Topological quantum field theories & homotopy cobordisms

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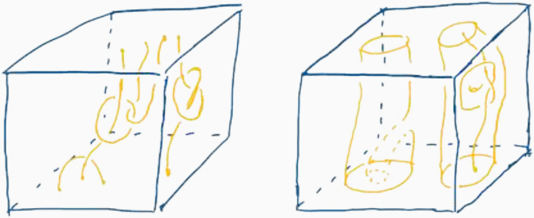

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- Such functors may factor through other categories that may be easier to work with - I will give a construction of a category of *cofibrant cospans* of topological spaces. Functors into this category are obtained roughly by taking the complement of particle trajectories.
- I will also show that Yetter's TQFTs associated to finite groups generalise to explicitly calculable functors from this category.

Talk Plan

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3. Family of functors $Z_G\colon \mathrm{HomCob} \to \mathsf{Vect}_{\mathbb{C}}$

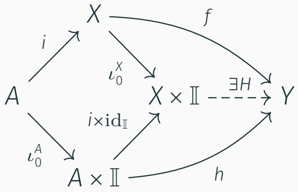
Cofibrant cospans and homotopy cobordisms

Definition
Let X, Y and M be spaces. A <u>cofibrant cospan</u> from X to Y is a diagram $i : X \to M \leftarrow Y : j$ such that $\langle i, j \rangle : X \sqcup Y \to M$ is a closed cofibration.
For spaces $X, Y \in \mathbf{Top}$, we define the set of all cofibrant cospans

$$\mathrm{CofCos}(X, Y) = \left\{ \begin{matrix} X & & & & Y \\ & {}_i\searrow & & \swarrow_j & \\ & & M & & \end{matrix} \;\middle|\; \langle i, j \rangle \text{ is a closed cofibration} \right\}.$$

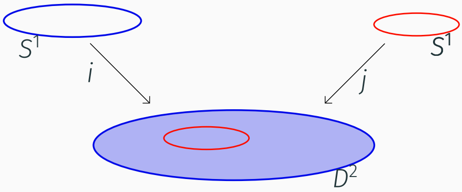
Definition

Let A and X be spaces. A map $i\colon A \to X$ has the homotopy extension property, with respect to the space Y, if for any pair of a homotopy $h\colon A \times \mathbb{I} \to Y$ and a map $f\colon X \to Y$ satisfying $(f \circ i)(a) = h(a, 0)$, there exists a homotopy $H : X \times \mathbb{I} \to Y$, extending h, with $H(x, 0) = f(x)$ and $H(i(a), t) = h(a, t)$. This is illustrated by the following diagram.



(Where for any space X, $\iota_0^X\colon X \to X \times \mathbb{I}$ is the map $x \mapsto (x, 0)$.)
We say that $i\colon A \to X$ is a cofibration if i satisfies the homotopy extension property for all spaces Y.

Cofibrant cospans



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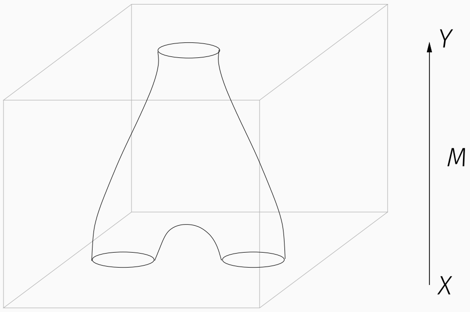
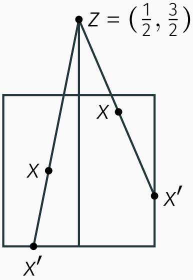


Example
Let X be a space. The cospan $\mathrm{id}_X \colon X \to X \leftarrow X \colon \mathrm{id}_X$ is not a cofibrant cospan, unless $X = \varnothing$.

Cofibrant cospans

Proposition
For X a topological space, the cospan $\iota_0^X\colon X \to X \times \mathbb{I} \leftarrow X : \iota_1^X$ is a cofibrant cospan (where $\iota_a^X\colon X \to X \times \mathbb{I}$ is the map $x \mapsto (x, a)$).

Proof sketch
Suppose there exists a homotopy $h\colon (X \sqcup X) \times \mathbb{I} \to K$, and a map $f\colon X \times \mathbb{I} \to K$, such that $h((x,0),0) = f(x,0)$ and $h((x,1),0) = f(x,1)$. Composition with below retraction gives homotopy $H\colon (X \times \mathbb{I}) \times \mathbb{I} \to K$.



Proposition

A concrete cobordism canonically defines a cofibrant cospan.

Precisely, let X, Y and M be smooth oriented manifolds, and let M be a concrete cobordism from X to Y. Hence there exists a diffeomorphism $\phi\colon \bar{X} \sqcup Y \to \partial M$. Define maps $i(x) = \phi(x, 0)$ and $j(y) = \phi(y, 1)$. Then, using X, Y and M to denote the underlying topological spaces, $i\colon X \to M \leftarrow Y : j$ is a cofibrant cospan.

Example

Any CW complex together with a pair of disjoint subcomplexes and inclusions gives a cofibrant cospan.

Composition of cofibrant cospans

Lemma

(I) For any spaces X, Y and Z in $Ob(\mathbf{Top})$ there is a composition of cofibrant cospans

$$\bullet : \mathrm{CofCos}(X,Y) \times \mathrm{CofCos}(Y,Z) \to \mathrm{CofCos}(X,Z)$$

$$\left(X \xrightarrow{i} M \xleftarrow{j} Y \;,\; Y \xrightarrow{k} N \xleftarrow{l} Z \right) \mapsto X \xrightarrow{\tilde{i}} M \sqcup_Y N \xleftarrow{\tilde{l}} Z$$

where $\tilde{i} = p_M \circ i$ and $\tilde{l} = p_N \circ l$ are obtained via the following diagram

$$\begin{array}{ccccccc} X & & & Y & & & Z \\ & \searrow^{i} & \swarrow_{j} & & \searrow^{k} & \swarrow_{l} & \\ & M & & & & N & \\ & & \searrow^{p_M} & & \swarrow_{p_N} & & \\ & & & M \sqcup_Y N, & & & \end{array}$$

the middle square of which is the pushout of $j : M \leftarrow Y \to N : k$ in $\mathbf{Top}$.

(II) Hence there is a magmoid $\mathrm{CofCos} = (Ob(\mathbf{Top}), \mathrm{CofCos}(-,-), \bullet)$.

Equivalence classes cofibrant cospans

Lemma

For each pair $X, Y \in Ob(\text{CofCos})$, we define a relation on $\text{CofCos}(X, Y)$ by

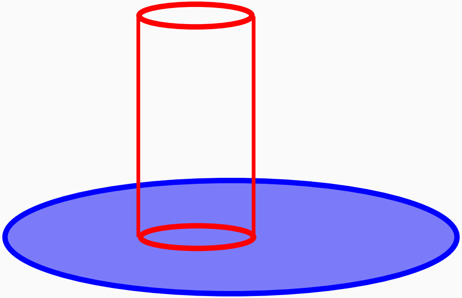
$$\begin{pmatrix} X & & & & Y \\ & {}_i\searrow & & \swarrow_j & \\ & & M & & \end{pmatrix} \overset{ch}{\sim} \begin{pmatrix} X & & & & Y \\ & {}_{i'}\searrow & & \swarrow_{j'} & \\ & & N & & \end{pmatrix}$$

if there exists a commuting diagram

$$\begin{array}{ccccc} & & M & & \\ & \overset{i}{\nearrow} & & \overset{j}{\nwarrow} & \\ X & & \downarrow \psi & & Y \\ & \underset{i'}{\searrow} & & \underset{j'}{\swarrow} & \\ & & M' & & \end{array}$$

where ψ is a homotopy equivalence. For each pair $X, Y \in$ **Top** the relations $(\text{CofCos}(X, Y), \overset{ch}{\sim})$ are a congruence on CofCos.

Equivalence classes of cofibrant cospans



Equivalence classes of cofibrant cospans

Proof uses classical theorem (E.g. Brown06, Thm7.2.8):

If $X \xrightarrow{i} M \xleftarrow{j} Y$, $X \xrightarrow{i'} N \xleftarrow{j'} Y$ are cospans such that $\langle i, j \rangle : X \sqcup Y \to M$ and $\langle i', j' \rangle : X \sqcup Y \to N$ are cofibrations, then the set of homotopy equivalences ψ such that

$$\begin{array}{ccccc} & & M & & \\ & \overset{i}{\nearrow} & & \overset{j}{\nwarrow} & \\ X & & \downarrow{\scriptstyle \psi} & & Y \\ & \underset{i'}{\searrow} & & \underset{j'}{\swarrow} & \\ & & M' & & \end{array}$$

commutes, is in bijective correspondence with the set of ψ' such that there exists $\phi : N \to M$ with $\psi' \circ \phi$ and $\phi \circ \psi'$ homotopic to identity through maps commuting with cospans.

Theorem (T.)
The quadruple

$$\mathrm{CofCos} = \left(Ob(\mathtt{Top})\ ,\ \mathrm{CofCos}(X,Y)/\overset{ch}{\sim}\ ,\ \bullet\ ,\ \left[\begin{array}{ccc} X & & X \\ & {\scriptstyle \iota_0^X} \searrow \quad \swarrow {\scriptstyle \iota_1^X} & \\ & X \times \mathbb{I} & \end{array} \right]_{\mathrm{ch}} \right)$$

is a category.

There is a functor $\Phi\colon \mathbf{Top}^h \to \mathrm{CofCos}$ which sends a homeomorphism $f\colon X \to Y$ to the cospan
$$X \xrightarrow{\iota_0^Y \circ f} Y \times \mathbb{I} \xleftarrow{\iota_1^Y} Y.$$

Theorem (T.)
There is a symmetric monoidal category $(\mathrm{CofCos}, \otimes, \varnothing, \alpha_{X,Y,Z}, \lambda_X, \rho_X, \beta_{X,Y})$ where

$$\left[W \xrightarrow{i} M \xleftarrow{j} X \right]_{\mathrm{ch}} \otimes \left[Y \xrightarrow{k} N \xleftarrow{l} Z \right]_{\mathrm{ch}} = \left[W \sqcup Y \xrightarrow{i \sqcup k} M \sqcup N \xleftarrow{j \sqcup l} X \sqcup Z \right]_{\mathrm{ch}}.$$

All other maps are the images of the corresponding maps in $(\mathbf{Top}, \sqcup)$.

Definition
A space X is called *homotopically* 1-*finitely generated* if $\pi(X,A)$ is finitely generated for all finite sets of basepoints A.
Let χ denote the class of all homotopically 1-finitely generated spaces.

Theorem (T.)
There is a (symmetric monoidal) subcategory of CofCos

$$\mathrm{HomCob} = \left(\chi, \mathrm{HomCob}(X,Y), \bullet, \begin{bmatrix} X & & & & X \\ & \overset{\iota_0^X}{\searrow} & & \overset{\iota_1^X}{\swarrow} & \\ & & X \times \mathbb{I} & & \end{bmatrix}_{\mathrm{ch}} \right).$$

Motion groupoids

Definition
Fix a manifold, submanifold pair $\underline{M} = (M, A)$. A flow in $\underline{M}$ is a map $f \in \mathsf{Top}(\mathbb{I}, \mathsf{TOP}^h_A(M,M))$ with $f_0 = \mathrm{id}_M$. Define,

$$\mathrm{Flow}_{\underline{M}} = \{f \in \mathsf{Top}(\mathbb{I}, \mathsf{TOP}^h_A(M,M)) \mid f_0 = \mathrm{id}_M\}.$$

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Example
For any manifold M the path $f_t = \mathrm{id}_M$ for all t, is a flow. We will denote this flow Id_M.

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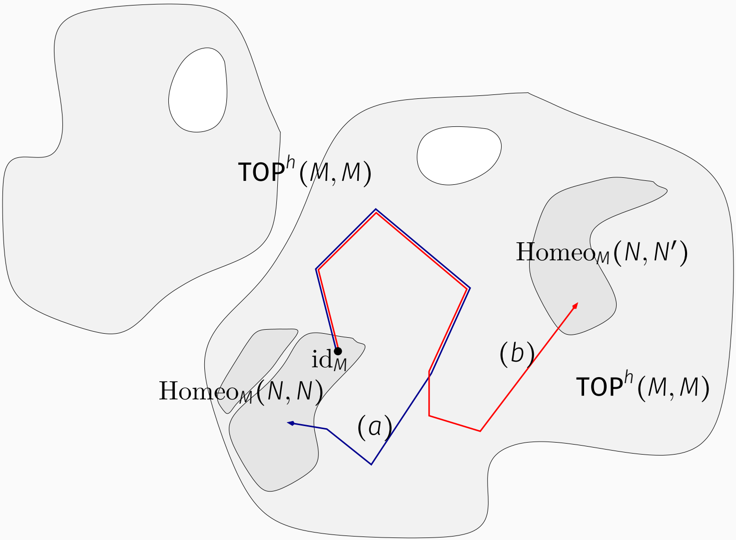
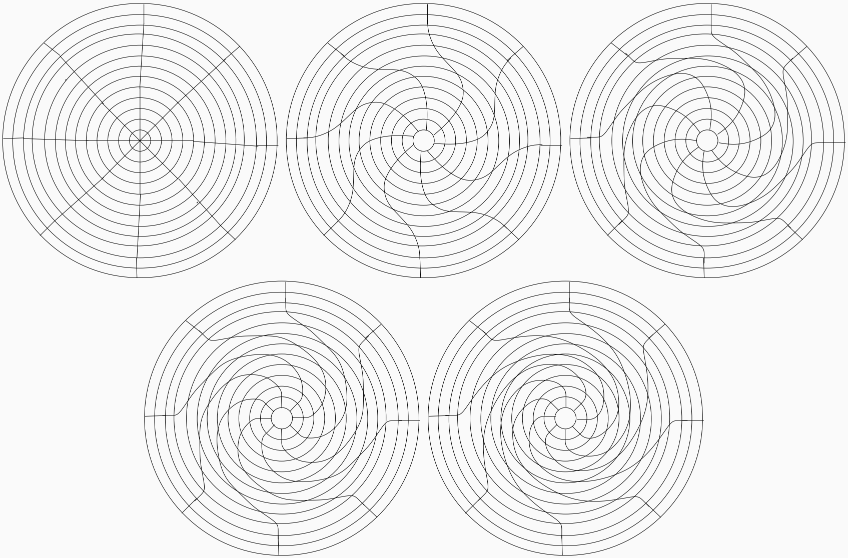
Example
For $M = S^1$ (the unit circle) we may parameterise by $\theta \in \mathbb{R}/2\pi$ in the usual way. Consider the functions $\tau_\phi : S^1 \to S^1$ ($\phi \in \mathbb{R}$) given by $\theta \mapsto \theta + \phi$, and note that these are homeomorphisms. Then consider the path $f_t = \tau_{t\pi}$ (‘half-twist’). This is a flow.

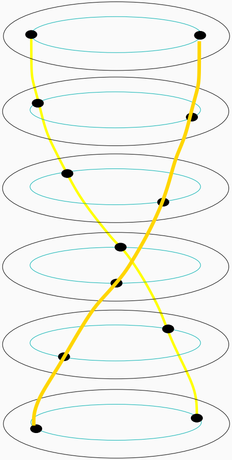
Definition
Fix a $\underline{M} = (M, A)$. A motion in $\underline{M}$ is a triple $f\colon N \rightarrowtail N'$ consisting of a flow $f \in \mathrm{Flow}_{\underline{M}}$, a subset $N \subseteq M$ and the image of N at the endpoint of f, $f_1(N) = N'$.


TOP$^h(M,M)$
Homeo$_M(N,N')$
id$_M$
Homeo$_M(N,N)$
(a)
(b)
TOP$^h(M,M)$

Example $M = D^2$



Motion groupoids



Theorem (.T, Faria Martins, Martin)
Let $\underline{M} = (M, A)$ where M is a manifold and $A \subset M$ a subset. There is a groupoid

$$\mathrm{Mot}_{\underline{M}} = (\mathcal{P}M,\ \mathrm{Mt}_{\underline{M}}(N, N')/\overset{m}{\sim}, *, [\mathrm{Id}_M]_{\mathrm{m}},\ [f]_{\mathrm{m}} \mapsto [\bar{f}]_{\mathrm{m}}).$$

where

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where

(I) objects are subsets of M;

(II) composition of representative morphisms is given by

$$g\colon N' \rightarrowtail N'' * f\colon N \rightarrowtail N' = g * f\colon N \rightarrowtail N''.$$

where

$$(g * f)_t = \begin{cases} f_{2t} & 0 \le t \le 1/2, \\ g_{2(t-1/2)} \circ f_1 & 1/2 \le t \le 1; \end{cases} \tag{1}$$

(III) the inverse for each morphism $[f\colon N \rightarrowtail N']_{\mathrm{m}}$ is the motion-equivalence class of $\bar{f}\colon N' \rightarrowtail N$ where $\bar{f}_t = f_{(1-t)} \circ f_1^{-1}$.

(III) the inverse for each morphism $[f\colon N \rightarrowtail N']_{\mathrm{m}}$ is the motion-equivalence class of $\bar{f}\colon N' \rightarrowtail N$ where $\bar{f}_t = f_{(1-t)} \circ f_1^{-1}$.

(IV) morphisms between subsets N, N' are motion-equivalence classes $[f\colon N \rightarrowtail N']_{\mathrm{m}}$ of motions; explicitly

$$f\colon N \rightarrowtail N' \overset{m}{\sim} g\colon N \rightarrowtail N' \text{ if } \bar{g} * f \overset{p}{\sim} h;$$

where $h_t(N) = N$ for all t;

(III) the inverse for each morphism $[f\colon N \rightarrowtail N']_m$ is the motion-equivalence class of $\bar{f}\colon N' \rightarrowtail N$ where $\bar{f}_t = f_{(1-t)} \circ f_1^{-1}$.

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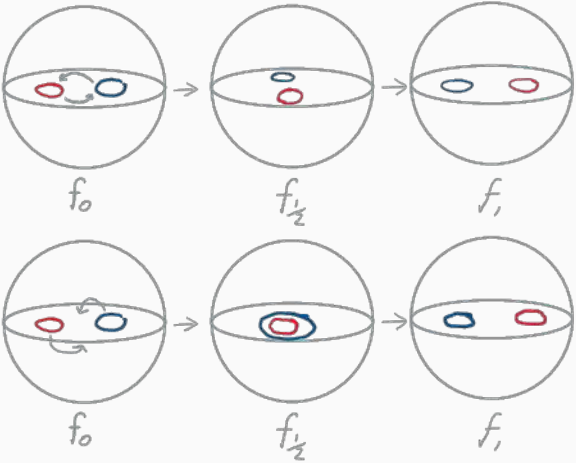
(V) the identity at each object N is the motion-equivalence class of $\mathrm{Id}_M\colon N \rightarrowtail N$, $(\mathrm{Id}_M)_t(m) = m$ for all $m \in M$.

- The motion subgroupoid of a configuration of n points in the disk is isomorphic to the n strand Artin braid group.

- The motion subgroupoid of a configuration of n unknotted unlinked loops in the 3-ball is isomorphic to the loop braid group with n loops.



Definition
The <u>worldline</u> of a motion $f\colon N \rightarrowtail N'$ in a manifold M is

$$\mathsf{W}\left(f\colon N \rightarrowtail N'\right) = \bigcup_{t\in[0,1]} f_t(N) \times \{t\} \subseteq M \times \mathbb{I}.$$

Let $\mathsf{W}'(f\colon N \rightarrowtail N') = (M \times \mathbb{I}) \smallsetminus \left(\mathsf{W}\left(f\colon N \rightarrowtail N'\right)\right).$

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The worldline of a motion $f\colon N \rightsquigarrow N'$ in a manifold M is

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Let $\mathsf{W}'(f\colon N \rightsquigarrow N') = (M \times \mathbb{I}) \smallsetminus (\mathsf{W}\,(f\colon N \rightsquigarrow N'))$.

Homotopy finite version of $\mathrm{Mot}_{\underline{M}}$
Let M be a homotopy finite space. Let $\mathrm{hfMot}_{\underline{M}}$ be the full subgroupoid of $\mathrm{Mot}_{\underline{M}}$ such that the complement of each object is a homotopy finite space.

Theorem (T.)
Let M be a manifold. There is a well-defined functor

$$\mathcal{MOT}_M^A\colon \mathrm{hfMot}_{\underline{M}} \to \mathrm{HomCob}$$

which sends an object $N \in Ob(\mathrm{hfMot}_{\underline{M}})$ to $M \setminus N$, and which sends a morphism $[f\colon N \rightsquigarrow N']_m$ to the cospan homotopy equivalence class of

$$M \setminus N \xrightarrow{\iota_{f_0}} \mathsf{W}'(f\colon N \rightsquigarrow N') \xleftarrow{\iota_{f_1}} M \setminus N'$$

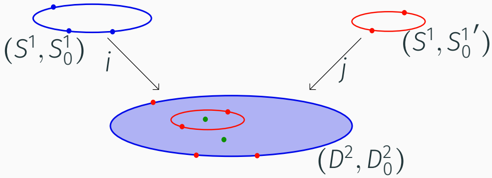
where $\iota_{f_t}\colon M \setminus f_t(N) \to \mathsf{W}'(f\colon N \rightsquigarrow N')$, $m \mapsto (m, t)$.

$Z_G\colon \mathrm{HomCob} \to \mathsf{Vect}_{\mathbb{C}}$

Definition
Let $\boldsymbol{\chi}$ be the set of pairs (X, X_0) such that X is in χ and X_0 is a finite representative subset.
Let (X, X_0), (Y, Y_0) and (M, M_0) be in $\boldsymbol{\chi}$.

Definition
Let $\boldsymbol{\chi}$ be the set of pairs (X, X_0) such that X is in χ and X_0 is a finite representative subset.
Let (X, X_0), (Y, Y_0) and (M, M_0) be in $\boldsymbol{\chi}$. A *based homotopy cobordism* from (X, X_0) to (Y, Y_0) is a diagram $i\colon (X, X_0) \to (M, M_0) \leftarrow (Y, Y_0) :j$ such that:

1. $i\colon X \to M \to Y\colon j$ is a homotopy cobordism.
2. i and j are maps of pairs.
3. $M_0 \cap i(X) = i(X_0)$ and $M_0 \cap j(Y) = j(Y_0)$.


(S^1, S^1_0)
i
j
$(S^1, {S^1_0}')$
(D^2, D^2_0)

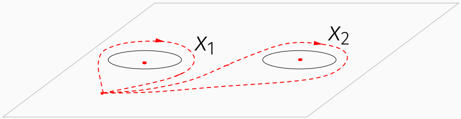
Let G be a group.

For a pair $(X, X_0) \in \boldsymbol{\chi}$, define

$$Z_G^!(X, X_0) = \mathbb{C}\left(\mathsf{Grpd}\left(\pi(X, X_0), G\right)\right).$$

Example

$\pi(X, X_0) \cong (\mathbb{Z} * \mathbb{Z}) \sqcup \{*\} \sqcup \{*\}$. Maps from $\pi(X, X_0)$ to G are determined by pairs in $G \times G$, whose elements respectively denote the images of the equivalence classes of the loops marked x_1 and x_2 in the figure, so we have $Z^!_G(X, X_0) \cong \mathbb{C}(G \times G)$.



Let $i\colon (X, X_0) \to (M, M_0) \leftarrow (Y, Y_0) : j$ be a based homotopy cobordism, we define a matrix

$$Z_G^{!}\left(\begin{array}{ccc}(X,X_0) & & (Y,Y_0)\\ & {}_i\searrow \quad \swarrow_j & \\ & (M,M_0) & \end{array}\right) : Z_G^{!}(X, X_0) \to Z_G^{!}(Y, Y_0)$$

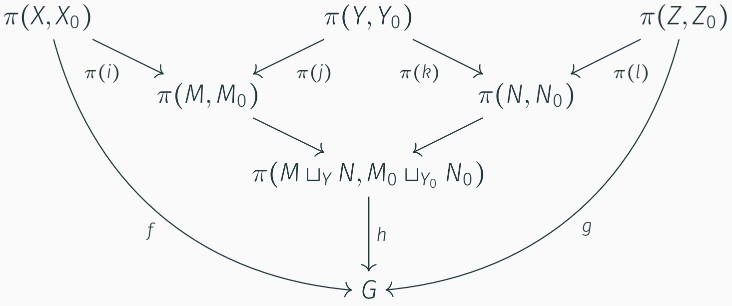
as follows. Let $f \in Z_G^{!}(X, X_0)$ and $g \in Z_G^{!}(Y, Y_0)$ be basis elements, then

$$\left\langle g \,\middle|\, Z_G^{!}\left(\begin{array}{ccc}(X,X_0) & & (Y,Y_0)\\ & {}_i\searrow \quad \swarrow_j & \\ & (M,M_0) & \end{array}\right) \middle|\, f \right\rangle = \left| \left\{ h : \pi(M, M_0) \to G \;\middle|\; \begin{array}{ccc} \pi(X,X_0) & & \pi(Y,Y_0) \\ & \overset{\pi(i)}{\searrow} \quad \overset{\pi(j)}{\swarrow} & \\ & \pi(M,M_0) & \\ {}_f\searrow & \downarrow h & \swarrow_g \\ & G & \end{array} \right\} \right|$$

$Z_G\colon \mathrm{HomCob} \to \mathbf{Vect}_{\mathbb{C}}$

Lemma

The map $Z_G^!$ preserves composition, extended in the obvious way to a composition of based cospans.



Proof

Thm.9.1.2, Topology and Groupoids, Brown gives that middle square is a push out.

Lemma
Let X be a topological space, G a group, $X_0 \subseteq X$ a finite representative subset and $y \in X$ a point with with $y \notin X_0$. There is a non-canonical bijection of sets

$$\Theta_\gamma\colon \mathbf{Grpd}(\pi(X, X_0), G) \times G \to \mathbf{Grpd}(\pi(X, X_0 \cup \{y\}), G)$$
$$(f, g) \mapsto F$$

where γ is a choice of a path from some $x \in X_0$ to y and F is the extension along γ and g.

Consider a concrete homotopy cobordism, $i\colon (X, X_0) \to (M, M_0) \leftarrow (Y, Y_0) :j$. It follows

$$Z^!_G(M, M_0 \cup \{m\}) = |G|\, Z^!_G(M, M_0).$$

It follows that for all M_0' and M_0, we can write

$$Z^!_G(M, M_0' \cup M_0) = |G|^{(|M_0' \cup M_0| - |M_0|)} Z^!_G(M, M_0)$$

and

$$Z^!_G(M, M_0' \cup M_0) = |G|^{(|M_0' \cup M_0| - |M_0'|)} Z^!_G(M, M_0')$$

which together imply

$$|G|^{-|M_0|} Z^!_G(M, M_0) = |G|^{-|M_0'|} Z^!_G(M, M_0')$$

and that

$$|G|^{-(|M_0| - |X_0|)} Z^!_G(M, M_0) = |G|^{-(|M_0'| - |X_0|)} Z^!_G(M, M_0').$$

Lemma

We redefine the linear map we assign to a concrete based homotopy cobordisms as

$$Z_G^{!!}\left((X,X_0) \xrightarrow{i} (M,M_0) \xleftarrow{j} (Y,Y_0) \right) = |G|^{-(|M_0|-|X_0|)} Z_G^{!}\left((X,X_0) \xrightarrow{i} (M,M_0) \xleftarrow{j} (Y,Y_0) \right).$$

The map $Z_G^{!!}$ does not depend on the choice of subset $M_0 \subseteq M$, and this preserves composition. When the relevant cospan is clear, we will refer to this as $Z_G^{!!}(M, X_0, Y_0)$ to highlight the dependence on X_0 and Y_0.

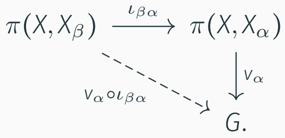
Z_G: HomCob → $\mathbf{Vect}_{\mathbb{C}}$

Lemma

There is a contravariant functor

$$\mathcal{V}_X : \mathbf{FinSet}^*(X) \to \mathbf{Set}$$

constructed as follows. Let $X_\alpha, X_\beta \in Ob(\mathbf{FinSet}^*(X))$ with $X_\beta \subseteq X_\alpha$. Let $\mathcal{V}_X(X_\alpha) = \mathbf{Grpd}(\pi(X, X_\alpha), G)$. For any $v_\alpha \in \mathcal{V}_X(X_\alpha)$ we have a commuting triangle



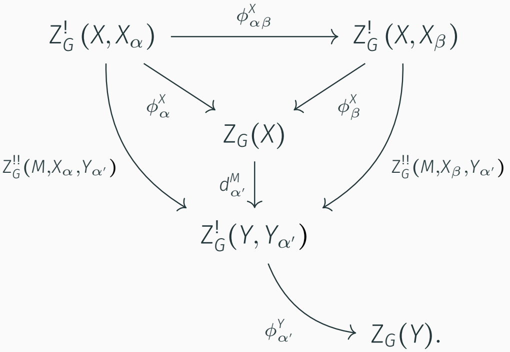
Now let $\mathcal{V}_X(\iota_{\beta\alpha}: X_\beta \to X_\alpha) = \phi_{\alpha\beta}$ where $\phi_{\alpha\beta} : \mathcal{V}_X(X_\alpha) \to \mathcal{V}_X(X_\beta)$, $v_\alpha \mapsto v_\alpha \circ \iota_{\alpha\beta}$.

Definition
For $X \in \chi$ define

$$Z_G(X) = \mathrm{colim}(\mathcal{V}'_X) = \mathbb{C}(\mathrm{colim}(\mathcal{V}_X))$$

where $\mathcal{V}'_X = F_{V_{\mathbb{C}}} \circ \mathcal{V}_X$ and $\mathcal{V}_X\colon \mathbf{FinSet}^*(X) \to \mathbf{Set}$.

Let $i\colon X \to M \leftarrow Y : j$ be a concrete homotopy cobordism. Fix a choice of $Y_{\alpha'} \subseteq Y$ such that $(Y, Y_{\alpha'}) \in \boldsymbol{\chi}$. For each pair $X_\alpha, X_\beta \subseteq X$ such that $(X, X_\alpha), (X, X_\beta) \in \boldsymbol{\chi}$ we have the following diagram



(2)

Lemma
The assignment

$$Z_G\begin{pmatrix} X & & & & Y \\ & {}_i\searrow & & \swarrow_j & \\ & & M & & \end{pmatrix} = \phi^Y_{\alpha'} d^M_{\alpha'}$$

does not depend on the choice of $Y_{\alpha'}$.

Theorem (T.)
Z_G is a functor.

Lemma

Let $i\colon X \to M \leftarrow Y : j$ be a concrete homotopy cobordism, $i\colon (X, X_0) \to (M, M_0) \leftarrow (Y, Y_0) : j$ a choice of concrete based homotopy cobordism, and $[f] \in Z_G(X)$ and $[g] \in Z_G(Y)$ be basis elements (so $[f]$, for example, is an equivalence class in $\mathrm{colim}(\mathcal{V}_X)$), then

$$\begin{aligned}\langle [g] | Z_G(M) | [f] \rangle &= |G|^{-(|M_0|-|X_0|)} \sum_{g \in \phi_0^{Y-1}([g])} \left| \left\{ h\colon \pi(M, M_0) \to G \,\middle|\, h|_{\pi(X,X_0)} = f \wedge h|_{\pi(Y,Y_0)} = g \right\} \right| \\ &= |G|^{-(|M_0|-|X_0|)} \sum_{g \in \phi_0^{Y-1}([g])} \left\langle g \,\middle|\, Z_G^!(M, M_0) \,\middle|\, f \right\rangle\end{aligned}$$

where $\phi_0^Y\colon Z_G^!(Y, Y_0) \to Z_G(Y)$ is the map into $\mathrm{colim}(\mathcal{V}'_Y)$.

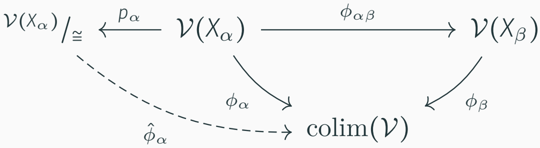
Lemma
Let $i\colon X \to M \leftarrow Y : j$ be a concrete homotopy cobordism, $i\colon (X, X_0) \to (M, M_0) \leftarrow (Y, Y_0) : j$ a choice of concrete based homotopy cobordism, and $[f] \in Z_G(X)$ and $[g] \in Z_G(Y)$ be basis elements (so $[f]$, for example, is an equivalence class in $\mathrm{colim}(\mathcal{V}_X)$), then

$$\begin{aligned}\langle [g] | Z_G(M) | [f] \rangle &= |G|^{-(|M_0|-|X_0|)} \sum_{g \in \phi_0^{Y-1}([g])} \left| \left\{ h\colon \pi(M, M_0) \to G \,|\, h|_{\pi(X,X_0)} = f \wedge h|_{\pi(Y,Y_0)} = g \right\} \right| \\ &= |G|^{-(|M_0|-|X_0|)} \sum_{g \in \phi_0^{Y-1}([g])} \left\langle g \,\middle|\, Z_G^{!}(M, M_0) \,\middle|\, f \right\rangle\end{aligned}$$

where $\phi_0^Y\colon Z_G^{!}(Y, Y_0) \to Z_G(Y)$ is the map into $\mathrm{colim}(\mathcal{V}'_Y)$. Equivalently

$$\langle [g] | Z_G(M) | [f] \rangle = |G|^{-(|M_0|-|X_0|)} \left| \left\{ h : \pi(M, M_0) \to G \,|\, h|_{\pi(X,X_0)} = f \wedge h|_{\pi(Y,Y_0)} \sim g \right\} \right|$$

$Z_G\colon \mathrm{HomCob} \to \mathbf{Vect}_{\mathbb{C}}$

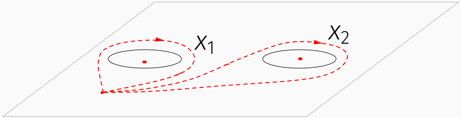


Theorem (T.)
For X a space, the map $\hat{\phi}_\alpha$ is an isomorphism. Hence, for a homotopically 1-finitely generated space $X \in \chi$

$$Z_G(X) = \mathbb{C}((\mathbf{Grpd}(\pi(X, X_0), G)/\cong),$$

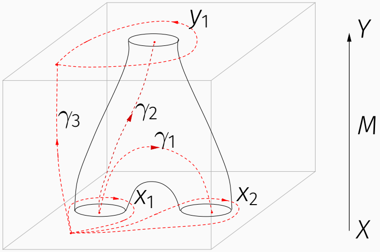
for any choice $X_0 \subset X$ of finite representative subset, where $\cong$ denotes taking maps up to natural transformation.
Further,

$$Z_G(X) = \mathbb{C}((\mathbf{Grpd}(\pi(X), G)/\cong).$$



Let X be the complement of the embedding of two circles shown. Letting $X_0 \subset X$ be the subset shown, $\mathsf{Grpd}(\pi(X, X_0), G) = G \times G$ as discussed previously. Since all objects are mapped to the unique object in G, taking maps up to natural transformation is means taking maps up to conjugation by elements of G at each basepoint, hence in this case maps are labelled by pairs of elements of G, up to simultaneous conjugation, so we have $Z_G(X) = \mathbb{C}((G \times G)/G)$.

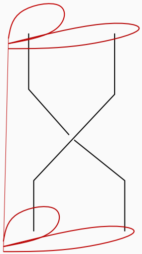
Example



Basis elements in $Z_G(X)$ are given by equivalence classes $[(f_1, f_2)]$ where $f_1, f_2 \in G$ and $[]$ denotes simultaneous conjugation by the same element of G. Basis elements in $Z_G(Y)$ are given by elements of g taken up to conjugation, denoted $[g_1]$. We have

$$
\begin{aligned}
\langle [g_1] | Z_G(M) | [(f_1, f_2)] \rangle &= |G|^{-2} \left\{ a, b, c, d, e \in G \mid a = f_1, b = f_2, g_1 \sim ebae^{-1} \right\} \\
&= \left\{ e \in G \mid g_1 \sim ef_1 f_2 e^{-1} \right\} \\
&= \begin{cases} |G| & \text{if } g_1 \sim f_1 f_2 \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}
$$

Undercrossing braid



$$\langle [g_1, g_2] | Z_G M | [f_1, f_2] \rangle = |G|^{-1} \{a, b, c | a = f_1, b = f_2, c f_1 c^{-1} \sim g_2, c f_1^{-1} f_2 f_1 c^{-1} = g_1\}$$
$$= \begin{cases} 1 & [g_1, g_2] = [f_1^{-1} f_2 f_1, f_1] \\ 0 & \text{otherwise} \end{cases}$$

Undercrossing

$$\langle [g_1, g_2] | Z_G M | [f_1, f_2] \rangle = \begin{cases} 1 & [g_1, g_2] = [f_1^{-1} f_2 f_1, f_1] \\ 0 & \text{otherwise} \end{cases}$$

Overcrossing

$$\langle [g_1, g_2] | Z_G M | [f_1, f_2] \rangle = \begin{cases} 1 & [g_1, g_2] = [f_2, f_2^{-1} f_1 f_2] \\ 0 & \text{otherwise} \end{cases}$$

Topological quantum field theories & homotopy cobordisms

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