A SURVEY OF TOPOS THEORY

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It is the purpose of this survey to provide some preliminary insight into the connections between topoi and the following fields of endeavour.

(a) The theory of sheaves.

(b) General topology.

(c) Algebraic geometry.

(d) Classical set theory.

(e) Variable set theory (or higher-order intuitionistic logic) as a framework for non-standard analysis, independence results, representations of rings.

(f) Geometric theories.

(g) Local structures.

(h) Cohomology.

1. The space of germs of analytic functions

Let X be an open subset of the complex numbers $\mathbb{C}$ (or, more generally, any complex manifold). For each point $x \in X$, let E_x be the set of power series a about x convergent in some neighbourhood of x : that is,

$$a(y) \;=\; a_0 + a_1(y-x) + a_2(y-x)^2 + \ldots, \qquad a_i \in \mathbb{C}$$

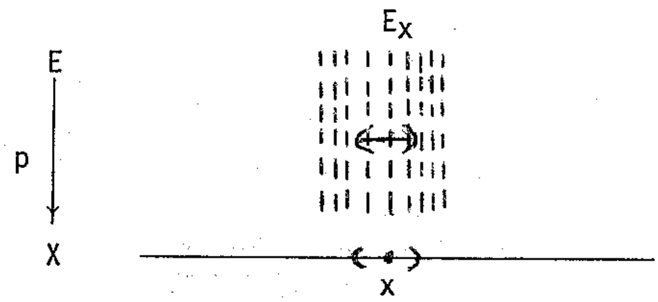
convergent for y in some open neighbourhood of x. Let E be the disjoint union of the E_x ; that is,

$$E \;=\; \{ (x,a) \mid x \in X,\ a \in E_x \}.$$

There is an obvious topology on E: (x,a) is close to (y,b) when x is close to y and a, b are power series for the same analytic functions. Then we have a continuous function $p: E \longrightarrow X$ given by $p(x,a) = x$. Yet p has a further property. For each $(x,a) \in E$, put $N = \{ (y,b) \mid a$ is convergent at y and b is the power series expansion of a about $y \}$; then N is an open neighbourhood of (x,a) in E which is mapped homeomorphically by p to $p(N)$.

2. Local homeomorphisms

A *local homeomorphism* is a continuous function $p: E \longrightarrow X$ between topological spaces E, X such that, for each $e \in E$, there exists an open neighbourhood N of e which is mapped homeomorphically by p onto an open neighbourhood $p(N)$ of $p(e)$. Clearly local homeomorphisms take open sets to open sets, and any homeomorphism is a local homeomorphism. For each $x \in X$, the space $E_x = \{ e \in E \mid p(e) = x \}$ is called the *fibre* or the *stalk* over x. As subspaces of E the stalks are all discrete (for $e \in E_x$, choose an open nbd N of e such that p restricted to N is a homeomorphism onto $p(N)$, then $e' \in E_x \cap N$ implies $p(e') = x = p(e)$, which implies $e' = e$ since p is one-to-one on N; so $\{e\} = E_x \cap N$ is an open subset of E_x). Given X, to give a local homeomorphism into it is to give all the stalks E_x pictured as dotted vertical lines, and a topology on the disjoint union E such that small horizontal cuts look like their projections onto X.



Examples of local homeomorphisms are $p: \mathbb{R} \longrightarrow S^1$ given by $p(x) = e^{2\pi i x}$, and $p: E \longrightarrow X$ as in 1. above. For any open subset U of a space X, the inclusion $i_U: U \longrightarrow X$ is a local homeomorphism; we shall see that all local homeomorphisms are obtained by "gluing together" inclusions of open subsets.

3. Sections

Suppose $p: E \longrightarrow X$ is any continuous function and U is an open subset of X. *A section of* p *over* U is a continuous function $s: U \longrightarrow E$ for which $p(s(x)) = x$ for all $x \in U$. Write $E(U)$ for the set of sections of p over U. If $V \subseteq U$ notice that restriction gives us a function $\rho_U^V: E(U) \longrightarrow E(V)$; that is, $\rho_U^V(s)$ is s restricted to V. These functions satisfy $\rho_U^U = 1_{E(U)}$, $\rho_V^W \rho_U^V = \rho_U^W$ which we shall see means that we have a set-valued functor. A section s of a local homeomorphism over U can be nicely pictured as a continuous path picking out a point in each stalk over points in U.

If $p: E \to X$ is an inclusion of an open subset of X then there is precisely one section over U when $U \subset E$ and none otherwise.

If $p: E \to X$ is as in 1., sections $s: U \to E$ are in bijection with analytic functions $U \to \mathbb{C}$.

4. Categories and functors

A *category* $\mathcal{C}$ consists of objects $A, B, C, \ldots$, arrows $f: A \to B, \ldots$, and an associative composition of arrows $\underbrace{A \xrightarrow{f} B \xrightarrow{g} C}_{gf}$ which has identities $1_A: A \to A$. Write $\mathcal{C}(A,B)$ for the set of arrows in $\mathcal{C}$ from A to B; such sets are called *hom-sets* of $\mathcal{C}$.

A *functor* $T: \mathcal{C} \to \mathcal{D}$ assigns to each object A of $\mathcal{C}$ an object TA of $\mathcal{D}$ and, to each arrow $f: A \to B$, an arrow $Tf: TA \to TB$ such that composition and identities are preserved; that is, $T(gf) = (Tg)(Tf)$ and $T\,1_A = 1_{TA}$.

A *natural transformation* $\alpha: T \to S$ between functors $T,S: \mathcal{C} \to \mathcal{D}$ assigns to each object A of $\mathcal{C}$ an arrow $\alpha A: TA \to SA$ of $\mathcal{D}$ such that, for all arrows $f: A \to B$ in $\mathcal{C}$, the square

$$\begin{array}{ccc} TA & \xrightarrow{\alpha A} & SA \\ {\scriptstyle Tf}\downarrow & & \downarrow{\scriptstyle Sf} \\ TB & \xrightarrow[\alpha B]{} & SB \end{array}$$

commutes; that is, $(Sf)(\alpha A) = (\alpha B)(Tf)$. Natural transformations can be composed $T \xrightarrow{\alpha} S \xrightarrow{\beta} R$ in the obvious way: componentwise in $\mathcal{D}$. We obtain a category $[\mathcal{C},\mathcal{D}]$ whose objects are the functors from $\mathcal{C}$ to $\mathcal{D}$ and whose arrows are natural transformations.

For any category $\mathcal{C}$, write $\mathcal{C}^{op}$ for the category with the same objects as $\mathcal{C}$ but with the arrows in the reverse direction.

The basic example of a big category is *the category* $\mathcal{S}$ *of small sets*; that is, the objects are sets in some universe and the arrows are functions.

5. Two fundamental categorical theorems.

Suppose $T: \mathcal{C} \longrightarrow \mathcal{D}$ is a functor. Each object D of $\mathcal{D}$ gives rise to a functor $T \pitchfork D: \mathcal{C}^{op} \longrightarrow \mathcal{S}$ as follows:

$$\begin{array}{ccc} \mathcal{C}^{op} & & A \xleftarrow{\quad f \quad} B \\ {\scriptstyle T \pitchfork D}\big\downarrow & & \Big\downarrow \\ \mathcal{S} & & \mathcal{D}(TA,D) \xrightarrow[\mathcal{D}(Tf,D)]{} \mathcal{D}(TB,D) \end{array}$$

$$TA \xrightarrow{u} D \rightsquigarrow TB \xrightarrow{u.Tf} D.$$

Each arrow $h: D \longrightarrow D'$ of $\mathcal{D}$ gives rise to a natural transformation $T \pitchfork h: T \pitchfork D \longrightarrow T \pitchfork D'$ given by:

$$\mathcal{D}(TA,D) \xrightarrow{\quad (T \pitchfork h)A \quad} \mathcal{D}(TA,D')$$

$$TA \xrightarrow{u} D \rightsquigarrow TA \xrightarrow{hu} D'.$$

What we have now described is a functor

$$\begin{array}{rl} T \pitchfork -: \mathcal{D} & \longrightarrow [\mathcal{C}^{op},\mathcal{S}] \\ D & \rightsquigarrow T \pitchfork D \\ h & \rightsquigarrow T \pitchfork h. \end{array}$$

Starting with T as the identity functor 1_C of C, we obtain *the Yoneda embedding* $1_C \pitchfork - = Y_C$ (or just Y when C is understood): $C \longrightarrow [C^{op}, S]$.

First theorem (Yoneda) *Suppose* K,D *are objects of* C, D *and* $T: C \longrightarrow D$ *is a functor. For each arrow* $h: TK \longrightarrow D$, *there is a unique natural transformation* $\alpha: Y_C K \longrightarrow T \pitchfork D$ *such that* $(\alpha K)1_K = h$.

As a particular case, suppose $F: C^{op} \longrightarrow S$ is a functor which can also be considered as a functor $F^{op}: C \longrightarrow S^{op}$. Apply the theorem with $D = S^{op}$, $T = F^{op}$ and D the set $1 = \{0\}$. This gives that natural transformations $Y_C K \longrightarrow F$ are in bijection with elements of FK.

As another more particular case, take $T = Y_C L$. We see that natural transformations $\alpha: Y_C K \longrightarrow Y_C L$ are in bijection with arrows $h: K \longrightarrow L$. Consequently the Yoneda embedding allows us to regard C as that part of $[C^{op}, S]$ consisting of the objects of the form $Y_C K$, called *representable functors*.

Second theorem (D. Kan) *Suppose* C *is a small category and* D *is a cocomplete category with small hom-sets. Then, for any functor* $T: C \longrightarrow D$, *there is a functor*

$$- \otimes T: [C^{op}, S] \longrightarrow D$$

and a natural bijection

$$D(P \otimes T, D) \cong [C^{op}, S](P, T \pitchfork D).$$

The situations we have in mind for this theorem are as follows. The small category C consists of certain simple model objects of some bigger nice category E which lies somewhere between C and $[C^{op}, S]$. If we have a way T of regarding the models as objects of $\mathcal{D}$ then the theorem gives us a useful extension so that the general objects of E give rise to objects of $\mathcal{D}$.

6. Two categories associated with a space.

Let X denote a topological space. First we have a small category $\mathcal{O}X$ whose objects are the open subsets of X and the arrows are the inclusions. Note that $\mathcal{O}X$ is really a preordered set in that each hom-set $(\mathcal{O}X)(V,U)$ has at most one element (one when $V \subset U$, none otherwise).

The second category Top/X is cocomplete and not small. The objects are pairs (E,p) where $p: E \longrightarrow X$ is a continuous function, and the arrows $f: (E,p) \longrightarrow (F,q)$ are continuous functions $f: E \longrightarrow F$ such that $qf = p$. Sometimes we just write E for (E,p) and refer to it as a *space over* X.

There is an obvious functor $I: \mathcal{O}X \longrightarrow Top/X$ given by $IU = (U,i_U)$ where i_U is the inclusion of U in X, and $I\, i_V^U = i_V^U$ where $i_V^U: V \longrightarrow U$ is an arrow of $\mathcal{O}X$.

By 5. we obtain a functor

$$I \pitchfork -: Top/X \longrightarrow [(\mathcal{O}X)^{op}, \mathcal{S}]$$

which is given by

$$\begin{aligned}(I \pitchfork (E,p))U &= (Top/X)(IU,(E,p)) \\ &= E(U), \text{ the set of sections of } p.\end{aligned}$$

Also, by the theorem of Kan we have a functor

$$- \otimes I: [(\mathcal{O}X)^{op}, \mathcal{S}] \longrightarrow Top/X.$$

7. The etale space of a presheaf.

A functor $P: (\mathcal{O}X)^{op} \longrightarrow \mathcal{S}$ is called a *presheaf on* X. We saw in 3. that each space E over X gives rise to a presheaf which we have now seen is $I \pitchfork E$. For a presheaf P we shall now describe the space $P \otimes I$ over X, called the *etale space of* P. First form the disjoint union of all the sets $PU \times U$ as U runs over the open subsets of X; that is,

$$\{ (U,s,x) \mid x \in U \in \mathcal{O}X,\ s \in PU \}.$$

Next consider the equivalence relation generated by identifying the elements (U,s,x), $(V,(Pi_V^U)s,x)$ for each arrow $i_V^U: V \longrightarrow U$ in $\mathcal{O}X$ and $x \in V$. The elements of $P \otimes I$ are the equivalence classes $[U,s,x]$ of elements of the above disjoint union under this equivalence relation. For each $U \in \mathcal{O}X$ and $s \in PU$ we have a function $U \longrightarrow P \otimes I$ taking x to $[U,s,x]$; the topology on $P \otimes I$ is the one with the largest set of open sets for which all these functions are continuous. The continuous function $p: P \otimes I \longrightarrow X$ is given by $p[U,s,x] = x$.

<u>*Proposition.*</u> *The function* $p: P \otimes I \longrightarrow X$ *is a local homeomorphism for all presheaves* P *on* X.

<u>*Exercises.*</u> A) Let E be the space of germs of analytic functions over X (see 1.) and let $P = I \pitchfork E$ be the presheaf of sections of E. Prove that E is homeomorphic to $P \otimes I$ as spaces over X.

B) For each presheaf P on a space X and each space E over X, show that there is a bijection between continuous functions over X from the etale space of P to E and natural transformations from P to the presheaf of sections of E. (This shows that formation of the etale space is an instance of Kan's theorem.)

C) Let $\mathcal{U}$ be a family of open sets of a space X and let $U = \cup\,\mathcal{U}$ be the union of the sets in $\mathcal{U}$; so $\mathcal{U}$ is an open cover of the open subset U of X. Observe that the following describes a presheaf $R_{\mathcal{U}}$ on X:

$$R_{\mathcal{U}}V = \begin{cases} 1 & \text{when } \exists\ W \in \mathcal{U} \text{ with } W \supset V, \\ 0 & \text{otherwise.} \end{cases}$$

Describe the etale space of $R_{\mathcal{U}}$.

8. Sheaves on a space.

A presheaf $F: (\mathcal{O}X)^{op} \longrightarrow \mathcal{S}$ on a space X is called a *sheaf* when it satisfies the following condition:

- *for each family* $\mathcal{U}$ *of open subsets of* X *and each family of elements* $s_V \in FV$, $V \in \mathcal{U}$, *such that*

$$(*) \qquad (F\, i^V_{V \cap W}) s_V = (F\, i^W_{V \cap W}) s_W \quad \textit{for all } V, W \in \mathcal{U}$$

there exists a unique $s \in FU$ *where* $U = \cup\,\mathcal{U}$ *such that*

$$(F\, i^U_V) s = s_V \quad \textit{for all } V \in \mathcal{U}.$$

<u>*Theorem.*</u> *The following conditions on a presheaf on a space* X *are equivalent:*

(a) F *is a sheaf;*

(b) there exists a space E *over* X *and a natural isomorphism* $F \cong I \pitchfork E$*;*

(c) there exists a local homeomorphism $p: E \longrightarrow X$ *and a natural isomorphism* $F \cong I \pitchfork E$*;*

(d) for each open subset U *of* X *and each open cover* $\mathcal{U}$ *of* U*, each natural transformation* $\alpha: R_{\mathcal{U}} \longrightarrow F$ *extends uniquely to a natural transformation* $\beta: YU \longrightarrow F$ (see Exercise C) above).

Outline of proof.

(a) $\Leftrightarrow$ (d) The Yoneda theorem yields that natural transformations $\beta: YU \to F$ amount to elements s of FU. It is easily seen that natural transformations $\alpha: R_{\mathcal{U}} \to F$ amount to families $s_V \in FV$, $V \in \mathcal{U}$, satisfying (*).

(c) $\Rightarrow$ (b) is trivial.

(b) $\Rightarrow$ (a) We must see that $I \pitchfork E$ has the sheaf property for any space E over X. Given $\mathcal{U}$ and sections s_V of E over V, we can define $sx = s_V$ when $x \in V$ provided condition (*) is satisfied.

(a) $\Rightarrow$ (c) For any presheaf F we have a canonical natural transformation $F \to I \pitchfork (F \otimes I)$ which can be shown to be an isomorphism if and only if F satisfies the sheaf condition. □

Actually more is true. Consider the pair of functors:

$$\mathit{Top}/X \underset{I \pitchfork -}{\overset{- \otimes I}{\rightleftarrows}} [(\mathcal{O}X)^{op}, \mathcal{S}].$$

On the one hand we have the subcategory $\mathit{Et}(X)$ of Top/X consisting of the local homeomorphisms into X and all arrows between them. On the other hand we have the subcategory $\mathit{Sh}(X,\mathcal{S})$ of $[(\mathcal{O}X)^{op},\mathcal{S}]$ consisting of the sheaves and all natural transformations between them. The above functors induce an equivalence of categories between these two subcategories $\mathit{Et}(X) \simeq \mathit{Sh}(X,\mathcal{S})$. In particular, this means that a bijection is induced between the natural transformations $F \to F'$ where F, F' are sheaves and arrows $F \otimes I \to F' \otimes I$ in Top/X between the corresponding etale spaces.

In dealing with sheaves on a space X we freely pass back and forth from the sheaf itself to its corresponding etale space.

9. Sheaves of functions.

Let X be a space and M a set. For each open subset U of X, let F_1U be the set of functions from U to M. Together with the restriction functions this defines a sheaf F_1 on X called *the sheaf of M-valued functions on* X.

With more structure on M and X, there are subsheaves of F_1 arising from any locally defined class of functions. For example, if M is a topological space we can take F_2U to be the set of *continuous functions* from U to M. Or, if S, M are differentiable manifolds, we can take F_3U to be the set of *differentiable functions* from U to M. Etc.

Indeed, the space of germs of analytic functions is just the etale space of the sheaf of $\mathbb{C}$-valued analytic functions on the complex manifold X. But this example is a little nicer as an etale space than the sheaves of all, continuous, differentiable or even smooth functions; in particular, the etale space is hausdorff whereas it is not in the other examples.

However, not all the interesting examples of sheaves arise as subfunctors of sheaves of functions.

10. The spectrum of a ring.

Let A denote any commutative ring with an identity element 1. A subset P of A is called an *ideal* when:

$$a,\ a' \in P \implies a+a' \in P$$

$$a \in A,\ b \in P \implies ab \in P.$$

An ideal P of A is called *prime* when $1 \notin P$ and

$$a,b \in A,\ ab \in P \implies a \in P \text{ or } b \in P.$$

For example, if A is the ring $\mathbb{Z}$ of integers, each integer a gives an ideal $\{ ab \mid b \in \mathbb{Z} \} = a\mathbb{Z}$; as a runs over all the non-negative integers $a\mathbb{Z}$ runs over all the ideals of $\mathbb{Z}$. Of course, $a\mathbb{Z}$ is prime if and only if $a = 0$, or a is prime.

Let *SpecA* denote the set of prime ideals of A. For each $a \in A$, put $D(a) = \{ P \in SpecA \mid a \notin P \}$ and note that $D(a) \cap D(b) = D(ab)$. A topology on *SpecA* is obtained by taking the open subsets to be arbitrary unions of subsets of the form $D(a)$.

A space K is said to be *reducible* when it can be written as a union of two proper closed subsets. A space X is said to be *sober* when, for each irreducible closed subspace K of X, there exists a unique point $x \in K$ such that K is the closure of $\{x\}$. Any Hausdorff space is sober. Points in *SpecA* are not closed in general, yet:

<u>*Theorem.*</u> *For each commutative ring* A, *the space* SpecA *is compact and sober.* □

A Boolean algebra A gives a ring with operations $a + b = (a \wedge \neg b) \vee (\neg a \wedge b)$, $ab = a \wedge b$. Then *SpecA* is the *Stone space* of the Boolean algebra; it is totally disconnected, compact and hausdorff.

For any family Λ of polynomials over $\mathbb{C}$ in indeterminates $x_1, \ldots, x_n$, let V be the set of points in $\mathbb{C}^n$ which are zeros for all the polynomials in Λ. Such a V is called a *complex affine variety.* Let A be the set of functions a from V to $\mathbb{C}$ for which there exists a polynomial p over $\mathbb{C}$ in n indeterminates such that $a(v) = p(v)$ for all $v \in V$. These are called the *regular functions* from V to $\mathbb{C}$. Pointwise addition, multiplication and scalar multiplication make A an algebra over $\mathbb{C}$. For each $v \in V$, the subset $P_v = \{ a \in A \mid a(v) = 0 \}$ is a prime ideal of A. The assignment $v \mapsto P_v$ is one-to-one and so allows us to regard V as a subset of *SpecA*. The topology induced on V by that on *SpecA* is called the *Zariski topology*; a subset C of V is closed if and only if there is a family of polynomials in n indeterminates for which C is precisely the set of common zeros. There are thus far fewer closed subsets than in the usual topology that V inherits as a subset of $\mathbb{C}^n$, yet the usual topology taken in isolation from the embedding $V \to \mathbb{C}^n$ contains little information regarding the algebraic geometry of V.

11. Rings of quotients

Let A be a commutative ring with an identity 1 and let S be a subset of A which contains 1 and is closed under multiplication. Define an equivalence relation of $A \times S$ by:

$$(a,s) \sim (b,t) \Longleftrightarrow \exists\ w \in S \text{ with } w(sb - at) = 0.$$

Let $A[S^{-1}]$ denote the set of equivalence classes a/s of elements (a,s) of $A \times S$ under this relation. With the usual formulas for addition and multiplication of fractions $A[S^{-1}]$ becomes a ring. There is a ring homomorphism $A \to A[S^{-1}]$ given by $a \mapsto a/1$ which provides inverses for all elements of S, and is universal amongst all ring homomorphisms out of A which do this.

As a particular case, take $S = \{ s^n \mid n \in N \}$ where s is a given element of A. Then we denote $A[S^{-1}]$ by $A[s^{-1}]$. Note that, even if s is *nilpotent* (that is, $s^n = 0$ for some $n \in N$) then $A[s^{-1}]$ still makes sense, but it reduces to the ring with one element.

The other important special case is when S is the complement $\neg P$ of a prime ideal P of A. Then $A[S^{-1}]$ is denoted by A_P and is called *the localization of* A *at* P. A ring B is said to be *local* when, for all $x \in B$, either x or $1-x$ is invertible, *and* $1 \neq 0$.

Proposition. *For each prime ideal* P *of* A, *the ring* A_P *is local.*

Proof. The elements of the set M of non-invertible elements in A_P have the form a/s where $a \in P$, $s \notin P$. It follows that M is an ideal of A_P not containing 1. But if $x, 1-x \in M$ then $1 = x+1-x \in M$, a contradiction. □

12. The structure sheaf of a ring.

It is a theorem of Stone that there is a duality between Boolean algebras and totally-disconnected compact hausdorff spaces; the Boolean algebra A gives rise to the Stone space and A can be recaptured from the space. We have also given the example of a complex affine variety V and mentioned that the algebra A of regular functions from V to $\mathbb{C}$ leads to a space *Spec*A which contains V as a subspace; in fact, V amounts to the subspace of *Spec*A consisting of the maximal ideals.

It is not however the case that A can be recaptured from the topological space *Spec*A alone. The extra structure needed turns out to be a sheaf $\tilde{A}$ on *Spec* A. It is this sheaf we wish now to describe.

Let A be a commutative ring with a 1. Let $\mathcal{D}_A$ denote the category whose objects are elements a of A and for which there is precisely one arrow $b \to a$ when $\mathcal{D}(b) \subset \mathcal{D}(a)$ and none otherwise. Let $D: \mathcal{D}_A \to \mathcal{O}(\mathit{Spec}A)$ be the functor taking a to $D(a)$.

To say $D(b) \subset D(a)$ is to say every prime ideal containing a contains b. It follows that the coset $b+(a)$ is contained in every prime ideal of the factor ring $A/(a)$. The intersection of all prime ideals of a ring is the set of nilpotent elements of the ring. So $D(b) \subset D(a)$ means precisely that $b^n = ac$ for some $n > 0$, $c \in A$. Thus the homomorphism $A \longrightarrow A[b^{-1}]$ which inverts b also inverts a and so induces a homomorphism $A[a^{-1}] \longrightarrow A[b^{-1}]$.

What we have described in the last paragraph is a functor $A[-^{-1}]: \mathcal{D}_A^{op} \longrightarrow \mathcal{S}$ (in fact it lands in the category of rings, not just sets). Using the theorem of Kan, we see that there exists a functor $\tilde{A}: \mathcal{O}(SpecA)^{op} \longrightarrow \mathcal{S}$, unique up to isomorphism, such that there is a natural bijection between natural transformations $P \longrightarrow \tilde{A}$ and natural transformations $PD^{op} \longrightarrow A[-^{-1}]$ (where P is an arbitrary presheaf on $SpecA$ and $D^{op}: \mathcal{D}_A^{op} \longrightarrow \mathcal{O}(SpecA)^{op}$ is induced by D).

__Theorem.__ *The presheaf* $\tilde{A}$ *on* $SpecA$ *described above is a sheaf.* □

__*Exercises.*__ D) Prove that the stalk above $P \in SpecA$ for the etale space associated with $\tilde{A}$ is A_P.

E) When A is a Boolean algebra, $\tilde{A}$ is the sheaf of 2-valued Boolean homomorphisms on $SpecA$.

F) When A is the ring of regular functions from a complex affine variety V to $\mathbb{C}$, then $\tilde{A}$ "pulls back" to the sheaf of regular $\mathbb{C}$-valued functions on V.

13. Spatial topoi

A category $\mathcal{E}$ is said to be a *spatial topos* when there exists a topological space X and an equivalence of categories $\mathcal{E} \simeq \mathcal{Sh}(X,\mathcal{S})$.

Suppose $K\colon \mathcal{D} \to \mathcal{E}$ is any functor. A *limit for* K is an object L of $\mathcal{E}$ together with a natural transformation $\lambda\colon L! \to K$, where $L!$ denotes the constant functor at L, such that, if $\alpha\colon A! \to K$ is any natural transformation from a constant functor, then there exists a unique arrow $u\colon A \to L$ such that $\alpha D = \lambda D . u$ for all D in $\mathcal{D}$.

When $\mathcal{D}$ is the empty category, a limit of the unique functor $\mathcal{D} \to \mathcal{E}$ is called a *terminal object* of $\mathcal{E}$; that is, an object 1 for which there is precisely one arrow $A \to 1$ for each object A of $\mathcal{E}$.

When $\mathcal{D}$ is the category with precisely two non-identity arrows and these having the same target but different sources (there being three objects in all), then a functor $K\colon \mathcal{D} \to \mathcal{E}$ amounts to a diagram $A \xrightarrow[f]{} C \xleftarrow[g]{} B$ in $\mathcal{E}$ and a limit for K is called a *pullback of* f,g; that is, a universal commuting diagram

$$\begin{array}{ccc} P & \xrightarrow{\;q\;} & B \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle g} \\ A & \xrightarrow[\;f\;]{} & C \end{array}$$

A category $\mathcal{E}$ is called *finitely complete* when, for all finite categories $\mathcal{D}$, all functors $K\colon \mathcal{D} \to \mathcal{E}$ have limits. If $\mathcal{E}$ has pullbacks and a terminal object then $\mathcal{E}$ is finitely complete.

A category $\mathcal{E}$ is called *complete* (relative to a given category $\mathcal{S}$ of sets) when, for each category $\mathcal{D}$ whose set of arrows is in $\mathcal{S}$, all functors $K\colon \mathcal{D} \to \mathcal{E}$ have limits.

<u>*Proposition.*</u> *Spatial topoi are complete.*

Proof. Any category equivalent to a complete category is complete. Given any functor $K: \mathcal{D} \to Sh(X,S)$, each open subset U of X gives us a functor $K(-): \mathcal{D} \to S$. If, as usual, 1 denotes a set with one element, put LU equal to the set of natural transformations from $1^!$ to $K(-)U$. When $\mathcal{D}$ is small this set is in S. The restriction functions of the KD induce restriction functions between the LU determining a presheaf L. The sheaf condition is satisfied by L since it is for each KD. One checks readily that there is an obvious natural transformation $\lambda: L^! \to K$ yielding a limit for K. In other words, limits in $Sh(X,S)$ are formed pointwise. □

In a finitely complete category, a particular choice of a pullback of $A \to 1 \leftarrow B$ is denoted by $A \xleftarrow{p} A \times B \xrightarrow{q} B$; call $A \times B$ the *product* of A and B, and call p, q the *projections*.

A finitely complete category E is called *cartesian closed* when, for each pair of objects B, C, there exists an object $[B,C]$ and an arrow $\varepsilon: [B,C] \times B \to C$ such that, for each arrow $f: A \times B \to C$, there exists a unique arrow $g: A \to [B,C]$ such that the following commutes.

$$\begin{array}{ccc} A \times B & \xrightarrow{\quad f \quad} & C \\ {\scriptstyle g \times 1_B}\downarrow & \nearrow_{\varepsilon} & \\ [B,C] \times B & & \end{array}$$

Proposition. *Spatial topoi are cartesian closed.*

Proof. For any object B of $Sh(X,S)$ and any open subset U of B, we obtain a subsheaf B_U of B by defining $B_U V = BV$ when $V \subset U$, and $B_U V = 0$ otherwise. For sheaves B, C, we now define $[B,C]$ to be the sheaf whose sections over U are arrows $B_U \to C_U$ in $Sh(X,S)$, and whose restriction functions are given by restriction. Then $\varepsilon U: [B,C]U \times BU \longrightarrow CU$ is given by evaluation. □

A *subobject classifier* in a category E is an object Ω together with an arrow $t: 1 \longrightarrow \Omega$ such that, for each monomorphism $m: A' \longrightarrow A$, there exists a unique arrow $\chi_m: A \longrightarrow \Omega$ such that the following square is a pullback.

$$\begin{array}{ccc} A' & \xrightarrow{\ m\ } & A \\ \downarrow & & \downarrow{\scriptstyle \chi_m} \\ 1 & \xrightarrow[\ t\ ]{} & \Omega \end{array}$$

Recall that an arrow $m: A' \longrightarrow A$ is a *monomorphism* when the following diagram is a pullback

$$\begin{array}{ccc} A' & \xrightarrow{\ 1_{A'}\ } & A' \\ {\scriptstyle 1_{A'}}\downarrow & & \downarrow{\scriptstyle m} \\ A' & \xrightarrow[\ m\ ]{} & A \end{array} .$$

<u>*Proposition.*</u> *Spatial topoi have subobject classifiers.*

<u>*Proof.*</u> $\Omega U = \{ V \in \mathcal{O}X \mid V \subset U \}$, $(tU)0 = U$. □

This leads us to define an *elementary topos* to be a finitely-complete cartesian-closed category with a subobject classifier. We do <u>not</u> ask for completeness since the concept would then depend on an external category of sets. It turns out that elementary topoi are *complete in an internal sense* which we shall probably not have time to discuss. (My 1972 lectures gave some indication.)

14. Topoi as generalized spaces

Suppose $f: X \rightarrow Y$ is a continuous function between topological spaces.

If we take a local homeomorphism $p: E \rightarrow Y$ and form the pullback

$$\begin{array}{ccc} f^*E & \longrightarrow & E \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ X & \xrightarrow[f]{} & Y \end{array}$$

(so that $f^*E = \{ (x,e) \mid fx = pe \}$ as a subspace of $X \times E$) then q is also a local homeomorphism. This gives a functor $f^*: \mathcal{E}t(Y) \rightarrow \mathcal{E}t(X)$ which takes E to f^*E. By 8. we have a functor $\mathcal{S}h(Y,\mathcal{S}) \rightarrow \mathcal{S}h(X,\mathcal{S})$ which we denote by the same symbol f^*. Pulling back along f preserves finite limits, so f^* does. A functor which preserves finite limits is called *left exact.*

On the other hand, f induces a functor $f^{-1}: \mathcal{O}Y \rightarrow \mathcal{O}X$ given by $f^{-1}V = \{ x \in X \mid fx \in V \}$. This leads to a functor $f_*: [(\mathcal{O}X)^{op},\mathcal{S}] \rightarrow [(\mathcal{O}Y)^{op},\mathcal{S}]$ given by $(f_*P)V = P(f^{-1}V)$. If F is a sheaf on X it is easily seen that f_*F is a sheaf on Y. So f_* restricts to a functor $f_*: \mathcal{S}h(X,\mathcal{S}) \rightarrow \mathcal{S}h(Y,\mathcal{S})$.

The functors f^*, f_* between $\mathcal{S}h(X,\mathcal{S})$ and $\mathcal{S}h(Y,\mathcal{S})$ in opposite directions are related. Suppose $p: E \rightarrow Y$ is a local homeomorphism and F is a sheaf on X. Each section $s: V \rightarrow E$ over $V \subset Y$ induces a section $\hat{s}: f^{-1}V \rightarrow f^*E$ over $f^{-1}V \subset X$ via the formula $\hat{s}(x) = (x, sfx)$. Thus each arrow of sheaves $\alpha: I \pitchfork f^*E \rightarrow F$ induces an arrow of sheaves $\hat{\alpha}: I \pitchfork E \rightarrow f_*E$ given by $(\hat{\alpha}V)s = \alpha(f^{-1}V)(\hat{s}) \in F(f^{-1}V)$. It is readily checked that the assignment $\alpha \mapsto \hat{\alpha}$ is an isomorphism between the set of arrows $I \pitchfork f^*E \rightarrow F$ and the set of arrows $I \pitchfork E \rightarrow f_*F$.

A functor $S: \mathcal{F} \to \mathcal{E}$ is said to be *left adjoint* to a functor $T: \mathcal{E} \to \mathcal{F}$ when, for each pair of objects E, F of $\mathcal{E}$, $\mathcal{F}$, respectively, there is an isomorphism between the set of arrows $u: SF \to E$ in $\mathcal{E}$ and the set of arrows $v: F \to TE$ in $\mathcal{F}$ such that, if u, v correspond and $m: E \to E'$, $n: F' \to F$ are any arrows in $\mathcal{E}$, $\mathcal{F}$, respectively, then $mu(Sn)$ and $(Tm)vn$ also correspond. Write $S \dashv T$. Adjoints are uniquely determined up to isomorphism by the other functor. Left adjoints preserve all colimits which exist and right adjoints (in the above T is a right adjoint for S) preserve all limits which exist. If $S \dashv T$ and $S' \dashv T'$ and $T'T$ is defined then $SS' \dashv T'T$. Note that the theorem of Kan can be restated as: $- \otimes T \dashv T \pitchfork -$.

<u>*Proposition.*</u> *For a continuous function* $f: X \to Y$, *the functor* $f^*: Sh(Y,S) \to Sh(X,S)$ *is a left exact left-adjoint for* $f_*: Sh(X,S) \to Sh(Y,S)$. □

missing material

<u>*Proof.*</u> A geometric morphism $M: S \to Sh(Y,S)$ is determined up to isomorphism by its left-exact left-adjoint M^*. The etale space construction shows that every local homeomorphism is a colimit of open subsets. Since M^* preserves colimits, it suffices to know M^* on the open subsets of Y. Since M^* preserves products, for any open V, the diagonal function $M^*V = M^*(V \cap V) \to M^*V \times M^*V$ is an isomorphism. So M^*V has at most one element. So it suffices to know for which V the set M^*V is empty. Let W denote the union of the open V for which M^*V is empty. Then W is the largest open for which M^*W is empty (the arrows $V \to M0$ corresponding to $M^*V = 0$ are compatible on intersections and so induce an arrow $W \to M0$, so $M^*W = 0$). Thus $M^*V = 0$ if and only if $V \subset W$. So M is uniquely determined up to isomorphism by W. We claim $\neg W$ is irreducible. For suppose $\neg W = C \cup D$ where C, D are closed. Then

$W = \neg C \cap \neg D$ which is a product in $Et(Y)$. So $0 = M^*W \cong M^*(\neg C) \times M^*(\neg D)$. So $\neg C \subset W$. So $C = \neg W$ is not a proper subset of $\neg W$.

Thus we have described a monomorphism which takes the isomorphism class of M to $\neg W$. Suppose K is an irreducible closed subspace of Y. Let $K^*\colon \mathcal{O}Y \longrightarrow S$ denote the functor given by $K^*U = 1$ when $K \cap U \neq 0$ and $K^*U = 0$ otherwise. Irreducibility of K amounts precisely to the statement: $K^*(U \cap V) \cong K^*U \times K^*V$. By the theorem of Kan, K^* extends to a functor $M^*\colon Et(Y) \longrightarrow S$ such that functions $M^*E \longrightarrow S$ are in bijection with natural transformations $I \pitchfork E \longrightarrow K^* \pitchfork S$. For general reasons, left exactness of K^* implies that of M^*. Let M be the composite functor:

$$S \xrightarrow{K^* \pitchfork -} [(\mathcal{O}Y)^{op}, S] \xrightarrow{I \otimes -} Et(Y).$$

Then $M^* \dashv M$, so M is a geometric morphism. Moreover, $\bigcup\{V \mid M^*V = 0\} = \bigcup\{V \mid V \cap K = 0\} = \bigcup\{V \mid V \subset \neg K\} = \neg K$. So we have the desired isomorphism. □

Given any topological space X, let σX denote the set of irreducible closed subspaces of X. We have an inclusion $X \longrightarrow \sigma X$ which takes x to the closure of $\{x\}$. Give σX the topology with the largest set of open sets such that this inclusion is continuous.

Exercise G. Prove that $X \longrightarrow \sigma X$ induces an isomorphism of categories $\mathcal{O}(\sigma X) \cong \mathcal{O}X$ and that σX is sober. Any continuous function $X \longrightarrow Y$ with Y sober uniquely factors through $X \longrightarrow \sigma X$.

Corollary. The category of sober spaces and continuous functions is equivalent to the category of spatial topoi and isomorphism classes of geometric morphisms.

Proof. If E is a spatial topos equivalent to $Sh(X,S)$ then, by

Exercise G, it is also equivalent to $Sh(\sigma X,S)$ and σX is sober.

Suppose X, Y are sober and $M: Sh(X,S) \longrightarrow Sh(Y,S)$ is a geometric morphism. Each point $x: 1 \longrightarrow X$ of X yields a geometric morphism $x_*: S \longrightarrow Sh(X,S)$; it corresponds to the irreducible closure of $\{x\}$. Composing we obtain a geometric morphism $Mx_*: S \longrightarrow Sh(Y,S)$ which by the above Proposition yields an irreducible closed subspace of Y which must be the closure of a unique singleton $\{fx\}$ since Y is sober. Thus we have described a function $f: X \longrightarrow Y$; we leave the proof of continuity and $f^* \cong M^*$ to the reader. So isomorphism classes of geometric morphisms $Sh(X,S) \longrightarrow Sh(Y,S)$ are in bijection with continuous functions $X \longrightarrow Y$. □

15. First-order language and interpretation.

A great deal of what follows applies to categories E more general than elementary topoi. For illustrative purposes we shall suppose E is the category $Sh(X,S)$ for some fixed topological space X.

For each object A of E we suppose we have a supply of symbols $a, a', \ldots$ called *variables of type* A.

A term of type A is an expression $f(a_1,\ldots,a_n)$ where $f: A_1 \times \ldots \times A_n \longrightarrow A$ is an arrow of E and $a_1,\ldots,a_n$ are variables of type $A_1,\ldots,A_n$, respectively. We make the obvious substitution conventions. For example, given $g: B \times C \longrightarrow A_1$, $h: D \longrightarrow A_2$, $f: A_1 \times A_2 \longrightarrow A$ and variables b, c, d of type B, C, D, we write $f(g(b,c),h(d))$ for the term $((fk)(b,c,d))$ of type A where $k: B \times C \times D \longrightarrow A_1 \times A_2$ is the unique arrow whose first projection is g and second projection is h.

We define inductively what is meant by *a formula in the (free) variables* $a_1,\ldots,a_m$.

(i) For terms $f(a_1,\ldots,a_m)$, $g(b_1,\ldots,b_n)$ of the *same type*, $f(a_1,\ldots,a_m) = g(b_1,\ldots,g_n)$ is a formula in the variables $a_1,\ldots,a_m$, $b_1,\ldots,b_n$.

(ii) For a term $f(a_1,\ldots,a_m)$ of type A and a subsheaf A' of A, $f(a_1,\ldots,a_m) \in A'$ is a formula in the variables $a_1,\ldots,a_m$.

(iii) If ϕ, ψ are formulas with no constrained variable of one (see (iv)) appearing in the other, then $\phi \wedge \psi$, $\phi \vee \psi$, $\phi \Rightarrow \psi$ are each formulas whose variables are the union of those for ϕ and ψ, and $\neg\phi$ is a formula with the same variables as ϕ.

(iv) If ϕ is a formula in the distinct variables $a, b_1,\ldots,b_n$ where a is of type A, then $\exists_{a\in A}$ and $\forall_{a\in A}$ are formulas in the variables $b_1,\ldots,b_n$; also a is called a *dummy* or *constrained variable* in any formula of which these formulas are a part.

Next we shall explain how to interpret formulas. The interpretation of a formula ϕ in the variables $a_1,\ldots,a_m$ of type $A_1,\ldots,A_m$ is a subsheaf of $A_1 \times\ldots\times A_m$ which we shall describe inductively and denote by:

$$\{ (a_1,\ldots,a_m) \in A_1 \times\ldots\times A_m \mid \phi(a_1,\ldots,a_m) \}.$$

(i) The interpretation of $f(a_1,\ldots,a_m) = g(b_1,\ldots,b_n)$ is the equalizer of the two arrows $P \to A_1 \times\ldots\times A_m \xrightarrow{f} C$, $P \to B_1 \times\ldots\times B_n \xrightarrow{g} C$ where P is the product of the types of the *distinct* variables.

(ii) The interpretation of the formula $f(a_1,\ldots,a_m) \in A'$ is the pull-back of the inclusion $A' \to A$ along f.

(iii) Suppose the interpretations of ϕ, ψ are the subsheaves A, B of C. Then the interpretations of $\phi\wedge\psi$, $\phi\vee\psi$, $\phi\Rightarrow\psi$, $\neg\phi$ are the subsheaves of C given as follows. For each open subset U of X:

$\{ c \in C \mid \phi \wedge \psi \}U = \{ s \in CU \mid s \in AU$ and $s \in BU \}$,

$\{ c \in C \mid \phi \vee \psi \}U = \{ s \in CU \mid$ there exists an open covering (U_λ) of U such that $s|_{U_\lambda}$ belongs to either AU_λ or BU_λ for each $\lambda \}$,

$\{ c \in C \mid \phi \Longrightarrow \psi \}U = \{ s \in CU \mid$ if $s|_V \in AV$ for $V \subset U$ open then $s|_V \in BV \}$,

$\{ c \in C \mid \neg\phi \}U = \{ s \in CU \mid s|_V \notin AV$ for all $V \subset U$ open $\}$.

(iv) Suppose the interpretation of ϕ is the subsheaf A of $B \times C$. Then the interpretations of $\exists_{b\in B}\phi$, $\forall_{b\in B}\phi$ are given as follows. For open $U \subset X$:

$\{ c \in C \mid \exists_{b\in B}\phi(b,c) \}U = \{ s \in CU \mid$ there exist an open cover (U_λ) of U and, for each λ, a $t_\lambda \in BU_\lambda$ such that $(t_\lambda, s|_{U_\lambda}) \in AU_\lambda \}$,

$\{ c \in C \mid \forall_{b\in B}\phi(b,c) \}U = \{ s \in CU \mid (t, s|_V) \in AV$ for all $V \subset U$ open and all $t \in BV \}$.

A formula ϕ is said to be *valid* (in this interpretation) when its interpretation $\{ c \in C \mid \phi \}$ is C (that is, the whole sheaf C and not a proper subsheaf of it).

For $U \subset X$ open, let's agree to write U for the sheaf of sections of the inclusion $U \rightarrow X$. In particular, the terminal object 1 of E is denoted by X. *Subsheaves of* X *are precisely the open subsets* U. The interpretation of the formula $x \in U$ (where x has type X) is just U. The interpretation of $\neg(x \in U)$ can be seen to be the interior of the complement of U. Now the interior of the complement of the interior of the complement of an open subset U of X need *not* be U (for example, $X = [0,1]$, $U = [0,\frac{1}{2}) \cup (\frac{1}{2},1]$). Hence the formulas ϕ, $\neg\neg\phi$ do not generally have the same interpretation. The formulas $\phi \Longrightarrow \neg\neg\phi$ and $\neg\phi \Longleftrightarrow \neg\neg\neg\phi$ are valid however. In classical logic we have the law of the excluded middle: $\phi \dot{\vee} \neg\phi$ is always valid. This law does *not* hold in a general topos; the logic of a topos is *intuitionistic*, not classical. In classical logic $\exists$ can be defined in terms of $\forall$ by means of $\exists = \neg\forall\neg$. In intuitionistic logic the validity of $\forall_{b\in B}\neg\phi(b,c)$ does not imply the validity of $\neg\exists_{b\in B}\phi(b,c)$. Of course there are spaces X in which every open subset is also closed (for example, the Stone space of a Boolean algebra); then the first-order logic of $Sh(X,S)$ is classical.

16. Sheaves of models of geometric theories

A *presheaf on a space* X *with values in an arbitrary category* is a functor $P: (OX)^{op} \longrightarrow C$. So we have a category $[(OX)^{op},C]$ of C-valued presheaves on X. Refer to the definition of a sheaf on page 8. This definition cannot be transferred *verbatim et litteratum* to C-valued presheaves since it involves elements s_V of FV and for a C-valued presheaf F we just have that FV is an object of C. As usual in categories we replace the elements in the set case by arrows. So a C-valued presheaf F is said to be a *sheaf* when, for each family $\mathcal{U}$ of open subsets of X, each object C of C, and each family of arrows $s_V: C \longrightarrow FV$, $V \in \mathcal{U}$, such that ... there exists a unique arrow $s: C \longrightarrow FU$ ... . Write $Sh(X,C)$ for the category of C-valued sheaves and all natural transformations between them.

In particular we have the notions of *group-valued*, *ring-valued*, and *small-category-valued sheaves*.

On the other hand, in any category E with finite limits we can speak of models of such finitary algebraic theories. For example, a *ring* *in* E is an object A together with arrows $0\colon 1 \longrightarrow A$, $\eta\colon 1 \longrightarrow A$, $\iota\colon A \longrightarrow A$, $\alpha\colon A\times A \longrightarrow A$, $\mu\colon A\times A \longrightarrow A$ (thought of as "zero, one, minus, add, multiply") satisfying certain axioms which can all be expressed by commuting diagrams. In particular, the axioms $a+(b+c) = (a+b)+c$, $a+(-a) = 0$ for rings become the commuting diagrams:

$$
\begin{array}{ccc}
A\times A\times A & \xrightarrow{1\times\alpha} & A\times A \\
{\scriptstyle \alpha\times 1}\downarrow & & \downarrow{\scriptstyle \alpha} \\
A\times A & \xrightarrow[\alpha]{} & A
\end{array}
\quad , \qquad
\begin{array}{ccccc}
A & \xrightarrow{\text{diag}} & A\times A & \xrightarrow{1\times\iota} & A\times A \\
\downarrow & & & & \downarrow{\scriptstyle \alpha} \\
1 & & \xrightarrow[0]{} & & A
\end{array}
\quad .
$$

As another example, *a category* C *in* E consists of an object C_0 ("of objects"), an object C ("of arrows"), arrows $d_0, d_1 : C_1 \longrightarrow C_0$, $i\colon C_0 \longrightarrow C_1$, $c\colon C_2 \longrightarrow C_1$ ("which assign domains, codomains, identity arrows, composites") where C_2 is the pullback of d_0, d_1 ("the object of composable pairs") satisfying certain commutative diagrams which involve a further pullback.

Models of such an algebraic theory in a category are taken to models of the same theory in another category by any left-exact functor between the two categories.

Homomorphisms of models of a theory in E are defined in the obvious way as structure-preserving families of arrows between the families of objects involved in the models. Write $Mod(\mathbb{T},E)$ for the category whose objects are models of the theory $\mathbb{T}$ in E and whose arrows are homomorphisms. For example, if $\mathbb{T}$ is the theory of rings and E is the category of sets then $Mod(\mathbb{T},E)$ is the category of rings.

Theorem. For any algebraic theory $\mathbb{T}$ *and topological space* X, *the categories* $Sh(X, Mod(\mathbb{T},S))$ *and* $Mod(\mathbb{T}, Sh(X,S))$ *are equivalent.* □

In particular, a ring-valued sheaf is essentially the same thing as a ring in the category of (set-valued) sheaves. However, if non-equational axioms are allowed in the theory this happy situation is upset. In 11. we met the concept of a local ring; the extra axiom is *not* equational. The stalks of the sheaf $\tilde{A}$ on *Spec*A are local rings A_p so one would like to say that $\tilde{A}$ is a local ring in the category of sheaves on *Spec*A. However, $\tilde{A}\ D(a) = A[a^{-1}]$ is certainly not a local ring in general (for example, $\mathbb{Z}[2^{-1}]$ is the ring of rationals of the form $n/2^m$ where m, n are integers, and neither $3/2^m$ nor $1-\frac{3}{2^m}$ is invertible); so $\tilde{A}$ *is not a local-ring-valued sheaf on* *Spec*A.

The logic of a topos E allows us to define local rings in it. For a ring A in E, the terms of type A corresponding to 0, η are denoted by 0, 1, and the terms a, α(a,b), μ(a,b) are denoted by -a, a+b, ab. We define A to be *local* when the formula

$$(\exists_{b\in A} ab = 1) \vee (\exists_{c\in A}(1-a)c = 1),$$

is valid for a of type A and the formula $\neg(1=0)$ interprets as the terminal object 1. It can be shown that for a geometric morphism $M: F \rightarrow E$ between topoi, if A is a local ring in E then M^*A is a local ring in F. For each point $x: 1 \rightarrow X$ of a space X, the functor $x^*: Sh(X,S) \rightarrow S$ assigns to each sheaf the stalk at x of the corresponding etale space. So if A is a local ring in $Sh(X,S)$ then the stalks of the etale space of A are all local rings. The converse is actually also true; however, for elementary topoi we must rely on the definition of local ring as given in terms of formulas.

<u>*Theorem.*</u> *For any commutative ring* A *with a* 1, *the structure sheaf* $\tilde{A}$ *is a local ring in* *Sh*(*Spec*A,*S*). □

A ring homomorphism $f: A \longrightarrow A'$ in E is called *local* when the following formula is valid:

$$(\exists_{b\in A} ab = 1) \Longleftrightarrow (\exists_{a'\in A'} f(a)a' = 1).$$

It is not our purpose here (however, see 22.) to describe general *geometric theories*. The logical operations allowed in their description are true, false, $\wedge$, $\vee$, $\exists$; the axioms allowable are assertions of validity of formulas of the form $\phi \Rightarrow \psi$ where ϕ, ψ are formulas involving only true, false, $\wedge$, $\vee$, $\exists$. The symbols $\neg$, $\forall$ are not allowed and $\Rightarrow$ is only allowed in the axioms in the manner explained. It is then true that left adjoints of geometric morphisms take models of a geometric theory to models of the same theory.

17. Higher-order language

First-order language involves the symbols τ, ϕ, $=$, $\wedge$, $\vee$, $\neg$, $\Rightarrow$, $\exists$, $\forall$ read "true, false, equals, and, or, not, implies, there exists, for all". To exemplify the distinction between this and a higher-order language we shall look at the definition of a preordered set and related concepts. A *poset* P consists of *element symbols* x, y, z, ... and *predicate symbols* $P(x,y)$ satisfying certain axioms the first of which is:

$$\forall_x \forall_y \big(P(x,y) = \tau\big) \vee \big(P(x,y) = \phi\big).$$

Define $(x \le y) = \big(P(x,y) = \tau\big)$; then the other axioms are:

$$\forall_x (x \le x), \qquad \forall_x \forall_y \forall_z (x \le y) \wedge (y \le z) \Rightarrow (x \le z).$$

We say that P has a *least element* when $\exists_0 (\forall_x 0 \le x)$. We say that P is an *upper semi-lattice* when

$$\forall_x \forall_y \exists_u \big((x \le u) \wedge (y \le u)\big) \wedge \forall_z \big((x \le z) \wedge (y \le z) \Rightarrow u \le z\big).$$

All statements in the language must be expressible by a finite string of symbols. An upper semi-lattice with a least element has a supremum for any given finite string of elements $x_1,\ldots,x_n$; that is, for all intents and purposes, for any natural number n, we have

$$\forall_{x_1}\ldots\forall_{x_n}\exists_u((x_1 \le u)\wedge\ldots\wedge(x_n \le u)) \wedge \forall_z((x_1 \le z)\wedge\ldots\wedge(x_n \le z) \Rightarrow u \le z).$$

Universal quantification over a finite string of elements $x_1,\ldots,x_n$ is defined by $\forall_{x_1,\ldots,x_n} = \forall_{x_1}\ldots\forall_{x_n}$. A *complete lattice* is a poset P satisfying:

$$\forall_{S\subset P}\exists_u\Big[\forall_x(x \in S \Rightarrow x \le u) \wedge \forall_z\big(\forall_x(x \in S \Rightarrow x \le z) \Rightarrow u \le z\big)\Big].$$

The quantifier $\forall_{S\subset P}$ cannot be defined in the first-order language; we cannot quantify over an infinite "set" of element symbols. Higher-order languages allow us to do this. Given the first-order language and a model of set theory of course we can take the elements of P to form a set in the set theory and then $\forall_{S\subset P}$ can be defined. But this is more data than is really required.

Let us return to the language and its interpretation in $Sh(X,S)$ as described in 15. A *sentence* is a formula with no free variables. The interpretation of a sentence is thus a subobject of the empty product (= terminal object) X; that is, an open subset U of X, called *the truth-value* of the sentence. A sentence is *true* when its interpretation is X and *false* when its interpretation is the empty subset 0 of X; but these are not the only possibilities.

An *element* of an object A is an arrow $X \longrightarrow A$ from the terminal object to A. This amounts to a section of the sheaf A over the whole space; that is, a *global section* of A. In general, objects are not determined by their elements (as in the case of sets); sheaves are not determined by their global sections.

Elements of Ω correspond to subsheaves of X; that is, to open subsets of X. So Ω is regarded as *the object of truth-values*. For a formula ϕ, the interpretation of ϕ is a subobject $\{ c \in C \mid \phi \}$ of C and so yields an arrow $|\phi|: C \longrightarrow \Omega$; arrows into Ω are called *predicates* and $|\phi|$ is *the predicate associated with* ϕ.

Elements of $[A,B]$ correspond to arrows $A \longrightarrow B$. For a variable a of type A and a variable f of type $[A,B]$, the term $\varepsilon(f,a)$ of type B is denoted by $f(a)$ where $\varepsilon: [A,B] \times A \longrightarrow B$ is the evaluation arrow.

It can be shown that an arrow $f: A \longrightarrow B$ is a monomorphism if and only if the sentence

$$\forall_{a \in A} \forall_{a' \in A} \, f(a) = f(a') \Rightarrow a = a'$$

is true. Also $f: A \longrightarrow B$ is an epimorphism (that is, right cancellable) if and only if the sentence

$$\forall_{b \in B} \exists_{a \in A} f(a) = b$$

is true. But now with the notation of the last paragraph we can take f to be a variable of type $[A,B]$ so that the two sentences displayed above become formulas with free variable f; their interpretations give subobjects of $[A,B]$ called *the object of monomorphisms* and *the object of epimorphisms* from A to B.

Elements of $[A,\Omega]$ correspond to predicates $A \longrightarrow \Omega$ which correspond to subobjects of A. So we think of $[A,\Omega]$ as *the object of subobjects of* A, or *the power object of* A.

We now *extend the language* by admitting as a formula the expression

$$a \in A'$$

where a is a variable of type A and A' is a variable of type $[A,\Omega]$. The interpretation of this formula is the subobject of $[A,\Omega] \times A$ corresponding to the evaluation arrow $\varepsilon\colon [A,\Omega] \times A \longrightarrow \Omega$. For a formula ϕ with free variables a, b, we define $\exists_{a \in A'} \phi(a,b)$ to be $\exists_{a \in A} a \in A' \wedge \phi(a,b)$, and we define $\forall_{a \in A'} \phi(a,b)$ to be $\forall_{a \in A} a \in A' \Rightarrow \phi(a,b)$. Thus we have extended quantification to "quantification over subobjects".

A *relation* from A to B is a subobject R of $A \times B$. A relation R from A to B is said to be *functional* when the following two sentences are true:

$$\forall_{a \in A} \forall_{b \in B} \forall_{b' \in B} (a,b) \in R \wedge (a,b') \in R \Rightarrow b = b'$$

$$\forall_{a \in A} \exists_{b \in B} (a,b) \in R.$$

It can be proved that R from A to B is functional if and only if there is an arrow $f\colon A \longrightarrow B$ whose graph is R (that is, the monomorphisms $R \longrightarrow A \times B$ and $\begin{pmatrix} 1_A \\ f \end{pmatrix}\colon A \longrightarrow A \times B$ are isomorphic). But now with the extension of the last paragraph we can take R to be a variable of type $[A \times B, \Omega]$ so that the two sentences displayed above become formulas in R. It can be proved that the interpretation of their conjunction is the subobject $[A,B]$ of $[A \times B, \Omega]$.

18. The language for elementary topoi.

Let E denote an elementary topos. For each object A of E, let *Sub*A denote the preordered set of monomorphisms into A. We often write A' instead of the monomorphism $A' \longrightarrow A$. It can be shown that *Sub*A is a lattice with smallest element 0 and largest element A. Moreover, for each pair of elements A', A'', there is an element $A'' \Rightarrow A''$ such that $A''' \leq A' \Rightarrow A''$ if and only if $A''' \wedge A' \leq A''$. This means *Sub*A is a *Heyting algebra*. Define $\neg A'$ to be $A' \Rightarrow 0$. Then $A' \leq \neg\neg A'$ but we do *not* have $\neg\neg A' \leq A'$.

For any arrow $f\colon A \longrightarrow B$, pullback along f gives an order preserving function $f^{*}\colon \mathit{Sub}B \longrightarrow \mathit{Sub}A$. It can be shown that there are also order preserving functions $\exists_f, \forall_f\colon \mathit{Sub}A \longrightarrow \mathit{Sub}B$ determined by the following conditions:

$$\exists_f A' \leq B' \quad \text{if and only if} \quad A' \leq f^{*}B',$$

$$B' \leq \forall_f A' \quad \text{if and only if} \quad f^{*}B' \leq A'.$$

Exercises *H*) Describe $\exists_f, \forall_f$ in the case where E is the category of sets.

I) Modify the interpretation of formulas as given in 15. so that only the Heyting algebra structures on the ordered sets $\mathit{Sub}A$ together with the functions $\exists_f, \forall_f$ are used. With this modification we get the language and interpretation for an elementary topos.

J) The predicate $|a \in A' \wedge a \in A''|\colon [A,\Omega]\times[A,\Omega]\times A \longrightarrow \Omega$ corresponds to an arrow $\cap\colon [A,\Omega]\times[A,\Omega] \longrightarrow [A,\Omega]$. For variables A', A'' of type $[A,\Omega]$ define the formula $A' \leq A''$ to mean $\cap(A',A'') = A'$. The interpretation of $A' \leq A''$ defines *the order relation on* $[A,\Omega]$. Prove that the following sentences are true:

$$\forall_{B\in[A,\Omega]}\forall_{A'\in[A,\Omega]}\forall_{A''\in[A,\Omega]}(B \leq \cap(A',A'')) \Leftrightarrow (B \leq A' \wedge B \leq A''),$$

$$\forall_{A'\in[A,\Omega]}\forall_{A''\in[A,\Omega]}\exists_{C\in[A,\Omega]}\forall_{B\in[A,\Omega]} B \leq C \Leftrightarrow \cap(B,A') \leq A'',$$

$$\forall_{F\in[[A,\Omega],\Omega]}\exists_{S\in[A,\Omega]}((\forall_{A'\in F} A' \leq S) \wedge \forall_{B\in[A,\Omega]}(\forall_{A'\in F} A' \leq B \Rightarrow S \leq B)).$$

These sentences express the internal sense in which $[A,\Omega]$ is a complete Heyting algebra (it is not in general Boolean).

19. Intuitionistic mathematics

"All the theorems of intuitionistic mathematics are true when interpreted in an elementary topos E." A little care must be taken: the early writers on intuitionism did not take into account the empty type 0, however, their work can be easily modified to do this. The solution is not to leave 0 out of the category since quite respectable sheaves can have some empty stalks and these cause the same difficulty; the solution is to modify the rules of deduction. Of course, for special topoi E there will be *more* theorems. When E is the category of sheaves on a space, the validity of a formula can be tested by reducing it (using the interpretations of 15.) to a statement about sheaves and then using the methods of classical mathematics. The alternative is to establish a suitable list of basic theorems (*axioms*) and a list of basic allowable deductions (*rules of inference*) which then allow theorems to be proved within the language by a sequence of deductions from the axioms.

Suppose we work now in a fixed elementary topos E. A *theorem* is a valid formula. For all formulas ϕ, ψ, χ and all terms t in which a is not a variable, the following are theorems:

$$a = a, \qquad a = a' \Rightarrow \phi(a) = (a')$$

$$\phi \Rightarrow \phi, \quad \phi \wedge \psi \Rightarrow \phi, \quad \phi \wedge \psi \Rightarrow \psi \wedge \phi, \quad \phi \Rightarrow \phi \vee \psi,$$

$$(\chi \Rightarrow \phi) \wedge (\chi \Rightarrow \psi) \Rightarrow (\chi \Rightarrow \phi \wedge \psi), \qquad ((\phi \Rightarrow \chi) \wedge (\psi \Rightarrow \chi)) \Rightarrow (\phi \vee \psi \Rightarrow \chi),$$

$$(\phi \wedge \psi) \wedge \chi \Rightarrow \phi \wedge (\psi \wedge \chi), \qquad (\chi \Rightarrow (\phi \Rightarrow \psi)) \Longleftrightarrow (\chi \wedge \phi \Rightarrow \psi),$$

$$\phi(t) \Rightarrow \exists_{a \in A} \phi(a), \qquad \forall_{a \in A} \phi(a) \Rightarrow \phi(t).$$

These formulas will be called *axioms*.

Given formulas $\phi_1, \ldots, \phi_n$, ϕ, if under the assumption that $\phi_1, \ldots, \phi_n$ are valid it can be proved in the topos E that ϕ is valid, we write

$$\frac{\phi_1,\ldots,\phi_n}{\phi}\,E.$$

For formulas ϕ, ψ such that the free variables of ϕ are also free in ψ and for a formula χ in which a is not a free variable, we have the following for any E:

$$\frac{\phi,\ \phi\Rightarrow\psi}{\psi},\qquad \frac{\chi\Rightarrow\phi(a)}{\chi\Rightarrow\forall_{a\in A}\phi(a)},\qquad \frac{\phi(a)\Rightarrow\chi}{\exists_{a\in A}\phi(a)\Rightarrow\chi}.$$

In each case we shall say the formula on the bottom is obtained from those or that on the top by a *rule of inference*; these are the three rules of inference.

A *deduction* of the formula ψ from the formulas $\phi_1,\ldots,\phi_n$ is a finite sequence $\Gamma_1,\ldots,\Gamma_m$ of finite sets of formulas Γ_i such that $\Gamma_1 = \{\phi_1,\ldots,\phi_n\}$, $\Gamma_m = \{\psi\}$, and each formula in Γ_i is either an axiom, an element of some Γ_j with $j < i$, or obtained from elements of Γ_{i-1} by a rule of inference. Write $\phi_1,\ldots,\phi_n \vdash \psi$ when there exists a deduction of ψ from $\phi_1,\ldots,\phi_n$. Clearly $\phi_1,\ldots,\phi_n \vdash \psi$ implies

$$\frac{\phi_1,\ldots,\phi_n}{\psi}\,E\ .$$

<u>*Completeness theorem.*</u> *If, for all elementary topoi* E,

$$\frac{\phi_1,\ldots,\phi_n}{\psi}\,E$$

then $\phi_1,\ldots,\phi_n \vdash \psi$. □

This formalizes the connection between elementary topoi and intuitionistic logic.

Given a theorem of classical mathematics, we try to adjust the proof into intuitionistic deductions. This may not be possible if an essential use is made of the axiom of choice or even of proof by contradiction. If it is possible we obtain a theorem in any topos.

20. Coordinate-free definitions of local structures.

A local-ringed space is a pair (X,R) where X is a topological space and R is a local ring in $Sh(X,S)$. *A morphism of local-ringed spaces* $(f,\phi): (X,R) \longrightarrow (Y,S)$ consists of a continuous function $f: X \longrightarrow Y$ and a local ring homomorphism $\phi: f^*S \longrightarrow R$ in $Sh(X,S)$.

Let us formulate the definition of a differentiable manifold. For each open subset G of $\mathbb{R}^n$, we have a local-ringed space (G,D_G) where D_G is the sheaf of differentiable $\mathbb{R}$-valued functions on G. *A differentiable manifold* is a local-ringed space (X,R) such that each point of X has an open neighbourhood U for which (U,i_U^*R) is isomorphic to a local-ringed space of the form (G,D_G) where G is an open subset of some $\mathbb{R}^n$. Then of course, R is the sheaf of differentiable $\mathbb{R}$-valued functions on X. *A differentiable function* between differential manifolds is precisely a morphism of the corresponding local-ringed spaces.

In the above, the important property that the model local-ringed spaces (G,D_G) have is that, for each open $H \subset G$, we obtain a monomorphism $(H,D_H) \longrightarrow (G,D_G)$.

For each ring A (commutative with a 1), the local-ringed space $(Spec A,\tilde{A})$ is called an *affine scheme*. Given an open subset $H = \bigcup_\lambda D(a_\lambda)$ of $Spec A$, let I denote the ideal of A generated by all the a_λ, and let B denote the quotient ring A/I. It can be shown that there is a homeomorphism $\mathit{Spec}B \cong H$ given by $P/I \longmapsto P$ (where P is a prime ideal of A containing I), and we obtain a monomorphism $(\mathit{Spec}B,\tilde{B}) \longrightarrow (\mathit{Spec}A,\tilde{A})$.

Theorem. The dual of the category of commutative rings with 1 and homomorphisms of rings which preserve 1 is equivalent to the category of affine schemes and local-ringed space morphisms.

Proof. Given $(\mathit{Spec}A,\tilde{A})$, we can recapture A as the ring of global sections of $\tilde{A}$ (the continuous function $\mathit{Spec}A \longrightarrow 1$ induces a geometric morphism from sheaves on $\mathit{Spec}A$ to S which takes the local ring $\tilde{A}$ to the local ring A).

Given a ring homomorphism $\theta: B \longrightarrow A$, we obtain a continuous function $f: \mathit{Spec}A \longrightarrow \mathit{Spec}B$ which takes P to $\theta^{-1}P$. For each $b \in B$, the composite $B \xrightarrow{\theta} A \longrightarrow A[\theta(b)^{-1}]$ inverts b and so induces a homomorphism $B[b^{-1}] \longrightarrow A[\theta(b)^{-1}]$ which is a homomorphism $\tilde{B}\,D(b) \longrightarrow \tilde{A}\,D(\theta b) = (f_*\tilde{A})D(b)$. These components extend to an arrow $\phi.: \tilde{B} \longrightarrow f_*\tilde{A}$ of sheaves which corresponds to an arrow $\phi: f^*\tilde{B} \longrightarrow \tilde{A}$. The assignment $\theta \longmapsto (f,\phi)$ is an isomorphism between the set of ring homomorphisms $\theta: B \longrightarrow A$ and the set of morphisms of local-ringed spaces $(f,\phi): (\mathit{Spec}A,\tilde{A}) \longrightarrow (\mathit{Spec}B,\tilde{B})$. □

A scheme is a local-ringed space (X,R) such that each point of X has an open neighbourhood U for which (U,i_U^*R) is isomorphic to an affine scheme. *A morphism of schemes* is just a morphism of the corresponding local-ringed spaces. An affine variety gives rise to an affine scheme which is a scheme, and the regular functions between affine varieties correspond to morphisms of schemes. However, the notion of scheme includes other concepts of algebraic geometry, for example, the notion of *projective variety* (= subset of projective space given by zeros of a family of homogeneous polynomials).

In keeping with the spirit that a topos is a generalized space, one is led to define a *local-ringed topos* to be a pair (E,R) where E is a topos and R is a local ring in E. *A morphism of local-ringed topoi* is a pair $(f,\phi)\colon (E,R) \to (F,S)$ where $f\colon E \to F$ is a geometric morphism and $\phi\colon f^{*}S \to R$ is a local ring homomorphism in E. A *transformation* $\alpha\colon (f,\phi) \to (f',\phi')$ between such morphisms is a natural transformation $\alpha\colon f \to f'$ such that the diagram

$$\begin{array}{ccc} S & \xrightarrow{\ \phi.\ } & fR \\ & \underset{\phi'.}{\searrow} & \downarrow{\scriptstyle \alpha R} \\ & & f'R \end{array}$$

commutes where $\phi.$, $\phi'.$ correspond to ϕ, ϕ' under the adjunctions $f^{*} \dashv f$, $f'^{*} \dashv f'$. The category of local-ringed topoi and isomorphism classes of morphisms contains "all categories of local structures" as full subcategories.

21. Grothendieck topologies.

We saw in the last section that an open subset of *Spec*A could be regarded as a special kind of monomorphism $(\mathit{Spec}B,\tilde{B}) \to (\mathit{Spec}A,\tilde{A})$ in the category of affine schemes. The topology on *Spec*A can thus be described by the specification of certain monomorphisms into $(\mathit{Spec}A,\tilde{A})$ in the category of affine schemes. Sheaves on *Spec*A can then be identified with functors from the dual of a subcategory of the category of affine subschemes of $(\mathit{Spec}A,\tilde{A})$ into S. It was found by Grothendieck in his work on *descent* and the *étale fundamental group* that these sheaves did not provide all the information required and that the restriction to "monomorphism" above could be relaxed without affecting the sheaf condition provided one had a good notion of *covering*; then, by looking at the category of affine schemes with an "étale morphism" into $(\mathit{Spec}A,\tilde{A})$ and a suitable notion of cover, he was able to define "sheaves" containing precisely the right information.

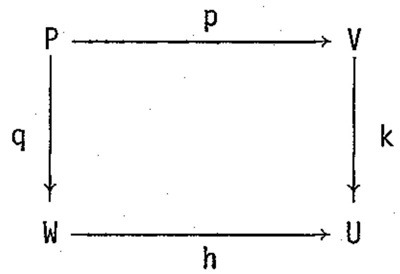
Let C denote a category. *A Grothendieck pretopology* on C is a function J which assigns to each object U of C a set $J(U)$ whose elements are sets of arrows into U, such that

- for all U, $\{1_U: U \to U\} \in J(U)$

- for all $f: V \to U$ and $\mathcal{U} \in J(U)$, each arrow in $\mathcal{U}$ has a pullback along f and the set $f^*\mathcal{U}$ of arrows into V so obtained is in $J(V)$;

- given $\mathcal{U} \in J(U)$ and $\mathcal{U}_k \in J(V)$ for each $k: V \to U$ in $\mathcal{U}$, the set $\{ kh \mid k \in \mathcal{U}, h \in \mathcal{U}_k \}$ is in $J(U)$.

A functor $F: C^{op} \to S$ is called a *sheaf for the pretopology* J when for each object U of C, each $\mathcal{U} \in J(U)$, and each family of elements $s_k \in FV$ where $k: V \to U$ runs over $\mathcal{U}$ satisfying the conditions $(Fp)s_k = (Fq)s_h$ where the square



is a pullback and $h,k \in \mathcal{U}$, there exists a unique element $s \in FU$ such that $(Fk)s = s_k$ for all $k \in \mathcal{U}$.

For any category C, a set R of arrows into an object U is said to be a U-*crible* when, for all $h: V \to U$ in R and all arrows $f: W \to V$, the composite hf is in R. We can identify a U-crible R with the subfunctor of $Y_C U: C^{op} \to S$ (see 5.) whose value at V is the set of arrows in R from V to U.

When C has pullbacks, *a Grothendieck topology* J on C is a pretopology such that the elements of each J(U) are all U-cribles. When one looks at the axioms for a pretopology in the case where each $U \in$ J(U) is a crible one finds that they can be expressed without the need for pullbacks in C. This is left to the reader. Moreover, in this case, one does not need pullbacks to define sheaf. A functor $F: C^{op} \to S$ is a *sheaf for the topology* J when for each object U of C and each $R \in$ J(U), each natural transformation $\alpha: R \to F$ extends uniquely to a natural transformation $\beta: Y_C U \to F$ (compare the theorem in 8.).

Each set U of arrows into an object U of any category C generates the crible consisting of those arrows into U which factor through some arrow in U. *The topology* $\overline{J}$ *generated by a pretopology* J on C is given by taking $\overline{J}(U)$ to be the set of cribles generated by the elements of J(U). Then *the sheaves for* $\overline{J}$ *are precisely the same as the sheaves for* J. A set of arrows into U is said to be a *covering* of U in the topology J when the crible generated by it is in $\overline{J}(U)$.

For a topological space X, we obtain a Grothendieck topology on $\mathcal{O}X$ by requiring that a set of arrows into an object U should cover precisely when the union of the sources of the arrows is all of U. The sheaves for this topology are the sheaves on X as defined previously in 8.

On any category C we have the *chaotic topology* for which the only covering crible for each object U is the set of all arrows into U. Then every functor $F: C^{op} \to S$ is a sheaf.

A category C together with a Grothendieck topology is called a *site* which we often denote also by C. Write $Sh(C,S)$ for the category of sheaves for the topology and all natural transformations between them. A site is called *small* when the underlying category is small.

Proposition. *For each small site, the category* $Sh(C,S)$ *is a complete elementary topos.*

A Grothendieck topos is a category E for which there exists a small site C and an equivalence $E \simeq Sh(C,S)$. It can be shown that C can always be taken to be finitely complete.

Since we have the chaotic topology on any category, for each small category C, the functor category $[C^{op},S]$ is a Grothendieck topos. In fact, the subobject classifier Ω in this topos is the functor $\Omega\colon C^{op} \longrightarrow S$ whose value at U is the set ΩU of U-cribles. A Grothendieck topology J on C can be thought of as a subobject of Ω in the category $[C^{op},S]$. Since Ω is the subobject classifier, the subobject J of Ω corresponds to an arrow $j\colon \Omega \longrightarrow \Omega$. In fact, Grothendieck topologies J on C are in bijection with order-preserving arrows $j\colon \Omega \longrightarrow \Omega$ such that $j^2 = j$ and $1 \leq j$. Notice that in any elementary topos we could consider an arrow $j\colon \Omega \longrightarrow \Omega$ satisfying these conditions; this is important in the development of elementary topoi.

Given a group G, we obtain a category C with only one object U, with $C(U,U) = G$, and with composition group multiplication. Then a functor $F\colon C^{op} \longrightarrow S$ amounts to a set FU on which G acts; so $[C^{op},S]$ is the category of G-sets, a Grothendieck topos.

For a topological space X, we have the Grothendieck topoi $Sh(X,S)$ and $[(OX)^{op},S]$. The inclusion of the first in the second has a left adjoint $I \,\phi\, (- \otimes I)$ given by taking a presheaf to the sheaf of sections of the associated etale space. It can be shown that this left adjoint is left exact. More generally, we have the following result.

<u>*Theorem*</u> (Grothendieck). *For each small site* C*, the inclusion of* $Sh(C,S)$ *in* $[C^{op},S]$ *is a geometric morphism.* □

For a small category C, each functor $C^{op} \longrightarrow S$ has a limit, so we have a functor

$$lim: [C^{op},S] \longrightarrow S$$

which assigns to each functor its limit. This functor has a left adjoint which preserves all limits (a set S is taken to the functor which is constant at S) and so *lim* is a geometric morphism. For a small site C, the composite

$$\Gamma : Sh(C,S) \longrightarrow [C^{op},S] \xrightarrow[\text{lim}]{} S$$

is called the *global sections functor*, although there is no analogue for a site to the etale space construction for a space.

22. Classifying topoi

For each of the algebraic theories discussed in 16. there is a finitely complete category C which we think of as *the theory* for that type of algebraic structure such that *a model* of the theory in a category E amounts (up to isomorphism) to a left exact functor $M: C \longrightarrow E$.

For example, for the theory of groups, take C to be the dual of the category of finitely presented groups and homomorphisms between them. (*A finitely presented group* is a group given by a finite number of generators and relations.) Since C^{op} has pushouts and an initial object (the group with one element), this C is finitely complete. Every finitely presented group is a finite colimit of copies of the infinite cyclic group $\langle x \rangle$, so a left exact functor $M: C \to E$ is determined on objects by $M\langle x \rangle = G$, say. Moreover, G becomes a group in E by taking the multiplication arrow $G \times G \to G$ to be the image under M of the homomorphism $\langle x \rangle \to \langle x,y \rangle$ which takes x to xy (note that the free two generator group $\langle x,y \rangle$ is $\langle x \rangle \times \langle y \rangle$ in C). Conversely, each group in E determines a left exact functor $C \to E$; also homomorphisms of groups in E correspond to natural transformations between the associated left exact functors.

Consequently, for a finitely complete category C, we write $Mod(C,E)$ for the category of left exact functors from C to E and natural transformations between them.

For elementary topoi E, F, we write $Geom(E,F)$ for the category of geometric morphisms from E to F and natural transformations between their left adjoints.

For a small category C and a cocomplete category E, we saw in 5. that each functor $T: C \to E$ gives rise to a functor $T \pitchfork - : E \to [C^{op},S]$ which has a left adjoint $- \otimes T$. Each functor $C^{op} \to S$ is a colomit of representables, and so $- \otimes T$ is determined by T (up to isomorphic) via the formula $Y_C U \otimes T \cong TU$. If E is a Grothendieck topos, T is left exact if and only if $- \otimes T$ is left exact. There is in fact an equivalence of categories:

$$(*) \qquad Mod(C,E) \simeq Geom(E,[C^{op},S]).$$

Let us look at some special cases of this equivalence. Take C^{op} to be the category of finite sets. A left exact functor $C \to E$ is determined by its value at the set 1 with one element (C is "the theory of objects"); so $Mod(C,E) \simeq E$. So geometric morphisms $E \longrightarrow [C^{op},S]$ are determined up to isomorphism by objects of E. We learnt in 14. that, for a spatial topos, geometric morphisms from S amount to points of the space. With the point of view that topoi are generalized spaces and that a *point* of a generalized space is a geometric morphism from S we see that $[C^{op},S]$ is a topos whose points are sets. So $[C^{op},S]$ *is the generalized space of all sets.*

As another example, take C to be the theory of groups as described above. A geometric morphism $E \longrightarrow [C^{op},S]$ amounts to a group in E. In particular, taking the identity geometric morphism of $[C^{op},S]$ we obtain a group G in $[C^{op},S]$ called *the generic group*. Any group in any Grothendieck topos E can be obtained as the image of G under a left adjoint for a geometric morphism (unique up to isomorphism). It may be profitable to study G; it is a special group in a particular (intuitionistic) set theory, yet any theorems holding for G which are formulas preserved by left adjoints to geometric morphisms will hold for all groups in all Grothendieck topoi.

Next we shall consider the refinement of the above which is needed to account for geometric theories.

First observe that a Grothendieck topos E can be made into a site in a canonical way. A set U of arrows into an object A is said to be *jointly epimorphic* when uk = vk for all $k \in U$ implies u = v. Put the smallest Grothendieck topology on E for which the jointly epimorphic sets are coverings. The sheaves for this site turn out to be precisely the functors which are isomorphic to representables:

$$Sh(E,S) \simeq E.$$

This means that E can be regarded as a defining site for itself; however, of course, E is not small.

If C is a finitely-complete small site and E is a Grothendieck topos, define $Mod(C,E)$ to be the category of left-exact covering-preserving functors from C to E. This equivalence of categories (*) restricts to an equivalence:

$$\boxed{Mod(C,E) \simeq Geom(E,Sh(C,S))}\ .$$

When the topology of C is chaotic this gives back (*). We shall give evidence below that a finitely-complete small site C is a good notion of *geometric theory*. We call $Sh(C,S)$ *the classifying topos* for the theory; models of the theory correspond to geometric morphisms into it. We have the result that *every Grothendieck topos is the classifying topos for a geometric theory.*

Let C denote the dual of the category of finitely presented rings (as always, commutative with 1). The objects of C are quotients of polynomial rings $\mathbb{Z}[x_1,\ldots,x_n]$ by finitely generated ideals. This C is the (algebraic) theory of rings. A left exact functor $M\colon C \to E$ amounts to a ring in E whose underlying object is $M\,\mathbb{Z}[x] = R$. Now enrich C to a site by taking the smallest Grothendieck topology such that the one-point ring has no coverings and the pair of arrows into $\mathbb{Z}[x]$ corresponding to the two homomorphisms

$$\begin{array}{ccc} \mathbb{Z}[x] & \longrightarrow & \mathbb{Z}[x,y]/(1-xy) \\ \big\downarrow & & \\ \mathbb{Z}[x,y]/(y(1-x)-1) & & \end{array}$$

(which each take x to the coset of x) is a covering. To ask that M preserve the latter covering is to ask that the two first projections

$$\begin{array}{ccc} & & \{ (a,b) \mid ab=1 \} \\ & & \downarrow \\ \{ (a,b) \mid b(1-a)=1 \} & \longrightarrow & R \end{array}$$

should be jointly epimorphic. So we see that *Mod*(C,E) is the category of local rings and homomorphisms (not just local ones!) between them in E. The topos $Sh(C,S)$ is called the *Zariski topos* ; it is the classifying topos for local rings. This topos was originally constructed by the French School using as a site a small subcategory of the category of affine schemes. The relationship can be seen using the Theorem in 10.

Just as we argued that we may as well forget the space once we know the category of sheaves, we now argue that we may as well forget the site once we have its category of sheaves. It is not easy to see in general when two sites determine equivalent topoi and hence the same geometric theory. The real invariant of the geometric theory is the classifying topos itself.

23. Analysis in a topos.

An elementary topos is said to satisfy the *axiom of infinity* or have a *natural numbers object* when:

(NNO) *there exist an object* N *and arrows* $0\colon 1 \longrightarrow N$, $suc\colon N \longrightarrow N$ *such that, for all arrows* $x_0\colon 1 \longrightarrow A$, $u\colon A \longrightarrow A$, *there exists a unique arrow* $s\colon N \longrightarrow A$ *such that* $s0 = x_0$ *and* $s\ suc = us$.

One can deduce the usual Peano axioms for N: namely, $suc\colon N \longrightarrow N$ is a monomorphism, $\exists_{n\in N}(suc(n)=0)$ is false, and, for all formulas $\phi(n)$ when n is of type N, the following is a theorem:

$$\phi(0) \wedge \forall_{m\in N}(\phi(m) \Rightarrow \phi(m+1)) \Longrightarrow \phi(n).$$

This last theorem can be added to the list of axioms in 19. and we obtain the corresponding completeness theorem for elementary topoi satisfying (NNO).

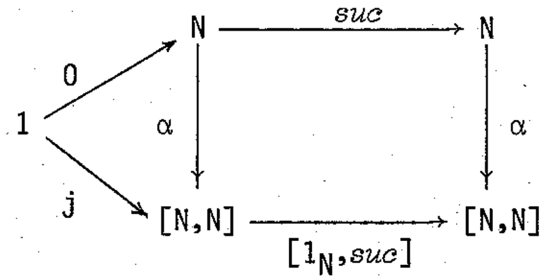
For a space X, the topos $Sh(X,S)$ has a natural numbers object N which is the sheaf of sections of the local homeomorphism $X \times \mathbb{N} \longrightarrow X$ given by first projection (where $\mathbb{N}$ is the set of natural numbers). So NU is the set of locally constant functions from the open subset U of X to $\mathbb{N}$.

For a small category C, the functor N which is constant at the set $\mathbb{N}$ is a natural numbers object for $[C^{op},S]$.

If $M: F \longrightarrow E$ is a geometric morphism between topoi and N is a natural numbers object in E then M^*N is a natural numbers object in F. This implies that each Grothendieck topos satisfies (NNO).

The categories of finite sets and of (finite) permutation representations of a group are examples of elementary topoi not satisfying (NNO).

For a topos E satisfying (NNO), the usual operations can be defined for N. For example, the operation $N \times N \longrightarrow N$ of addition is defined to correspond under cartesian closedness to the unique arrow $\alpha: N \longrightarrow [N,N]$ such that the diagram



commutes, where j corresponds to the projection $1 \times N \longrightarrow N$. For variables m,n of type N, we define the formula $m \leq n$ to be

$\exists_{p \in N}(m+p=n)$. We also have multiplication and exponentiation arrows $\mu, \xi \colon N \times N \longrightarrow N$ and we write mn and m^n for the terms $\mu(m,n)$ and $\xi(m,n)$ of type N. The familiar properties hold; for example, we have the theorem

$$\forall_{m,n,p \in N}(0 \le p,\ m \le n \Longrightarrow pm \le pn).$$

The usual construction of the rationals as equivalence classes of pairs of integers leads to an ordered ring Q in $\mathcal{E}$. In $Sh(X,S)$ where X is a space, Q is the sheaf of locally constant rational-valued functions on X.

The construction of the reals as equivalence classes of Cauchy sequences of rationals leads to an ordered ring R_c in $\mathcal{E}$. In $Sh(X,S)$, R_c is the sheaf of locally constant real-valued functions on X.

The more interesting construction is the one of Dedekind involving *cuts*. Let R_ℓ denote the subobject of $[Q,\Omega]$ obtained as the interpretation of the formula

$$\neg(L=0) \wedge \neg(L=Q) \wedge \forall_{q \in Q} \forall_{p \in L} (q<p \Rightarrow q \in L) \wedge \forall_{p \in L} \exists_{r \in L}\, p < r,$$

where L is a variable of type $[Q,\Omega]$. So R_ℓ is the object of *lower cuts* of Q. In $Sh(X,S)$, R_ℓ is the sheaf of lower-semi-continuous real-valued functions on X.

Alternatively we could use *upper cuts* or *upper and lower cuts* and so obtain objects R_u, R_d. In $Sh(X,S)$, R_u is the sheaf of upper-semi-continuous real-valued functions on X, and R_d is the sheaf of continuous real-valued functions on X.

The usual proof shows that R_ℓ, R_u, R_d are ordered rings. They are fields in the sense that the following formula is valid:

$$\neg(\exists_{y\in R_d} xy=1) \Longrightarrow x = 0.$$

However the following are *not* valid:

$$\neg(x=0) \Longrightarrow \exists_{y\in R_d}\, xy = 1,$$

$$x=0 \vee \exists_{y\in R_d}\, xy = 1.$$

The first of these three formulas is not geometric; the third is. The sense in which R_d is a field is not a geometric one. It is however true that R_ℓ, R_u, R_d are all local rings in E.

Leaving off the formulas $\neg(L=0)$, $\neg(L=Q)$ in the formula for lower cuts, we obtain an object R_ℓ^∞ which is a complete lattice. Similarly for R_u^∞, R_d^∞.

24. Representation theory of rings.

A representation of a ring A is a homomorphism from A into some nice ring of functions on a space. More precisely, a *representation of* A is a space X, a sheaf F of rings on X and a homomorphism from A to the ring of global sections of F. Note that the global sections functor preserves models of algebraic theories (rings in particular) but not models of general geometric theories.

A classical example is the case where X is a compact hausdorff space and F is the ring R_d in $Sh(X,S)$. The problem was not classically phrased in terms of sheaves but rather in terms of the ring $\Gamma R_d = Top(X,\mathbb{R})$ of real-valued continuous functions on X. However, as a ring $Top(X,\mathbb{R})$ is not very interesting; it has none of the special properties that allow the deep results of ring theory to be applied to it. It has been persuasively argued by Chris Mulvey that the object of study in this context is the representing sheaf of rings F. After all, as we have pointed out, R_d is for example a local ring in $Sh(X,S)$. So any intuitionistically valid results of ring theory about local rings should be applicable to R_d. As an example, Mulvey takes the theorem:

Theorem: (Kaplansky). *Any finitely generated projective module over a local ring in S admits a finite basis.*

The proof can be shown to be intuitionistic and so remains valid in any topos. For a compact hausdorff space X it is easily seen that the global sections functor Γ induces an equivalence between the category of R_d-modules in $Sh(X,S)$ and the category of ΓR_d-modules in S; moreover, finitely generated projective modules correspond under this equivalence. For any space X, finite dimensional vector spaces over R_d in $Sh(X,S)$ are precisely *finite dimensional real vector bundles on* X. Thus we obtain a more conceptual proof of the following fundamental result of K-theory.

Theorem (Swan). *For any compact hausdorff space X, the category of finite dimensional real vector bundles on X is equivalent to the category of finitely generated projective modules over the ring of real-valued continuous functions on X.*

The structure sheaf $\tilde{A}$ provides a representation of an arbitrary commutative ring A as a ring of global sections of a local ring in $Sh(SpecA,S)$. This gives another view-point to the theorem in 20.: any commutative ring can be represented as a local ring by a "change of set theory" from S to $Sh(SpecA,S)$. This illustrates the general principle that for geometric theories "free constructions" may require a change of set theory. It is well known that for two algebraic theories $\mathbb{T}, \mathbb{T}'$ with the first richer than the second (for example, $\mathbb{T}$ the theory of rings and $\mathbb{T}'$ the theory of monoids) it is possible to construct the free model M of $\mathbb{T}$ in S on any model M' of $\mathbb{T}'$ in (the monoid ring $M = \mathbb{Z}(M')$ on the monoid M'). For geometric theories the free model M of $\mathbb{T}$ on the model M' of $\mathbb{T}'$ in S will in general be a model of $\mathbb{T}$ in some topos other than S.

25. Classical mathematics.

In 19. we claimed that mathematics in a general topos could not be taken to be anything more specific than intuitionistic mathematics. We shall now consider elementary conditions on a topos which eventually pin it down to a category of sets in some (slightly weakened) model of Zermelo-Fraenkel set theory (ZF).

There is a simple elementary property satisfied by a spatial topos $Sh(X,S)$. Two arrows of sheaves $f,g\colon A \longrightarrow B$ are equal if and only if $(fU)s = (gU)s$ in BU for all open subsets U of X and all $s \in AU$. Open subsets U of X amount to subsheaves of the terminal object 1. It follows that spatial topoi have the property (OG) that "opens generate":

(OG) *for all arrows* $f,g\colon A \longrightarrow B$, *if* $fs = gs$ *for all arrows* $s\colon U \longrightarrow A$ *such that* $U \longrightarrow 1$ *is a monomorphism, then* $f = g$.

Any partially ordered set H can be regarded as a site by giving it the smallest Grothendieck topology for which sets of elements with $U \in H$ as supremum are coverings of U. The topos $Sh(H,S)$ then satisfies (OG).

Theorem. An elementary topos E *with a geometric morphism* $E \longrightarrow S$ *(for example, any Grothendieck topos) satisfies* (OG) *if and only if there exists a complete Heyting algebra* H *in* S *such that* $E \simeq Sh(H\ S)$.

Not every complete Heyting algebra is the partially ordered set $\mathcal{O}X$ for some space X. Each poset $\mathcal{O}X$ is isomorphic to one for which X is sober. Then the points of X are in bijection with arbitrary-supremum-preserving finite-infirmum-preserving functions $\mathcal{O}X \to \mathcal{O}1$ (where $\mathcal{O}1$ is just the ordinal 2). Thus we define the *points* of a Heyting algebra H (as against its "elements") to be arbitrary-sup-preserving finite-inf-preserving maps $H \to \mathcal{O}1$. However, a Heyting algebra, as we have said, may not have "enough points" to make it the poset of opens on a space.

A topos E is said to have *enough points* when, for all arrows $f,g: A \to B$, if $M^*f = M^*g$ for all geometric morphisms (points) $M: S \to E$, then $f = g$. This is not an elementary condition on E however it can be made elementary in terms of a given geometric morphism $E \to S$.

<u>*Theorem*</u>. *A topos* E *is spatial if and only if there is a geometric morphism* $E \to S$, *condition* (OG) *is satisfied, and* E *has enough points.*

A topos E is said to be *Boolean* when the Heyting algebra Ω in E is a Boolean algebra. This is an elementary condition on E: the arrow $|\neg(x \in \Omega)|: \Omega \to \Omega$ composes with itself to yield the identity of Ω. In a Boolean topos, for all formulas ϕ, the formula $\neg\neg\phi \Leftrightarrow \phi$ is valid. The first-order logic of a Boolean topos is classical.

A topos E is said to be *two-valued* when the terminal object 1 has only two subobjects (more precisely, if $U \to 1$ is a monomorphism and U is not initial then it is an isomorphism). For a two-valued topos the condition (OG) becomes the condition that 1 should generate: distinct arrows can be detected by arrows from 1.

A two-valued Boolean topos E satisfying (NNO) and (OG) has all the elementary properties of a category of sets satsifying (ZF) (omitting the axiom of choice). The *replacement schema* of (ZF) is not an elementary condition. It can be shown that E gives a model of set theory satisfying all of the other axioms of (ZF) together with a "bounded" replacement schema. If E *has* a geometric morphism into S then the second last theorem above yields that $E \simeq S$ (for this (NNO) and (AC) are not needed in E nor S).

Now consider the axiom of choice. A topos E is said to satisfy "the axiom of choice" when:

(AC) *for all epimorphisms* $e: A \longrightarrow C$, *there exists an arrow* $m: C \longrightarrow A$ *such that* $em = 1_C$.

In S, to give an epimorphism $e: A \longrightarrow C$ is to give the family of non-empty disjoint sets $A_c = \{ a \in A \mid ea = c \}$ indexed by $c \in C$, and an arrow m such that $em = 1_C$ is a choice of an element out of each A_c. We shall suppose that the sets in S are a model of (ZFC) (that is, (ZF) plus choice) so that S satisfies (AC).

Theorem. An elementary topos E *with a geometric morphism* $E \longrightarrow S$ *satisfies* (AC) *if and only if* E *is Boolean and satisfies* (OG).

Classical mathematics takes place in a two-valued elementary topos satisfying (NNO) and (AC).

16. Completeness for geometric theories.

As was remarked in 22., a finitely-complete small site C is a good notion of geometric theory. The theory is called *finitary* when the topology on C is generated by a pretopology J in which the elements of each $J(U)$ are all finite sets of arrows into U. Categories which are equivalent to categories of sheaves on such a site C are called *coherent topoi*.

<u>*Theorem*</u> (Deligne). *Coherent topoi have enough points.*

A *geometric formula* for a geometric theory C is a formula τ in the "language of C" which is of the type allowable as axioms for a geometric theory (see end 16.). To say τ *is a theorem of* C is to say τ is valid in every model of C in every Grothendieck topos E.

<u>*Completeness theorem.*</u> *For any finitary geometric theory* C*, a geometric formula* τ *is a theorem of* C *if and only if* τ *is valid for every model of* C *in* S.

<u>*Proof.*</u> Geometric formulas τ are preserved by left adjoints of geometric morphisms and so, to say τ is valid for every model of C in every Grothendieck topos is to say it is valid for the generic model A of C in the classifying topos $Sh(C,S)$. By Deligne's theorem $Sh(C,S)$ has enough points so τ is valid for A in $Sh(C,S)$ if it is valid for all models of C in S (the question of validity of τ in A amounts to the question of whether or not two arrows in $Sh(C,S)$ are equal). □

If a Grothendieck topos E has enough points then there is a set Λ (= a category with only identity arrows) and a geometric morphism $M: [\Lambda,S] \longrightarrow E$ such that M^* is faithful (that is, $M^*f = M^*g$ implies $f = g$). The category $[\Lambda,S]$ is a Grothendieck topos satisfying (AC). A general Grothendieck topos need not have enough points yet we do have:

<u>*Theorem*</u> (Barr). *For each Grothendieck topos* E *there exist a Grothendieck topos* B *satisfying* (AC) *and a geometric morphism* $M: B \longrightarrow E$ *such that* M^* *is faithful.* □

This theorem cannot be refined to the extent of replacing B by a product of *two-valued* Grothendieck topoi satisfying (AC) (that is, a product $[\Lambda,S]$ of copies of S). The reason for this lies in the work of Kripke who observed that, for the intuitionistic first-order predicate calculus (Heyting algebra), a two-valued semantics (true-false) is inadequate, and one needs the semantics of arbitrary complete Boolean algebras ($\mathcal{B}$ in the last theorem is equivalent to the category of sheaves on a complete Boolean algebra by the theorems of 25.). The theorem of Barr can be regarded as a completeness theorem for general geometric theories.

7. Non-standard analysis (of Robinson)

Recall that a *filter* ∇ on a lower semilattice P is a subset such that $1 \in \nabla$, if $x,y \in \nabla$ then $x \wedge y \in \nabla$, and, if $x \leq y$ and $x \in \nabla$ then $y \in \nabla$. In other words, the characteristic function $\chi_\nabla \colon P \longrightarrow 2$ of ∇ is a lower semilattice homomorphism. A filter is *proper* when $\nabla \neq P$. An *ultrafilter* on P is a maximal proper filter. Zorn's lemma implies that each proper filter is contained in an ultrafilter.

A filter ∇ on a Heyting algebra P determines an equivalence relation on P given by: $x \sim y$ if and only if $(x \Rightarrow y) \wedge (y \Rightarrow x) \in \nabla$. There is a unique Heyting algebra structure on the set P/∇ of equivalence classes such that the canonical epimorphism $P \longrightarrow P/\nabla$ is a Heyting algebra homomorphism.

<u>*Theorem.*</u> *For a proper filter* ∇ *on a Heyting algebra* P *the following conditions are equivalent:*

(a) ∇ *is an ultrafilter;*

(b) $\chi_\nabla \colon P \longrightarrow 2$ *is a Heyting algebra homomorphism;*

(c) $P/\nabla \simeq 2$;

(d) *for all* $x \in P$, *either* $x \in \nabla$ *or* $\neg x \in \nabla$.

We shall now describe a procedure for cutting down a topos to a two-valued topos without damaging the logic.

Let E be a topos and let $H = Sub_E 1$ be the Heyting algebra of subobjects of the terminal object. Let *Set* denote a category of sets large enough to contain the set of arrows of E, and write *Cat* for the category of models of the theory of categories in *Set*. It is possible to define a functor $\underline{E}: H^{op} \longrightarrow Cat$ such that $\underline{E}U \simeq E/U$ and, for $i: V \to U$, the functor $\underline{E}i: \underline{E}U \longrightarrow \underline{E}V$ corresponds to the functor $i^*: E/U \longrightarrow E/V$ given by pullback along i. (The reason for not taking $\underline{E}U = E/U$ is that the assignment $i \longmapsto i^*$ only preserves composition up to isomorphism and so is technically not a functor.) Each of the categories E/U is an elementary topos and i^* is a homomorphism of elementary topoi (i.e preserves all the data involved in the definition of an elementary topos; such functors are called *logical morphisms* . So $\underline{E}$ lands in the category of models of the theory of elementary topoi in *Set*. It can be shown that the theory of elementary topoi is algebraic in the sense needed for the first theorem of 16. So $\underline{E}$ is actually a model of the theory of elementary topoi in the category $[H^{op}, Set]$.

A filter ∇ on H yields a left exact characteristic functor $\chi_\nabla: H \longrightarrow Set$ ($\chi_\nabla U$ has one element when $U \in \nabla$ and none otherwise). Thus we obtain a left exact functor

$$- \otimes \chi_\nabla : [H^{op}, Set] \longrightarrow Set \, .$$

The elementary topos $\underline{E}$ in $[H^{op}, Set]$ yields an elementary topos $\underline{E} \otimes \chi_\nabla$ in *Set* since left exact functors preserve models of algebraic theories. We denote $\underline{E} \otimes \chi_\nabla$ by E/∇.

More explicitly, the objects of E/∇ are the same as the objects of E, and $(E/\nabla)(A,B)$ is the direct limit of the sets $E(A \times U,B)$ as U runs over the poset ∇. It can be shown directly that E/∇ is an elementary topos with a logical morphism (the canonical projection) $E \longrightarrow E/\nabla$ which collapses all $U \in \nabla$ to terminal objects. This allows those who will to ignore the last two paragraphs! If E is Boolean, satisfies (NNO), or satisfies (AC) then so does E/∇. There is an equivalence between the Heyting algebra of subobjects of 1 in E/∇ and the Heyting algebra H/∇. So, *when* ∇ *is an ultrafilter,* E/∇ *is two-valued;* it is called the *ultrapower of* E *modulo* ∇. In this case, if E satisfies (AC), E/∇ is Grothendieck if and only if $E/\nabla \simeq S$ (a two-valued Grothendieck topos satisfying (AC) is equivalent to S). So either this construction always yields S itself or else it produces elementary topoi which are not Grothendieck. The latter is the case.

The special topos which leads via the above construction to "non-standard analysis" is simply the category $E = [\mathbb{N},S]$ of sequences of sets. The objects of E are sequences $A = (A_n)$ of sets and the arrows are sequences $f = (f_n): A \longrightarrow B$ of functions $f_n: A_n \longrightarrow B_n$. The terminal object 1 of E is the constant sequence at the one-point set. A subobject U of 1 is a sequence of sets with at most one element; each such U can be identified with the subset of $\mathbb{N}$ consisting of those n for which U_n is non-empty. Thus $Sub_E 1 \simeq P\mathbb{N}$, the Boolean algebra of subsets of the natural numbers $\mathbb{N}$. A filter on the Boolean algebra $P\mathbb{N}$ is also known as a filter ∇ on the *set* $\mathbb{N}$. The objects of E/∇ are sequences of sets. An arrow $[U,f]: A \longrightarrow B$ in E/∇ is an equivalence class of pairs (U,f) where $U \in \nabla$ and f is a family of functions $f_n: A_n \longrightarrow B_n$ indexed by $n \in U$; pairs (U,f), (V,g) are equivalent when there exists $W \in \nabla$ such that $W \subset U \cap V$ and $f_n = g_n$ for all $n \in W$.

When ∇ is a *principal ultrafilter* (that is, $\nabla = \{ U \in \mathbb{N} \mid m \in U \}$ for some fixed $m \in \mathbb{N}$) then $[\mathbb{N},S]/\nabla$ is equivalent to S which is certainly two-valued, and the logical morphism $[\mathbb{N},S] \longrightarrow S$ is evaluation at m.

Interest is in the case where ∇ is a non-principal ultrafilter (they exist by Zorn's lemma which holds in some model of set theory if there are any models at all!). It can be shown that $[\mathbb{N},S]/\nabla$ is not equivalent to S; yet it is a model of set theory - a two-valued elementary topos satisfying (AC) and (NNO), not Grothendieck.

The "global sections" functor $\Gamma: [\mathbb{N},S]/\nabla \longrightarrow S$ which assigns to each object its set of arrows from 1 is not a logical (nor a geometric) morphism when ∇ is not principal. However, *it does preserve all the first-order logic* (and all limits). Note that ΓA is the set of equivalence classes of elements of $\prod_{n \in \mathbb{N}} A_n$ where $a \sim b$ when $\{n \in \mathbb{N} \mid a_n = b_n \} \in \nabla$; this set A is called the *ultraproduct* of the sets A_n modulo ∇.

Regarding each set as a constant sequence we obtain a logical morphism $S \longrightarrow [\mathbb{N},S]$, and hence a logical morphism $S \longrightarrow [\mathbb{N},S]/\nabla$ which is in fact faithful. So $[\mathbb{N},S]/\nabla$ is a logical extension of S. For each set A there is a monomorphism $A \longrightarrow \Gamma A$ which is not in general an isomorphism. In particular, $\Gamma\mathbb{R}$ is *the set of non-standard real numbers* and the elements of $\Gamma\mathbb{R}$ not in the image of $\mathbb{R} \longrightarrow \Gamma\mathbb{R}$ are the *infinitesimals*. Of course, $\mathbb{R}$ is the real-numbers object $(R_c = R_d)$ in $[\mathbb{N},S]/\nabla$, so we have represented the set of non-standard real numbers as the global sections of a real-numbers object in a non-standard model of set theory. Now $\mathbb{R}$ in $[\mathbb{N},S]/\nabla$ has all the properties of the real number field (the higher-order logic *is* that of set theory), and since Γ preserves the first-order logic, $\Gamma\mathbb{R}$ has all the first-order properties of the real number field (it is an ordered field for example).

28. Independence results in set theory.

In the last part we saw how to force a topos to become two-valued by stopping its variation at an ultrafilter. It is also possible to force a topos to be Boolean by taking the "sheaves for the double negation topology" which we shall now explain in a special case general enough for our purpose here.

Let P be a poset. Then $[P^{op}, S]$ is not in general Boolean. We put a Grothendieck topology on P by taking as the covering cribles of $x \in P$ the cribles C at x such that $y \leq x \Rightarrow \exists\, z \in C$ with $z \leq y$. This is called the *double negation topology on* P. Write $Sh_{\neg\neg}(P,S)$ for the category of sheaves on P with this topology. In this case, not only is $Sh_{\neg\neg}(P,S)$ Boolean, it satisfies (AC).

The hard part of independence proofs is to find an appropriate poset P such that the axiom in question fails in $Sh_{\neg\neg}(P,S)$. Then we just take an ultrapower of $Sh_{\neg\neg}(P,S)$ to obtain a model of set theory in which the axiom fails. This is the technique of *forcing* developed by P.J. Cohen (1963).

For example, to show the independence of *the continuum hypothesis*, one constructs a poset P for which there is an object A in $Sh_{\neg\neg}(P,S)$ with monomorphisms $\mathbb{N} \rightarrowtail A$, $A \rightarrowtail [\mathbb{N},\Omega]$ and yet the objects of epimorphisms from N to A and from A to $[\mathbb{N},\Omega]$ are initial.

29. Cohomology.

Since we have shown that a topos E is a generalized space (amongst other things), it is reasonable to expect there to be a good notion of cohomology for E. This is indeed the case.

Suppose the homsets of E are in S. Write $\Gamma: E \to S$ for the "global sections" functor $\Gamma A = E(1,A)$. Since Γ is left exact it takes abelian groups in E to abelian groups.

Recall that an abelian group D is *injective* when, for every monomorphism $A \to B$ between abelian groups, each homomorphism $A \to D$ extends to a homomorphism $B \to D$. This can be said in E too, and it can be shown that each abelian group A in E admits a monomorphism $A \to D$ into an injective abelian group D. It follows that we can find an exact sequence

$$1 \longrightarrow A \longrightarrow D_0 \xrightarrow{d_0} D_1 \xrightarrow{d_1} D_2 \xrightarrow{d_2} \cdots$$

of abelian groups in E for each A in which each D_n is injective; this is called an *injective resolution* of A. Thus we obtain a complex of abelian groups

$$1 \longrightarrow D_0 \xrightarrow{d_0} D_1 \xrightarrow{d_1} D_2 \xrightarrow{d_2} \cdots .$$

Put $H^n(E;A)$ equal to the factor group of the kernel of Γd_n by the image of Γd_{n-1}. In particular, $H^0(E;A)$ is the kernel of Γd_0, and since Γ is left exact, we have an isomorphism

$$H^0(E;A) \cong \Gamma A.$$

It is a standard result of homological algebra hat any other choice of injective resolution would yield isomorphic groups $H^n(E;A)$, and so we call these groups the *cohomology groups of* E *with coefficients in* A.

As one would hope, when $E = Sh(X,S)$ where X is a space and A is a sheaf of abelian groups on X, then

$$H^n(E;A) \cong H^n(X;A),$$

where the right-hand side is the usual cohomology of X with values in A (see Godement's "Topologie Algébrique et Théorie des Faisceux").

As another example, take G to be a group regarded as a category with one object and with arrows the elements. Then $[G,S]$ is the category of sets on which G-acts. The functor $\Gamma: [G,S] \to S$ assigns the set of fixed points to each set on which G-acts. An abelian group A in $[G,S]$ is precisely an abelian group A together with a group homomorphism $G \to Hom(A,A)$; that is, A is a $\mathbb{Z}G$-module where $\mathbb{Z}G$ is the group-ring of G. There is an isomorphism

$$H^n([G,S];A) \cong H^n(G;A)$$

where the right-hand side is the usual cohomology of G with coefficients A (see Mac Lane's "Homology").

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