Lenses: applications and generalizations

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Outline

An agent in an environment

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- The agent has an effect on the environment and vice versa.
- What does that mean?
- It means agent and environment are communicating somehow.
    - The agent *observes* the environment and *acts* on it.
    - The agent's state affects that of the environment and vice versa.
        - Agent affects environment through *action*.
        - Environment affects agent through *observation*.
        - Each is affected in that it undergoes a change of *state*.

How shall we model this mathematically?

A formalization of agent/environment interaction

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- Agent's state is updated by the observation via $S_{\mathrm{Ag}} \times \mathit{Obs} \to S_{\mathrm{Ag}}$.
- Observation is dictated by environment's state via $S_{\mathrm{En}} \to \mathit{Obs}$.
- Environment's state is updated by the action via $S_{\mathrm{En}} \times \mathit{Act} \to S_{\mathrm{En}}$.

How to organize all this stuff?

We have sets $S_{\mathrm{Ag}}, S_{\mathrm{En}}, \mathit{Act}, \mathit{Obs}$ and functions

$$S_{\mathrm{Ag}} \to \mathit{Act} \qquad S_{\mathrm{En}} \to \mathit{Obs}$$

$$S_{\mathrm{Ag}} \times \mathit{Obs} \to S_{\mathrm{Ag}} \qquad S_{\mathrm{En}} \times \mathit{Act} \to S_{\mathrm{En}}$$

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- Each pair of functions is a special case of what are called *lenses*.
- Lenses are the morphisms in a cat **Lens**, whose objects are pairs $\binom{X}{Y}$.
  - The lenses from our agent/environment setup would be denoted:
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Lenses have been coming up in the ACT community a lot lately.

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- Bidirectional transformations (Oles),
- dialectica categories and linear logic (de Paiva),
- the view-update problem in databases (Hoffman, Pierce),
- functional programming (Gibbons, Oliveira, Palmer, Kmett),
- wiring diagrams, discrete and continuous dynamical systems (Spivak),
- open economic games (Ghani-Hedges),
- supervised learning (Fong-Spivak-Tuyéras).

I'll explain a few of these as we go, especially the ones I've worked on.

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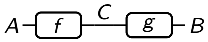
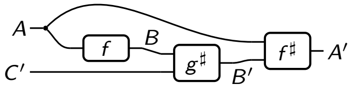
For simplicity, let's take $\mathcal{C} = \mathbf{Set}$ and just write **Lens** for $\mathbf{Lens}_{\mathbf{Set}}$.

- $\mathrm{Ob}(\mathbf{Lens}) := \left\{ \binom{A}{A'} \;\middle|\; A, A' \in \mathrm{Ob}(\mathbf{Set}) \right\}$
- Monoidal unit: $\binom{1}{1}$; monoidal product: $\binom{A}{A'} \otimes \binom{B}{B'} := \binom{A \times B}{A' \times B'}$
- $\mathbf{Lens}\left(\binom{A}{A'}, \binom{B}{B'}\right) := \left\{ \binom{f}{f^\sharp} \;\middle|\; \begin{array}{r} f\colon A \to B \\ f^\sharp\colon A \times B' \to A' \end{array} \right\}$.
- $\mathrm{id}_{\binom{A}{A'}} = \binom{\mathrm{id}_A}{\pi}$, where $\pi\colon A \times A' \to A'$ is the projection.
- $\binom{f}{f^\sharp} \mathbin{⨟} \binom{g}{g^\sharp} = \binom{f \mathbin{⨟} g}{(a,c') \mapsto f^\sharp(a, g^\sharp(f(a), c'))}$





Bringing lenses into the fold

I found the formula for lenses and their composition kinda weird:

$$\mathbf{Lens}\left(\begin{pmatrix} A \\ A' \end{pmatrix}, \begin{pmatrix} B \\ B' \end{pmatrix}\right) := \left\{ \begin{pmatrix} f \\ f^\sharp \end{pmatrix} \;\middle|\; \begin{array}{r} f\colon A \to B \\ f^\sharp\colon A \times B' \to A' \end{array} \right\}.$$

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- Today: we'll first see **Lens** as part of a larger category that
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- Today: we'll first see **Lens** as part of a larger category that
  - provides a sort of geometrical perspective,
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  - has better formal properties.
- We then generalize further to pick up some close cousins of lenses.

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- Kmett, Riley, etc. have generalized lenses to *optics*.
  - Briefly: for any monoidal category $(\mathcal{C}, I, \otimes)$, ...
  - an optic $\binom{A}{A'} \to \binom{B}{B'}$ can be identified with an element of

$$\int^{M \in \mathcal{C}} C(A, M \otimes B) \times C(M \otimes B', A').$$

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- This can be generalized even further using Tambara modules.
- However, it's not the direction I want to go today.

Plan of the talk

Plan for the rest of the talk:

- Some applications of lenses
- Generalizing lens categories

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We began with an agent and an environment interacting.

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- Idea: if we tensor $\otimes$ these lenses we get:

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- Composing, we get a single lens $\binom{S}{S} \to \binom{1}{1}$, where $S = S_{\text{Ag}} \times S_{\text{En}}$.
  - It's just a set S and a map $S \to S$: a *discrete dynamical system*.

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We can see this as part of a bigger picture.

The agent-environment system

So what were we doing when we:

- started with lenses $\binom{S}{S} \to \binom{Act}{Obs}$ and $\binom{S'}{S'} \to \binom{Obs}{Act}$,
- multiplied them together to get a map $\binom{S \times S'}{S \times S'} \to \binom{Act \times Obs}{Obs \times Act}$, and then
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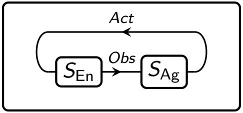

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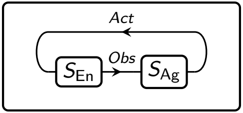
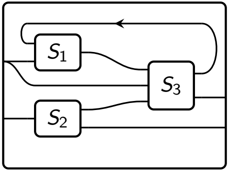


More generally we can consider open systems with many interacting agents



Wiring diagrams

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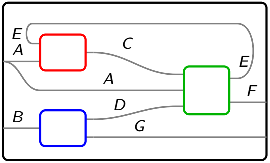

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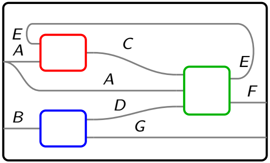

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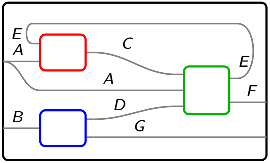

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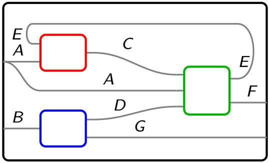

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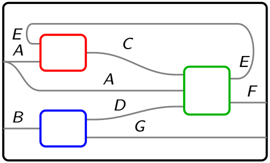

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- Both maps are just projections and diagonals:

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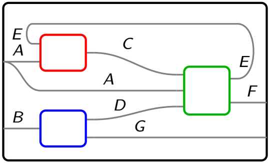

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Every wiring diagram gives a lens made of projections and diagonals.

WDs and discrete dynamical systems

A *discrete dynamical system* of type $\binom{A}{A'}$ consists of

- A set S
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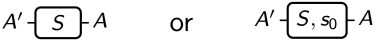

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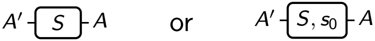

- A wiring diagram is a lens $\binom{A_1}{A_1'} \otimes \cdots \otimes \binom{A_n}{A_n'} \to \binom{B}{B'}$, and
- Each dyn'l system is a lens $\binom{S_i}{S_i} \to \binom{A_i}{A_i'}$. Composing and multiplying...
- We get a dynamical system $\binom{S_1 \times \cdots \times S_n}{S_1 \times \cdots \times S_n} \to \binom{B}{B'}$ in outer box.

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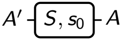
or



- A wiring diagram is a lens $\binom{A_1}{A_1'} \otimes \cdots \otimes \binom{A_n}{A_n'} \to \binom{B}{B'}$, and
- Each dyn'l system is a lens $\binom{S_i}{S_i} \to \binom{A_i}{A_i'}$. Composing and multiplying...
- We get a dynamical system $\binom{S_1 \times \cdots \times S_n}{S_1 \times \cdots \times S_n} \to \binom{B}{B'}$ in outer box.

This story of DS's and WD's existed years before I knew about lenses.

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Our category **Learn** is just **Para**(**Lens**).

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Can we do better?

Continuous dynamical systems?

Recall that a discrete dynamical system with inputs A' and outputs A is:

- A set S
- A function $f^{\mathrm{rdt}} \colon S \to A$ called "readout"
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The two notions are quite similar, but can we see the latter as a lens?

Outline

So how should I think about an object in Lens?

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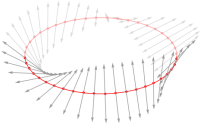
Suggestion: think of objects as "bundles."

What are bundles?



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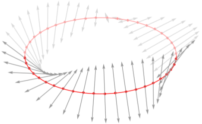

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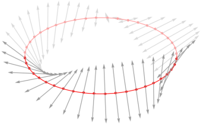

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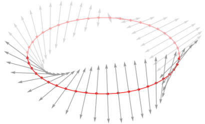

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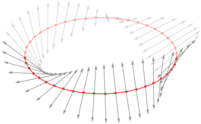

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A *trivial bundle* is one of the form $\pi_1\colon B \times B' \to B$ for some B'.

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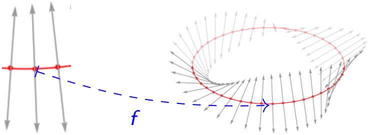
The fiber over any $b_1 : B_1$ is that over its image, $(f^*E_2)(b_1) = E_2(f(b_1))$.



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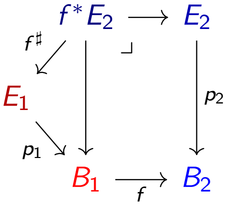

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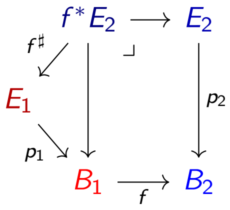

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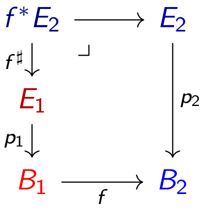

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There's a strong relationship between the AG-style maps and lenses.

Example

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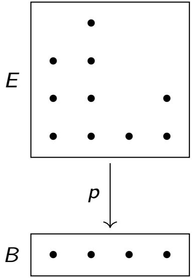

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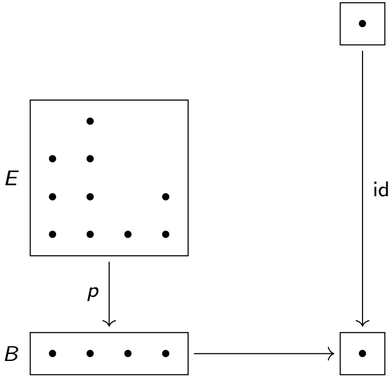

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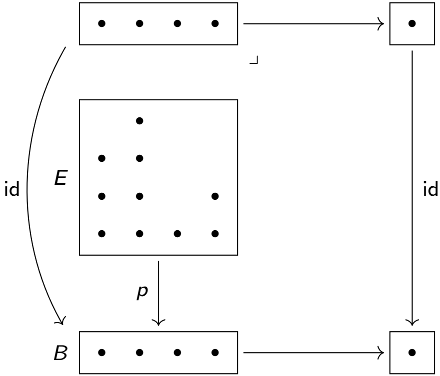

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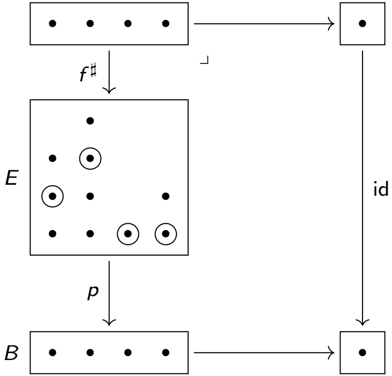

Interpretation of bimorphic lenses as trivial bundles

We will see that **Lens** sits inside this category **Bund** of bundles.

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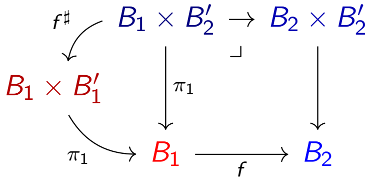
- Send morphism $\binom{f}{f^\sharp} \colon \binom{B_1}{B_1'} \to \binom{B_2}{B_2'}$ to the bundle morphism:



Such a map $f^\sharp \colon B_1 \times B_2' \to B_1 \times B_1'$, — in order to commute with π_1 — has no choice on the B_1 factor. Thus it can be identified with a map $f^\sharp \colon B_1 \times B_2' \to B_1'$.

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  - objects $\left\{ \begin{bmatrix} E \\ B \end{bmatrix} \;\middle|\; B : \mathcal{B}, E : \mathcal{E}(B) \right\}$
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How to think about Lens

This suggests the following way of thinking of (generalized) lenses.

- An object $\begin{bmatrix} A' \\ A \end{bmatrix}$ consists of contexts and actions: $\begin{bmatrix} \text{actions} \\ \text{contexts} \end{bmatrix}$
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Examples: ringed spaces, cts dynamical systems, functorial view-update.

Ringed spaces

In algebraic geometry they study *ringed spaces* $(X, \mathcal{O}_X)$.

- Here X is a topological space and $\mathcal{O}_X$ is a sheaf of rings on it.
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A *morphism* of ringed spaces $\binom{f}{f^\sharp}\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is:

- A continuous map $f\colon X \to Y$
- A map of sheaves $f^*\mathcal{O}_Y \to \mathcal{O}_X$.

That is, it's a map $\begin{bmatrix} \mathcal{O}_X \\ X \end{bmatrix} \to \begin{bmatrix} \mathcal{O}_Y \\ Y \end{bmatrix}$.

Continuous dynamical systems

Recall: if A', A are manifolds, a *continuous dynamical system* is:

- A manifold S, (tangent bundle TS),
- A differentiable map $f^{\mathrm{rdt}} \colon S \to A$,
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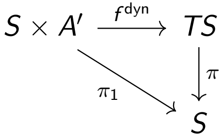

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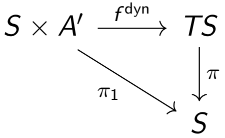
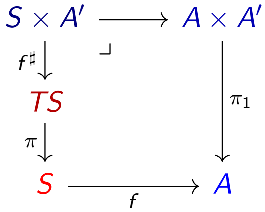


But from the bundle perspective that commutative diagram is baked in.



In other words the dynamical system is just a lens map $\begin{bmatrix} TS \\ S \end{bmatrix} \to \begin{bmatrix} A' \\ A \end{bmatrix}$

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\begin{array}{ccc}
\displaystyle\sum_{I_1 : B_1\textbf{-Inst}} Q_* I_1/B_2\textbf{-Inst} & \longrightarrow & \displaystyle\sum_{I_2 : B_2\textbf{-Inst}} I_2/B_2\textbf{-Inst} \\
\text{univ. construction above}\downarrow & \lrcorner & \\
\displaystyle\sum_{I_1 : B_1\textbf{-Inst}} I_1/B_1\textbf{-Inst} & & \downarrow \pi_1 \\
\pi_1 \downarrow & & \\
B_1\textbf{-Inst} & \xrightarrow[Q_*]{} & B_2\textbf{-Inst}
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This lens $\begin{bmatrix} -/B_1\textbf{-Inst} \\ B_1\textbf{-Inst} \end{bmatrix} \to \begin{bmatrix} -/B_2\textbf{-Inst} \\ B_2\textbf{-Inst} \end{bmatrix}$ does the expected view-update.

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- Take $\mathcal{B} := \{\text{comonoids } (c, \epsilon, \delta) \text{ in } \mathcal{M}\}$
- Take $\mathcal{E}(c) := \mathbf{coKl}(c \otimes -)$, the coKleisli cat. of comonad $x \mapsto c \otimes x$.
- In $\begin{bmatrix} m \\ c \end{bmatrix}$, think of m as the product coalgebra $c \otimes m$, "trivial bundle".

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A morphism $\begin{bmatrix} E_1 \\ B_1 \end{bmatrix} \to \begin{bmatrix} E_2 \\ B_2 \end{bmatrix}$ in the twisted arrow category.

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- ... then $\mathbf{Lens}_{\mathcal{B}/-}$ has excellent formal properties.
  - Complete, cocomplete, cartesian closed.
  - Initial alg's and final coalg's for polynomial endofunctors.
  - Another fact'n system: $\begin{bmatrix} f^\sharp \\ f \end{bmatrix}$ factors as $\begin{bmatrix} E_1 \\ B_1 \end{bmatrix} \to \begin{bmatrix} f_* E_1 \\ B_2 \end{bmatrix} \to \begin{bmatrix} E_2 \\ B_2 \end{bmatrix}$

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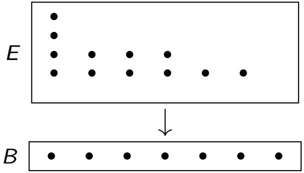

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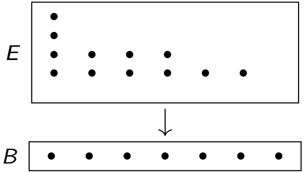

- And they have the same morphisms too: $\mathbf{Poly}_{\mathcal{B}} \cong \mathbf{Lens}_{\mathcal{B}/-}$.

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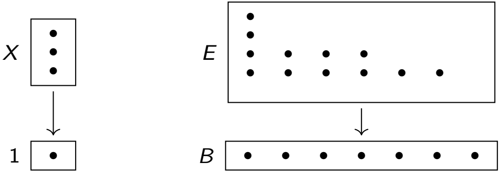

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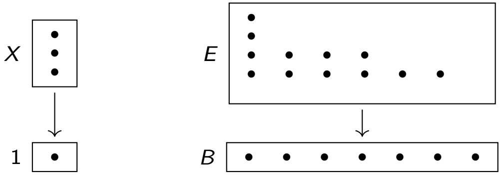

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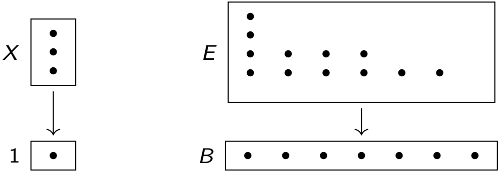

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- The functor acts on a lens $\left[\begin{smallmatrix} E \\ B \end{smallmatrix}\right] \to \left[\begin{smallmatrix} E' \\ B' \end{smallmatrix}\right]$ by composing with it.

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Thanks; comments and questions welcome!