From Langlands Duality of Holomorphic Invariants to Mirror Symmetry of Quasi-Topological Strings via D-branes

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Outline of Talk

- Introduction and Motivation
- Summary of Results
- Main Body of the Talk
- Conclusion

In this talk, we will discuss a **topological-holomorphic twist** of a 4d $\mathcal{N}=4$ SYM gauge theory on $M_4 = \Sigma_1 \times \Sigma_2$, which for unitary gauge groups, is realized by the worldvolume theory of the Euclidean D3-brane in the following type IIB string theory background:

	Σ_1		Σ_2		$NM_4 \subset T^*\Sigma_1$		$NM_4 \subset T^*\Sigma_2$			
	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
D3	$\times$	$\times$	$\times$	$\times$						

(For other gauge groups, one can add O-planes etc.)

Introduction and Motivation

Considering the cohomology of different linear combinations $Q = \mathcal{Q}_a + t\mathcal{Q}_b$ of the resulting 4 scalar supercharges $\mathcal{Q}_a$, $\mathcal{Q}_b$ (i.e., the different BPS sectors of the D3-brane worldvolume theory), allows us to have either of the following:

- a topological theory on all of M_4
- a topological-holomorphic theory that is topological on Σ_1 and holomorphic on Σ_2
- a theory holomorphic on both Σ_1 and Σ_2

The motivations for doing so are to

- Derive **novel** topological and holomorphic invariants of M_4.
- Relate them to the 2d invariants of topological and **quasi-topological** strings on Hitchin moduli space via an equivalent 2d $\mathcal{N} = (4, 4)$ sigma-model.
- Obtain a Langlands dual of these invariants, and a resulting mirror symmetry of the aforementioned strings.

This talk is based on

- Tan, Meng-Chwan et al., “Topological-Holomorphic N=4 Gauge Theory: From Langlands Duality of Holomorphic Invariants to Mirror Symmetry of Quasi-Topological Strings”. arXiv: 2305.15965.

Built on earlier insights in

- Bershadsky, Michael, et al, “Topological reduction of 4D SYM to 2D σ-models”, Nuclear Physics B 448.1-2, 166-186 (1995).
- Kapustin, Anton. “Holomorphic reduction of N= 2 gauge theories, Wilson-'t Hooft operators, and S-duality”. arXiv: hep-th/0612119.
- Tan, Meng-Chwan. "Two-dimensional twisted sigma models and the theory of chiral differential operators." Advances in Theoretical and Mathematical Physics 10.6 (2006): 759-851.
- Kapustin, Anton and Witten, Edward. “Electric-magnetic duality and the geometric Langlands program”, Communications in Number Theory and Physics Volume 1, Number 1, (2007).

Topological theory on M_4

1. For gauge group G and complex coupling τ, the **novel** 4d topological invariant is the correlation function of operators $\mathcal{O}$ in the Q-cohomology:

$$\langle \Pi_i \mathcal{O}_i \rangle_{4d}(\tau, G) = \int_{\mathcal{M}} D\phi \, \Pi_i \mathcal{O}_i e^{-S}$$

2. Compactify $M_4 = \Sigma_1 \times \Sigma_2$ along Σ_1, where both Σ_1 and Σ_2 are closed

Riemann surfaces, and Σ_1 has a genus $g \geq 2$. We arrive at an A-model in complex structure I on Σ_2 with $\mathcal{N} = (4,4)$ supersymmetry and target space $\mathcal{M}^G_{\text{Higgs}}(\Sigma_1)$, the moduli space of Higgs Bundles on Σ_1. Then, **Topological invariance implies a 4d-2d correspondence** of correlation functions and thus invariants:

$$\langle \Pi_i \mathcal{O}_i \rangle_{4d}(\tau, G) = \langle \Pi_i \tilde{\mathcal{O}}_i \rangle_{2d}(\tau, \mathcal{M}^G_{\text{Higgs}}(\Sigma_1))$$

3. If $\Sigma_2 = \mathbb{R} \times I$, we have a category of A-branes of type $(A, *, *)$ in

$\mathcal{M}^G_{\text{Higgs}}(\Sigma_1)$.

Summary of Results

Topological-holomorphic theory on M_4

4. Since the theory is topological along Σ_1 and holomorphic along Σ_2, whose coordinates are $(z, \bar{z})$ and $(w, \bar{w})$, respectively, correlation functions will have a **holomorphic dependence on the coordinates of Σ_2**:

$$\langle \Pi_i \mathcal{O}'_i \rangle_{4d}(w, G) = \int_{\mathcal{M}'} D\phi' \; \Pi_i \mathcal{O}'_i e^{-S}$$

This defines a **novel** 4d holomorphic invariant in w.

5. Compactify $M_4 = \Sigma_1 \times \Sigma_2$ along Σ_1, we get a holomorphic or **quasi-topological** sigma-model in complex structure $J + \alpha K$ on Σ_2 with $\mathcal{N} = (4, 4)$ supersymmetry and target space $\mathcal{M}^{G_{\mathbb{C}}}_{\text{flat}}(\Sigma_1)$, the moduli space of flat complexified connections on Σ_1. **Topological invariance along Σ_1 implies a 4d-2d correspondence** of correlation functions and thus invariants:

$$\langle \Pi_i \mathcal{O}'_i \rangle_{4d}(w, G) = \langle \Pi_i \tilde{\mathcal{O}}'_i \rangle_{2d}(w, \mathcal{M}^{G_{\mathbb{C}}}_{\text{flat}}(\Sigma_1))$$

Holomorphic theory on M_4 (both Σ_1 and Σ_2)

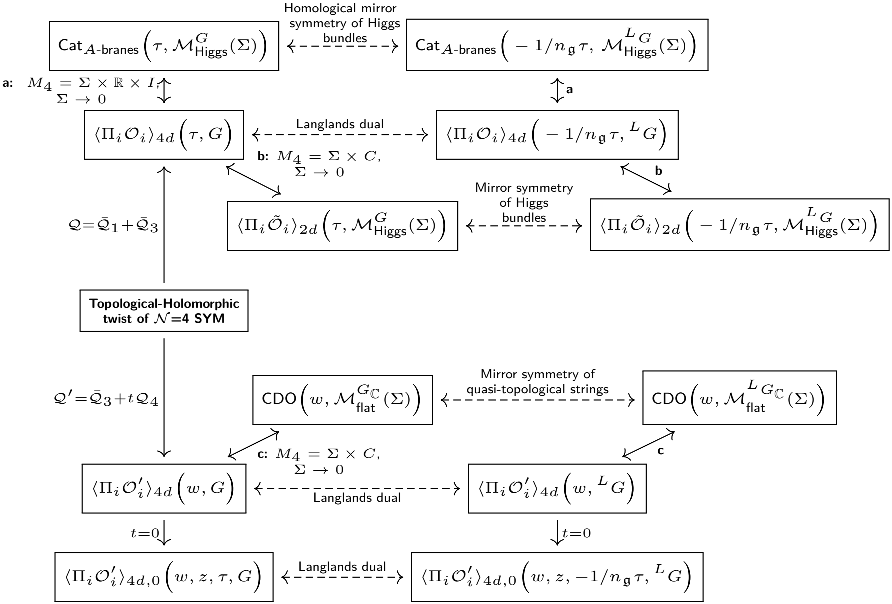
6. If we consider the cohomology of only ONE scalar supercharge, i.e., $t=0$, we can obtain a theory that is holomorphic on M_4 (holomorphic on both Σ_1 and Σ_2). Correlation functions have a dependence on both z and w

$$\langle \Pi_i \mathcal{O}'_{0,i} \rangle_{4d,0}(w, z, \tau, G) = \int_{\mathcal{M}'_0} D\phi' \, \Pi_i \mathcal{O}'_{0,i} e^{-S}$$

This defines a **novel** 4d holomorphic invariant in z and w.

$\mathcal{N}=4$ S-duality

7. S-duality implies a Langlands duality and thus mirror symmetry of the aforementioned invariants and 2d sigma-models (and therefore strings).


$\mathrm{Cat}_{A\text{-branes}}\left(\tau, \mathcal{M}^{G}_{\text{Higgs}}(\Sigma)\right)$
Homological mirror symmetry of Higgs bundles
$\mathrm{Cat}_{A\text{-branes}}\left(-1/n_{\mathfrak{g}}\tau, \mathcal{M}^{^LG}_{\text{Higgs}}(\Sigma)\right)$
a: $M_4 = \Sigma \times \mathbb{R} \times I$, $\Sigma \to 0$
a
$\langle \Pi_i \mathcal{O}_i \rangle_{4d}\left(\tau, G\right)$
Langlands dual
$\langle \Pi_i \mathcal{O}_i \rangle_{4d}\left(-1/n_{\mathfrak{g}}\tau, {}^LG\right)$
b: $M_4 = \Sigma \times C$, $\Sigma \to 0$
b
$\langle \Pi_i \tilde{\mathcal{O}}_i \rangle_{2d}\left(\tau, \mathcal{M}^{G}_{\text{Higgs}}(\Sigma)\right)$
Mirror symmetry of Higgs bundles
$\langle \Pi_i \tilde{\mathcal{O}}_i \rangle_{2d}\left(-1/n_{\mathfrak{g}}\tau, \mathcal{M}^{^LG}_{\text{Higgs}}(\Sigma)\right)$
$\mathcal{Q} = \bar{\mathcal{Q}}_1 + \bar{\mathcal{Q}}_3$
Topological-Holomorphic twist of $\mathcal{N}=4$ SYM
$\mathcal{Q}' = \bar{\mathcal{Q}}_3 + t\mathcal{Q}_4$
$\mathrm{CDO}\left(w, \mathcal{M}^{G_{\mathbb{C}}}_{\text{flat}}(\Sigma)\right)$
Mirror symmetry of quasi-topological strings
$\mathrm{CDO}\left(w, \mathcal{M}^{^LG_{\mathbb{C}}}_{\text{flat}}(\Sigma)\right)$
c: $M_4 = \Sigma \times C$, $\Sigma \to 0$
c
$\langle \Pi_i \mathcal{O}'_i \rangle_{4d}\left(w, G\right)$
Langlands dual
$\langle \Pi_i \mathcal{O}'_i \rangle_{4d}\left(w, {}^LG\right)$
$t=0$
$t=0$
$\langle \Pi_i \mathcal{O}'_i \rangle_{4d,0}\left(w, z, \tau, G\right)$
Langlands dual
$\langle \Pi_i \mathcal{O}'_i \rangle_{4d,0}\left(w, z, -1/n_{\mathfrak{g}}\tau, {}^LG\right)$

LET'S EXPLAIN HOW WE GOT THESE RESULTS

The Topological-Holomorphic Twist

- The (Euclidean) spacetime group of $M_4 = \Sigma_1 \times \Sigma_2$ is $E = U(1)_{E_1} \times U(1)_{E_2}$.
- Embed the three different $U(1)_R$ subgroups of the $SU(4)_R$ of $\mathcal{N} = 4$ SUSY in E so as to shift the spins of the (R-charged) fermions - the first, second and third $U(1)_R$ in $U(1)_{E_1}$, $U(1)_{E_2}$, respectively.
- We end up with four scalar supercharges $\bar{\mathcal{Q}}_1$, $\mathcal{Q}_2$, $\bar{\mathcal{Q}}_3$, $\mathcal{Q}_4$ with the following anti-commutator relations:

$$
\begin{aligned}
\{\bar{\mathcal{Q}}_1, \mathcal{Q}_{1z}\} &\propto P_z, & \{\mathcal{Q}_2, \bar{\mathcal{Q}}_{2\bar{z}}\} &\propto P_{\bar{z}}, \\
\{\bar{\mathcal{Q}}_1, \mathcal{Q}_{1w}\} &\propto P_w, & \{\mathcal{Q}_2, \bar{\mathcal{Q}}_{2w}\} &\propto P_w, \\
\{\bar{\mathcal{Q}}_3, \mathcal{Q}_{3\bar{w}}\} &\propto P_{\bar{w}}, & \{\mathcal{Q}_4, \bar{\mathcal{Q}}_{4z}\} &\propto P_z, \\
\{\bar{\mathcal{Q}}_3, \mathcal{Q}_{3\bar{z}}\} &\propto P_{\bar{z}} & \{\mathcal{Q}_4, \bar{\mathcal{Q}}_{4\bar{w}}\} &\propto P_{\bar{w}}.
\end{aligned}
\tag{3.1}
$$

- For **different linear combinations** Q of these scalar supercharges, different components of P_μ will be Q-exact. As such, the corresponding Q-cohomologies will have different properties on M_4.

A Topological Theory on M_4

- We first consider the cohomology of

$$\mathcal{Q} = \bar{\mathcal{Q}}_1 + \bar{\mathcal{Q}}_3. \tag{3.2}$$

Bearing in mind the (twisted) relation $\{\mathcal{Q}_i, \bar{\mathcal{Q}}_j\}_\mu = \delta_{ij} P_\mu$, we can see from (3.1) that

$$\begin{aligned} \{\mathcal{Q}, \mathcal{Q}_{1z}\} &\propto P_z, & \{\mathcal{Q}, \mathcal{Q}_{1w}\} &\propto P_w, \\ \{\mathcal{Q}, \mathcal{Q}_{3\bar{w}}\} &\propto P_{\bar{w}}, & \{\mathcal{Q}, \mathcal{Q}_{3\bar{z}}\} &\propto P_{\bar{z}}. \end{aligned} \tag{3.3}$$

Alll components of the four-momentum are $\mathcal{Q}$-exact, i.e., **the theory is topological over** M_4 with respect to the $\mathcal{Q}$-cohomology.

- The action can be written in the following form:

$$S = \frac{1}{e^2} \int_{M_4} d^2z d^2w \sqrt{g}\ \mathrm{Tr}\{\mathcal{Q}, V\} - \frac{i\tau}{4\pi} \int_{M_4} \mathrm{Tr} F \wedge F, \tag{3.4}$$

where V is a gauge fermion. The energy-momentum tensor $T_{\mu\nu}$ is $\mathcal{Q}$-exact, and **correlation functions of operators in the $\mathcal{Q}$-cohomology are independent of spacetime coordinates**.

A Topological Theory on M_4

- The path integral is **independent of the coupling constant** e, so we can set $e \to 0$ whence the path integral localizes on $\mathcal{M}$, the moduli space of field configurations satisfying the BPS equations:

$$
\begin{aligned}
(F_{z\bar{z}} - i[B_z, B_{\bar{z}}])g^{z\bar{z}} - i[C, C^\dagger] + (F_{w\bar{w}} + i[B_{\bar{w}}, B_w])g^{w\bar{w}} = 0, \\
F_{\bar{w}\bar{z}} + i[B_{\bar{z}}, B_{\bar{w}}] = 0, \qquad F_{wz} - i[B_z, B_w] = 0, \\
g^{z\bar{z}} D_{\bar{z}} B_z + g^{w\bar{w}} D_w B_{\bar{w}} = 0, \qquad D_{\bar{w}} B_z + D_z B_{\bar{w}} = 0, \\
g^{w\bar{w}} D_{\bar{w}} B_w + g^{z\bar{z}} D_z B_{\bar{z}} = 0, \qquad D_{\bar{z}} B_w + D_w B_{\bar{z}} = 0, \\
[C, B_w] = [C, B_{\bar{w}}] = [C, B_{\bar{z}}] = [C, B_z] = D_\mu C = 0.
\end{aligned}
\tag{3.5}
$$

- C is a scalar field generating gauge transformations. Set $C = 0$ to have irreducible connections.
- The general form of a 4d topological correlation function is

$$
\langle \Pi_i \mathcal{O}_i \rangle_{4d}(\tau, G) = \int_{\mathcal{M}} D\phi \, \Pi_i \mathcal{O}_i e^{-S} \tag{3.6}
$$

where $D\phi$ represents the integration measure over all field configurations, and $\mathcal{O}_i$ is an operator in the $\mathcal{Q}$-cohomology.

$\mathcal{N} = (4,4)$ A-model, Higgs Bundles and GW Theory

- Introduce ϵ, a small parameter to rescale Σ_1. The metric becomes

$$g = \text{diag}(\epsilon g_{\Sigma_1}, g_{\Sigma_2}). \tag{3.7}$$

- When $\epsilon \to 0$, in order for the action to remain well-defined, i.e. finite, we obtain the following conditions along Σ_1:

$$F_{z\bar{z}} - i[B_z, B_{\bar{z}}] = D_{\bar{z}} B_z = 0 \tag{3.8}$$

  Here, A_{Σ_1} and a section $B_{\Sigma_1} \in \Omega^1(\Sigma_1)$ modulo gauge transformations span **Hitchin's moduli space** $\mathcal{M}_H^G(\Sigma_1)$.

- We get a sigma-model on Σ_2 with a map $\Phi(X,Y) : \Sigma_2 \to \mathcal{M}_H^G(\Sigma_1)$, where the bosonic scalars (X,Y) on Σ_2 correspond to $(A_{\Sigma_1}, B_{\Sigma_1})$.

- The sigma-model is an A-model, where the BPS equations of the sigma model are **holomorphic maps**, obtained from the dimensional reduction of (3.5):

$$\partial_{\bar{w}} X^i = \partial_{\bar{w}} Y^i = 0. \tag{3.9}$$

$\mathcal{N} = (4,4)$ A-model, Higgs Bundles and GW Theory

- The A-model symplectic form is in complex structure I, whence the target space $\mathcal{M}_H^G(\Sigma_1) = \mathcal{M}_{\text{Higgs}}^G(\Sigma_1)$. The path integral localizes to an integral over the **moduli space of holomorphic maps**

$$\mathcal{M}_{\text{maps}} = \{\Phi(X^i, Y^i) : \Sigma_2 \to \mathcal{M}_{\text{Higgs}}^G(\Sigma_1) \mid \partial_{\bar{w}} X^i = \partial_{\bar{w}} Y^i = 0\} \tag{3.10}$$

- 2d topological correlation functions of the $\mathcal{Q}$-cohomology of operators $\tilde{\mathcal{O}}_i$ correspond to **Gromov-Witten (GW) invariants** of $\mathcal{M}_{\text{Higgs}}^G(\Sigma_1)$:

$$\langle \Pi_i \tilde{\mathcal{O}}_i \rangle_{2d}(\tau, \mathcal{M}_{\text{Higgs}}^G(\Sigma_1)) = \int_{\mathcal{M}_{\text{maps}}} D\tilde{\phi}\, \Pi_i \tilde{\mathcal{O}}_i e^{-S_1}, \tag{3.11}$$

  where S_1 is the action of the $\mathcal{N} = (4,4)$ A-model on Σ_2.
- Topological invariance along Σ_1 implies a **4d-2d correspondence**:

$$\boxed{\langle \Pi_i \mathcal{O}_i \rangle_{4d}(\tau, G) = \langle \Pi_i \tilde{\mathcal{O}}_i \rangle_{2d}(\tau, \mathcal{M}_{\text{Higgs}}^G(\Sigma_1))} \tag{3.12}$$

- If $\Sigma_2 = \mathbb{R} \times I$, we have a category of A-branes of type $(A, *, *)$ on $\mathcal{M}_{\text{Higgs}}^G(\Sigma_1)$.

A Topological-Holomorphic Theory on M_4

- We can also consider the cohomology of
$$\mathcal{Q}' = \bar{\mathcal{Q}}_3 + t\mathcal{Q}_4 \tag{3.13}$$
where $t \in \mathbb{C}$, and $t \neq 0, \infty$. Using the (twisted) relations $\{\mathcal{Q}_i, \bar{\mathcal{Q}}_j\}_\mu = \delta_{ij} P_\mu$ and $\{\mathcal{Q}_i, \mathcal{Q}_j\}_\mu = \{\bar{\mathcal{Q}}_i, \bar{\mathcal{Q}}_j\}_\mu = 0$, we can see from (3.1) that
$$\begin{aligned} \{\mathcal{Q}', \mathcal{Q}_{3\bar{w}}\} \propto P_{\bar{w}}, \qquad \{\mathcal{Q}', \mathcal{Q}_{3\bar{z}}\} &\propto P_{\bar{z}} \\ \{\mathcal{Q}', \bar{\mathcal{Q}}_{4z}\} \propto P_z, \qquad \{\mathcal{Q}', \bar{\mathcal{Q}}_{4\bar{w}}\} &\propto P_{\bar{w}}, \end{aligned} \tag{3.14}$$
- The theory is **topological along Σ_1 and holomorphic** along Σ_2 with respect to the $\mathcal{Q}'$-cohomology, since there is now a dependence on the w-coordinate.
- The action can be written in the following form:
$$S = \frac{1}{e^2}\int_{M_4} d^2z d^2w \sqrt{g}\, \mathrm{Tr}\{\mathcal{Q}', V'\} - \frac{i\tau}{4\pi}\int_{M_4} \mathrm{Tr} F \wedge F + \ldots \tag{3.15}$$
where "…" represents fermionic terms that can be interpreted as a wedge product of differential forms, and V' is a gauge fermion.

A Topological-Holomorphic Theory on M_4

- The coupling constant e appears only in the $\mathcal{Q}$'-exact part of S, so again, the path integral is **independent of the coupling constant**. Setting $e \to 0$, the path integral localizes onto a moduli space $\mathcal{M}'$ of field configurations satisfying the BPS equations.
- For a complex gauge connection $\mathcal{A} \in \Omega^1(M_4)$, where

$$\begin{aligned} \mathcal{A} &= \mathcal{A}_z dz + \mathcal{A}_{\bar{z}} d\bar{z} + \mathcal{A}_w dw + \mathcal{A}_w d\bar{w} \\ &= (A_z + tB_z)dz + (A_{\bar{z}} - t^{-1}B_{\bar{z}})d\bar{z} + A_w dw + A_{\bar{w}} d\bar{w}, \end{aligned} \tag{3.16}$$

the BPS equations are

$$\begin{aligned} \mathcal{F}_{\bar{z}z} = \mathcal{F}_{wz} = \mathcal{F}_{w\bar{z}} &= 0, & D_{\bar{z}}B_{\bar{w}} - it^{-1}[B_{\bar{z}}, B_{\bar{w}}] &= 0, \\ D_z B_{\bar{w}} + it[B_z, B_{\bar{w}}] &= 0, & D_{\bar{z}}C - it^{-1}[B_{\bar{z}}, C] &= 0, \\ D_z C + it[B_z, C] &= 0, & D_w B_{\bar{w}} = D_w C &= 0, \\ [B_{\bar{w}}, C] &= 0. \end{aligned} \tag{3.17}$$

- Again, we can set $C = 0$ to have irreducible connections.

A Topological-Holomorphic Theory on M_4

- The 4d (partially) holomorphic correlation functions are of the form

$$\langle \Pi_i \mathcal{O}'_i \rangle_{4d}(w, G) = \int_{\mathcal{M}'} D\phi' \, \Pi_i \mathcal{O}'_i e^{-S} \tag{3.18}$$

where $D\phi'$ represents the integration measure over all field configurations, and $\mathcal{O}'_i = \mathcal{O}'_i(w)$ is an operator in the $\mathcal{Q}'$-cohomology.

- τ appears only in the instanton term, and

$$\frac{i\tau}{4\pi} \int_{M_4} \mathrm{Tr} F \wedge F \sim \frac{i\tau}{4\pi} \int_{M_4} \mathrm{Tr} \mathcal{F} \wedge \mathcal{F} \tag{3.19}$$

where " $\sim$" in the above equation indicates that the expressions are $\mathcal{Q}'$-cohomologous.

- BPS equations (3.17) gives $\mathcal{F}_{wz} = \mathcal{F}_{w\bar{z}} = \mathcal{F}_{z\bar{z}} = 0$. Thus, only the zero-instanton sector contributes to (3.18), whence **the correlation functions are independent of τ.**

A Quasi-Topological $\mathcal{N} = (4, 4)$ Sigma-Model and CDO's

- Introduce ϵ, a small parameter to rescale Σ_1. The metric becomes

$$g = \text{diag}(\epsilon g_{\Sigma_1}, g_{\Sigma_2}). \tag{3.20}$$

- When $\epsilon \to 0$, in order for the action to remain well-defined, i.e. finite, we obtain the following conditions along Σ_1:

$$F_{z\bar{z}} - i[B_z, B_{\bar{z}}] = D_{\bar{z}} B_z = 0 \tag{3.21}$$

  Here, A_{Σ_1} and a section $B_{\Sigma_1} \in \Omega^1(\Sigma_1)$ modulo gauge transformations span **Hitchin's moduli space** $\mathcal{M}_H^G(\Sigma_1)$.

- We get a sigma model on Σ_2 where the fields $(A_{\Sigma_1}, B_{\Sigma_1})$ define a map $\Phi : \Sigma_2 \to \mathcal{M}_H^G(\Sigma_1)$.

- The BPS equations of the sigma model are obtained from the dimensional reduction of (3.17):

$$\partial_w \mathcal{A}_z = \partial_w \mathcal{A}_{\bar{z}} = 0. \tag{3.22}$$

- Identify (X, Y) with $(\mathcal{A}_z, \mathcal{A}_{\bar{z}})$ as holomorphic coordinates on $\mathcal{M}_H^G(\Sigma_1)$.
- $\mathcal{F}_{z\bar{z}} = 0$ in the BPS equations imply flat complexified connections on Σ_1, i.e., the target space $\mathcal{M}_H^G(\Sigma_1)$ can be identified as $\mathcal{M}_{\text{flat}}^{G_{\mathbb{C}}}(\Sigma_1)$, the **moduli space of flat complexified connections on** Σ_1. The relevant complex structure of the sigma model must then be $J + \alpha K$.
- With $Z^I = X^i \oplus Y^i$, the BPS equation (3.22) becomes

$$\partial_w Z^I = 0 \tag{3.23}$$

and the map $Z : \Sigma_2 \to \mathcal{M}_{\text{flat}}^{G_{\mathbb{C}}}(\Sigma_1)$ is anti-holomorphic. The path integral thus localizes to an integral over the moduli space of anti-holomorphic maps

$$\mathcal{M}'_{\text{maps}} = \{Z : \Sigma_2 \to \mathcal{M}_{\text{flat}}^{G_{\mathbb{C}}}(\Sigma_1) \mid \partial_w Z^I = 0\} \tag{3.24}$$

A Quasi-Topological $\mathcal{N} = (4,4)$ Sigma-Model and CDO's

- The topological term in the 2d action is

$$i\tau \int_C \Phi^*(\omega_t) \tag{3.25}$$

  where ω_t is the symplectic form of Hitchin's moduli space in complex structure $J + \alpha K$.

- ω_t is $\mathcal{Q}'$-exact. Thus, the sigma-model correlation functions of operators $\tilde{\mathcal{O}}'_i = \tilde{\mathcal{O}}'_i(w)$ in the $\mathcal{Q}'$-cohomology will be **independent of** τ.

- Only degree-zero maps of $Z : \Sigma_2 \to \mathcal{M}^{G_\mathbb{C}}_{\text{flat}}(\Sigma_1)$ contribute, where $\mathcal{M}'_{\text{maps}}$ is simply $\mathcal{M}^{G_\mathbb{C}}_{\text{flat}}(\Sigma_1)$.

- The 2d holomorphic correlation functions will thus be of the form

$$\langle \Pi_i \tilde{\mathcal{O}}'_i \rangle_{2d}(w, \mathcal{M}^{G_\mathbb{C}}_{\text{flat}}(\Sigma_1)) = \int_{\mathcal{M}^{G_\mathbb{C}}_{\text{flat}}} D\tilde{\phi}' \, \Pi_i \tilde{\mathcal{O}}'_i e^{-S} \tag{3.26}$$

A Quasi-Topological $\mathcal{N}=(4,4)$ Sigma-Model and CDO's

- This holomorphic or **quasi-topological** $\mathcal{N}=(4,4)$ sigma-model is similar to the half-twisted A-model. Just as the usual A-model defines a topological string, our sigma-model defines a quasi-topological string with worldsheet Σ_2.
- One can show, via a Čech-Dolbeault isomorphism, that

$$\tilde{\mathcal{O}}' \cong H^*_{\check{\text{C}}\text{ech}}(\mathcal{M}^{G_{\mathbb{C}}}_{\text{flat}}(\Sigma_1), \Omega), \tag{3.27}$$

where $H^*_{\check{\text{C}}\text{ech}}(\mathcal{M}^{G_{\mathbb{C}}}_{\text{flat}}(\Sigma_1), \Omega)$ is the **Čech cohomology of the sheaf Ω of chiral differential operators (CDO's) on $\mathcal{M}^{G_{\mathbb{C}}}_{\text{flat}}(\Sigma_1)$**.

- Topological invariance along Σ_1 implies a **4d-2d correspondence**:

$$\boxed{\langle \Pi_i \mathcal{O}'_i \rangle_{4d}(w, G) = \text{CDO}(w, \mathcal{M}^{G_{\mathbb{C}}}_{\text{flat}}(\Sigma_1))} \tag{3.28}$$

where $\text{CDO}(\dots)$ is the evaluation over $\mathcal{M}^{G_{\mathbb{C}}}_{\text{flat}}(\Sigma_1)$ of a cup product of the classes in $H^*_{\check{\text{C}}\text{ech}}$ corresponding to the operators $\tilde{\mathcal{O}}'_i(w)$.

Holomorphic Theory on M_4

- Lastly, consider $t = 0$, whence $\mathcal{Q}' = \bar{\mathcal{Q}}_3$. Then, from (3.14), we have

$$\{\mathcal{Q}', \mathcal{Q}_{3\bar{w}}\} \propto P_{\bar{w}}, \qquad \{\mathcal{Q}', \mathcal{Q}_{3\bar{z}}\} \propto P_{\bar{z}}. \tag{3.29}$$

- The theory is holomorphic on $M_4 = \Sigma_1 \times \Sigma_2$ with respect to the $\mathcal{Q}'$-cohomology, where correlation functions of operators $\mathcal{O}'$ in the $\mathcal{Q}'$-cohomology can have a dependence on z and w.
- The action can be written in the following form:

$$S = \frac{1}{e^2} \int_{M_4} d^2z d^2w \sqrt{g}\ \mathrm{Tr}\{\mathcal{Q}', V''\} - \frac{i\tau}{4\pi} \int_{M_4} \mathrm{Tr} F \wedge F + \dots \tag{3.30}$$

where "$\dots$" represent fermionic terms that can be interpreted as a wedge product of differential forms, and V'' is a gauge fermion.

- As before, the path integral is **independent of the coupling constant** e. Setting $e \to 0$, the path integral localizes onto a moduli space $\mathcal{M}'_0$ of field configurations satisfying the BPS equations.

- The BPS equations are:

$$\begin{aligned} g^{z\bar{z}}(F_{z\bar{z}} - i[B_z, B_{\bar{z}}]) + g^{w\bar{w}}(F_{w\bar{w}} + i[B_{\bar{w}}, B_w]) - i[C, C^\dagger] = 0, \\ D_w C = D_z C = g^{w\bar{w}} D_w B_{\bar{w}} = D_z B_{\bar{w}} = D_w B_{\bar{z}} = g^{z\bar{z}} D_z B_{\bar{z}} = 0, \\ F_{wz} = [B_{\bar{z}}, B_{\bar{w}}] = [B_{\bar{z}}, C] = 0. \end{aligned} \tag{3.31}$$

  We can again set $C = 0$ if we want only irreducible connections.

- In contrast to the case where $t \neq 0, \infty$, (3.31) indicates that we will *not* have $\int \mathrm{Tr} F \wedge F = 0$. **There will be nonzero instanton contributions, and correlation functions will have a τ dependence.**
- The 4d holomorphic correlation functions are of the form

$$\langle \Pi_i \mathcal{O}'_{0,i} \rangle_{4d,0}(w, z, \tau, G) = \int_{\mathcal{M}'_0} D\phi' \, \Pi_i \mathcal{O}'_{0,i} e^{-S} \tag{3.32}$$

  where the subscript 0 is to indicate that $t = 0$.

- With $\mathcal{N}=4$ supersymmetry, the 4d theory posseses an $SL(2,\mathbb{Z})$ symmetry, where S-duality maps $G \to {}^LG$, $\tau \to -1/n_{\mathfrak{g}}\tau$, and $n_{\mathfrak{g}}$ is the lacing number of G.
- This gives the following dualities in $\mathcal{Q}$-cohomology

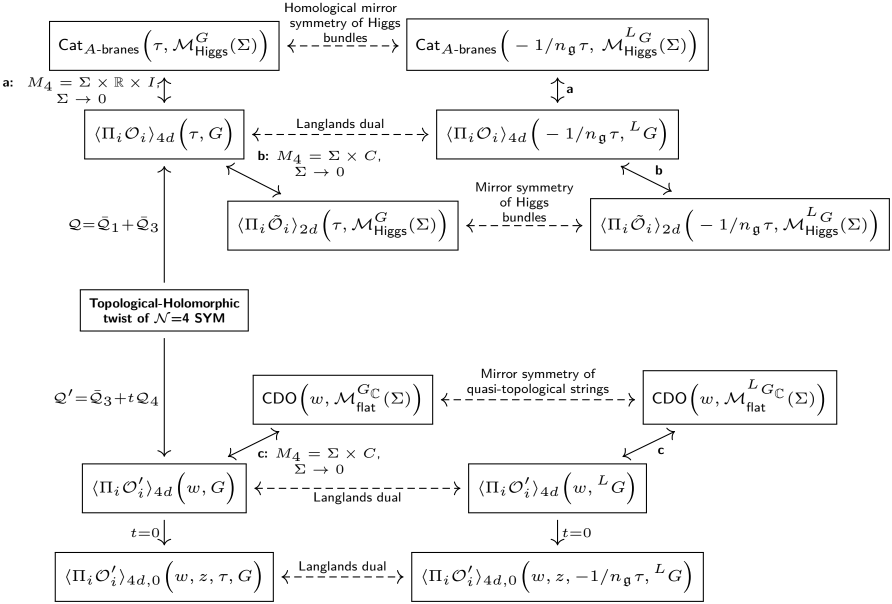
$$\begin{gathered}\langle \Pi_i \mathcal{O}_i \rangle_{4d}(\tau, G) \longleftrightarrow \langle \Pi_i \mathcal{O}_i \rangle_{4d}(-1/n_{\mathfrak{g}}\tau, {}^LG) \\ \langle \Pi_i \tilde{\mathcal{O}}_i \rangle_{2d}(\tau, \mathcal{M}^G_{\text{Higgs}}(\Sigma_1)) \longleftrightarrow \langle \Pi_i \tilde{\mathcal{O}}_i \rangle_{2d}(-1/n_{\mathfrak{g}}\tau, \mathcal{M}^{{}^LG}_{\text{Higgs}}(\Sigma_1)) \\ \text{Cat}_{A\text{-branes}}(\tau, \mathcal{M}^G_{\text{Higgs}}(\Sigma_1)) \longleftrightarrow \text{Cat}_{A\text{-branes}}(-1/n_{\mathfrak{g}}\tau, \mathcal{M}^{{}^LG}_{\text{Higgs}}(\Sigma_1))\end{gathered} \tag{3.33}$$

- In $\mathcal{Q}$'-cohomology ($t \neq 0, \infty$)

$$\begin{gathered}\langle \Pi_i \mathcal{O}'_i \rangle_{4d}(w, G) \longleftrightarrow \langle \Pi_i \mathcal{O}'_i \rangle_{4d}(w, {}^LG) \\ \text{CDO}(w, \mathcal{M}^{G_{\mathbb{C}}}_{\text{flat}}(\Sigma_1)) \longleftrightarrow \text{CDO}(w, \mathcal{M}^{{}^LG_{\mathbb{C}}}_{\text{flat}}(\Sigma_1))\end{gathered} \tag{3.34}$$

- In $\mathcal{Q}$'-cohomology ($t = 0$)

$$\langle \Pi_i \mathcal{O}'_i \rangle_{4d,0}(w, z, \tau, G) \longleftrightarrow \langle \Pi_i \mathcal{O}'_i \rangle_{4d,0}(w, z, -1/n_{\mathfrak{g}}\tau, {}^LG) \tag{3.35}$$



Homological mirror symmetry of Higgs bundles
$\mathrm{Cat}_{A\text{-branes}}\Big(\tau, \mathcal{M}^{G}_{\mathrm{Higgs}}(\Sigma)\Big)$
$\mathrm{Cat}_{A\text{-branes}}\Big(-1/n_{\mathfrak{g}}\tau, \mathcal{M}^{^LG}_{\mathrm{Higgs}}(\Sigma)\Big)$
a: $M_4 = \Sigma \times \mathbb{R} \times I$, $\Sigma \to 0$
a
Langlands dual
$\langle \Pi_i \mathcal{O}_i \rangle_{4d}\Big(\tau, G\Big)$
$\langle \Pi_i \mathcal{O}_i \rangle_{4d}\Big(-1/n_{\mathfrak{g}}\tau, {}^LG\Big)$
b: $M_4 = \Sigma \times C$, $\Sigma \to 0$
b
Mirror symmetry of Higgs bundles
$\mathcal{Q} = \bar{\mathcal{Q}}_1 + \bar{\mathcal{Q}}_3$
$\langle \Pi_i \tilde{\mathcal{O}}_i \rangle_{2d}\Big(\tau, \mathcal{M}^{G}_{\mathrm{Higgs}}(\Sigma)\Big)$
$\langle \Pi_i \tilde{\mathcal{O}}_i \rangle_{2d}\Big(-1/n_{\mathfrak{g}}\tau, \mathcal{M}^{^LG}_{\mathrm{Higgs}}(\Sigma)\Big)$
Topological-Holomorphic twist of $\mathcal{N}$=4 SYM
Mirror symmetry of quasi-topological strings
$\mathcal{Q}' = \bar{\mathcal{Q}}_3 + t\mathcal{Q}_4$
$\mathrm{CDO}\Big(w, \mathcal{M}^{G_{\mathbb{C}}}_{\mathrm{flat}}(\Sigma)\Big)$
$\mathrm{CDO}\Big(w, \mathcal{M}^{^LG_{\mathbb{C}}}_{\mathrm{flat}}(\Sigma)\Big)$
c: $M_4 = \Sigma \times C$, $\Sigma \to 0$
c
$\langle \Pi_i \mathcal{O}'_i \rangle_{4d}\Big(w, G\Big)$
$\langle \Pi_i \mathcal{O}'_i \rangle_{4d}\Big(w, {}^LG\Big)$
Langlands dual
$t=0$
$t=0$
Langlands dual
$\langle \Pi_i \mathcal{O}'_i \rangle_{4d,0}\Big(w, z, \tau, G\Big)$
$\langle \Pi_i \mathcal{O}'_i \rangle_{4d,0}\Big(w, z, -1/n_{\mathfrak{g}}\tau, {}^LG\Big)$

Conclusion

- By considering the cohomology of different linear combinations of scalar supercharges from the topological-holomorphic twist on $M_4 = \Sigma_1 \times \Sigma_2$, we have either
  1. a topological theory on M_4 ($\mathcal{Q}$-cohomology)
  2. a theory topological on Σ_1 and holomorphic on Σ_2 ($\mathcal{Q}$'-cohomology, $t \neq 0, \infty$)
  3. a holomorphic theory on M_4 ($\mathcal{Q}$'-cohomology, $t = 0$)
- Via dimensional reduction on Σ_1, we have a 4d-2d correspondence between the 4d invariants and 2d invariants.
- We obtained a fundamental 4d understanding of why CDO's describe only the worldshet instanton-free quasi-topological strings.
- S-duality of $\mathcal{N} = 4$ supersymmetry allows us to obtain Langlands duals of all the discussed invariants, branes and CDO's.

THANKS FOR LISTENING!