).$$

Moreover, $\phi(t)$ can be analytically continued into the upper half plane:

$$\phi_+(z) = \int_{\overline{V}_+} e^{\sqrt{-1}\, z \langle \overline{w},p \rangle}\, d\langle M\Omega_0, E_p\, M\Omega_0 \rangle \quad (\underline{\text{Im}}\, z > 0).$$

Since $\phi_+(z)$ is continuous on $\underline{\text{Im}}\, z \geq 0$ with boundary values $\phi(t)$, the Schwarz reflection principle implies that the prescription

$$\phi_-(z) = \overline{\phi_+(\overline{z})} \quad (\underline{\text{Im}}\, z < 0)$$

is an analytic continuation of $\phi_+(z)$ into the lower half plane. The resulting function is bounded and entire, hence by Liouville, is a constant.

<u>Wightman's Inequality</u> Suppose given a weakly additive PTV, where $\mathfrak{M}$ is not abelian. Fix O,P:

$$O \subset P \text{ and } \underline{\text{dis}}(O, \underline{\text{fr}}\, P) > 0.$$

Then $\mathfrak{M}(O)$ is properly contained in $\mathfrak{M}(P)$.

To prove this, we shall argue by contradiction and assume that $\mathfrak{M}(O) = \mathfrak{M}(P)$. Choose $\varepsilon > 0$:

$$\|a\| < \varepsilon \Rightarrow O + a \subset P$$

$$\Rightarrow \mathfrak{M}(O+a) \subset \mathfrak{M}(P).$$

Take any b: $\|b\| < \varepsilon$ -- then

$$\begin{aligned}\mathfrak{M}(O + a + b) &= U(I,b)\,\mathfrak{M}(O + a)U(I,b)^{-1} \\ &\subset U(I,b)\,\mathfrak{M}(P)U(I,b)^{-1} \\ &= U(I,b)\,\mathfrak{M}(O)U(I,b)^{-1} \\ &= \mathfrak{M}(O + b) \\ &\subset \mathfrak{M}(P),\end{aligned}$$

from which, by iteration,

$$\mathfrak{M}(O + \sum_{k=1}^{n} a_k) \subset \mathfrak{M}(P) \quad (\|a_k\| < \varepsilon, k=1,\ldots,n),$$

so $\forall\ a \in \underset{\sim}{R}^{1,d}$,

$$\mathfrak{M}(O + a) \subset \mathfrak{M}(P)$$

$$\Rightarrow$$

$$\bigcup_a \mathfrak{M}(O + a) \subset \mathfrak{M}(P)$$

$\Rightarrow$

$$\mathfrak{M} = \Big(\bigcup_a \mathfrak{M}(O + a) \Big)'' \subset \mathfrak{M}(P)''$$

$$= \mathfrak{M}(P) \subset \mathfrak{M}$$

$$\Rightarrow$$

$$\mathfrak{M}(O) = \mathfrak{M} .$$

On the other hand, $\exists\, a : O + a \perp O$ (choose a spacelike with $\| a \| \gg 0$), thus, from PT_2,

$$\mathfrak{M}(O + a) \subset \mathfrak{M}(O)'.$$

But

$$U(I,a)\,\mathfrak{M}(O)U(I,a)^{-1}$$

$$= U(I,a)\,\mathfrak{M}\,U(I,a)^{-1}$$

$$= \mathfrak{M} \qquad (U(I,a) \in \mathfrak{M})$$

$$\Rightarrow$$

$$\mathfrak{M} \subset \mathfrak{M}'.$$

I.e.: $\mathfrak{M}$ is abelian, contrary to assumption.

Here is an application: Each $\mathfrak{M}(O)$ is infinite dimensional provided that $\mathfrak{M}$ is not abelian. Suppose false: $\exists\, O$: $\underline{\dim}\ \mathfrak{M}(O) < +\infty$. Choose $O_1 \supset O_2 \supset \cdots : O \supset O_n \ \forall\, n$ and $\underline{\mathrm{dis}}(O_{n+1}, \underline{\mathrm{fr}}\ O_n) > 0$ -- then $\mathfrak{M}(O_n)$ is properly contained in $\mathfrak{M}(O)$ $\forall\, n$, hence

$$\underline{\dim}\ \mathfrak{M}(O) > \underline{\dim}\ \mathfrak{M}(O_1) > \underline{\dim}\ \mathfrak{M}(O_2) \quad \cdots,$$

and this is plainly impossible.

Therefore $\mathfrak{M}$ not abelian $\Rightarrow$ $\underline{\dim}\ \mathcal{H} = +\infty$. If in addition, our weakly additive PTV has a unique vacuum, then $\mathfrak{M} = \mathcal{B}(\mathcal{H})$, so under these circumstances $\mathfrak{M}$ is not abelian if $\underline{\dim}\ \mathcal{H} > 1$ and the $\mathfrak{M}(O)$ are necessarily infinite dimensional.

<u>A Theorem of Borchers</u> Suppose that $t \to U(t) = e^{\sqrt{-1}\,tH}$ is a one parameter unitary group, where the generator H is nonnegative: $H \geq 0$. Let $E, F \in \mathcal{B}(\mathcal{H})$ be projections with $EF = FE = 0$. Assume: $\exists\ \varepsilon > 0$ such that $|t| < \varepsilon \Rightarrow$

$$U(t)EU(-t)\cdot F = F\cdot U(t)EU(-t).$$

Then $\forall\ t$,

$$FU(t)EU(-t) = 0.$$

To prove this, introduce

$$e^{-H}\ FU(t)EU(-t)e^{-H}.$$

Due to the assumption on H, e^{-H} is invertible and its range is dense. Given $\psi \in \mathcal{H}$, put

$$f_{\psi}(t) = \langle \psi\ ,\ e^{-H}\ FU(t)EU(-t)e^{-H}\ \psi \rangle\ ,$$

our objective being to establish that f_{ψ} is identically zero.

We may assume that $\varepsilon = 1$. Since

$$U(z) = \int_0^{\infty} e^{\sqrt{-1}\,z\lambda}\ dE_{\lambda}$$

is holomorphic in $\underline{\text{Im}}\ z > 0$, the function defined by

$$f_{\psi}(z) = \begin{cases} \langle \psi,\ e^{-H}FU(z)EU(-z)e^{-H}\ \psi \rangle & (0 < \underline{\text{Im}}\ z < 1) \\ \\ \langle \psi,\ e^{-H}U(z)EU(-z)Fe^{-H}\ \psi \rangle & (-1 < \underline{\text{Im}}\ z < 0) \end{cases}$$

is holomorphic in the unit disk and vanishes at the origin (the boundary values coincide at real points $|x| < 1$ and

$\begin{cases} U(-z)e^{-H} \\ e^{-H}U(z) \end{cases}$ is holomorphic in $\begin{cases} \underline{\text{Im}}\ z < 1 \\ -1 < \underline{\text{Im}}\ z \end{cases}$). Fix $\delta > 0$: $0 < \delta < 1/2$

and for $n > 1$, put

$$\begin{cases} \Phi^+_\psi(z;h) = \langle \psi, e^{-H} F U(z) E_{h_1} \cdots E_{h_n} U(-z) e^{-H} \psi \rangle & (0 < \underline{\mathrm{Im}}\, z < 1) \\ \Phi^-_\psi(z;h) = \langle \psi, e^{-H} U(z) E_{h_1} \cdots E_{h_n} U(-z) F e^{-H} \psi \rangle & (-1 < \underline{\mathrm{Im}}\, z < 0). \end{cases}$$

Here $h=(h_1,\ldots,h_n)$ ($|h_i| < \delta$ & $h_i \neq h_j$ $(i \neq j)$) and $E_{h_i} = U(h_i) E U(-h_i)$.

Obviously,

$$\underline{\lim}_{\substack{y \to 0 \\ y > 0}} \Phi^+_\psi(x+\sqrt{-1}\, y;h) = \underline{\lim}_{\substack{y \to 0 \\ y < 0}} \Phi^-_\psi(x+\sqrt{-1}\, y;h)$$

provided $|x| < 1-\delta$. Therefore the function

$$\Phi_\psi(z;h) = \begin{cases} \Phi^+_\psi(z;h) & (0 < \underline{\mathrm{Im}}\, z < 1) \\ \Phi^-_\psi(z;h) & (-1 < \underline{\mathrm{Im}}\, z < 0) \end{cases}$$

is holomorphic in $|z| < 1-\delta$. An easy calculation shows that

$$\Phi_\psi(z;-h_i) = 0 \quad (i=1,\ldots,n),$$

thus

$$\frac{\Phi_\psi(z;h)}{\prod_{i=1}^{n} (z+h_i)}$$

is also holomorphic for $|z| < 1-\delta$. Since $\|U(z)\| \leq 1$ and $|z+h_i| \geq |z| - |h_i| \geq 1-\delta-\delta = 1-2\delta$ if $|z| = 1-\delta$, it follows from the maximum modulus principle that

$$|\Phi_\psi(z;h)| \leq \frac{1}{(1-2\delta)^n} \left(\prod_{i=1}^{n} |z+h_i| \right) \cdot \|\psi\|^2$$

if $|z| < 1-\delta$. Now let $\delta \to 0$ to get

$$|f_\psi(z)| \leq |z|^n \cdot \|\psi\|^2 \quad (|z| < 1).$$

But n is arbitrary, hence f_ψ vanishes identically in the unit disk.

Finally, f_ψ is actually holomorphic in $\begin{cases} 0 \leq \underline{\text{Im}}\, z < 1 \\ -1 < \underline{\text{Im}}\, z \leq 0 \end{cases}$ less

$]-\infty,-1] \cup [+1,+\infty[$, so f_ψ vanishes identically in this region as well. Therefore

$$0 = f_\psi(t \pm \sqrt{-1}\, 0) = f_\psi(t),$$

from which the result.

Remark: This proof makes no use of the assumption that $E,F \in \mathfrak{B}(\mathcal{H})$ are projections.

with a unique vacuum.

The Schlieder Property Suppose given a weakly additive PTV Λ. Fix O,P for which $\exists\ \varepsilon > 0$: $\|a\| < \varepsilon \Rightarrow O + a \perp P$. Take nonzero projections $E \in \mathfrak{M}(O)$, $F \in \mathfrak{M}(P)$ -- then their product EF is nonzero.

To prove this, assume instead that EF=0. Consider the one parameter group of time translations: $U(t) = e^{\sqrt{-1}\, tH}$. Since H is ≥ 0, Borchers theorem is applicable, thus $\forall$ t,

$$FU(t)EU(-t)=0$$

or, since the situation is symmetric,

$$EU(t)FU(-t)=0$$

$$\Rightarrow$$

$$EU(t)F=0$$

$$\Rightarrow$$

$$E\hat{f}(H)F=0,$$

where $f \in L^1(\underset{\sim}{R})$ and $\underline{\text{spt}}\ \hat{f}$ is compact. Choose $\psi_0 : F\psi_0 \neq 0$ and choose $f : \hat{f}(H)F\psi_0 \neq 0$. Put $\Psi_0 = \hat{f}(H)F\psi_0$ -- then Ψ_0 is analytic for the energy and $E\Psi_0 = 0$. But Ψ_0 is separating for $\mathfrak{M}(O)$, hence E=0, a contradiction.

[Note: The Paley-Wiener space contains an approximate identity (i.e., $\exists\ \{f_n\} : f_n \rightarrow \delta$), thus $\hat{f}_n(H) \rightarrow I$ weakly, so $\forall\ \psi$, $\psi' \in \mathcal{H}$, we have

$$\langle \hat{f}_n(H)\psi\ ,\ \psi' \rangle$$

$$= \int_{-\infty}^{\infty} f_n(t) \langle U(t)\psi\ ,\ \psi' \rangle\, dt$$

$$\rightarrow \langle U(0)\psi\ ,\ \psi' \rangle = \langle \psi, \psi' \rangle$$

$\Rightarrow\ \exists\ n$:

$$\hat{f}_n(H)\psi \neq 0 \quad (\psi \neq 0).]$$

with a unique vacuum.

The Borchers Property Suppose given a weakly additive PTV Λ. Fix O,P:

$$O \subset P \text{ and } \underline{\text{dis}}(O, \underline{\text{fr}}\ P) > 0$$

and for which $\exists\, O_0$:

$$O_0 \subset O^{\perp} \cap P.$$

Then $\forall$ nonzero projection $E \in \mathfrak{M}(O)$, $\exists$ a partial isometry $V \in \mathfrak{M}(P)$ such that $V^*V = I$ & $VV^* = E$.

Remark: If O,P are both double cones, then

$$O \subset P \text{ and } \underline{\text{dis}}(O, \underline{\text{fr}}\ P) > 0$$

$$\Rightarrow \exists\, O_0 : O_0 \subset O^{\perp} \cap P.$$

In fact O, being a double cone, is connected with $O^{\perp}$, i.e., $\bar{O} \cap \overline{(\bar{O})^{\perp}} \neq \emptyset$, hence (see the causality notes) $(\bar{O})^{\perp} \cap P \neq \emptyset$. But $(\bar{O})^{\perp}$ is open and $(\bar{O})^{\perp} \cap P \subset O^{\perp} \cap P$, so we can take

$$O_0 = (\bar{O})^{\perp} \cap P.$$

Rappel: Let $V \in \mathfrak{B}(\mathcal{H})$ -- then V is said to be a partial isometry if $\exists$ closed subspaces $\mathcal{H}'$ and $\mathcal{H}''$ such that V restricted to $\mathcal{H}'$ is an isometry from $\mathcal{H}'$ onto $\mathcal{H}''$ while V restricted to $\mathcal{H}'^{\perp}$ vanishes identically. One calls $\mathcal{H}'$ the initial space of V, $\mathcal{H}''$ the final space of V. The adjoint V^* is then a partial isometry with initial space $\mathcal{H}''$ and final space $\mathcal{H}'$. Moreover, V^*V is the orthogonal projection of $\mathcal{H}$ onto $\mathcal{H}'$ and VV^* is the orthogonal projection of $\mathcal{H}$ onto $\mathcal{H}''$.

[Note: Any one of the following conditions is necessary and sufficient that $V \in \mathfrak{B}(\mathcal{H})$ be a partial isometry: (i) $VV^*V=V$; (ii) V^*V is a projection; (iii) $V^*VV^*=V^*$; (iv) VV^* is a projection.]

<u>LEMMA</u> Ω_0 is cyclic for $\mathfrak{M}(P)'$ and $E\Omega_0$ is separating for $\mathfrak{M}(P)'$.

[According to Reeh-Schlieder, Ω_0 is separating for $\mathfrak{M}(P)$, i.e., Ω_0 is separating for $(\mathfrak{M}(P)')'$, hence is cyclic for $\mathfrak{M}(P)'$. Fix $O_0 : O_0 \subset O^\perp \cap P \Rightarrow \mathfrak{M}(O_0) \subset \mathfrak{M}(O)'$ & $\mathfrak{M}(O_0) \subset \mathfrak{M}(P)$. Suppose that $M'E\Omega_0 = 0$, where $M' \in \mathfrak{M}(P)'$ -- then $\forall M \in \mathfrak{M}(O_0)$, $M'EM\Omega_0 = M'ME\Omega_0 = MM'E\Omega_0 = 0$. But Ω_0 is cyclic for $\mathfrak{M}(O_0)$ (Reeh-Schlieder again), so $M'E=0$. Therefore $M'=0$ (apply the Schlieder property), which proves that $E\Omega_0$ is separating for $\mathfrak{M}(P)'$.]

[Note: Since $E \in \mathfrak{M}(O) \subset \mathfrak{M}(P)$, we have $M'E=EM'$, hence $M'E=0 \Rightarrow EM'=0$. This said, to draw the conclusion that $M'=0$, look at the argument used to establish the Schlieder property. Here, $\|a\| < \varepsilon \Rightarrow O + a \subset P \Rightarrow U(a)EU(-a) \in \mathfrak{M}(P)$. In particular: $|t| < \varepsilon \Rightarrow U(t)EU(-t) \in \mathfrak{M}(P) \Rightarrow M' \cdot U(t)EU(-t) = U(t)EU(-t) \cdot M'$, which sets the stage for an application of the Borchers theorem.]

Define now a positive linear functional on $\mathfrak{M}(P)'$ by the prescription

$$M' \to \langle E\Omega_0, M'E\Omega_0 \rangle.$$

Because

$$\begin{aligned} &\langle E\Omega_0, M'^*M'E\Omega_0 \rangle \\ &\quad = \langle M'E\Omega_0, M'E\Omega_0 \rangle \\ &\quad > 0 \end{aligned}$$

if $M' \neq 0$, this functional is faithful (by the lemma, $E\Omega_0$ is separating for $\mathfrak{M}(P)'$), so $\exists$ a vector Ψ_0 cyclic for $\mathfrak{M}(P)'$ such that $\forall M' \in \mathfrak{M}(P)'$,

$$\langle E\Omega_0, M'E\Omega_0\rangle = \langle \Psi_0, M'\Psi_0\rangle \ .$$

Definition: Write

$$VM'\Psi_0 = M'E\Omega_0.$$

[Note: This makes sense. In fact, $M'\Psi_0 = 0 \Rightarrow \langle E\Omega_0, M'E\Omega_0\rangle = 0 \Rightarrow M' = 0$ (a faithful state is injective.]

To see that $V \in \mathfrak{M}(P)$, let $N' \in \mathfrak{M}(P)'$ -- then $N'VM'\Psi_0 = N'M'E\Omega_0$. On the other hand, $VN'M'\Psi_0 = N'M'E\Omega_0$. Therefore $N'V = VN' \Rightarrow V \in (\mathfrak{M}(P)')' = \mathfrak{M}(P)$.

<u>V*V=I</u>: We have

$$\begin{aligned} &\langle V^*VM'\Psi_0, N'\Psi_0\rangle \\ &= \langle V^*M'E\Omega_0, N'\Psi_0\rangle \\ &= \langle M'E\Omega_0, VN'\Psi_0\rangle \\ &= \langle M'E\Omega_0, N'E\Omega_0\rangle \\ &= \langle E\Omega_0, M'^*N'E\Omega_0\rangle \\ &= \langle \Psi_0, M'^*N'\Psi_0\rangle \\ &= \langle M'\Psi_0, N'\Psi_0\rangle \end{aligned}$$

$\Rightarrow$

$$V^*V=I.$$

<u>VV*=E</u>: We have

$$V^*VM'\Psi_0 = V^*M'E\Omega_0 = M'\Psi_0$$

$\Rightarrow$

$$VV^*M'E\Omega_0 = VM'\Psi_0 = M'E\Omega_0.$$

I.e.:

$$VV^*EM'\Omega_0 = EM'\Omega_0$$

$$\Longrightarrow$$

$$VV^*E = E,$$

Ω_0 being cyclic for $\mathcal{M}(P)'$. Since V is a partial isometry, VV^* is a projection, thus

$$E \leq VV^*.$$

To establish equality, it suffices to show that the range of VV^* is contained in the range of E or still, that $EVx = Vx \ \forall\, x \in \mathcal{H}$. Indeed,

$$\begin{aligned} EVM'\psi_0 &= EM'E\Omega_0 \\ &= M'E^2\Omega_0 \\ &= M'E\Omega_0 \\ &= VM'\psi_0 \end{aligned}$$

$$\Longrightarrow$$

$$EV = V$$

$$\Longrightarrow$$

$$EVV^* = VV^*$$

$$\Longrightarrow$$

$$VV^* \leq E.$$

Simplicity of the Quasilocal Algebra Suppose given a weakly additive PTV with a unique vacuum -- then the quasilocal algebra

$$\mathfrak{A} = C^* \left(\bigcup_{O} \mathfrak{M}(O) \right)$$

is simple, i.e., has no nontrivial closed ideals.

Thus let $\mathfrak{I} \neq \{0\}$ be a closed ideal of $\mathfrak{A}$, $\pi : \mathfrak{A} \rightarrow \mathfrak{A}/\mathfrak{I}$ the canonical projection.

LEMMA $\exists\, O$, $\mathfrak{I} \cap \mathfrak{M}(O) \neq \{0\}$.

[Assume the opposite -- then $\forall\, O$, the restriction of π to $\mathfrak{M}(O)$ is one-to-one, hence isometric, so by continuity, π is an isomorphism, which implies that $\mathfrak{I} = \{0\}$, a contradiction.]

We can, of course, take the O of the lemma to be a double cone. This done, fix another double cone P:

$$O \subset P \text{ and } \underline{\text{dis}}(O, \underline{\text{fr}}\, P) > 0.$$

Choose a positive selfadjoint $M \in \mathfrak{I} \cap \mathfrak{M}(O)$:

$$M = \int_0^{\|M\|} \lambda \, dE_\lambda .$$

Consider the projection

$$E = \int_\varepsilon^{\|M\|} dE_\lambda \quad (\varepsilon > 0).$$

Since $E \in \mathfrak{M}(O)$, $\exists$ a partial isometry $V \in \mathfrak{M}(P)$ such that $V^*V=I$ & $VV^*=E$ (Borchers).

LEMMA We have $M \geq \varepsilon E$.

[In fact, $\forall\, \psi \in \mathcal{H}$,

$$\langle \psi , M\psi \rangle = \int_0^{\|M\|} \lambda \; d\langle \psi , E_\lambda \psi \rangle$$

$$= \int_0^{\varepsilon} \lambda \, d\langle \psi , E_\lambda \psi \rangle + \int_{\varepsilon}^{\|M\|} \lambda \; d\langle \psi , E_\lambda \psi \rangle$$

$$\geq \int_{\varepsilon}^{\|M\|} \lambda \, d\langle \psi , E_\lambda \psi \rangle$$

$$\geq \varepsilon \int_{\varepsilon}^{\|M\|} d\langle \psi , E_\lambda \psi \rangle$$

$$= \varepsilon \langle \psi , E \psi \rangle = \langle \psi , \varepsilon E \psi \rangle .]$$

Accordingly,

$$V^*MV \geq \varepsilon V^*EV.$$

But

$$VV^* = E \Rightarrow V^*VV^* = V^*E$$

$$\Rightarrow V^*VV^*V = V^*EV$$

$$\Rightarrow I = V^*EV.$$

Therefore

$$V^*MV \geq \varepsilon I,$$

which implies that $V^*MV \in \mathfrak{S}$ is invertible. But this means that $\mathfrak{S} = \mathfrak{A}$.

<u>The Totality Lemma</u> Suppose given a weakly additive PTV with a unique vacuum. Fix a nonzero projection $E_0 \in \mathfrak{M}(O_0)$ -- then the set

$$\{ U(I,a)E_0U(I,-a)\psi : a \in \underset{\sim}{R}^{1,d}, \ \psi \in \mathcal{H} \}$$

is total in $\mathcal{H}$.

To prove this, put $U(a) = U(I,a)$ and suppose $\langle \Psi_0, U(a)E_0U(-a)\Psi \rangle = 0 \ \forall a \ \& \ \forall \Psi$ -- then we have to prove that $\Psi_0 = 0$ or, recast, we have to prove that $E_a\Psi_0 = 0 \ \forall a \Rightarrow \Psi_0 = 0$, where $E_a = U(a)E_0U(-a)$. Let P_0 be the orthogonal projection of $\mathcal{H}$ onto $\{\Psi_0 : E_a\Psi_0 = 0 \ \forall a\}$. Fix O: $\exists \ \varepsilon > 0$ for which $\|a\| < \varepsilon \Rightarrow O_0 + a \perp O$ -- then

$$E_aM = ME_a \quad (M \in \mathfrak{M}(O), \ \|a\| < \varepsilon).$$

Let P be the orthogonal projection of $\mathcal{H}$ onto $\overline{\mathfrak{M}(O)P_0\mathcal{H}}$. Obviously, $\|a\| < \varepsilon \Rightarrow E_aP = 0$, thus by an analytic continuation argument, $E_aP = 0 \ \forall a$, so $P \leq P_0$. On the other hand, it is clear that $P_0 \leq P$. Therefore

$$P = P_0.$$

But $P \in \mathfrak{M}(O)'$. Indeed, $\overline{\mathfrak{M}(O)P_0\mathcal{H}}$ is invariant w.r.t. $\mathfrak{M}(O)$, hence $\forall \ M \in \mathfrak{M}(O)$, $MP = PMP \Rightarrow PM^* = PM^*P$. Since $\mathfrak{M}(O)$ is generated by its selfadjoint elements, it follows that $\forall M \in \mathfrak{M}(O)$, $MP = PM$, i.e., $P \in \mathfrak{M}(O)'$ or still, $P_0 \in \mathfrak{M}(O)'$. To finish the proof, we shall show that $P_0 = 0$. First, $E_aP_0\Omega_0 = 0$, hence $\forall \ a \neq 0$,

$$0 = \langle \Omega_0, E_aP_0\Omega_0 \rangle$$

$$= \langle \Omega_0, U(a)E_0U(-a)P_0\Omega_0 \rangle$$

$$= \langle E_0\Omega_0, U(-a)P_0\Omega_0 \rangle$$

$\Longrightarrow$

$$0 = \varliminf_{t \to +\infty} \langle E_0\Omega_0, U(-ta)P_0\Omega_0 \rangle$$

$$= \langle E_0\Omega_0, \langle\Omega_0, P_0\Omega_0\rangle \Omega_0 \rangle$$

$$= \langle \Omega_0, P_0\Omega_0 \rangle \langle E_0\Omega_0, \Omega_0 \rangle$$

$$= \| P_0\Omega_0 \|^2 \| E_0\Omega_0 \|^2.$$

But $E_0 \neq 0 \Longrightarrow E_0\Omega_0 \neq 0$, Ω_0 being separating for $\mathfrak{M}(O_0)$. This means that $P_0\Omega_0 = 0$. But Ω_0 is separating for $\mathfrak{M}(O)'$ (being cyclic for $\mathfrak{M}(O)$), which implies that $P_0 = 0$, as contended.

<u>Uniqueness of the Translation Representation</u> Suppose given a weakly additive PTV -- then PT_3 provides us with a unitary representation U of $\widetilde{\mathcal{P}}_+^\uparrow$ on $\mathcal{H}$ such that

$$U(\widetilde{\Lambda},a)\mathfrak{M}(O)U(\widetilde{\Lambda},a)^{-1} = \mathfrak{M}((\widetilde{\Lambda},a)\cdot O),$$

where

$$\underline{\text{spec}}(U \mid \underset{\sim}{R}^{1,d}) \subset \overline{V}_+.$$

Question: Does the assignment $O \rightarrow \mathfrak{M}(O)$ determine U uniquely? While the answer in general is "no", what can be said is this: The restriction $U \mid \underset{\sim}{R}^{1,d}$ is unique.

The proof depends on the following considerations.

Definition: An <u>inner symmetry</u> of a PTV is a unitary operator $\mathfrak{T}:\mathcal{H} \rightarrow \mathcal{H}$ such that $\mathfrak{T}\Omega_0 = \Omega_0$ and

$$\mathfrak{T}\,\mathfrak{M}(O)\,\mathfrak{T}^{-1} = \mathfrak{M}(O) \quad \forall\, O.$$

Example: Let U_1, U_2 be two unitary representations of $\widetilde{\mathcal{P}}_+^\uparrow$ on $\mathcal{H}$ attached to a PTV. Consider the restrictions $\begin{cases} U_1 \mid \underset{\sim}{R}^{1,d} \\ U_2 \mid \underset{\sim}{R}^{1,d} \end{cases}$ -- then for any inner symmetry $\mathfrak{T}$,

$$\mathfrak{T}_a = U_1(I,a)\,\mathfrak{T}\,U_2(I,-a)$$

is again an inner symmetry.

THEOREM Suppose given a weakly additive PTV -- then every inner symmetry $\mathfrak{T}$ intertwines $\begin{cases} U_1 \mid \underset{\sim}{R}^{1,d} \\ U_2 \mid \underset{\sim}{R}^{1,d} \end{cases}$, i.e., $\forall\, a$,

$$U_1(I,a)\zeta = \zeta\, U_2(I,a).$$

Application: Choose $\zeta = I$ to see that $\forall a$,

$$U_1(I,a) = U_2(I,a).$$

To prove the theorem, take an $f \in L^1(\underset{\sim}{R}^{1,d})$ and let

$$A = \int_{\underset{\sim}{R}^{1,d}} f(x)U_1(I,x)\zeta\, U_2(I,-x)dx.$$

Write

$$\begin{cases} U_1(I,x) = \displaystyle\int_{\underset{\sim}{R}^{1,d}} e^{\sqrt{-1}\langle x,p\rangle}\, dE_p \\ U_2(I,x) = \displaystyle\int_{\underset{\sim}{R}^{1,d}} e^{\sqrt{-1}\langle x,q\rangle}\, dF_q \end{cases}$$

and, ignoring constants, put

$$\hat{f}(x) = \int_{\underset{\sim}{R}^{1,d}} e^{\sqrt{-1}\langle x,y\rangle}\, f(y)dy.$$

Then

$$A = \iint_{\Delta_f} \hat{f}(p-q)dE_p\,\zeta\, dF_q,$$

where

$$\Delta_f = \{(p,q):\ p\in \overline{V}_+, q\in \overline{V}_+, p-q \in \underline{\text{spt}}\ \hat{f}\}.$$

(B) Let

$$B(q) = \int_{\Delta_f(B)} \hat{f}(p-q)dE_p,$$

where

$$\Delta_f(B) = \{ p : p \in \overline{V}_+, p-q \in \underline{\text{spt}}\ \hat{f} \}.$$

Then

$$A = \int_{\Delta_{1,f}} B(q) \zeta \, dF_q.$$

Here

$$\Delta_{1,f} = \overline{V}_+ \cap (\overline{V}_+ - \underline{\text{spt}}\ \hat{f}).$$

(C) Let

$$C(p) = \int_{\Delta_f(C)} \hat{f}(p-q) dF_q,$$

where

$$\Delta_f(C) = \{ q : q \in \overline{V}_+, p-q \in \underline{\text{spt}}\ \hat{f} \}.$$

Then

$$A = \int_{\Delta_{2,f}} dE_p \zeta C(p).$$

Here

$$\Delta_{2,f} = \overline{V}_+ \cap (\overline{V}_+ + \underline{\text{spt}}\ \hat{f}).$$

Given $M_1, M_2 \in \mathcal{M}(O)$, introduce

$$F_1(x,y) = \langle \Omega_0, \zeta_y M_1 \zeta_y^{-1} U_1(I,x) M_2 U_1(I,-x) \Omega_0 \rangle$$

$$= \langle M_1^* \Omega_0, \zeta_y^{-1} U_1(I,x) M_2 \Omega_0 \rangle$$

and

$$F_2(x,y) = \langle \Omega_0, U_1(I,x) M_2 U_1(I,-x) \zeta_y M_1 \zeta_y^{-1} \Omega_0 \rangle$$

$$= \langle M_2^* \Omega_0, U_1(I,-x) \zeta_y M_1 \Omega_0 \rangle.$$

Now multiply through by f(y) and then integrate w.r.t. y. After some manipulation, we find that

$$F_1(x;f) \equiv \int_{\underset{\sim}{R}^{1,d}} F_1(x,y)f(y)dy$$

$$= \int_{\overline{V}_+} e^{\sqrt{-1}\langle x,p\rangle} \langle M_1^* \Omega_0, \int_{\underset{\sim}{R}^{1,d}} f(y)\zeta_y^{-1}\, dy dE_p M_2 \Omega_0 \rangle$$

and

$$F_2(x;f) \equiv \int_{\underset{\sim}{R}^{1,d}} F_2(x,y)f(y)dy$$

$$= \int_{\overline{V}_-} e^{\sqrt{-1}\langle x,q\rangle} \langle M_2^* \Omega_0, dE_{-q} \int_{\underset{\sim}{R}^{1,d}} f(y)\zeta_y dy M_1 \Omega_0 \rangle$$

or still,

$$F_1(x;f) = \int_{\overline{V}_+} e^{\sqrt{-1}\langle x,p\rangle} d\mu_{1,f}(p)$$

and

$$F_2(x;f) = \int_{\overline{V}_-} e^{\sqrt{-1}\langle x,q\rangle} d\mu_{2,f}(q),$$

where for Borel sets Δ :

$$\mu_{1,f}(\Delta) = \langle M_1^* \Omega_0, \int_{\underset{\sim}{R}^{1,d}} f(y)\zeta_y^{-1}\, dy\, E(\Delta)M_2 \Omega_0 \rangle$$

and

$$\mu_{2,f}(\Delta) = \langle M_2^* \Omega_0, E(-\Delta) \int_{\underset{\sim}{R}^{1,d}} f(y)\zeta_y dy M_1 \Omega_0 \rangle .$$

Using the formulas for A in terms of B and C, it is not difficult to check that

$$\begin{cases} \underline{\text{spt}}\,\mu_{1,f} \subset \Delta_{1,f} \\ \underline{\text{spt}}\,\mu_{2,f} \subset -\Delta_{2,f}. \end{cases}$$

At this point, f is an arbitrary L^1-function, a fact which we shall take advantage of in a moment. But first let's indicate how the proof of the theorem is going to be concluded.

The function

$$y \longrightarrow \langle M_2^* \Omega_0, \zeta_y M_1 \Omega_0 \rangle$$

$$= \langle M_2^* \Omega_0, U_1(I,y) \zeta U_2(I,-y) M_1 \Omega_0 \rangle$$

is bounded and continuous, thus defines a tempered distribution T. The machinery developed above will then be employed to establish that the support of $\check{T}$ is the origin, so $\check{T}$ is a finite linear combination of derivatives of the Dirac delta. Our function is therefore a polynomial in y, hence is a constant (being bounded). I.e.: $\forall$ y,

$$\langle M_2^* \Omega_0, U_1(I,y) \zeta U_2(I,-y) M_1 \Omega_0 \rangle$$

$$= \langle M_2^* \Omega_0, \zeta M_1 \Omega_0 \rangle .$$

By Reeh-Schlieder, the $M_2^* \Omega_0$ are dense in $\mathcal{H}$, hence

$$U_1(I,y) \zeta U_2(I,-y) M_1 \Omega_0 = \zeta M_1 \Omega_0.$$

But again by Reeh-Schlieder, the $M_1 \Omega_0$ are dense in $\mathcal{H}$, hence

$$U_1(I,y) \zeta U_2(I,-y) = \zeta$$

or still,

$$U_1(I,y) \zeta = \zeta U_2(I,y),$$

as desired.

[Note: O will be suitably specialized below.]

Consider now any double cone

$$D(a,b) = V_+(a) \cap V_-(b) \quad (b \in V_+(a))$$

subject to

$$0 \notin D(a,b).$$

Work with any f: $\underline{\mathrm{spt}}\ \hat{f} \subset D(a,b)$. Recalling that

$$\begin{cases} \Delta_{1,f} = \bar{V}_+ \cap (\bar{V}_+ - \underline{\mathrm{spt}}\ \hat{f}) \\ \Delta_{2,f} = \bar{V}_+ \cap (\bar{V}_+ + \underline{\mathrm{spt}}\ \hat{f}), \end{cases}$$

we have

$$\begin{aligned} \Delta_{1,f} &\subset \bar{V}_+ \cap \{\bar{V}_+ - D(a,b)\} \\ &= \bar{V}_+ \cap \bar{V}_+ (b_0) \qquad (b_0 = -b) \end{aligned}$$

and

$$\begin{aligned} -\Delta_{2,f} &\subset \bar{V}_- \cap \{\bar{V}_- - D(a,b)\} \\ &= \bar{V}_- \cap \bar{V}_- (a_0) \qquad (a_0 = -a). \end{aligned}$$

In this connection, note that

$$\begin{aligned} -D(a,b) &= V_-(-a) \cap V_+(-b) \\ &= D(b_0,a_0). \end{aligned}$$

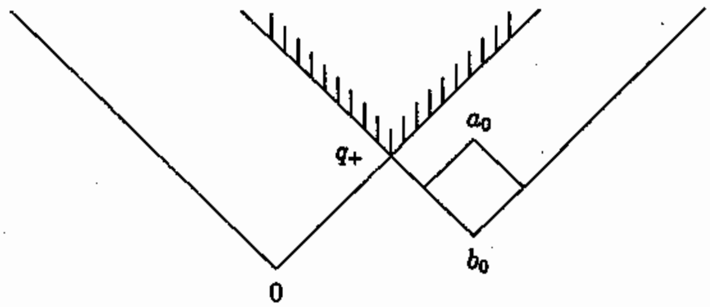
Therefore

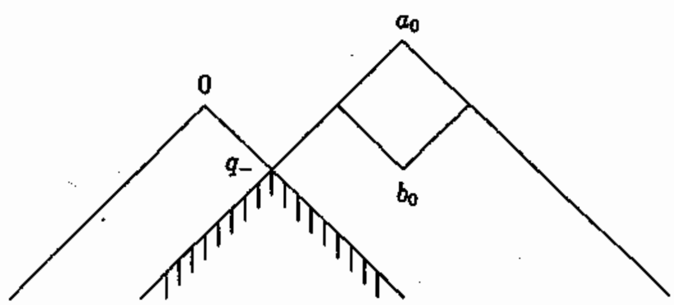
$$\begin{cases} \underline{\mathrm{spt}}\ \mu_{1,f} \subset \Delta_{1,f} \subset \bar{V}_+ (q_+) \\ \underline{\mathrm{spt}}\ \mu_{2,f} \subset -\Delta_{2,f} \subset \bar{V}_- (q_-), \end{cases}$$

where

$$\bar{V}_+(q_+) \cap \bar{V}_-(q_-) = \emptyset.$$

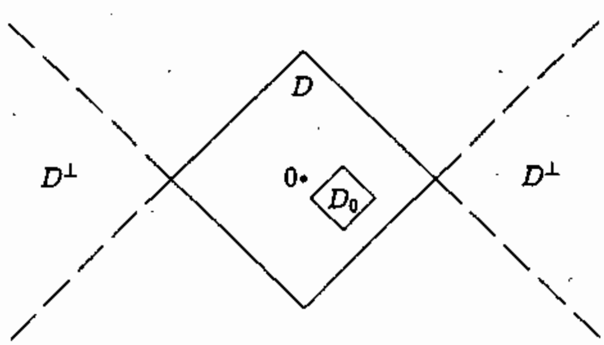
Picture:





The last step is to select O judiciously. Start by choosing a double cone D centered at the origin and then take for O a double cone $D_0 \subset D$: $(D_0 + x) \perp D_0 \ \forall \ x \in D^{\perp}$.

Picture:



Since ζ_y is an inner symmetry,

$$\zeta_y \mathfrak{M}(0) \zeta_y^{-1} = \mathfrak{M}(0)$$

$$\Rightarrow$$

$$\zeta_y M_1 \zeta_y^{-1} \in \mathfrak{M}(0) \quad \forall M_1.$$

On the other hand, $\forall x \in D^{\perp}$,

$$U_1(I,x) \mathfrak{M}(0) U_1(I,-x) = \mathfrak{M}(0 + x)$$

and, by construction, the elements of $\mathfrak{M}(0)$ commute with those of $\mathfrak{M}(0 + x)$. Therefore

$$\zeta_y M_1 \zeta_y^{-1} \cdot U_1(I,x) M_2 U_1(I,-x)$$

$$= U_1(I,x) M_2 U_1(I,-x) \cdot \zeta_y M_1 \zeta_y^{-1}$$

$$\Rightarrow$$

$$F_1(x,y) = F_2(x,y) \quad (x \in D^{\perp}, y \in \underset{\sim}{R}^{1,d})$$

$$\Rightarrow$$

$$F_1(x;f) = F_2(x;f) \quad (x \in D^{\perp}).$$

Let

$$\begin{cases} \mu_f(\Delta) = \mu_{1,f}(\Delta) - \mu_{2,f}(\Delta) \\ F(x;f) = F_1(x;f) - F_2(x;f). \end{cases}$$

Then

$$F(x;f) = \int_{\underset{\sim}{R}^{1,d}} e^{\sqrt{-1}\langle x,p \rangle} d\mu_f(p).$$

Moreover,

$$F(x;f) = 0 \quad (x \in D^{\perp})$$

and

$$\underline{\text{spt}}\, \mu_f \subset \overline{V_+(q_+)} \cup \overline{V_-(q_-)}.$$

LEMMA Let

$$F(x) = \int_{\mathbb{R}^{1,d}} e^{\sqrt{-1}\langle x,p\rangle} \, d\mu(p)$$

be the Fourier transform of a complex measure μ of finite total variation with

$$\underline{\text{spt}}\, \mu \subset \overline{V_+(q_+)} \cup \overline{V_-(q_-)}.$$

Assume that

$$F(x) = 0 \quad (x \in D^{\perp}).$$

Then

$$\mu = 0.$$

Accordingly,

$$\mu_f = 0$$

$$\Rightarrow$$

$$\mu_{1,f} = \mu_{2,f}$$

$$\Rightarrow \mu_{1,f} = 0 \ \& \ \mu_{2,f} = 0,$$

the last step because the supports of $\mu_{1,f}$ and $\mu_{2,f}$ are disjoint.

In particular:

$$0 = \mu_{2,f}(\mathbb{R}^{1,d})$$

$$= \langle M_2^* \Omega_0, \int_{\underset{\sim}{R}^{1,d}} f(y) \zeta_y dy M_1 \Omega_0 \rangle$$

$$= \int_{\underset{\sim}{R}^{1,d}} f(y) \langle M_2^* \Omega_0, U_1(I,y) \zeta U_2(I,-y) M_1 \Omega_0 \rangle dy$$

$$= \langle T, f \rangle .$$

Suppose that $\phi \in C_c^\infty(\underset{\sim}{R}^{1,d} - \{0\})$ is arbitrary. Choose D(a,b): $\underline{\text{spt}}\ \phi \subset D(a,b)$ $(0 \notin D(a,b))$ (this is permissible (use a partition of unity argument)). Write $\phi = \hat{f}$ -- then

$$\langle \check{T}, \phi \rangle = \langle T, \check{\phi} \rangle$$

$$= \langle T, \check{\hat{f}} \rangle$$

$$= \langle T, f \rangle$$

$$= 0$$

$\Longrightarrow$

$$\underline{\text{spt}}\ \check{T} = \{0\} .$$

<u>Inner Symmetries</u> Suppose given a weakly additive PTV -- then the <u>gauge group</u> G of the theory is its group of inner symmetries, i.e., the unitary operators $\mathfrak{I}: \mathcal{H} \rightarrow \mathcal{H}$ such that $\mathfrak{I}\Omega_0 = \Omega_0$ and

$$\mathfrak{I}\,\mathfrak{M}(O)\,\mathfrak{I}^{-1} = \mathfrak{M}(O) \quad \forall\, O.$$

Rappel: Let U_1, U_2 be two unitary representations of $\widetilde{\mathcal{P}}_+^\uparrow$ on $\mathcal{H}$ attached to the theory -- then every $\mathfrak{I} \in G$ intertwines $\begin{cases} U_1 \mid \underset{\sim}{R}^{1,d} \\ U_2 \mid \underset{\sim}{R}^{1,d} \end{cases}$, i.e., $\forall$ a,

$$U_1(I,a)\mathfrak{I} = \mathfrak{I}\,U_2(I,a).$$

Taking $\mathfrak{I} = I$, it follows that

$$U_1(I,a) = U_2(I,a) \quad \forall\, a.$$

However, it need not be true that

$$U_1(\widetilde{\Lambda},a) = U_2(\widetilde{\Lambda},a) \quad \forall\, (\widetilde{\Lambda},a).$$

Definition: G is said to satisfy the <u>gauge condition</u> if G is compact in the strong operator topology and commutes with the $U(\widetilde{\Lambda},a)$:

$$\mathfrak{I}\,U(\widetilde{\Lambda},a) = U(\widetilde{\Lambda},a)\mathfrak{I} \quad (\mathfrak{I} \in G)$$

for any U fixing Ω_0 such that

$$U(\widetilde{\Lambda},a)\,\mathfrak{M}(O)\,U(\widetilde{\Lambda},a)^{-1} = \mathfrak{M}((\widetilde{\Lambda},a)\cdot O) \quad \forall\, O.$$

Remark: There are conditions which guarantee that G satisfies the gauge condition, one being the <u>split property</u>: $\mathfrak{M}$ is not abelian and $\forall$ O,P:

$$O \subset P \text{ and } \underline{\text{dis}}(O, \underset{\sim}{\text{fr}}\, P) > 0,$$

$\exists$ a type I factor between $\mathfrak{M}(0)$ and $\mathfrak{M}(P)$.

[Note: Recall that $\mathfrak{M}(0)$ is necessarily a proper subset of $\mathfrak{M}(P)$.]

LEMMA Suppose that G satisfies the gauge condition -- then U is unique.

[Given U_1, U_2 fixing Ω_0 such that $\forall$ 0,

$$\left\{\begin{array}{l} U_1(\tilde{\Lambda},a)\,\mathfrak{M}(0)U_1(\tilde{\Lambda},a)^{-1} = \mathfrak{M}((\tilde{\Lambda},a)\cdot 0) \\ \\ U_2(\tilde{\Lambda},a)\,\mathfrak{M}(0)U_2(\tilde{\Lambda},a)^{-1} = \mathfrak{M}((\tilde{\Lambda},a)\cdot 0) \end{array}\right. \quad \forall\, (\tilde{\Lambda},a),$$

it is clear that

$$U_1(\tilde{\Lambda},a)U_2(\tilde{\Lambda},a)^{-1} \in G.$$

This said, define a unitary representation U of $\tilde{\mathcal{P}}_+^\uparrow$ on $\mathcal{H}$ by

$$(\tilde{\Lambda},a) \longrightarrow U_1(\tilde{\Lambda},a)U_2(\tilde{\Lambda},a)^{-1}.$$

Thus, on the one hand,

$$\begin{aligned} &U((\tilde{\Lambda}_1,a_1)(\tilde{\Lambda}_2,a_2)) \\ &= U_1((\tilde{\Lambda}_1,a_1)(\tilde{\Lambda}_2,a_2))U_2((\tilde{\Lambda}_2,a_2)^{-1}(\tilde{\Lambda}_1,a_1)^{-1}) \\ &= U_1(\tilde{\Lambda}_1,a_1)U_1(\tilde{\Lambda}_2,a_2)U_2(\tilde{\Lambda}_2,a_2)^{-1}U_2(\tilde{\Lambda}_1,a_1)^{-1} \end{aligned}$$

while, on the other,

$$\begin{aligned} &U(\tilde{\Lambda}_1,a_1)U(\tilde{\Lambda}_2,a_2) \\ &= U_1(\tilde{\Lambda}_1,a_1)U_2(\tilde{\Lambda}_1,a_1)^{-1}\left(U_1(\tilde{\Lambda}_2,a_2)U_2(\tilde{\Lambda}_2,a_2)^{-1}\right) \\ &= U_1(\tilde{\Lambda}_1,a_1)\left(U_1(\tilde{\Lambda}_2,a_2)U_2(\tilde{\Lambda}_2,a_2)^{-1}\right)U_2(\tilde{\Lambda}_1,a_1)^{-1}. \end{aligned}$$

Here we have used the fact that

$$U_1(\tilde{\Lambda}_2,a_2)U_2(\tilde{\Lambda}_2,a_2)^{-1} \in G,$$

hence commutes with $U_2(\tilde{\Lambda}_1,a_1)^{-1}$. It follows that U is indeed a homomorphism. Recall now that the group of unitary operators on $\mathcal{H}$ is a topological group in the strong operator topology. Therefore G is a topological group. But G acts on $\mathcal{H}$ by itself: $\Pi(\zeta)x = \zeta x$ $(\zeta \in G)$. Accordingly, thanks to compactness, $\exists$ cardinal numbers n_π $(\pi \in \hat{G})$ such that $\Pi = \bigoplus_{\pi \in \hat{G}} n_\pi \pi$. Since $\tilde{\mathcal{P}}_+^\uparrow$ has no nontrivial finite dimensional unitary representations, the action of $\tilde{\mathcal{P}}_+^\uparrow$ on each π is trivial. This in turn implies that the action of $\tilde{\mathcal{P}}_+^\uparrow$ on all of $\mathcal{H}$ is trivial:

$$\begin{aligned} U(\tilde{\Lambda},a) &= U_1(\tilde{\Lambda},a)U_2(\tilde{\Lambda},a)^{-1} \\ &= I \end{aligned}$$

$$\Rightarrow$$

$$U_1(\tilde{\Lambda},a) = U_2(\tilde{\Lambda},a).]$$

Remark: Since G is compact, the assumption that

$$\zeta U(\tilde{\Lambda},a) = U(\tilde{\Lambda},a)\zeta \quad (\zeta \in G)$$

is actually automatic.

Tools from Harmonic Analysis Let G be a LCA ~~abelian~~ group, U a unitary representation of G on $\mathcal{H}$ -- then the generalization of Stone's theorem to G is the assertion that $\exists$ a projection valued measure E on the Borel subsets of the dual $\hat{G}$ such that $\forall\ \sigma \in G$,

$$U(\sigma) = \int_{\hat{G}} \langle \sigma, \hat{\sigma} \rangle \, dE(\hat{\sigma}),$$

the spectrum of U being by definition the support of E.

Remark: $\forall\ f \in L^1(G)$, we have

$$\begin{aligned}
&\int_G f(\sigma)U(\sigma)d\sigma \\
&\quad = \int_G f(\sigma)\left(\int_{\hat{G}} \langle \sigma, \hat{\sigma} \rangle \, dE(\hat{\sigma})\right) d\sigma \\
&\quad = \int_{\hat{G}} \left(\int_G f(\sigma)\langle \sigma, \hat{\sigma} \rangle \, d\sigma\right) dE(\hat{\sigma}) \\
&\quad = \int_{\hat{G}} \hat{f}(\hat{\sigma}) dE(\hat{\sigma}).
\end{aligned}$$

Suppose now that $\mathfrak{M}$ is a W*-algebra. Make the following assumptions:

(1) $\forall\ \sigma \in G$, $U(\sigma)\mathfrak{M}U(\sigma)^{-1} \subset \mathfrak{M}$;

(2) $\exists\ \Omega_0 \in \mathcal{H}$ ($\|\Omega_0\| = 1$):

$$U(\sigma)\Omega_0 = \Omega_0 \quad (\sigma \in G) \ \& \ \overline{\mathfrak{M}\Omega_0} = \mathcal{H} \ ;$$

(3) $(\underline{\text{spec}}\ U) \cap (\underline{\text{spec}}\ U)^{-1} = \{\hat{e}\}$.

Then it can be shown that $U(G) \subset \mathfrak{M}$.

Example: Suppose given a PTV -- then $\forall\ a \in \underset{\sim}{R}^{1,d}$,

$$U(I,a)\mathfrak{M}\,U(I,a)^{-1} \subset \mathfrak{M} .$$

By assumption,

$$\underline{\mathrm{spec}}(U|\mathbb{R}^{1,d}) \subset \overline{V}_+$$

$$\Longrightarrow$$

$$\underline{\mathrm{spec}}(U|\mathbb{R}^{1,d}) \cap -\underline{\mathrm{spec}}(U|\mathbb{R}^{1,d})$$

$$\subset \overline{V}_+ \cap -\overline{V}_+ = \overline{V}_+ \cap \overline{V}_- = \{0\} .$$

Therefore

$$U(\mathbb{R}^{1,d}) \subset \mathfrak{M} .$$

Returning to our W*-algebra $\mathfrak{M}$, fix a unit vector $\Omega_0 \in \mathcal{H}$.

Definition: The centralizer $\mathfrak{M}_{\Omega_0}$ of $\mathfrak{M}$ w.r.t. Ω_0 is

$$\{ A \in \mathfrak{M}: \langle \Omega_0, AM\Omega_0 \rangle = \langle \Omega_0, MA\Omega_0 \rangle \ \forall M \in \mathfrak{M} \} .$$

Supposing still that G is a LCA group and U is a unitary representation of G on $\mathcal{H}$, impose the following conditions on $(\mathfrak{M}, G, U, \Omega_0)$:

(1) Ω_0 is separating for $\mathfrak{M}$;

(2) Ω_0 is invariant for G;

(3) $\mathbb{C}\,\Omega_0 = \mathcal{H}^G$;

(4) $(\underline{\mathrm{spec}}\ U) \cap (\underline{\mathrm{spec}}\ U)^{-1} = \{\hat{e}\}$;

(5) $G = \mathcal{S} \cup \mathcal{S}^{-1}$, where

$$\mathcal{S} = \{ \sigma \in G: U(\sigma)\mathfrak{M}U(\sigma)^{-1} \subset \mathfrak{M} \} .$$

THEOREM Under the preceding conditions,

$$\mathfrak{M}_{\Omega_0} = \mathbb{C}I.$$

[Let $A \in \mathfrak{M}_{\Omega_0}$ -- then $\forall\, \sigma \in \mathcal{S}$ & $\forall\, M \in \mathfrak{M}$, we have

$$\langle \Omega_0, AU(\sigma)MU(\sigma)^{-1}\Omega_0 \rangle$$

$$= \langle \Omega_0, U(\sigma)MU(\sigma)^{-1}A\,\Omega_0 \rangle$$

$\Rightarrow$

$$\langle A^*\Omega_0, U(\sigma)M\,\Omega_0 \rangle$$

$$= \langle M^*\Omega_0, U(\sigma)^{-1}A\,\Omega_0 \rangle .$$

Specialize and take M=A and A=A* (this can be done without loss of generality; see below). Assuming that $A\,\Omega_0 \neq 0$ ($A\,\Omega_0 = 0 \Rightarrow A = 0$ by (1)), put

$$f(\sigma) = \langle A\,\Omega_0, U(\sigma)A\,\Omega_0 \rangle \quad (\sigma \in G).$$

Then

$$\sigma \in \mathcal{S} \Rightarrow f(\sigma) = f(\sigma^{-1}).$$

But $G = \mathcal{S} \cup \mathcal{S}^{-1}$, so

$$f(\sigma) = f(\sigma^{-1}) \qquad \forall\, \sigma \in G.$$

We have

$$f(\sigma) = \langle A\,\Omega_0, \int_{\hat{G}} \langle \sigma, \hat{\sigma} \rangle \, dE(\hat{\sigma})A\,\Omega_0 \rangle$$

$$= \int_{\hat{G}} \langle \sigma, \hat{\sigma} \rangle \langle A\,\Omega_0, dE(\hat{\sigma})A\,\Omega_0 \rangle .$$

Here the assignment

$$\Delta \rightarrow \langle A\,\Omega_0, E(\Delta)A\,\Omega_0 \rangle$$

defines a positive measure μ on $\hat{G}$ of total mass

$$\langle A\,\Omega_0, A\,\Omega_0 \rangle$$

with

$$\underline{\text{spt}}\,\mu = (\underline{\text{spt}}\,\mu)^{-1}.$$

But

$$\underline{\text{spt}}\,\mu \subset \underline{\text{spec}}\ U$$

$\Rightarrow$

$$\underline{\text{spt}}\,\mu = \{\hat{e}\}$$

$\Rightarrow$

$$\mu = K\ \delta_{\hat{e}} \quad (K > 0)$$

$\Rightarrow$

$$f(\sigma) = \int_{\hat{G}} \langle \sigma, \hat{\sigma} \rangle d\mu(\hat{\sigma}) = K\langle \sigma, \hat{e} \rangle = K$$

$\Rightarrow$

$$f(\sigma) = f(e) \qquad \forall\, \sigma \in G$$

$\Rightarrow$

$$\langle A\,\Omega_0, U(\sigma) A\,\Omega_0 \rangle$$

$$= \langle A\,\Omega_0, A\,\Omega_0 \rangle$$

$$= \| A\,\Omega_0 \|^2$$

$$= \| A\,\Omega_0 \| \cdot \| A\,\Omega_0 \|$$

$$= \| U(\sigma) A\,\Omega_0 \| \cdot \| A\,\Omega_0 \|$$

$\Rightarrow$

$$U(\sigma) A\,\Omega_0 = c_\sigma A\,\Omega_0 \qquad (\exists\, c_\sigma)$$

$\Rightarrow$

$$\langle A\,\Omega_0, U(\sigma) A\,\Omega_0 \rangle$$

$$= \langle A\,\Omega_0, c_\sigma\ A\,\Omega_0 \rangle$$

$$= C_\sigma \langle A\Omega_0, A\Omega_0\rangle$$

$$= \langle A\Omega_0, A\Omega_0\rangle$$

$\Rightarrow$

$$C_\sigma = 1$$

$\Rightarrow$

$$A\Omega_0 = C\Omega_0 \quad (\exists C)$$

$\Rightarrow$

$$(A-CI)\Omega_0 = 0$$

$\Rightarrow$

$$A-CI = 0$$

$\Rightarrow$

$$\mathfrak{M}_{\Omega_0} = \underset{\sim}{C}I.]$$

[Note: To explicate the detail omitted above, observe first that $A \in \mathfrak{M}_{\Omega_0} \Rightarrow A^* \in \mathfrak{M}_{\Omega_0}$: $\forall\, M \in \mathfrak{M}$,

$$\langle \Omega_0, A^*M\Omega_0\rangle$$

$$= \langle A\Omega_0, M\Omega_0\rangle$$

$$= \langle M^*A\Omega_0, \Omega_0\rangle$$

$$= \overline{\langle \Omega_0, M^*A\Omega_0\rangle}$$

$$= \overline{\langle \Omega_0, AM^*\Omega_0\rangle}$$

$$= \langle AM^*\Omega_0, \Omega_0\rangle$$

$$= \langle M^*\Omega_0, A^*\Omega_0\rangle$$

$$= \langle \Omega_0, MA^*\Omega_0\rangle.$$

This said, write

$$A = \frac{A+A^*}{2} + \frac{\sqrt{-1}(A-A^*)}{2\sqrt{-1}}$$

$$= E + F.$$

Both E and F are selfadjoint and in $\mathfrak{M}_{\Omega_0}$, hence

$$\left\{ \begin{array}{l} E = C_E I \\ F = C_F I \end{array} \right. \Longrightarrow A = (C_E + C_F)I.]$$

Remark: The case when $G = \mathcal{S}$ can be treated differently. Since Ω_0 is separating for $\mathfrak{M}$, it is cyclic for $\mathfrak{M}'$. And: $\forall\, \sigma \in G$,

$$U(\sigma)\mathfrak{M}U(\sigma)^{-1} \subset \mathfrak{M} \Rightarrow U(\sigma)\mathfrak{M}'U(\sigma)^{-1} \subset \mathfrak{M}'.$$

Therefore

$$U(G) \subset \mathfrak{M}'.$$

This in turn implies that

$$U(\sigma)MU(\sigma)^{-1} = M \quad \forall\, M \in \mathfrak{M}'' = \mathfrak{M}$$

$$\Longrightarrow \quad U(\sigma)M\Omega_0 = M\Omega_0$$

$$\Longrightarrow \quad M = CI$$

$$\Longrightarrow \quad \mathfrak{M} = \underset{\sim}{C}I$$

$$\Longrightarrow \quad \mathfrak{M}_{\Omega_0} = \underset{\sim}{C}I.$$

Observation: A consequence of the theorem is that $\mathfrak{M}$ must be a factor. In fact, $Z_{\mathfrak{M}} = \mathfrak{M} \cap \mathfrak{M}' \subset \mathfrak{M}_{\Omega_0} = \underset{\sim}{C}I$.

Projections and Classification In this section, we shall assume that the underlying Hilbert space $\mathcal{H}$ is separable (but this restriction is not essential).

Two projections E,F in a W*-algebra $\mathfrak{M}$ are said to be equivalent (written $E \sim F$) if $\exists$ a partial isometry $V \in \mathfrak{M}$ such that $E = V^*V$ and $F = VV^*$.

Example: Suppose given a weakly additive PTV. Fix double cones O,P for which $\exists\ \varepsilon > 0$: $\|a\| < \varepsilon \Rightarrow O + a \subset P$ -- then according to Borchers, $\forall$ nonzero projection $E \in \mathfrak{M}(O)$, $\exists$ a partial isometry $V \in \mathfrak{M}(P)$ such that $V^*V = I$ & $VV^* = E$. Therefore the nonzero projections in $\mathfrak{M}(O)$, when viewed in $\mathfrak{M}(P)$, are equivalent to the identity.

Define now a partial order on the projections in $\mathfrak{M}$ by writing $E \precsim F$ if E is equivalent to a subprojection of F (i.e., $E \sim F' \leq F$).

LEMMA Suppose that $E \precsim F$ and $F \precsim E$ -- then $E \sim F$.

Remark: If $\mathfrak{M}$ is a factor, then any two projections in $\mathfrak{M}$ are comparable: $E \precsim F$ or $F \precsim E$.

A projection in $\mathfrak{M}$ is said to be finite if it is not equivalent to a proper subprojection of itself; otherwise, it is infinite.

Example: Minimal projections are finite.

[If E is minimal, then its only proper subprojection is 0 and only 0 is equivalent to 0.]

Facts:

(1) If E is finite and if $E' \leq E$, then E' is finite.

(2) If E is infinite and if $E' \geq E$, then E' is infinite.

(3) If $E \sim F$ and if E is infinite, then F is infinite.

(4) If $E \sim F$ and if E is finite, then F is finite.

LEMMA Suppose that E and F are finite -- then $E \vee F$ is finite.

[Note: This is the most delicate point in the comparison theory of projections.]

Remark: If $\mathfrak{M}$ is a factor, then any two infinite projections in $\mathfrak{M}$ are equivalent.

Definition: $\mathfrak{M}$ is finite if all its projections are finite; otherwise, $\mathfrak{M}$ is infinite.

Criterion: Let $\mathfrak{M}$, $\mathfrak{N}$ be W*-algebras. Suppose that $\mathfrak{M} \subset \mathfrak{N}$ is a proper inclusion. Let $\Omega_0 \in \mathcal{H}$ be cyclic and separating for both $\mathfrak{M}$ and $\mathfrak{N}$ -- then $\mathfrak{N}$ is infinite.

THEOREM Suppose given a weakly additive PTV. Assume: $\mathfrak{M}$ is not abelian -- then $\forall$ O, the W*-algebra $\mathfrak{M}(O)$ is infinite.

[Choose $O_0 \subset O$: $\underline{\text{dis}}(O_0, \underline{\text{fr}}\ O) > 0$. Thanks to Wightman's inequality, $\mathfrak{M}(O_0)$ is properly contained in $\mathfrak{M}(O)$. But, by Reeh-Schlieder, Ω_0 is cyclic and separating for both algebras, therefore $\mathfrak{M}(O)$ is infinite.]

Terminology: Let $\mathfrak{M}$ be a W*-algebra.

(1) $\mathfrak{M}$ is properly infinite if all nonzero projections in $Z_{\mathfrak{M}}$ are infinite.

(2) $\mathfrak{M}$ is purely infinite if all nonzero projections in $\mathfrak{M}$ are infinite.

[Note: Obviously, $\mathfrak{M}$ purely infinite $\Rightarrow \mathfrak{M}$ properly infinite.]

Example: Suppose given a weakly additive PTV with a unique vacuum. Let W be a wedge -- then it will be shown in due course that

$\mathfrak{M}(W)$ is purely infinite.

<u>THEOREM</u> Suppose given a weakly additive PTV with a unique vacuum -- then $\forall$ O, the W*-algebra $\mathfrak{M}(O)$ is properly infinite.

[Let $E \in Z_{\mathfrak{M}(O)}$ be a nonzero central projection -- then $\exists\ O_0 \subset O$ $(O_0 \neq O)$ such that the containment

$$\mathfrak{M}(O_0)|E\mathcal{H} \subset \mathfrak{M}(O)|E\mathcal{H}$$

is strict. Granted this, it follows that $\mathfrak{M}(O)|E\mathcal{H}$ is infinite, hence E is infinite (here it is necessary to observe that $E\Omega_0$ is cyclic and separating for both $\mathfrak{M}(O_0)|E\mathcal{H}$ and $\mathfrak{M}(O)|E\mathcal{H}$). To establish our contention, let us suppose that the contrary held: $\mathfrak{M}(O_0)|E\mathcal{H} = \mathfrak{M}(O)|E\mathcal{H} \quad \forall\ O_0 \subset O$. Obviously,

$$\underset{\sim}{C}I \subset \bigcap_{O_0 \subset O} \mathfrak{M}(O_0).$$

On the other hand, in view of the intersection property, per any $a \in O$,

$$\bigcap_{\substack{O_0 \subset O \\ O_0 \ni a}} \mathfrak{M}(O_0) = \underset{\sim}{C}I.$$

Therefore

$$\bigcap_{O_0 \subset O} \mathfrak{M}(O_0) = \underset{\sim}{C}I$$

$$\Rightarrow$$

$$\mathfrak{M}(O)|E\mathcal{H} = \underset{\sim}{C}I|E\mathcal{H}$$

$$\Rightarrow$$

$$E\mathcal{H} = \underset{\sim}{C}E\Omega_0,$$

which implies that E is one dimensional. But $\exists$ a wedge W: $O \subset W \Rightarrow$ $\mathfrak{M}(O) \subset \mathfrak{M}(W)$ and, as has been noted above, $\mathfrak{M}(W)$ is purely infinite, thus contains no finite projections. This contradiction establishes the existence of O_0.]

Let E be a projection in $\mathfrak{M}$ -- then E is abelian if $E\mathfrak{M}E$ is abelian.

[Note: If $E \sim F$ and if E is abelian, then F is abelian.]

Example: Minimal projections are abelian.

[If E is minimal, then each projection in $E\mathfrak{M}E$ is either E or 0. Since $E\mathfrak{M}E$ is a W*-algebra, it is generated by its projections. Therefore $E\mathfrak{M}E$ consists of scalar multiples of E.]

Fact: An abelian projection is finite.

W*-algebras are classified into types depending on the kinds of projections which they contain.

Type I: A W*-algebra $\mathfrak{M}$ is of type I if each nonzero projection in $\mathfrak{M}$ majorizes a nonzero abelian projection.

Type II: A W*-algebra $\mathfrak{M}$ is of type II if each nonzero projection in $\mathfrak{M}$ majorizes a nonzero finite projection but $\mathfrak{M}$ contains no nonzero abelian projections.

Type III: A W*-algebra $\mathfrak{M}$ is of type III if it is purely infinite.

LEMMA We have:

(i) $\mathfrak{M}$ is of type I iff $\mathfrak{M}'$ is of type I;

(ii) $\mathfrak{M}$ is of type II iff $\mathfrak{M}'$ is of type II;

(iii) $\mathfrak{M}$ is of type III iff $\mathfrak{M}'$ is of type III.

It is a fact that every W*-algebra $\mathfrak{M}$ is uniquely decomposable as a direct sum

$$\mathfrak{M} = \mathfrak{M}_{I} \oplus \mathfrak{M}_{II} \oplus \mathfrak{M}_{III}$$

of distinct types.

Each of the three types can be classified further, the focus being on factors.

If $\mathfrak{M}$ is a factor of type I, then $\mathfrak{M}$ is isomorphic to the algebra of bounded linear operators on some Hilbert space. If that Hilbert space is finite dimensional and of dimension n, $\mathfrak{M}$ is said to be of <u>type I_n</u>; otherwise, $\mathfrak{M}$ is said to be of <u>type I_∞</u>.

The existence of factors of type II or type III is not obvious. To begin with, a finite factor of type II is called a factor of <u>type II_1</u>; an infinite factor of type II is called a factor of <u>type II_∞</u>. There is also a further subdivision of factors of type III but we shall put this off for now. Examples realizing the various possibilities can be obtained by suitably specializing the following construction.

Let $\alpha : G \rightarrow \underline{\text{Aut}}\,\mathfrak{M}$ be a representation of G on $\mathfrak{M}$ -- then the triple $\{L^2(G;\mathcal{H}),\pi,U\}$ implements α, where

$$\begin{cases} \pi(M)f(\sigma) = (\alpha(\sigma^{-1})M)f(\sigma) \\ U(\sigma)f(\tau) = f(\sigma^{-1}\tau). \end{cases}$$

I.e.:

$$U(\sigma)\pi(M)U(\sigma)^{-1} = \pi(\alpha(\sigma)M) \qquad (M \in \mathfrak{M},\ \sigma \in G).$$

Indeed,

$$(U(\sigma)\pi(M)U(\sigma)^{-1}f)\ (\tau)$$
$$= (\pi(M)U(\sigma)^{-1}f)\ (\sigma^{-1}\tau)$$

$$= (\alpha(\tau^{-1}\sigma)M)\ (U(\sigma^{-1})f)\ (\sigma^{-1}\tau)$$

$$= (\alpha(\tau^{-1}\sigma)M)f(\tau),$$

while

$$(\pi(\alpha(\sigma)M)f)\ (\tau)$$

$$= (\alpha(\tau^{-1})\alpha(\sigma)M)f(\tau)$$

$$= (\alpha(\tau^{-1}\sigma)M)f(\tau).$$

Definition: The W*-algebra generated by $\pi(\mathfrak{M})$ and $U(G)$ is called the <u>crossed product of $\mathfrak{M}$ by G w.r.t. α</u> and is denoted by $\mathfrak{R}(\mathfrak{M},G,\alpha)$.

Examples:

(1) Let $\mathcal{H} = L^2([0,1])$, $\mathfrak{M} = L^\infty([0,1])$, and $G = \mathbb{Q}$ -- then G operates on $[0,1]$, viz. $x \rightarrow \{x+\sigma\}$ (= the fractional part of $x+\sigma$). This action lifts to an action α of G on $\mathfrak{M}$ and $\mathfrak{R}(G,\mathfrak{M},\alpha)$ is a type II_1 factor.

(2) Let $\mathcal{H} = L^2(\mathbb{R})$, $\mathfrak{M} = L^\infty(\mathbb{R})$, and $G = \mathbb{Q}$ -- then G operates on $\mathbb{R}$, viz. $x \rightarrow x+\sigma$. This action lifts to an action α of G on $\mathfrak{M}$ and $\mathfrak{R}(G,\mathfrak{M},\alpha)$ is a type II_∞ factor.

(3) Let $\mathcal{H} = L^2(\mathbb{R})$, $\mathfrak{M} = L^\infty(\mathbb{R})$, and $G = \mathbb{Z} \ltimes \mathbb{Q}$, where $(m,\tau)\cdot(n,\sigma) = (m+n,\ 2^m\sigma+\tau)$ -- then G operates on $\mathbb{R}$, viz. $x \rightarrow 2^n x + \sigma$. This action lifts to an action α of G on $\mathfrak{M}$ and $\mathfrak{R}(G,\mathfrak{M},\alpha)$ is a type III factor.

<u>Type III</u> Recall that a W*-algebra $\mathfrak{M}$ is said to be of type III if it is purely infinite, i.e., if all the nonzero projections in $\mathfrak{M}$ are infinite.

Criterion: Suppose that $\mathfrak{M}$ admits a cyclic and separating unit vector Ω_0. Assume further that $\mathfrak{M}_{\Omega_0} = \underset{\sim}{C}I$ and $\mathfrak{M} \neq \underset{\sim}{C}I$ -- then $\mathfrak{M}$ is a factor of type III.

THEOREM Suppose given a weakly additive PTV with a unique vacuum. Let $W \in \mathcal{W}$ be a wedge in M -- then $\mathfrak{M}(W)$ is a type III factor.

[Since $W = (\Lambda,a)\cdot W_R$ for some $(\Lambda,a) \in \mathcal{P}_+^\uparrow$, it will be enough to consider $W_R(= \{x \in M: |x_0| < x_1\})$. Let $a = \{1,1,0,\ldots,0\} \in \underset{\sim}{R}^{1,d}$ -- then

$$W_R + \lambda a \subset W_R \quad (\lambda \geq 0).$$

This is because

$$|x_0 + \lambda| \leq |x_0| + \lambda < x_1 + \lambda.$$

Now put

$$\begin{cases} G = \underset{\sim}{R}a \\ \mathcal{S} = \underset{\sim}{R}_{\geq 0}a. \end{cases}$$

Owing to the uniqueness of the vacuum,

$$\underline{\dim}\{x \in \mathcal{H} : U(I,ta)x = x \quad \forall\, t \in \underset{\sim}{R}\} = 1$$

which implies that $\underset{\sim}{C}\,\Omega_0 = \mathcal{H}^G$. Next,

$$\lambda \geq 0 \implies$$

$$U(\lambda)\mathfrak{M}(W_R)U(\lambda)^{-1} = \mathfrak{M}(W_R + \lambda a)$$
$$\subset \mathfrak{M}(W_R).$$

Furthermore, by Reeh-Schlieder, Ω_0 is cyclic and separating for $\mathfrak{M}(W_R)$. Therefore

$$\overline{\mathfrak{M}(W_R)\,\Omega_0} = \underset{\sim}{C}I.$$

On the other hand, it is clear that $\mathfrak{M}(W_R) \neq \underset{\sim}{C}I$, thus $\mathfrak{M}(W_R)$ is a factor of type III.]

Remark: Let $\mathfrak{M}$ be a type III factor -- then one can attach to $\mathfrak{M}$ a closed subgroup $\Gamma(\mathfrak{M})$ of $\underset{\sim}{R}_{>0}$, which can be labeled by a parameter $\lambda \in [0,1]$:

$$(0)\quad \lambda = 0,\ \Gamma(\mathfrak{M}) = \{1\};$$

$$(\lambda)\quad 0 < \lambda < 1,\ \Gamma(\mathfrak{M}) = \lambda^{\underset{\sim}{Z}};$$

$$(1)\quad \lambda = 1,\ \Gamma(\mathfrak{M}) =]0, +\infty[,$$

$\mathfrak{M}$ being called type III_0, III_λ, or III_1 as the case may be.

Example: $\mathfrak{M}(W_R)$ is a type III_1 factor.

[Note: The details will be provided later on.]

Observation: Suppose that $\mathfrak{M}$ is infinite and each nonzero projection in $\mathfrak{M}$ is equivalent to the identity -- then $\mathfrak{M}$ is type III.

[Fix $E_0 \in \mathfrak{M}$: E_0 infinite. Consider now any nonzero projection $E \in \mathfrak{M}$. We have: $E \sim I$ & $E_0 \sim I \Rightarrow E \sim E_0$, hence E is infinite. Therefore $\mathfrak{M}$ is purely infinite, i.e., $\mathfrak{M}$ is type III.]

Suppose given a weakly additive PTV with a unique vacuum -- then $\mathfrak{M} = \mathcal{B}(\mathcal{H})$ is not abelian, so $\forall$ O, $\mathfrak{M}(O)$ is infinite. Moreover, $\mathfrak{M}(O)$ is "almost type III" in the following sense: Fix a pair

(P, ε): $\| a \| < \varepsilon \Rightarrow O + a \subset P$ -- then the nonzero projections in $\mathcal{M}(O)$, when viewed in $\mathcal{M}(P)$, are equivalent to the identity if O,P are both double cones.

<u>The Split Property</u> Let E,F be Banach spaces, $\Theta : E \longrightarrow F$ a bounded linear transformation -- then Θ is said to be <u>nuclear</u> if $\exists$ sequences

$$\begin{cases} \{\varphi_i\} \subset E^* \\ \{\psi_i\} \subset F \end{cases} : \sum_i \|\varphi_i\| \cdot \|\psi_i\| < +\infty$$

such that

$$\Theta(x) = \sum_i \varphi_i(x)\, \psi_i \qquad (x \in E).$$

The <u>nuclearity index</u> of Θ is then

$$\|\Theta\|_1 = \underline{\inf} \sum_i \|\varphi_i\| \cdot \|\psi_i\|,$$

where the <u>inf</u> is taken over all such realizations of Θ.

Let $\mathfrak{M}$ be a W*-algebra. Suppose given a selfadjoint nonnegative operator H which admits 0 as an eigenvalue of multiplicity 1, say $H\Omega_0 = 0$ ($\|\Omega_0\| = 1$). Suppose further that Ω_0 is cyclic and separating for $\mathfrak{M}$.

Definition: The triple $(\mathfrak{M}, H, \Omega_0)$ satisfies the <u>nuclearity condition</u> if $\forall\, \beta > 0$, the map

$$\begin{cases} \Theta_\beta : \mathfrak{M} \rightarrow \mathcal{H} \\ \qquad M \rightarrow e^{-\beta H} M\Omega_0 \end{cases}$$

is nuclear.

Consider now a weakly additive PTV with a unique vacuum -- then the energy operator H of the theory is selfadjoint and nonnegative. In addition, $H\Omega_0 = 0$ with multiplicity 1.

Rappel: The <u>split property</u> obtains if $\mathfrak{M}$ is not abelian and $\forall$ O,P:

$$O \subset P \text{ and } \underline{\text{dis}}(O, \underline{\text{fr}}\ P) > 0,$$

$\exists$ a type I factor between $\mathfrak{M}(O)$ and $\mathfrak{M}(P)$.

[Note: The uniqueness of the vacuum implies that $\mathfrak{M} = \mathcal{B}(\mathcal{H})$, which is not abelian if $\underline{\dim}\ \mathcal{H} > 1$, thus $\mathfrak{M}(O)$ is necessarily a proper subset of $\mathfrak{M}(P)$.]

Assume: $\forall$ O, the triple $(\mathfrak{M}(O), H, \Omega_0)$ satisfies the nuclearity condition. And: $\exists$ $n \in \underset{\sim}{N}$, $\beta_0 > 0$ (both depending on O) such that

$$\|\Theta_\beta\|_1 \leq e^{(\beta_0/\beta)^n} \quad (0 < \beta < 1).$$

THEOREM Under these assumptions, the theory possesses the split property.

Example: One can associate with the free relativistic particle of spin 0 and mass $m > 0$ a weakly additive PTV with a unique vacuum. It is a fact (nontrivial) that the assumptions of the theorem are met, hence the split property holds.

A W*-algebra $\mathfrak{M}$ is said to be <u>hyperfinite</u> if $\exists$ an increasing sequence of finite dimensional W*-algebras $\mathfrak{M}_n \subset \mathfrak{M}$ such that $\left(\bigcup_n \mathfrak{M}_n\right)'' = \mathfrak{M}$. So, e.g., a type I factor is hyperfinite.

Example: Suppose that the split property obtains and the $\mathfrak{M}(O)$ are continuous from the inside -- then the $\mathfrak{M}(O)$ are hyperfinite.

[Fix O and choose an increasing sequence $O_1 \subset O_2 \subset \cdots$: $\underline{\text{dis}}(O_n, \underline{\text{fr}}\ O_{n+1}) > 0$ & $\bigcup_n O_n = O$, thus $\mathfrak{M}(O) = \left(\bigcup_n \mathfrak{M}(O_n)\right)''$. Choose a type I factor $\mathfrak{N}_n : \mathfrak{M}(O_n) \subset \mathfrak{N}_n \subset \mathfrak{M}(O_{n+1})$ -- then $\mathfrak{N}_1 \subset \mathfrak{N}_2$ ... and $\left(\bigcup_n \mathfrak{N}_n\right)'' = \mathfrak{M}(O)$. But $\mathfrak{N}_n = \left(\bigcup_k \mathfrak{N}_{nk}\right)''$, where $\mathfrak{N}_{n1} \subset \mathfrak{N}_{n2} \subset \ldots$ and $\mathfrak{N}_{nk}$ is finite dimensional. Therefore $\mathfrak{M}(O) = \left(\bigcup_{n=1}^{\infty} \bigcup_{k=n}^{\infty} \mathfrak{N}_{nk}\right)'' = \left(\bigcup_{k=1}^{\infty} \bigcup_{n=1}^{k} \mathfrak{N}_{nk}\right)''$, which implies that $\mathfrak{M}(O)$ is hyperfinite.]

A W*-algebra $\mathfrak{M}$ is said to be __injective__ if $\exists$ an idempotent map $r: \mathfrak{B}(\mathcal{H}) \to \mathfrak{B}(\mathcal{H})$ with the following properties:

$$\begin{cases} r(\mathfrak{B}(\mathcal{H})) \subset \mathfrak{M} \\ \qquad r(M) = M \quad (M \in \mathfrak{M}), \end{cases} \quad \begin{cases} r(A^*) = r(A)^* \\ \| r(A) \| \leq \| A \| \end{cases} \quad (A \in \mathfrak{B}(\mathcal{H})).$$

Fact: Suppose that $\mathfrak{M}$ is a factor -- then $\mathfrak{M}$ is hyperfinite iff $\mathfrak{M}$ is injective.

It is a theorem that up to isomorphism, $\exists$ a unique injective type III_1 factor, call it $\mathfrak{R}$.

__UNIVERSAL STRUCTURE OF LOCAL ALGEBRAS__ Suppose that the split property obtains and the $\mathfrak{M}(O)$ are continuous from the inside. Assume in addition that $\forall$ O, the Connes invariant $S(\mathfrak{M}(O))$ of $\mathfrak{M}(O)$ is $[0, +\infty[$ -- then

$$\mathfrak{M}(O) \approx \mathfrak{R} \otimes \mathfrak{Z}(O),$$

where $\mathfrak{Z}(O)$ is the center of $\mathfrak{M}(O)$.

<u>The Modular Theory</u> Let $T: \mathcal{H} \to \mathcal{H}$ be a densely defined linear operator. Assume: T is closed -- then there is a representation $T = VA$, where V is a partial isometry (hence is bounded) and A is positive and selfadjoint (hence, in general, is unbounded).

[Note: This representation is called the <u>polar decomposition</u> of T. It is unique. Furthermore, the initial space of V is the closure of the range of T^*, the final space of V is the closure of the range of T, and $A = \sqrt{T^*T} = |T|$.]

Definition: A <u>standard W*-algebra</u> is a pair $(\mathfrak{M}, \Omega_0)$, where $\mathfrak{M}$ is a W*-algebra and Ω_0 is a cyclic and separating unit vector for $\mathfrak{M}$.

Given a standard W*-algebra $(\mathfrak{M}, \Omega_0)$, define two conjugate linear operators by

$$\begin{cases} S_0 M\Omega_0 = M^*\Omega_0 & (M \in \mathfrak{M}) \\ F_0 M'\Omega_0 = M'^*\Omega_0 & (M' \in \mathfrak{M}'). \end{cases}$$

Then S_0 and F_0 are welldefined on $\mathfrak{M}\Omega_0$ and $\mathfrak{M}'\Omega_0$.

[Note: Recall that Ω_0 is cyclic for $\mathfrak{M}$ iff Ω_0 is separating for $\mathfrak{M}'$ and Ω_0 is cyclic for $\mathfrak{M}'$ iff Ω_0 is separating for $\mathfrak{M}'' = \mathfrak{M}$. Therefore both $\mathfrak{M}\Omega_0$ and $\mathfrak{M}'\Omega_0$ are dense in $\mathcal{H}$.]

Fact: S_0 and F_0 admit closure and

$$\begin{cases} S_0^* = \overline{F}_0 \\ F_0^* = \overline{S}_0 . \end{cases}$$

Put

$$\begin{cases} S = \overline{S}_0 \\ F = \overline{F}_0 . \end{cases}$$

[Note: Since $\begin{cases} S_0 \\ F_0 \end{cases}$ is densely defined and admits closure, we have

$$\begin{cases} \bar{S}_0^* = S_0^* \\ \bar{F}_0^* = F_0^* \end{cases}, \text{ hence } \begin{cases} S^* = \bar{S}_0^* = S_0^* = \bar{F}_0 = F \\ F^* = \bar{F}_0^* = F_0^* = \bar{S}_0 = S \end{cases}.]$$

Fact: Each of the operators S and F has range the same as its domain, is invertible, and coincides with its inverse.

Let $\Delta = S^*S$ (= FS) -- then Δ has inverse $\Delta^{-1} = SS^*$ (= SF). The polar decomposition of S is

$$S = J\sqrt{S^*S} = J\Delta^{1/2}.$$

The initial space of J is $\overline{\underline{\mathrm{Ran}}}_{S^*} = \overline{\underline{\mathrm{Ran}}}_F = \mathcal{H}$, while the final space of J is $\overline{\underline{\mathrm{Ran}}}_S = \mathcal{H}$. Therefore J is a conjugate linear isometry of $\mathcal{H}$ onto $\mathcal{H}$. It is not difficult to check that

$$J = J^*, \ J^2 = I, \ J\Omega_0 = \Omega_0$$

$$\begin{cases} \Delta^{-1/2} = J\Delta^{1/2}J \\ \Delta^{-1} = J\Delta J \end{cases}$$

$\Rightarrow$

$$\begin{aligned} S^* &= \Delta^{1/2}J^* \\ &= JJ\Delta^{1/2}J \\ &= J\Delta^{-1/2}. \end{aligned}$$

Definition: Per the pair $(\mathfrak{M}, \Omega_0)$, J is called the <u>modular conjugation</u> and Δ is called the <u>modular operator</u>.

Example: Let $\mathcal{H} = L^2([0,1])$ and take $\mathfrak{M} = L^\infty([0,1])$ (the set of all multiplication operators M on $\mathcal{H}$: $(Mf)(x) = m(x)f(x)$) -- then $\mathfrak{M}$ is an abelian W*-algebra with multiplication defined pointwise, the involution being complex conjugation and $1 \leftrightarrow m(x) \equiv 1$. The vector $\Omega_0 = 1$ is cyclic and separating for $\mathfrak{M}$. Here $SM\Omega_0 = M^*\Omega_0 \leftrightarrow (Sm)(x) = \overline{m(x)}$, i.e., S is complex conjugation, J = S, and $\Delta = I$.

Example: Let $\mathcal{H} = \underset{\sim}{M}_n$ equipped with the Hilbert-Schmidt inner product (i.e., $\langle x,y \rangle = \underline{\text{tr}}(x^*y)$). Take for $\mathfrak{M}$ the algebra of left multiplication operators: $\mathfrak{M} = \{L_a : a \in \underset{\sim}{M}_n\}$ (so $\mathfrak{M}'$ is the algebra of right multiplication operators: $\mathfrak{M}' = \{R_a : a \in \underset{\sim}{M}_n\}$). Fix a nonsingular matrix Ω_0 of norm one and put $c = \Omega_0\Omega_0^*$, $d = \Omega_0^*\Omega_0$ -- then

$$\Delta x = cxd^{-1}.$$

Write $\Omega_0 = \sqrt{c}\,v = v\sqrt{d}$, where v is a unitary matrix such that $c = vdv^{-1}$ -- then

$$Jx = vx^*v.$$

Obviously, $J^2 = I$ and

$$\begin{aligned} J\Omega_0 &= v\Omega_0^*v \\ &= v\,(\sqrt{c}\,v)^*\,v \\ &= vv^*\sqrt{c}\,v \\ &= \Omega_0. \end{aligned}$$

(1) $J\mathfrak{M}J = \mathfrak{M}'$. In fact,

$$
\begin{aligned}
(JL_aJ)(x) &= (JL_a)(vx^*v) \\
&= J(avx^*v) \\
&= v(avx^*v)^*v \\
&= vv^*xv^*a^*v \\
&= xv^*a^*v \\
&= R_{v^*a^*v}(x).
\end{aligned}
$$

(2) $J \Delta J = \Delta^{-1}$. In fact,

$$
\begin{aligned}
(J \Delta J)(x) &= J \Delta (vx^*v) \\
&= J(cvx^*vd^{-1}) \\
&= v(cvx^*vd^{-1})^*v \\
&= v(d^{-1}v^*xv^*c)v \\
&= (vd^{-1}v^*)x(v^*cv) \\
&= c^{-1}xd \\
&= \Delta^{-1}(x).
\end{aligned}
$$

Write

$$
\Delta^{\sqrt{-1}\, t} = \underline{\exp}(\sqrt{-1}\, t \,\underline{\log}\, \Delta) \qquad (t \in \mathbb{R}),
$$

where "$\underline{\log}$" is the Borel function on $\mathbb{C}$ defined as $\underline{\mathrm{Log}}\, z$ if $z \neq -|z|$ and as 0 if $z = -|z|$.

Observation: We have

$$
\begin{cases}
\Delta^{\sqrt{-1}\,(t - \frac{\sqrt{-1}}{2})} M\Omega_0 = \Delta^{\sqrt{-1}\, t} JM^*\Omega_0 & (M \in \mathcal{M}) \\
\Delta^{\sqrt{-1}\,(t + \frac{\sqrt{-1}}{2})} M'\Omega_0 = \Delta^{\sqrt{-1}\, t} JM'^*\Omega_0 & (M' \in \mathcal{M}').
\end{cases}
$$

[In fact,

$$\Delta^{\sqrt{-1}\,(t - \frac{\sqrt{-1}}{2})} M\Omega_0 = \Delta^{\sqrt{-1}\,t} \Delta^{1/2} M\Omega_0$$

$$= \Delta^{\sqrt{-1}\,t} JJ\, \Delta^{1/2} M\Omega_0$$

$$= \Delta^{\sqrt{-1}\,t} JSM\Omega_0$$

$$= \Delta^{\sqrt{-1}\,t} JM^*\Omega_0.$$

Ditto for the second relation.]

Suppose that $\{E_\lambda\}$ is the spectral resolution of Δ -- then $\{JE_\lambda J\}$ is the spectral resolution of Δ^{-1} and $\forall\, x \in \mathcal{H}$,

$$\langle JE_\lambda Jx, x\rangle = \langle JE_\lambda Jx, JJx\rangle$$

$$= \langle Jx, E_\lambda Jx\rangle$$

$$= \langle E_\lambda Jx, Jx\rangle$$

$\Rightarrow$

$$\langle \Delta^{-\sqrt{-1}\,t} x, x\rangle = \int_{-\infty}^{\infty} \lambda^{\sqrt{-1}\,t}\, d\langle JE_\lambda Jx, x\rangle$$

$$= \int_{-\infty}^{\infty} \lambda^{\sqrt{-1}\,t}\, d\langle E_\lambda Jx, Jx\rangle$$

$$= \langle \Delta^{\sqrt{-1}\,t} Jx, Jx\rangle$$

$$= \langle Jx, \Delta^{-\sqrt{-1}\,t} Jx\rangle$$

$$= \langle Jx, JJ\,\Delta^{-\sqrt{-1}\,t} Jx\rangle$$

$$= \langle J\,\Delta^{-\sqrt{-1}\,t} Jx, x\rangle$$

$\Rightarrow$

$$\Delta^{-\sqrt{-1}\,t} = J\,\Delta^{-\sqrt{-1}\,t} J$$

$$\Longrightarrow \qquad J\,\Delta^{\sqrt{-1}\,t} = \Delta^{\sqrt{-1}\,t}\,J \qquad (t \in \underset{\sim}{R}).$$

THEOREM Under the preceding assumptions and conditions, we have

$$J\mathfrak{M}J = \mathfrak{M}'$$

and

$$\Delta^{\sqrt{-1}\,t}\,\mathfrak{M}\,\Delta^{-\sqrt{-1}\,t} = \mathfrak{M} \qquad (t \in \underset{\sim}{R}).$$

In particular, the theorem provides us with an arrow

$$\begin{cases} \underset{\sim}{R} \longrightarrow \underline{\mathrm{Aut}}\ \mathfrak{M} \\ t \longrightarrow \sigma_t, \end{cases}$$

where

$$\sigma_t(M) = \Delta^{\sqrt{-1}\,t}\,M\,\Delta^{-\sqrt{-1}\,t}.$$

One calls $\{\sigma_t\}$ the <u>modular automorphism group</u> of the pair $(\mathfrak{M}, \Omega_0)$.

[Note: The state associated with Ω_0, i.e., $\omega_0(M) = \langle \Omega_0, M\Omega_0 \rangle$, is invariant w.r.t. σ_t: $\omega_0(\sigma_t(M)) = \langle \Omega_0, \sigma_t(M)\Omega_0 \rangle =$ $\langle \Omega_0, \Delta^{\sqrt{-1}\,t} M \Delta^{-\sqrt{-1}\,t} \Omega_0 \rangle = \langle \Omega_0, M\Omega_0 \rangle = \omega_0(M)$ ($\Delta^{\sqrt{-1}\,t}\Omega_0 = 1^{\sqrt{-1}\,t}\,\Omega_0$ by the spectral theorem)†.]

Remark: Recall that $\mathfrak{M}$ is said to be purely infinite or type III if all the nonzero projections in $\mathfrak{M}$ are infinite. It is then a fact

---

† If A is selfadjoint, if f is Borel, and if $Ax = \lambda x$ $(x \neq 0)$, then $f(A)x = f(\lambda)x$.

that $\mathfrak{M}$ is <u>not</u> purely infinite iff $\exists$ an invertible positive selfadjoint operator H such that $H^{\sqrt{-1}t} \in \mathfrak{M}\ \forall\ t$ and

$$\sigma_t(M) = H^{\sqrt{-1}t} M H^{-\sqrt{-1}t},$$

i.e., the σ_t are inner.

Example: Take $\mathcal{H} = \underset{\sim}{M}_n$ and let $\mathfrak{M}$ be as above -- then

$$\Delta^{\sqrt{-1}t} = L_{c^{\sqrt{-1}t}} R_{d^{-\sqrt{-1}t}}.$$

Since

$$\Delta^{\sqrt{-1}t} L_a \Delta^{-\sqrt{-1}t} = L_{c^{\sqrt{-1}t} a c^{-\sqrt{-1}t}},$$

it follows that

$$\sigma_t(\mathfrak{M}) = \mathfrak{M} \quad \forall\ t.$$

Definition: A one parameter group $\{\alpha_t : t \in \underset{\sim}{R}\}$ of automorphisms of a W*-algebra $\mathfrak{M}$ satisfies the <u>modular condition</u> relative to a state ω of $\mathfrak{M}$ if given $X, Y \in \mathfrak{M}$, there is a complex valued function $f_{X,Y}$ which is bounded and continuous on $0 \leq \underline{\text{Im}}\ z \leq 1$, holomorphic in $0 < \underline{\text{Im}}\ z < 1$, with the property that

$$\left\{\begin{array}{l} f_{X,Y}(t) = \omega(\alpha_t(X)Y) \\ \\ f_{X,Y}(t+\sqrt{-1}) = \omega(Y\alpha_t(X)). \end{array}\right. \qquad (t \in \underset{\sim}{R})$$

[Note: More generally, if $\mathfrak{A}$ is a unital C*-algebra and $\{\alpha_t : t \in \underset{\sim}{R}\}$ is a one parameter group of automorphisms of $\mathfrak{A}$, then a state ω is said to be a KMS state w.r.t. α at inverse temperature $\beta > 0$ provided that for all $A,B \in \mathfrak{A}$, there is a complex valued function $f_{A,B}$ which is bounded and continuous on $0 \leq \underline{\mathrm{Im}}\ z \leq \beta$, holomorphic in $0 < \underline{\mathrm{Im}}\ z < \beta$, with the property that

$$\begin{cases} f_{A,B}(t) = \omega(\alpha_t(A)B) \\ f_{A,B}(t + \sqrt{-1}\,\beta) = \omega(B\alpha_t(A)). \end{cases} \qquad (t \in \underset{\sim}{R})$$

Such a state is necessarily α-invariant, i.e., $\forall$ t,

$$\omega(\alpha_t(X)) = \omega(X) \qquad (X \in \mathfrak{A}).]$$

LEMMA If $x,y \in \underline{\mathrm{Dom}}_{\Delta^{1/2}}$, then the function

$$f_{x,y}(z) = \langle x, \Delta^z y \rangle$$

is a bounded continuous function of z on $0 \leq \underline{\mathrm{Re}}\ z \leq 1/2$ and is a holomorphic function of z in $0 < \underline{\mathrm{Re}}\ z < 1/2$.

[By polarization, it suffices to consider the case when x=y. To see that $x \in \underline{\mathrm{Dom}}_{\Delta^z}$ and $\| \Delta^z x \|$ stays bounded, proceed as follows. With $\{E_\lambda\}$ the spectral resolution of Δ, we have

$$\| x \|^2 = \int_0^\infty d\langle x, E_\lambda x\rangle, \quad \| \Delta^{1/2} x \|^2 = \int_0^\infty \lambda \, d\langle x, E_\lambda x\rangle.$$

When $0 \leq \underline{\mathrm{Re}}\ z \leq 1/2$ and $\lambda > 0$,

$$| \lambda^z |^2 = \lambda^{2 \underline{\mathrm{Re}}\ z} \leq \underline{\max}\ \{1, \lambda\} < 1 + \lambda$$

$\Rightarrow$

$$\int_0^{\infty} |\lambda^z|^2 \, d\langle x, E_\lambda x\rangle$$

$$\leq \int_0^{\infty} (1+\lambda) d\langle x, E_\lambda x\rangle$$

$$\leq \|x\|^2 + \|\Delta^{1/2} x\|^2 < +\infty$$

$\Rightarrow$

$$x \in \underline{\text{Dom}} \; \Delta^z$$

$\Rightarrow$

$$|\langle x, \Delta^z x\rangle| \leq \|x\| \cdot \|\Delta^z x\|$$

$$\leq \|x\| \cdot (\|x\|^2 + \|\Delta^{1/2} x\|^2)^{1/2}$$

$$< +\infty .$$

And:

$$f_{x,x}(z) = \int_0^{\infty} \lambda^z \, d\langle x, E_\lambda x\rangle .$$

For n = 1,2,..., put

$$\phi_n(z) = \int_{1/n}^{n} \lambda^z \, d\langle x, E_\lambda x\rangle .$$

Upon expanding $\lambda^z = \underline{\exp}(z \, \underline{\log} \, \lambda)$ as a power series in z and noting that it converges uniformly for λ in $[1/n, n]$, we conclude that ϕ_n is a holomorphic function of z. Finally, if $0 \leq \underline{\text{Re}} \; z \leq 1/2$, then

$$|f_{x,x}(z) - \phi_n(z)|$$

$$= \left| \left(\int_0^{1/n} + \int_n^{\infty} \right) \lambda^z \, d\langle x, E_\lambda x\rangle \right|$$

$$\leq \left(\int_0^{1/n} + \int_n^{\infty} \right) (1+\lambda) d \langle x, E_\lambda x \rangle$$

$$\longrightarrow 0 \qquad (n \longrightarrow \infty).$$

Therefore $\phi_n \longrightarrow f$ uniformly on $0 \leq \underline{\mathrm{Re}}\ z \leq 1/2$, thus f is continuous on $0 \leq \underline{\mathrm{Re}}\ z \leq 1/2$ and holomorphic in its interior.]

Rappel: Suppose that $a < b < c$ and f is a complex valued function which is bounded and continuous on $\{ z : a \leq \underline{\mathrm{Im}}\ z \leq c \}$ and holomorphic in

$$\begin{cases} \{ z : a < \underline{\mathrm{Im}}\ z < b \} \\ \{ z : b < \underline{\mathrm{Im}}\ z < c \}. \end{cases}$$

Then f is holomorphic in $\{ z : a < \underline{\mathrm{Im}}\ z < c \}$.

THEOREM The modular automorphism group $\{ \sigma_t \}$ satisfies the modular condition relative to the state ω_0 associated with Ω_0.

[Fix $X, Y \in \mathfrak{M}$. Since $\mathfrak{M}\,\Omega_0 \subset \underline{\mathrm{Dom}}\ \Delta^{1/2}$, the functions

$$\begin{cases} g(z) = \langle X^* \Omega_0, \Delta^{-\sqrt{-1}\,z}\, Y \Omega_0 \rangle & (0 \leq \underline{\mathrm{Im}}\ z \leq 1/2) \\ h(z) = \langle Y^* \Omega_0, \Delta^{1+\sqrt{-1}\,z}\, X \Omega_0 \rangle & (1/2 \leq \underline{\mathrm{Im}}\ z \leq 1) \end{cases}$$

are bounded and continuous on their strips of definition and holomorphic in their interior. But

$$g\left(t + \frac{\sqrt{-1}}{2}\right) = \langle X^* \Omega_0, \Delta^{-\sqrt{-1}\,t}\, \Delta^{1/2}\, Y \Omega_0 \rangle$$

$$= \langle SX\Omega_0,\ \Delta^{-\sqrt{-1}\,t}\, JJ\, \Delta^{1/2}\, Y\Omega_0 \rangle$$

$$= \langle J\,\Delta^{1/2}\, X\Omega_0,\ J\,\Delta^{-\sqrt{-1}\,t}\, SY\Omega_0 \rangle$$

$$= \langle \Delta^{-\sqrt{-1}\,t}\, Y^*\Omega_0,\ \Delta^{1/2}\, X\Omega_0 \rangle$$

$$= \langle Y^*\Omega_0,\ \Delta^{\sqrt{-1}\,t + 1/2}\, X\Omega_0 \rangle$$

$$= \quad h(t + \frac{\sqrt{-1}}{2}),$$

so it follows that g and h can be combined into a single function f which is bounded and continuous on $0 \leq \underline{\mathrm{Im}}\ z \leq 1$ and holomorphic in $0 < \underline{\mathrm{Im}}\ z < 1$. On the other hand,

$$\omega_0(\sigma_t(X)Y) = \langle \Omega_0,\ \Delta^{\sqrt{-1}\,t}\, X\, \Delta^{-\sqrt{-1}\,t}\, Y\Omega_0 \rangle$$

$$= \langle \Delta^{-\sqrt{-1}\,t}\,\Omega_0, X\,\Delta^{-\sqrt{-1}\,t}\, Y\Omega_0 \rangle$$

$$= \langle \Omega_0,\ X\,\Delta^{-\sqrt{-1}\,t}\, Y\Omega_0 \rangle$$

$$= \langle X^*\Omega_0,\ \Delta^{-\sqrt{-1}\,t}\, Y\Omega_0 \rangle$$

$$= \quad g(t)$$

and similarly,

$$\omega_0(Y\sigma_t(X)) = h(t + \sqrt{-1}).$$

Finally, let

$$f_{X,Y} = f.$$

Then

$$\begin{cases} f_{X,Y}(t) = g(t) = \omega_0(\sigma_t(X)Y) \\ f_{X,Y}(t+\sqrt{-1}) = h(t+\sqrt{-1}) = \omega_0(Y\sigma_t(X)), \end{cases}$$

from which the theorem.]

[Note: Suppose that $\{\alpha_t : t \in \underset{\sim}{R}\}$ is a one parameter group of automorphisms of $\mathfrak{M}$ which satisfies the modular condition relative to ω_0 -- then it can be shown that $\alpha_t = \sigma_t \ \forall t$.]

Remark: The equation $\sigma_t(M) = M$ is valid for all $t \in \underset{\sim}{R}$ iff $M \in \mathfrak{M}_{\Omega_0}$.

[Assuming that $\sigma_t(M) = M \ \forall t$, take an arbitrary $N \in \mathfrak{M}$ and consider $f_{M,N}$ -- then

$$\begin{aligned} f_{M,N}(t) &= \omega_0(\sigma_t(M)N) \\ &= \omega_0(MN) \end{aligned}$$

and

$$\begin{aligned} f_{M,N}(t+\sqrt{-1}) &= \omega_0(N\sigma_t(M)) \\ &= \omega_0(NM). \end{aligned}$$

A certain nonzero multiple of $f_{M,N}$ has constant <u>real</u> value on the real axis, thus has an analytic continuation across the real axis (Schwarz reflection principle). This extension is holomorphic in a domain D that contains $\underset{\sim}{R}$ and $0 < \underline{\text{Im}}\ z < 1$. It is constant on $\underset{\sim}{R}$, hence constant on D. Therefore $f_{M,N}$ is constant in $0 < \underline{\text{Im}}\ z < 1$, thus by continuity is constant on its closure. But this implies that

$$\omega_0(MN) = \omega_0(NM).$$

Turning to the converse, fix $M \in \mathfrak{M}_{\Omega_0}$ -- then for any $N \in \mathfrak{M}$ and all t,

$$\omega_0(\sigma_t(N)M) = \omega_0(M\sigma_t(N))$$

$\Longrightarrow$

$$f_{N,M}(t) = f_{N,M}(t + \sqrt{-1}),$$

so $f_{N,M}$ can be extended by periodicity to a bounded continuous function on $\underset{\sim}{C}$ which is holomorphic in each of the strips $\{z: n < \underline{\text{Im}}\, z < n + 1\}$ $(n = 0, \pm 1, \ldots)$. This extended function is entire, hence by Liouville, is a constant. In particular, $f_{N,M}$ is a constant

$\Longrightarrow$

$$\omega_0(\sigma_t(N)M) = \omega_0(NM)$$

$\Longrightarrow$

$$\begin{aligned} \omega_0(N\sigma_t(M)) &= \omega_0(\sigma_{-t}(N\sigma_t(M))) \\ &= \omega_0(\sigma_{-t}(N)M) = \omega_0(NM) \end{aligned}$$

$\Longrightarrow$

$$\omega_0(N(M - \sigma_t(M))) = 0 \quad \forall t.$$

Take now $N = (M - \sigma_t(M))^*$ to get

$$\begin{aligned} 0 &= \omega_0((M - \sigma_t(M))^* (M - \sigma_t(M))) \\ &= \langle \Omega_0, (M - \sigma_t(M))^* (M - \sigma_t(M))\Omega_0 \rangle \\ &= \| (M - \sigma_t(M)) \Omega_0 \|^2 \end{aligned}$$

$$\Rightarrow \qquad (M - \sigma_t(M))\Omega_0 = 0$$

$$\Rightarrow \qquad M = \sigma_t(M),$$

Ω_0 being separating.]

Definition: The natural cone $\mathcal{P}$ of the pair $(\mathfrak{M}, \Omega_0)$ is the closure of the set

$$\{ MJMJ\,\Omega_0 : M \in \mathfrak{M} \}.$$

Properties:

(1) $\Delta^{\sqrt{-1}\,t}\mathcal{P} = \mathcal{P} \quad \forall\, t;$

(2) $J\xi = \xi \quad \forall\, \xi \in \mathcal{P};$

(3) $MJMJ\,\mathcal{P} \subset \mathcal{P} \quad \forall\, M \in \mathfrak{M};$

(4) $\mathcal{P} \cap (-\mathcal{P}) = \{0\};$

(5) $\mathcal{P} - \mathcal{P} = \{x \in \mathcal{H} : Jx = x\};$

(6) $\mathcal{P} = \check{\mathcal{P}}$ $(= \{x \in \mathcal{H} : \langle \xi, x \rangle \geq 0 \quad \forall\, \xi \in \mathcal{P}\}).$

Fact: $\mathcal{H}$ is linearly spanned by $\mathcal{P}$.

[For suppose that x is orthogonal to the linear span of $\mathcal{P}$ -- then $x \in \check{\mathcal{P}} = \mathcal{P}$, hence $\langle x, x \rangle = 0 \Rightarrow x = 0$.]

Observation: Let $\xi \in \mathcal{P}$ -- then ξ is cyclic for $\mathfrak{M}$ iff ξ is separating for $\mathfrak{M}$.

[If $\xi \in \mathcal{P}$ is cyclic for $\mathfrak{M}$, then $\xi = J\xi$ is cyclic for $\mathfrak{M}' = J\mathfrak{M}J$, so ξ is separating for $\mathfrak{M}$ and conversely.]

[Note: Let $\xi_0 \in \mathcal{P}$ be a cyclic unit vector for $\mathfrak{M}$. Since ξ_0 is necessarily separating, the pair $(\mathfrak{M}, \xi_0)$ is standard. Denote by J_0 its modular conjugation -- then

$$J_0 = J.$$

Call $\mathcal{P}_0$ the associated natural cone. Since $\mathcal{P}_0$ is generated by elements of the form

$$MJ_0MJ_0\xi_0 = MJMJ\xi_0,$$

property (3) implies that $\mathcal{P}_0 \subset \mathcal{P}$. On the other hand, in view of property (6),

$$\mathcal{P} = \check{\mathcal{P}} \subset \check{\mathcal{P}}_0 = \mathcal{P}_0.$$

Therefore

$$\mathcal{P}_0 = \mathcal{P}.]$$

Since Ω_0 is, in particular, separating, the positive elements ω of $\mathfrak{M}_*$ are representable, i.e., are the $M \rightarrow \langle x, Mx \rangle$ $(x \in \mathcal{H})$. The correspondence $\omega \rightarrow x$ is one to many but if we work with $\mathcal{P}$ instead of $\mathcal{H}$, then matters can be made precise: Given any positive element ω of $\mathfrak{M}_*$, there exists a unique $\xi_\omega \in \mathcal{P}$ such that $\omega(M) = \langle \xi_\omega, M\xi_\omega \rangle$. And: The map $\omega \rightarrow \xi_\omega$ is a homeomorphism because

$$\| \xi_{\omega_1} - \xi_{\omega_2} \|^2 \leq \| \omega_1 - \omega_2 \| \leq \| \xi_{\omega_1} - \xi_{\omega_2} \| \cdot \| \xi_{\omega_1} + \xi_{\omega_2} \| .$$

Remark: This machinery can be used to establish that every automorphism α of $\mathfrak{M}$ is implementable by a unitary operator $U(\alpha)$:

$$\alpha M = U(\alpha) M U(\alpha)^{-1} \quad (M \in \mathfrak{M}).$$

One can even arrange that

$$U(\alpha)\mathcal{P} = \mathcal{P}, \quad JU(\alpha) = U(\alpha)J.$$

In fact,

$$U(\alpha)\xi_\omega = \xi_{(\alpha^{-1})^T\omega}.$$

[Note: The assignment $\alpha \longrightarrow U(\alpha)$ is a representation of Aut $\mathfrak{M}$ on $\mathcal{H}$.]

Let ω be a faithful normal state on $\mathfrak{M}$, $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$ the associated GNS data (so $\omega(M) = \langle \Omega_\omega, \pi_\omega(M)\Omega_\omega \rangle$) -- then $(\pi_\omega(\mathfrak{M}), \Omega_\omega)$ is a standard W*-algebra.

[Note: The normality of ω implies that $\pi_\omega(\mathfrak{M})$ is a W*-algebra. Of course, Ω_ω is automatically cyclic for $\pi_\omega(\mathfrak{M})$. To see that Ω_ω is also separating for $\pi_\omega(\mathfrak{M})$, suppose that $\pi_\omega(M)\Omega_\omega = 0$ -- then $\omega(M^*M) = \|\pi_\omega(M)\Omega_\omega\|^2 = 0 \Rightarrow M^*M = 0 \Rightarrow \|M^*M\| = \|M\|^2 = 0 \Rightarrow M = 0.$]

The preceding theory thus furnishes us with a modular operator Δ_ω and by definition

$$\sigma_t^\omega(M) = \pi_\omega^{-1}\left(\Delta_\omega^{\sqrt{-1}\,t}\,\pi_\omega(M)\,\Delta_\omega^{-\sqrt{-1}\,t}\right)$$

is the <u>modular automorphism group</u> of the pair $(\mathfrak{M}, \omega)$.

[Note: Suppose that $\{\alpha_t : t \in \mathbb{R}\}$ is a one parameter group of automorphisms of $\mathfrak{M}$ which satisfies the modular condition relative to ω -- then it can be shown that $\alpha_t = \sigma_t^\omega \ \forall t$.]

THEOREM Suppose that ω', ω'' are two faithful normal states on $\mathfrak{M}$ -- then $\exists$ a one parameter family of unitary operators $U_t \in \mathfrak{M}$ such that

$$\begin{cases} \sigma_t^{\omega''}(M) = U_t \, \sigma_t^{\omega'}(M) U_t^{-1} \\ U_{t+s} = U_t \, \sigma_t^{\omega'}(U_s). \end{cases}$$

It is customary to incorporate ω', ω'' into the notation and write

$$(D_{\omega''}:D_{\omega'})(t) = U_t,$$

the so-called Radon-Nikodym cocycle.

[Note: We have

$$(D_{\omega''}:D_{\omega'})(t)^{-1} = (D_{\omega'}:D_{\omega''})(t).]$$

Since ω', ω'' are faithful and normal, $\exists$ cyclic and separating unit vectors Ω', Ω'' such that

$$\begin{cases} \omega'(M) = \langle \Omega', M\Omega' \rangle \\ \omega''(M) = \langle \Omega'', M\Omega'' \rangle \end{cases} \quad (M \in \mathfrak{M}).$$

Define a conjugate linear operator $S_{\Omega',\Omega''}$ by

$$S_{\Omega',\Omega''} M\Omega' = M^*\Omega'' \quad (M \in \mathfrak{M}).$$

As before, $S_{\Omega',\Omega''}$ admits closure, hence there is a polar decomposition

$$\bar{S}_{\Omega',\Omega''} = J_{\Omega',\Omega''} \Delta_{\Omega',\Omega''}^{1/2} .$$

Definition: Per the triple $(\mathfrak{M}, \Omega', \Omega'')$, $J_{\Omega',\Omega''}$ is called the <u>relative modular conjugation</u> and $\Delta_{\Omega',\Omega''}$ is called the <u>relative modular operator</u>.

Fact: We have

$$(D\omega'':D\omega')(t) = \Delta_{\Omega',\Omega''}^{\sqrt{-1}\,t}\,\Delta_{\Omega'}^{-\sqrt{-1}\,t}\ .$$

<u>Type III (bis)</u> It has been mentioned earlier that if $\mathfrak{M}$ is a type III factor, then one can attach to $\mathfrak{M}$ a closed subgroup $\Gamma(\mathfrak{M})$ of $\underset{\sim}{R}_{>0}$. We shall now make matters more precise.

Assume: $(\mathfrak{M}, \Omega_0)$ is a standard W*-algebra, hence possesses a modular theory.

Fact: If $\mathfrak{M}$ is a type III factor and if $\mathfrak{M}_{\Omega_0} = \underset{\sim}{C}I$, then

$$\Gamma(\mathfrak{M}) = \underline{\text{spec}}\,\Delta - \{0\}.$$

Furthermore, given an isolated element λ of $\underline{\text{spec}}\,\Delta - \{0\}$, $\exists$ a nonzero $M \in \mathfrak{M}$ such that

$$\Delta^{\sqrt{-1}\,t} M\Omega_0 = \lambda^{\sqrt{-1}\,t} M\Omega_0 \quad \forall t.$$

<u>LEMMA</u> If $\mathfrak{M}_{\Omega_0} = \underset{\sim}{C}I$ and $\mathfrak{M} \neq \underset{\sim}{C}I$, then $\mathfrak{M}$ is a factor of type III_1.

[The hypotheses imply that $\mathfrak{M}$ is a factor of type III, thus the strategy is to eliminate the cases $\lambda = 0$ and $0 < \lambda < 1$.

<u>Ad $\lambda = 0$</u>: Here, $\underline{\text{spec}}\,\Delta - \{0\} = \{1\}$, hence either $\underline{\text{spec}}\,\Delta = \{1\}$ or $\underline{\text{spec}}\,\Delta = \{0,1\}$.

(i) $\underline{\text{spec}}\,\Delta = \{1\} \Rightarrow \Delta = I \Rightarrow \sigma_t = I \ \forall t \Rightarrow \sigma_t(M) = M \ \forall t$ & $\forall M \Rightarrow \mathfrak{M} = \mathfrak{M}_{\Omega_0} \Rightarrow \mathfrak{M} = \underset{\sim}{C}I$, a contradiction.

(ii) $\underline{\text{spec}}\,\Delta = \{0,1\} \Rightarrow \exists x \in \mathcal{H} : x \neq 0$ & $\Delta x = 0$, contradicting the invertibility of Δ.

<u>Ad $0 < \lambda < 1$</u>: Fix $M \neq 0$ in $\mathfrak{M}$:

$$\Delta^{\sqrt{-1}\,t} M\Omega_0 = \lambda^{\sqrt{-1}\,t} M\Omega_0 \quad \forall t.$$

Since

$$\Delta^{\sqrt{-1}\, t} M \Omega_0 = \Delta^{\sqrt{-1}\, t} M \Delta^{-\sqrt{-1}\, t} \Omega_0,$$

it follows that

$$\Delta^{\sqrt{-1}\, t} M \Delta^{-\sqrt{-1}\, t} = \lambda^{\sqrt{-1}\, t} M,$$

Ω_0 being separating for $\mathcal{M}$. Therefore

$$\begin{aligned} f_{M,N}(t) &= \langle \Omega_0, \sigma_t(M) N \Omega_0 \rangle \\ &= \lambda^{\sqrt{-1}\, t} \langle \Omega_0, MN \Omega_0 \rangle \end{aligned}$$

and

$$\begin{aligned} f_{M,N}(t + \sqrt{-1}) &= \langle \Omega_0, N \sigma_t(M) \Omega_0 \rangle \\ &= \lambda^{\sqrt{-1}\, t} \langle \Omega_0, NM \Omega_0 \rangle . \end{aligned}$$

From the first relation,

$$\begin{aligned} f_{M,N}(t + \sqrt{-1}) &= \lambda^{\sqrt{-1}(t + \sqrt{-1})} \langle \Omega_0, MN \Omega_0 \rangle \\ &= \lambda^{\sqrt{-1}\, t - 1} \langle \Omega_0, MN \Omega_0 \rangle \end{aligned}$$

$\Longrightarrow$

$$\lambda^{\sqrt{-1}\, t} \langle \Omega_0, NM \Omega_0 \rangle = \lambda^{\sqrt{-1}\, t - 1} \langle \Omega_0, MN \Omega_0 \rangle$$

$\Longrightarrow$

$$\langle \Omega_0, MN \Omega_0 \rangle = \lambda \langle \Omega_0, NM \Omega_0 \rangle .$$

On the other hand,

$$\Delta^{\sqrt{-1}\,t} M^* \Delta^{-\sqrt{-1}\,t} = \lambda^{-\sqrt{-1}\,t} M^*$$

$\Longrightarrow$

$$\Delta^{\sqrt{-1}\,t} M^*M \Delta^{-\sqrt{-1}\,t}$$

$$= \Delta^{\sqrt{-1}\,t} M^* \Delta^{-\sqrt{-1}\,t} \cdot \Delta^{\sqrt{-1}\,t} M \Delta^{-\sqrt{-1}\,t}$$

$$= M^*M$$

$\Longrightarrow$

$$\sigma_t(M^*M) = M^*M \quad \forall t$$

$\Longrightarrow$

$$M^*M \in \mathfrak{M}_{\Omega_0} = \underset{\sim}{C}I$$

$\Longrightarrow$

$$M^*M = \mu I \quad (\exists\, \mu > 0).$$

The same argument applies to MM*:MM* = ν I ($\exists\, \nu >$ 0). Normalize M so that M*M = I -- then (MM*)(MM*) = MM* $\Rightarrow \nu^2 = \nu \Rightarrow \nu = 1$. In other words: M is unitary. Taking N = M* in the above then gives

$$\langle \Omega_0, MM^* \Omega_0 \rangle = \lambda \langle \Omega_0, M^*M \Omega_0 \rangle$$

$$\Longrightarrow \lambda = 1,$$

a contradiction.]

<u>Characteristic Functions</u> Suppose given two standard W*-algebras $(\mathfrak{M}, \Omega_0)$ and $(\mathfrak{N}, \Omega_0)$ -- then the symbols

$$\begin{cases} S_{\mathfrak{M}}, J_{\mathfrak{M}}, \Delta_{\mathfrak{M}}, \sigma_{\mathfrak{M}} = \sigma^{\mathfrak{M}} \\ S_{\mathfrak{N}}, J_{\mathfrak{N}}, \Delta_{\mathfrak{N}}, \sigma_{\mathfrak{N}} = \sigma^{\mathfrak{N}} \end{cases}$$

are to be assigned the obvious interpretations.

<u>LEMMA</u> Suppose that $\mathfrak{N} \subset \mathfrak{M}$ -- then the following are equivalent:

(a) $\mathfrak{N} = \mathfrak{M}$; (b) $S_{\mathfrak{N}} = S_{\mathfrak{M}}$; (c) $\Delta_{\mathfrak{N}} = \Delta_{\mathfrak{M}}$; (d) $J_{\mathfrak{N}} = J_{\mathfrak{M}}$.

[The implications (a) $\Rightarrow$ (b) $\Rightarrow$ (c) and (b) $\Rightarrow$ (d) are obvious. To prove that (d) $\Rightarrow$ (a), note that

$$\mathfrak{N}' = J_{\mathfrak{N}} \mathfrak{N} J_{\mathfrak{N}} \subset J_{\mathfrak{N}} \mathfrak{M} J_{\mathfrak{N}} = J_{\mathfrak{M}} \mathfrak{M} J_{\mathfrak{M}} = \mathfrak{M}'$$

$$\Rightarrow \qquad \mathfrak{M}'' = \mathfrak{M} \subset \mathfrak{N}'' = \mathfrak{N}.$$

To finish, it suffices to show that (c) $\Rightarrow$ (a) but I shall omit the details.]

[Note: On general grounds, $\mathfrak{N} \subset \mathfrak{M} \Rightarrow \Delta_{\mathfrak{M}} \leq \Delta_{\mathfrak{N}}$, so $\underline{\text{Dom}}\, \Delta_{\mathfrak{N}} \subset \underline{\text{Dom}}\, \Delta_{\mathfrak{M}}$ and $\forall\, \Psi \in \underline{\text{Dom}}\, \Delta_{\mathfrak{N}}$, $\langle \Psi, \Delta_{\mathfrak{M}} \Psi \rangle \leq \langle \Psi, \Delta_{\mathfrak{N}} \Psi \rangle$.]

Remark: Another useful fact is this. Suppose that

$$\begin{cases} (\mathfrak{M}_1, \Omega_1) & (\mathfrak{M}_1 \subset \mathcal{B}(\mathcal{H}_1)) \\ (\mathfrak{M}_2, \Omega_2) & (\mathfrak{M}_2 \subset \mathcal{B}(\mathcal{H}_2)) \end{cases}$$

are standard with modular objects $\begin{cases} \Delta_1, J_1 \\ \Delta_2, J_2 \end{cases}$. Let $U: \mathcal{H}_1 \to \mathcal{H}_2$ be a unitary operator: $U \mathfrak{M}_1 U^* = \mathfrak{M}_2$ & $U \Omega_1 = \Omega_2$ -- then

$$\begin{cases} U \Delta_1 U^* = \Delta_2 \\ U J_1 U^* = J_2 . \end{cases}$$

Example: Take $\begin{cases} \mathfrak{M}_1 = \mathfrak{M} \\ \mathfrak{M}_2 = \mathfrak{M}' \end{cases}$ -- then $J \mathfrak{M} J = \mathfrak{M}'$ $(J = J^*)$, hence

$$\begin{cases} \Delta' = J \Delta J = \Delta^{-1} \\ J' = JJJ = J. \end{cases}$$

Assuming that $\mathfrak{N}$ is a W*-subalgebra of $\mathfrak{M}$, put

$$D_{\mathfrak{M},\mathfrak{N}}(t) = \Delta_{\mathfrak{M}}^{-\sqrt{-1}\,t} \; \Delta_{\mathfrak{N}}^{\sqrt{-1}\,t}.$$

Then the function

$$D(t) \equiv D_{\mathfrak{M},\mathfrak{N}}(t)$$

has the following properties:

(1) $D(0) = I$;

(2) $D(t)$ is unitary and strongly continuous in t;

(3) $D(t)\Omega_0 = \Omega_0$;

(4) $D(t)$ has a bounded analytic continuation into the strip $\{z : 0 < \underline{\text{Im}}\, z < 1/2\}$;

(5) $D(t + \frac{\sqrt{-1}}{2})$ is unitary and strongly continuous in t;

(6) $D(s + t) = \sigma_{\mathfrak{M}}^{-t}(D(s))D(t)$;

(7) $D(t + \frac{\sqrt{-1}}{2})^* J_{\mathfrak{M}} D(t) = D(t)^* J_{\mathfrak{M}} D(t + \frac{\sqrt{-1}}{2})$ is independent of t;

(8) $D(t)D(\frac{\sqrt{-1}}{2})^* \mathfrak{M} (D(t)D(\frac{\sqrt{-1}}{2})^*)^{-1} \subset \mathfrak{M}$.

[Properties (1) - (3) are obvious, while (4) and (5) are variations on the usual theme. As for (6), we have

$$D(s + t) = \Delta_{\mathfrak{M}}^{-\sqrt{-1}(s+t)} \Delta_{\mathfrak{N}}^{\sqrt{-1}(s+t)}$$

$$= \Delta_{\mathfrak{M}}^{-\sqrt{-1}t} \Delta_{\mathfrak{M}}^{-\sqrt{-1}s} \Delta_{\mathfrak{N}}^{\sqrt{-1}s} \Delta_{\mathfrak{M}}^{\sqrt{-1}t} \Delta_{\mathfrak{M}}^{-\sqrt{-1}t} \Delta_{\mathfrak{N}}^{\sqrt{-1}t}$$

$$= \Delta_{\mathfrak{M}}^{-\sqrt{-1}t} D(s) \Delta_{\mathfrak{M}}^{\sqrt{-1}t} D(t)$$

$$= \sigma_{\mathfrak{M}}^{-t}(D(s))D(t).$$

The proof of (7) is based on the fact that

$$D(t + \frac{\sqrt{-1}}{2}) = J_{\mathfrak{M}} D(t) J_{\mathfrak{N}}.$$

Thus $\forall\, N' \in \mathfrak{N}'$,

$$D(t + \frac{\sqrt{-1}}{2})N'\Omega_0$$

$$= \Delta_{\mathfrak{M}}^{-\sqrt{-1}(t + \frac{\sqrt{-1}}{2})} \Delta_{\mathfrak{N}}^{\sqrt{-1}(t + \frac{\sqrt{-1}}{2})} N'\Omega_0$$

$$= \Delta_{\mathfrak{M}}^{-\sqrt{-1}(t + \frac{\sqrt{-1}}{2})} \Delta_{\mathfrak{N}}^{\sqrt{-1}t} \Delta_{\mathfrak{N}}^{-1/2} N'\Omega_0$$

$$= \Delta_{\mathfrak{m}}^{-\sqrt{-1}(t+\frac{\sqrt{-1}}{2})} \Delta_{\mathfrak{n}}^{\sqrt{-1}\,t} J_{\mathfrak{n}} S^{*}N' \Omega_0$$

$$= \Delta_{\mathfrak{m}}^{-\sqrt{-1}(t+\frac{\sqrt{-1}}{2})} \Delta_{\mathfrak{n}}^{\sqrt{-1}\,t} J_{\mathfrak{n}} FN' \Omega_0$$

$$= \Delta_{\mathfrak{m}}^{-\sqrt{-1}(t+\frac{\sqrt{-1}}{2})} \Delta_{\mathfrak{n}}^{\sqrt{-1}\,t} J_{\mathfrak{n}} N'^{*} \Omega_0$$

$$= \Delta_{\mathfrak{m}}^{-\sqrt{-1}(t+\frac{\sqrt{-1}}{2})} \Delta_{\mathfrak{n}}^{\sqrt{-1}\,t} J_{\mathfrak{n}} N'^{*} J_{\mathfrak{n}} J_{\mathfrak{n}} \Omega_0$$

$$= \Delta_{\mathfrak{m}}^{-\sqrt{-1}(t+\frac{\sqrt{-1}}{2})} \Delta_{\mathfrak{n}}^{\sqrt{-1}\,t} (J_{\mathfrak{n}} N'^{*} J_{\mathfrak{n}}) \Omega_0$$

$$= \Delta_{\mathfrak{m}}^{-\sqrt{-1}(t+\frac{\sqrt{-1}}{2})} \Delta_{\mathfrak{n}}^{\sqrt{-1}\,t} (J_{\mathfrak{n}} N'^{*} J_{\mathfrak{n}}) \Delta_{\mathfrak{n}}^{-\sqrt{-1}\,t} \Omega_0$$

$$= \Delta_{\mathfrak{m}}^{-\sqrt{-1}\,t} \Delta_{\mathfrak{m}}^{1/2} \Delta_{\mathfrak{n}}^{\sqrt{-1}\,t} (J_{\mathfrak{n}} N'^{*} J_{\mathfrak{n}}) \Delta_{\mathfrak{n}}^{-\sqrt{-1}\,t} \Omega_0$$

$$= \Delta_{\mathfrak{m}}^{-\sqrt{-1}\,t} J_{\mathfrak{m}} J_{\mathfrak{m}} \Delta_{\mathfrak{m}}^{1/2} \Delta_{\mathfrak{n}}^{\sqrt{-1}\,t} (J_{\mathfrak{n}} N'^{*} J_{\mathfrak{n}}) \Delta_{\mathfrak{n}}^{-\sqrt{-1}\,t} \Omega_0$$

$$= \Delta_{\mathfrak{m}}^{-\sqrt{-1}\,t} J_{\mathfrak{m}} S_{\mathfrak{m}} \Delta_{\mathfrak{n}}^{\sqrt{-1}\,t} (J_{\mathfrak{n}} N'^{*} J_{\mathfrak{n}}) \Delta_{\mathfrak{n}}^{-\sqrt{-1}\,t} \Omega_0$$

$$= \Delta_{\mathfrak{m}}^{-\sqrt{-1}\,t} J_{\mathfrak{m}} (\Delta_{\mathfrak{n}}^{\sqrt{-1}\,t} (J_{\mathfrak{n}} N'^{*} J_{\mathfrak{n}}) \Delta_{\mathfrak{n}}^{-\sqrt{-1}\,t})^{*} \Omega_0$$

$$= \Delta_{\mathfrak{m}}^{-\sqrt{-1}\,t} J_{\mathfrak{m}} \Delta_{\mathfrak{n}}^{\sqrt{-1}\,t} (J_{\mathfrak{n}} N' J_{\mathfrak{n}}) \Delta_{\mathfrak{n}}^{-\sqrt{-1}\,t} \Omega_0$$

$$= \Delta_{\mathfrak{m}}^{-\sqrt{-1}\,t} J_{\mathfrak{m}} \Delta_{\mathfrak{n}}^{\sqrt{-1}\,t} J_{\mathfrak{n}} N' \Omega_0$$

$$= J_{\mathfrak{m}} \Delta_{\mathfrak{m}}^{-\sqrt{-1}\,t} \Delta_{\mathfrak{n}}^{\sqrt{-1}\,t} J_{\mathfrak{n}} N' \Omega_0$$

$$= J_{\mathfrak{m}} D(t) J_{\mathfrak{n}} N' \Omega_0 .$$

Since the $N'\Omega_0$ are dense in $\mathcal{H}$, it follows that

$$D(t + \frac{\sqrt{-1}}{2}) = J_{\mathcal{M}} D(t) J_{\mathcal{N}} .$$

Accordingly,

$$D(t + \frac{\sqrt{-1}}{2})^* J_{\mathcal{M}} D(t)$$

$$= J_{\mathcal{N}} D(t)^* J_{\mathcal{M}} J_{\mathcal{M}} D(t) = J_{\mathcal{N}}$$

and

$$D(t)^* J_{\mathcal{M}} D(t + \frac{\sqrt{-1}}{2})$$

$$= D(t)^* J_{\mathcal{M}} J_{\mathcal{M}} D(t) J_{\mathcal{N}} = J_{\mathcal{N}} ,$$

which implies (7). Finally,

$$D(\frac{\sqrt{-1}}{2}) = J_{\mathcal{M}} J_{\mathcal{N}} .$$

Therefore conjugation by $D(t)D(\frac{\sqrt{-1}}{2})^*$ is conjugation by

$\Delta_{\mathcal{M}}^{-\sqrt{-1}\, t} \; \Delta_{\mathcal{N}}^{\sqrt{-1}\, t} \; J_{\mathcal{N}} J_{\mathcal{M}}$. But

$$\Delta_{\mathcal{N}}^{\sqrt{-1}\, t} J_{\mathcal{N}} J_{\mathcal{M}} \mathcal{M} J_{\mathcal{M}} J_{\mathcal{N}} \; \Delta_{\mathcal{N}}^{-\sqrt{-1}\, t}$$

$$= \Delta_{\mathcal{N}}^{\sqrt{-1}\, t} J_{\mathcal{N}} \mathcal{M}' J_{\mathcal{N}} \; \Delta_{\mathcal{N}}^{-\sqrt{-1}\, t}$$

$$\subset \Delta_{\mathcal{N}}^{\sqrt{-1}\, t} J_{\mathcal{N}} \mathcal{N}' J_{\mathcal{N}} \; \Delta_{\mathcal{N}}^{-\sqrt{-1}\, t}$$

$$= \Delta_{\mathcal{N}}^{\sqrt{-1}\, t} \; \mathcal{N} \; \Delta_{\mathcal{N}}^{-\sqrt{-1}\, t}$$

$$= \mathfrak{N} \subset \mathfrak{M},$$

which proves (8).]

An operator valued function D(t) possessing properties (1) - (8) is called a <u>characteristic function</u> of $\mathfrak{M}$.

So, to each W*-subalgebra $\mathfrak{N} \subset \mathfrak{M}$ admitting Ω_0 as a cyclic vector, one can attach a characteristic function, viz. $D_{\mathfrak{M},\mathfrak{N}}$. And: $D_{\mathfrak{M},\mathfrak{N}_1} = D_{\mathfrak{M},\mathfrak{N}_2} \Rightarrow \mathfrak{N}_1 = \mathfrak{N}_2$. In fact,

$$\begin{cases} D_{\mathfrak{M},\mathfrak{N}_1}(\frac{\sqrt{-1}}{2}) = J_{\mathfrak{M}} J_{\mathfrak{N}_1} \\ D_{\mathfrak{M},\mathfrak{N}_2}(\frac{\sqrt{-1}}{2}) = J_{\mathfrak{M}} J_{\mathfrak{N}_2} \end{cases}$$

$$\Rightarrow \quad J_{\mathfrak{N}_1} = J_{\mathfrak{N}_2} \Rightarrow \mathfrak{N}_1 = \mathfrak{N}_2.$$

On the other hand, it can be shown that for any characteristic function D(t), there is a W*-subalgebra $\mathfrak{N} \subset \mathfrak{M}$ admitting Ω_0 as a cyclic vector with $D(t) = \Delta_{\mathfrak{M}}^{-\sqrt{-1}\,t} \Delta_{\mathfrak{N}}^{\sqrt{-1}\,t}$.

<u>LEMMA</u> Suppose that $(\mathfrak{M}, \Omega_0)$ and $(\mathfrak{N}, \Omega_0)$ are standard W*-algebras with $\mathfrak{N} \subset \mathfrak{M}$. Assume: $\Delta_{\mathfrak{M}}$ and $\Delta_{\mathfrak{N}}$ commute -- then $\mathfrak{N} = \mathfrak{M}$.

[We have

$$D(\frac{\sqrt{-1}}{2}) = \Delta_{\mathfrak{M}}^{1/2} \Delta_{\mathfrak{N}}^{-1/2} = J_{\mathfrak{M}} J_{\mathfrak{N}}.$$

Therefore

$$(\Delta_{\mathfrak{M}}^{1/2}\,\Delta_{\mathfrak{N}}^{-1/2})^* = (J_{\mathfrak{M}} J_{\mathfrak{N}})^*.$$

But

$$\Delta_{\mathfrak{M}}^{1/2}\,\Delta_{\mathfrak{N}}^{-1/2} = \Delta_{\mathfrak{N}}^{-1/2}\,\Delta_{\mathfrak{M}}^{1/2}$$

$$\Rightarrow$$

$$(\Delta_{\mathfrak{M}}^{1/2}\,\Delta_{\mathfrak{N}}^{-1/2})^* = \Delta_{\mathfrak{M}}^{1/2}\,\Delta_{\mathfrak{N}}^{-1/2}$$

$$\Rightarrow$$

$$(J_{\mathfrak{M}} J_{\mathfrak{N}})^* = J_{\mathfrak{M}} J_{\mathfrak{N}}.$$

I.e.:

$$J_{\mathfrak{N}} J_{\mathfrak{M}} = J_{\mathfrak{M}} J_{\mathfrak{N}}.$$

Now introduce

$$\begin{cases} F^+(t) = \langle \Omega_0, M'\,\Delta_{\mathfrak{M}}^{-\sqrt{-1}\,t}\,\Delta_{\mathfrak{N}}^{\sqrt{-1}\,t}\,N\,\Omega_0 \rangle \\ F^-(t) = \langle \Omega_0, N\,\Delta_{\mathfrak{N}}^{-\sqrt{-1}\,t}\,\Delta_{\mathfrak{M}}^{\sqrt{-1}\,t}\,M'\,\Omega_0 \rangle, \end{cases}$$

where $M' \in \mathfrak{M}'$, $N \in \mathfrak{N}$. Since $\mathfrak{N} \subset \mathfrak{M} \Rightarrow \mathfrak{M}' \subset \mathfrak{N}'$,

$$\langle \Omega_0, M'\,\Delta_{\mathfrak{M}}^{-\sqrt{-1}\,t}\,\Delta_{\mathfrak{N}}^{\sqrt{-1}\,t}\,N\,\Omega_0 \rangle$$

$$= \langle \Omega_0, \Delta_{\mathfrak{M}}^{-\sqrt{-1}\,t}(\Delta_{\mathfrak{M}}^{\sqrt{-1}\,t}\,M'\,\Delta_{\mathfrak{M}}^{-\sqrt{-1}\,t})(\Delta_{\mathfrak{N}}^{\sqrt{-1}\,t}\,N\,\Delta_{\mathfrak{N}}^{-\sqrt{-1}\,t})\,\Delta_{\mathfrak{N}}^{\sqrt{-1}\,t}\,\Omega_0 \rangle$$

$$= \langle \Omega_0, (\Delta_{\mathfrak{M}}^{\sqrt{-1}\,t}\,M'\,\Delta_{\mathfrak{M}}^{-\sqrt{-1}\,t})(\Delta_{\mathfrak{N}}^{\sqrt{-1}\,t}\,N\,\Delta_{\mathfrak{N}}^{-\sqrt{-1}\,t})\,\Omega_0 \rangle$$

$$= \langle \Omega_0, (\Delta_{\mathfrak{N}}^{\sqrt{-1}\,t}\,N\,\Delta_{\mathfrak{N}}^{-\sqrt{-1}\,t})(\Delta_{\mathfrak{M}}^{\sqrt{-1}\,t}\,M'\,\Delta_{\mathfrak{M}}^{-\sqrt{-1}\,t})\,\Omega_0 \rangle$$

$$= \langle \Omega_0, N \Delta_{\mathcal{N}}^{-\sqrt{-1}\, t} \Delta_{\mathcal{M}}^{\sqrt{-1}\, t} M' \Omega_0 \rangle ,$$

and so

$$F^+(t) = F^-(t).$$

Next,

$$\begin{cases} J_{\mathcal{N}} \Delta_{\mathcal{M}}^{\sqrt{-1}\, t} M' \Delta_{\mathcal{M}}^{-\sqrt{-1}\, t} J_{\mathcal{N}} \in \mathcal{N} \\ J_{\mathcal{M}} \Delta_{\mathcal{N}}^{\sqrt{-1}\, t} N \Delta_{\mathcal{N}}^{-\sqrt{-1}\, t} J_{\mathcal{M}} \in \mathcal{M}'. \end{cases}$$

Therefore

$$F^-(t - \tfrac{\sqrt{-1}}{2}) = \langle \Omega_0, N \Delta_{\mathcal{N}}^{-\sqrt{-1}\, t} \Delta_{\mathcal{N}}^{-1/2} \Delta_{\mathcal{M}}^{1/2} \Delta_{\mathcal{M}}^{\sqrt{-1}\, t} M' \Omega_0 \rangle$$

$$= \langle \Omega_0, \Delta_{\mathcal{N}}^{\sqrt{-1}\, t} N \Delta_{\mathcal{N}}^{-\sqrt{-1}\, t} J_{\mathcal{M}} J_{\mathcal{N}} \Delta_{\mathcal{M}}^{\sqrt{-1}\, t} M' \Delta_{\mathcal{M}}^{-\sqrt{-1}\, t} \Omega_0 \rangle$$

$$= \langle \Omega_0, (J_{\mathcal{M}} \Delta_{\mathcal{N}}^{\sqrt{-1}\, t} N \Delta_{\mathcal{N}}^{-\sqrt{-1}\, t} J_{\mathcal{M}})(J_{\mathcal{N}} \Delta_{\mathcal{M}}^{\sqrt{-1}\, t} M' \Delta_{\mathcal{M}}^{-\sqrt{-1}\, t} J_{\mathcal{N}}) \Omega_0 \rangle$$

$$= \langle \Omega_0, (J_{\mathcal{N}} \Delta_{\mathcal{M}}^{\sqrt{-1}\, t} M' \Delta_{\mathcal{M}}^{-\sqrt{-1}\, t} J_{\mathcal{N}})(J_{\mathcal{M}} \Delta_{\mathcal{N}}^{\sqrt{-1}\, t} N \Delta_{\mathcal{N}}^{-\sqrt{-1}\, t} J_{\mathcal{M}}) \Omega_0 \rangle$$

$$= \langle \Omega_0, M' \Delta_{\mathcal{M}}^{-\sqrt{-1}\, t} J_{\mathcal{N}} J_{\mathcal{M}} \Delta_{\mathcal{N}}^{\sqrt{-1}\, t} N \Omega_0 \rangle$$

$$= \langle \Omega_0, M' \Delta_{\mathcal{M}}^{-\sqrt{-1}\, t} J_{\mathcal{M}} J_{\mathcal{N}} \Delta_{\mathcal{N}}^{\sqrt{-1}\, t} N \Omega_0 \rangle$$

$$= \langle \Omega_0, M' \Delta_{\mathcal{M}}^{-\sqrt{-1}\, t} \Delta_{\mathcal{M}}^{1/2} \Delta_{\mathcal{N}}^{\sqrt{-1}\, t} \Delta_{\mathcal{N}}^{-1/2} N \Omega_0 \rangle$$

$$= F^+(t + \tfrac{\sqrt{-1}}{2}).$$

Applying the usual argument, we conclude that

$$F^{+}(t) = F^{-}(t) = F(0)$$

$\Rightarrow$

$$\langle M'^{*}\Omega_0, \Delta_{\mathcal{M}}^{-\sqrt{-1}\,t} \Delta_{\mathcal{N}}^{\sqrt{-1}\,t} N\Omega_0 \rangle$$

$$= \langle M'^{*}\Omega_0, N\Omega_0 \rangle$$

$\Rightarrow$

$$\Delta_{\mathcal{M}}^{-\sqrt{-1}\,t} \Delta_{\mathcal{N}}^{\sqrt{-1}\,t} N\Omega_0 = N\Omega_0$$

$\Rightarrow$

$$\Delta_{\mathcal{M}}^{-\sqrt{-1}\,t} \Delta_{\mathcal{N}}^{\sqrt{-1}\,t} = I$$

$\Rightarrow$

$$\Delta_{\mathcal{M}}^{\sqrt{-1}\,t} = \Delta_{\mathcal{N}}^{\sqrt{-1}\,t}$$

$\Rightarrow$

$$\Delta_{\mathcal{M}} = \Delta_{\mathcal{N}} .]$$

<u>The Fundamental Lemma</u> Suppose given two standard W*-algebras $(\mathfrak{M}, \Omega_0)$ and $(\mathfrak{N}, \Omega_0)$.

Let $W(t) \in \mathcal{B}(\mathcal{H})$ $(t \in \mathbb{R})$ be an indexed family of bounded linear transformations with the following properties:

(1) $W(t)\Omega_0 = \Omega_0$;

(2) W(t) is unitary and strongly continuous in t;

(3) W(t) has a bounded analytic continuation into the strip $\{z : 0 < \operatorname{Im} z < 1/2\}$;

(4) $W(t + \frac{\sqrt{-1}}{2})$ is unitary and strongly continuous in t;

(5) $W(t)\mathfrak{N}W(t)^* \subset \mathfrak{M}$;

(6) $W(t + \frac{\sqrt{-1}}{2})\mathfrak{N}'W(t + \frac{\sqrt{-1}}{2})^* \subset \mathfrak{M}'$.

Then the relations

$$\begin{cases} \Delta_{\mathfrak{M}}^{\sqrt{-1}\,t}\, W(s)\, \Delta_{\mathfrak{N}}^{-\sqrt{-1}\,t} = W(s-t) \\ J_{\mathfrak{M}}\, W(t)\, J_{\mathfrak{N}} = W(t + \frac{\sqrt{-1}}{2}) \end{cases}$$

constitute the fundamental lemma.

Here is a sketch of the proof. Given $N \in \mathfrak{N}$, $M' \in \mathfrak{M}'$, fix s and define two functions of t:

$$\begin{cases} F^+(t) = \langle \Omega_0, M'\, \Delta_{\mathfrak{M}}^{\sqrt{-1}\,t}\, W(s+t)\, \Delta_{\mathfrak{N}}^{-\sqrt{-1}\,t}\, N\Omega_0 \rangle \\ F^-(t) = \langle \Omega_0, N\, \Delta_{\mathfrak{N}}^{\sqrt{-1}\,t}\, W^*(s+t)\, \Delta_{\mathfrak{M}}^{-\sqrt{-1}\,t}\, M'\Omega_0 \rangle . \end{cases}$$

From the assumptions, $F^+(t)$ has a bounded analytic continuation into the strip $\{z: 0 < \underline{\text{Im}}\ z < 1/2\}$ and $F^-(t)$ has a bounded analytic continuation into the strip $\{z: -1/2 < \underline{\text{Im}}\ z < 0\}$. It is easy to check that

$$F^+(t) = F^-(t) \text{ and } F^+(t + \frac{\sqrt{-1}}{2}) = F^-(t - \frac{\sqrt{-1}}{2}).$$

The data therefore produces, via periodicity, a bounded entire function, which is thus a constant, so

$$\langle \Omega_0, M' \Delta_{\mathfrak{M}}^{\sqrt{-1}\, t} W(s + t) \Delta_{\mathfrak{N}}^{-\sqrt{-1}\, t} N\Omega_0 \rangle$$

$$= \langle \Omega_0, M'W(s)N\Omega_0 \rangle$$

$\Longrightarrow$

$$\Delta_{\mathfrak{M}}^{\sqrt{-1}\, t} W(s + t) \Delta_{\mathfrak{N}}^{-\sqrt{-1}\, t} = W(s)$$

$\Longrightarrow$

$$\Delta_{\mathfrak{M}}^{\sqrt{-1}\, t} W(s) \Delta_{\mathfrak{N}}^{-\sqrt{-1}\, t} = W(s - t).$$

This establishes the first relation. As for the second, its verification proceeds along standard lines and can be omitted (see the preceding section (proof of property (7))).

Example: Take $\mathfrak{M} = \mathfrak{N}$. Suppose that $U(t) = \underline{\exp}(\sqrt{-1}\, t\, H)$ $(H \geq 0)$ is a one parameter unitary group which leaves Ω_0 fixed with

$$U(t)\mathfrak{M}U(t)^{-1} \subset \mathfrak{M} \quad (t \geq 0).$$

Then

$$\Delta^{\sqrt{-1}\, t} U(s) \Delta^{-\sqrt{-1}\, t} = U(s\, e^{-2\pi t})$$

and

$$JU(t)J = U(-t).$$

[Apply the above to $W(t) = U(e^{2\pi t})$. This gives

$$\begin{cases} \Delta^{\sqrt{-1}\,t}\, U(e^{2\pi s})\, \Delta^{-\sqrt{-1}\,t} = U(e^{2\pi s} e^{-2\pi t}) \\ JU(e^{2\pi t})J = U(-e^{2\pi t}). \end{cases}$$

Here

$$W\left(t + \frac{\sqrt{-1}}{2}\right) = U\left(e^{2\pi\left(t + \frac{\sqrt{-1}}{2}\right)}\right)$$

$$= U\left(e^{2\pi t} e^{\pi\sqrt{-1}}\right)$$

$$= U(-e^{2\pi t}).$$

We now claim that

$$\Delta^{\sqrt{-1}\,t}\, U(-e^{2\pi s})\, \Delta^{-\sqrt{-1}\,t} = U(-e^{2\pi s} e^{-2\pi t}).$$

In fact,

$$JU(e^{2\pi s})J = U\left(e^{2\pi\left(s + \frac{\sqrt{-1}}{2}\right)}\right)$$

$$\Rightarrow$$

$$\Delta^{\sqrt{-1}\,t}\, JU(e^{2\pi s})J\, \Delta^{-\sqrt{-1}\,t} = \Delta^{\sqrt{-1}\,t}\, U(-e^{2\pi s})\, \Delta^{-\sqrt{-1}\,t}$$

$$\Rightarrow$$

$$\Delta^{\sqrt{-1}\,t}\, U(-e^{2\pi s})\, \Delta^{-\sqrt{-1}\,t} = J\, \Delta^{\sqrt{-1}\,t}\, U(e^{2\pi s})\, \Delta^{-\sqrt{-1}\,t}\, J$$

$$= J\, U(e^{2\pi s} e^{-2\pi t}) J$$

$$= J\, U(e^{2\pi (s-t)}) J$$

$$= U(-e^{2\pi (s-t)})$$

$$= U(-e^{2\pi s} e^{-2\pi t}).$$

Hence the claim.]

[Note: If instead

$$U(t)\mathfrak{M}U(t)^{-1} \subset \mathfrak{M} \quad (t \leq 0),$$

then

$$\Delta^{\sqrt{-1}\,t} U(s) \Delta^{-\sqrt{-1}\,t} = U(se^{2\pi t})$$

and

$$JU(t)J = U(-t).]$$

Remark: It is too much to require that

$$U(t)\mathfrak{M}U(t)^{-1} \subset \mathfrak{M} \ \forall\, t.$$

Indeed:

$\underline{\text{spec}}\ U \cap -\underline{\text{spec}}\ U = \{0\} \Rightarrow U(t) \in \mathfrak{M}\ \forall\, t \Rightarrow U(t)\mathfrak{M}U(-t) = \mathfrak{M}\ \forall\, t.$

Therefore $U(t)JU(-t) = J \Rightarrow JU(t)J = U(t) \Rightarrow U(t) = U(-t) \Rightarrow I = U(0)$ $= U(t-t) = U(t)U(-t) = U(t)U(t) = U(2t)$, i.e., U is trivial.

The machinery set forth above can be used to give another proof that the translation representation associated with a weakly additive PTV is unique.

Abbreviate $U(I,a)$ to $U(a)$ $(a \in \underset{\sim}{R}^{1,d})$.

LEMMA Let W be a wedge. Suppose that

$$\begin{cases} 0 \neq a_+ \in \overline{V}_+ \ \&\ W + a_+ \subset W \\ 0 \neq a_- \in \overline{V}_- \ \&\ W + a_- \subset W. \end{cases}$$

Then

$$\begin{cases} J_{\mathfrak{M}(W)}\, U(ta_+) J_{\mathfrak{M}(W)} = U(-ta_+) \\ J_{\mathfrak{M}(W)}\, U(ta_-) J_{\mathfrak{M}(W)} = U(-ta_-). \end{cases}$$

[Since

$$\begin{cases} \mathfrak{M}(W + a_+) = U(a_+)\, \mathfrak{M}(W) U(-a_+) \\ \mathfrak{M}(W + a_-) = U(a_-)\, \mathfrak{M}(W) U(-a_-), \end{cases}$$

the relevant one parameter unitary groups are

$$\begin{cases} t \longrightarrow U(ta_+) \\ t \longrightarrow U(-ta_-).] \end{cases}$$

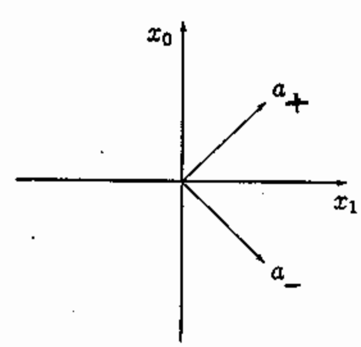
If, as usual,

$$W_R = \{ x \in M: |x_0| < x_1 \},$$

then we can take

$$\begin{cases} a_+ = \{1,1,0,\ldots,0\} \\ a_- = \{-1,1,0,\ldots,0\}. \end{cases}$$

Picture:



Given $x \in M$, write

$$x = x_0 e_0 + x_1 e_1 + y,$$

where $y=(0,0,x_2,\ldots,x_d)$. Since $W_R = W_R + y$, we have

$$\mathfrak{M}(W_R) = U(y)\,\mathfrak{M}(W_R)\,U(y)^{-1}$$

$$\Rightarrow$$

$$J_{\mathfrak{M}(W_R)} = U(y)\,J_{\mathfrak{M}(W_R)}\,U(y)^{-1}$$

$$\Rightarrow$$

$$J_{\mathfrak{M}(W_R)}\,U(y) = U(y)\,J_{\mathfrak{M}(W_R)}.$$

On the other hand, it is clear that a_+ and a_- span the 2-plane $x_0 e_0 + x_1 e_1$. In fact,

$$x_0 e_0 + x_1 e_1 = \left(\frac{x_0 + x_1}{2}\right) a_+ \;+ \left(\frac{x_1 - x_0}{2}\right) a_- .$$

Therefore

$$J_{\mathfrak{M}(W_R)}\,U(x) = J_{\mathfrak{M}(W_R)}\,U(x_0 e_0 + x_1 e_1 + y)$$

$$= J_{\mathfrak{M}(W_R)}\,U\left(\left(\frac{x_0 + x_1}{2}\right) a_+\right) U\left(\left(\frac{x_1 - x_0}{2}\right) a_-\right) U(y)$$

$$= U\left(-\left(\frac{x_0 + x_1}{2}\right) a_+\right) U\left(-\left(\frac{x_1 - x_0}{2}\right) a_-\right) U(y)\,J_{\mathfrak{M}(W_R)}$$

$$= U(-x_0 e_0 - x_1 e_1 + y)\,J_{\mathfrak{M}(W_R)}.$$

So, when $y = 0$,

$$J_{\mathfrak{M}(W_R)}\,U(x) = U(-x)\,J_{\mathfrak{M}(W_R)}.$$

LEMMA $\forall\, a \in \underset{\sim}{R}^{1,d}$, $\exists$ a wedge W_a such that

$$J_{\mathfrak{M}(W_a)}\, U(a) = U(-a) J_{\mathfrak{M}(W_a)}.$$

[Let $a = (a_0, \underset{\sim}{a})$, $\underset{\sim}{a} \in \underset{\sim}{R}^d$. If $\underset{\sim}{a} = 0$, then $W_a = W_R$ will work. Suppose now that $\underset{\sim}{a} \neq 0$. Fix a rotation R_a:

$$R_a^{-1} a = (a_0, |\underset{\sim}{a}|, 0, \ldots, 0)$$

and put

$$W_a = R_a W_R.$$

Then

$$\mathfrak{M}(W_a) = \mathfrak{M}(R_a W_R)$$

$$= U(R_a, 0)\, \mathfrak{M}(W_R) U(R_a, 0)^{-1}$$

$$= U(R_a)\, \mathfrak{M}(W_R) U(R_a)^{-1}$$

$\Rightarrow$

$$J_{\mathfrak{M}(W_a)} = U(R_a) J_{\mathfrak{M}(W_R)}\, U(R_a)^{-1}$$

$\Rightarrow$

$$J_{\mathfrak{M}(W_a)}\, U(a) = U(R_a) J_{\mathfrak{M}(W_R)}\, U(R_a)^{-1}\, U(a)$$

$$= U(R_a) J_{\mathfrak{M}(W_R)}\, U(R_a)^{-1}\, U(a) U(R_a) U(R_a)^{-1}$$

$$= U(R_a) J_{\mathfrak{M}(W_R)}\, U((R_a^{-1}, 0)(I, a)(R_a, 0)) U(R_a)^{-1}$$

$$= U(R_a) J_{\mathfrak{M}(W_R)}\, U((R_a^{-1}, R_a^{-1} a)(R_a, 0)) U(R_a)^{-1}$$

$$= U(R_a)J_{\mathfrak{M}(W_R)}\, U(I, R_a^{-1}a)U(R_a)^{-1}$$

$$= U(R_a)J_{\mathfrak{M}(W_R)}\, U(R_a^{-1}a)U(R_a)^{-1}$$

$$= U(R_a)U(-\, R_a^{-1}a)J_{\mathfrak{M}(W_R)}\, U(R_a)^{-1}$$

$$= U(R_a)U(-\, R_a^{-1}a)U(R_a)^{-1}U(R_a)J_{\mathfrak{M}(W_R)}U(R_a)^{-1}$$

$$= U(-\, R_a R_a^{-1}a)J_{\mathfrak{M}(W_a)}$$

$$= U(-a)J_{\mathfrak{M}(W_a)}.]$$

Application: We have

$$U(2a) = J_{\mathfrak{M}(W_a + a)}\, J_{\mathfrak{M}(W_a)}.$$

[In fact,

$$J_{\mathfrak{M}(W_a + a)} = U(a)J_{\mathfrak{M}(W_a)}\, U(a)^{-1}$$

$\Rightarrow$

$$J_{\mathfrak{M}(W_a + a)}J_{\mathfrak{M}(W_a)} = U(a)J_{\mathfrak{M}(W_a)}U(a)^{-1}J_{\mathfrak{M}(W_a)}$$

$$= U(a)J_{\mathfrak{M}(W_a)}\, U(-a)J_{\mathfrak{M}(W_a)}$$

$$= U(a)U(a)J_{\mathfrak{M}(W_a)}J_{\mathfrak{M}(W_a)}$$

$$= U(2a)J^2_{\mathfrak{M}(W_a)} = U(2a).]$$

It therefore follows that the modular conjugations attached to the wedges determines the translation representation of our weakly additive PTV.

[Note: The assumption of weak additivity figures in the proof of Reeh-Schlieder, which in turn implies that the $(\mathfrak{M}(W), \Omega_0)$ are standard.]

Let $(\mathfrak{M}, \Omega_0)$ be a standard W*-algebra.

LEMMA Suppose that $U: \mathcal{H} \to \mathcal{H}$ is a unitary operator which fixes Ω_0 and has the property that $U \mathfrak{M} U^{-1} \subset \mathfrak{M}$ -- then the operator valued function

$$\Delta^{\sqrt{-1}\, t} \, U \, \Delta^{-\sqrt{-1}\, t}$$

has a bounded analytic continuation into the strip $\{z: -\frac{1}{2} < \underline{\mathrm{Im}}\, z < 0\}$ with continuous boundary values at $\underline{\mathrm{Im}}\, z = -\frac{1}{2}$. Moreover,

$$\| \Delta^{\sqrt{-1}\, z} \, U \, \Delta^{-\sqrt{-1}\, z} \| \leq 1.$$

This lemma can be used to give a quick proof of the converse to the claim of our basic example. Thus let $U(t) = \underline{\exp}\,(\sqrt{-1}\, t H)$ be a one parameter unitary group subject to:

$$\begin{cases} U(t)\, \Omega_0 = \Omega_0 \\ U(t) \mathfrak{M}\, U(t) \subset \mathfrak{M} \quad (t \geq 0). \end{cases}$$

Assume:

$$\Delta^{\sqrt{-1}\, t} \, U(s) \, \Delta^{-\sqrt{-1}\, t} = U(se^{-2\pi t}).$$

Then

$$H \geq 0.$$

[Taking $s = 1$, the operator valued function $t \longrightarrow U(e^{-2\pi t})$, when continued, admits the bound $\| U(e^{-2\pi z}) \| \leq 1$. Now put $z = -\frac{\sqrt{-1}}{4}$ to get

$$\| U(e^{\frac{\pi}{2}\sqrt{-1}}) \| = \| U(\sqrt{-1}) \| = \| e^{-H} \| \leq 1.$$

But this implies that the spectrum of H is nonnegative, hence H is ≥ 0.]

[Note: We have

$$0 \leq \underline{\inf}\ \sigma(e^{-H}) = \underset{\|x\| \leq 1}{\underline{\inf}} \langle x, e^{-H} x \rangle$$

$$\leq \underset{\|x\| \leq 1}{\underline{\sup}} \langle x, e^{-H} x \rangle = \underline{\sup}\ \sigma(e^{-H}) \leq 1.$$

And:

$$\sigma(e^{-H}) = \overline{\left\{ e^{-\lambda} : \lambda \in \sigma(H) \right\}}$$

$$\Longrightarrow \quad e^{-\lambda} \leq 1$$

$$\Longrightarrow \quad \lambda \geq 0$$

$$\Longrightarrow \quad H \geq 0.]$$

<u>The Bisognano-Wichmann Property</u> In this section we shall take $d = 3$ and work in $\mathbb{R}^4 = \mathbb{R}^{1,3}$.

Example: Put

$$\Lambda(t) = \begin{pmatrix} \cosh 2\pi t & -\sinh 2\pi t & 0 & 0 \\ -\sinh 2\pi t & \cosh 2\pi t & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Then $t \longrightarrow \Lambda(t)$ is a one parameter group of boosts taking W_R into itself. Obviously,

$$\Lambda(t)\cdot a_{\pm} = e^{-2\pi t} a_{\pm} .$$

On the other hand, the theory tells us that

$$\Delta_{\mathfrak{M}(W_R)}^{\sqrt{-1}\,t} U(a_{\pm}) \Delta_{\mathfrak{M}(W_R)}^{-\sqrt{-1}\,t} = U(e^{-2\pi t} a_{\pm}) \quad (s = 1).$$

Given $x \in \mathbb{R}^4$, write $x = x_0e_0 + x_1e_1 + x_2e_2 + x_3e_3$ and set $y = x_2e_2 + x_3e_3$ -- then $W_R = W_R + y$, hence

$$\mathfrak{M}(W_R) = U(y)\,\mathfrak{M}(W_R)U(y)^{-1}$$

$$\Rightarrow$$

$$\Delta_{\mathfrak{M}(W_R)}^{\sqrt{-1}\,t} = U(y)\,\Delta_{\mathfrak{M}(W_R)}^{\sqrt{-1}\,t}\,U(y)^{-1}$$

$$\Rightarrow \Delta_{\mathcal{M}(W_R)}^{\sqrt{-1}\,t}\, U(y) = U(y)\, \Delta_{\mathcal{M}(W_R)}^{\sqrt{-1}\,t} .$$

Therefore

$$\Delta_{\mathcal{M}(W_R)}^{\sqrt{-1}\,t}\, U(x)\, \Delta_{\mathcal{M}(W_R)}^{-\sqrt{-1}\,t}$$

$$= \Delta_{\mathcal{M}(W_R)}^{\sqrt{-1}\,t}\, U(x_0 e_0 + x_1 e_1 + y)\, \Delta_{\mathcal{M}(W_R)}^{-\sqrt{-1}\,t}$$

$$= \Delta_{\mathcal{M}(W_R)}^{\sqrt{-1}\,t}\, U\left(\left(\frac{x_0 + x_1}{2}\right) a_+\right) U\left(\left(\frac{x_1 - x_0}{2}\right) a_-\right) \Delta_{\mathcal{M}(W_R)}^{-\sqrt{-1}\,t}\, U(y)$$

$$= U(\Lambda(t)(x_0 e_0 + x_1 e_1))U(y)$$

$$= U(\Lambda(t)(x_0 e_0 + x_1 e_1))U(\Lambda(t)y)$$

$$= U(\Lambda(t)x).$$

For any wedge $W \in \mathcal{W}$, there is a one parameter group $t \to \Lambda_W(t)$ of boosts which maps the wedge into itself.

Definition: A weakly additive PTV satisfies the <u>Bisognano-Wichmann (B-W) property</u> if $\forall W \in \mathcal{W}$,

$$\Delta_{\mathcal{M}(W)}^{\sqrt{-1}\,t} = U(\Lambda_W(t)).$$

[Note: Here,

$$U(\Lambda_W(t)) \equiv U(\Lambda_W(t), 0).$$

And:

$$\Delta_{\mathfrak{M}(W)}^{\sqrt{-1}\,t} U(x) \Delta_{\mathfrak{M}(W)}^{-\sqrt{-1}\,t}$$

$$= U(\Lambda_W(t),0)U(I,x)U(\Lambda_W(t)^{-1},0)$$

$$= U(\Lambda_W(t),\Lambda_W(t)x)U(\Lambda_W(t)^{-1},0)$$

$$= U(I,\Lambda_W(t)x)$$

$$= U(\Lambda_W(t)x).$$

In addition:

$$\Delta_{\mathfrak{M}(W)}^{\sqrt{-1}\,t} \mathfrak{M}(O) \Delta_{\mathfrak{M}(W)}^{-\sqrt{-1}\,t}$$

$$= U(\Lambda_W(t))\mathfrak{M}(O)U(\Lambda_W(t))^{-1}$$

$$= \mathfrak{M}(\Lambda_W(t)\cdot O).]$$

THEOREM Suppose given a weakly additive PTV -- then the B-W property obtains iff the theory satisfies wedge duality and the reality condition.

The proof of this theorem is lengthy so I am going to omit some of it. However, let's at least get the definitions straight.

Ad Wedge Duality: It will be convenient to abuse notation and write $S^{\perp}$ when we really mean the interior of the causal complement of S. By definition, $\forall\, W \in \mathcal{W}$,

$$\mathfrak{M}(W) = \Big(\bigcup_{O \subset W} \mathfrak{M}(O) \Big)'' .$$

Now

$$O \subset W \Rightarrow W^{\perp} \subset O^{\perp}$$

$$\Rightarrow \quad \mathfrak{M}(W^{\perp}) \subset \mathfrak{M}(O^{\perp})$$

$$\Rightarrow \quad \mathfrak{M}(O^{\perp})' \subset \mathfrak{M}(W^{\perp})'.$$

But

$$\mathfrak{M}(O) \subset \mathfrak{M}(O^{\perp})'$$

$$\Rightarrow \quad \bigcup_{O \subset W} \mathfrak{M}(O) \subset \bigcup_{O \subset W} \mathfrak{M}(O^{\perp})' \subset \mathfrak{M}(W^{\perp})'$$

$$\Rightarrow \quad \Big(\bigcup_{O \subset W} \mathfrak{M}(O) \Big)'' \subset \mathfrak{M}(W^{\perp})'''$$

$$= \mathfrak{M}(W^{\perp})'.$$

I.e.:

$$\mathfrak{M}(W) \subset \mathfrak{M}(W^{\perp})'.$$

Definition: A weakly additive PTV satisfies <u>wedge duality</u> if $\forall\, W \in \mathcal{W}$, $\mathfrak{M}(W) = \mathfrak{M}(W^{\perp})'$.

Remark: Examples are known of theories which do not satisfy wedge duality, hence do not satisfy the B-W property.

Observation: To verify wedge duality, it suffices to check that

$$\mathfrak{M}(W_R) = \mathfrak{M}(W_R^{\perp})' \quad (= \mathfrak{M}(W_L)').$$

In fact, if $W \in \mathcal{W}$, then $W = (\Lambda, a) \cdot W_R$ and

$$\begin{aligned}\mathfrak{M}(W) &= \mathfrak{M}((\Lambda,a)\cdot W_R)\\ &= U(\Lambda,a)\,\mathfrak{M}(W_R)U(\Lambda,a)^{-1}\\ &= U(\Lambda,a)\,\mathfrak{M}(W_R^{\perp})'\,U(\Lambda,a)^{-1}\\ &= U(\Lambda,a)\,\mathfrak{M}(W_L)'\,U(\Lambda,a)^{-1}\\ &= \mathfrak{M}((\Lambda,a)\cdot W_L)'\\ &= \mathfrak{M}(W^{\perp})'.\end{aligned}$$

Let K be a double cone centered at the origin and symmetric w.r.t. the (x_0,x_1)-plane. Given $A \in \mathfrak{M}(K)$, put $A(K,x) = U(x)AU(-x)$.

[Note: Consider those x such that $K + x \subset W_R$ -- then

$$\bigcup_x \mathfrak{M}(K + x)$$

generates $\mathfrak{M}(W_R)$, i.e., the $A(K,x)$ are dense in $\mathfrak{M}(W_R)$.]

Notation: By $\mathfrak{A}$ we shall understand the set of all $A \in \mathfrak{M}(K)$ with the following properties:

(i) $\forall\, x$: $K + x \subset W_R$, the function

$$U(\Lambda(t))A(K,x)\Omega_0$$

has a bounded analytic continuation into the strip $\left\{ z: -\frac{1}{2} < \underline{\text{Im}}\, z < 0 \right\}$

with continuous boundary values at $\underline{\text{Im}}\ z = -\frac{1}{2}$.

(ii) $\forall$ x: $K + x \subset W_L$, the function

$$U(\Lambda(t))A^*(K,x)\Omega_0$$

has a bounded analytic continuation into the strip $\{z: 0 < \underline{\text{Im}}\ z < \frac{1}{2}\}$ with continuous boundary values at $\underline{\text{Im}}\ z = \frac{1}{2}$.

[Note: If $K + x \subset W_R$, then $K - x \subset W_L$. In fact,

$$K + x \subset W_R \Rightarrow - K - x \subset - W_R = W_L.$$

But K is symmetric, hence K = - K.]

THEOREM The condition

$$\mathfrak{M}(W_R) = \mathfrak{M}(W_L)'$$

is equivalent to

(R) The set

$$\{A(K,x) : A \in \mathfrak{A},\ K + x \subset W_R\}$$

is dense in $\mathfrak{M}(W_R)$;

AND

(L) The set

$$\{A(K,x) : A \in \mathfrak{A}^*,\ K + x \subset W_L\}$$

is dense in $\mathfrak{M}(W_L)$.

Rappel: Given a standard W*-algebra $(\mathfrak{M}, \Omega_0)$, the function

$$t \to \Delta^{\sqrt{-1}\,t} M\Omega_0 \qquad (M \in \mathfrak{M})$$

has a bounded analytic continuation into the strip $\{z: -\frac{1}{2} < \underline{\mathrm{Im}}\, z < 0\}$ with continuous boundary values at $\underline{\mathrm{Im}}\, z = -\frac{1}{2}$ and the function

$$t \to \Delta^{\sqrt{-1}\,t} M'\Omega_0 \qquad (M' \in \mathfrak{M}')$$

has a bounded analytic continuation into the strip $\{z: 0 < \underline{\mathrm{Im}}\, z < \frac{1}{2}\}$ with continuous boundary values at $\underline{\mathrm{Im}}\, z = \frac{1}{2}$.

The B-W property implies that

$$\mathfrak{M}(W_R) = \mathfrak{M}(W_L)'.$$

Indeed, in this situation, $\mathfrak{A} = \mathfrak{M}(K) = \mathfrak{A}^*$ and R + L holds. For suppose that $K + x \subset W_R$ -- then $\forall A \in \mathfrak{M}(K)$, $A(K,x) \in \mathfrak{M}(W_R)$ and

$$U(\Lambda(t))A(K,x)\Omega_0$$

$$= \Delta_{\mathfrak{M}(W_R)}^{\sqrt{-1}\,t} A(K,x)\Omega_0$$

has the required continuation properties. Similar comments apply if $K + x \subset W_L$. Therefore condition R is satisfied. Ditto for condition L.

LEMMA Suppose that

$$\mathfrak{M}(W_R) = \mathfrak{M}(W_L)'.$$

Then $\forall A \in \mathfrak{A}$ & $\forall x$: $K + x \subset W_R$, $\exists\ \hat{A} \in \mathfrak{A}^*$ such that

$$U(\Lambda(-\frac{\sqrt{-1}}{2}))A(K,x)\Omega_0 = \hat{A}(K,-x)\Omega_0 .$$

Suppose that

$$\mathfrak{M}(W_R) = \mathfrak{M}(W_L)'.$$

Then the theory is said to satisfy the reality condition if

$\forall\ A \in \mathfrak{A} \cap \mathfrak{A}^*\ \&\ \forall x\colon K + x \subset W_R$,

$$\hat{A^*}(K, -x) = \hat{A}(K, -x)^*$$

and

$$\{A(K,x)\Omega_0\colon A \in \mathfrak{A} \cap \mathfrak{A}^*,\ K + x \subset W_R\}$$

is dense in $\mathcal{H}$.

[Note: Since $A \in \mathfrak{A}$, it makes sense to consider $\hat{A}(K, -x)$. But $A \in \mathfrak{A}^* \Leftrightarrow A^* \in \mathfrak{A}$, thus it also makes sense to consider $\hat{A^*}(K, -x)$.]

Assume now that the B-W property obtains -- then, as we have seen above, $\mathfrak{A} = \mathfrak{M}(K) = \mathfrak{A}^*$. In addition,

$$\mathfrak{M}(W_R) = \mathfrak{M}(W_L)'.$$

Therefore the preceding lemma is applicable, hence

$$\begin{aligned}\hat{A^*}(K, -x)\Omega_0 &= U(\Lambda(-\frac{\sqrt{-1}}{2}))A^*(K,x)\Omega_0 \\ &= \Delta^{1/2}_{\mathfrak{M}(W_R)} A^*(K,x)\Omega_0 \\ &= \Delta^{1/2}_{\mathfrak{M}(W_R)} SA(K,x)\Omega_0 \\ &= \Delta^{1/2}_{\mathfrak{M}(W_R)} J_{\mathfrak{M}(W_R)} \Delta^{1/2}_{\mathfrak{M}(W_R)} A(K,x)\Omega_0 \\ &= \Delta^{1/2}_{\mathfrak{M}(W_R)} \Delta^{-1/2}_{\mathfrak{M}(W_R)} J_{\mathfrak{M}(W_R)} A(K,x)\Omega_0\end{aligned}$$

$$= J_{\mathfrak{M}(W_R)} A(K,x) J_{\mathfrak{M}(W_R)} \Omega_0.$$

On the other hand,

$$\hat{A}(K, -x)^* \Omega_0 = S^* \hat{A}(K, -x) \Omega_0$$

$$= J_{\mathfrak{M}(W_R)} \Delta^{-1/2}_{\mathfrak{M}(W_R)} \hat{A}(K, -x) \Omega_0$$

$$= J_{\mathfrak{M}(W_R)} \Delta^{-1/2}_{\mathfrak{M}(W_R)} U(\Lambda(-\frac{\sqrt{-1}}{2})) A(K,x) \Omega_0$$

$$= J_{\mathfrak{M}(W_R)} \Delta^{-1/2}_{\mathfrak{M}(W_R)} \Delta^{1/2}_{\mathfrak{M}(W_R)} A(K,x) \Omega_0$$

$$= J_{\mathfrak{M}(W_R)} A(K,x) J_{\mathfrak{M}(W_R)} \Omega_0.$$

From this, it follows that the reality condition is in force.

[Note: In making the calculation, we have used the fact that $J \Delta^{1/2} = \Delta^{-1/2} J$.]

LEMMA Let s,t be real variables and let σ, τ be complex variables. Suppose given two bounded continuous functions $\begin{cases} F^+(s,t) \\ F^-(s,t) \end{cases}$ with $F^+(s,t) = F^-(s,t)\ \forall\ (s,t) \in \underline{R}^2$. Assume:

(+) $F^+(s,t)$ can be analytically continued into

$$\{\sigma : 0 < \operatorname{Im} \sigma < 1/2\} \times \{\tau : -1/2 < \operatorname{Im} \tau < 0\} \ ;$$

(-) $F^-(s,t)$ can be analytically continued into

$$\{\sigma : -1/2 < \underline{\mathrm{Im}}\,\sigma < 0\} \times \{\tau : 0 < \underline{\mathrm{Im}}\,\tau < 1/2\};$$

$$(\pm)\quad F^{+}\left(s + \frac{\sqrt{-1}}{2}, t - \frac{\sqrt{-1}}{2}\right) = F^{-}\left(s - \frac{\sqrt{-1}}{2}, t + \frac{\sqrt{-1}}{2}\right).$$

Then $\exists$ a function F holomorphic in

$$\left\{ -\tfrac{1}{2} < \underline{\mathrm{Im}}\,\sigma + \underline{\mathrm{Im}}\,\tau < \tfrac{1}{2} \right\}$$

which analytically continues F^{+} and F^{-}. Moreover, $\forall\ z \in \underset{\sim}{C}$,

$$F(\sigma, \tau) = F(\sigma + z, \tau - z).$$

[Note: Therefore

$$F(s,t) = F(s + z, t - z)$$

$\Rightarrow$

$$F(s,t) = F(s + t, 0) \qquad (z = t)$$

$\Rightarrow$

$$F(s, -s) = F(0,0) \qquad (t = -s).]$$

Suppose that the theory satisfies wedge duality and the reality condition -- then we claim that the B-W property obtains. Thus let $A \in \mathfrak{A} \cap \mathfrak{A}^*$, $K + x \subset W_R$, and $B \in \mathfrak{M}(W_L)$. Simplify notation and drop the subscript $\mathfrak{M}(W_R)$ from the modular objects. Put

$$\begin{cases} F^{+}(s,t) = \langle \Omega_0, B\, \Delta^{\sqrt{-1}\, s}\, U(\Lambda(t)) A(K,x)\, \Omega_0 \rangle \\ F^{-}(s,t) = \langle \Omega_0, A(K,x) U(\Lambda(-t))\, \Delta^{-\sqrt{-1}\, s}\, B\, \Omega_0 \rangle \end{cases}.$$

It is easy to see that $\begin{cases} F^{+}(s,t) \\ F^{-}(s,t) \end{cases}$ satisfies condition $\begin{cases} (+) \\ (-) \end{cases}$ of the lemma. Since $\mathfrak{M}(W_L) = \mathfrak{M}(W_R)'$,

$$F^{+}(s,t) = \langle \Omega_0, B\, \Delta^{\sqrt{-1}\, s}\, U(\Lambda(t))A(K,x)\,\Omega_0\rangle$$

$$= \langle \Omega_0, \Delta^{\sqrt{-1}\, s}\, (\Delta^{-\sqrt{-1}\, s}\, B\, \Delta^{\sqrt{-1}\, s})(U(\Lambda(t))A(K,x)U(\Lambda(-t)))U(\Lambda(t))\Omega_0\rangle$$

$$= \langle \Omega_0, (\Delta^{-\sqrt{-1}\, s}\, B\, \Delta^{\sqrt{-1}\, s})(U(\Lambda(t))A(K,x)U(\Lambda(-t)))\,\Omega_0\rangle$$

$$= \langle \Omega_0, (U(\Lambda(t))A(K,x)U(\Lambda(-t)))(\Delta^{-\sqrt{-1}\, s}\, B\, \Delta^{\sqrt{-1}\, s})\,\Omega_0\rangle$$

$$= \langle \Omega_0, A(K,x)U(\Lambda(-t))\,\Delta^{-\sqrt{-1}\, s}\, B\,\Omega_0\rangle = F^{-}(s,t).$$

To apply the lemma, it remains to show

$$(\pm)\quad F^{+}(s + \tfrac{\sqrt{-1}}{2}, t - \tfrac{\sqrt{-1}}{2}) = F^{-}(s - \tfrac{\sqrt{-1}}{2}, t + \tfrac{\sqrt{-1}}{2}).$$

Let's start with the LHS:

$$F^{+}(s + \tfrac{\sqrt{-1}}{2}, t - \tfrac{\sqrt{-1}}{2})$$

$$= \langle \Omega_0, B\, \Delta^{\sqrt{-1}\, s}\, \Delta^{-1/2}\, U(\Lambda(t))U(\Lambda(-\tfrac{\sqrt{-1}}{2}))A(K,x)\,\Omega_0\rangle$$

$$= \langle \Omega_0, B\, \Delta^{-1/2}\, \Delta^{\sqrt{-1}\, s}\, U(\Lambda(t))\hat{A}(K,-x)\,\Omega_0\rangle$$

$$= \langle \Delta^{-1/s}\, B^{*}\,\Omega_0, \Delta^{\sqrt{-1}\, s}\, U(\Lambda(t))\hat{A}(K,-x)\,\Omega_0\rangle$$

$$= \langle \Delta^{-1/2}\, S^{*}B\,\Omega_0, \Delta^{\sqrt{-1}\, s}\, U(\Lambda(t))\hat{A}(K,-x)\,\Omega_0\rangle$$

$$= \langle \Delta^{-1/2}\, J\, \Delta^{-1/2}\, B\,\Omega_0, \Delta^{\sqrt{-1}\, s}\, U(\Lambda(t))\hat{A}(K,-x)\,\Omega_0\rangle$$

$$= \langle JBJ\,\Omega_0, \Delta^{\sqrt{-1}\,s} U(\Lambda(t))\hat{A}(K,-x)\,\Omega_0 \rangle$$

$$= \langle \Omega_0, JB^*J\,\Delta^{\sqrt{-1}\,s} U(\Lambda(t))\hat{A}(K,-x)\,\Omega_0 \rangle .$$

Turning to the RHS, proceed from

$$F^-(s,t) = \langle \Omega_0, A(K,x)U(\Lambda(-t))\,\Delta^{-\sqrt{-1}\,s} B\,\Omega_0 \rangle$$

$$= \langle U(\Lambda(t))A(K,x)^*\,\Omega_0, \Delta^{-\sqrt{-1}\,s} B\,\Omega_0 \rangle$$

to

$$F^-(s - \frac{\sqrt{-1}}{2}, t + \frac{\sqrt{-1}}{2})$$

$$= \langle U(\Lambda(t))U(\Lambda(\frac{\sqrt{-1}}{2}))A(K,x)^*\,\Omega_0, \Delta^{-\sqrt{-1}\,s}\,\Delta^{-1/2} B\,\Omega_0 \rangle$$

$$= \langle U(\Lambda(t))U(\Lambda(-\frac{\sqrt{-1}}{2}))A(K,x)^*\,\Omega_0, \Delta^{-\sqrt{-1}\,s}\,\Delta^{-1/2} B\,\Omega_0 \rangle$$

$$= \langle U(\Lambda(t))\hat{A}^*(K,-x)\,\Omega_0, \Delta^{-\sqrt{-1}\,s}\,\Delta^{-1/2} B\,\Omega_0 \rangle$$

$$= \langle U(\Lambda(t))\hat{A}(K,-x)^*\,\Omega_0, \Delta^{-\sqrt{-1}\,s}\,\Delta^{-1/2} B\,\Omega_0 \rangle$$

$$= \langle U(\Lambda(t))\hat{A}(K,-x)^*\,\Omega_0, \Delta^{-\sqrt{-1}\,s} JJ\,\Delta^{-1/2} B\,\Omega_0 \rangle$$

$$= \langle U(\Lambda(t))\hat{A}(K,-x)^*\,\Omega_0, \Delta^{-\sqrt{-1}\,s} JS^*B\,\Omega_0 \rangle$$

$$= \langle \Omega_0, \hat{A}(K,-x)U(\Lambda(-t))\,\Delta^{-\sqrt{-1}\,s} JB^*J\,\Omega_0 \rangle .$$

But

$$J\,\mathfrak{M}(W_R)'J = \mathfrak{M}(W_R).$$

I.e.:

$$J\,\mathfrak{M}(W_L)J = \mathfrak{M}(W_R)$$

$$\Longrightarrow$$

$$JB^*J \in \mathfrak{M}(W_R).$$

Therefore

$$F^-(s - \tfrac{\sqrt{-1}}{2},\ t + \tfrac{\sqrt{-1}}{2})$$

$$= \langle \Omega_0, U(\Lambda(t))\hat{A}(K,-x)U(\Lambda(-t))\ \Delta^{-\sqrt{-1}\,s} JB^*J\,\Omega_0 \rangle$$

$$= \langle \Omega_0,\ \Delta^{\sqrt{-1}\,s} U(\Lambda(t))\hat{A}(k,-x)U(\Lambda(-t))\ \Delta^{-\sqrt{-1}\,s} JB^*J\,\Omega_0 \rangle$$

$$= \langle \Omega_0, JB^*J\ \Delta^{\sqrt{-1}\,s} U(\Lambda(t))\hat{A}(K,-x)\,\Omega_0 \rangle .$$

Accordingly, condition ($\pm$) is satisfied, hence

$$F(s, -s) = F(0,0)$$

$$\Longrightarrow$$

$$\langle \Omega_0, B\ \Delta^{\sqrt{-1}\,s} U(\Lambda(-s))A(K,x)\,\Omega_0 \rangle$$

$$= \langle \Omega_0, BA(K,x)\,\Omega_0 \rangle .$$

Since $\mathfrak{M}(W_L)\,\Omega_0$ and $\{ A(K,x)\,\Omega_0 \}$ are dense, we then conclude that

$$\Delta^{\sqrt{-1}\,s} U(\Lambda(-s)) = I \quad \forall s,$$

which is equivalent to the B-W property.

<u>Haag Duality</u> Suppose given a weakly additive PTV. Fix a bounded open set O_0 -- then

$$\mathfrak{M}(O_0) \subset \bigcap_{O \perp O_0} \mathfrak{M}(O)'$$

$$= \Big(\bigcup_{O \perp O_0} \mathfrak{M}(O)\Big)'.$$

On the other hand,

$$\mathfrak{M}(O_0^\perp) = \Big(\bigcup_{O \subset O_0^\perp} \mathfrak{M}(O)\Big)''$$

$$= \Big(\bigcup_{O \perp O_0} \mathfrak{M}(O)\Big)''$$

$\Rightarrow$

$$\mathfrak{M}(O_0^\perp)' = \Big(\bigcup_{O \perp O_0} \mathfrak{M}(O)\Big)'''$$

$$= \Big(\bigcup_{O \perp O_0} \mathfrak{M}(O)\Big)'.$$

Definition: O_0 is <u>dual</u> if

$$\mathfrak{M}(O_0) = \mathfrak{M}(O_0^\perp)'.$$

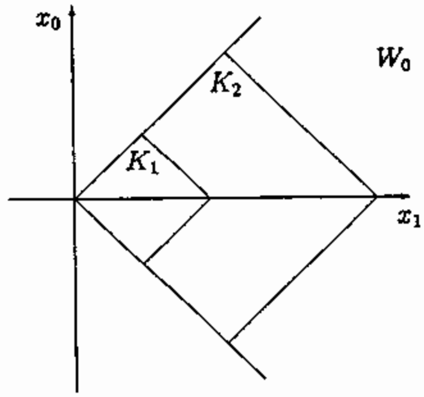
The theory is then said to satisfy <u>Haag duality</u> if each double cone K is dual.

Fact: In the presence of Haag duality, $\forall$ double cone K,

$$\bigcap_{W \supset K} \mathfrak{M}(W) = \bigcap_{W \supset K} \mathfrak{M}(W^\perp)'.$$

<u>LEMMA</u> Suppose that the theory satisfies Haag duality -- then the theory satisfies wedge duality.

[Fix $W_0 \in \mathcal{W}$. Choose an increasing sequence of double cones K_n such that $\bigcup_1^\infty K_n = W_0$.



Let $\mathcal{S}_n = \{(\Lambda,a) : (\Lambda,a)\cdot W_0 \supset K_n\}$ and put

$$\mathcal{S}_\infty = \bigcap_1^\infty \mathcal{S}_n \equiv \{(\Lambda,a) : (\Lambda,a)\cdot W_0 \supset W_0\}.$$

By the above, for n = 1,2,..., we have

$$\bigcap_{W \supset K_n} \mathfrak{M}(W) = \bigcap_{W \supset K_n} \mathfrak{M}(W^\perp)'.$$

But $\forall\, W \in \mathcal{W}$, $\exists\, (\Lambda,a) \in \mathcal{P}_+^\uparrow : (\Lambda,a)\cdot W_0 = W$, hence

$$\bigcap_{\mathcal{S}_n} \mathfrak{M}((\Lambda,a)\cdot W_0) = \bigcap_{\mathcal{S}_n} \mathfrak{M}((\Lambda,a)\cdot W_0^\perp)'$$

$\Rightarrow$

$$\bigcap_1^\infty \bigcap_{\mathcal{S}_n} \mathfrak{M}((\Lambda,a)\cdot W_0) = \bigcap_1^\infty \bigcap_{\mathcal{S}_n} \mathfrak{M}((\Lambda,a)\cdot W_0^\perp)'$$

$\Longrightarrow$

$$\bigcap_{\mathscr{L}_\infty} \mathfrak{M}((\Lambda,a)\cdot W_0) = \bigcap_{\mathscr{L}_\infty} \mathfrak{M}((\Lambda,a)\cdot W_0^\perp)'.$$

But

$$(\Lambda,a)\cdot W_0 \supset W_0$$

$\Longrightarrow$

$$(\Lambda,a)\cdot W_0^\perp \subset W_0^\perp$$

$\Longrightarrow$

$$\mathfrak{M}((\Lambda,a)\cdot W_0^\perp) \subset \mathfrak{M}(W_0^\perp)$$

$\Longrightarrow$

$$\mathfrak{M}(W_0^\perp)' \subset \mathfrak{M}((\Lambda,a)\cdot W_0^\perp)'$$

$\Longrightarrow$

$$\begin{aligned} \mathfrak{M}(W_0^\perp)' &\subset \bigcap_{\mathscr{L}_\infty} \mathfrak{M}((\Lambda,a)\cdot W_0^\perp)' \\ &= \bigcap_{\mathscr{L}_\infty} \mathfrak{M}((\Lambda,a)\cdot W_0) \\ &\subset \mathfrak{M}(W_0) \quad ((I,0) = \underline{\mathrm{id}} \in \mathscr{L}_\infty). \end{aligned}$$

Since the opposite containment

$$\mathfrak{M}(W_0) \subset \mathfrak{M}(W_0^\perp)'$$

is always true, it follows that wedge duality is satisfied.]

<u>Half-Sided Modular Inclusions</u> Suppose given a standard W*-algebra $(\mathfrak{M}, \Omega_0)$.

Notation:

(1) <u>hsmi</u>$(\mathfrak{M})^-$ is the set of W*-subalgebras $\mathfrak{N} \subset \mathfrak{M}$ for which Ω_0 is cyclic and

$$\Delta_{\mathfrak{M}}^{\sqrt{-1}\, t}\, \mathfrak{N}\, \Delta_{\mathfrak{M}}^{-\sqrt{-1}\, t} \subset \mathfrak{N} \quad (t \leq 0).$$

(2) <u>hsmi</u>$(\mathfrak{M})^+$ is the set of W*-subalgebras $\mathfrak{N} \subset \mathfrak{M}$ for which Ω_0 is cyclic and

$$\Delta_{\mathfrak{M}}^{\sqrt{-1}\, t}\, \mathfrak{N}\, \Delta_{\mathfrak{M}}^{-\sqrt{-1}\, t} \subset \mathfrak{N} \quad (t \geq 0).$$

[Note: If $\mathfrak{N} \in$ <u>hsmi</u>$(\mathfrak{M})^{\pm}$, then it is automatic that the pair $(\mathfrak{N}, \Omega_0)$ is standard.]

Remark: The condition

$$\Delta_{\mathfrak{M}}^{\sqrt{-1}\, t}\, \mathfrak{N}\, \Delta_{\mathfrak{M}}^{-\sqrt{-1}\, t} \subset \mathfrak{N} \quad \forall\, t$$

implies that $\mathfrak{N} = \mathfrak{M}$. Indeed, under these circumstances, $\Delta_{\mathfrak{M}}^{\sqrt{-1}\, t}$ is a one parameter group of automorphisms of $\mathfrak{N}$ satisfying the modular condition per ω_0, hence, as the proof of uniqueness shows,

$$\Delta_{\mathfrak{N}} = \Delta_{\mathfrak{M}} \Rightarrow \mathfrak{N} = \mathfrak{M}.$$

Example: Suppose that $U(t) = \underline{\exp}(\sqrt{-1}\, t\, H)$ $(H \geq 0)$ is a one parameter unitary group which leaves Ω_0 fixed with

$$U(t)\mathfrak{M}\, U(t)^{-1} \subset \mathfrak{M} \quad (t \geq 0).$$

Let

$$\mathcal{N}_s = U(s)\mathcal{M}\,U(s)^{-1}.$$

Then $\forall\ s \geq 0$,

$$\mathcal{N}_s \in \underline{\text{hsmi}}(\mathcal{M})^{-}.$$

[In fact, $\forall\ t \geq 0$,

$$\Delta_{\mathcal{M}}^{-\sqrt{-1}\,t}\,\mathcal{N}_s\,\Delta_{\mathcal{M}}^{\sqrt{-1}\,t}$$

$$= \Delta_{\mathcal{M}}^{-\sqrt{-1}\,t}\,U(s)\,\Delta_{\mathcal{M}}^{\sqrt{-1}\,t}\,\Delta_{\mathcal{M}}^{-\sqrt{-1}\,t}\,\mathcal{M}\,\Delta_{\mathcal{M}}^{\sqrt{-1}\,t}\,\Delta_{\mathcal{M}}^{-\sqrt{-1}\,t}\,U(-s)\,\Delta_{\mathcal{M}}^{\sqrt{-1}\,t}$$

$$= U(se^{2\pi t})\,\Delta_{\mathcal{M}}^{-\sqrt{-1}\,t}\,\mathcal{M}\,\Delta_{\mathcal{M}}^{\sqrt{-1}\,t}\,U(-se^{2\pi t})$$

$$= U(s)U((e^{2\pi t}-1)s)\,\mathcal{M}\,U(-(e^{2\pi t}-1)s)U(-s)$$

$$\subset U(s)\mathcal{M}U(-s) = \mathcal{N}_s.]$$

THEOREM Let $\mathcal{N} \in \underline{\text{hsmi}}(\mathcal{M})^{-}$ -- then $\exists$ a one parameter unitary group $U(t) = \underline{\exp}(\sqrt{-1}\,t\,H)$ $(H \geq 0)$ which leaves Ω_0 fixed with

$$U(t)\mathcal{M}\,U(t)^{-1} \subset \mathcal{M} \quad (t \geq 0)$$

such that

$$\mathcal{N} = U(1)\mathcal{M}\,U(-1).$$

To see how U is going to be produced, assume the truth of the theorem -- then

$$\mathfrak{n} = U(1)\,\mathfrak{m}\,U(-1)$$

$\Rightarrow$

$$\Delta_{\mathfrak{n}} = U(1)\,\Delta_{\mathfrak{m}}\,U(-1)$$

$\Rightarrow$

$$D_{\mathfrak{m},\mathfrak{n}}(t) = \Delta_{\mathfrak{m}}^{-\sqrt{-1}\,t}\,\Delta_{\mathfrak{n}}^{\sqrt{-1}\,t}$$

$$= \Delta_{\mathfrak{m}}^{-\sqrt{-1}\,t}\,U(1)\,\Delta_{\mathfrak{m}}^{\sqrt{-1}\,t}\,U(-1)$$

$$= U(e^{2\pi t})U(-1)$$

$$= U(e^{2\pi t}-1).$$

And:

$$D_{\mathfrak{m},\mathfrak{n}}\left(t + \frac{\sqrt{-1}}{2}\right)$$

$$= J_{\mathfrak{m}} D_{\mathfrak{m},\mathfrak{n}}(t) J_{\mathfrak{n}}$$

$$= J_{\mathfrak{m}} U(e^{2\pi t}-1) J_{\mathfrak{n}}$$

$$= J_{\mathfrak{m}}\,U(e^{2\pi t}-1) J_{\mathfrak{m}} J_{\mathfrak{m}} J_{\mathfrak{n}}$$

$$= U(-e^{2\pi t}+1) J_{\mathfrak{m}} J_{\mathfrak{n}}$$

$$= U(-e^{2\pi t}+1) J_{\mathfrak{m}} U(1) J_{\mathfrak{m}}\,U(-1)$$

$$= U(-e^{2\pi t}+1) U(-1) U(-1)$$

$$= U(-e^{2\pi t}-1).$$

It is therefore a question of showing that this data can be used to define a one parameter unitary group with the desired properties.

What we shall do is show that the characteristic function

$$D(t) \equiv D_{\mathcal{M},\mathcal{N}}(t)$$

commutes for different values of the arguments:

$$D(t)D(t') = D(t')D(t).$$

This will prove that U is additive for positive arguments and the rest will follow.

The key technical point is to apply the fundamental lemma to

$$W(t) = D\left(\frac{1}{2\pi} \underline{\log}(e^{2\pi t} + 1)\right),$$

working, however, with $\mathcal{N}$ alone. Therefore one has to check assumptions (1)-(6). Of these, assumptions (1)-(4) are clear (being properties of characteristic functions). Since $\mathcal{N} \in \underline{\text{hsmi}}(\mathcal{M})^{-}$,

$$D(t)\mathcal{N} D(t)^* \subset \mathcal{N} \quad \forall\, t \geq 0$$

$$\Rightarrow$$

$$W(t)\mathcal{N} W(t)^* \subset \mathcal{N} \quad \forall\, t,$$

which verifies assumption (5). Next,

$$D(t + \frac{\sqrt{-1}}{2}) = J_{\mathcal{M}} D(t) J_{\mathcal{N}}$$

$$\Rightarrow$$

$$J_{\mathcal{M}} D(t) J_{\mathcal{N}} \mathcal{N}' J_{\mathcal{N}} D(t)^* J_{\mathcal{M}}$$

$$= J_{\mathcal{M}} D(t) \mathcal{N} D(t)^* J_{\mathcal{M}}$$

$$\subset J_{\mathcal{M}} \mathcal{M} J_{\mathcal{M}} = \mathcal{M}' \subset \mathcal{N}'$$

$\Rightarrow$

$$W(t + \tfrac{\sqrt{-1}}{2})\, \mathcal{N}' W(t + \tfrac{\sqrt{-1}}{2})^* \subset \mathcal{N}',$$

which verifies assumption (6).

So:

$$\Delta_{\mathcal{N}}^{\sqrt{-1}\,t}\, W(s)\, \Delta_{\mathcal{N}}^{-\sqrt{-1}\,t} = W(s-t)$$

or still,

$$\Delta_{\mathcal{N}}^{\sqrt{-1}\,t}\, D\left(\frac{1}{2\pi}\, \underline{\log}\, (e^{2\pi s} + 1)\right) \Delta_{\mathcal{N}}^{-\sqrt{-1}\,t}$$

$$= D\left(\frac{1}{2\pi}\, \underline{\log}\, (e^{2\pi (s-t)} + 1)\right) .$$

Multiply this equation on the left by $\Delta_{\mathcal{M}}^{-\sqrt{-1}\,t}$ and on the right by $\Delta_{\mathcal{N}}^{\sqrt{-1}\,t}$ -- then the LHS becomes

$$\Delta_{\mathcal{M}}^{-\sqrt{-1}\,t}\ \Delta_{\mathcal{N}}^{\sqrt{-1}\,t}\ \Delta_{\mathcal{M}}^{-\sqrt{-1}\,t'}\ \Delta_{\mathcal{N}}^{\sqrt{-1}\,t'},$$

where

$$t' = \frac{1}{2\pi}\, \underline{\log}(e^{2\pi s} + 1).$$

As for the RHS, write

$$t = \frac{1}{2\pi}\, \underline{\log}\ e^{2\pi t}$$

and note that

$$\frac{1}{2\pi}\left[\underline{\log}(e^{2\pi(s-t)}+1)+\underline{\log}\, e^{2\pi t}\right]$$

$$=\frac{1}{2\pi}\,\underline{\log}\,((e^{2\pi(s-t)}+1)e^{2\pi t})$$

$$=\frac{1}{2\pi}\,\underline{\log}\,(e^{2\pi s}+e^{2\pi t})$$

$$=\frac{1}{2\pi}\,\underline{\log}\,(e^{2\pi t}+e^{2\pi t'}-1).$$

We thus end up with

$$\Delta_m^{-\sqrt{-1}\,t}\,\Delta_n^{\sqrt{-1}\,t}\,\Delta_m^{-\sqrt{-1}\,t'}\,\Delta_n^{\sqrt{-1}\,t'}$$

$$=D\left(\frac{1}{2\pi}\,\underline{\log}(e^{2\pi t}+e^{2\pi t'}-1)\right),$$

an expression which is symmetric in t and t'.

And:

$$U(e^{2\pi t}-1)U(e^{2\pi t'}-1)$$

$$=D(t)D(t')$$

$$=D\left(\frac{1}{2\pi}\,\underline{\log}\,(e^{2\pi t}+e^{2\pi t'}-1)\right)$$

$$=U\left(e^{2\pi\left(\frac{1}{2\pi}\,\underline{\log}\,(e^{2\pi t}+e^{2\pi t'}-1)\right)}-1\right)$$

$$=U(e^{2\pi t}+e^{2\pi t'}-2).$$

This establishes additivity for positive arguments. Easy manipulations then lead to additivity for arbitrary arguments.

Remark: The generator H is positive. Indeed, $\Delta_{\mathfrak{N}} \geq \Delta_{\mathfrak{M}} \Rightarrow$ $\underline{\log}\, \Delta_{\mathfrak{N}} \geq \underline{\log}\, \Delta_{\mathfrak{M}}$. But

$$\frac{d}{dt}\left(\Delta_{\mathfrak{M}}^{-\sqrt{-1}\,t}\,\Delta_{\mathfrak{N}}^{\sqrt{-1}\,t}\right)\Big|_{t=0}$$

$$= \sqrt{-1}\,(\underline{\log}\,\Delta_{\mathfrak{N}} - \underline{\log}\,\Delta_{\mathfrak{M}})$$

and H is the closure of

$$\frac{1}{2\pi}\,(\underline{\log}\,\Delta_{\mathfrak{N}} - \underline{\log}\,\Delta_{\mathfrak{M}}).$$

So

$$U(t) = \underline{\exp}(\sqrt{-1}\,tH) \quad (H \geq 0)$$

is a one parameter unitary group which leaves Ω_0 fixed. In addition,

$$U(t)\,\mathfrak{N}\,U(t)^{-1} \subset \mathfrak{N} \quad (t \geq 0).$$

This is obvious if t=0. Suppose, therefore, that t is positive -- then

$$U(e^{2\pi t}-1)\,\mathfrak{N}\,U(e^{2\pi t}-1)^{-1}$$

$$= \Delta_{\mathfrak{M}}^{-\sqrt{-1}\,t}\,\Delta_{\mathfrak{N}}^{\sqrt{-1}\,t}\,\mathfrak{N}\,\Delta_{\mathfrak{N}}^{-\sqrt{-1}\,t}\,\Delta_{\mathfrak{M}}^{\sqrt{-1}\,t}$$

$$= \Delta_{\mathfrak{M}}^{-\sqrt{-1}\,t}\,\mathfrak{N}\,\Delta_{\mathfrak{M}}^{\sqrt{-1}\,t} \subset \mathfrak{N},$$

$\mathfrak{N}$ being by assumption in $\underline{\text{hsmi}}(\mathfrak{M})^-$. Consequently,

$$\Delta_{\mathfrak{N}}^{\sqrt{-1}\,t}\,U(s)\,\Delta_{\mathfrak{N}}^{-\sqrt{-1}\,t} = U(se^{-2\pi t})$$

and

$$J_{\mathcal{N}} U(t) J_{\mathcal{N}} = U(-t).$$

LEMMA We have

$$\Delta_{\mathcal{N}}^{\sqrt{-1}\, t} = U(1)\, \Delta_{\mathcal{M}}^{\sqrt{-1}\, t}\, U(-1).$$

[Take s=1 to get

$$\Delta_{\mathcal{N}}^{\sqrt{-1}\, t}\, U(1)\, \Delta_{\mathcal{N}}^{-\sqrt{-1}\, t} = U(e^{-2\pi t})$$

$\Rightarrow$

$$\begin{aligned} U(-1)\, \Delta_{\mathcal{N}}^{\sqrt{-1}\, t}\, U(1)\, \Delta_{\mathcal{N}}^{-\sqrt{-1}\, t} &= U(-1) U(e^{-2\pi t}) \\ &= U(e^{-2\pi t} - 1) \end{aligned}$$

$\Rightarrow$

$$\begin{aligned} U(-1)\, \Delta_{\mathcal{N}}^{\sqrt{-1}\, t}\, U(1) &= U(e^{-2\pi t} - 1)\, \Delta_{\mathcal{N}}^{\sqrt{-1}\, t} \\ &= \Delta_{\mathcal{M}}^{\sqrt{-1}\, t}\, \Delta_{\mathcal{N}}^{-\sqrt{-1}\, t}\, \Delta_{\mathcal{N}}^{\sqrt{-1}\, t} \\ &= \Delta_{\mathcal{M}}^{\sqrt{-1}\, t} \end{aligned}$$

$\Rightarrow$

$$\Delta_{\mathcal{N}}^{\sqrt{-1}\, t} = U(1)\, \Delta_{\mathcal{M}}^{\sqrt{-1}\, t}\, U(-1).]$$

Let $N \in \mathcal{N}$ -- then $\forall\, t$

$$\Delta_{\mathcal{M}}^{\sqrt{-1}\, t}\, \Delta_{\mathcal{N}}^{-\sqrt{-1}\, t}\, N\, \Delta_{\mathcal{N}}^{\sqrt{-1}\, t}\, \Delta_{\mathcal{M}}^{-\sqrt{-1}\, t} \in \mathcal{M}.$$

I.e.: $\forall$ t,

$$U(e^{-2\pi t}-1)NU(1-e^{-2\pi t}) \in \mathcal{M} .$$

But

$$\underset{t \to +\infty}{\underline{\lim}} \; U(e^{-2\pi t}-1)NU(1-e^{-2\pi t})$$

$$= U(-1)NU(1) \quad \text{(weak operator topology)}$$

$\Rightarrow$

$$U(-1)NU(1) \in \mathcal{M} .$$

Therefore

$$\mathcal{N} \subset U(1)\mathcal{M}U(-1).$$

On the other hand, thanks to the lemma, the modular groups of $(\mathcal{N}, \Omega_0)$ and $(U(1)\mathcal{M}U(-1), \Omega_0)$ are one and the same, hence

$$\mathcal{N} = U(1)\mathcal{M}\,U(-1).$$

It remains to establish that

$$U(t)\mathcal{M}\,U(t)^{-1} \subset \mathcal{M} \quad (t \geq 0).$$

But

$$\Delta_{\mathcal{M}}^{-\sqrt{-1}\,t}\,\Delta_{\mathcal{N}}^{\sqrt{-1}\,t} = U(e^{2\pi t}-1)$$

$\Rightarrow$

$$U(e^{2\pi t})U(-1) = \Delta_{\mathcal{M}}^{-\sqrt{-1}\,t}\,\Delta_{\mathcal{N}}^{\sqrt{-1}\,t}$$

$\Rightarrow$

$$U(e^{2\pi t}) = \Delta_{\mathcal{M}}^{-\sqrt{-1}\,t}\,\Delta_{\mathcal{N}}^{\sqrt{-1}\,t}\,U(1)$$

$\Rightarrow$

$$U(e^{2\pi t})\,\mathcal{M}\,U(e^{-2\pi t})$$

$$= \Delta_{\mathcal{M}}^{-\sqrt{-1}\,t}\,\Delta_{\mathcal{N}}^{\sqrt{-1}\,t}\,U(1)\,\mathcal{M}\,U(-1)\,\Delta_{\mathcal{N}}^{-\sqrt{-1}\,t}\,\Delta_{\mathcal{M}}^{\sqrt{-1}\,t}$$

$$= \Delta_{\mathcal{M}}^{-\sqrt{-1}\,t}\,\Delta_{\mathcal{N}}^{\sqrt{-1}\,t}\,\mathcal{N}\,\Delta_{\mathcal{N}}^{-\sqrt{-1}\,t}\,\Delta_{\mathcal{M}}^{\sqrt{-1}\,t}$$

$$= \Delta_{\mathcal{M}}^{-\sqrt{-1}\,t}\,\mathcal{N}\,\Delta_{\mathcal{M}}^{\sqrt{-1}\,t} \subset \Delta_{\mathcal{M}}^{-\sqrt{-1}\,t}\,\mathcal{M}\,\Delta_{\mathcal{M}}^{\sqrt{-1}\,t} = \mathcal{M}.$$

Observation: In the relation

$$D\left(t + \frac{\sqrt{-1}}{2}\right) = J_{\mathcal{M}}\,D(t)\,J_{\mathcal{N}}$$

take $t = 0$ to get

$$U\left(e^{2\pi\left(\frac{\sqrt{-1}}{2}\right)} - 1\right) = J_{\mathcal{M}}\,J_{\mathcal{N}}\ .$$

I.e.:

$$U(-2) = J_{\mathcal{M}}\,J_{\mathcal{N}}$$

or still,

$$U(2) = J_{\mathcal{N}}\,J_{\mathcal{M}}.$$

LEMMA Suppose that $\mathfrak{N} \in \underline{\text{hsmi}}(\mathfrak{M})^{\pm}$ and $\mathfrak{N} \neq \mathfrak{M}$ -- then $\not\exists$ a type I factor $\mathfrak{L}$:

$$\mathfrak{N} \subset \mathfrak{L} \subset \mathfrak{M} .$$

Example: Suppose given a weakly additive PTV -- then $\forall\, t \geq 0$,

$$\begin{aligned} &\mathfrak{M}(W_R + ta_+) \\ &\quad = U(ta_+)\,\mathfrak{M}(W_R)U(-ta_+) \\ &\quad \subset \mathfrak{M}(W_R). \end{aligned}$$

Therefore (s=1)

$$\mathfrak{M}(W_R + a_+) \in \underline{\text{hsmi}}(\mathfrak{M}(W_R))^{-}.$$

Consequently, the inclusion

$$\mathfrak{M}(W_R + a_+) \longrightarrow \mathfrak{M}(W_R)$$

is not split.

<u>C*-Categories</u> Let $\mathfrak{T}$ be a category.

Notation: Elements of $\underline{\text{Ob}}\,\mathfrak{T}$ will be denoted by $\rho, \sigma, \ldots$ and elements of $\underline{\text{Mor}}\,\mathfrak{T}$ will be denoted by $R, S, \ldots$.

Definition: $\mathfrak{T}$ is a <u>C*-category</u> if the following conditions are satisfied.

(1) The morphism sets are complex Banach spaces, composition of arrows is bilinear, and

$$\| R \circ S \| \leq \| R \| \cdot \| S \| .$$

(2) There is a conjugate linear involutive contravariant functor $*: \mathfrak{T} \to \mathfrak{T}$ which is the identity on objects such that $\| R^* \circ R \| = \| R \|^2$.

[Note: If $R: \rho \to \sigma$, then $R^*: \sigma \to \rho$, thus it makes sense to form $R^* \circ R$.]

Example: A C*-category with a single object is just a C*-algebra.

Remark: In general, $\forall\, \rho \in \underline{\text{Ob}}\,\mathfrak{T}$, $\underline{\text{Mor}}(\rho, \rho)$ is a unital C*-algebra.

[Note: It is also necessary to postulate that $\forall\, R$, $R^* \circ R$ is a positive element of $\underline{\text{Mor}}(\rho, \rho)$, so $R^* \circ R = X^* \circ X$ for some $X \in \underline{\text{Mor}}(\rho, \rho)$.]

Suppose that $\mathfrak{T}$ is a C*-category -- then

(i) $\mathfrak{T}$ is said to have <u>subobjects</u> if given a projection $E \in \underline{\text{Mor}}(\rho, \rho)$ there is a $V \in \underline{\text{Mor}}(\sigma, \rho)$:

$$V^* \circ V = 1_\sigma \ \& \ V \circ V^* = E.$$

(ii) $\mathfrak{T}$ is said to have <u>finite direct sums</u> if given $\rho, \sigma \in \underline{\text{Ob}}\,\mathfrak{T}$ there are $V \in \underline{\text{Mor}}(\rho, \mathfrak{T})$, $W \in \underline{\text{Mor}}(\sigma, \tau)$:

$$\begin{cases} V^* \circ V = 1_\rho \\ W^* \circ W = 1_\sigma \end{cases} \quad \& \ V \circ V^* + W \circ W^* = 1_\tau \ .$$

Suppose that $\mathfrak{T}$ is a C*-category -- then $\mathfrak{T}$ is said to be a <u>strict monoidal</u> C*-category if there is an associative bilinear bifunctor $\otimes: \mathfrak{T} \times \mathfrak{T} \to \mathfrak{T}$ which commutes with $*$ and admits a unit $\iota \in \underline{\text{Ob}}\, \mathfrak{T}$.

In detail: Associativity means that the functors

$$\begin{cases} \otimes(\otimes \times 1): (\mathfrak{T} \times \mathfrak{T}) \times \mathfrak{T} \to \mathfrak{T} \\ \otimes(1 \times \otimes): \mathfrak{T} \times (\mathfrak{T} \times \mathfrak{T}) \to \mathfrak{T} \end{cases}$$

are equal, while the condition on the unit translates to

$$\otimes(\iota \times 1) = \underline{\text{id}}_{\mathfrak{T}} = \otimes(1 \times \iota),$$

where

$$\begin{cases} \iota \times 1: \mathfrak{T} \to \mathfrak{T} \times \mathfrak{T} \\ \rho \to (\iota, \rho) \end{cases} , \quad \begin{cases} 1 \times \iota : \mathfrak{T} \to \mathfrak{T} \times \mathfrak{T} \\ \rho \to (\rho, \iota) \end{cases} .$$

Bifunctoriality says that $1_\rho \otimes 1_\sigma = 1_{\rho \otimes \sigma}$ and

$$(R \otimes S) \circ (R' \otimes S') = (R \circ R') \otimes (S \circ S')$$

whenever the composites $R \circ R'$ and $S \circ S'$ are defined.

[Note: The bifunctor $\otimes$ assigns to each pair of objects ρ, σ an object $\rho \otimes \sigma$ and to each pair of arrows $R: \rho \to \sigma$, $R': \rho' \to \sigma'$

an arrow $R \otimes R': \rho \otimes \rho' \to \sigma \otimes \sigma'$ with

$$R \otimes R' = (1_\sigma \otimes R') \circ (R \otimes 1_{\rho'})$$

or still,

$$R \otimes R' = (R \otimes 1_{\sigma'}) \circ (1_\rho \otimes R').$$

The bilinear operation $\otimes$ is associative on both objects and arrows. In addition,

$$\begin{cases} \rho \otimes \iota = \iota \otimes \rho = \rho \\ 1_\iota \otimes R = R \otimes 1_\iota = R . \end{cases}$$

Finally,

$$(R \otimes R')^* = R^* \otimes R'^*.]$$

Remark: Mor(ι,ι) is an abelian C*-algebra. In fact, if $R, R' \in$ Mor(ι,ι), then

$$\begin{cases} R \otimes R' = (1_\iota \otimes R') \circ (R \otimes 1_\iota) = R' \circ R \\ R \otimes R' = (R \otimes 1_\iota) \circ (1_\iota \circ R') = R \circ R' \end{cases}$$

$$\Longrightarrow$$

$$R' \circ R = R \circ R'.$$

[Note: Mor(ρ,σ) has the structure of a Mor(ι,ι)-bimodule and $\left\{ \begin{matrix} \circ \\ \otimes \end{matrix} \right.$ are Mor(ι,ι) compatible maps.]

Example: Let $\mathfrak{A}$ be a unital C*-algebra -- then by End $\mathfrak{A}$ we shall understand the C*-category whose objects are the unital endomorphisms $\rho : \mathfrak{A} \to \mathfrak{A}$ and whose arrows $\rho \to \sigma$ are the intertwiners, i.e.,

$$\underline{\mathrm{Mor}}(\rho, \sigma) = \{ T \in \mathfrak{A} : T\rho(A) = \sigma(A)T \ \ \forall A \in \mathfrak{A} \} .$$

Here, the composition of arrows, when defined, is given by the product in $\mathfrak{A}$ and $1_{\mathfrak{A}} \in \underline{\mathrm{Mor}}(\rho, \rho)$ serves as the identity: $1_\rho \equiv 1_{\mathfrak{A}}$.

Now $\underline{\mathrm{End}}\ \mathfrak{A}$, in and of itself, is a semigroup: $(\rho \circ \sigma)(A) = \rho(\sigma(A))$.

On the other hand, if $R \in \underline{\mathrm{Mor}}(\rho, \rho')$ and $S \in \underline{\mathrm{Mor}}(\sigma, \sigma')$, then $R\rho(S)$ $(= \rho'(S)R) \in \underline{\mathrm{Mor}}(\rho \circ \sigma, \rho' \circ \sigma')$. In fact,

$$\begin{aligned} R\rho(S)\rho(\sigma(A)) &= R\rho(S\sigma(A)) \\ &= \rho'(S\sigma(A))R \\ &= \rho'(\sigma'(A)S)R \\ &= \rho'(\sigma'(A))\rho'(S)R. \end{aligned}$$

Agreeing to write $R \times S = R\rho(S)$, put

$$\begin{cases} \rho \otimes \sigma = \rho \circ \sigma \\ R \otimes S = R \times S \end{cases}, \quad \iota = \underline{\mathrm{id}}_{\mathfrak{A}} .$$

Then it is clear that with these operations, $\underline{\mathrm{End}}\ \mathfrak{A}$ is a strict monoidal C*-category.

[Note: By definition, $\underline{\mathrm{Mor}}(\iota, \iota) = \{ T \in \mathfrak{A} : TA = AT \ \ \forall A \in \mathfrak{A} \}$, i.e., $\underline{\mathrm{Mor}}(\iota, \iota) = Z_{\mathfrak{A}} \subset \underline{\mathrm{Mor}}(\rho, \rho) \ \ \forall \rho$.]

Suppose that $\mathfrak{T}$ is a strict monoidal C*-category -- then a <u>permutation structure</u> on $\mathfrak{T}$ is a function which assigns to each pair

$(\rho,\sigma) \in \underline{\mathrm{Ob}}\,\mathfrak{T} \times \underline{\mathrm{Ob}}\,\mathfrak{T}$ a unitary element $\varepsilon(\rho,\sigma) \in \underline{\mathrm{Mor}}(\rho \otimes \sigma, \sigma \otimes \rho)$ such that

$$\varepsilon(\sigma,\rho) \circ \varepsilon(\rho,\sigma) = 1_{\rho\otimes\sigma},$$

$$\varepsilon(\iota,\rho) = \varepsilon(\rho,\iota) = 1_{\rho},$$

$$\varepsilon(\rho\otimes\sigma,\tau) = (\varepsilon(\rho,\tau)\otimes 1_{\sigma}) \circ (1_{\rho} \otimes \varepsilon(\sigma,\tau)),$$

$$\varepsilon(\rho',\sigma') \circ (R\otimes S) = (S\otimes R) \circ \varepsilon(\rho,\sigma),$$

where in the last line $R \in \underline{\mathrm{Mor}}(\rho,\rho')$, $S \in \underline{\mathrm{Mor}}(\sigma,\sigma')$.

[Note: The conditions

$$\begin{cases} \varepsilon(\rho,\sigma)^* \circ \varepsilon(\rho,\sigma) = 1_{\rho\otimes\sigma} \\ \varepsilon(\rho,\sigma) \circ \varepsilon(\rho,\sigma)^* = 1_{\sigma\otimes\rho} \end{cases}$$

are tantamount to the unitarity of $\varepsilon(\rho,\sigma)$.]

Remark: A <u>braided structure</u> on $\mathfrak{T}$ is defined by a similar set of axioms except the requirement

$$\varepsilon(\sigma,\rho) \circ \varepsilon(\rho,\sigma) = 1_{\rho\otimes\sigma}$$

is dropped and the assumption

$$\varepsilon(\rho,\sigma\otimes\tau) = (1_{\sigma} \otimes \varepsilon(\rho,\tau)) \circ (\varepsilon(\rho,\sigma)\otimes 1_{\tau})$$

is added to the list.

Suppose that $\mathfrak{T}$ is a strict monoidal C*-category which admits a permutation structure -- then a <u>conjugation structure</u> on $\mathfrak{T}$ is an assignment $\begin{cases} \underline{\mathrm{Ob}}\,\mathfrak{T} \rightarrow \underline{\mathrm{Ob}}\,\mathfrak{T} \\ \rho \rightarrow \bar{\rho} \end{cases}$ together with arrows

$$\begin{cases} R_\rho \in \underline{\text{Mor}}(\iota, \bar{\rho} \otimes \rho) \\ \bar{R}_\rho \in \underline{\text{Mor}}(\iota, \rho \otimes \bar{\rho}), \end{cases}$$

where

$$\bar{R}_\rho = \epsilon(\bar{\rho}, \rho) \circ R_\rho ,$$

subject to

$$\begin{cases} (\bar{R}^*_\rho \otimes 1_\rho) \circ (1_\rho \otimes R_\rho) = 1_\rho \\ (R^*_\rho \otimes 1_{\bar{\rho}}) \circ (1_{\bar{\rho}} \otimes \bar{R}_\rho) = 1_{\bar{\rho}} , \end{cases}$$

the <u>conjugate equations</u>.

[Note: One calls $\bar{\rho}$ a <u>conjugate</u> for ρ .]

<u>LEMMA</u> Suppose that $\rho \in \underline{\text{Ob}}\ \mathcal{T}$ -- then there are natural isomorphisms

$$\begin{cases} \underline{\text{Mor}}(\rho \otimes \sigma, \tau) \longrightarrow \underline{\text{Mor}}(\sigma, \bar{\rho} \otimes \tau) \\ \underline{\text{Mor}}(\sigma \otimes \rho, \tau) \longrightarrow \underline{\text{Mor}}(\sigma, \tau \otimes \bar{\rho}), \end{cases}$$

viz.

$$S \longrightarrow (1_{\bar{\rho}} \otimes S) \circ (R_\rho \otimes 1_\sigma)$$

with inverse

$$S' \longrightarrow (\bar{R}^*_\rho \otimes 1_\tau) \circ (1_\rho \otimes S')$$

and

$$T \longrightarrow (T \otimes 1_{\bar{\rho}}) \circ (1_\sigma \otimes \bar{R}_\rho)$$

with inverse

$$T' \longrightarrow (1_\tau \otimes R^*_\rho) \circ (T' \otimes 1_\rho).$$

[Consider, e.g., the first assertion:

$$1_{\bar\rho} \otimes \Big((\bar{R}^*_{\rho} \otimes 1_\tau) \circ (1_\rho \otimes S') \Big) \circ (R_\rho \otimes 1_\sigma)$$

$$= (1_{\bar\rho} \otimes \bar{R}^*_{\rho} \otimes 1_\tau) \circ (1_{\bar\rho \otimes \rho} \otimes S') \circ (R_\rho \otimes 1_\sigma)$$

$$= (1_{\bar\rho} \otimes \bar{R}^*_{\rho} \otimes 1_\tau) \circ (1_{\bar\rho \otimes \rho} \circ R_\rho) \otimes (S' \circ 1_\sigma)$$

$$= (1_{\bar\rho} \otimes \bar{R}^*_{\rho} \otimes 1_\tau) \circ (R_\rho \otimes S')$$

$$= (1_{\bar\rho} \otimes \bar{R}^*_{\rho} \otimes 1_\tau) \circ (R_\rho \otimes 1_{\bar\rho \otimes \tau}) \circ (1_\tau \otimes S')$$

$$= (1_{\bar\rho} \otimes \bar{R}^*_{\rho} \otimes 1_\tau) \circ (R_\rho \otimes 1_{\bar\rho} \otimes 1_\tau) \circ S'.$$

By hypothesis,

$$(R^*_\rho \otimes 1_{\bar\rho}) \circ (1_{\bar\rho} \otimes \bar{R}_\rho) = 1_{\bar\rho}$$

$$\Longrightarrow$$

$$(1_{\bar\rho} \otimes \bar{R}_\rho)^* \circ (R^*_\rho \otimes 1_{\bar\rho})^* = 1^*_{\bar\rho} = 1_{\bar\rho}$$

$$\Longrightarrow$$

$$(1_{\bar\rho} \otimes \bar{R}^*_\rho) \circ (R_\rho \otimes 1_{\bar\rho}) = 1_{\bar\rho}$$

$$\Longrightarrow$$

$$\Big((1_{\bar\rho} \otimes \bar{R}^*_\rho) \circ (R_\rho \otimes 1_{\bar\rho})\Big) \otimes (1_\tau \circ 1_\tau) = 1_{\bar\rho} \otimes 1_\tau$$

$$\Longrightarrow$$

$$(1_{\bar\rho} \otimes \bar{R}^*_\rho \otimes 1_\tau) \circ (R_\rho \otimes 1_{\bar\rho} \otimes 1_\tau) = 1_{\bar\rho \otimes \tau}.$$

Therefore, upon precomposing with S', we end up with

$$1_{\bar\rho \otimes \tau} \circ S' = S'.$$

Ditto for the other direction.]

[Note: This lemma admits an obvious interpretation in terms of adjoint functors.]

It follows that

$$\underline{\mathrm{Mor}}(\rho,\rho) \approx \underline{\mathrm{Mor}}(\iota,\bar{\rho}\otimes\rho) \approx \underline{\mathrm{Mor}}(\iota,\rho\otimes\bar{\rho}) \approx \underline{\mathrm{Mor}}(\bar{\rho},\bar{\rho}).$$

[In the relation

$$\underline{\mathrm{Mor}}(\rho\otimes\sigma,\tau) \approx \underline{\mathrm{Mor}}(\sigma,\bar{\rho}\otimes\tau)$$

take $\sigma = \iota$ and $\tau = \rho$ to get

$$\underline{\mathrm{Mor}}(\rho,\rho) \approx \underline{\mathrm{Mor}}(\iota,\bar{\rho}\otimes\rho),$$

hence, by symmetry,

$$\underline{\mathrm{Mor}}(\bar{\rho},\bar{\rho}) \approx \underline{\mathrm{Mor}}(\iota,\rho\otimes\bar{\rho}).$$

But from

$$\underline{\mathrm{Mor}}(\sigma\otimes\rho,\tau) \approx \underline{\mathrm{Mor}}(\sigma,\tau\otimes\bar{\rho})$$

with $\sigma = \iota$ and $\tau = \rho$, we also have

$$\underline{\mathrm{Mor}}(\rho,\rho) \approx \underline{\mathrm{Mor}}(\iota,\rho\otimes\bar{\rho}).]$$

LEMMA Fix $\rho \in \underline{\mathrm{Ob}}\,\mathfrak{T}$. Assume: $\exists\, R_1 \in \underline{\mathrm{Mor}}(\iota,\rho_1\otimes\rho)$ such that

$$\begin{cases} (\bar{R}_1^*\otimes 1_\rho)\circ(1_\rho\otimes R_1) = 1_\rho \\ (R_1^*\otimes 1_{\rho_1})\circ(1_{\rho_1}\otimes\bar{R}_1) = 1_{\rho_1} \end{cases} \qquad (\bar{R}_1 = \varepsilon(\rho_1,\rho)\circ R_1).$$

Then there is a unique unitary $U \in \underline{\mathrm{Mor}}(\bar{\rho}, \rho_1)$ with

$$\begin{cases} R_1 = (U \otimes 1_\rho) \circ R_\rho \\ \bar{R}_1 = (1_\rho \otimes U) \circ \bar{R}_\rho \end{cases} .$$

[This is an easy consequence of the preceding lemma.]

<u>Dimension Theory</u> Let $\mathcal{J}$ be a strictly monoidal C*-category having subobjects and finite direct sums as well as a permutation structure and a conjugation structure.

Assumption: $\underline{\text{Mor}}(\iota, \iota) = \underset{\sim}{C} 1_\iota$.

[Note: This implies that $\forall\ \rho \in \underline{\text{Ob}}\ \mathcal{J}$, $\underline{\text{Mor}}(\rho, \rho)$ is finite dimensional.]

Definition: Let $\rho \in \underline{\text{Ob}}\ \mathcal{J}$ -- then by the <u>dimension</u> of ρ, written $d(\rho)$, we understand the complex number

$$R_\rho^* \circ R_\rho \in \underline{\text{Mor}}(\iota, \iota).$$

[Note: Recall that $R_\rho^* \circ R_\rho$ is the composite $\iota \xrightarrow{R_\rho} \bar{\rho} \otimes \rho \xrightarrow{R_\rho^*} \iota$. Observe too that the definition is independent of the choice of R_ρ: $\forall$ unitary $U \in \underline{\text{Mor}}(\bar{\rho}, \rho_1)$,

$$((U \otimes 1_\rho) \circ R_\rho)^* \circ ((U \otimes 1_\rho) \circ R_\rho)$$

$$= R_\rho^* \circ (U \otimes 1_\rho)^* \circ (U \otimes 1_\rho) \circ R_\rho$$

$$= R_\rho^* \circ (U^* \otimes 1_\rho) \circ (U \otimes 1_\rho) \circ R_\rho$$

$$= R_\rho^* \circ [(U^* \circ U) \otimes (1_\rho \otimes 1_\rho)] \circ R_\rho$$

$$= R_\rho^* \circ (1_{\bar{\rho}} \otimes 1_\rho) \circ R_\rho$$

$$= R_\rho^* \circ 1_{\bar{\rho} \otimes \rho} \circ R_\rho$$

$$= R_\rho^* \circ R_\rho .]$$

The dimension function

$$d : \underline{\mathrm{Ob}}\, \mathcal{T} \longrightarrow \mathbf{C}$$

has the following properties:

(1) $d(\rho) \geq 0$;

(2) $d(\rho) = d(\bar{\rho})$;

(3) $d(\rho \otimes \sigma) = d(\rho) d(\sigma)$;

(4) $d(\rho_1 \oplus \rho_2) = d(\rho_1) + d(\rho_2)$.

Remark: It is a fact that the set of unitary equivalence classes of objects in $\mathcal{T}$ with $d(\rho) = 1$ form an abelian group under the monoidal product, the <u>Picard group</u> of $\mathcal{T}$.

Let $\mathbf{P}$ be the strict symmetric monoidal category whose objects are the nonnegative integers, its morphisms being

$$\begin{cases} \underline{\mathrm{Mor}}(n,n) = \mathbf{P}_n & (\underline{\mathrm{Mor}}(0,0) = 1_0) \\ \underline{\mathrm{Mor}}(m,n) = \emptyset & (m \neq n). \end{cases}$$

Here the monoidal structure on objects is defined by addition (i.e., $m \otimes n = m+n$), while on ~~objects~~ morphisms it is given by

$$p \otimes q = \begin{pmatrix} 1 & 2 \ldots m & m+1 \ \ldots \ m+n \\ p(1) & p(2) \ldots p(m) & m+q(1) \ \ldots \ m+q(n) \end{pmatrix} \quad \begin{cases} p \in \mathbf{P}_m \\ q \in \mathbf{P}_n. \end{cases}$$

The symmetry $T : \mathbf{P} \longrightarrow \mathbf{P}$ is the natural isomorphism

$$T_{m,n} : m+n \longrightarrow m+n$$

specified by the permutation

$$(m,n) = \begin{pmatrix} 1 & 2\ldots m & m+1 & m+2\ldots m+n \\ n+1 & n+2\ldots n+m & 1 & 2\ldots n \end{pmatrix}.$$

Fix now an object $\rho \in \underline{\text{Ob}}\ \mathcal{T}$ -- then there is one and only one strict monoidal functor $\varepsilon_\rho : \underset{\sim}{P} \to \mathcal{T}$ with $\varepsilon_\rho(1) = \rho$ such that

$$\begin{aligned} \varepsilon_\rho(\mathsf{T}_{m,n}) &= \mathsf{T}_{\varepsilon_\rho(m),\ \varepsilon_\rho(n)} \\ &\equiv \varepsilon(\varepsilon_\rho(m),\ \varepsilon_\rho(n)) \\ &= \varepsilon(\rho^m, \rho^n). \end{aligned}$$

First, ε_ρ is uniquely defined on objects since we must have $\varepsilon_\rho(0) = \iota$ and $\varepsilon_\rho(n) = \varepsilon_\rho(1 \otimes \cdots \otimes 1) = \rho \otimes \cdots \otimes \rho \equiv \rho^n$. Obviously, $\varepsilon_\rho(1_0) = 1_\iota$ and $\varepsilon_\rho(1_1) = 1_\rho$. Moreover, ε_ρ is uniquely defined on $\underset{\sim}{P}_2$. In fact, $\varepsilon_\rho(1_2) = 1_{\rho^2}$ and $\varepsilon_\rho(\mathsf{T}_{1,1})$ $(= \varepsilon_\rho((1,1))) = \varepsilon(\varepsilon_\rho(1),\ \varepsilon_\rho(1)) = \varepsilon(\rho, \rho)$, as ε_ρ must preserve the symmetry.

Next, for a given $p \in \underset{\sim}{P}_n$ $(n \geq 2)$, there are two possibilities, viz:

(1) $p(1) = 1$ and $p = 1_1 \otimes p'$, where $p' \in \underset{\sim}{P}_{n-1}$;

(2) $p(1) \neq 1$ and $p = (1_1 \otimes p') \circ ((1,1) \otimes 1_{n-2}) \circ (1_1 \otimes q')$,

where $p', q' \in \underset{\sim}{P}_{n-1}$.

Accordingly, if ε_ρ has been defined on $\underset{\sim}{P}_{n-1}$, then it is necessary to set

$$\varepsilon_\rho(p) = 1_\rho \otimes \varepsilon_\rho(p')$$

or

$$\mathcal{E}_\rho(p) = (1_\rho \otimes \mathcal{E}_\rho(p')) \circ (\mathcal{E}(\rho,\rho) \otimes 1_{\rho^{n-2}}) \circ (1_\rho \otimes \mathcal{E}_\rho(q'))$$

as the case may be. So, granted existence, uniqueness follows.

To establish existence, we shall use these formulas to define $\mathcal{E}_\rho$ inductively and claim:

$(a)_n$ $\mathcal{E}_\rho$ is welldefined on $\underset{\sim}{P}_n$;

$(b)_n$ $\mathcal{E}_\rho(p) \circ \mathcal{E}_\rho(q) = \mathcal{E}_\rho(p \circ q)$ $(p,q \in \underset{\sim}{P}_n)$;

$(c)_n$ $\mathcal{E}_\rho(p \otimes q) = \mathcal{E}_\rho(p) \otimes \mathcal{E}_\rho(q)$ $(p \otimes q \in \underset{\sim}{P}_n)$;

$(d)_n$ $\mathcal{E}_\rho((r,s)) = \mathcal{E}_\rho(\rho^r, \rho^s)$ $(r+s = n)$.

[Note: In succession, $(b)_n$ says that $\mathcal{E}_\rho$ is a functor, $(c)_n$ says that $\mathcal{E}_\rho$ is monoidal, and $(d)_n$ says that $\mathcal{E}_\rho$ preserves the symmetry.]

These statements are proved by induction. They are, of course, trivial for $n = 1,2$. Suppose that they have been verified for $k < n$ $(n > 2)$. It is clear that $(a)_n$ is valid on $1_1 \otimes \underset{\sim}{P}_{n-1}$. Assume then that

$$(1_1 \otimes p') \circ ((1,1) \otimes 1_{n-2}) \circ (1_1 \otimes q')$$

$$= (1_1 \otimes p'') \circ ((1,1) \otimes 1_{n-2}) \circ (1_1 \otimes q''),$$

where $p',p'',q',q'' \in \underset{\sim}{P}_{n-1}$. Multiply on the left by $(1_1 \otimes p'')^{-1}$ and on the right by $(1_1 \otimes q')^{-1}$. Changing the notation, we are thus reduced to showing that

$$1_1 \otimes p = ((1,1) \otimes 1_{n-2}) \circ (1_1 \otimes q) \circ ((1,1) \otimes 1_{n-2})$$

$\Rightarrow$

$$1_\rho \otimes \varepsilon_\rho(p) = (\varepsilon(\rho,\rho)\otimes 1_{\rho^{n-2}}) \circ (1_\rho \otimes \varepsilon_\rho(q)) \circ (\varepsilon(\rho,\rho)\otimes 1_{\rho^{n-2}}).$$

Here, $p,q \in \underset{\sim}{P}_{n-1}$ and it is easy to check that actually p=q with $p(1) = q(1) = 1$, so $q = 1_1 \otimes q'$ ($q' \in \underset{\sim}{P}_{n-2}$). Therefore

$$(\varepsilon(\rho,\rho)\otimes 1_{\rho^{n-2}}) \circ (1_\rho \otimes \varepsilon_\rho(q)) \circ (\varepsilon(\rho,\rho)\otimes 1_{\rho^{n-2}})$$

$$= (\varepsilon(\rho,\rho)\otimes 1_{\rho^{n-2}}) \circ (1_\rho \otimes 1_\rho \otimes \varepsilon_\rho(q')) \circ (\varepsilon(\rho,\rho)\otimes 1_{\rho^{n-2}})$$

$$= (\varepsilon(\rho,\rho)\otimes 1_{\rho^{n-2}}) \circ (1_{\rho^2} \otimes \varepsilon_\rho(q')) \circ (\varepsilon(\rho,\rho)\otimes 1_{\rho^{n-2}})$$

$$= (\varepsilon(\rho,\rho)\otimes 1_{\rho^{n-2}}) \circ [(1_{\rho^2} \circ \varepsilon(\rho,\rho))\otimes(\varepsilon_\rho(q') \circ 1_{\rho^{n-2}})]$$

$$= (\varepsilon(\rho,\rho)\otimes 1_{\rho^{n-2}}) \circ (\varepsilon(\rho,\rho) \otimes \varepsilon_\rho(q'))$$

$$= (\varepsilon(\rho,\rho) \circ \varepsilon(\rho,\rho))\otimes(1_{\rho^{n-2}} \circ \varepsilon_\rho(q'))$$

$$= 1_{\rho^2} \otimes \varepsilon_\rho(q')$$

$$= 1_\rho \otimes 1_\rho \otimes \varepsilon_\rho(q')$$

$$= 1_\rho \otimes \varepsilon_\rho(q)$$

$$= 1_\rho \otimes \varepsilon_\rho(p).$$

Consequently, ε_ρ is welldefined. The proofs of the other properties are similar and will be omitted.

Remark: By construction, $\varepsilon_\rho \mid \underset{\sim}{P}_n \equiv \varepsilon_\rho^{(n)}$ is a representation by unitary elements in $\underline{\text{Mor}}(\rho^n, \rho^n)$.

Define now a linear map

$$\Psi_\rho : \underline{\text{Mor}}(\rho^n, \rho^n) \longrightarrow \underline{\text{Mor}}(\rho^{n-1}, \rho^{n-1}) \quad (n \geq 1)$$

by the prescription

$$T \longrightarrow (R^*_\rho \otimes 1_{\rho^{n-1}}) \circ (1_{\bar\rho} \otimes T) \circ (R_\rho \otimes 1_{\rho^{n-1}}).$$

Example: We have

$$\Psi_\rho(1_{\rho^n}) = (R^*_\rho \otimes 1_{\rho^{n-1}}) \circ (1_{\bar\rho} \otimes 1_{\rho^n}) \circ (R_\rho \otimes 1_{\rho^{n-1}})$$

$$= (R^*_\rho \otimes 1_{\rho^{n-1}}) \circ 1_{\bar\rho \otimes \rho^n} \circ (R_\rho \otimes 1_{\rho^{n-1}})$$

$$= (R^*_\rho \otimes 1_{\rho^{n-1}}) \circ (R_\rho \otimes 1_{\rho^{n-1}})$$

$$= (R^*_\rho \circ R_\rho) \otimes (1_{\rho^{n-1}} \circ 1_{\rho^{n-1}})$$

$$= d(\rho)\, 1_\iota \otimes 1_{\rho^{n-1}}$$

$$= d(\rho)\, 1_{\rho^{n-1}}.$$

Example: We have

$$\Psi_\rho((1_\rho \otimes T') \circ T \circ (1_\rho \otimes T'')) = T' \circ \Psi_\rho(T) \circ T''.$$

LEMMA Ψ_ρ is positive: $\forall\, T,\ \Psi_\rho(T^* \circ T) \geq 0$.

[In fact,

$$\Psi_\rho(T^* \circ T) = (R^*_\rho \otimes 1_{\rho^{n-1}}) \circ (1_{\bar\rho} \otimes (T^* \circ T)) \circ (R_\rho \otimes 1_{\rho^{n-1}})$$

$$= (R^*_\rho \otimes 1_{\rho^{n-1}}) \circ (1_{\bar\rho} \otimes T^*) \circ (1_{\bar\rho} \otimes T) \circ (R_\rho \otimes 1_{\rho^{n-1}})$$

and

$$((1_{\bar\rho} \otimes T) \circ (R_\rho \otimes 1_{\rho^{n-1}}))^*$$

$$= (R_\rho \otimes 1_{\rho^{n-1}})^* \circ (1_{\bar\rho} \otimes T)^*$$

$$= (R_\rho^* \otimes 1_{\rho^{n-1}}) \circ (1_{\bar\rho} \otimes T^*).]$$

Iteration

$$\underline{\mathrm{Mor}}(\rho^n, \rho^n) \to \underline{\mathrm{Mor}}(\rho^{n-1}, \rho^{n-1}) \to \cdots \to \underline{\mathrm{Mor}}(\rho, \rho) \to \underline{\mathrm{Mor}}(\iota, \iota)$$

then leads to a positive linear map

$$\Psi_\rho^{(n)}: \underline{\mathrm{Mor}}(\rho^n, \rho^n) \to \underline{\mathrm{Mor}}(\iota, \iota).$$

Since ε_ρ is a functor with $\varepsilon_\rho(n) = \rho^n$, of necessity $\varepsilon_\rho(\underline{\mathrm{Mor}}(n,n)) \subset \underline{\mathrm{Mor}}(\rho^n, \rho^n)$, i.e., $\varepsilon_\rho^{(n)}(\underset{\sim}{P}_n) \subset \underline{\mathrm{Mor}}(\rho^n, \rho^n)$.

It turns out that one can compute $\Psi_\rho(\varepsilon_\rho^{(n)}(p)) \; \forall \, p \in \underset{\sim}{P}_n$. Indeed,

$$\Psi_\rho(\varepsilon_\rho^{(n)}(p)) = \begin{cases} d(\rho)\, \varepsilon_\rho^{(n-1)}(p') & (p(1)=1) \\ \varepsilon_\rho^{(n-1)}(p') & (p(1) \neq 1). \end{cases}$$

Here $p' \in \underset{\sim}{P}_{n-1}$ and $p = 1_1 \otimes p'$ if $p(1) = 1$ while if $p(1) \neq 1$, then p' is that element of $\underset{\sim}{P}_{n-1}$ obtained by dropping the number 1 in the decomposition of p into cycles and replacing the remaining k by k-1 $(k \neq 1)$.

To illustrate, consider the first possibility:

$$\Psi_\rho(\varepsilon_\rho^{(n)}(p)) = \Psi_\rho(1_\rho \otimes \varepsilon_\rho^{(n-1)}(p'))$$

$$= \Psi_\rho((1_\rho \otimes \varepsilon_\rho^{(n-1)}(p')) \circ 1_{\rho^n} \circ (1_\rho \otimes 1_{\rho^{n-1}}))$$

$$= \varepsilon_\rho^{(n-1)}(p') \circ \Psi_\rho(1_{\rho^n}) \circ 1_{\rho^{n-1}}$$

$$= \varepsilon_\rho^{(n-1)}(p') \circ d(\rho) 1_{\rho^{n-1}}$$

$$= d(\rho)\, \varepsilon_\rho^{(n-1)}(p').$$

One may attach to $\varepsilon_\rho^{(n)}$ two canonical projections, namely

$$\left\{ \begin{array}{l} S_\rho^{(n)} = \dfrac{1}{n!} \displaystyle\sum_{p \in \underset{\sim}{P}_n} \varepsilon_\rho^{(n)}(p) \\ \\ A_\rho^{(n)} = \dfrac{1}{n!} \displaystyle\sum_{p \in \underset{\sim}{P}_n} \underline{\text{sgn}}\, p \;\varepsilon_\rho^{(n)}(p), \end{array} \right. \quad \in \underline{\text{Mor}}(\rho^n, \rho^n)$$

the symmetric and antisymmetric projections, respectively.

Rappel: For any real number d,

$$\binom{d}{n} \equiv d(d-1)\cdots(d-n+1)$$

$$= \frac{1}{n!} \sum_{p \in \underset{\sim}{P}_n} \underline{\text{sgn}}\, p \cdot d^{m(p)},$$

where m(p) is the number of cycles in p.

Since

$$\Psi_\rho^n(\varepsilon_\rho^{(n)}(p)) = d(\rho)^{m(p)},$$

it follows that

$$\Psi_{\rho}^{(n)}(A_{\rho}^{(n)})$$

$$= \frac{1}{n!} \sum_{p \in \underset{\sim}{P}_n} \underline{\text{sgn}}\ p \cdot \Psi_{\rho}^{(n)}(\varepsilon_{\rho}^{(n)}(p))$$

$$= \frac{1}{n!} \sum_{p \in \underset{\sim}{P}_n} \underline{\text{sgn}}\ p \cdot d(\rho)^{m(p)}$$

$$= \binom{d(\rho)}{n} .$$

But $A_{\rho}^{(n)}$ is a positive element of $\underline{\text{Mor}}(\rho^n, \rho^n)$, hence $\Psi_{\rho}^{(n)}(A_{\rho}^{(n)})$ is a positive element of $\underline{\text{Mor}}(\iota, \iota)$. In other words: $\forall\ n \geq 1$,

$$d(\rho)(d(\rho)-1)\cdots(d(\rho)-n+1) \geq 0,$$

which is possible only if $d(\rho)$ is a nonnegative integer.

<u>DHR Theory</u> Suppose given a weakly additive PTV with a unique vacuum which satisfies Haag duality. Let $\mathfrak{A}$ be the quasilocal algebra of the theory, $\mathfrak{M}$ the global algebra of the theory -- then $\mathfrak{M} = \mathfrak{A}''$, i.e., $\mathcal{B}(\mathcal{H}) = \mathfrak{A}''$, hence $\mathfrak{A}' = \mathbb{C}I \Rightarrow Z_{\mathfrak{A}} = \mathbb{C}I$.

[Note: Recall that Haag duality means that each double cone K is dual, i.e., $\mathfrak{M}(K) = \mathfrak{M}(K^{\perp})'$.]

Notation: $\forall$ open subset G of $\mathbb{R}^{1,d}$ (bounded or unbounded), put

$$\mathfrak{A}(G) = C^*\Big(\bigcup_{O \subset G} \mathfrak{M}(O)\Big) .$$

Then

$$\mathfrak{A}(G)'' = \mathfrak{M}(G).$$

[Note: So, for a double cone K, $\mathfrak{M}(K) = \mathfrak{A}(K^{\perp})''' = \mathfrak{A}(K^{\perp})'$.]

Consider now <u>End</u> $\mathfrak{A}$, the unital endomorphisms of $\mathfrak{A}$ -- then, as has been seen earlier, <u>End</u> $\mathfrak{A}$ carries the structure of a strict monoidal C*-category.

[Note: A unital endomorphism $\rho : \mathfrak{A} \to \mathfrak{A}$ is necessarily injective. Proof: The kernel of ρ is a closed ideal and $\mathfrak{A}$ is simple. Accordingly, $\forall\ A \in \mathfrak{A}$, $\|\rho(A)\| = \|A\|$.]

Definition: Objects $\rho, \sigma \in$ <u>End</u> $\mathfrak{A}$ are said to be <u>unitarily equivalent</u> if $\exists$ a unitary $U \in$ <u>Mor</u>(ρ, σ).

[Note: U is unitary if $U^*U = 1_{\mathfrak{A}} = UU^*$.]

This notion splits the objects of <u>End</u> $\mathfrak{A}$ into equivalence classes $[\rho]$.

Example: $U \in$ <u>Mor</u>$(\iota, \rho) \Rightarrow UA = \rho(A)U\ \ \forall\ A \in \mathfrak{A} \Rightarrow UAU^{-1} = \rho(A)\ \ \forall\ A \in \mathfrak{A}$, so $[\iota]$ consists of the inner automorphisms of $\mathfrak{A}$

corresponding to the unitary elements in $\mathfrak{A}$.

We shall now single out a full C*-subcategory $\mathfrak{T} \subset \underline{\text{End}}\, \mathfrak{A}$ with the following properties:

$$\begin{cases} \iota \in \underline{\text{Ob}}\, \mathfrak{T} \\ \rho, \sigma \in \underline{\text{Ob}}\, \mathfrak{T} \Rightarrow \rho \circ \sigma \in \underline{\text{Ob}}\, \mathfrak{T} \\ \rho \in \underline{\text{Ob}}\, \mathfrak{T} \Rightarrow [\rho] \subset \underline{\text{Ob}}\, \mathfrak{T}. \end{cases}$$

In particular, therefore, $\mathfrak{T}$ is strict monoidal.

Definition: A unital endomorphism ρ of $\mathfrak{A}$ is said to be <u>localizable</u> if $\exists\, O_0$: $\rho(A) = A \ \forall A \in \mathfrak{A}(O_0^{\perp})$, in which case ρ is <u>localized in O_0</u>.

[Note: Accordingly, ρ is also localizable in O if $O_0 \subset O$, thus a localizable ρ can always be localized in a double cone K_0.]

Definition: A unital endomorphism ρ of $\mathfrak{A}$ is said to be <u>transportable</u> if $\forall\, O_0\ \exists\, \rho_0 \in [\rho]$: ρ_0 is localized in O_0.

A <u>DHR endomorphism</u> is a unital ρ which is localizable and transportable.

Example: The unit ι is DHR.

Remark: Let π_0 be the vacuum representation of $\mathfrak{A}$ ($\pi_0(A) = A \ \forall A \in \mathfrak{A}$ which, up to unitary equivalence, is the GNS representation attached to the vacuum state ω_0) -- then by DHR(π_0) we understand the set whose elements are those representations π of $\mathfrak{A}$ such that $\forall$ double cone K, $\pi \mid \mathfrak{A}(K^{\perp})$ is unitarily equivalent to $\pi_0 \mid \mathfrak{A}(K^{\perp})$

or still, those representations π of $\mathfrak{A}$ such that $\exists$ a DHR endomorphism ρ with the property that π is unitarily equivalent to $\pi_0 \circ \rho$.

Notation: $\mathfrak{I}$ is the full subcategory of <u>End</u> $\mathfrak{A}$ whose objects are the DHR endomorphisms.

<u>LEMMA</u> <u>Ob</u> $\mathfrak{I}$ is closed w.r.t. products: $\rho, \sigma \in \underline{\text{Ob}}\ \mathfrak{I} \Rightarrow \rho \circ \sigma \in \underline{\text{Ob}}\ \mathfrak{I}$.

[Suppose that ρ is localized in K and σ is localized in L. Choose a double cone D: $D \supset K \cup L$ -- then $D^\perp \subset K^\perp \cap L^\perp$ and $\mathfrak{A}(D^\perp) \subset \mathfrak{A}(K^\perp) \cap \mathfrak{A}(L^\perp)$, thus $\forall\ A \in \mathfrak{A}(D^\perp)$, $(\rho \circ \sigma)(A) = \rho(\sigma(A)) = \rho(A) = A$, so $\rho \circ \sigma$ is localized in D. It remains to prove that $\rho \circ \sigma$ is transportable. For this purpose, let K_0 be an arbitrary double cone and let $\rho_0 \in [\rho]$, $\sigma_0 \in [\sigma]$ be localized in K_0. Write $\rho_0 = U_0 \rho U_0^{-1}$, $\sigma_0 = V_0 \sigma V_0^{-1}$ $(U_0, V_0$ unitary) -- then $\forall\ A \in \mathfrak{A}(K_0^\perp)$, $(\rho_0 \circ \sigma_0)(A) = \rho_0(\sigma_0(A)) = \rho_0(A) = A$, hence $\rho_0 \circ \sigma_0$ is localized in K_0. On the other hand, $\forall\ A \in \mathfrak{A}$,

$$
\begin{aligned}
(\rho_0 \circ \sigma_0)(A) &= \rho_0(\sigma_0(A)) \\
&= \rho_0(V_0 \sigma(A) V_0^{-1}) \\
&= (U_0 \rho(V_0))\ (\rho \circ \sigma)(A) (U_0 \rho(V_0))^{-1}
\end{aligned}
$$

$$\Rightarrow$$

$$\rho_0 \circ \sigma_0 \in [\rho \circ \sigma],$$

$U_0 \rho(V_0)$ being unitary.]

[Note: Maintaining the above notation, assume in addition that $K \perp L$ -- then $\rho \circ \sigma = \sigma \circ \rho$.]

Observation: Let $\rho, \sigma \in \underline{\text{Ob}}\, \mathfrak{T}$ and let $T \in \underline{\text{Mor}}(\rho, \sigma)$ -- then $\exists\, K: T \in \mathfrak{M}(K)$.

[Suppose that ρ is localized in O and σ is localized in P. Since $\forall\, A \in \mathfrak{A}$, $T\rho(A) = \sigma(A)T$, it follows that $\forall\, A \in \mathfrak{A}(O^{\perp}) \cap \mathfrak{A}(P^{\perp})$, $TA = AT \Rightarrow T \in (\mathfrak{A}(O^{\perp}) \cap \mathfrak{A}(P^{\perp}))'$. Fix a double cone $K: O \cup P \subset K \Rightarrow K^{\perp} \subset O^{\perp} \cap P^{\perp} \Rightarrow \mathfrak{A}(K^{\perp}) \subset \mathfrak{A}(O^{\perp}) \cap \mathfrak{A}(P^{\perp}) \Rightarrow (\mathfrak{A}(O^{\perp}) \cap \mathfrak{A}(P^{\perp}))' \subset \mathfrak{A}(K^{\perp})' \Rightarrow T \in \mathfrak{A}(K^{\perp})' = \mathfrak{M}(K)$.]

[Note: This argument does not require that $T \in \mathfrak{A}$: It is valid for any $T \in \mathfrak{B}(\mathcal{H})$ with $T\rho(A) = \sigma(A)T \quad \forall\, A \in \mathfrak{A}$.]

Fact: $\mathfrak{T}$ is closed w.r.t. the formation of subobjects.

[Let $E \in \underline{\text{Mor}}(\rho, \rho)$ be a projection: $E\rho(A) = \rho(A)E \quad \forall\, A \in \mathfrak{A}$. Choose $K: E \in \mathfrak{M}(K)$ -- then, thanks to the Borchers property, $\exists\, L \supset K$ and a partial isometry $V \in \mathfrak{M}(L)$ such that $V^*V = 1_{\mathfrak{A}}$ & $VV^* = E$. Define σ by

$$\sigma(A) = V^* \rho(A) V.$$

Since $VV^* = E \Rightarrow V = VV^*V = EV$, $\forall\, A, B \in \mathfrak{A}$, we have

$$\begin{aligned}\sigma(AB) &= V^* \rho(AB) V \\ &= V^* \rho(A) \rho(B) V \\ &= V^* \rho(A) \rho(B) E V \\ &= V^* \rho(A) E \rho(B) V\end{aligned}$$

$$= V^* \rho(A) V \cdot V^* \rho(B) V$$

$$= \sigma(A)\, \sigma(B).$$

And:

$$\sigma(1_{\mathfrak{A}}) = V^* \rho(1_{\mathfrak{A}}) V$$

$$= V^* V$$

$$= 1_{\mathfrak{A}}.$$

Therefore σ is a unital endomorphism of $\mathfrak{A}$. But

$$\sigma(A) = V^* \rho(A) V$$

$$\Rightarrow$$

$$V\sigma(A) = VV^* \rho(A) V$$

$$= E\, \rho(A) V$$

$$= \rho(A) E V = \rho(A) V$$

$$\Rightarrow$$

$$V \in \underline{\mathrm{Mor}}(\sigma, \rho).$$

So, to complete the proof, one has to check that σ is DHR. First, σ is localized in L. In fact, $L^{\perp} \subset K^{\perp} \Rightarrow \mathfrak{A}(L^{\perp}) \subset \mathfrak{A}(K^{\perp})$, thus $\forall\, A \in \mathfrak{A}(L^{\perp})$,

$$\sigma(A) = V^* \rho(A) V = V^* A V = V^* V A = A,$$

because $V \in \mathfrak{M}(L)$ and $\mathfrak{M}(L)' = \mathfrak{M}(L^{\perp}) \supset \mathfrak{A}(L^{\perp})$. Second, σ is transportable. To see this, given K_0, choose K':

$$K' \subset K_0 \text{ and } \underline{\mathrm{dis}}(K', \underline{\mathrm{fr}}\, K_0) > 0.$$

Since ρ is transportable, $\exists\, U$: $\rho' = U\rho U^{-1}$ is localized in K'. Put $E' = UEU^{-1}$ -- then another appeal to the Borchers property produces a double cone L': $K' \subset L' \subset K_0$ and a partial isometry $V' \in \mathfrak{M}(L')$: $V'^*V' = 1_{\mathfrak{A}}$ & $V'V'^* = E'$. Let

$$\sigma'(A) = V'^*\rho'(A)V' \quad (A \in \mathfrak{A}).$$

Then σ' is a unital endomorphism of $\mathfrak{A}$ which is localized in L', hence in K_0. Moreover, $\forall\, A \in \mathfrak{A}$,

$$\begin{aligned}
\sigma'(A) &= V'^*U\,\rho(A)U^{-1}V' \\
&= V'^*U\,\rho(A)U^{-1}E'V' \\
&= V'^*U\,\rho(A)U^{-1}UEU^{-1}V' \\
&= V'^*U\,\rho(A)EU^{-1}V' \\
&= V'^*U\,\rho(A)EEU^{-1}V' \\
&= V'^*UE\,\rho(A)EU^{-1}V' \\
&= V'^*UVV^*\,\rho(A)VV^*U^{-1}V' \\
&= V'^*UV\,\sigma(A)V^*U^{-1}V'.
\end{aligned}$$

Finally, V'^*UV is unitary:

$$\begin{cases} V'^*UVV^*U^{-1}V' = V'^*UEU^{-1}V' = V'^*E'V' = V'^*V' = 1_{\mathfrak{A}} \\ V^*U^{-1}V'V'^*UV = V^*U^{-1}E'UV = V^*EV = V^*V = 1_{\mathfrak{A}} \end{cases}$$

$$\Longrightarrow$$

$$\sigma' \in [\sigma].$$

Therefore σ is transportable.]

Fact: $\mathfrak{T}$ is closed w.r.t. the formation of finite direct sums.

Rappel: Given $R \in \underline{\mathrm{Mor}}(\rho, \rho')$ and $S \in \underline{\mathrm{Mor}}(\sigma, \sigma')$, by definition

$$R \times S = R\,\rho(S) \quad (= \rho'(S)R)$$

is the monoidal operation on arrows (on objects, $\otimes$ is simply composition of unital endomorphisms).

Remark: Suppose that $d \geq 2$ ($\Rightarrow \underline{\dim}\, R^{1,d} \geq 3$) -- then $\forall$ double cone K, $K^{\perp}$ is path connected. This said, assume that

$$\begin{cases} \rho \text{ is localized in } K \\ \rho' \text{ is localized in } K' \end{cases} \quad \text{and} \quad \begin{cases} \sigma \text{ is localized in } L \\ \sigma' \text{ is localized in } L'. \end{cases}$$

Then if

$$K \perp L \text{ and } K' \perp L',$$

we have

$$R \times S = S \times R.$$

Our next objective is to equip $\mathfrak{T}$ with a permutation structure, i.e., to construct a function $\mathcal{E}$ which assigns to each pair $(\rho, \sigma) \in \underline{\mathrm{Ob}}\,\mathfrak{T} \times \underline{\mathrm{Ob}}\,\mathfrak{T}$ a unitary element $\mathcal{E}(\rho, \sigma) \in \underline{\mathrm{Mor}}(\rho \circ \sigma, \sigma \circ \rho)$ satisfying the usual conditions. To this end, it will be necessary to slightly restrict the generality.

Convention: Assume henceforth that $d \geq 2$.

Suppose given DHR endomorphisms ρ and σ. Choose double cones $K_{\rho_0}, L_{\sigma_0}: K_{\rho_0} \perp L_{\sigma_0}$ and choose $\rho_0 \in [\rho]$, $\sigma_0 \in [\sigma]$: ρ_0 is localized in K_{ρ_0} and σ_0 is localized in L_{σ_0} ($\Rightarrow \rho_0 \circ \sigma_0 = \sigma_0 \circ \rho_0$). Choose unitary elements $U_\rho \in \underline{\mathrm{Mor}}(\rho, \rho_0)$, $U_\sigma \in \underline{\mathrm{Mor}}(\sigma, \sigma_0)$.

Definition: Put

$$\varepsilon(\rho,\sigma) = (U_\sigma^* \times U_\rho^*)(U_\rho \times U_\sigma)$$

$$\in \underline{\mathrm{Mor}}(\rho \circ \sigma, \sigma \circ \rho).$$

[Note: This makes sense. In fact,

$$\begin{cases} U_\rho \times U_\sigma \in \underline{\mathrm{Mor}}(\rho \circ \sigma, \rho_0 \circ \sigma_0) \\ U_\sigma^* \times U_\rho^* \in \underline{\mathrm{Mor}}(\sigma_0 \circ \rho_0, \sigma \circ \rho) \end{cases}$$

and $\rho_0 \circ \sigma_0 = \sigma_0 \circ \rho_0$.]

It is a formality to check that $\varepsilon(\rho,\sigma)$ is independent of the choice of $\rho_0, \sigma_0, U_\rho, U_\sigma$ within the given restrictions.

LEMMA ε defines a permutation structure on $\mathcal{T}$.

[This is established by straightforward manipulations. When $d = 1$, the definition of $\varepsilon(\rho,\sigma)$ is different and it is no longer necessarily true that

$$\varepsilon(\sigma,\rho)\,\varepsilon(\rho,\sigma) = 1_{\rho\circ\sigma},$$

i.e., in this case, the monodromy is nontrivial.]

Consequently, $\forall\, \rho \in \underline{\mathrm{Ob}}\,\mathcal{T}$, there is a unitary representation $\varepsilon_\rho^{(n)}$ of $\underset{\sim}{P}_n$ on the Hilbert space $\mathcal{H}$ of the vacuum representation π_0 of $\mathfrak{A}$.

[Note: Recall that $\varepsilon_\rho^{(n)}(\underset{\sim}{P}_n) \subset \underline{\mathrm{Mor}}(\rho^n, \rho^n)$.]

Remark: Suppose that $\rho_0 \in [\rho]$ and let $U \in \underline{\text{Mor}}(\rho, \rho_0)$ be unitary -- then

$$\varepsilon_{\rho_0}^{(n)} = U^{(n)} \varepsilon_{\rho}^{(n)} U^{(-n)} ,$$

where

$$U^{(n)} = U \times \cdots \times U \in \underline{\text{Mor}}(\rho^n, \rho_0^n).$$

This shows that the equivalence class of $\varepsilon_{\rho}^{(n)}$ depends only on the equivalence class of ρ .

Notation: $\hat{\underset{\sim}{P}}_n$ is the unitary dual of $\underset{\sim}{P}_n$.

Since $\underset{\sim}{P}_n$ is compact, $\varepsilon_{\rho}^{(n)}$ is discretely decomposable:

$$\varepsilon_{\rho}^{(n)} = \bigoplus_{D \in \hat{\underset{\sim}{P}}_n} n_D D,$$

so

$$\mathcal{H} = \bigoplus_{D \in \hat{\underset{\sim}{P}}_n} \mathcal{H}_D$$

with isotypic projection $E_D : \mathcal{H} \longrightarrow \mathcal{H}_D$.

[Note: On general grounds, the nonzero E_D generate the center of $\varepsilon_{\rho}^{(n)}(\underset{\sim}{P}_n)''$.]

LEMMA Suppose that ρ is localized in K -- then $\forall$ D, $E_D \in \mathcal{M}(K)$.

[$\forall$ $p \in \underset{\sim}{P}_n$, we have

$$\varepsilon_{\rho}^{(n)}(p) \in \underline{\text{Mor}}(\rho^n, \rho^n)$$

$\Rightarrow$

$$\varepsilon_\rho^{(n)}(p)\,\rho^n(A) = \rho^n(A)\,\varepsilon_\rho^{(n)}(p) \qquad (A \in \mathfrak{A})$$

$\Rightarrow$

$$\varepsilon_\rho^{(n)}(p)A = A\,\varepsilon_\rho^{(n)}(p) \qquad (A \in \mathfrak{A}(K^\perp))$$

$\Rightarrow$

$$\varepsilon_\rho^{(n)}(p) \in \mathfrak{A}(K^\perp)' = \mathfrak{M}(K).$$

Therefore

$$\varepsilon_\rho^{(n)}(\underset{\sim}{P}_n)'' \subset \mathfrak{M}(K)$$

$\Rightarrow$

$$E_D \in \mathfrak{M}(K).]$$

It follows from the lemma that if E_D is nonzero, then n_D is infinite. In fact, thanks to the Borchers property, $\exists\ L \supset K$ and a partial isometry $V \in \mathfrak{M}(L)$: $V^*V = 1_{\mathfrak{A}}$ & $VV^* = E_D$, thus in $\mathfrak{M}(L)$, E_D is equivalent to the identity, so the dimension of $\mathcal{H}_D$ is infinite.

We shall now take up dimension theory. Here, it will be best to proceed directly, putting aside any consideration of a conjugation structure until later.

Definition: A DHR endomorphism ρ is <u>irreducible</u> if <u>Mor</u>$(\rho, \rho) = \underset{\sim}{C}1_\rho$.

[Note: A <u>sector</u> of the theory is the unitary equivalence class of an irreducible $\pi \in \mathrm{DHR}(\pi_0)$. In view of the correspondence $\pi \leftrightarrow \pi_0 \circ \rho$, sectors are parameterized by the $[\rho]$, where ρ is irreducible. Example: $[\iota]$ picks off the <u>vacuum sector</u>.]

Suppose that ρ is irreducible -- then one can attach to ρ an element $\lambda_\rho \in [-1, +1]$, the <u>statistics parameter</u> of ρ (definition omitted). It depends only on $[\rho]$.

<u>$\lambda_\rho = 0$:</u> In this case, every $D \in \hat{\underset{\sim}{P}}_n$ occurs in the decomposition

$$\varepsilon_\rho^{(n)} = \bigoplus_{D \in \hat{\underset{\sim}{P}}_n} n_D D,$$

i.e., $\forall D, n_D \neq 0$.

<u>$\lambda_\rho \neq 0$:</u> Put

$$d_\rho = \frac{1}{|\lambda_\rho|} .$$

Then it turns out that $d_\rho \in \underset{\sim}{N}$, a fact which places an a priori restriction on those $D \in \hat{\underset{\sim}{P}}_n$ which can occur in the decomposition of $\varepsilon_\rho^{(n)}$, namely:

(i) If $\lambda_\rho = \frac{1}{d_\rho}$, then $n_D \neq 0$ iff the lengths of the columns of the Young diagram attached to D are $\leq d_\rho$.

(ii) If $\lambda_\rho = -\frac{1}{d_\rho}$, then $n_D \neq 0$ iff the lengths of the rows of the Young diagram attached to D are $\leq d_\rho$.

Remark: $[\rho]$ is a so-called irreducible <u>sector</u> of the theory. There is

<u>infinite statistics</u> if $\lambda_\rho = 0$,

<u>para-Bose statistics</u> if $\lambda_\rho = \frac{1}{d_\rho}$,

<u>para-Fermi statistics</u> if $\lambda_\rho = -\frac{1}{d_\rho}$.

An irreducible ρ is said to be <u>finite</u> if $\lambda_\rho \neq 0$.

Definition: Let $\rho \in \underline{\mathrm{Ob}}\,\mathfrak{T}$ be arbitrary -- then ρ is said to be <u>finite</u> if it can be written as a finite direct sum of finite irreducibles.

Denote by $\mathfrak{T}_f$ the full C*-subcategory of $\mathfrak{T}$ whose objects are the finite ρ -- then

$$\begin{cases} \iota \in \underline{\mathrm{Ob}}\ \mathfrak{T}_f \\ \rho, \sigma \in \underline{\mathrm{Ob}}\ \mathfrak{T}_f \Rightarrow \rho \circ \sigma \in \underline{\mathrm{Ob}}\ \mathfrak{T}_f \\ \rho \in \underline{\mathrm{Ob}}\ \mathfrak{T}_f \Rightarrow [\rho] \subset \underline{\mathrm{Ob}}\ \mathfrak{T}_f . \end{cases}$$

In particular, therefore, $\mathfrak{T}_f$ is strict monoidal. As such, it inherits a permutation structure from $\mathfrak{T}$.

[Note: By contrast with $\mathfrak{T}$, $\mathfrak{T}_f$ can also be equipped with a conjugation structure (cf. infra).]

Fact: $\mathfrak{T}_f$ is closed w.r.t. the formation of subobjects.

Fact: $\mathfrak{T}_f$ is closed w.r.t. the formation of finite direct sums.

Suppose that ρ is finite -- then $\underline{\mathrm{Mor}}(\rho, \rho)$ is finite dimensional. Its center is generated by minimal central projections E_i $(i = 1,\ldots,n)$, hence

$$[\rho] = \bigoplus_{i=1}^{n} m_i [\rho_i] \qquad (m_i \in \underset{\sim}{N}),$$

where $\rho_i \leftrightarrow E_i$ and

$$\underline{\mathrm{Mor}}(\rho_i, \rho_j) = \begin{cases} \underset{\sim}{C} & (i=j) \\ 0 & (i \neq j). \end{cases}$$

LEMMA Let $\rho \to \mu(\rho)$ be a complex valued function on the finite irreducibles which is an equivalence class invariant. Extend μ to arbitrary finite ρ by writing

$$\mu(\rho) = \sum_{i=1}^{n} \mu(\rho_i)E_i \quad (\in \underline{\mathrm{Mor}}(\rho,\rho)).$$

Then $\forall\ \rho_1, \rho_2 \in \underline{\mathrm{Ob}}\ \mathfrak{T}_f$,

$$T\mu(\rho_1) = \mu(\rho_2)T \quad (T \in \underline{\mathrm{Mor}}(\rho_1,\rho_2)).$$

Example: Given a finite irreducible ρ, put

$$\lambda_\rho = \frac{\varkappa_\rho}{d_\rho} \; (\varkappa_\rho = \pm 1,\ d_\rho = 1,2,\ldots).$$

Then for arbitrary finite ρ,

$$\varkappa(\rho) = E_+(\rho) - E_-(\rho),$$

where

$$\begin{cases} E_+(\rho) = \sum_{\varkappa_i = +1} E_i \\ E_-(\rho) = \sum_{\varkappa_i = -1} E_i . \end{cases}$$

Therefore

$$\varkappa(\rho)^2 = 1_\rho \quad \& \quad \varkappa(\rho)^* = \varkappa(\rho).$$

One calls ρ bosonic if $E_-(\rho) = 0$, fermionic if $E_+(\rho) = 0$. Of course, if ρ is a finite irreducible, then ρ is either bosonic or fermionic. In general, ρ can be expressed as the direct sum of a bosonic and a fermionic endomorphism:

$$\rho = \rho_+ \oplus \rho_-$$

with

$$\begin{cases} \rho_+ \leftrightarrow E_+(\rho) \\ \rho_- \leftrightarrow E_-(\rho) . \end{cases} \in \underline{\text{Mor}}(\rho, \rho)$$

[Note: $\varkappa$ respects composition in the sense that $\varkappa(\rho_1 \circ \rho_2) = \varkappa(\rho_1) \times \varkappa(\rho_2)$.]

Definition: The <u>statistical dimension</u> of a $\rho \in \underline{\text{Ob}}\ \mathcal{T}_f$ is

$$d(\rho) = \sum_{i=1}^{n} m_i\, d_{\rho_i}.$$

It is easy to see that the statistical dimension is additive on finite direct sums. Furthermore, $d(\rho_1 \circ \rho_2) = d(\rho_1)d(\rho_2)$.

Fact: Suppose that ρ is finite -- then ρ is an automorphism iff $d(\rho) = 1$.

We have yet to consider the existence of a conjugation structure. In point of fact, there is an assignment

$$\begin{cases} \underline{\text{Ob}}\ \mathcal{T}_f \rightarrow \underline{\text{Ob}}\ \mathcal{T}_f \\ \quad \rho \rightarrow \bar{\rho} \end{cases}$$

together with arrows

$$\begin{cases} R_\rho \in \underline{\text{Mor}}(\iota, \bar{\rho} \circ \rho) \\ \bar{R}_\rho \in \underline{\text{Mor}}(\iota, \rho \circ \bar{\rho}) \end{cases}$$

such that

$$\begin{cases} (R^*_\rho \times 1_\rho)\ (1_{\bar\rho} \times \varepsilon(\rho,\rho))\ (R_\rho \times 1_\rho) = \varkappa(\rho) \\ (\bar{R}^*_\rho \times 1_{\bar\rho})\ (1_\rho \times \varepsilon(\bar\rho,\bar\rho))\ (\bar{R}_\rho \times 1_{\bar\rho}) = \varkappa(\bar\rho) \end{cases}$$

or still,

$$\begin{cases} R^*_\rho\ \bar\rho(\varepsilon(\rho,\rho))R_\rho = \varkappa(\rho) \\ \bar{R}^*_\rho\ \rho(\varepsilon(\bar\rho,\bar\rho))\bar{R}_\rho = \varkappa(\bar\rho). \end{cases}$$

These, however, are not the conjugate equations.

[Note: Actually, the conjugate equations are valid. What breaks down is the relation $\bar{R}_\rho = \varepsilon(\bar\rho,\rho) \circ R_\rho$, which is valid iff ρ is bosonic.]

Remark: Return for the moment to the abstract picture -- then

$$\begin{cases} (R^*_\rho \otimes 1_\rho) \circ (1_{\bar\rho} \otimes \varepsilon(\rho,\rho)) \circ (R_\rho \otimes 1_\rho) = 1_\rho \\ (\bar{R}^*_\rho \otimes 1_{\bar\rho}) \circ (1_\rho \otimes \varepsilon(\bar\rho,\bar\rho)) \circ (\bar{R}_\rho \otimes 1_{\bar\rho}) = 1_{\bar\rho}. \end{cases}$$

For example, let us derive the first relation, starting from the conjugate equation

$$(\bar{R}^*_\rho \otimes 1_\rho) \circ (1_\rho \otimes R_\rho) = 1_\rho .$$

Thus

$$\varepsilon(\iota,\rho) = 1_\rho$$

$$\Longrightarrow$$

$$(\bar{R}^*_\rho \otimes 1_\rho) \circ (1_\rho \otimes R_\rho) \circ \varepsilon(\iota,\rho) = 1_\rho .$$

But

$$\begin{cases} 1_\rho \in \underline{\mathrm{Mor}}(\rho,\rho) \\ R_\rho \in \underline{\mathrm{Mor}}(\iota,\bar\rho\otimes\rho) \end{cases}$$

$\Longrightarrow$

$$(1_\rho\otimes R_\rho)\circ\varepsilon(\iota,\rho) = \varepsilon(\bar\rho\otimes\rho,\rho)\circ(R_\rho\otimes 1_\rho).$$

And

$$\bar R_\rho = \varepsilon(\bar\rho,\rho)\circ R_\rho .$$

Therefore

$$1_\rho = ((\varepsilon(\bar\rho,\rho)\circ R_\rho)^*\otimes 1_\rho)\circ\varepsilon(\bar\rho\otimes\rho,\rho)\circ(R_\rho\otimes 1_\rho)$$

$$= ((R^*_\rho\circ\varepsilon(\rho,\bar\rho))\otimes 1_\rho)\circ\varepsilon(\bar\rho\otimes\rho,\rho)\circ(R_\rho\otimes 1_\rho).$$

But

$$\varepsilon(\bar\rho\otimes\rho,\rho) = (\varepsilon(\bar\rho,\rho)\otimes 1_\rho)\circ(1_{\bar\rho}\otimes\varepsilon(\rho,\rho))$$

$\Longrightarrow$

$$((R^*_\rho\circ\varepsilon(\rho,\bar\rho))\otimes 1_\rho)\circ\varepsilon(\bar\rho\otimes\rho,\rho)$$

$$= ((R^*_\rho\circ\varepsilon(\rho,\bar\rho))\otimes 1_\rho)\circ(\varepsilon(\bar\rho,\rho)\otimes 1_\rho)\circ(1_{\bar\rho}\otimes\varepsilon(\rho,\rho)).$$

And

$$((R^*_\rho\circ\varepsilon(\rho,\bar\rho))\otimes 1_\rho)\circ(\varepsilon(\bar\rho,\rho)\otimes 1_\rho)$$

$$= (R^*_\rho\circ\varepsilon(\rho,\bar\rho)\circ\varepsilon(\bar\rho,\rho))\otimes(1_\rho\circ 1_\rho)$$

$$= (R^*_\rho\circ 1_{\bar\rho\otimes\rho})\otimes 1_\rho$$

$$= R^*_\rho\otimes 1_\rho .$$

Consequently,

$$1_\rho = (R^*_\rho \otimes 1_\rho) \circ (1_{\bar\rho} \otimes \varepsilon(\rho,\rho)) \circ (R_\rho \otimes 1_\rho),$$

as claimed.

The escape from this difficulty is simple: Alter the given permutation structure on $\mathfrak{T}_f$ in such a way that in the new permutation structure a legitimate conjugation structure can be defined, thereby paving the way for an application of dimension theory. Fortunately, no ambiguities arise: The two potential meanings of $d(\rho)$ are the same.

Definition: Given $\rho, \sigma \in \underline{\mathrm{Ob}}\ \mathfrak{T}_f$, put

$$\delta(\rho,\sigma) = \frac{1}{2}\,(1_\rho \times 1_\sigma + 1_\rho \times \varkappa(\sigma) + \varkappa(\rho) \times 1_\sigma - \varkappa(\rho) \times \varkappa(\sigma))$$

$$\in \underline{\mathrm{Mor}}(\rho \circ \sigma, \rho \circ \sigma).$$

Properties:

(1) $\varepsilon(\rho,\sigma)\,\delta(\rho,\sigma) = \delta(\sigma,\rho)\,\varepsilon(\rho,\sigma)$;

(2) $\delta(\rho,\sigma)\,\delta(\rho,\sigma) = 1_{\mathfrak{A}}$;

(3) $$\delta(\rho \circ \sigma, \tau) = \frac{1}{2}\,(1_\rho \times 1_\sigma \times 1_\tau + \varkappa(\rho) \times 1_\sigma \times 1_\tau + 1_\rho \times 1_\sigma \times \varkappa(\tau) - \varkappa(\rho) \times 1_\sigma \times \varkappa(\tau))\,(1_\rho \times \delta(\sigma,\tau)).$$

Definition: Given $\rho, \sigma \in \underline{\mathrm{Ob}}\ \mathfrak{T}_f$, put

$$\hat{\varepsilon}(\rho,\sigma) = \varepsilon(\rho,\sigma)\,\delta(\rho,\sigma).$$

LEMMA $\hat{\varepsilon}$ defines a permutation structure on $\mathfrak{T}_f$.

[The verification is purely computational.]

Remark: The passage $\varepsilon \to \hat{\varepsilon}$ is called a Klein transformation.

To see that $(\mathcal{T}_f, \hat{\varepsilon})$ admits a conjugation structure, it suffices to modify $\bar{R}_\rho$, leaving R_ρ as is:

$$\begin{cases} \hat{R}_\rho \equiv R_\rho \\ \hat{\bar{R}}_\rho \equiv \hat{\varepsilon}(\bar{\rho}, \rho)\hat{R}_\rho . \end{cases}$$

We then claim that the pair $(\hat{R}_\rho, \hat{\bar{R}}_\rho)$ satisfies the conjugate equations, i.e.,

$$\begin{cases} (\hat{\bar{R}}^*_\rho \times 1_\rho)(1_\rho \times \hat{R}_\rho) = 1_\rho \\ (\hat{R}^*_\rho \times 1_{\bar{\rho}})(1_{\bar{\rho}} \times \hat{\bar{R}}_\rho) = 1_{\bar{\rho}} \end{cases}$$

or still,

$$\begin{cases} \hat{\bar{R}}^*_\rho \, \rho(\hat{R}_\rho) = 1_\rho \\ \hat{R}^*_\rho \, \bar{\rho}(\hat{\bar{R}}_\rho) = 1_{\bar{\rho}} . \end{cases}$$

To illustrate, consider the first relation. On general grounds, we have

$$T\,\varkappa(\rho_1) = \varkappa(\rho_2)T \quad (T \in \underline{\mathrm{Mor}}(\rho_1, \rho_2)).$$

So, since $\varkappa(\iota) = 1_\iota$,

$$(\varkappa(\bar{\rho}) \times \varkappa(\rho))R_\rho = \varkappa(\bar{\rho} \circ \rho)R_\rho = R_\rho\, \varkappa(\iota) = R_\rho .$$

But

$$\varkappa(\bar{\rho}) \times \varkappa(\rho) = (1_{\bar{\rho}} \times \varkappa(\rho))(\varkappa(\bar{\rho}) \times 1_\rho)$$

$\Longrightarrow$

$$(1_{\bar\rho} \times \varkappa(\rho))(\varkappa(\bar\rho)\times 1_\rho)R_\rho = R_\rho$$

$\Longrightarrow$

$$(\varkappa(\bar\rho)\times 1_\rho)R_\rho = (1_{\bar\rho} \times \varkappa(\rho))R_\rho$$

$\Longrightarrow$

$$\hat{\bar{R}}_\rho = \hat{\varepsilon}(\bar\rho,\rho)\hat{R}_\rho$$

$$= \varepsilon(\bar\rho,\rho)\ \delta(\bar\rho,\rho)R_\rho$$

$$= \frac{1}{2}\ \varepsilon(\bar\rho,\rho)\Big(1_{\mathfrak{A}} + 1_{\bar\rho}\times\varkappa(\rho) + \varkappa(\bar\rho)\times 1_\rho - \varkappa(\bar\rho\circ\rho)\Big)\ R_\rho$$

$$= \varepsilon(\bar\rho,\rho)(1_{\bar\rho}\times\varkappa(\rho))R_\rho\ .$$

Therefore

$$(\hat{\bar{R}}^*\times 1_\rho)(1_\rho\times\hat{R}_\rho)$$

$$= \Big((\varepsilon(\bar\rho,\rho)(1_{\bar\rho}\times\varkappa(\rho))R_\rho)^*\times 1_\rho\Big)\ \ (1_\rho\times R_\rho)$$

$$= \Big((R^*_\rho\ \ (1^*_{\bar\rho}\times\varkappa(\rho)^*)\,\varepsilon(\bar\rho,\rho)^*)\times 1_\rho\Big)\ \ (1_\rho\times R_\rho)$$

$$= \Big((R^*_\rho\ \ (1_{\bar\rho}\times\varkappa(\rho))\,\varepsilon(\rho,\bar\rho))\times 1_\rho\Big)\ \ (1_\rho\times R_\rho)$$

$$= (R^*_\rho\times 1_\rho)(1_{\bar\rho}\times\varkappa(\rho)\times 1_\rho)(\varepsilon(\rho,\bar\rho)\times 1_\rho)(1_\rho\times R_\rho).$$

But

$$\begin{cases} 1_\rho\times R_\rho = \varepsilon(\bar\rho\circ\rho,\rho)R_\rho \\ \varepsilon(\rho,\bar\rho)\,\varepsilon(\bar\rho\circ\rho,\rho) = \bar\rho\,(\varepsilon(\rho,\rho)). \end{cases}$$

Therefore

$$(\hat{\bar{R}}^*_\rho \times 1_\rho)(1_\rho \times \hat{R}_\rho)$$

$$= R^*_\rho \, \bar{\rho}(\varkappa(\rho))\, \varepsilon(\rho,\bar{\rho})\, \varepsilon(\bar{\rho}\circ\rho,\rho) R_\rho$$

$$= R^*_\rho \, \bar{\rho}(\varkappa(\rho))\, \bar{\rho}(\varepsilon(\rho,\rho)) R_\rho$$

$$= R^*_\rho \, \bar{\rho}(\varkappa(\rho)\, \varepsilon(\rho,\rho)) R_\rho .$$

But

$$\varepsilon(\rho_1,\rho_2)(1_{\rho_1} \times \varkappa(\rho_2)) = (\varkappa(\rho_2) \times 1_{\rho_1})\, \varepsilon(\rho_1,\rho_2)$$

$\Longrightarrow$

$$\varkappa(\rho)\, \varepsilon(\rho,\rho) = (\varkappa(\rho) \times 1_\rho)\, \varepsilon(\rho,\rho)$$

$$= \varepsilon(\rho,\rho)(1_\rho \times \varkappa(\rho))$$

$$= \varepsilon(\rho,\rho)\, \rho(\varkappa(\rho)).$$

Therefore

$$(\hat{\bar{R}}^*_\rho \times 1_\rho)(1_\rho \times \hat{R}_\rho)$$

$$= R^*_\rho \, \bar{\rho}(\varepsilon(\rho,\rho))\, \bar{\rho}(\rho(\varkappa(\rho))) R_\rho$$

$$= (R^*_\rho \times 1_\rho)(1_{\bar{\rho}} \times \varepsilon(\rho,\rho))(1_{\bar{\rho}} \times \rho(\varkappa(\rho)))(R_\rho \times 1_\rho)$$

$$= (R^*_\rho \times 1_\rho)(1_{\bar{\rho}} \times \varepsilon(\rho,\rho))(1_{\bar{\rho}} \times 1_\rho \times \varkappa(\rho))(R_\rho \times 1_\rho)$$

$$= (R^*_\rho \times 1_\rho)(1_{\bar{\rho}} \times \varepsilon(\rho,\rho))(1_{\bar{\rho}\circ\rho} \times \varkappa(\rho))(R_\rho \times 1_\rho)$$

$$= (R^*_\rho \times 1_\rho)(1_{\bar{\rho}} \times \varepsilon(\rho,\rho))(R_\rho \times \varkappa(\rho))$$

$$= (R^*_{\rho} \times 1_{\rho})(1_{\bar\rho} \times \varepsilon(\rho,\rho))(R_{\rho} \times 1_{\rho})(1_{\iota} \times \varkappa(\rho))$$

$$= (R^*_{\rho} \times 1_{\rho})(1_{\bar\rho} \times \varepsilon(\rho,\rho))(R_{\rho} \times 1_{\rho})\varkappa(\rho)$$

$$= \varkappa(\rho)\varkappa(\rho)$$

$$= \varkappa(\rho)^2 = 1_{\rho}.$$

<u>The Field Algebra</u> Suppose given a weakly additive PTV with a unique vacuum which satisfies Haag duality. Maintain the convention that $d \geq 2$ and append a subzero to the Hilbert space underlying the vacuum representation of $\mathfrak{A}$: I.e., write $\mathcal{H}_0$ in place of $\mathcal{H}$. Finally, let $\mathcal{K}$ be the set of double cones $K \subset M$.

The ingredients underlying the theory of the field algebra are the following entities.

(a) A representation π of $\mathfrak{A}$ on a Hilbert space $\mathcal{H}$ containing π_0 as a subrepresentation on $\mathcal{H}_0 \subset \mathcal{H}$.

(b) A compact group G and a faithful unitary representation U of G on $\mathcal{H}$ leaving $\mathcal{H}_0$ pointwise fixed.

(c) An inclusion preserving map $K \rightarrow \mathfrak{F}(K)$ from $\mathcal{K}$ to the W*-algebras on $\mathcal{H}$.

THEN:

(1) There is a one-to-one correspondence

$$\begin{cases} \underline{\mathrm{Irr}}\ \mathfrak{T}_f/\sim \ \leftrightarrow \hat{G} \\ \qquad [\rho] \leftrightarrow \xi \end{cases}$$

between the sectors of the theory and the unitary dual of G with $d(\rho) = d(\xi)$.

(2) There is an orthogonal decomposition

$$\mathcal{H} = \bigoplus_{\xi \in \hat{G}} \mathcal{H}_\xi \otimes \underset{\sim}{C}^{d(\xi)}.$$

(3) U operates on $\mathcal{H}$ as

$$U(\sigma) = \bigoplus_{\xi \in \hat{G}} 1 \otimes U_\xi(\sigma) \quad (\sigma \in G),$$

where, $\forall \xi$, U_ξ is an irreducible unitary representation of G of dimension $d(\xi)$ and distinct ξ give inequivalent U_ξ.

(4) Π operates on $\mathcal{H}$ as

$$\Pi(A) = \bigoplus_{\xi \in \hat{G}} \pi_\xi(A) \otimes 1 \quad (A \in \mathfrak{A}),$$

where $\forall \xi$, π_ξ is an irreducible DHR representation of $\mathfrak{A}$ ($\pi_\xi \leftrightarrow \pi_0 \circ \rho$ under $[\rho] \leftrightarrow \xi$) and distinct ξ give inequivalent π_ξ.

(5) $\forall\ K \in \mathcal{K}$ & $\forall \sigma \in G$,

$$U(\sigma)\, \mathfrak{F}(K) U(\sigma)^{-1} = \mathfrak{F}(K).$$

(6) $\forall\ K \in \mathcal{K}$, $\mathfrak{F}^G(K)$ ($= \mathfrak{F}(K) \cap U(G)'$) leaves each $\mathcal{H}_\xi$ invariant and

$$\mathfrak{F}^G(K) \mid \mathcal{H}_\xi = \pi_\xi(\mathfrak{A}(K)).$$

In particular:

$$\mathfrak{F}^G(K) \mid \mathcal{H}_0 = \mathfrak{A}(K).$$

(7) $\forall\ K \in \mathcal{K}$, $\mathfrak{F}(K)\mathcal{H}_0$ spans a dense subspace of $\mathcal{H}$.

(8) If K and L are spacelike separated, then the elements of $\mathfrak{F}(K)$ commute with the elements of $\Pi(\mathfrak{A}(L))$.

Remark: The left regular representation of G on $L^2(G)$ is unitarily equivalent to

$$\bigoplus_{\xi \in \hat{G}} d(\xi) U_\xi \ .$$

On the other hand, by the above,

$$\Pi = \bigoplus_{\xi \in \hat{G}} d(\xi)\pi_\xi .$$

The <u>field algebra</u> $\mathfrak{F}$ of a particle theory is the norm closure of $\bigcup_K \mathfrak{F}(K)$.

One has

$$\mathfrak{F}' = \underset{\sim}{C}I = \Pi(\mathfrak{A})' \cap \mathfrak{F},$$

where

$$\Pi(\mathfrak{A})' = U(G)''.$$

Remark: Let $\gamma \in$ <u>Aut</u> $\mathfrak{F}$ -- then $\gamma =$ <u>Ad</u> $U(\sigma)$ for some $\sigma \in G$ iff γ acts trivially on $\Pi(\mathfrak{A})$.

There is an additional element of structure that reflects the Bose-Fermi alternative, namely $\exists$ an element $\zeta \in Z_G$ (the center of G): $\zeta^2 = e$, which controls the commutativity relation in $\mathfrak{F}$.

Thus let $\Gamma = U(\zeta)$, so

$$\Gamma^2 = I, \ \Gamma = \Gamma^{-1} = \Gamma^*.$$

[Note: One can be precise. Indeed, $U_\xi(\zeta) = \varkappa_\xi$, hence

$$\Gamma = \bigoplus_{\xi \in \hat{G}} 1 \otimes \varkappa_\xi .$$

Traditionally, $\varkappa_\xi$ is referred to as the <u>statistics phase</u>.]

Put

$$X_{\pm} = \frac{1}{2}\,(X \pm \Gamma X \Gamma^{-1}).$$

Definition: X is <u>bosonic</u> if $\Gamma X \Gamma^{-1} = X$, <u>fermionic</u> if $\Gamma X \Gamma^{-1} = -X$.

[Note: X_+ is bosonic, X_- is fermionic, and $X = X_+ + X_-$.]

Fact: If K and L are spacelike separated, then

$$\begin{cases} X \in \mathfrak{F}(K) \\ Y \in \mathfrak{F}(L) \end{cases}$$

$$\Rightarrow$$

$$\begin{cases} X_+Y_+ = Y_+X_+ \\ X_+Y_- = Y_-X_+ \ \& \ X_-Y_+ = Y_+X_- \\ X_-Y_- + Y_-X_- = 0. \end{cases}$$

[Note: In suggestive notation, this says that

$$\begin{cases} \mathfrak{F}_+(K) \text{ commutes with } \mathfrak{F}(L) \\ \mathfrak{F}_+(L) \text{ commutes with } \mathfrak{F}(K) \\ \mathfrak{F}_-(K) \text{ anticommutes with } \mathfrak{F}_-(L).] \end{cases}$$

Now set

$$X^t = ZXZ^*,$$

where

$$Z = \frac{1}{1+\sqrt{-1}}\,(\underline{id}_{\mathcal{H}} + \sqrt{-1}\,\Gamma) \quad (\Rightarrow Z^2 = \Gamma).$$

Then

$$X^t = X_+ + \sqrt{-1}\,\Gamma X_-.$$

Put

$$\mathfrak{F}^t(K) = Z\,\mathfrak{F}(K)Z^*.$$

Then twisted Haag duality obtains:

$$\mathfrak{F}^t(K) = \mathfrak{F}(K^\perp)'.$$

[Note: It is not difficult to check that

$$\mathfrak{F}^t(K) \subset \mathfrak{F}(K^\perp)'.$$

For, by definition, $\mathfrak{F}(K^\perp)$ is generated by the $\mathfrak{F}(L)$ with $K \perp L$. So fix such an L and consider $[X^t,Y]$ $(X \in \mathfrak{F}(K),\ Y \in \mathfrak{F}(L))$ -- then

$$[X^t,Y] = [X_+ + \sqrt{-1}\,\Gamma X_-,\ Y_+ + Y_-]$$

$$= \sqrt{-1}\,([\Gamma X_-,Y_+] + [\Gamma X_-,Y_-]).$$

And:

$$[\Gamma X_-,Y_+] = \Gamma X_-Y_+ - Y_+\Gamma X_-$$

$$= \Gamma X_-Y_+ - \Gamma^2 Y_+\Gamma X_-$$

$$= \Gamma X_-Y_+ - \Gamma\Gamma Y_+\Gamma X_-$$

$$= \Gamma(X_-Y_+ - \Gamma Y_+\Gamma X_-)$$

$$= \Gamma(X_-Y_+ - Y_+X_-)$$

$$= \Gamma 0 = 0.$$

By the same token,

$$[\Gamma X_-, Y_-] = \Gamma(X_-Y_- - \Gamma Y_-\Gamma X_-)$$

$$= \Gamma(X_-Y_- + Y_-X_-)$$

$$= \Gamma 0 = 0.]$$

Remark: Recall that Haag duality $\Rightarrow$ wedge duality. Here, twisted Haag duality $\Rightarrow$ twisted wedge duality: $\forall\ W \in \mathcal{W}$,

$$\mathfrak{F}^t(W) = \mathfrak{F}(W^\perp)'.$$

By definition, our PTV comes equipped with a unitary representation U_0 of $\widetilde{\mathcal{P}}_+^\uparrow$ on $\mathcal{H}_0$ such that

$$U_0(\widetilde{\Lambda}, a)\mathfrak{M}(O)U_0(\widetilde{\Lambda}, a)^{-1} = \mathfrak{M}((\widetilde{\Lambda}, a)\cdot O),$$

where

$$\underline{\text{spec}}(U_0 \mid \underset{\sim}{R}^{1,d}) \subset \overline{V}_+.$$

It remains to incorporate this fact into the theory of the field algebra.

Definition: Suppose that ρ is finite -- then ρ is said to be <u>covariant</u> if $\exists$ a unitary representation U_ρ of $\widetilde{\mathcal{P}}_+^\uparrow$ on $\mathcal{H}_0$ such that

$$\rho(U_0(\widetilde{\Lambda}, a)AU_0(\widetilde{\Lambda}, a)^{-1}) = U_\rho(\widetilde{\Lambda}, a)\,\rho(A)U_\rho(\widetilde{\Lambda}, a)^{-1}.$$

[Note: It can be shown that

$$\underline{\text{spec}}(U_\rho \mid \underset{\sim}{R}^{1,d}) \subset \overline{V}_+.]$$

Example: The unit ι is covariant.

[Simply take $U_\iota = U_0$.]

<u>LEMMA</u> Suppose that ρ, ρ' are covariant and let $R \in \underline{\text{Mor}}(\rho, \rho')$ -- then $\forall(\tilde{\Lambda}, a)$,

$$RU_\rho(\tilde{\Lambda},a) = U_{\rho'}(\tilde{\Lambda},a)R.$$

[Define an action α of $\tilde{\mathcal{P}}_+^\uparrow$ on $\underline{\text{Mor}}(\rho, \rho')$ by the prescription

$$\alpha(\tilde{\Lambda},a)R = U_{\rho'}(\tilde{\Lambda},a)RU_\rho(\tilde{\Lambda},a)^*.$$

This makes sense, i.e.,

$$U_{\rho'}(\tilde{\Lambda},a)RU_\rho(\tilde{\Lambda},a)^* \rho(A)$$
$$= \rho'(A)U_{\rho'}(\tilde{\Lambda},a)RU_\rho(\tilde{\Lambda},a)^*$$

or still,

$$U_{\rho'}(\tilde{\Lambda},a)RU_\rho(\tilde{\Lambda},a)^* \rho(A)U_\rho(\tilde{\Lambda},a)$$
$$= \rho'(A)U_{\rho'}(\tilde{\Lambda},a)R.$$

Indeed, the LHS equals

$$U_{\rho'}(\tilde{\Lambda},a)R\,\rho(U_0(\tilde{\Lambda},a)^*AU_0(\tilde{\Lambda},a))$$
$$= U_{\rho'}(\tilde{\Lambda},a)\,\rho'(U_0(\tilde{\Lambda},a)^*AU_0(\tilde{\Lambda},a))R$$
$$= U_{\rho'}(\tilde{\Lambda},a)U_{\rho'}(\tilde{\Lambda},a)^*\,\rho'(A)U_{\rho'}(\tilde{\Lambda},a)R$$
$$= \rho'(A)U_{\rho'}(\tilde{\Lambda},a)R,$$

which is the RHS. But $\underline{\text{Mor}}(\rho, \rho')$ admits an inner product with the property that the $\alpha(\tilde{\Lambda},a)$ are unitary. Since $\underline{\text{Mor}}(\rho, \rho')$ is finite dimensional, it follows that $\forall(\tilde{\Lambda},a)$, $\alpha(\tilde{\Lambda},a) = I$, hence the lemma.]

Application: U_ρ is unique.

[In the lemma, take $\rho = \rho'$ and $R = 1_\rho$.]

[Note: Another corollary is the containment

$$U_\rho(\tilde{\mathcal{P}}_+^\uparrow) \subset \rho(\mathfrak{A})''.$$

In fact,

$$\underline{\text{Mor}}(\rho,\rho) = \rho(\mathfrak{A})'$$

and the $U_\rho(\tilde{\Lambda},a)$ commute with the elements of $\underline{\text{Mor}}(\rho,\rho)$, so $\forall\ (\tilde{\Lambda},a),\ U_\rho(\tilde{\Lambda},a) \in \rho(\mathfrak{A})''$.]

Denote by $\mathcal{T}_c$ the full C*-subcategory of $\mathcal{T}_f$ whose objects are covariant -- then

$$\begin{cases} \iota \in \underline{\text{Ob}}\ \mathcal{T}_c \\ \rho,\sigma \in \underline{\text{Ob}}\ \mathcal{T}_c \implies \rho\circ\sigma \in \underline{\text{Ob}}\ \mathcal{T}_c \\ \rho \in \underline{\text{Ob}}\ \mathcal{T}_c \implies [\rho] \subset \underline{\text{Ob}}\ \mathcal{T}_c. \end{cases}$$

Fact: $\mathcal{T}_c$ is closed w.r.t. the formation of subobjects.

Fact: $\mathcal{T}_c$ is closed w.r.t. the formation of finite direct sums.

In addition, it can be shown that

$$\rho \in \underline{\text{Ob}}\ \mathcal{T}_c \implies \bar{\rho} \in \underline{\text{Ob}}\ \mathcal{T}_c.$$

Therefore $\mathcal{T}_c$ has the same structure as $\mathcal{T}_f$.

Remark: It is conceivable that every finite ρ is covariant but this is an open question.

The theory of the field algebra can then be written in terms of $\mathcal{T}_c$ as opposed to $\mathcal{T}_f$. Of course, in this situation,

$$\begin{cases} \underline{\text{Irr}}\ \mathcal{T}_c/\!\sim\ \longleftrightarrow \hat{G} \\ \qquad [\rho] \longleftrightarrow \xi\ . \end{cases}$$

But there is also a new ingredient, viz. a unitary representation U of $\widetilde{\mathcal{P}}_+^\uparrow$ on $\mathcal{H}$ with the following properties:

(i) $U(\widetilde{\Lambda},a)\,\Omega_0 = \Omega_0$;

(ii) $\underline{\mathrm{spec}}(U|\underset{\sim}{R}^{1,d}) \subset \overline{V}_+$;

(iii) $U(\widetilde{\mathcal{P}}_+^\uparrow) \subset \Pi(\mathfrak{A})''$;

(iv) $U(\widetilde{\Lambda},a)\,\mathfrak{F}(K)U(\widetilde{\Lambda},a)^{-1} = \mathfrak{F}((\widetilde{\Lambda},a)\cdot K)$.

Remark: Per the orthogonal decomposition

$$\mathcal{H} = \bigoplus_{\xi\in\hat{G}} \mathcal{H}_\xi \otimes \underset{\sim}{C}^{d(\xi)},$$

we have

$$U(\widetilde{\Lambda},a) = \bigoplus_{\xi\in\hat{G}} U_\xi(\widetilde{\Lambda},a)\otimes 1,$$

where $U_\xi \leftrightarrow U_\rho$ under $[\rho] \leftrightarrow \xi$, thus

$$U_\xi(\widetilde{\Lambda},a)\;\pi_\xi(A)U_\xi(\widetilde{\Lambda},a)^{-1} = \pi_\xi(U_0(\widetilde{\Lambda},a)AU_0(\widetilde{\Lambda},a)^{-1}).$$

[Note: The symbol U has two meanings:

(1) U is a unitary representation of $\widetilde{\mathcal{P}}_+^\uparrow$ on $\mathcal{H}$;

(2) U is a unitary representation of G on $\mathcal{H}$.

Obviously,

$$U(\widetilde{\Lambda},a)U(\sigma) = U(\sigma)U(\widetilde{\Lambda},a).]$$

<u>The Statistics Theorem</u> Suppose given a weakly additive PTV with a unique vacuum which satisfies Haag duality, where $d \geq 2$.

Assumption: The inner symmetry group of the theory is compact, i.e., satisfies the gauge condition.

For any wedge $W \in \mathcal{W}$, there is a reflection $j(W)$ in the characteristic two-plane of the wedge which leaves the apex of the wedge unchanged. For example,

$$j(W_R)(x_0, x_1, x_2, \ldots, x_d) = (-x_0, -x_1, x_2, \ldots, x_d).$$

Remark: $j(W) \in \mathcal{P}_+$ (but $j(W) \notin \mathcal{P}_+^\uparrow$). Moreover,

$$\Lambda_{W^\perp}(t) = j(W)\Lambda_W(t)j(W) = \Lambda_W(-t).$$

Definition: A weakly additive PTV satisfies the <u>modular conjugation principle</u> (MCP) if $\forall\, W \in \mathcal{W}$,

$$J_{\mathfrak{M}(W)}\mathfrak{M}(K)J_{\mathfrak{M}(W)} = \mathfrak{M}(j(W)\cdot K) \quad (K \in \mathcal{K}).$$

[Note: It can be shown that the B-W property implies the MCP.]

We shall assume that the MCP is in force throughout the remainder of this section.

The adjoint action $\underline{\mathrm{Ad}}(j(W))$ of $j(W)$ on $\mathcal{P}_+^\uparrow$ has a unique lift to a homomorphism $\widetilde{\underline{\mathrm{Ad}}}(j(W))$ of $\widetilde{\mathcal{P}}_+^\uparrow$:

$$\begin{array}{ccc} \widetilde{\mathcal{P}}_+^\uparrow & \cdots\cdots\cdots\rightarrow & \widetilde{\mathcal{P}}_+^\uparrow \\ \downarrow & & \downarrow \\ \mathcal{P}_+^\uparrow & \xrightarrow[\underline{\mathrm{Ad}}(j(W))]{} & \mathcal{P}_+^\uparrow \end{array}$$

LEMMA. We have

$$J_{\mathfrak{M}(W)} U_0(\widetilde{\Lambda},a) J_{\mathfrak{M}(W)} = U_0(\widetilde{\underline{Ad}}(j(W))(\widetilde{\Lambda},a)).$$

[The prescription

$$(\widetilde{\Lambda},a) \rightarrow J_{\mathfrak{M}(W)} U_0(\widetilde{\underline{Ad}}(j(W))(\widetilde{\Lambda},a)) J_{\mathfrak{M}(W)}$$

defines a unitary representation of $\widetilde{\mathcal{P}}_+^{\uparrow}$ on $\mathcal{H}_0$ that leaves Ω_0 invariant. In addition, $\forall\ K \in \mathcal{K}$,

$$J_{\mathfrak{M}(W)} U_0(\widetilde{\underline{Ad}}(j(W))(\widetilde{\Lambda},a)) J_{\mathfrak{M}(W)}\ \mathfrak{M}(K) J_{\mathfrak{M}(W)} U_0(\widetilde{\underline{Ad}}(j(W))(\widetilde{\Lambda},a))^{-1} J_{\mathfrak{M}(W)}$$

$$= J_{\mathfrak{M}(W)} U_0(\underline{Ad}(j(W))(\widetilde{\Lambda},a))\ \mathfrak{M}\ (j(W)\cdot K) U_0(\widetilde{\underline{Ad}}(j(W))(\widetilde{\Lambda},a))^{-1} J_{\mathfrak{M}(W)}$$

$$= J_{\mathfrak{M}(W)}\ \mathfrak{M}\ (\underline{Ad}(j(W))(\Lambda,a) j(W)\cdot K) J_{\mathfrak{M}(W)}$$

$$= \mathfrak{M}\ (j(W)\underline{Ad}(j(W))(\Lambda,a) j(W)\cdot K)$$

$$= \mathfrak{M}\ (\underline{Ad}(j(W)^2)(\Lambda,a)\cdot K)$$

$$= \mathfrak{M}((\Lambda,a)\cdot K) \equiv \mathfrak{M}((\widetilde{\Lambda},a)\cdot K).$$

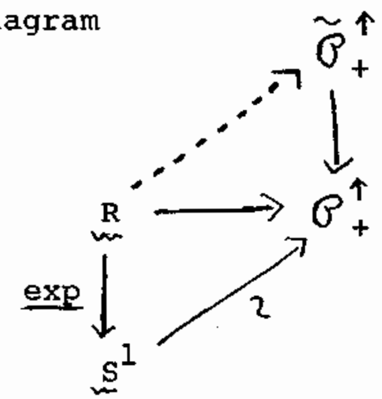
So, on the basis of the gauge condition,

$$U_0(\widetilde{\Lambda},a) = J_{\mathfrak{M}(W)} U_0(\widetilde{\underline{Ad}}(j(W))(\widetilde{\Lambda},a)) J_{\mathfrak{M}(W)},$$

from which the lemma.]

There is a homomorphism $\underset{\sim}{S}^1 \xrightarrow{\iota} \mathcal{P}_+^{\uparrow}$ embedding $\underset{\sim}{S}^1$ as the group of rotations in the 1-2 plane. Let $\underline{\exp}: \underset{\sim}{R} \longrightarrow \underset{\sim}{S}^1$ be the covering map --

then the diagram



admits a filler $r:\underset{\sim}{R} \longrightarrow \widetilde{\mathcal{P}}_+^\uparrow$ which is a homomorphism of groups.

Given $\theta \in [0,2\pi]$, let $W_R(\theta)$ be the image of W_R under the rotation by the angle θ in the 1-2 plane ($\Rightarrow W_L = W_R(\pi)$). Put

$$R_0(t) = U_0(r(t)) \quad (t \in \underset{\sim}{R}).$$

Then

$$R_0(\theta)\mathcal{M}(W_R)R_0(-\theta) = \mathcal{M}(W_R(\theta))$$

$$\Rightarrow$$

$$R_0(\theta)J_{\mathcal{M}(W_R)}R_0(-\theta) = J_{\mathcal{M}(W_R(\theta))}$$

$$\Rightarrow$$

$$J_{\mathcal{M}(W_R(\theta/2))}J_{\mathcal{M}(W_R)}$$

$$= R_0(\theta/2)J_{\mathcal{M}(W_R)}R_0(-\theta/2)J_{\mathcal{M}(W_R)}$$

$$= R_0(\theta/2)U_0(\widetilde{\underline{\mathrm{Ad}}}(j(W_R))r(-\theta/2))J^2_{\mathcal{M}(W_R)}$$

$$= R_0(\theta/2)R_0(\theta/2)$$

$$= R_0(\theta).$$

Example: We have

$$R_0(2\pi) = J_{\mathfrak{M}(W_R(\pi))} J_{\mathfrak{M}(W_R)}$$

$$= J_{\mathfrak{M}(W_L)} J_{\mathfrak{M}(W_R)}$$

$$= J_{\mathfrak{M}(W_R)} J_{\mathfrak{M}(W_R)}$$

$$= \underline{\mathrm{id}}_{\mathcal{H}_0}.$$

[Note: In general, $J_{\mathfrak{M}} = J_{\mathfrak{M}'}$. Bear in mind that Haag duality $\Rightarrow$ wedge duality, hence $\mathfrak{M}(W_R)' = \mathfrak{M}(W_L)$.]

The statistics theorem is the assertion that in the covariant situation,

$$U(r(2\pi)) = \Gamma,$$

thus $\forall$ irreducible $\rho \in \underline{\mathrm{Ob}}\,\mathcal{T}_c$,

$$U_\rho(r(2\pi)) \otimes 1 = 1 \otimes \varkappa_\rho$$

$$\Rightarrow \quad U_\rho(r(2\pi)) = \varkappa_\rho\, \underline{\mathrm{id}}_{\mathcal{H}_\rho}.$$

[Note: This agrees with the preceding deduction: Take $\rho = \iota$, so $U_\rho = U_0$ -- then $U_0(r(2\pi)) = \varkappa_\iota\, \underline{\mathrm{id}}_{\mathcal{H}_0}$ ($\varkappa_\iota = +1$).]

Facts:

(1) $\forall\ W \in \mathcal{W}$, $(\mathfrak{F}(W), \Omega_0)$ is a standard W*-algebra;

(2) $\forall\ W \in \mathcal{W}$, $J_{\mathfrak{F}(W)} | \mathcal{H}_0 = J_{\mathfrak{M}(W)}$;

(3) $\forall\ W \in \mathcal{W}$,

$$J_{\mathfrak{F}(W)} \Pi(A) J_{\mathfrak{F}(W)} = \Pi(J_{\mathfrak{M}(W)} A J_{\mathfrak{M}(W)}) \quad (A \in \mathfrak{A});$$

(4) $\forall\, W \in \mathcal{W}$,

$$J_{\mathfrak{F}(W)} U(\tilde{\Lambda}, a) J_{\mathfrak{F}(W)} = U(\underline{\widetilde{\mathrm{Ad}}}(j(W))(\tilde{\Lambda}, a)).$$

LEMMA $\forall\, W \in \mathcal{W}$,

$$Z J_{\mathfrak{F}(W)} = J_{\mathfrak{F}(W)} Z^*.$$

[Put $J = J_{\mathfrak{F}(W)}$ -- then

$$ZJ = \frac{1}{1 + \sqrt{-1}} (\underline{\mathrm{id}}_{\mathcal{H}} + \sqrt{-1}\,\Gamma) J.$$

But

$$\Gamma \mathfrak{F}(W) \Gamma = \Gamma \mathfrak{F}(W) \Gamma^{-1}$$

$$= U(\zeta)\, \mathfrak{F}(W) U(\zeta)^{-1}$$

$$= \mathfrak{F}(W)$$

$\Rightarrow$

$$\Gamma J \Gamma = J$$

$\Rightarrow$

$$\Gamma J = J \Gamma.$$

Therefore

$$ZJ = \frac{1}{1 + \sqrt{-1}} (J + \sqrt{-1}\, J \Gamma)$$

$$= J\left(\frac{1}{1 - \sqrt{-1}}\, \underline{\mathrm{id}}_{\mathcal{H}} - \sqrt{-1}\,\Gamma\right)$$

$$= JZ^*.]$$

Observation: Recall that here twisted wedge duality obtains: $\forall\, W \in \mathcal{W}$,

$$\mathfrak{F}^{t}(W) = \mathfrak{F}(W^{\perp})'$$

or still,

$$Z\,\mathfrak{F}(W)Z^{*} = \mathfrak{F}(W^{\perp})'.$$

Since Z is unitary, it follows that

$$ZJ_{\mathfrak{F}(W)}Z^{*} = J_{\mathfrak{F}(W^{\perp})'} = J_{\mathfrak{F}(W^{\perp})}.$$

In particular:

$$ZJ_{\mathfrak{F}(W_R)}Z^{*} = J_{\mathfrak{F}(W_L)}.$$

Proceeding as before, we find that

$$J_{\mathfrak{F}(W_R(\theta/2))}J_{\mathfrak{F}(W_R)} = U(r(\theta)).$$

Consequently,

$$\begin{aligned}
U(r(2\pi)) &= J_{\mathfrak{F}(W_R(\pi))}J_{\mathfrak{F}(W_R)} \\
&= J_{\mathfrak{F}(W_L)}J_{\mathfrak{F}(W_R)} \\
&= ZJ_{\mathfrak{F}(W_R)}Z^{*}J_{\mathfrak{F}(W_R)} \\
&= ZZJ_{\mathfrak{F}(W_R)}J_{\mathfrak{F}(W_R)} \\
&= Z^{2}J^{2}_{\mathfrak{F}(W_R)} \\
&= Z^{2} \\
&= \Gamma.
\end{aligned}$$

To interpret the statistics theorem, specialize

$$U_\xi(\tilde{\Lambda},a)\ \pi_\xi(A)U_\xi(\tilde{\Lambda},a)^{-1} = \pi_\xi(U_0(\tilde{\Lambda},a)AU_0(\tilde{\Lambda},a)^{-1})$$

to

$$U_\xi(r(2\pi))\ \pi_\xi(A)U_\xi(r(2\pi))^{-1} = \pi_\xi(U_0(r(2\pi))AU_0(r(2\pi))^{-1})$$

or still,

$$U_\xi(r(2\pi))\ \pi_\xi(A)U_\xi(r(2\pi))^{-1} = \pi_\xi(A).$$

Since π_ξ is irreducible, it follows that

$$U_\xi(r(2\pi)) = s_\xi \underline{id}_{\mathcal{H}_\xi} ,$$

where s_ξ is a complex number of modulus one.

Definition: The <u>spin</u> of the sector represented by π_ξ is s_ξ .

In this terminology, we have therefore proved that the spin s_ξ is precisely the statistics phase $\varkappa_\xi$.

<u>Normality</u> Suppose given a weakly additive PTV with a unique vacuum which satisfies Haag duality.

Fix:

(1) An increasing sequence $\{O_n\}$ of double cones such that $\underset{\sim}{R}^{1,d} = \bigcup_1^{\infty} O_n$;

(2) A sequence $\{K_n\}$ of double cones such that $\forall\, n$, $K_n \perp O_n$.

Then $\forall\, K \in \mathcal{K}$, $\exists\, N(K)$: $n \geq N(K) \Rightarrow K \perp K_n$.

[Choose $N(K): K \subset O_{N(K)}$, hence $K \subset O_{N(K)}^{\perp\perp} \subset K_{N(K)}^{\perp} \Rightarrow K \perp K_{N(K)}$. Since $n > N(K) \Rightarrow O_n \supset O_{N(K)}$, it follows that $K \perp K_n$ as well.]

<u>LEMMA</u> Suppose that ρ is a DHR endomorphism which is localized in $K \in \mathcal{K}$ -- then $\exists$ a unitary $U \in \mathfrak{A}$:

$$\rho(M) = UMU^{-1} \quad (M \in \mathfrak{M}(K)).$$

[Choose $\rho_n \in [\rho]$: ρ_n is localized in K_n, say $\rho_n = U_n \cdot \rho \cdot U_n^{-1}$, where $U_n \in \underline{\text{Mor}}(\rho, \rho_n)$ is unitary -- then $\forall\, n \geq N(K)$, we have

$$\begin{aligned}
K \perp K_n &\Rightarrow K \subset K_n^{\perp} \\
&\Rightarrow \mathfrak{M}(K) \subset \mathfrak{A}(K_n^{\perp}) \\
&\Rightarrow \rho_n(M) = M \\
&\Rightarrow U_n \rho(M) U_n^{-1} = M \\
&\Rightarrow \rho(M) = U_n^{-1} M U_n.
\end{aligned}$$

And, as we know, the $U_n \in \mathfrak{A}$.]

[Note: By definition, $\mathfrak{A}$ is the norm closure of $\bigcup_{O} \mathfrak{M}(O)$, where O ranges over $\mathcal{K}$. From the above, it follows that $\forall M \in \mathfrak{M}(O)$, $\rho_n(M) = M$ eventually. If now $A \in \mathfrak{A}$ is arbitrary, then $\exists$ a sequence $M_k \in \mathfrak{M}(O_k)$:

$$\underline{\lim}_{k \to \infty} \| M_k - A \| = 0.$$

Fix $\varepsilon > 0$ and choose $k(\varepsilon)$:

$$\| M_k - A \| < \varepsilon \qquad (k \geq k(\varepsilon)).$$

Choose $N = N(\varepsilon)$:

$$\rho_n(M_{k(\varepsilon)}) = M_{k(\varepsilon)} \quad (n \geq N).$$

Then $\forall n \geq N$,

$$\begin{aligned} &\| \rho_n(A) - A \| \\ &= \| \rho_n(A) - \rho_n(M_{k(\varepsilon)}) + \rho_n(M_{k(\varepsilon)}) - A \| \\ &\leq \| \rho_n(A - M_{k(\varepsilon)}) \| + \| \rho_n(M_{k(\varepsilon)}) - A \| \\ &= \| A - M_{k(\varepsilon)} \| + \| M_{k(\varepsilon)} - A \| < 2\varepsilon . \end{aligned}$$

Therefore

$$\underline{\lim}_{n \to \infty} \| \rho_n(A) - A \| = 0.$$

This implies that ρ is the strong pointwise limit of DHR inner automorphisms, viz.:

$$\lim_{n\to\infty} \| \rho(A) - \gamma_n(A) \| = 0,$$

where

$$\gamma_n(A) = U_n^{-1} A U_n \quad (A \in \mathfrak{A}).]$$

Remark: $\exists$ a double cone $L \supset K$ and a unitary $U \in \mathfrak{M}(L)$:

$$\rho(M) = UMU^{-1} \quad (M \in \mathfrak{M}(K)).$$

[From the preceding considerations,

$$\rho(M) = U_n^{-1} M U_n \quad (M \in \mathfrak{M}(K)).$$

Choose a double cone $L: L \supset K \cup K_n$ -- then $\forall\, A \in \mathfrak{A}(L^\perp)$,

$$\begin{cases} \rho(A) = A \\ \rho_n(A) = A \end{cases}$$

$\Rightarrow$

$$U_n \rho(A) U_n^{-1} = U_n A U_n^{-1} = A$$

$\Rightarrow$

$$U_n A = A U_n$$

$\Rightarrow$

$$U_n \in \mathfrak{A}(L^\perp)' = \mathfrak{M}(L),$$

so we can take $U = U_n^{-1}$.]

A <u>left inverse</u> for ρ is a bounded linear map $\Phi : \mathfrak{A} \to \mathcal{B}(\mathcal{H})$ with the following properties:

(i) $\Phi(1_{\mathfrak{A}}) = \underline{\mathrm{id}}_{\mathcal{H}}$;

(ii) $A \geq 0 \Rightarrow \Phi(A) \geq 0$;

(iii) $\Phi(A^*) = \Phi(A)^*$;

(iv) $\Phi(\rho(A)B) = A\,\Phi(B)$.

One then shows without difficulty that $\Phi(\mathfrak{A}) \subset \mathfrak{A}$ and

$$\Phi(\underline{\mathrm{Mor}}(\rho^n, \rho^n)) \subset \underline{\mathrm{Mor}}(\rho^{n-1}, \rho^{n-1}) \quad (n=1,2,\ldots).$$

[Note: By definition,

$$\|\Phi\| = \underline{\sup}_{\|A\| \leq 1} \|\Phi(A)\| .$$

Therefore

$$\|\Phi\| \geq 1 \quad (\|\Phi(1_{\mathfrak{A}})\| = \|\underline{\mathrm{id}}_{\mathcal{H}}\| = 1)$$

and we claim that $\|\Phi\| = 1$. Using the positivity of Φ, one checks that

$$\Phi(A^*A) \geq \Phi(A)^* \Phi(A) \geq 0$$

$$\Rightarrow$$

$$\|\Phi(A^*A)\| \geq \|\Phi(A)^* \Phi(A)\| = \|\Phi(A)\|^2.$$

This said, to get a contradiction assume that $\exists\, A: \|A\| \leq 1$ & $\|\Phi(A)\| \geq 1$ -- then $\|A^*A\| = \|A\|^2 \leq 1$

$$\Rightarrow$$

$$\|\Phi\| \geq \|\Phi(A)\|^2.$$

Proceed from here by iteration, replacing A by B = A*A:

$$\|\Phi\| \geq \|\Phi(A^*A)\|^2$$
$$\geq \|\Phi(A)\|^4.$$

So, $\forall\, n > 0$,

$$\|\Phi\| \geq \|\Phi(A)\|^{2n},$$

an impossibility.]

To establish the existence of a left inverse for ρ, introduce the set $\mathcal{L}(\mathfrak{A}, \mathcal{B}(\mathcal{H}))$ of all bounded linear maps $T: \mathfrak{A} \longrightarrow \mathcal{B}(\mathcal{H})$ and let $\underset{\sim}{B} = \{T: \|T\| \leq 1\}$. Equip $\mathcal{L}(\mathfrak{A}, \mathcal{B}(\mathcal{H}))$ with the pointwise σ-weak topology, thus a net $\{T_i\}$ converges to T if $T_i(A) \longrightarrow T(A)$ in the σ-weak topology $\forall A \in \mathfrak{A}$ -- then in this topology, $\underset{\sim}{B}$ is compact. Consider now the sequence $\{\gamma_n^{-1}\}: \gamma_n^{-1}(A) = U_n A U_n^{-1} \Rightarrow \gamma_n^{-1} \in \mathcal{L}(\mathfrak{A}, \mathcal{B}(\mathcal{H}))$. Since U_n is unitary, it is clear that $\{\gamma_n^{-1}\} \subset \underset{\sim}{B}$, so $\{\gamma_n^{-1}\}$ has a limit point Φ. Any such limit point is a left inverse for ρ.

Remark: The significance of the existence of a left inverse Φ for ρ is simply this: Take ρ irreducible -- then

$$\Phi(\varepsilon(\rho, \rho)) \in \underline{\mathrm{Mor}}(\rho, \rho) = \underset{\sim}{C} 1_\rho$$

is a scalar which turns out to be independent of the choice of Φ and is in fact the statistics parameter λ_ρ.

Review:

$$\begin{cases} \mathfrak{M}(K) = \mathfrak{M}(K^\perp)' \\ \mathfrak{A}(K^\perp)'' = \mathfrak{M}(K^\perp) \end{cases}$$

$\Rightarrow$

$$\mathfrak{A}(K^\perp) \subset \mathfrak{A}(K^\perp)''$$

$$= \mathfrak{M}(K^{\perp})$$

$$= \mathfrak{M}(K^{\perp})''$$

$$= \mathfrak{M}(K)'.$$

Suppose that ρ is a DHR endomorphism which is localized in $K \in \mathcal{K}$ -- then $\rho(\mathfrak{M}(K))$ is contained in $\mathfrak{M}(K)$. Thus let

$$\begin{cases} M \in \mathfrak{M}(K) \\ M' \in \mathfrak{A}(K^{\perp}). \end{cases}$$

Then

$$\begin{aligned} \rho(M)M' &= \rho(M)\rho(M') \\ &= \rho(MM') \\ &= \rho(M'M) \\ &= \rho(M')\rho(M) \\ &= M'\rho(M). \end{aligned}$$

Therefore

$$\rho(\mathfrak{M}(K)) \subset \mathfrak{A}(K^{\perp})' = \mathfrak{M}(K).$$

Now fix a wedge $W: K \subset W$ -- then it follows that $\rho(\mathfrak{A}(W))$ is contained in $\mathfrak{A}(W)$.

[Note: Bear in mind that if $O \supset K$, then ρ is necessarily localized in O.]

LEMMA $\exists$ a unitary $U \in \mathfrak{A}$:

$$\rho(A) = UAU^{-1} \quad (A \in \mathfrak{A}(W)).$$

[Choose $L \in \mathcal{K} : L \subset W^{\perp}$ ($\Rightarrow W \subset L^{\perp}$) -- then $\exists\, \sigma \in [\rho] : \sigma$ is localized in L, hence $\exists$ a unitary $V \in \mathfrak{A} : \sigma = V \cdot \rho \cdot V^{-1}$. Since $\mathfrak{A}(W) \subset \mathfrak{A}(L^{\perp})$, $\forall\, A \in \mathfrak{A}(W)$:

$$\sigma(A) = A$$

$$\Rightarrow$$

$$V\,\rho(A)V^{-1} = A$$

$$\Rightarrow$$

$$\rho(A) = UAU^{-1} \quad (U = V^{-1}).]$$

Rappel: If $\mathfrak{M}$ and $\mathfrak{N}$ are W*-algebras, then a morphism $\phi : \mathfrak{M} \to \mathfrak{N}$ of the underlying C*-algebras is said to be normal if it is σ-weakly continuous. When this is so, the image $\phi(\mathfrak{M})$ is a W*-subalgebra of $\mathfrak{N}$.

Since by the lemma, the restriction $\rho\,|\mathfrak{A}(W)$ is unitarily implementable, it extends to a normal endomorphism of $\mathfrak{M}(W)$, call it ρ_W, thus $\rho_W(\mathfrak{M}(W))$ is a W*-subalgebra of $\mathfrak{M}(W)$.

[Note: To check continuity in the σ-weak sense, simply observe that $\forall\, T \in \mathcal{B}_1(\mathcal{H})$,

$$|\underline{tr}(UAU^{-1}T)| = |\underline{tr}(UAU^{-1}TUU^{-1})|$$

$$= |\underline{tr}(AU^{-1}TU)|$$

and $U^{-1}TU \in \mathcal{B}_1(\mathcal{H})$. Accordingly, if $A_i \to M$ σ-weakly ($A_i \in \mathfrak{A}(W)$, $M \in \mathfrak{M}(W)$), then $UA_iU^{-1} \to UMU^{-1}$ σ-weakly, hence, via extension by continuity, $\rho_W(M) = UMU^{-1}\ \forall\, M \in \mathfrak{M}(W)$.]

LEMMA The pair $(\rho_W(\mathfrak{M}(W)), U\Omega_0)$ is a standard W*-algebra.

[$U\Omega_0$ is cyclic for $\rho_W(\mathfrak{M}(W))$. Proof: Given $\Psi \in \mathcal{H}$, choose $M_n \in \mathfrak{M}(W): M_n\Omega_0 \to U^{-1}\Psi$ -- then $UM_nU^{-1}\cdot U\Omega_0 \to \Psi$.

$U\Omega_0$ is separating for $\rho_W(\mathfrak{M}(W))$. Proof: For $UMU^{-1}\cdot U\Omega_0 = 0$ $\Rightarrow UM\Omega_0 = 0 \Rightarrow M\Omega_0 = 0 \Rightarrow M = 0 \Rightarrow UMU^{-1} = 0$.]

LEMMA $\rho_W(\mathfrak{M}(W))$ is a factor.

[Suppose that $Z\rho_W(M) = \rho_W(M)Z \ \forall M \in \mathfrak{M}(W)$, where Z is in the center of $\rho_W(\mathfrak{M}(W))$. Write $Z = \rho_W(M_0)$ -- then

$$\rho_W(M_0)\rho_W(M) = \rho_W(M)\rho_W(M_0)$$

$$\Rightarrow$$

$$\rho_W(M_0M) = \rho_W(MM_0)$$

$$\Rightarrow$$

$$M_0M = MM_0.$$

Since this is true for all $M \in \mathfrak{M}(W)$ and $\mathfrak{M}(W)$ is a factor, it follows that $M_0 = CI$ $(\exists C) \Rightarrow \rho(M_0) = CI$.]

Rappel: Multiplication on the left or right by a fixed element of $\mathcal{B}(\mathcal{H})$ defines a map $\mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ which is continuous in the σ-weak topology.

Notation: Put

$$\underline{\mathrm{Mor}}(\rho_W, \rho_W) = \{T \in \mathfrak{M}(W) : T\rho_W(M) = \rho_W(M)T \ \forall M \in \mathfrak{M}(W)\}.$$

LEMMA We have

$$\underline{\mathrm{Mor}}(\rho,\rho) \subset \underline{\mathrm{Mor}}(\rho_W,\rho_W).$$

[Let $T \in \underline{\mathrm{Mor}}(\rho,\rho)$ -- then by definition, $T\rho(A) = \rho(A)T$ $\forall A \in \mathfrak{A}$, hence in particular, $T\rho(A) = \rho(A)T$ $\forall A \in \mathfrak{A}(W)$. I.e.: $\forall A \in \mathfrak{A}(W)$,

$$TUAU^{-1} = UAU^{-1}T.$$

So, if $A_i \to M$ σ-weakly ($A_i \in \mathfrak{A}(W)$, $M \in \mathfrak{M}(W)$), then

$$\begin{cases} TUA_iU^{-1} \to TUMU^{-1} \\ \\ UA_iU^{-1}T \to UMU^{-1}T \end{cases}$$

$$\Rightarrow$$

$$T\rho_W(M) = \rho_W(M)T \quad \forall M \in \mathfrak{M}(W).$$

Therefore

$$\underline{\mathrm{Mor}}(\rho,\rho) \subset \underline{\mathrm{Mor}}(\rho_W,\rho_W),$$

as contended.]

Fact: Suppose that ρ is finite -- then

$$\underline{\mathrm{Mor}}(\rho,\rho) = \underline{\mathrm{Mor}}(\rho_W,\rho_W).$$

Remark: When ρ is finite, one can attach to the inclusion $\rho_W(\mathfrak{M}(W)) \to \mathfrak{M}(W)$ a nonnegative real number $\underline{\mathrm{Ind}}(\rho)$, the <u>Jones index</u>. On general grounds,

$$\underline{\mathrm{Ind}}(\rho) \in \left\{ 4\,\underline{\cos}^2\left(\frac{\pi}{k}\right) : k \in \underset{\sim}{N},\ k \geq 3 \right\} \cup [4, +\infty].$$

But, in the case at hand, Longo has made matters precise:

$$\underline{\mathrm{Ind}}(\rho) = d(\rho)^2,$$

where, as usual, $d(\rho)$ is the statistical dimension of ρ.

<u>Some Results of Longo</u> In this section we shall take d = 3 and work in $\underset{\sim}{R}^4 = \underset{\sim}{R}^{1,3}$.

Suppose given a weakly additive PTV with a unique vacuum which satisfies Haag duality and has the B-W property. In a change of notation, put

$$\Lambda(t) = \begin{pmatrix} \cosh t & \sinh t & 0 & 0 \\ \sinh t & \cosh t & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Then

$$\Delta^{\sqrt{-1}\, t} = U(\Lambda(-2\pi t)) \quad (\Delta \equiv \Delta_{\mathfrak{M}(W_R)}).$$

[Note: Since the field algebra is not going to play a role, there is no need to append a subzero to U.]

Now fix a DHR endomorphism ρ which is localized in $K \subset W_R$. Assume:

(i) ρ is finite;

(ii) ρ is covariant.

LEMMA Put

$$z_\rho(t) = U_\rho(\Lambda(t))U(\Lambda(-t)).$$

Then

$$z_\rho(t) \in \mathfrak{M}(W_R).$$

[To begin with,

$$U(\Lambda(t))MU(\Lambda(-t)) \in \mathfrak{M}(W_R) \quad (M \in \mathfrak{M}(W_R))$$

$\Rightarrow$

$$U(\Lambda(t))M'U(\Lambda(-t)) \in \mathfrak{M}(W_R)' \quad (M' \in \mathfrak{M}(W_R)').$$

In addition, $K \subset W_R \Rightarrow W_R^{\perp} = W_L \subset K^{\perp}$, so ρ is the identity on $\mathfrak{A}(W_L) \subset \mathfrak{A}(K^{\perp})$. This persists to $\mathfrak{A}(W_L)'' = \mathfrak{M}(W_L)$, i.e., to $\mathfrak{M}(W_R)'$ (wedge duality). Therefore

$$\rho(U(\Lambda(t))M'U(\Lambda(-t))) = U(\Lambda(t))M'U(\Lambda(-t)).$$

But

$$\rho(U(\Lambda(t))M'U(\Lambda(-t)))$$

$$= U_\rho(\Lambda(t))\,\rho(M')U_\rho(\Lambda(-t))$$

$$= U_\rho(\Lambda(t))M'U_\rho(\Lambda(-t))$$

$\Rightarrow$

$$U(\Lambda(t))M'U(\Lambda(-t)) = U_\rho(\Lambda(t))M'U_\rho(\Lambda(-t))$$

$\Rightarrow$

$$U(\Lambda(t))^{-1}\,U_\rho(\Lambda(t)) \in \mathfrak{M}(W_R)'' = \mathfrak{M}(W_R).$$

Replacing t by -t and taking adjoints serves to complete the proof.]

Application: The prescription

$$\alpha_t^{\rho}(M) = U_\rho(\Lambda(t))MU_\rho(\Lambda(-t)) \quad (M \in \mathfrak{M}(W_R))$$

defines a one parameter group of automorphisms of $\mathfrak{M}(W_R)$.

[One has only to observe that

$$\alpha_t^{\rho}(\mathfrak{M}(W_R)) = z_{\rho}(t)U(\Lambda(t))\mathfrak{M}(W_R)U(\Lambda(-t))z_{\rho}(-t)$$

$$= z_{\rho}(t)\,\mathfrak{M}(W_R)z_{\rho}(-t)$$

$$= \mathfrak{M}(W_R).]$$

Define H_{ρ} by writing

$$U_{\rho}(\Lambda(t)) = \underline{\exp}(\sqrt{-1}\, t\, H_{\rho}).$$

THEOREM We have

$$d(\rho) = \langle \Omega_0, e^{-2\pi H_{\rho}}\Omega_0\rangle.$$

Example: Let's run a reality check and take $\rho = \iota$ -- then

$$\Delta^{\sqrt{-1}\, t} = \underline{\exp}(\sqrt{-1}\, t\, \underline{\log}\,\Delta)$$

$$= U(\Lambda(-2\pi t))$$

$$= \underline{\exp}(\sqrt{-1}(-2\pi t)H_{\iota})$$

$\Rightarrow$

$$\underline{\log}\,\Delta = -2\pi H_{\iota}$$

$\Rightarrow$

$$e^{\underline{\log}\,\Delta}\Omega_0 = e^{-2\pi H_{\iota}}\Omega_0.$$

But

$$\Delta\Omega_0 = \Omega_0 \Rightarrow e^{\underline{\log}\,\Delta}\Omega_0 = e^{\underline{\log}\, 1}\Omega_0 = \Omega_0$$

$$\Longrightarrow \qquad e^{-2\pi H_\iota}\,\Omega_0 = \Omega_0$$

$$\Longrightarrow$$

$$\langle \Omega_0, e^{-2\pi H_\iota}\,\Omega_0 \rangle = \langle \Omega_0, \Omega_0 \rangle = 1,$$

which agrees with the fact that $d(\iota) = 1$.

LEMMA The formula

$$\omega_\rho(X^*Y) = \frac{1}{d(\rho)} \langle e^{-\pi H_\rho} X\Omega_0, e^{-\pi H_\rho} Y\Omega_0 \rangle$$

determines a faithful normal state on $\mathfrak{M}(W_R)$ such that

$$(D\omega_\rho : D\omega_0)(t) = d(\rho)^{-\sqrt{-1}\,t} z_\rho(-2\pi t).$$

[Note: The modular automorphism group of the pair $(\mathfrak{M}(W_R), \omega_\rho)$ can be explicated:

$$\sigma_t^{\omega_\rho}(M) = U_t \sigma_t(M) U_t^{-1}$$

$$= (D\omega_\rho : D\omega_0)(t)\ \Delta^{\sqrt{-1}\,t} M \Delta^{-\sqrt{-1}\,t}\ (D\omega_\rho : D\omega_0)(t)^{-1}$$

$$= U_\rho(\Lambda(-2\pi t))\ \Delta^{-\sqrt{-1}\,t}\ \Delta^{\sqrt{-1}\,t} M \Delta^{-\sqrt{-1}\,t}\ \Delta^{\sqrt{-1}\,t}\ U_\rho(\Lambda(2\pi t))$$

$$= U_\rho(\Lambda(-2\pi t)) M U_\rho(\Lambda(2\pi t))$$

$$= \alpha^\rho_{-2\pi t}(M).]$$

Choose a cyclic and separating unit vector Ω_ρ such that

$$\omega_\rho(M) = \langle \Omega_\rho, M\Omega_\rho \rangle \quad (M \in \mathcal{M}(W_R)).$$

Then

$$(D\omega_\rho : D\omega_0)(t) = \Delta^{\sqrt{-1}\,t}_{\Omega_0,\Omega_\rho} \Delta^{-\sqrt{-1}\,t}_{\Omega_0} \quad (\Delta_{\Omega_0} = \Delta).$$

But

$$z_\rho(-2\pi t) = d(\rho)^{\sqrt{-1}\,t} (D\omega_\rho : D\omega_0)(t)$$

$\Rightarrow$

$$U_\rho(\Lambda(-2\pi t))U(\Lambda(2\pi t)) = d(\rho)^{\sqrt{-1}\,t} \Delta^{\sqrt{-1}\,t}_{\Omega_0,\Omega_\rho} \Delta^{-\sqrt{-1}\,t}_{\Omega_0}$$

$\Rightarrow$

$$U_\rho(\Lambda(-2\pi t)) = d(\rho)^{\sqrt{-1}\,t} \Delta^{\sqrt{-1}\,t}_{\Omega_0,\Omega_\rho}$$

$\Rightarrow$

$$\underline{\exp}(\sqrt{-1}\,(-2\pi t)H_\rho) = d(\rho)^{\sqrt{-1}\,t} \Delta^{\sqrt{-1}\,t}_{\Omega_0,\Omega_\rho}$$

$\Rightarrow$

$$2\pi H_\rho = -\underline{\log}\,\Delta_{\Omega_0,\Omega_\rho} - \underline{\log}\,d(\rho).$$

Definition: The <u>free energy</u> of ω_ρ relative to ω_0 is

$$F(\omega_\rho \mid \omega_0) = \langle \Omega_\rho, (H_\rho + \frac{1}{2\pi}\underline{\log}\,\Delta_{\Omega_0,\Omega_\rho})\Omega_\rho \rangle.$$

Therefore

$$F(\omega_\rho \mid \omega_0) = -\langle \Omega_\rho , (\frac{1}{2\pi} \log d(\rho))\Omega_\rho \rangle$$

$$= -\frac{1}{2\pi} \log d(\rho).$$

Consequently, the possible values of $F(\omega_\rho \mid \omega_0)$ are quantized:

$$F(\omega_\rho \mid \omega_0) \in -\frac{1}{2\pi} \{ \log n : n=1,2,\ldots \} .$$

Remark: Recall that ρ determines an endomorphism ρ_{W_R} of $\mathfrak{M}(W_R)$, thus one can introduce the conditional entropy of ρ_{W_R}: $S_c(\rho)$, which, by the Pimsner-Popa theorem, is

$$\log \mathrm{Ind}(\rho)$$

or still,

$$2 \log d(\rho).$$

<u>Past and Future</u> In all that follows, (M,g) denotes a space-time. So:

(1) M is a connected C^{∞} manifold of dimension 1 + d ($d \geq 1$);

(2) g is a lorentzian metric of signature (1,d) (+, -,...,-);

(3) M is time orientable, i.e., admits a timelike vector field.

Remark: The tangent space M_x at a given $x \in M$ is Minkowski space-time. Therefore a vector $X \in M_x$ is <u>timelike</u> if $g(X,X) > 0$, <u>lightlike</u> if $g(X,X) = 0$, and <u>spacelike</u> if $g(X,X) < 0$. The complement in M_x of the closure of the spacelike points has two components ("timecones") and there is no intrinsic way to distinguish them. If one of these cones is singled out and called the future cone $V_+(x)$, then M_x is said to be time oriented. A timelike or lightlike vector in^or on $V_+(x)$ is said to be <u>future directed</u>. The other cone is denoted by $V_-(x)$. A timelike or lightlike vector in^or on $V_-(x)$ is said to be <u>past directed</u>.

[Note: If τ is a timelike vector field, then M_x can be time oriented by specifying the timecone containing τ_x.]

Remark: Suppose that g_1, g_2 are lorentzian metrics on M. Assume: $\forall\, x \in M$ & $\forall\, X \in M_x$,

$$g_1(X,X) = 0 \text{ iff } g_2(X,X) = 0.$$

Then $\exists$ a C^{∞} function $\Omega : M \to \underset{\sim}{R}_{>0}$ such that

$$g_2 = \Omega g_1.$$

A curve in M is timelike, lightlike, or spacelike if its tangent vectors are timelike, lightlike, or spacelike.

[Note: If γ is a geodesic, then $g(\dot{\gamma},\dot{\gamma})$ is a constant, hence a geodesic which is timelike, lightlike, or spacelike for some value of its parameter is timelike, lightlike, or spacelike for all values of its parameter.]

A curve in M is <u>causal</u> if its tangent vectors are timelike or lightlike. A causal curve is <u>future directed</u> (<u>past directed</u>) if its tangent vectors have this property.

A future directed causal curve $\gamma : I \to M$ is said to have a <u>future endpoint</u> (<u>past endpoint</u>) if $\gamma(t)$ converges to some point in M as $t \uparrow \underline{\sup}\ I$ ($t \downarrow \underline{\inf}\ I$).

A past directed causal curve $\gamma : I \to M$ is said to have a <u>past endpoint</u> (<u>future endpoint</u>) if $\gamma(t)$ converges to some point in M as $t \uparrow \underline{\sup}\ I$ ($t \downarrow \underline{\inf}\ I$).

A future (past) directed causal curve γ is said to <u>start at a point $p \in M$</u> provided that p is the past (future) endpoint of γ.

A future (past) directed causal curve γ is said to be <u>future (past) inextendible</u> if it possesses no future (past) endpoint.

Notation: $\forall\ p \neq q$ in M,

<u>$p \ll q$</u>: $\exists$ a future directed timelike curve from p to q.

<u>$p < q$</u>: $\exists$ a future directed causal curve from p to q.

[Note: Obviously, $p \ll q \Rightarrow p < q$. Write $p \leq q$ if either $p < q$ or $p = q$.]

Definition: The <u>chronological future of p</u> is

$$I^+(p) = \{q : p \ll q\}$$

and the causal future of p is

$$J^{+}(p) = \{q : p \leq q\}.$$

The chronological past of p is

$$I^{-}(p) = \{q : q << p\}$$

and the causal past of p is

$$J^{-}(p) = \{q : q \leq p\}.$$

[Note: For a nonempty subset $S \subset M$, the sets $I^{\pm}(S)$, $J^{\pm}(S)$ are defined analogously. E.g.: $I^{+}(S) = \{q : p << q \ (\exists p \in S)$ and $J^{+}(S) = \{q : p \leq q \ (\exists p \in S)\}$. Obviously, $I^{+}(S) = \bigcup_{p \in S} I^{+}(p)$ and $J^{+}(S) = \bigcup_{p \in S} J^{+}(p)$. Furthermore, $J^{+}(S) \supset S \cup I^{+}(S)$.]

LEMMA If $x << y$ and $y \leq z$ or if $x \leq y$ and $y << z$, then $x << z$.

Application: We have

$$I^{+}(S) = I^{+}(I^{+}S) = I^{+}(J^{+}S)$$
$$= J^{+}(I^{+}S) \subset J^{+}(J^{+}S) = J^{+}(S).$$

Rappel: An open subset $\mathcal{C}$ of M is geodesically convex provided $\mathcal{C}$ is a normal neighborhood of each of its points. Accordingly, given $q', q'' \in \mathcal{C}$, $\exists$ a unique geodesic segment $\gamma : [0,1] \longrightarrow M$ such that $\gamma(0) = q'$, $\gamma(1) = q''$ with $\gamma[0,1] \subset \mathcal{C}$. Furthermore, it can be shown that $q' << q''$ iff γ is future directed and timelike.

[Note: $\mathcal{C}$ is a space-time in its own right. And, in obvious notation,

(1) $q \in I^+(p, \mathcal{C}) \iff q = \underline{\exp}_p(X)$, where $X \in V_+(p)$;

(2) $q \in J^+(p, \mathcal{C}) \iff q = \underline{\exp}_p(X)$, where $X \in \overline{V_+(p)}$.

It then follows that $I^+(p, \mathcal{C})$ is open and $J^+(p, \mathcal{C}) = \overline{I^+(p, \mathcal{C})}$.]

LEMMA If $p \ll q$, then $\exists$ neighborhoods N_p of p and N_q of q such that

$$\left\{\begin{array}{l} p' \in N_p \\ \\ q' \in N_q \end{array}\right. \Rightarrow p' \ll q'.$$

[Suppose that $\gamma : [a,b] \rightarrow M$ is a future directed timelike curve with $\gamma(a) = p$ & $\gamma(b) = q$. Choose a geodesically convex neighborhood $\mathcal{C}_q$ of q and fix a point $q^- \in \mathcal{C}_q$ on γ before q: $q^- \ll q$. Choose a geodesically convex neighborhood $\mathcal{C}_p$ of p and fix $p^+ \in \mathcal{C}_p$ on γ between p and q^-: $p \ll p^+ \ll q^-$. Now put $N_p = I^-(p^+, \mathcal{C}_p)$ and

$$N_q = I^+(q^-, \mathcal{C}_q) : \left\{\begin{array}{l} p' \in N_p \\ \\ q' \in N_q \end{array}\right. \Rightarrow p' \ll p^+ \ll q^- \ll q'.]$$

Application: $\forall\ p \in M$, $I^+(p)$ is open.

[Note: It is obvious that $I^+(p)$ is nonempty and connected.]

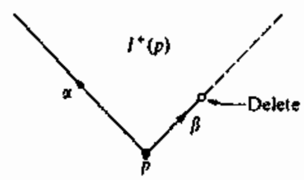
Facts: $\forall\ p \in M$,

(1) $\underline{\text{int}}\ \overline{I^+(p)} = I^+(p)$;

(2) $\overline{I^+(p)} = \{x : I^+(x) \subset I^+(p)\}$;

(3) $\underline{\text{fr}}\ I^+(p) = \{x : x \notin I^+(p)\ \&\ I^+(x) \subset I^+(p)\}$.

In Minkowski space-time, $J^+(p)$ is closed but this fails to be true in general. For example, let $M = \underset{\sim}{R}^{1,1}$ with the point (1,1) deleted and take for p the origin:



No causal curve from p can reach points on the dotted line, thus $J^+(p)$ consists of $I^+(p)$ together with the lightlike geodesic rays α and β . In particular: $J^+(p)$ is a proper subset of $\overline{I^+(p)}$.

LEMMA If γ is a future directed causal curve from S to a point $q \in J^+(S) - I^+(S)$, then γ is a lightlike geodesic that does not meet $I^+(S)$.

Facts: $\forall\ S \subset M$,

(1) $\underline{\text{int}}\ J^+(S) = I^+(S)$;

(2) $J^+(S) \subset \overline{I^+(S)}$.

Definition: A space-time (M,g) is said to be chronological if M contains no closed timelike curves, i.e., $\forall$ p, $p \notin I^+(p)$.

LEMMA A compact space-time (M,g) contains a closed timelike curve, hence is not chronological.

[Since the $I^+(p)$ are open and $M = \bigcup_{p \in M} I^+(p)$, $\exists$ points $p_1, \ldots, p_n$ such that $M = I^+(p_1) \cup \cdots \cup I^+(p_n)$. And: $\exists$ $i(1)$: $p_1 \in I^+(p_{i(1)})$ $(1 \leq i(1) \leq n)$, $\exists$ $i(2)$: $p_{i(1)} \in I^+(p_{i(2)})$ $(1 \leq i(2) \leq n)$ etc. This leads to an infinite sequence $\cdots << p_{i(k+1)} << p_{i(k)} << \cdots << p_1$. Since n is finite, there are a finite number of distinct $p_{i(k)}$, so there are repetitions: $p_{i(k)} = p_{i(\ell)}$ $(i(k) < i(\ell))$: $p_{i(\ell)} << p_{i(k)}$ $\Rightarrow$ $p_{i(k)} \in I^+(p_{i(k)})$, which means that (M,g) contains a closed timelike curve.]

Definition: A space-time (M,g) is said to be <u>causal</u> if M contains no closed causal curves.

Of course, "causal" $\Rightarrow$ "chronological" (but the converse is false).

Definition: A space-time (M,g) is said to be <u>strongly causal at p</u> if given any neighborhood O of p $\exists$ a neighborhood $O' \subset O$ of p such that every causal curve segment with endpoints in O' lies entirely in O.

If (M,g) is strongly causal at p, then there does not exist a causal curve segment $\gamma : [a,b] \rightarrow M$ with $\gamma(a) = p = \gamma(b)$. Thus choose $t_0 \in [a,b]$: $\gamma(t_0) \neq p$ and O: $\gamma(t_0) \notin O$. Take O' per the definition -- then $\gamma[a,b] \subset O$, a contradiction.

Fact: The set of points at which (M,g) is strongly causal is open.

Definition: A space-time (M,g) is said to be <u>strongly causal</u> if it is strongly causal at every point.

Example: Suppose that (M,g) is strongly causal and $K \subset M$ is compact. Let $\gamma : [0, +\infty[\longrightarrow M$ be a causal curve with image in K -- then

$$\lim_{t \to +\infty} \gamma(t)$$

exists.

[By the compactness of K, we can assume that $\lim p_n = p$ exists, where $p_n = \gamma(n)$. Claim: $\lim_{t \to +\infty} \gamma(t) = p$. Suppose false, so $\exists$ a neighborhood O of p such that $\forall t_0, \exists\, t > t_0: \gamma(t) \not\in O$. Choose a neighborhood O' of p per strong causality. Fix $n_0 >> 0: n \geq n_0 \Rightarrow$ $p_n = \gamma(n) \in O'$ -- then $\exists\, t > n_0: \gamma(t) \not\in O$. But $\exists\, m > t$ & $\gamma(m) \in O'$, thus $\gamma[n_0, m] \subset O$, a contradiction.]

Remark: Suppose that (M,g) is strongly causal -- then the $I^+(p) \cap I^-(q)$ $(p,q \in M)$ are a basis for the topology on M.

Globally Hyperbolic Space-Times A space-time (M,g) is said to be globally hyperbolic if it is strongly causal and $\forall\, p,q \in M$, the set $J^+(p) \cap J^-(q)$ is compact.

[Note: This implies that $J^+(K) \cap J^-(L)$ is compact whenever K and L are compact.]

LEMMA If (M,g) is globally hyperbolic, then $\forall\, p$, $J^+(p)$ is closed.

[To get a contradiction, let us suppose that $\exists\, q \in \overline{J^+(p)} - J^+(p)$. Choose a sequence $\{q_n\} \subset J^+(p) : q_n \rightarrow q$ and fix an $x \in I^+(q)$. So:

$$p \leq q_n \ \& \ q \ll x.$$

Since $q \in I^-(x)$ and $I^-(x)$ is open, it follows that $\exists\, N : n \geq N \Rightarrow q_n \in I^-(x)$. But $q \notin J^+(p) \Rightarrow q \notin J^+(p) \cap J^-(x)$. On the other hand, $q_n \in J^+(p)$ (by assumption) and $q_n \in I^-(x) \subset J^-(x)$ (if $n \geq N$), hence $\forall\, n \geq N$, $q_n \in J^+(p) \cap J^-(x)$, which implies that $q \in J^+(p) \cap J^-(x)$, an impossibility.]

[Note: More generally, K compact $\Rightarrow J^+(K)$ closed.]

The following space-times are globally hyperbolic: Minkowski, Robertson-Walker, Schwarzschild-Kruskal.

Example: Let (M,g) be an arbitrary space-time. Suppose given a subset S of M -- then the future development $D^+(S)$ of S is the set of $p \in M$ such that every past inextendible causal curve starting at p meets S ($\Rightarrow S \subset D^+(S) \subset J^+(S)$). Assume now that S is achronal, i.e., no timelike curve meets S more than once -- then int $D(S)$, if nonempty, is globally hyperbolic.

[Note: The definition of $D^-(S)$ is dual. The union $D(S) = D^+(S) \cup D^-(S)$ is the domain of dependence of S.]

Definition: A Cauchy surface is a closed, connected hypersurface $\Sigma \subset M$ which is intersected exactly once by each inextendible causal curve in M.

So, e.g., in $\underset{\sim}{R}^{1,d}$, the hyperplanes x_0 = constant are Cauchy surfaces.

Remark: Let $\Sigma \subset M$ be a Cauchy surface. Fix a timelike vector field X on M -- then the trajectories of X partition M, so $\forall\ p \in M$, the trajectory of X through p meets Σ at a unique point r(p) (trajectories are necessarily inextendible). Using Brouwer's theorem on invariance of domain, one can show that $r: M \to \Sigma$ is a continuous open map which leaves Σ pointwise fixed. Consequently, any two Cauchy surfaces in M are homeomorphic (if Σ_1, Σ_2 are Cauchy and if r_1, r_2 are the corresponding retractions, then $r_2 | \Sigma_1$, $r_1 | \Sigma_2$ are mutually inverse).

[Note: It is clear that $D(\Sigma) = M$, hence $M = \underline{\text{int}}\ D(\Sigma)$ is globally hyperbolic.]

STRUCTURE THEOREM Suppose that (M,g) is globally hyperbolic -- then there exists a d-dimensional manifold Σ and a diffeomorphism $\Psi: \underset{\sim}{R} \times \Sigma \to M$ such that $\forall t$, $\Sigma_t = \Psi(\{t\} \times \Sigma)$ is a Cauchy surface in M, hence

$$M = \coprod_t \Sigma_t$$

is foliated by a C^∞-family $\{\Sigma_t\}$ of Cauchy surfaces.

Addendum: $\forall\ \sigma \in \Sigma$, the map $t \to \Psi(t,\sigma)$ is a future directed timelike inextendible curve. Accordingly, if $\Sigma_0 \subset M$ is a Cauchy surface, then $\Psi(t,\sigma)$ intersects Σ_0 exactly once at the parameter value $t = \tau_{\Sigma_0}(\sigma)$. We call $\tau_{\Sigma_0} : \Sigma \to \mathbb{R}$ the <u>time level function</u> of Σ_0. It is C^∞ and $\Sigma_0 = \{\Psi(\tau_{\Sigma_0}(\sigma),\sigma) : \sigma \in \Sigma\}$. Moreover, if Σ_1, Σ_2 are Cauchy, then the map $\Sigma_1 \to \Sigma_2$ defined by the rule

$$\Psi(\tau_{\Sigma_1}(\sigma),\sigma) \to \Psi(\tau_{\Sigma_2}(\sigma),\sigma)$$

is a diffeomorphism.

LEMMA A globally hyperbolic space-time (M,g) admits a <u>time function</u>, i.e., a C^∞ function $T: M \to \mathbb{R}$ whose gradient is a future directed timelike vector field.

Klein-Gordon Let (M,g) be a globally hyperbolic space-time.

Notation:

(1) $\square^2$ is the d'Alembertian on M per g, thus in a coordinate neighborhood U,

$$\square^2 = |G|^{-\frac{1}{2}} \sum_k \partial_k \left(\sum_\ell g^{k\ell} |G|^{\frac{1}{2}} \partial_\ell \right),$$

where

$$|G| = |\underline{\det}(g_{ij})|.$$

(2) μ is the positive measure on M determined by g, thus in a coordinate neighborhood U,

$$\mu(f) = \int_U f|G|^{\frac{1}{2}} dx.$$

THEOREM Fix $m > 0$ -- then $\exists$ continuous linear maps

$$E^{\pm} : C_c^{\infty}(M) \longrightarrow C^{\infty}(M)$$

such that

$$\begin{cases} E^{\pm}(\square^2 + m^2)f = f \\ (\square^2 + m^2)E^{\pm} f = f. \end{cases}$$

Furthermore,

$$\underline{\text{spt}}\ E^{\pm} f \subset J^{\pm}(\underline{\text{spt}}\ f).$$

It turns out that $E^{\pm}$ are integral operators:

$$E^{\pm} f(x) = \int_M E^{\pm}(x,y) f(y) d\mu(y).$$

Let

$$E(x,y) = E^{+}(x,y) - E^{-}(x,y).$$

Then

$$Ef(x) = \int_M E(x,y) f(y) d\mu(y)$$

and

$$\underline{\text{spt}}\, E^{\pm} f \subset J^{\pm}(\underline{\text{spt}}\, f)$$

$$\Longrightarrow$$

$$\underline{\text{spt}}\, Ef \subset J(\underline{\text{spt}}\, f).$$

Remark: Take $M = \mathbf{R}^{1,3}$. In this setting it is then customary to deal with the

- causal Green's function E^{c}
- retarded Green's function E^{+}
- advanced Green's function E^{-}

as well as the

- positive frequency function D^{+}
- negative frequency function D^{-} .

The connections between them are:

$$\begin{cases} E^+ = E^C + D^+ \\ E^- = E^C - D^- \end{cases}$$

$$\Rightarrow$$

$$E^+ - E^- = D^+ + D^- = E.$$

Here

$$\begin{cases} D^+(x) = -\dfrac{\sqrt{-1}}{(2\pi)^3}\displaystyle\int_{X_m} e^{\sqrt{-1}\langle p,x\rangle}\, d\mu_m(p) \\ D^-(x) = \dfrac{\sqrt{-1}}{(2\pi)^3}\displaystyle\int_{X_m} e^{-\sqrt{-1}\langle p,x\rangle}\, d\mu_m(p) \end{cases}$$

$$\Rightarrow$$

$$E(x,y) = \frac{\sqrt{-1}}{(2\pi)^3}\int_{X_m}\left(e^{-\sqrt{-1}\langle p,x-y\rangle} - e^{\sqrt{-1}\langle p,x-y\rangle}\right) d\mu_m(p)$$

or still,

$$E(x,y) = \frac{2}{(2\pi)^3}\int_{X_m} \underline{\sin}\langle p,x-y\rangle\, d\mu_m(p).$$

Let

$$\sigma(f_1,f_2) = \int_M \int_M f_1(x)E(x,y)f_2(y)\,d\mu(x)\,d\mu(y).$$

Then σ is a nondegenerate alternating bilinear form on $C_c^\infty(M;\underset{\sim}{R})/\underline{\ker}\, E$.

[Note: It is a fact that

$$E(x,y) = -E(y,x).$$

Accordingly,

$$\sigma(f_2,f_1) = \int_M \int_M f_2(x)E(x,y)f_1(y)d\mu(x)d\mu(y)$$

$$= \int_M \int_M f_2(y)E(y,x)f_1(x)d\mu(y)d\mu(x)$$

$$= - \int_M \int_M f_1(x)E(x,y)f_2(y)d\mu(x)d\mu(y) = - \sigma(f_1,f_2).]$$

Definition: The <u>Klein-Gordon algebra</u> of the pair (M,g) is

$$\mathfrak{A}_g = \text{CCR}(C_c^{\infty}(M;\underset{\sim}{R})/\underline{\ker}\ E,\ \sigma).$$

There is an evident assignment

$$O \longrightarrow \mathfrak{A}_g(O) = C^*\{W([f]): \underline{\text{spt}}\ f \subset O\}$$

from the bounded open subsets of M to subalgebras of $\mathfrak{A}_g$ and

$$\mathfrak{A}_g = C^*(\bigcup_O \mathfrak{A}_g(O)).$$

I_g: Suppose that O_1 is contained in O_2 -- then $\mathfrak{A}_g(O_1)$ is contained in $\mathfrak{A}_g(O_2)$.

II_g: Suppose that O_1 is contained in $M-J(O_2)$ (meaning that O_1 and O_2 are spacelike separated). Let

$$\begin{cases} f_1 \in C_c^\infty(M;\underset{\sim}{R}): \underline{\text{spt}}\ f_1 \subset O_1 \\ f_2 \in C_c^\infty(M;\underset{\sim}{R}): \underline{\text{spt}}\ f_2 \subset O_2. \end{cases}$$

Then

$$\sigma(f_1,f_2) = \int_M \int_M f_1(x)E(x,y)f_2(y)d\mu(x)d\mu(y)$$

$$= \int_M f_1(x)Ef_2(x)d\mu(x)$$

$$= 0.$$

In this connection, recall that

$$\underline{\text{spt}}\ Ef_2 \subset J(\underline{\text{spt}}\ f_2) \quad (\subset J(O_2)).$$

Therefore the elements of $\mathfrak{A}_g(O_1)$ commute with the elements of $\mathfrak{A}_g(O_2)$.

III_g: Suppose that P is contained in the domain of dependence of O -- then $\mathfrak{A}_g(P)$ is contained in $\mathfrak{A}_g(O)$. To see this, take a $\psi \in C_c^\infty(P;\underset{\sim}{R})$ -- then $\exists\ \phi \in C_c^\infty(O;\underset{\sim}{R}): E\phi = E\psi$ (proof omitted), hence $\phi - \psi \in \underline{\ker}\ E \Rightarrow [\phi] = [\psi] \Rightarrow W([\phi]) = W([\psi]) \Rightarrow$ $\mathfrak{A}_g(P) \subset \mathfrak{A}_g(O)$.

IV_g: Suppose that $\zeta : M \rightarrow M$ is a diffeomorphism -- then ζ defines a symplectic isomorphism

$$(E_g, \sigma_g) \rightarrow (E_{\zeta^* g}, \sigma_{\zeta^* g}),$$

viz. $[f] \rightarrow [f \circ \zeta]$. Here we have put

$$E_g = C_c^{\infty}(M;\underset{\sim}{R})/\underline{\ker}\ E, \quad \sigma_g = \sigma \text{ etc.}$$

So, by Bogolubov, $\exists$ an isomorphism

$$\alpha_\zeta : \mathfrak{A}_g \rightarrow \mathfrak{A}_{\zeta * g}$$

such that

$$\alpha_\zeta(W[f]) = W[f \circ \zeta].$$

It is not difficult to check that $\forall$ O, α_ζ sends $\mathfrak{A}_g(O)$ to $\mathfrak{A}_{\zeta * g}(\zeta^{-1}O)$.

THEOREM Suppose that $\Sigma \subset M$ is a Cauchy surface. Fix $u, u' \in C_c^{\infty}(\Sigma)$ -- then there is a unique $f \in C^{\infty}(M)$ such that $(\Box^2 + m^2)f = 0$ and

$$f \mid \Sigma = u, \quad \frac{\partial f}{\partial n} = u'.$$

[Note: Tacitly, Σ is spacelike, i.e., $g \mid T_\sigma \Sigma \times T_\sigma \Sigma$ is negative definite $\forall \sigma \in \Sigma$. In addition, $\frac{\partial}{\partial n}$ is defined using the future directed unit normal along Σ.]

Quasifree States Let $E \neq 0$ be a real linear space equipped with a nondegenerate alternating bilinear form σ.

Notation: Given a state ω on $\underline{CCR}(E,\sigma)$, put

$$\phi_\omega(f) = \omega(W(f)) \quad (f \in E).$$

LEMMA Suppose given a complex valued function ϕ on E -- then $\phi = \phi_\omega$ for some state ω on $CCR(E,\sigma)$ if $\phi(0) = 1$, $\lambda \to \phi(\lambda f)$ $(\lambda \in \mathbb{R})$ is continuous, and

$$\sum_{i,j=1}^{n} z_i \bar{z}_j \, \underline{\exp}\left(-\frac{\sqrt{-1}}{2}\sigma(f_i,f_j)\right) \phi(f_i - f_j) \geq 0.$$

Example: Let $\langle \, , \rangle$ be a real valued inner product on E with

$$|\sigma(f,g)| \leq \langle f,f\rangle^{1/2} \langle g,g\rangle^{1/2} \quad (f,g \in E).$$

Then the assignment

$$f \to \underline{\exp}\left(-\frac{1}{4}\langle f,f\rangle\right)$$

satisfies the conditions of the lemma so $\exists$ a state ω on $CCR(E,\sigma)$ such that

$$\omega(W(f)) = \underline{\exp}\left(-\frac{1}{4}\langle f,f\rangle\right).$$

[Note: States of this form are said to be quasifree.]

Notation: $IP(E,\sigma)$ is the set of real valued inner products μ on E which dominate σ in the sense that

$$|\sigma(f,g)|^2 \leq \mu(f,f)\,\mu(g,g) \quad (f,g \in E).$$

Accordingly, there is a one-to-one correspondence $\mu \longrightarrow \omega_\mu$ between the elements of IP(E,σ) and the quasifree states on CCR(E,σ).

Given $\mu \in$ IP(E,μ), let $\mathcal{H}_\mu$ be the completion of E w.r.t. the topology induced by μ and denote by σ_μ the μ-continuous extension of σ to $\mathcal{H}_\mu$ -- then there exists a unique bounded linear operator $A_\mu : \mathcal{H}_\mu \longrightarrow \mathcal{H}_\mu$ such that

$$\sigma_\mu(x,y) = \mu(x,A_\mu y) \quad (x,y \in \mathcal{H}_\mu).$$

It is easy to check that

$$A_\mu^* = -A_\mu, \quad \|A_\mu\| \leq 1.$$

[Note: In general, $A_\mu E \not\subset E$.]

Example: Take for E a complex Hilbert space, view E as a real linear space via restriction of scalars, and let

$$\sigma(x,y) = \underline{\text{Im}}\langle x,y\rangle .$$

Then

$$\mu(x,y) = \underline{\text{Re}}\langle x,y\rangle$$

is a real valued inner product on E. Moreover

$$|\langle x,y\rangle| \leq \|x\| \cdot \|y\|$$

$$\Longrightarrow$$

$$|\sigma(x,y)|^2 \leq \mu(x,x)\mu(y,y).$$

In addition,

$$\langle x,y\rangle = \underline{\text{Re}}\langle x,y\rangle + \sqrt{-1}\,\underline{\text{Im}}\langle x,y\rangle$$

$$\langle x, \sqrt{-1}\, y \rangle = \sqrt{-1}\, \underline{\mathrm{Re}} \langle x,y \rangle - \underline{\mathrm{Im}} \langle x,y \rangle$$

$$\Rightarrow$$

$$\underline{\mathrm{Im}} \langle x,y \rangle = - \underline{\mathrm{Re}} \langle x, \sqrt{-1}\, y \rangle = \underline{\mathrm{Re}} \langle x, -\sqrt{-1}\, y \rangle .$$

I.e.:

$$\sigma(x,y) = \mu(x, -\sqrt{-1}\, y).$$

Therefore A_μ is multiplication by $-\sqrt{-1}$.

LEMMA σ_μ is nondegenerate iff A_μ is injective.

[Note: Suppose that σ_μ is nondegenerate -- then the range of A_μ is dense ($\mu(x,A_\mu y) = 0\ \forall\, y \Rightarrow \sigma_\mu(x,y) = 0\ \forall\, y \Rightarrow x = 0$), hence A^{-1} is densely defined (but possibly unbounded).]

Let

$$A_\mu = J_\mu |A_\mu|$$

be the polar decomposition of A_μ. Since $A_\mu^* = -A_\mu$, A_μ is normal, hence J_μ and $|A_\mu|$ commute. In addition,

$$A_\mu^* = |A_\mu| J_\mu^* = - A_\mu = - J_\mu |A_\mu|$$

$$\Rightarrow$$

$$J_\mu |A_\mu| J_\mu^* = - J_\mu^2 |A_\mu| .$$

But $J_\mu |A_\mu| J_\mu^*$ is positive, so the uniqueness of the polar decomposition gives

$$J_\mu^2 = - I.$$

Definition: Let $\mu \in IP(E,\sigma)$ -- then μ is said to be <u>pure</u> if $\forall f \in E$,

$$\mu(f,f) = \sup_{g \in E - \{0\}} \frac{|\sigma(f,g)|^2}{\mu(g,g)} .$$

Example: Consider again the case where E is a complex Hilbert space, so

$$\begin{cases} \sigma(x,y) = \underline{\mathrm{Im}}\langle x,y\rangle \\ \mu(x,y) = \underline{\mathrm{Re}}\langle x,y\rangle . \end{cases}$$

Then $\forall\ x \neq 0$,

$$\sigma(x, \sqrt{-1}\,x) = \mu(x,(-\sqrt{-1})\sqrt{-1}\,x) = \mu(x,x) = \|x\|^2$$

$$\Rightarrow$$

$$\frac{|\sigma(x,\sqrt{-1}\,x)|^2}{\mu(\sqrt{-1}\,x,\sqrt{-1}\,x)} = \frac{\|x\|^4}{\|x\|^2} = \|x\|^2.$$

Therefore μ is pure.

LEMMA μ is pure iff $|A_\mu| = I$.

[Note: It follows that if μ is pure, then σ_μ is nondegenerate.]

Remark: Suppose that ω is quasifree: $\omega = \omega_\mu$ -- then ω_μ is pure iff μ is pure.

[Note: Recall that the GNS representation π_{ω_μ} associated with ω_μ is irreducible iff ω_μ is pure.]

Fact: Given $\mu \in IP(E,\sigma)$, put

$$\mu_p(f,g) = \mu(f,|A_\mu|g).$$

Then $\mu_p \in \mathrm{IP}(E,\sigma)$ is pure.

[Note: μ_p is called the purification of μ.]

Example: Consider again the case where E is a complex Hilbert space, so

$$\begin{cases} \sigma(x,y) = \underline{\mathrm{Im}}\langle x,y\rangle \\ \mu(x,y) = \underline{\mathrm{Re}}\langle x,y\rangle \end{cases} .$$

Fix $\lambda > 1$ -- then $\nu = \lambda\mu \in \mathrm{IP}(E,\sigma)$. But

$$\sigma(x,y) = \mu(x, -\sqrt{-1}\,y) = \lambda\mu(x,\frac{-\sqrt{-1}}{\lambda}y) = \nu(x,\frac{-\sqrt{-1}}{\lambda}y)$$

$$\Rightarrow \quad A_\nu = \frac{-\sqrt{-1}}{\lambda} I \Rightarrow |A_\nu| = \frac{1}{\lambda} I.$$

Therefore

$$\nu_p(x,y) = \nu(x,\frac{1}{\lambda}y) = \mu(x,y).$$

Assuming that μ is pure, put

$$\mathcal{H}_\mu^{\underline{C}} = \mathcal{H}_\mu + \sqrt{-1}\,\mathcal{H}_\mu$$

and extend σ_μ, μ to all of $\mathcal{H}_\mu^{\underline{C}}$ by taking them conjugate linear in the first variable, linear in the second variable. Viewing J_μ as an element of $\mathcal{B}(\mathcal{H}_\mu^{\underline{C}})$, write

$$\mathcal{H}_\mu^{\underline{C}} = \mathcal{H}^+ \oplus \mathcal{H}^-,$$

where

$$\mathcal{H}^{\pm} = \{x : J_\mu x = \pm\sqrt{-1}\,x\}.$$

Let $P^{\pm}$ be the associated orthogonal projection. Define a real linear map $K:E \to \mathcal{H}^+$ by setting

$$K = P^+|E.$$

Call $\langle\ ,\ \rangle_{\mu}^{C}$ the complex inner product on $\mathcal{H}_{\mu}^{C}$ -- then $\forall f,g \in E$, we have

$$\begin{aligned}\langle Kf,Kg\rangle_{\mu}^{C} &= \langle P^+f,P^+g\rangle_{\mu}^{C}\\ &= \mu(P^+f,P^+g)\\ &= \mu(P^+f, \frac{1}{\sqrt{-1}} J_{\mu} P^+g)\\ &= \frac{1}{\sqrt{-1}} \mu(P^+f,J_{\mu} P^+g)\\ &= -\sqrt{-1}\ \sigma_{\mu}(P^+f,P^+g)\\ &= \sigma_{\mu}(\sqrt{-1}\ P^+f,P^+g).\end{aligned}$$

But $\forall x \in \mathcal{H}_{\mu}^{C}$,

$$x = P^+x + P^-x$$

$$\Rightarrow$$

$$\begin{aligned}J_{\mu}x &= J_{\mu}P^+x + J_{\mu}P^-x\\ &= \sqrt{-1}\ P^+x - \sqrt{-1}\ P^-x\end{aligned}$$

$$\Rightarrow$$

$$P^+x = \frac{J_{\mu}x + \sqrt{-1}\ x}{2\sqrt{-1}}.$$

Therefore

$$\sigma_{\mu}(\sqrt{-1}\ P^+f,P^+g)$$

$$= \sigma_{\mu}\left(\sqrt{-1}\,\frac{J_{\mu} f + \sqrt{-1}\, f}{2\sqrt{-1}}, \frac{J_{\mu} g + \sqrt{-1}\, g}{2\sqrt{-1}}\right)$$

$$= \frac{1}{4\sqrt{-1}}\,\sigma_{\mu}(J_{\mu} f + \sqrt{-1}\, f, J_{\mu} g + \sqrt{-1}\, g)$$

$$= \frac{1}{4\sqrt{-1}}(\sigma_{\mu}(J_{\mu} f, J_{\mu} g) + \sqrt{-1}\,\sigma_{\mu}(J_{\mu} f, g) - \sqrt{-1}\,\sigma_{\mu}(f, J_{\mu} g) + \sigma_{\mu}(f,g))$$

$$= \frac{1}{4}(\sigma_{\mu}(J_{\mu} f, g) - \sigma_{\mu}(f, J_{\mu} g)) + \frac{1}{4\sqrt{-1}}(\sigma_{\mu}(J_{\mu} f, J_{\mu} g) + \sigma(f,g)).$$

Since

$$\begin{cases} \sigma_{\mu}(J_{\mu} f, J_{\mu} g) = \mu(J_{\mu} f, J_{\mu} J_{\mu} g) = \mu(f, J_{\mu} g) \\ \sigma_{\mu}(J_{\mu} f, g) = \mu(J_{\mu} f, J_{\mu} g) = \mu(f,g) \\ \sigma_{\mu}(f, J_{\mu} g) = \mu(f, J_{\mu} J_{\mu} g) = -\mu(f,g), \end{cases}$$

it follows that

$$\langle Kf, Kg \rangle_{\mu}^{\mathbb{C}} = \frac{\mu(f,g)}{2} + \frac{\sigma(f,g)}{2\sqrt{-1}}.$$

Accordingly, K is one-to-one ($Kf = 0 \Rightarrow \mu(f,f) = 0 \Rightarrow f = 0$). Finally, KE is dense in $\mathcal{H}^{+}$. Indeed, E is dense in $\mathcal{H}_{\mu}$, thus it need only be shown that $P^{+}\mathcal{H}_{\mu}$ is dense in $\mathcal{H}^{+}$. So fix $y \in \mathcal{H}^{+}$ and suppose that $\langle P^{+}x, y\rangle_{\mu}^{\mathbb{C}} = 0 \ \forall\, x \in \mathcal{H}_{\mu}$ -- then $0 = \langle x, P^{+}y\rangle_{\mu}^{\mathbb{C}} = \langle x, y\rangle_{\mu}^{\mathbb{C}} \Rightarrow$ $y \perp \mathcal{H}_{\mu} \Rightarrow y \perp \mathcal{H}_{\mu}^{\mathbb{C}} \Rightarrow y = 0$.

[Note: One could equally well have used P^{-}. This would give

$$\langle Kf, Kg \rangle_{\mu}^{\mathbb{C}} = \frac{\mu(f,g)}{2} + \sqrt{-1}\,\frac{\sigma(f,g)}{2},$$

which is actually more convenient in the applications.]

Summary: Let $\mu \in IP(E,\sigma)$ be pure -- then $\exists$ a complex Hilbert space $(\mathcal{H}, \langle\,,\,\rangle)$ and a real linear map $K:E \rightarrow \mathcal{H}$ such that

(1) K is one-to-one and has a dense range;

(2) $\forall$ $f,g \in E$,

$$\langle Kf,Kg\rangle = \frac{\mu(f,g)}{2} + \sqrt{-1}\,\frac{\sigma(f,g)}{2}.$$

[Note: It is not necessary to assume that μ is pure, the only change in the statement being that the complexified range $KE + \sqrt{-1}\,KE$ of K is dense in $\mathcal{H}$ (the proof is an elaboration on the preceding theme).]

Remark: If $(K',(\mathcal{H}',\langle\,,\,\rangle'))$ has the same properties, then $\exists$ a unitary $U:\mathcal{H} \rightarrow \mathcal{H}'$ such that $U \circ K = K'$.

[Put $\mathcal{S} = KE \subset \mathcal{H}$, $\mathcal{S}' = K'E \subset \mathcal{H}'$ and define $T: \mathcal{S} \rightarrow \mathcal{S}'$ by the prescription $T(Kf) = K'f$ -- then $\forall$ $x,y \in \mathcal{S}$,

$$\langle Tx,Ty\rangle' = \langle x,y\rangle.$$

Therefore T extends to a unitary $U:\mathcal{H} \rightarrow \mathcal{H}'$ such that $U \circ K = K'$.]

Maintaining the assumption that $\mu \in IP(E,\sigma)$ is pure, consider $\mathcal{F}_s(\mathcal{H})$ -- then, as we know, a CCR realization of $(\mathcal{H}, \underline{\text{Im}}\langle\,,\,\rangle)$ is the C*-subalgebra of $\mathcal{B}(\mathcal{F}_s(\mathcal{H}))$ generated by the $W(x)$ $(x \in \mathcal{H})$, where

$$W(x) = \underline{\exp}(\sqrt{-1}\;\overline{\Phi_S(x)})$$

and

$$\Phi_S(x) = \frac{1}{\sqrt{2}}(\underset{\sim}{a}(x) + \underset{\sim}{c}(x)).$$

LEMMA The assignment

$$W(f) \longrightarrow \underline{\exp}(\sqrt{-1}\,(\underset{\sim}{a}(Kf) + \underset{\sim}{c}(Kf)))$$

defines a representation of $CCR(E,\sigma)$ on $\mathcal{F}_s(\mathcal{H})$.

[In fact,

$$W(f)W(g) = \underline{\exp}\left(-\frac{\sqrt{-1}}{2}\sigma(f,g)\right)W(f+g).$$

On the other hand,

$$\underline{\exp}(\sqrt{-1}(\underset{\sim}{a}(Kf) + \underset{\sim}{c}(Kf)))\ \underline{\exp}(\sqrt{-1}(\underset{\sim}{a}(Kg) + \underset{\sim}{c}(Kg)))$$

$$= \underline{\exp}\left(-\frac{\sqrt{-1}}{2}\,\underline{\mathrm{Im}}\langle\sqrt{2}\,Kf,\sqrt{2}\,Kg\rangle\right)\underline{\exp}(\sqrt{-1}(\underset{\sim}{a}(K(f+g)) + \underset{\sim}{c}(K(f+g))))$$

$$= \underline{\exp}\left(-\frac{\sqrt{-1}}{2}\,\sigma(f,g)\right)\underline{\exp}(\sqrt{-1}\,(\underset{\sim}{a}(K(f+g)) + \underset{\sim}{c}(K(f+g)))).$$

The assertion is therefore manifest.

[Note: It is not difficult to see that this representation is equivalent to the GNS representation π_{ω_μ} associated with the state ω_μ .]

Let $\mu_1,\mu_2 \in IP(E,\sigma)$ be pure and let π_1, π_2 be the representations of $CCR(E,\sigma)$ constructed above.

Problem: Determine conditions under which $\exists$ a unitary $U: \mathcal{F}_s(\mathcal{H}_1) \rightarrow \mathcal{F}_s(\mathcal{H}_2)$ such that $U\pi_1 U^{-1} = \pi_2$.

First, it is easy to see that there is no such U unless μ_1,μ_2 are equivalent, i.e., $\exists$ $C>0$, $D>0$: $\forall$ $f\in E$,

$$C\mu_1(f,f) \leq \mu_2(f,f) \leq D\mu_1(f,f).$$

Therefore $\mathcal{H}_{\mu_1} = \mathcal{H}_{\mu_2}$, label it $\mathcal{H}_{\mu}$. Define a linear map $Q: \mathcal{H}_{\mu} \to \mathcal{H}_{\mu}$ by

$$\mu_1(x,Qy) = \mu_2(x,y) - \mu_1(x,y).$$

Then π_1 and π_2 are unitarily equivalent iff Q is of the trace class.

[Note: To explicate Q, observe that $A_{\mu_1}A_{\mu_2}^{-1}$ extends to a bounded linear operator on $\mathcal{H}_{\mu}$. And:

$$\mu_1(x,A_{\mu_1}A_{\mu_2}^{-1}y) = \mu_2(x,y)$$

$$\Rightarrow$$

$$\mu_1(x,(I - A_{\mu_1}A_{\mu_2}^{-1})y)$$

$$= \mu_1(x,y) - \mu_2(x,y)$$

$$\Rightarrow$$

$$Q = A_{\mu_1}A_{\mu_2}^{-1} - I.]$$

<u>Hadamard States</u> Let (M,g) be a globally hyperbolic space-time. Fix $m>0$ and consider the associated Klein-Gordon algebra $\mathfrak{A}_g$ -- then the issue is to isolate a physically relevant class of states on $\mathfrak{A}_g$.

Since it is a question of states on

$$\mathrm{CCR}(C_c^\infty(M;\underset{\sim}{R})/\underline{\ker}\ E,\sigma),$$

the generalities from the previous section are applicable, thus we shall restrict our attention to those states which are quasifree.

Suppose that ω_μ is a quasifree state on $\mathfrak{A}_g$. Given $f_1, f_2 \in C_c^\infty(M;\underset{\sim}{R})$, put

$$\Lambda_\mu(f_1,f_2) = \frac{\mu([f_1],[f_2])}{2} + \sqrt{-1}\,\frac{\sigma([f_1],[f_2])}{2}.$$

Then Λ_μ is a complex valued separately continuous bilinear form on $C_c^\infty(M;\underset{\sim}{R})$.

Definition: A separately continuous bilinear form

$$\Lambda : C_c^\infty(M;\underset{\sim}{R}) \times C_c^\infty(M;\underset{\sim}{R}) \to \underset{\sim}{C}$$

satisfies the <u>Hadamard condition</u> if

$$\Lambda(f_1,f_2) = \lim_{\varepsilon \downarrow 0} \int_{M\times M} \Lambda_\varepsilon(p,q) f_1(p) f_2(q)\, d\mu(p)\, d\mu(q).$$

[Note: Here the Λ_ε are certain kernels whose exact form we shall not insist upon at present ($\Lambda_\varepsilon = G_\varepsilon + H$, where G_ε is singular and depends on (M,g) while H is smooth and depends on Λ).]

A quasifree state ω_μ on $\mathfrak{A}_g$ is said to be a <u>Hadamard state</u> provided Λ_μ satisfies the Hadamard condition.

It is a fact that pure quasifree Hadamard states exist.

[Note: Such a state is sometimes called a Hadamard vacuum.]

LEMMA The set of quasifree Hadamard states on $\mathfrak{A}_g$ spans an infinite dimensional vector subspace of the topological dual of $\mathfrak{A}_g$.

Rappel: Given a C*-algebra $\mathfrak{A}$, representations π_1 and π_2 of $\mathfrak{A}$ are said to be quasiequivalent if every subrepresentation of π_1 contains a representation which is unitarily equivalent to a subrepresentation of π_2.

[Note: Equivalent conditions are:

(1) $\exists$ an isomorphism $\Phi : \pi_1(\mathfrak{A})'' \rightarrow \pi_2(\mathfrak{A})''$ such that $\Phi(\pi_1(A)) = \pi_2(A) \ \forall A \in \mathfrak{A}$;

(2) $\exists$ a cardinal number n such that $n\pi_1$ is unitarily equivalent to $n\pi_2$.]

Rappel: Given a C*-algebra $\mathfrak{A}$, let π be a representation of $\mathfrak{A}$ on $\mathcal{H}$ -- then the folium of π is the set of states on $\mathfrak{A}$ of the form

$$A \rightarrow \underline{tr}(\pi(A)W),$$

where W is a density operator on $\mathcal{H}$.

[Note: The folium of a faithful representation of $\mathfrak{A}$ is weak* dense in the set of all states on $\mathfrak{A}$.]

Fact: The folium of π determines its quasiequivalence class.

THEOREM Let ω_1, ω_2 be quasifree Hadamard states on $\mathfrak{A}_g$ -- then

$\forall$ bounded open $O \subset M$, $\pi_{\omega_1} | \mathfrak{A}_g(O)$ is quasiequivalent to $\pi_{\omega_2} | \mathfrak{A}_g(O)$.

[Note: As usual, π_{ω_i} $(i=1,2)$ is the GNS representation associated with ω_i $(i=1,2)$.]

Another point is this. Suppose that ω_μ is a quasifree Hadamard state on $\mathfrak{A}_g$ -- then $\forall\, p \in M$,

$$\bigcap_{O \ni p} \pi_\omega(\mathfrak{A}_g(O))'' = \underset{\sim}{C} I.$$

Appendix

The precise formulation of the Hadamard condition is complicated. So, for simplicity, we shall restrict ourselves to the case when $\underline{\dim}\, M = 4$.

Convention: All Cauchy surfaces are assumed to be spacelike.

Definition: Let $\Sigma \subset M$ be a Cauchy surface -- then an open set $N \subset M$ containing Σ is said to be a <u>causal normal neighborhood</u> of Σ if $\forall\, p,q \in N$: $q \in J^+(p) \Rightarrow \exists$ a geodesically convex set $\mathcal{C} \subset M$ such that $J^+(p) \cap J^-(q) \subset \mathcal{C}$.

[Note: It can be shown that every Cauchy surface admits a causal normal neighborhood.]

Notation: Let $\mathcal{C} \subset M$ be geodesically convex -- then for $p,q \in \mathcal{C}$, $\sigma(p,q)$ is the signed square of the geodesic distance from p to q:

$$\sigma(p,q) = \pm\left(\int_a^b |g_{\gamma(t)}(\dot\gamma(t),\dot\gamma(t))|^{1/2} dt\right)^2 \quad (\gamma(a)=p,\ \gamma(b)=q),$$

the plus sign being taken if γ is spacelike and the minus sign being taken if γ is timelike. E.g.: When $M = \underset{\sim}{R}^{1,3}$, $\sigma(p,q) = -(p-q)^2$.

Notation: $J \subset M \times M$ is the set of causally related points (p,q) and $\partial \subset J$ is the set of causally related points (p,q) such that $J^+(p) \cap J^-(q)$ and $J^-(p) \cap J^+(q)$ are contained in a geodesically convex subset of M.

LEMMA There is a neighborhood U_∂ of ∂ on which σ is welldefined and smooth.

[It suffices to let

$$U_\partial = \bigcup_{(p,q) \in \partial} \mathcal{C}_{(p,q)} \times \mathcal{C}_{(p,q)},$$

where $\mathcal{C}_{(p,q)}$ is a geodesically convex subset of M containing $J^+(p) \cap J^-(q)$ or $J^-(p) \cap J^+(q)$ (whichever is not empty) and thus containing p and q.]

Let Σ be a Cauchy surface. Fix a causal normal neighborhood $N \supset \Sigma$.

LEMMA In $N \times N$, there is an open set $O_N \supset J \cap (N \times N)$ such that the closure $\overline{O}_N$ of O_N in $N \times N$ is contained in $U_\partial \cap (N \times N)$.

[Using the definitions, it is easy to check that $J \cap (N \times N) \subset U_\partial$ is closed in $N \times N$. Accordingly, there is an open subset O_N of $N \times N$ for which

$$J \cap (N \times N) \subset O_N \subset \overline{O}_N \subset U_\partial \cap (N \times N).]$$

Fix a C^{∞} function $\chi : N \times N \to [0,1]$ such that

$$\begin{cases} \chi(p,q) = 1 & ((p,q) \in \overline{O}_N) \\ \chi(p,q) = 0 & ((p,q) \notin U_{\delta}). \end{cases}$$

Given a time function $T : M \to \mathbb{R}$, for each $n \in \mathbb{N}$ and $\varepsilon > 0$, define a function $G_{\varepsilon}^{T,n}$ on U_{δ} by

$$G_{\varepsilon}^{T,n}(p\ q) = \frac{1}{(2\pi)^2}\left(\frac{\Delta(p,q)^{1/2}}{\sigma_{\varepsilon}(p,q)} + v^{(n)}(p,q)\ \underline{\log}(\sigma_{\varepsilon}(p,q))\right).$$

Here

$$\sigma_{\varepsilon}(p,q) = \sigma(p,q) + 2\sqrt{-1}\ \varepsilon(T(p) - T(q)) + \varepsilon^2.$$

In addition, Δ is the Van Vleck-Morette determinant and

$$v^{(n)}(p,q) = \sum_{k=0}^{n} v_k(p,q)\ (\sigma(p,q))^k,$$

the v_k being expressible in terms of the Hadamard recursion relations.

Definition: A separately continuous bilinear form

$$\Lambda : C_c^{\infty}(M;\mathbb{R}) \times C_c^{\infty}(M;\mathbb{R}) \to \mathbb{C}$$

satisfies the <u>Hadamard condition</u> if for some choice of Σ, N, χ, and T, there is a sequence of functions $H_n \in C^n(N \times N)$ such that $\forall$ n and $\forall\ f_1, f_2 \in C_c^{\infty}(N;\mathbb{R})$, we have

$$\Lambda(f_1,f_2) = \lim_{\varepsilon \downarrow 0} \int_{N \times N} \Lambda_{\varepsilon}^{T,n}(p,q) f_1(p) f_2(q)\, d\mu(p)\, d\mu(q),$$

where

$$\Lambda_{\varepsilon}^{T,n}(p,q) = \chi(p,q) G_{\varepsilon}^{T,n}(p,q) + H_n(p,q).$$

[Note: χ is zero off of U_δ, hence $\Lambda_\varepsilon^{T,n}$ is defined on all of $N \times N$.]

Remark: If (Σ, N) is fixed and $\begin{cases} T \\ \chi \end{cases}$ is changed to $\begin{cases} T' \\ \chi' \end{cases}$, then the condition is still valid (the changes are compensated for by choosing another sequence $H'_n \in C^n(N \times N)$). On the other hand, if Λ satisfies the condition per one choice of (Σ, N), then it satisfies the condition per any other choice (Σ', N').

<u>The Vacuum</u> Let $\mathcal{H}$ be a Hilbert space -- then, as we know, the C*-subalgebra of $\mathcal{B}(\mathcal{F}_s(\mathcal{H}))$ generated by the W(f) is a CCR realization of $(\mathcal{H}, \underline{\text{Im}}\langle\,,\rangle)$.

Definition: The <u>vacuum</u> is the state ω_0 on $\text{CCR}(\mathcal{H}, \underline{\text{Im}}\langle\,,\rangle)$ characterized by

$$\omega_0(W(f)) = \underline{\exp}\left(-\frac{1}{4}\|f\|^2\right).$$

[Note: Therefore ω_0 is quasifree, and, in addition, pure.]

LEMMA We have

$$\omega_0(W(f)) = \langle\Omega_0, W(f)\Omega_0\rangle .$$

[Note: It is this result which justifies the terminology.]

To prove the lemma, a preliminary will be needed.

<u>Exponential Construction</u>: Given $x \in \mathcal{H}$, put

$$\underline{\exp}\, x = 1 \oplus x \oplus \frac{1}{\sqrt{2!}}\, x \otimes x \oplus \cdots \oplus \frac{1}{\sqrt{n!}}\, x \otimes \cdots \otimes x \oplus \cdots \in \mathcal{F}_s(\mathcal{H}).$$

Then

$$\langle \underline{\exp}\, x, \underline{\exp}\, y\rangle = e^{\langle x,y\rangle}$$

and the set $\underline{\exp}\,\mathcal{H}$ is total in $\mathcal{F}_s(\mathcal{H})$.

Using the definitions, one now finds that

$$W(f)\underline{\exp}\, x = \underline{\exp}\left\{-\frac{1}{4}\|f\|^2 - \frac{1}{\sqrt{2}}\langle f,x\rangle\right\}\ \underline{\exp}\left(\frac{f}{\sqrt{2}} + x\right).$$

Since $\Omega_0 = \underline{\exp}\, 0$, it follows that

$$\langle\Omega_0, W(f)\Omega_0\rangle$$

$$= \langle \underline{\exp}\ 0,\ \underline{\exp}(-\tfrac{1}{4}\|f\|^2)\ \underline{\exp}(\tfrac{f}{\sqrt{2}})\rangle$$

$$= \underline{\exp}(-\tfrac{1}{4}\|f\|^2)\langle \underline{\exp}\ 0\ ,\ \exp(\tfrac{f}{\sqrt{2}})\rangle$$

$$= \underline{\exp}(-\tfrac{1}{4}\|f\|^2) = \omega_0(W(f)).$$

Example: Take for M the Minkowski space-time $\underset{\sim}{R}^{1,3}$ and fix $m > 0$ -- then the associated Klein-Gordon algebra $\mathfrak{A}$ sits in

$$CCR(L^2(X_m, \mu_m),\ \underline{Im}\langle\ ,\ \rangle),$$

hence ω_0 determines by restriction a state on $\mathfrak{A}$ which will also be referred to as the vacuum. Explicitly: $\forall\ f \in C_c^\infty(\underset{\sim}{R}^{1,3};\underset{\sim}{R})$,

$$\omega_0(W(f)) = \underline{\exp}\left\{-\tfrac{1}{4}\|\hat{f}\,|\,X_m\|^2\right\}\ .$$

[Note: We shall see later that ω_0 is a Hadamard state.]

<u>The Wave Front Set</u> Let T be a distribution on $\underset{\sim}{R}^n$ -- then a point $(x,\xi) \in \underset{\sim}{R}^n \times \dot{\underset{\sim}{R}}^n$ is called a <u>regular directed point</u> for T if $\exists\, \phi \in C_c^\infty(\underset{\sim}{R}^n)$ with $\phi(x) \neq 0$ such that $\forall N\ \exists\, C_N > 0$:

$$|\widehat{\phi T}(\xi')| \leq C_N(1 + |\xi'|)^{-N}$$

for all ξ' in a conic neighborhood $\Gamma \subset \dot{\underset{\sim}{R}}^n$ of ξ.

[Note: As usual, $\dot{\underset{\sim}{R}}^n = \underset{\sim}{R}^n - \{0\}$. To say that Γ is conic means: $\xi' \in \Gamma \Rightarrow t\xi' \in \Gamma\ \forall t > 0$.]

The <u>wave front set</u> of T, denoted WF(T), is the complement in $\underset{\sim}{R}^n \times \dot{\underset{\sim}{R}}^n$ of the set of regular directed points of T.

[Note: Accordingly, WF(T) consists of the pairs (x,ξ) such that the Fourier transform of ϕT is not rapidly decreasing in the direction ξ no matter how closely ϕ is concentrated at x (bear in mind, however, that since ϕT is compactly supported, its Fourier transform $\widehat{\phi T}$ is necessarily a slowly increasing function). One interprets x as a "singularity" of T and ξ as a "direction of propagation" of this singularity.]

Rappel: Let T be a compactly supported distribution on $\underset{\sim}{R}^n$ -- then T is a C^∞ function iff its Fourier transform $\hat{T}$ is a rapidly decreasing function, i.e., $\forall\, N\ \exists C_N > 0$:

$$|\hat{T}(\xi)| \leq C_N(1 + |\xi|)^{-N}$$

for all $\xi \in \dot{\underset{\sim}{R}}^n$.

Let T be a distribution on $\underset{\sim}{R}^n$ -- then a point $x \in \underset{\sim}{R}^n$ is called a <u>regular point</u> for T if $\exists$ a neighborhood U of x and a function $F \in C^\infty(U)$

such that

$$T(f) = \int f(x)F(x)dx \quad \forall f: \underline{\text{spt}}\ f \subset U.$$

The <u>singular support</u> of T, denoted by <u>sing spt</u> T, is the complement in $\mathbb{R}^n$ of the set of regular points of T.

Remark: The singular support of T is the complement of the largest open set on which T is C^∞ or still, is the projection of WF(T) onto the first variable. Therefore the wave front set of a C^∞ function is empty.

Example: We have

$$WF(\delta) = \{(0,\xi): \xi \neq 0\}.$$

[In fact

$$\widehat{\phi\delta}(\xi) = (2\pi)^{-n/2}\ \delta(\phi\, e^{\sqrt{-1}\langle x,\xi\rangle})$$

$$\doteq \phi(0),$$

which is not rapidly decreasing in any direction.]

Example: Consider $\delta(x-y)$, the distribution on $\mathbb{R}^n \times \mathbb{R}^n$ which sends $f(x,y)$ to $\int f(u,u)du$ -- then

$$\widehat{\phi\delta(x-y)}\ (\xi,\eta) = (2\pi)^{-n}\ \delta(x-y)\ (\phi\, e^{\sqrt{-1}\langle (x,y),(\xi,\eta)\rangle})$$

$$= (2\pi)^{-n} \int \phi(u,u)\ e^{\sqrt{-1}\langle u,\xi+\eta\rangle}\ du$$

$$\doteq \Phi(\xi+\eta) \qquad (\Phi(u) = \phi(u,u)).$$

Therefore the singular support of $\delta(x-y)$ is the diagonal x=y and the directions in the wave front set of $\delta(x-y)$ are subject to the restriction $\xi + \eta = 0$.

Properties of WF:

(1) WF(T) is a closed subset of $\mathbb{R}^n \times \dot{\mathbb{R}}^n$;

(2) $\forall$ differential operator D, $WF(DT) \subset WF(T)$;

(3) $\forall$ compactly supported C^∞ function f, $WF(fT) \subset WF(T)$;

(4) $WF(T + S)$ is contained in $WF(T) + WF(S)$,

Let $\zeta : \mathbb{R}^n \to \mathbb{R}^n$ be a diffeomorphism -- then by definition

$$\langle \zeta T, f \rangle = \int f(\zeta(x)) \, | J_\zeta(x) | \, dT(x),$$

where $J_\zeta(x) = \underline{\det}\, d\zeta_x$, the Jacobian of ζ at x. Define $\zeta_*: \mathbb{R}^n \times \dot{\mathbb{R}}^n \to \mathbb{R}^n \times \dot{\mathbb{R}}^n$ by

$$\zeta_*(x, \xi) = (\zeta(x), d\zeta_x^*(\xi)).$$

Fact: We have

$$WF(\zeta T) = \zeta_* WF(T).$$

It is clear that all of the preceding discussion can be carried over to distributions on open subsets of $\mathbb{R}^n$. This, in conjunction with the last fact, then allows the theory to be written for distributions T on a C^∞ manifold M, thus now

$$WF(T) \subset \dot{T}^*(M),$$

where $\dot{T}^*M$ is the cotangent bundle of M with the zero section removed.

<u>The Theorem of Radzikowski</u> Suppose that (M,g) is a globally hyperbolic space-time with <u>dim</u> M = 4. Let

$$\mathcal{R} = \{(x_1, \xi_1; x_2, -\xi_2) \in \dot{T}^*(M \times M) : (x_1, \xi_1) \sim (x_2, \xi_2)\} \quad ,$$

where

$$(x_1, \xi_1) \sim (x_2, \xi_2)$$

is defined as follows.

(i) When $x_1 \neq x_2$, there is a future directed lightlike geodesic γ:

$$\begin{cases} \gamma(t_1) = x_1 \\ \gamma(t_2) = x_2 \end{cases} \quad \& \quad \begin{cases} \dot{\gamma}(t_1) \underset{g}{\longleftrightarrow} \xi_1 \\ \dot{\gamma}(t_2) \underset{g}{\longleftrightarrow} \xi_2 . \end{cases}$$

(ii) When $x_1 = x_2$, ξ_1 and ξ_2 are lightlike, equal, and in $\overline{V}_+ (x_1 = x_2)$.

Now fix $m > 0$ and consider the associated Klein-Gordon algebra $\mathfrak{A}_g$. Let ω_μ be a quasifree state on $\mathfrak{A}_g$ -- then the theorem in question says that Λ_μ satisfies the Hadamard condition iff $WF(\Lambda_\mu) = \mathcal{R}$.

Example: Take for M the Minkowski space-time $\underset{\sim}{R}^{1,3}$. Consider the vacuum ω_0 -- then $WF(\Lambda_0)$ equals

$$\{(x, \xi; y, -\xi) \in \dot{T}^*(M \times M) :$$
$$x \neq y, (x-y)^2 = 0, \ \xi \parallel (x-y), \ \xi_0 > 0\}$$
$$\cup$$
$$\{(x, \xi; x, -\xi) \in \dot{T}^*(M \times M) : \ \xi^2 = 0, \ \xi_0 > 0\} .$$

Therefore $WF(\Lambda_0)$ has the required form, thus ω_0 is a Hadamard state.