Spaces as infinity-groupoids

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1 Introduction

As the title of this chapter suggests, there are aspects of 'spaces' mirrored by things called '∞-groupoids'. There are also such ∞-groupoids that arise in 'nature' from other contexts. This raises a whole lot of questions. Some of these are obvious, for example: What 'aspects'? What on earth are ∞-groupoids? What job do they do? What sort of 'spaces' are we considering? These questions are, formally, easy to answer, but leave deeper, harder questions still to be considered. Informally, the initial vague idea is that an infinity groupoid model of a space should generalise the classical idea of a fundamental group or groupoid of a space that you may have met from algebraic topology textbooks, working not just with paths, but with higher dimensional analogues, 'paths between paths', 'paths between those things' and so on, all being considered up to some appropriate idea of homotopy or deformation. What it should do is, thus, to generalise the sort of things that the fundamental groupoid is good at doing, such as classifying covering spaces and other types of bundle-like objects[1]. That being said, how to generalise those properties is not always obvious and their formalisation can be tricky.

To continue with our questions, what is the conceptual advantage of working with things such as ∞-groupoids, –, whatever they might be (and there is more than one answer to that question)? Very importantly, what are the intuitions underpinning this formalisation? What are the limitations of the formalisation? How do these objects fit into the general scheme of things, say, from the perspective of algebraic topology? How 'practical' is it to try to 'calculate' with such models of spaces? If we model 'spaces' by ∞-groupoids, do we gain some new insights on the spaces? Turning that around, the opposite question is: how good are 'spaces', themselves, as models for the perhaps naive notions of ∞-groupoid that arise in other areas of mathematics? Is the spatial intuition, thus being invoked, a good one to use or is it too constraining or, alternatively, too wide?

We will attempt to answer some of these. To answer them all fully would take a lot more space. We will attempt to do this by looking back at the developments that led to the perception that there was a link between spatial phenomena of a more-or-less geometric nature and something that would be eventually called 'an ∞-groupoid' and which can be thought of as being more algebraic or 'categorical' in its inspiration rather than inherently 'geometric'. We will go right back to the beginnings of the use of algebraic tools to study topology so as to look, very briefly, at some of the ways that Poincaré, and some of those who came after him, looked at the fundamental groupoid of a space. (Here, thankfully, we can keep things quite brief as the detailed historical analysis of how the algebraic structure of paths in a space was first encoded, has been initiated by Krömer, [41]. That article also contains some very relevant passages quoted from Poincaré, and others.)

[1] We will be looking at precisely that very shortly as there are several types of property that we will be seeking to generalise. This will also serve as an *aide mémoire* on the fundamental group(oid) construction.

Right from the origins of the fundamental group(oid), the link with covering spaces was recognised as one of the important aspects, and that will be seen also in the motivation for the idea behind an ∞-groupoid approach to analogous higher order structures. We will briefly mention this with respect to Grothendieck's approach to the fundamental group in SGA1, [33]. That will lead us, naturally, to the letters from Alexander Grothendieck to Larry Breen, [27–29], of which, for us, the first is probably the most central for our purposes here, and also to the subsequent 'letter to Quillen', [30], in which a strong relationship between spaces and ∞-groupoids is more explicitly mentioned.

Before that, we will need to give some indication of what ∞-groupoids are, as there are several possible manifestations of the notion. (We will look at only two of them in any detail.)

In another thread of the chapter, we will try to show how J. H. C. Whitehead's 'combinatorial homotopy' or 'algebraic homotopy' fits into this theme. One of the test problems he considered was to model polyhedra by algebraic data. Here dimension is crucial, yet within the corresponding area of ∞-groupoid theory, this geometric aspect is less evident.[2] In such applications, it would seem that the combinatorial approach, via simplicial complexes and their combinatorial (Whitehead) homotopy theory, may be more directly useful than the fully ∞-groupoid one.

As the area is a huge one, we will tend to give brief descriptions rather than detailed definitions, directing the reader to the original literature where needs be.

2 The beginnings: recollections of Poincaré's fundamental groupoid.

The fundamental group of a space was introduced by Poincaré in 1895, [56]. At this point in time, the 'spaces' concerned arose as Riemann surfaces and thus naturally came with the insights and problems of analytic continuation of functions, integration along paths and over regions bounded by paths or collections of paths. As noted by Sarkaria, [62] (in I. M. James' *History of Topology*, [37], Chapter 6), Poincaré gave four approaches to the fundamental group(oid) of such a space, M; see also Krömer, [41]. These were, in modern terminology:

(i) as the group of deck transformations of covering spaces over M, thus implicitly involving a form of 'multiple valued function',

(ii) as the holonomy group of what would now be called an integrable connection on a vector bundle,[3]

(iii) as the set of homotopy classes of loops at a base point (or, more generally, of paths) in M, and finally

(iv) on any such space, M, obtained from a polytope or simplicial complex, as a group given by explicit generators and relations.

[2] but see Ara and Maltsiniotis, [2].

[3] We will not be following up on this approach here.

Here we will initially be looking at the third of these, but will also need to consider the covering space approach and that using combinatorial group theoretic methods, corresponding, usually, to some simplicial or CW-complex structure and thus, more or less, to triangulations[4].

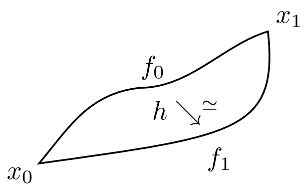
2.1 Homotopy classes of paths

Although this subsection will consist of well known standard material, in order to increase the accessibility of the account, it will be useful to recall and comment on some of the basic definitions and terminology relating to homotopy and the fundamental group(oid). This will also draw attention to certain aspects of this basic theory that we will need, but that are perhaps understressed in the standard accounts.

- Given a space, X, a *path* in X is a continuous map, $\alpha : I \to X$, where $I = [0, 1]$. The path has *source*, $\alpha(0)$, and *target*, $\alpha(1)$.

- Given two continuous maps, $f_0, f_1 : X \to Y$, a *homotopy* between them is a continuous map, $h : X \times I \to Y$, such that, for all $x \in X$, $h(x, 0) = f_0(x)$ and $h(x, 1) = f_1(x)$. The two maps are said to be *homotopic* if there is a homotopy between them and then are said '*to be in the same homotopy class*'. We write $h : f_0 \simeq f_1$ in this case.

Intuitions: With two homotopic maps, the interpretation is that each can be deformed continuously to the other. For instance, in the case of X being just a single point, each function from X to Y gives a point in Y and vice versa. A homotopy between two such maps gives a path 'deforming' one point in Y to the other. If X is a unit interval, then the two maps will just be paths and the homotopy deforms one into the other. An important case in this situation is when the paths share the same source, and also the same target: $x_0 = f_0(0) = f_1(0)$ and $x_1 = f_0(1) = f_1(1)$, then the homotopy may *fix end points*,



so that the map $h(0, -)$ is constant at $f_0(0)$ and $h(1, -)$ is constant at $f_0(1)$. By a *path class*, we will mean a fixed end point homotopy class of paths.

- Two spaces, X and Y, *have the same homotopy type* if there are continuous maps, $f : X \to Y$, and $g : Y \to X$, such that there are homotopies, $gf \simeq id_X$, and $fg \simeq id_Y$. We also say X and Y are *homotopy equivalent* and that f is a *homotopy equivalence* between them. When we refer to

[4]This latter situation relates strongly to certain constructions within theoretical physics as well as raising the question as to whether, in our discussion, 'spaces' should be just 'topological spaces' or should they come with additional structure such as that of a simplicial or CW-complex, or that of a manifold, etc. We will meet this several times later on and will give a more detailed account then.

a *homotopy type*, we thus mean a maximal family of 'spaces' all of which are homotopically equivalent to each other.

Intuitions: The idea is that if two spaces are homeomorphic (i.e., are essentially 'the same') then they will be homotopically equivalent. Often, however, the aim is to decide if two spaces are essentially different, and so should have different properties, behaviour etc., and, for that, one tries to find 'quantities' that are invariant under homeomorphism to test if the two spaces are, or are not, 'the same'. It is much easier to find invariants of homotopy type however, and if X and Y can be shown to differ on some homotopy invariant quantity, then, as they can then not be of the same homotopy type, they must also not be homeomorphic. Equally importantly, that same basic methodology often can be adapted to show whether some mapping has, or has not, some particular property. To do this, one searches for 'algebraic' invariants of homotopy types, ..., but we are getting ahead of ourselves here!

To return to describing Poincaré's constructions:

- The paths in X can be composed (concatenated) in more-or-less the same way as when integrating along paths in $\mathbb{R}^n$. A fairly obvious formula for this corresponds to concatenation (which defines the composite on an interval of length 2) followed by 'rescaling'. This gives:
  If $\alpha, \beta : I \to X$ are two paths such that $\alpha(1) = \beta(0)$, then $\alpha \cdot \beta : I \to X$ is defined by
  $$\alpha \cdot \beta(t) = \begin{cases} \alpha(2t) & 0 \leq t \leq \frac{1}{2} \\ \beta(2t-1) & \frac{1}{2} \leq t \leq 1 \end{cases}$$
  Although this is the 'obvious' composition, corresponding to the subdivision, $\{[0, \frac{1}{2}], [\frac{1}{2}, 1]\}$ of $[0, 1]$, it is not 'God given'. There are a whole lot of others that could have been used. For any subdivision $\{[0, r], [r, 1]\}$ of $[0, 1]$, we could have scaled α to fit on the first subinterval and scaled and shifted β to fit on the second one, before concatenating. All these composites would give homotopic paths however. (Even that does not exhaust the possible compositions, as we could have scaled non-linearly.)

- The composition we have chosen to give is not associative. If $\gamma : I \to X$ is such that $\gamma(0) = \beta(1)$, we can form both $(\alpha \cdot \beta) \cdot \gamma$ and $\alpha \cdot (\beta \cdot \gamma)$, but they are clearly not equal. They are however homotopic[5]. In fact, the usual homotopy given in texts just slides the 'middle' subinterval, $[\frac{1}{4}, \frac{1}{2}]$, on which β is used, along to the corresponding position, $[\frac{1}{2}, \frac{3}{4}]$, rescaling the other two subintervals accordingly. This means that it takes place *within* the 'track' of the composite, i.e., the image of the composite function within X. It is, thus, very 'thin' in the sense that whilst most homotopies can be thought of as 'sweeping out an area' within the space, here what is happening is more like a continuous reparametrisation of the function from one using the subdivision of $[0, 1]$ given by $\{[0, \frac{1}{4}], [\frac{1}{4}, \frac{1}{2}], [\frac{1}{2}, 1]\}$, to one using $\{[0, \frac{1}{2}], [\frac{1}{2}, \frac{3}{4}], [\frac{3}{4}, 1]\}$. We will return to ideas of thinness somewhat later.

[5] ... and this is where things start being interesting, as it is a natural occurrence of 'weak' structure rather than 'strict'.

- If we use the notation $[\alpha]$ for the class of a path, α, in X under *fixed end point* homotopies, then the set of such classes has an algebraic structure given by a partially defined composition, $[\alpha] \cdot [\beta] := [\alpha \cdot \beta]$, defined just on those pairs $([\alpha], [\beta])$ such that $\alpha(1) = \beta(0)$ - which is, of course, why we used fixed end point homotopies rather than 'free' homotopies in defining the path classes, $[\alpha]$. The algebraic structure we have here is a groupoid, i.e., a (small) category in which every morphism is invertible. It is the *fundamental groupoid*[6], $\Pi_1 X$, of X. That this all works is standard, but any reader who has not seen it spelt out in some detail should consult standard texts or look at Krömer's article that was mentioned earlier for a historical viewpoint. (It is of interest that the groupoid version is explicitly given by Schreier in 1927 and then later by Reidemeister, but then was not used for some time; see Krömer, [41], again. Their work is an early example of a structure, that could be considered algebraically, being thought of as a space.)

That completes a description of Poincaré's path based definition, except to note that he actually defines the fundamental *group* and not the more general, and more natural[7], groupoid version. For this 'fundamental group', one needs to choose a '*base-point*', $x_0 \in X$ and then the fundamental group, $\pi_1(X, x_0)$, of the *pointed space*, (X, x_0), is the vertex group of $\Pi_1 X$ at x_0, that is, $\Pi_1 X(x_0, x_0)$, so is the group of path classes of loops in X, based at x_0.

2.2 Covering spaces

If we now assume that X is 'sufficiently locally nice[8]', we can pass to another of Poincaré's definitions; (again we will only sketch the theory and more briefly than above, leaving more 'for the reader to check'). We will assume the space, X, is connected and will choose a base-point, x_0, in X. Furthermore, let $p : \tilde{X} \to X$ be a universal covering space for X. In other words, p is a local homeomorphism, so given any $y \in \tilde{X}$, if we look near enough to y, that is in a small enough neighbourhood of it, p behaves as a homeomorphism, mapping that neighbourhood to a neighbourhood of $p(y)$, and, moreover, a universality condition holds (that we will skate over; see standard algebraic topology texts for this, also for conditions on X for such a universal cover to exist, and, once again, Krömer, [41], for a historical perspective). This p will have a unique path lifting property: if we have a path, α, in X and pick a point, $x \in \tilde{X}$, such

[6] The objects of $\Pi_1 X$ are the points of X; the morphisms are the path classes, so that, between two points, x_0 and x_1, in X, the set $\Pi_1 X(x_0, x_1) = \{[\alpha] \mid \alpha(1) = x_0, \alpha(1) = x_1\}$. Composition is given as above and is associative; identities are given by the classes of constant paths at the points of X, and the inverse of $[\alpha]$ is $[\alpha^{(r)}]$, where $\alpha^{(r)}(t) = \alpha(1-t)$, the path, α traversed in the opposite direction.

[7] Recall that a monoid can be thought of as a category with a single object, and in the same way a group can be thought of as a groupoid again having just a single object. If G is a group, the corresponding groupoid, $\mathbb{G}$, has one object, denoted $*$, for the moment, and, $\mathbb{G}(*, *)$ is the group G with composition in $\mathbb{G}$ being just the multiplication in G. Note: we will not, in fact, use a different notation for a group and the corresponding one-object-groupoid in the main text.

[8] ... meaning that small enough neighbourhoods of each point are 'homotopically trivial', so, intuitively, nothing 'interesting' is happening at the very small scale!

that $p(x) = \alpha(0)$, then α lifts uniquely to a path, $\tilde{\alpha}$, in $\tilde{X}$, so that $p\tilde{\alpha} = \alpha$ and $\tilde{\alpha}(0) = x$.

Using this, one shows that the category of covering spaces of X is equivalent to the category of $\pi_1(X, x_0)$-sets, that is sets with an action of $\pi_1(X, x_0)$ on them:

$$Fibre : Cov/X \xrightarrow{\simeq} \pi_1(X, x_0){-}Sets,$$

where $Fibre(q : Y \to X)$ will be the set, $q^{-1}(x_0)$, with the action of $\pi_1(X, x_0)$ given by lifting paths[9]. This is almost Poincaré's deck transformation 'definition' of $\pi_1(X, x_0)$. (A deck transformation is simply an automorphism of a covering space, hence is compatible with the covering map.) Deck transformations of the universal cover of X give a group that is isomorphic to $\pi_1(X, x_0)$. To see why, we note that $Fibre$ sends the universal cover to the *set* of elements of $\pi_1(X, x_0)$ with the action of that same *group* given by multiplication. Any automorphism of the universal cover goes via the equivalence, $Fibre$, to an automorphism of that $\pi_1(X, x_0)$-set, and that gives an element of $\pi_1(X, x_0)$.

Comment: The above correspondence works for any 'nice topological space', but its importance for us is, also, that it acted as a key starting point for the definition by Grothendieck of the fundamental group of a scheme, the main algebraic geometric version of 'space'. This, in turn, used the exciting insight that this is a version of the fundamental theorem of Galois theory relating extension fields with actions of a Galois group; see SGA1, [33], for the basic source, and Douady and Douady, [24], for a neat treatment, but then there is an enormous literature on this theory as you would expect. Because of that link, it is useful to think of the above correspondence as being part of some more encompassing 'Galois-Poincaré theory'. The use of a *topos*, $\pi_1(X){-}Sets$, or $Sets^{\Pi_1(X)}$, to model aspects of the *topological properties* of X is, perhaps, to be noted for use elsewhere[10].

2.3 Complexes as 'presentations' of spaces and of groupoids

For the above theory, all that was needed was that X was a 'sufficiently nice' topological space. For the final approach to the fundamental group that we will look at, Poincaré assumed, in addition, that the space was specified as a 'complex' of some sort. When using 'path classes', one has to face the initially very large number of paths that there are in the 'usual' spaces that are considered. Although the ideas that Poincaré used were later extended and made much more exact, the intuitive ideas remain clear in what he introduced. These ideas were somewhat later applied by Schreier, Neilsen and others to problems in group theory. This illustrates well the somewhat symbiotic relationship between spaces and algebra, which relates to the main theme of this chapter. The initial development was topological and allowed one to encode spatial information in an algebraic form. The work in (combinatorial) group theory first encoded algebraic structure in combinatorial, and then in spatial, form, where

[9] This easily generalises to non-connected spaces. Replace $\pi_1(X, x_0)$ by $\Pi_1(X)$ and $\pi_1(X, x_0) - Sets$ by the category, $Sets^{\Pi_1(X)}$, of functors from $\Pi_1(X)$ to $Sets$. This also frees up the construction from needing to choose a base-point, x_0.

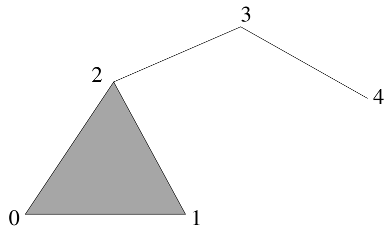
[10] An action of a group, G, on a set, S, is often called a *representation* of the group, as it *represents* the elements of the group as permutations of S. This also leads on to the subject of representation theory and the various categorical approaches to that subject area.

methods derived from algebraic topology could be applied to give new insights and results.

To start with, the 'complexes' that we will need are *simplicial complexes*, a notion which we will recall below. Later we will need simplicial sets, CW-complexes and various variants of such. We will not give fully formal definitions of these as they are readily available elsewhere, but the intuition for the simplicial setting is that of a triangulation of the space being studied[11]. Note, however, that each of these is an instance of a (topological) space *plus instructions on how it was built*, so is *not just a space.*

Simplicial complexes come in two flavours, one 'abstract' or combinatorial, the other 'geometric', or, perhaps in better terminology, 'spatial'. Abstractly, a simplicial complex, K, is specified by a set, $V(K)$, of 'vertices' and a set, $S(K)$, of 'simplices', which are non-empty finite subsets of vertices. Not every finite subset need be in $S(K)$, but all singletons, $\{v\}$, $v \in V(K)$, are considered to be simplices of K, and, very importantly, if a set of vertices gives a simplex, σ, of K, then *all* its non-empty subsets are also to be in $S(K)$. These subsets give the *faces* of σ.

As a simple example, suppose $V(K) = \{0, 1, 2, 3, 4\}$, whilst $S(K)$ consists of $\{0, 1, 2\}$, $\{2, 3\}$, $\{3, 4\}$ and all the non-empty subsets of these. We draw this schematically as:



The simplex, $\{0, 1, 2\}$, is a '2-simplex', and is pictured as being 2-dimensional. In general, an n-simplex of a K is a subset in $S(K)$ with $n + 1$ elements. We will write K_n for the set of n-simplices. Note that with the above example, K_n is empty for $n \geq 3$, and that any graph / network corresponds to a simplicial complex with no simplices in dimensions 2 or larger.

That gives a brief outline of the abstract form of the notion of simplicial complex. We can use such a specification as a plan for building a space, roughly as follows[12]. For each $n \geq 0$, we have a space, Δ^n, which is an n-simplex[13]. The idea is now to take, for each n, and each n-simplex, σ, in K, a copy, $K(\sigma)$, of Δ^n, then if τ is an $(n-1)$-simplex in K, which is a face of σ, we identify $K(\tau)$ with the corresponding face of $K(\sigma)$. Doing this for all the simplices gives a space, the geometric realisation of K, but this description suffers from being

[11] Think 'wire frame' image!

[12] This can be done in various equivalent ways, but we will just look at one which generalises easily to simplicial sets.

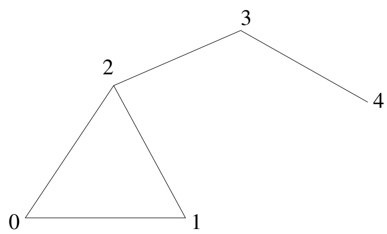
[13] The space, Δ^n, is given, for example, by

$$\{\underline{x} = (x_0, ..., x_n) \in \mathbb{R}^{n+1} | \sum_0^n x_i = 1; \text{ all } x_i \geq 0\}.$$

too informal[14]. Of course, for our example we need five Δ^0s, also five Δ^1s, i.e., geometric segments, and one Δ^2, and the resulting space, clearly, looks like we have pictured it! We think of the combinatorial gadget, K, as 'presenting' the space given by its 'geometric realisation'. Not all spaces can be represented in this way. Those that can are sometimes called 'polyhedra'.

We will return to this below, but now need to give how to go from such a 'presentation' of a space to a presentation of its fundamental groupoid.

Let K be a simplicial complex (and our notation will not distinguish between the combinatorial object and the corresponding space). If we restrict attention to the vertices and the 1-simplices of K, we obtain a graph, K^1, which forms the *1-skeleton*[15] of K. For our simple example, this has the same set of vertices, but $S(K^1)$ does not contain $\{0,1,2\}$, so the picture / space is:



in which there is now a hole, where, in our previous diagram, there was the 2-simplex corresponding to $\{0,1,2\}$.

Going back to the general abstract case, we can form the free groupoid, $F(K^1)$, on this graph. This has the vertices of K as its objects and between two such vertices, morphisms are reduced edge-paths between them. (An edge path is just a list of edges or their inverses in the graph, which are 'composable' so the target of each is the source of the next. We omit the detailed construction of this free groupoid as it is relatively well known and can easily be found in the literature. We note that it is important to choose a direction on each edge in K^1 and we will need a notation for such a directed edge. We will write $\langle v, v'\rangle$ for the edge with source, v, and target, v'. We will usually do more than just ordering the edges, rather we will pick a total order on $V(K)$ and then write $\langle v_0, \ldots, v_n\rangle$ for an n-simplex, $\{v_0, \ldots, v_n\} \in S(K)$, in which $v_0 < v_1 < \ldots < v_n$[16]. Returning to the groupoid, we now form the quotient of $F(K^1)$ by relations that come from the 2-simplices of K:

[14]Discussion of formal definitions of the geometric realisation of a simplicial complex can be found in many books on algebraic topology.

[15]This is not just the set of edges, but also involves the information on two ends of each edge.

[16]This gives a unique 'ordered simplex' representing each element of $S(K)$. It has the additional benefit of allowing us to talk of the k-th face, $d_k(\sigma)$, of a simplex, σ, just by deleting v_k from the simplex, thus in our example, if we take the obvious order on the vertices, $d_1\langle 0,1,2\rangle = \langle 0,2\rangle$, and so on, but note that if we change the total order, this will change the way the faces turn out. If we had ordered the vertices, $\{3 < 2 < 0 < 4 < 1\}$, then although $\{0,1,2\}$ still would be a simplex, now written $\langle 2,0,1\rangle$, we would have $d_1\langle 2,0,1\rangle = \langle 2,1\rangle$.

For each 2-simplex, $\langle v_0, v_1, v_2\rangle$, in K, which we picture as

$$\begin{tikzcd} & v_1 \arrow[dr] & \\ v_0 \arrow[ur] \arrow[rr] & & v_2 \end{tikzcd}$$

we form the relation

$$\langle v_0, v_1\rangle \cdot \langle v_1, v_2\rangle \cdot \langle v_0, v_2\rangle^{-1} = id_{v_0}.$$

If we write $R(K)$ for the set of such relations, then the groupoid, $\Pi_1^{comb}(K)$, with presentation, $\langle K^1 : R(K)\rangle$, is the combinatorial version of the fundamental groupoid of K. It will be isomorphic to the full subgroupoid of $\Pi_1(K)$ formed by the objects corresponding to vertices of the complex, K, and therefore is equivalent, as a groupoid, to $\Pi_1(K)$, but with far fewer objects. (To see, geometrically, why it is equivalent to $\Pi_1(K)$, first note that in $\Pi_1(K)$, any object is a point of K, so is in some simplex of K, and hence can be joined to a vertex of K by a path. We thus have that every such object is isomorphic to one in $\Pi_1^{comb}(K)$. Next given any path in K between two vertices, it will be homotopic[17] to one whose image is within the 1-skeleton of K, and which can be represented by an edge-path, thus by a morphism in $\Pi_1^{comb}(K)$. Finally any homotopy between paths can be replaced[18], up to a second level homotopy, i.e.. a homotopy between homotopies, by one within the '2-skeleton' of K, that is the subcomplex given just by the 0-, 1- and 2-simplices of K. What this implies is that homotopy can be mirrored, algebraically, by 'moves across 2-simplices' and thus by the *rewriting process* associated to the presentation that we gave.) Restricting attention to a single vertex, v_0, the vertex group of $\Pi_1^{comb}(K)$ at v_0 is isomorphic to $\pi_1(K, v_0)$, and gives us Poincaré's combinatorial form of his fundamental group.

This process shows some inadequacies in the simple combinatorial language given to us by simplicial complexes, at least when we want to build an algebraic object from it. For example, in the above description, we wrote id_{v_0} for the identity element at the vertex v_0, that is, the empty edge-path starting at v_0. In the simplicial complex, K, we do *not* have an edge $\langle v_0, v_0\rangle$. One way to get around this is to relax the condition $v_0 < v_1 < \ldots < v_n$ for this ordered set of vertices to be a simplex, replacing $<$ by $\leq$, thus allowing a vertex label to be repeated. For instance, if we take our example and order the vertices in the obvious way, then, as well as the 1-simplices, $\langle 0, 1\rangle$, etc., we would have 'degenerate' 1-simplices, such as $\langle 2, 2\rangle$, and degenerate 2-simplices, such as $\langle 2, 2, 3\rangle$ and $\langle 2, 3, 3\rangle$. We would have simplices in all (positive) dimensions, as a string with n copies of 2, followed by m copies of 3, would give us a degenerate $(n + m - 1)$-simplex. The rule for defining the faces of a simplex would still apply, so $d_0\langle 2, 2, 3\rangle = \langle 2, 3\rangle = d_1\langle 2, 2, 3\rangle$, whilst $d_2\langle 2, 2, 3\rangle = \langle 2, 2\rangle$, a degenerate 1-simplex, so an 'identity edge' at 2.

[17] To visualise this, think of a path in our example, going from 1 to 2, perhaps wandering around within the 2-simplex given by 0, 1, and 2. It could be pushed out (and thus 'homotoped') to the 1-skeleton, and this could be done in several different ways.

[18] The key to all this is a *simplicial approximation theorem*, which can be found in most books on basic algebraic topology.

Let us write K_n for the set of *all* n-simplices, now including the degenerate ones as well[19]. We now not only have the face operators, $d_k : K_n \to K_{n-1}$, $k = 0, \ldots, n$, but also some degeneracy operators, which, in the usual notation, are denoted $s_i : K_n \to K_{n+1}$, where, for instance,

$$s_1\langle v_0, v_1, \ldots, v_n\rangle = \langle v_0, v_1, v_1, \ldots, v_n\rangle,$$

so repeats the vertex label in position 1, whilst

$$s_0\langle v_0, v_1, \ldots, v_n\rangle = \langle v_0, v_0, v_1, \ldots, v_n\rangle,$$

and so on. In our calculation of the faces of $\langle 2, 2, 3\rangle$, which is, of course, $s_0\langle 2, 3\rangle$, we verified that, at least in this case, $d_0 s_0 = d_1 s_0$, but, of course, that is true in general as deleting either copy of a repeated label gets you the same result. Similar reasoning gives other such 'simplicial identities'[20] such as: if $i < j$, then $d_i d_j = d_{j-1} d_i$.

We need to abstract from this example. We here have a structure consisting of a family, $\{K_n\}_{n\geq 0}$, of sets, plus face and degeneracy operations which satisfy the simplicial identities. Such a structure, in general, is called a *simplicial set*[21] and is the second of our ways of 'presenting a space'. Simplicial complexes with a total order on their set of vertices gives just one example of such things, but there are other important examples that do not come from simplicial complexes. We will consider two such.

Example 1: Nerves of small categories. In the above, we could have used any partial order on $V(K)$ for which each simplex of each dimension was ordered. We did not really need a total order for the construction to work although that is the simplest type to work with. More generally, if we have a partially ordered set, $\mathcal{X} = (X, \leq)$, then we can form a simplicial set, $Ner(\mathcal{X})$, by taking its set of n-simplices to consist of all sequences, $x_0 \leq x_1 \leq \ldots \leq x_n$, and with face and degeneracy operators much as in the simplicial complex case we looked at before. A particular case which is very useful is $[n] = \{0, 1, \ldots, n\}$ with the usual order. The simplicial set, $Ner[n]$, is the standard model for the n-simplex, as is evident if you look at low values of n. This is usually written $\Delta[n]$.

The definition of $Ner(\mathcal{X})$ is a special case of the nerve of a (small) category[22]. If $\mathcal{C}$ is an arbitrary small category, we can define a simplicial set, $Ner(\mathcal{C})$, by taking its set of n-simplices to consist of all *composable* sequences of n-arrows in $\mathcal{C}$, that is of form:

$$\sigma = (x_0 \xrightarrow{c_1} x_1 \xrightarrow{c_2} \ldots \xrightarrow{c_{n-1}} x_{n-1} \xrightarrow{c_n} x_n).$$

The set of 0-simplices is simply the set of objects of $\mathcal{C}$. For the face and degeneracy operators, we will leave the details for the reader to search for in the

[19] so, slightly more formally, the basic set up of $V(K)$ and $S(K)$ is still the same, but now $\sigma = \langle v_0, \ldots, v_n\rangle$ stands for a set, after deleting any repetitions, $\{v_0, \ldots v_n\}$ in $S(K)$, with $v_0 \leq v_1 \leq \ldots \leq v_n$, and K_n is the set of all such σ.

[20] We will not give the usual complete list of these *simplicial identities* here, but refer the reader to the standard texts on simplicial sets and related homotopy theory. As useful if slightly old, brief introduction to this theory is to be found in Curtis's survey, [23].

[21] The category of simplicial sets will be denoted $\mathcal{S}$.

[22] Remember that any partially ordered set can be considered as a category with X being the set of objects and there being a single morphism from x to y if and only if $x \leq y$.

literature, but will rather look at the faces of a typical 2-simplex:

$$\sigma := (x_0 \xrightarrow{c_1} x_1 \xrightarrow{c_2} x_2),$$

or, sometimes, more conveniently, in the opposite order and for a general n, $(c_n, \ldots, c_2, c_1)$, recording just the morphisms[23], and which we draw as a triangle:

$$\begin{tikzcd} & x_1 \arrow[dr, "c_2"] & \\ x_0 \arrow[ur, "c_1"] \arrow[rr, "c_2c_1"'] & & x_2 \end{tikzcd}$$

(and the reversal of order, above, allows this to avoid a reversal here to get to c_2c_1). From this perspective, there is a clear idea of what the faces should be: $d_0(\sigma) = (c_2)$, the face opposite the vertex, 0; $d_2(\sigma) = (c_1)$, the face opposite vertex 2; and $d_1(\sigma) = (c_2c_1)$, so given by the composition[24] of the two arrows, and giving the face opposite vertex 1. The degeneracies insert identity maps in a fairly obvious way.

We can give another brief equivalent description of $Ner(\mathcal{C})$. We let $Ner(\mathcal{C})_n = Cat([n], \mathcal{C})$, which is easily seen to be the same as before, but in different language. The face and degeneracy maps are derived from functors / order preserving maps between the various $[n]$.

Example 2: The singular complex functor. We can use the same idea as above to obtain a simplicial set associated to a topological space, X. This is the classical *singular complex*, $Sing(X)$, of X. We use the topological simplices, Δ^n, that we have met earlier. There are face inclusions, $\delta_k : \Delta^{n-1} \to \Delta^n$, for $0 \leq k \leq n$, and some squashing maps, $\sigma_i : \Delta^{n+1} \to \Delta^n$, here given in footnote[25]. These induce face maps,

$$d_i : Sing(X)_n \to Sing(X)_{n-1}, \qquad 0 \leq i \leq n,$$

and degeneracy maps,

$$s_i : Sing(X)_n \to Sing(X)_{n+1}, \qquad 0 \leq i \leq n.$$

Remarks: (i) The singular complex construction is one of the key examples for our 'narrative' about ∞-groupoids and spaces. It is one of the main candidates for something worth calling an ∞-groupoid, and, most importantly, it is easy to construct from a space. We still have a way to go before giving a better idea of what an ∞-groupoid is, but we will be revisiting $Sing(X)$ several times later on.

This singular complex construction is one of several used to encode the results of 'probing' a space by nice 'test objects'. These test objects, in this

[23]The reason for the change in order in the symbol will be clear in a short while.

[24]A very important observation here is that the algebraically defined composition in $\mathcal{C}$ corresponds to the 'geometric' process of filling the (up-side down) V-shaped diagram consisting of the given two arrows. This V-shaped diagram is usually called a $(2,1)$-horn. It looks like the 1-dimensional skeleton of a 2-simplex with the d_1-face omitted.

[25]The face inclusion δ_k, sends $(x_0, \ldots, x_{n-1})$ to $(x_0, \ldots, 0, \ldots, x_{n-1})$, putting a 0 in the k^{th} position. The squashing map, σ_i, adds x_i and x_{i+1} together before placing the result in the i^{th} position then shifting each of the subsequent entries one place to the left.

case the topological simplices, Δ^n, are 'spaces' that are well understood, both in themselves individually and also in their relationship between each other[26].

(ii) We note that the notion of geometric realisation that we sketched for simplicial complexes, extends to one for simplicial sets, for which see any of the standard texts on simplicial homotopy theory. The geometric realisation of a simplicial set is an example of a type of space called a *CW-complex*[27].

(iii) We can also extend the idea of an n-skeleton from simplicial complexes to simplicial sets, but the construction is a little bit more subtle. Given an n and a simplicial set, K, we can form its n-skeleton, $sk_n(K)$, by throwing away the non-degenerate simplices in dimensions greater than n. There will still be simplices in those higher dimensions, but, if $\sigma \in sk_n(K)_m \subseteq K_m$ for $m > n$, then there will be an n-simplex, $\tau \in K$, and a sequence of degeneracy operators whose composite sends τ to σ.

(iv) We met in the footnote to the previous page the idea of a $(2,1)$-horn. This generalises to an (n,k)-horn in a simplicial set, K, for $0 \leq k \leq n$. The $(2,1)$-horn that we considered consisted of two 1-simplices that fitted together as if they formed all but one of the faces of a 2-simplex. (In the case, $K = Ner(\mathcal{C})$ that we looked at, there was a 2-simplex there 'filling the horn', but it is easy to see if we just had a simplicial set coming from a simplicial complex, for instance, the 'horn' might not have such a 'filler'.) A (n,k)-horn in K consists of a collection, $\underline{x} = (x_0, \ldots, x_{k-1}, -, x_{k+1}, \ldots, x_n)$, of $(n-1)$-simplices of K that fit together like all but the k-th face of a n-simplex. An n-simplex, $x \in K_n$ 'fills' the horn if for $j \neq k$, $d_j x = x_j$. The k-horn of a topological n-simplex, Δ^n, is defined in the analogous way as are the k-horns in $\Delta[n]$. In each case one takes the $(n-1)$-skeleton and then removes the k^{th}-face. We will write $\Lambda[k,n]$ for the corresponding simplicial subset of $\Delta[n]$ and note that $\underline{x}$ can be thought of as a simplicial morphism from $\Lambda[k,n]$ to K.

In the singular complex, $Sing(X)$, of a space, X, all horns have fillers since a topological n-simplex can easily be shown to retract onto any of its horns; see any introduction to simplicial sets for more discussion. (This means not only that there is a filler, but such fillers are 'thin' in the same intuitive sense as we mentioned earlier.) Those simplicial sets which satisfy the property of having fillers for all horns are called *Kan complexes* and will be very important later in our discussion. The nerve of a category is a Kan complex if, and only if, the category is a groupoid[28].

(v) An important related idea is that of simplicial object in a category, $\mathcal{C}$. For the case of $\mathcal{C}$ being the category of sets, the simplicial objects are just the

[26] We could have based the discussion on other categories of test objects, for instance, n-cubes or n-globes, of more generally multiprisms, that is, products of topological simplices, Δ^n. Each has nice properties but we will more or less restrict attention to the simplices partially for historical, and expositional reasons, but also because that theory is the most developed one.

[27] A CW-complex is built up from a discrete set of 'vertices' by progressively attaching n-dimensional cells to lower dimensional parts of the space, so we get a filtered space $X_0 \subseteq X_1 \subseteq \ldots \subseteq X_n \subseteq \ldots \subseteq X$ where X_0 is a discrete space, and, for each n, X_n is obtained from X_{n-1} by 'gluing' in some n-discs.

[28] In the nerve of an arbitrary category, all (n,k)-horns for $0 < k < n$, (the so-called 'inner horns'), have fillers given by the composition in the category, but the $(n,0)$- and (n,n)-horns, the 'outer horns', may not have fillers in general. Simplicial sets satisfying the weaker condition that all 'inner horns' have fillers were originally called 'weak Kan complexes', but now the term *quasi-category* is perhaps more often used. They are also one of the models for a class of ∞-category,..., but that is getting ahead of ourselves for the moment.

simplicial sets, of course, but taking $\mathcal{C}$ to be the category of groups, or Abelian groups, will give simplicial groups[29], and simplicial Abelian groups[30].

3 Whitehead's algebraic and combinatorial homotopy

On a seemingly tangential note, we will now consider the more general question of modelling homotopy types with algebraic data. We will see that this general idea lies in the background of our central theme, the point being that the idea of ∞-groupoids is 'algebraic' in some sense, at least in some of the interpretations of the term. This also asks when a term such as 'algebraically given data' can reasonably be thought of in spatial terms, as, for instance, group presentations can lead to spaces. Here we will very briefly set the scene for a fuller presentation of some of the ideas, but before that more detailed treatment, we will have to do some more groundwork introducing notions in the next section that illustrate the ideas here more fully and giving sketches of definitions, etc.

In his 1950 ICM address, J. H. C. Whitehead summarised his vision of what he called *Algebraic Homotopy*:

> *The ultimate aim of algebraic homotopy is to construct a purely algebraic theory, which is equivalent to homotopy theory in the same sort of way that 'analytic' is equivalent to 'pure' projective geometry.*
> J. H. C. Whitehead, [70], (quoted in Baues, [4])

A statement of the aims of 'algebraic homotopy' might thus include the following homotopy classification problems (from the same source, J. H. C. Whitehead, [70]):

> *Classify the homotopy types of polyhedra, X, Y, ..., by algebraic data.*
> *Compute the set of homotopy classes of maps, $[X,Y]$, in terms of the classifying data for X, Y.*

This nicely sets up the idea of modelling (nice) spaces by 'algebra', ..., but does not make precise what 'algebraic data' is to mean. In fact, when modelling spaces by algebraic data, there is nearly always a balance to be struck. More finely structured models are better for classifying the spaces and morphisms, but often the more structure there is, the harder it is to handle it all.[31] One question is thus what does an analysis of this set of problems look like in the context of the putative comparison:

$$\textit{spaces} \leftrightarrow \infty\textit{-groupoids}$$

and another query, again inspired by Whitehead's own list of problems, would

[29]... in which each K_n is a group, and each face and degeneracy maps is a group homomorphism.

[30]These latter objects form a category equivalent to that of chain complexes of Abelian groups, by the Dold-Kan theorem. This uses the Moore complex which is intersection of all the kernels of the face maps, d_k, $k > 0$, giving a chain complex from a simplicial Abelian group. This is closely related to the way we get from 2-groupoids to crossed modules (see footnote on page 24), and will be briefly examined in section 5.3.

[31]This is very well discussed in the early sections of Baues, [5].

be: if we have a finite dimensional space, say a k-dimensional CW, or simplicial, complex, how might that finite dimensionality be reflected in any associated ∞-groupoid?

As we said, detailed algebraic invariants become harder to 'calculate' the more 'detailed' they are! The exact sense of 'calculate' here is quite hard to pin down! Some of the meanings of 'detailed' are easier to explore and we will endeavour to do so. The overall aim is to find algebraic models for homotopy types in the above sense and in particular, here, to come up with a notion of ∞-groupoid which will fit into this Whitehead programme as a suitable form of 'algebraic data'. As a step in that direction we will look at various types of 'algebraic data' and explore their connection with this setting.

Clearly, from today's perspective, the assignment of 'algebraic data' to 'spaces' that Whitehead was proposing has to be functorial[32]. From that viewpoint, the overall aim of Whitehead's Algebraic Homotopy programme is to find natural algebraic models for homotopy types in such a way that the resulting 'functor' is as close to an equivalence of categories as possible. Any functorial homotopy invariant, $F : Top \to AlgebraicData$, will, however, determine a class of morphisms between spaces that become isomorphisms on application of F. By assumption, this class will contain that of homotopy equivalences, but may be much bigger. Controlling such a class for a given modelling functor F is where the more structured notions of homotopy theory come in.

4 Higher homotopy groups, weak homotopy types, truncation and connectedness

As an example of a type of algebraic data that is typical for Whitehead's setting, but one that is very 'minimal', in some sense, we could take a set, to represent the set of arcwise connected components of the space, plus, for each element in that set, an $\mathbb{N}$-indexed family of groups, and loosely take $F(X) = \{\pi_n(X, x_0) \mid x_0 \in X, n \in \mathbb{N}, n \geq 0\}$. The corresponding class of morphisms will be that of *weak homotopy equivalences* (often just called 'weak equivalences').

4.1 Higher homotopy groups, and weak homotopy types

To make sense of this, and it *has* several bits of terminology and notation that we have not yet formally met, we first need to set out the main ideas on the higher homotopy groups, $\pi_n(X, x_0)$, of a pointed space, (X, x_0). We first recall that one of the definitions of the fundamental group of a pointed topological space was as homotopy classes of loops at the base point. We can think of a loop as being a map from the circle, S^1, to X, and, as we want the loop 'at the base point', x_0, of X, we make S^1 into a pointed space by realising it as $\{\underline{x} = (x, y) \in \mathbb{R}^2 \mid ||\underline{x}|| = 1\}$, and then choosing $\underline{1} = (1, 0)$ as its base point. A loop at the base point of X is then a continuous function, $\gamma : S^1 \to X$, satisfying $\gamma(\underline{1}) = x_0$. The fundamental group, $\pi_1(X, x_0)$, is thus $[(S^1, \underline{1}), (X, x_0)]$, i.e., the set of pointed homotopy classes of pointed maps from the pointed circle to the pointed space, (X, x_0), and as, intuitively, this corresponds to studying 1-dimensional holes in X, it is natural to consider the

[32] ... and he was one of the first to adopt an overtly categorical view of such situations.

n-sphere, $S^n = \{\underline{x} \in \mathbb{R}^{n+1} \mid ||\underline{x}|| = 1|\}$, based at $\underline{1} = (1, 0, \ldots, 0)$ and to look at $\pi_n(X, x_0) := [(S^n, \underline{1}), (X, x_0)]$, in an attempt to capture something of the behaviour of the n-dimensional holes in X. This set has a natural group structure if $n > 0$ and that structure is Abelian if $n > 1$. The resulting group is called the n^{th} *homotopy group* of the pointed space, (X, x_0). Of course, if $f : X \to Y$ is a continuous map, then there is an induced group homomorphism,

$$\pi_n(f) : \pi_n(X, x_0) \to \pi_n(Y, f(x_0)).$$

If X is connected, then $\pi_n(X, x_0)$ does not really depend on x_0. If we choose a path from x_0 to some other point x_1, then there is an induced isomorphism from $\pi_n(X, x_0)$ to $\pi_n(X, x_1)$, which depends only on the homotopy class of the path. In fact, in this way we get a functor, $\pi_n(X)$, from the fundamental groupoid, $\Pi_1(X)$, to the category of groups (if $n > 0$) sending the point x, thought of as an object of $\Pi_1(X)$, to $\pi_n(X, x)$. Note that, as well as the case $n = 1$ corresponding to the fundamental group of (X, x_0), the case $n = 0$ gives the pointed set of connected components of (X, x_0), since S^0 is the two point discrete space, $\{-1, 1\}$.

Returning, now, to *weak homotopy equivalences*, these are central to a lot of what follows so we will give a slightly more formal definition:

A continuous map, $f : X \to Y$, between topological spaces is said to be a *weak equivalence* if f induces a bijection, $\pi_0(f) : \pi_0(X) \to \pi_0(Y)$, between the sets of arcwise connected components of the two spaces, and also, for each $x_0 \in X$ and each $n \geq 1$, the induced homomorphism, $\pi_n(f) : \pi_n(X, x_0) \to \pi_n(Y, f(x_0))$, is an isomorphism of groups.

Two spaces are said to have the same *weak homotopy type* if there is a zig-zag of maps between them, all of which maps being weak equivalences.[33]

The loose interpretation is that, if there is a weak equivalence between X and Y, then the set of invariants, π_n, cannot tell the two spaces, X and Y, apart. We note that any homotopy equivalence is a weak homotopy equivalence. In fact, if we restrict to CW-complexes, then weak equivalences *are* exactly homotopy equivalences, by a famous result of J. H. C. Whitehead, but, in general, there are pairs of topological spaces that are weakly equivalent without being homotopy equivalent[34].

Given any space, X, we can form $Sing(X)$ and then take its geometric realisation. The two constructions are adjoint functors and there is a natural map, $|Sing(X)| \to X$, which is a weak homotopy equivalence[35]. As $|Sing(X)|$ is a CW-space, we have any space is weakly equivalent to a CW-space; see below.

[33]We can form a new category from our category of spaces by 'formally inverting' the weak equivalences to form the corresponding *homotopy category*. Two spaces have the same weak homotopy type if they are isomorphic in that homotopy category.

[34]The space known as the Warsaw circle has the same weak homotopy type as the discrete space with two elements, but not the same homotopy type. The study of the algebraic topology of such more general spaces is known as 'shape theory' and that term is also applied, by generalisation, to handle toposes; see the various relevant entries in the nLab, [54].

[35]Strong homotopy equivalence, although useful, does not so readily lead to simply defined, good algebraic invariants. General spaces, i.e. ones that are not 'locally nice' need additional machinery for any effective study. This does relate both to methods of topos theory and of non-commutative geometry, but will not concern us in this chapter where we will almost always be restricting to CW-spaces.

Note that spaces with the extra 'combinatorial' information of being a CW-complex[36] make for a fairly well behaved setting, however even with such CW-complexes, and weak equivalences, this does not completely give the answer to Whitehead's idea of Algebraic Homotopy, since we do not know that there might not be two spaces, X and Y, with an isomorphism, $\theta : F(X) \to F(Y)$, yet there would be no continuous map, $f : X \to Y$, which satisfies $F(f) = \theta$. This is *Whitehead's realisation problem* and we will discuss it in slightly more detail later on. One has to realise both spaces *and morphisms*[37]. This is important for our main theme, as it asks whether the algebra *accurately models* homotopy aspects of the spatial structure!

Given any specific topological situation, the amount of information encoded in the weak homotopy type may be 'unnecessary' or 'unnatural' for the application that is in mind, or it may be impractical to calculate it[38], so it is quite natural to work with a subset of the possible dimensions, thus looking at the homotopy groups, say from 1 up to some given n, or, alternatively, from some integer n onwards to infinity, or over some other suitable range of values, say a segment, $[n, n+k]$, from n to $n+k$ for some integers n and k[39]. With regard to the original query of modelling 'spaces' by '∞-groupoids', these classes of weak homotopy types 'should' correspond to restricted classes of ∞-groupoids', hopefully ones that help one understand the general picture better as well as shedding light on the specific situation.

We will look at several such situations in a bit more detail. In each case, the corresponding F will be different and hence the notion of 'equivalence' being used will change. First, in subsection 4.2, we concentrate on the homotopy groups, π_k, for $k \in [n, \infty)$, and, in fact, will look at spaces that only have non-trivial homotopy groups in that range. The following subsection to that will look at the complementary setting, i.e., where the only non-trivial homotopy groups are concentrated in the range, $[1, n]$, and then will look at one or two classical situations in which there is some information giving the homotopy groups in all dimensions but not enough to determine the homotopy type. In each case there is a suitable choice of functor F, leading to a corresponding class of spaces.

4.2 n-connectedness

As was said above, when studying 'spaces' as (weak) homotopy types[40], one of the evident simplifications to make about the spaces is that some of that structure is trivial. This assumption can then be the starting point for attempts to decompose a given (general) homotopy type somehow into a part for which the assumption holds plus 'the rest'. To a minor extent, we can already see this idea in our restriction to considering arcwise connected spaces. For any space,

[36]..., or rather a *CW-space*, so the existence of a CW-complex structure on the space is required, but no choice of that structure is specified.

[37]... and perhaps homotopies between morphisms, and higher homotopies between them, ...

[38]e.g. even for spheres, it is not known how to work out a general formula for the homotopy groups or to what extent there is one.

[39]We will not follow up on these 'segments' here, but the interested reader can find some useful results in [38], which also puts this into the context of this chapter.

[40]via the information encoded in their homotopy groups

X, we can form $\pi_0(X)$ as a quotient of X, so getting $q : X \to \pi_0(X)$, and, trivially, each fibre of q is arcwise connected as it is a component of X.

There are higher forms of connectedness which can be used to get more useful and interesting instances of this sort of idea.

A space, X, is *n-connected* if $\pi_k(X)$ is trivial for $k = 0, 1, \ldots, n$.

What this means is that in an n-connected space, if we have any continuous map, $f : S^k \to X$, for $0 \leq k \leq n$, then as f must be homotopic to a constant map (at the unmentioned basepoint), we can extend f over the $(k+1)$-disc, D^{k+1}, in the sense that the $(k+1)$-disc, $D^{k+1} = \{\underline{x} \in \mathbb{R}^{k+1} \mid ||\underline{x}|| \leq 1\}$, has the k-sphere as its boundary, and there will be a map, $g : D^{k+1} \to X$, that restricts to f on the subspace S^k.

For $n = 0$, we retrieve the original idea, as this is just saying that X is arcwise connected, since if we have any two points $x_{-1}, x_1 \in X$, then we define a continuous map, $x : S^0 \to X$, by setting $x(-1) = x_{-1}$ and $x(1) = x_1$. If X is 0-connected, then this x extends to $y : D^1 \to X$, that is, as $D^1 \cong [0, 1]$, to an arc joining x_{-1}, and $x_1 \in X$, and conversely, thus '0-connected' = 'arcwise connected'[41].

For $n = 1$, '1-connectedness' is the same as what is often called 'simple connectedness'. It interprets as saying that any loop in X extends to a map of the disc, D^2. This notion is very important since, as we noted earlier, $\pi_1(X)$ acts on all the $\pi_n(X)$, $n \geq 2$, in fact making them into $\pi_1(X)$-modules. If X is simply connected, $\pi_1(X)$ is trivial, so the $\pi_n(X)$ are 'merely' Abelian groups, and so are much easier to classify and use. Simple connectedness connects up with universal covers as, if X has a universal cover, $\tilde{X}$, then the space, $\tilde{X}$, is simply connected[42]. More than this is true, in fact. If $p : Y \to X$ is any connected covering space of X, and Y is simply connected, then Y is a universal covering space for X.

This type of example is the $n = 1$ case of a much more general phenomenon that is quite central to our overall story, but, once again, needs expanding a little first. Recall (from page 7) that a covering space, $Y \to X$, has nice lifting properties. Generalising these is the notion of a fibration; see footnote[43] for the definition. In a fibration, if $b \in B$, the *fibre over* b is the subspace, $F_b := p^{-1}(b)$[44].

[41] To make the exposition slightly easier, we will usually assume in what follows that the spaces considered are arcwise connected, so will not need to mention the information on π_0.

[42] The fundamental group of X is 'still around' in this covering space as it acts on $\tilde{X}$ with X being the quotient space of 'orbits'. This is, of course, another of the classical Poincaré viewpoints.

[43] There are several versions, but roughly, a map $p : E \to B$ is a fibration if, for any CW complex, X and subcomplex, $A \subset X$, and for any commutative diagram

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "i"'] & E \arrow[d, "p"] \\ X \arrow[r, "v"'] \arrow[ur, dashed, "f"] & B \end{tikzcd}$$

there is a map, f, which 'lifts' v to E, (so $pf = v$) and extends u from A to X, (so $fi = u$).

[44] It is important to remember that changing b along a path makes the fibre change, so that,

Examples: a) A covering space is a fibration and its fibres will be discrete spaces.

b) Given any pointed space, $\underline{X} = (X, x_0)$, the set, $P(X)$, of all paths, $\alpha : [0,1] \to X$, which start at x_0, so $\alpha(0) = x_0$, can be given a natural topology so that the map, $p : P(X) \to X$, given by $p(\alpha) = \alpha(1)$, is a fibration, called the *path fibration of* $\underline{X}$, and the fibre at x_0 is the space, $\Omega(\underline{X})$, of loops at the base point of X. (It is usual to write $\Omega(X)$, for simplicity.)

c) The third example is rather a way of making more examples than one itself. If $p : E \to B$ is a fibration and $f : X \to B$ a continuous map, then in the pullback square:

$$\begin{tikzcd} f^*(E) \arrow[r] \arrow[d, "f^*(p)"'] & E \arrow[d, "p"] \\ X \arrow[r, "f"'] & B \end{tikzcd}$$

$f^*(p) : f^*(E) \to X$ is a fibration.

Using these we can shed more light on one version of higher dimensional analogues of covering spaces. Given any CW-space, X, and any $n > 0$, we can construct a space, $X(n)$, containing X as a (closed) subcomplex, and such that (i) for all k, $0 \leq k \leq n$, the induced homomorphism from $\pi_k(X)$ to $\pi_k(X(n))$ is an isomorphism[45] and (ii) $\pi_k(X(n))$ is trivial for $k > n$[46] Now take the path fibration, $P(X(n)) \to X(n)$, and restrict it to X by pulling back along $X \to X(n)$. The resulting fibration will be written $X\langle n\rangle \to X$, and it is easy to show that $X\langle n\rangle$ is n-connected. Its other higher homotopy groups are the same as those of X.

If we go back to the case $n = 1$ and assume that X is connected, then $X(1)$ will have just one non-trivial homotopy group, $\pi_1(X(1)) \cong \pi_1(X)$, and $\pi_k(X\langle 1\rangle) = 1$ if $k = 1$ and is isomorphic to $\pi_k(X)$ for $k > 1$. In other words, $X\langle 1\rangle \to X$ is like the universal cover, and, in fact, its fibre is homotopically equivalent to the underlying set of $\pi_1(X)$.

Often $X\langle n\rangle \to X$ is called *the n-connected cover* of X, but it is not a covering space, although it has many properties that are analogous to those of classical covering spaces, so perhaps a little care has to be taken with the terminology[47].

4.3 n-truncation

We now turn to the type of morphism exemplified by $X \to X(n)$ above. This mapping kills off any information encoded in the homotopy groups of X above

omitting the details, there is a 'homotopy' action of the fundamental groupoid of the base, B, on the set of fibres. In particular, $\pi_1(B, b)$ acts on $\pi_k(F_b)$ for all $k > 0$. The homotopy groups of a fibre, the total space and the base of a fibration are linked by a long exact sequence; see, for instance, Hatcher, [35], p. 376.

[45]We will meet such maps shortly and in more detail.

[46] ..., such spaces are said to be *n-coconnected* in some of the classical literature, but are also called n-truncated.

[47]The above construction is quite tricky to make into a functorial one as it involves constructing $X(n)$, which usually involves choices, but the corresponding problem for simplicial sets has a neat functorial solution using the idea of a coskeleton, that can be found in standard texts on simplicial homotopy.

level n. It *truncates* the homotopy type at dimension n.

Adapting the definition and terminology of weak equivalences, we say a continuous map, $f : X \to Y$, is a *homotopy n-equivalence* (or simply an *n-equivalence*) if it induces an isomorphism, $\pi_k(f) : \pi_k(X) \to \pi_k(Y)$ for $k = 1, \ldots, n$. Two spaces, X and Y, are said to *have the same n-type* if there is a zigzag of n-equivalences joining them. By this we mean that there is a diagram of form:

$$X = X^{(0)} \to X^{(1)} \leftarrow \ldots \leftarrow X^{(2k)} = Y$$

with all the maps n-equivalences.[48]

As we mentioned in the previous section, for any space X, we can build another space, which we denoted $X(n)$, together with an n-equivalence, $X \to X(n)$, such that $\pi_k(X(n)) = 0$ for $k > n$[49]. For this sort of space, it can be very much easier to understand what the 'spaces as ∞-groupoids' paradigm looks like, whilst our previous discussion indicates that there are ways of 'decomposing' a general CW-space, X, into an n-connected piece and an n-type, via a fibration. These ideas, in the main, were already present in Whitehead's conception of Algebraic Homotopy and do not, initially, seem that connected to the ideas of higher category theory. Turning to the models for homotopy n-types, however, we will start to see the beginnings of a link with ∞-groupoids via various types of higher dimensional groupoid encoding more information on a homotopy type, and, to aid this, we will explore n-types and their algebraic models in a bit more detail.

4.3.1 1-types and groupoids:

For $n = 1$, the only non-trivial homotopy group would be $\pi_1(X)$. A 1-type thus corresponds to an isomorphism class of groups. This is not quite all however. The spaces we are looking at are connected, so if we choose any base point, we will get the fundamental group of that pointed space, and if we change base point, we will get an isomorphic fundamental group. The isomorphism between them will be given by a path from one base point to the other, and different paths will usually give different isomorphisms. Because of this, it is better to use the fundamental *groupoid* of the space, $\Pi_1(X)$, even if the space is connected.

For any group (or groupoid), G, we can find a space, a *classifying space*, BG, with that group as its fundamental group and no other non-trivial homotopy groups, so a group(oid) yields a 1-type, modelling algebra by topology[50]; see, for instance, the treatment by Baues in [6], page 18. This illustrates an important aspect / theme of this general program: *to extract algebra from a space and to build a space from algebraic data.*

Looking back, we can see a vague idea emerging when going from the case $n = 0$ to $n = 1$. In forming $\pi_0(X)$, we put an equivalence relation on X, relating two points if there is a path joining them. It is thus the *existence* of a path that

[48] A weak equivalence is an n-equivalence for all n, so we can think of any n-type as being made up of lots of weak homotopy types. Each of these will consist of spaces with the same homotopy groups up to the n^{th} one but after that they will, in general, be different.

[49] Such a space with vanishing homotopy groups above the n^{th} one is itself often called a homotopy n-type, although strictly speaking from the definition that we have given that is an abuse of terminology.

[50] The classifying space of a groupoid is most usually taken to the geometric realisation of the nerve of that groupoid, but can also be constructed starting from a *presentation* of the group(oid).

counts at this stage, not the actual paths. In forming $\Pi_1(X)$, the paths joining the points are promoted to being of value *in themselves*. They become more 'centre stage'. We do not just consider two points of X equivalent because a path *exists*, rather we seriously look at all the paths between them[51] and ask when those paths are, themselves, to be thought of as being 'equivalent' by some 'higher path of paths', i.e., by the existence of a homotopy.

In general, an equivalence relation *encodes* a (special type of) groupoid, but general groupoids encode more information 'stored' in their vertex groups, telling us about the 'automorphisms' of the object at which one is looking[52]. Of course, in an equivalence relation thought of as a groupoid, those vertex groups are trivial.

We can probe that 'vague idea' a bit more. What would be the result if we took the equivalences between paths seriously as well, more-or-less considering them as 'reasons' that two paths are equivalent and thus would be 'identified' in the fundamental groupoid? We can find this sort of idea in algebra to some extent. In the theory of groups, it is not equivalence relations on a group, G, that are useful, but rather congruences. These have compatibility with the multiplication built in. They are 'internal' equivalences within the category of groups and, of course, can be usefully handled by looking at the subgroup of G consisting of those elements which are 'congruent' to the identity. In that way, the congruence is encoded by a *normal* subgroup. Our 'vague idea' about 'reasons between reasons' and also of replacing 'equivalence relations' by 'groupoids', so as to encode non-trivial automorphisms of objects, suggests that it is natural to extend 'internal equivalence relations' to 'internal groupoids', ..., but note that this is *quite natural* when coming from the situation with paths and is not just some abstract generalisation for the sake of it.

That 'natural' progression was not quite the way that the theory developed in the 1940s and 1950s. The reason would seem to be that the notion of groupoid was not that obviously useful for researchers having a good working knowledge of group theory. Groupoids had been introduced by Brandt in 1926, but his use was in other areas of algebra than those adjacent to topology. Schreier did make explicit use of them in topology in 1927 and Reidemeister included the construction of the fundamental groupoid in his book of 1932, but they were still not considered that useful by other topologists. The interaction of the fundamental group(oid) concept and spatial aspects of combinatorial *group theory*, as developed by Reidemeister for applications in knot theory[53] seems to have been central to that work. (This is explored by Krömer in the already cited paper, [41].) This work in combinatorial group theory mirrored, in an algebraic context, ideas that were emerging[54] in homotopy theory which directly related to the idea of calculating homotopy groups from combinatorial models of a space. These advances, however, used the 'normal subgroup' side of the picture rather than the 'congruence' one.

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[51]... perhaps thinking of the paths as the different 'reasons' that the points are to be 'equivalent'.

[52]This idea of a groupoid is thus very useful in classifying situations in which objects have important local symmetries. Often it is useful to replace a quotienting operation by the formation of a groupoid for this reason.

[53]Many early invariants of knots were derived from homotopical invariants of the complement of the knot, via group presentations of its fundamental group, such as those developed by Dehn and Wirtinger.

[54]e.g., Whitehead's paper, [67], of 1941.

4.3.2 2-types, crossed modules and 2-group(oid)s:

The developments that led to an understanding of models for 2-types came from three closely related areas: combinatorial group theory, homotopy theory itself and group cohomology. (We will not go into the third here, other than to say it relates to the homotopy theory of the classifying space of a group.) The first two of these correspond to ideas that were already there in Poincaré's approaches to the fundamental group that we mentioned earlier. We will explore a little the path that led from the case of '1-types / groups' to '2-types / crossed modules and 2-group(oid)s' as this shows the start of a shift of focus towards groupoid methods and then to 'higher dimensional groupoids'. The historical development of the 'new' concepts of crossed modules and 2-groupoids starting from more 'classical' notions shows clearly the beginnings of the progression from a low dimensional notion of 'space' encoding simple relationships between 'points' to one encompassing many dimensional relationships, i.e., higher categories and groupoids.

The nearest of these ideas to the further development of the 'vague idea' comes from Reidemeister, and later Peiffer[55], working on 'identities among relations' of presentations of groups. Given a group, G, it is often usual and useful to label the elements by some alphabet of generators, and then to say which 'words' in the symbols of the alphabet correspond to the same element of the group. In more usual terms, one gives a presentation, $\mathcal{P} = \langle X : R \rangle$, of G, where X is a set of generators, often thought of as a subset[56] of G, and R, a subset[57] of the free group, $F(X)$, on X. The elements of R are often called 'relators' or 'relations[58]'. Reidemeister started looking at 'identities among relations' in the following sense. There is a morphism, $\varphi : F(X) \to G$, given by evaluating each generator as an element of the group and the kernel of φ is the normal closure of R, so as was said above, the elements of $Ker\varphi$ are 'consequences' of R, and thus are words made up of conjugates of relators and their inverses. To study these, we could form a free group on symbols for these conjugates and then see if there were relations between them, that is, 'relations between the relations'. That sounds like our 'vague idea' coming near the surface again, and it is. It is also has a topological aspect to which we turn next.

We will first need another classical definition, namely that of the relative homotopy groups of a (pointed) pair of spaces. We postpone a more detailed description of these to an Appendix (starting on page 31) to this section, but for the moment it suffices to say they give homotopy groups for *pairs* of base-pointed spaces, (X, A), together with natural homomorphisms linking them with the homotopy groups of X and A; for a full treatment see Hatcher, [35], p. 343, or many other texts on homotopy theory.

In 1941, Whitehead examined a problem that is clearly related to the idea of specifying a space, or building it, by iteratively attaching cells to 'lower dimensional' parts as in a simplicial or CW-complex. He asked what would

[55] This research would seem to have been done early in the 1940s, but publication was delayed until 1949; see [61] and [55].

[56] ..., but that can sometimes be inconvenient,

[57] The congruence on $F(X)$ which identifies words, i.e., elements of $F(X)$, if they represent the same element of G, corresponds to the normal subgroup of $F(X)$ generated by the elements of R and all their conjugates. The words in the elements of R and their conjugates are called 'consequences' of R. The 'relators' in R thus help us to understand the congruence.

[58] ... although that latter term is a bit confusing in our context.

happen to the invariants, and in particular the homotopy groups, if extra cells were added to such a complex. The key set-up is thus a space, A, to which one attaches ('glues on') some cells to get a new space, X. Knowing information on the homotopy groups of A, and how the new cells were attached, what can be said about the homotopy type of X and the corresponding invariants? Before discussing the structure that he revealed, let us see why this is related to the question of identities among relations.

Suppose $\mathcal{P} = \langle Y : R \rangle$ is a presentation of a group, G, then we can form a CW-complex, $K(\mathcal{P})$, having a single vertex, with 1-skeleton consisting of a collection of pointed loops or circles, one for each generator (and indexed by the set, Y), all attached at their base points to that single vertex. (This will make up our space A, in this case.) The fundamental group of A is a free group[59] on the set Y. Each relation, $r \in R$, corresponds to (the homotopy class of) some map $f_r : S^1 \to A$, and we use f_r to attach a 2-cell to A. Doing this for all r gives us a 2-dimensional complex, $K(\mathcal{P})$, which is the X in our statement of the general problem that Whitehead considered. (This is the start of a process that can lead to a *small* complex having the homotopy type of the classifying space[60], BG, of G.) The relative homotopy group, $\pi_2(X, A)$, and the boundary map, $\partial : \pi_2(X, A) \to \pi_1(A)$, encode interesting and useful information about the presentation *and* the group, G. The kernel of ∂ is isomorphic to $\pi_2(X)$, and can be interpreted in the case of $K(\mathcal{P})$ as the G-module of identities amongst the relations of the presentation.

Whitehead's 1941 paper examined the general case and, in the situation, as here, of attaching *2-dimensional* discs to a complex A, he identified the structure of the algebraic object, $(\pi_2(X, A), \pi_1(A), \partial)$, encoding the way the cells were attached[61]. This algebraic structure was what is called a *crossed module*[62]. Just as replacing an equivalence relation by a groupoid encoded more of the spatial structure of paths in a space, so replacing a congruence in the form of a normal subgroup by a crossed module, continues that 'vague idea' corresponding to replacing the 'congruence', thought of as an internal 'equivalence relation' by an internal 'groupoid' and, yes, crossed modules *do* correspond to a form of '2-groupoid', as we will see.

In 1949 - 50, Mac Lane and Whitehead, [49], combined these ideas with some from group cohomology to show that 2-types corresponded to these *crossed modules*[63]. We will give the definition and some basic, but important, examples, but will not develop the theory here.

Definition: A *crossed module*, (C, G, δ), consists of groups, C and G, with a (left) action of G on C, written $(g, c) \to {}^g c$ for $g \in G$, $c \in C$, and a group homomorphism, $\delta : C \to G$, satisfying the following conditions:

(1) for all $c \in C$ and $g \in G$, $\delta({}^g c) = g\delta(c)g^{-1}$,

[59] The elements of $\pi_1(A)$ are homotopy classes of paths that go around the loops, and a word in Y can be used to encode the order in which this happens.

[60] The data needed to build such a small complex and thus to encode the homotopy type of BG may be finite, even when the group G is infinite. This is important for understanding properties of G.

[61] In [67], Whitehead considers and solves the problem for the case of general n-dimensional discs, but, for the immediate story here, the $n = 2$ case is the most important.

[62] The term seems first to have been used by Whitehead about that time.

[63] Warning: the definition of n-types has changed since that date, so their '3-types' are now usually called '2-types'.

and

(2) for all $c_1, c_2 \in C$, ${}^{\delta(c_2)}c_1 = c_2 c_1 c_2^{-1}$.[64]

Examples[65]: (i) Suppose that δ is just the inclusion of a subgroup of G, and that the action of G on that subgroup is by conjugation, then to say (C, G, ∂) is a crossed module just says that C is a *normal* subgroup of G, thus a normal subgroup yields a crossed module. (As we suggested above, a crossed module is, from this viewpoint, a generalisation of a normal subgroup in which we ditch the requirement of 'being a subgroup'!)

(ii) At the other extreme, if δ is the trivial homomorphism, then C will be Abelian and is just a G-module.

4.3.3 2-groupoids, crossed modules and the Mac Lane-Whitehead theorem.

Much later than the Mac Lane - Whitehead result on 2-types, some time about 1965, Verdier noticed that crossed modules corresponded to what would now be called '2-groupoids', that is, 2-categories[66] in which every 1-arrow and every 2-arrow is invertible. Crossed modules of groups, as we have defined them above, correspond to *2-groups*, that is 2-groupoids with exactly one object. This relationship was rediscovered by Brown and Spencer in 1972, see the introduction to [12], and a closely related result appeared, at about the same time, in the thesis, [63], of Grothendieck's student, Hoàng Xuân Sính. Her result, and Verdier's original one, relate to non-Abelian cohomology and the representation of cohomology classes, and this theme comes in later in some of Grothendieck's letters to Breen.

The method of going from crossed modules to 2-groups is quite simple. It is a simple extension of the way one replaces a congruence on a group by a normal subgroup, without loosing information. Here it will be relegated to the footnotes[67], so as not to interrupt the main flow of ideas. It does depend on

[64] There is a fairly obvious many object / non-connected version of this.

[65] We direct the reader to the literature for more examples and the development of the elementary theory of crossed modules. These can be found in the book, [11], and in numerous other sources, both surveys and original articles.

[66] We will look at 2-categories and 2-groupoids in a bit more detail very shortly in section 4.5. Here we only need the idea that 2-categories are like categories, but with objects, morphisms(= 1-arrows) between them and, in addition, 2-arrows between (parallel) 1-arrows.

[67] Given a crossed module, $\mathsf{C} = (C, G, \delta)$, (of groups), the corresponding 2-category, denoted $\mathcal{X}(\mathsf{C})$ here, has a single object, which we will denote $*$, the set of 1-arrows from $*$ to itself is the set of elements of G, with composition being its multiplication, the set of 2-arrows is the set of pairs (c, g), but with horizontal composition given by the multiplication of the semi-direct product group, $C \rtimes G$. The 1-source of a 2-arrow (c, g) is g, whilst its 1-target will be $\delta c.g$. This makes it look a bit like:

$$\begin{tikzcd} * \arrow[r, bend left=50, "g"] \arrow[r, bend right=50, "\delta c.g"'] \arrow[r, phantom, "\Downarrow (c,g)"] & * \end{tikzcd}.$$

(That this picture looks like two paths / loops and a homotopy between them is, of course, more than coincidental!)

Coming back from 2-group(oid)s to crossed modules is now easy. You look at the 1-arrows as the bottom group of the crossed module, and then the top group will be the group of 2-arrows with source the identity element of the group of 1-arrows. (That looks hopeful for a having some idea of what a 2-category is, but is otherwise quite simple. We will discuss various simple ideas about 2-categories and related structures slightly later, see section 4.5.

The result of Mac Lane and Whitehead thus says that 2-types correspond to 2-groupoids. There is, however, a 'but'. The classification of crossed modules / 2-groupoids that it needs is not that of 'up to isomorphism', rather it uses an algebraic form of homotopy equivalence, adapted for crossed modules. In fact, an even better way to think of it is that the category of 2-types should have a 2-category structure as should that of crossed modules, in which the 2-cells encode the homotopies.

We can use the relative homotopy groups that we met earlier to give the explicit functor from simplicial (or CW-) complexes to crossed modules, and which is the basis for the Mac Lane - Whitehead result. We assume that K is a simplicial or CW-complex and in our relative crossed module of a pair (X, A), we take A to be $K^{(1)}$, the 1-skeleton of K, and so get a crossed module, $(\pi_2(K, K^{(1)}), \pi_1(K^{(1)}), \partial)$, from the complex. This looks very good as an invariant of the CW-space, K, until one realises that it depends on the specified combinatorial structure of the complex and thus on the 'triangulation' or 'cellular decomposition' of the space, rather than just on the *topological* structure of the 'space', K. It is an analogue of Poincaré's combinatorial definition of the fundamental group, but does depend more on the combinatorial structure. A subdivision of the (simplicial) complex structure will give another *non-isomorphic* crossed module. What Mac Lane and Whitehead do in [49] is to analyse how such a combinatorial change is reflected by a 'combinatorial homotopy' of crossed modules[68].

Crossed modules are, thus, the 'slimmed down' encoding of a 2-group(oid), and we have associated a 2-groupoid to a (CW-) space, albeit by choosing a CW-structure on it.

4.3.4 n-types for $n \geq 3$

The story of 'modelling' n-types was really only continued in the 1980s by Loday, [46], (see also Bullejos, Cegarra, and Duskin, [13] and my own, [58]). Loday introduced generalisations of crossed modules and 2-groupoids valid 'for all n'. His models were *strict* n-fold groupoids, a slightly different form of 'multiple groupoid' than we will be considering, but still an indication of that somewhat elusive link between spaces and ∞-groupoids. Loday was able to use 'strict' objects because his models are very 'spread out'[69]. We have not the

generalisation to higher dimensions!)

[68]Note the date and title of Whitehead's two papers, [68, 69]. He envisaged, as part of algebraic homotopy, a 'combinatorial homotopy' which would extend ideas and methods of *combinatorial group theory* to higher dimensions. The introductions to these papers contain important reflections on homotopy types, and algebraic models for them.

[69]For instance, for 3-types, the structure corresponding to the extra data required for encoding 'weak' 3-groupoids, for instance, is given by his h-maps which are derived from commutators in the group structures.

Two 'criticisms' of Loday's models are that (i) from the point of view of 'spaces', their interpretation is, perhaps, less intuitive than one initially might hope for, and (ii) it is not clear if there is a way of adapting the theory to handle the case of $n = \infty$. (This may be just a question of looking at the structures in the 'right way', but that 'right way' is not yet obvious.)

space here to describe Loday's construction in more detail, although it leads to some interesting points relevant to our themes[70].

We should also mention Conduché's notion of 2-crossed module; see Conduché, [17], and the closely related notion of quadratic module, due to Baues, [5], but will not give details as that would require some bcakground that we have not assumed. Both provide models of connected 3-types.

4.4 Beyond 2-types towards infinity groupoids, from a classical/strict viewpoint

From our point of view, the Mac Lane - Whitehead result shows 2-types and 2-group(oid)s are closely linked. Whitehead's Combinatorial Homotopy papers, [68, 69], also talked of algebraic models that provide (usually incomplete) information in all dimensions. Surprisingly enough these also have an interpretation in terms of ∞-groupoids, but we will take a slightly leisurely approach using Whitehead's classical homotopy theoretic machinery rather than going directly to the ∞-groupoid model[71].

The use of the relative homotopy groups by Mac Lane and Whitehead, and the resulting crossed module used to model a 2-type, fits into another sequence of models which link into a classical construction due to Blakers, (1948), and which was developed further by Whitehead (1949) and then by Brown and Higgins, Baues and others[72] from the 1970s onwards.

As a first step towards them, we look at the chains on the universal cover of a CW-complex. This was one of the classical tools used by Whitehead in his key papers. This allows us to state a result that illustrates some of Whitehead's ideas simply and clearly.

First a little background, we mentioned the universal covering space, $\tilde{X}$, of a (nice) space, X. If X has a CW-complex structure, then the local homeomorphism property of covering maps allows one to obtain a CW-complex structure on $\tilde{X}$ for which the covering map, $p : \tilde{X} \to X$, is a cellular map.

Any CW-complex, X, gives rise to a complex, $C(X)$, of 'cellular chains', (see Hatcher, [35]), and, for the universal cover, $\tilde{X}$, the action of $\pi_1(X)$ on that space transfers to the chain complex, $C(\tilde{X})$, of chains on the universal cover, giving it the structure of a chain complex of modules over $\pi_1(X)$, that we will again use shortly. (We will denote the homology of this complex by $H_*(\tilde{X})$.)

To show the relevance of this for our 'quest' for algebra mirroring homotopy, we note the following theorem of Whitehead:

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[70] There *is* a topological interpretation of the construction that Loday uses and which provides an interesting insight on the whole question of what 'spaces' are. Loday does not work with a pair, (X, A), of spaces as such, rather he converts the inclusion $A \hookrightarrow X$ into a fibration, $\overline{A} \to X$, with $\overline{A}$ being the space of paths in X that start in A.

Given any (pointed) fibration $p : E \to B$, with fibre $F = p^{-1}(b_0)$, it is not hard to see that there is an action of $\pi_1(E)$ on $\pi_1(F)$, and that the inclusion $inc : F \hookrightarrow E$ induces a morphism, $\pi_1(inc) : \pi_1(F) \to \pi_1(E)$, which satisfies the crossed module axioms. Better than that, on converting this to the corresponding 2-groupoid, you get that there is a structure of a weak 2-groupoid on the pullback of E with itself (over B). (This seems to have been first noticed by Deligne, see Friedlander's paper, [26].)

Loday's general construction takes a (fibrant) $(n + 1)$-cube of fibrations and constructs an n-fold groupoid from it.

[71] Note that this is algebraic topology from the end of the 1940s, but is not that well known.

[72] For more detail on the history of this, see [11], p. 255, note 96. These models, which are called *crossed complexes*, are equivalent to a special class of ∞-groupoids.

A map, $f : X \to Y$, of (pointed connected) CW-complexes is a homotopy equivalence if, and only if, the induced homomorphisms, $\pi_1(f) : \pi_1(X) \to \pi_1(Y)$ and $H_n(\tilde{f}) : H_n(\tilde{X}) \to H_n(\tilde{Y})$, for all $n \geq 2$, are isomorphisms.

We thus have not only do the homotopy groups constitute a system of algebraic invariants sufficiently powerful to characterise the homotopy type of a CW-space, but so does the combination of π_1 and the homology of the universal covering space. Importantly, however, this does *not* mean that it solves all the basic problems of 'algebraic homotopy'. We could have isomorphisms, $\phi_n : \pi_n(X) \to \pi_n(Y)$, of all the homotopy groups of two spaces, or of the homology of their universal covers, but would not know if there was an $f : X \to Y$ realising these ϕ_n.

On the other hand, the question about how dimension of a complex might be reflected in the models is here very easy to resolve. If X is a CW-complex of dimension k, then so is $\tilde{X}$, and as the generators of $C(\tilde{X})$ in dimension n are the n-cells, we immediately have that $C(\tilde{X})$ is trivial in dimensions greater than k. Whitehead used this in his two papers on Combinatorial Homotopy, [68, 69], to show that the cellular chain complex of the universal covering does act as an 'algebraic equivalent' of a 3-dimensional polyhedron, so here the algebra does reflect a lot of the geometry of the space.[73]

The compatibility conditions that would be needed between the input data[74] must thus be part of the key to understanding the structure of homotopy types. Baues, in [6], calls the problem of finding necessary and sufficient conditions for this, the *realisation problem of Whitehead.* For any system of algebraic invariants, there will be a similar realisation problem[75]. In an attempt to study these, Whitehead introduced another algebraic model, which is nowadays called a *crossed complex.* These crossed complexes are equivalent to a special class of ∞-groupoids in the same way that crossed modules are an equivalent algebraic model to 2-groupoids. We will merely give the idea of the definition, referring to [11] for much fuller information.

A *crossed complex* is a chain complex of groups (or groupoids), (C_n, ∂), where C_n is defined for $n > 0$,

$$\ldots C_n \xrightarrow{\partial} C_{n-1} \xrightarrow{\partial} \ldots \xrightarrow{\partial} C_3 \xrightarrow{\partial} C_2 \xrightarrow{\partial} C_1,$$

in which there is an action of C_1 on all the terms, with ∂ respecting that action, (C_2, C_1, ∂) is a crossed module and, for $n \geq 3$, C_n is Abelian, and, in fact, is a $C_1/\partial C_2$-module.

This 'model' will therefore have something of a crossed module / 2-type in it and, as that crossed module has 'fundamental group' $C_1/\partial C_2$, it has something of the 'chains on the universal cover' model that we saw just now. It is a natural abstraction from the following motivating example in which the two parts fit exactly as required.

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[73]It is an interesting question to see the explicit relationship between these results and models of such spaces by ∞-groupoids. (Perhaps this is known, but I cannot recall seeing such a study.)

[74]in the above case, the ϕ_n and the actions,

[75]The problem is discussed in detail in that source, so we will not repeat here what is said there.

Example: Let K be a CW-complex and, for each n, let $K^{(n)}$ denote its n-skeleton[76]. We set, for $n \geq 2$, $C_n = \pi_n(K^{(n)}, K^{(n-1)}, x_0)$ and, when $n = 1$, we just take C_1 to be the fundamental group of $K^{(1)}$. The boundary ∂ will be the composite,

$$\pi_n(K^{(n)}, K^{(n-1)}, x_0) \to \pi_{n-1}(K^{(n-1)}, x_0) \to \pi_{n-1}(K^{(n-1)}, K^{(n-2)}, x_0).$$

As we have indicated earlier, $\pi_1(K^{(1)}, x_0)$ will act on all the higher relative homotopy groups[77] in the complex, and this gives a crossed complex.

This still seems very far from ∞-groupoids. (Historically we are still in the late 1940s or early 1950s!) The category of crossed complexes (over groupoids) is, however, equivalent to a category of *strict*[78] ∞-groupoids, [10][79]. We can think of these '∞-groupoids' as having not only objects and arrows, which will have 'inverses', as with an ordinary groupoid, but also having invertible 2-arrows / 2-cells joining the (1-)arrows, invertible 3-arrows joining certain pairs of 2-arrows, and so on. Once again our 'vague idea' is coming into evidence.

We cannot give the detailed description of the equivalence between these objects and crossed complexes here, but note that if we have a strict ∞-groupoid, H, then the module of n-cells in the associated crossed complex is the group of those n-cells of H, whose $(n-1)$-source is an $(n-1)$-fold identity. This thus generalises the way we mentioned of getting from a 2-group to a crossed module[80]. Note that these strict ∞-groupoids do *not* answer the general Whitehead's 'algebraic homotopy' problem however. The point is well made in the book by Brown, Higgins and Sivera, [11], ('Why crossed complexes?' page xxvii). Quoting that source:

> *Crossed complexes give a kind of linear model of homotopy types which includes all 2-types. Thus although they are not the most general model by any means (they do not contain quadratic information such as Whitehead products), this simplicity makes them easier to handle and to relate to classical tools. The new methods and results obtained for crossed complexes can be used as a model for more complicated situations.*

It is this very 'linearity' which means crossed complexes do not trap the whole of the information on the homotopy type. It also means that they correspond to a strict ∞-groupoid and not the fully general type of ∞-groupoids. For the relevance of this to Whitehead's realisation problem, this means that just as the chains on the universal cover do not capture the whole of the 2-type structure, although the fundamental crossed complex captures the 2-type, it does not capture the 3-type. Going further, Baues, [5], defines a 'quadratic'

[76] For simplicity, we will assume that K is 'reduced', so is connected and, in fact, $K^{(0)}$ consists of just one point, x_0. This means we can limit ourselves to groups rather than groupoids, which, although more natural, do require a bit more 'setting up' if they are to be handled well.

[77] Recall these are looked at in more detail in the Appendix to this section

[78] The meaning of *strict* should become apparent shortly.

[79] This was discovered by Brown and Higgins in their study of higher dimensional analogues of van Kampen's theorem; see [11] for a full treatment and discussion of that theorem.

[80] or from a congruence to the corresponding normal subgroup.

model (quadratic complexes), but whilst it captures the 3-type, ..., and so on[81].

4.5 From '2- ' heading to 'infinity', from Grothendieck's viewpoint.

To jump from the fundamental group and covering spaces, or from crossed complexes, all the way to

$$\textit{spaces} \leftrightarrow \infty\textit{-groupoids}$$

seems a very high step to jump. The next stage in understanding that link, both historically and conceptually, is via various intermediate stages that indicate that *there is something to jump to*! As a first small step, we need to gain some idea about infinity categories and groupoids in the sense needed later. In so doing, we will also encounter some other useful ideas.

The ∞-groupoids that we will be meeting later will not usually be 'strict', i.e., they will be infinite dimensional analogues of *bicategories*, with all arrows in all dimensions being 'weakly invertible' so we should first glance at bicategories as being the simplest such 'weak' context. (We will only give a sketch as usual, leaving the reader to follow up links to the literature. Introductions can be found in Leinster, [44], and Lack, [42], for instance.)

A (strict) 2-category, $\mathbb{A}$, is a category *enriched over* the category of small categories, so each 'hom' $\mathbb{A}(x, y)$ is a small category and the composition,

$$\mathbb{A}(x, y) \times \mathbb{A}(y, z) \to \mathbb{A}(x, z),$$

is a functor. Composition is associative, so given objects, w, x, y, z, the square of functors,

$$\begin{tikzcd}
\mathbb{A}(w,x) \times \mathbb{A}(x,y) \times \mathbb{A}(y,z) \arrow[r] \arrow[d] & \mathbb{A}(w,x) \times \mathbb{A}(x,z) \arrow[d] \\
\mathbb{A}(w,y) \times \mathbb{A}(y,z) \arrow[r] & \mathbb{A}(w,z),
\end{tikzcd}$$

is commutative; similarly for the identities. We have already seen some *2-groupoids* above, and they are, of course, the corresponding structures in which the $\mathbb{A}(x, y)$ are groupoids and also there is an inversion operation from each $\mathbb{A}(x, y)$ to the corresponding $\mathbb{A}(y, x)$ satisfying some hopefully obvious axioms.

In a *bicategory*[82], $\mathbb{B}$, although the basic structure looks the same, the corresponding diagrams are only required to commute up to specified natural isomorphisms. An important special case is that in which $\mathbb{B}$ has just one object. Such bicategories correspond to the monoidal categories that are wide-spread in algebraic contexts[83].

It is fairly easy to define the notion of a strict ∞-category, generalising that of strict 2-categories, and thus to define strict ∞-groupoids as being a special

[81] What is not always that clear is how the extra structure at each stage corresponds to adding in 'weakness' into a notion of ∞-groupoid, bit by bit.

[82] Note that the terminology is perhaps slowly changing from that used initially in this area and, as these bicategories are seen as being more natural and significant than the strict form, the term 'bicategory' is now often replaced by '2-category' with the older meaning of that latter term corresponding to 'strict 2-category'.

[83] As was mentioned earlier, a category with one object is essentially just a monoid. A bicategory, $\mathbb{B}$, with just one object, $*$, will have a category, $\mathbb{B}(*, *)$, forming the endomorphisms

class of such. Such strict ∞-groupoids correspond to crossed complexes[84]. We will need the weak form of ∞-category, and that, in some sense, is much harder to 'get right'. There are, perhaps, two main problems here: (i) how 'weak' should this be? Do we just weaken things like associativity, and the need for composition to be a functor, or do we also weaken composition to being 'existence of a composite' up to higher homotopies? and (ii) do we base things on globes, simplices, cubes or what?

We must now look to the 1970s when Grothendieck exchanged a series of letters, [27–29], with Larry Breen. In these he sketched out a theory of objects that he called 'n-stacks' and of the possible analogue of the Galois-Poincaré theory, (see [33]), in higher dimensions. We will return to that later, but, for the moment, we will just note that this series of letters resurfaced in February 1983, being mentioned in the 'letter to Quillen', which formed the first six pages of Grothendieck's epic manuscript 'Pursuing Stacks', [32][85].

To set the scene for that, Ronnie Brown had just sent Grothendieck some of the preprints produced in Bangor, in which ideas on (strict) infinity groupoid models (thus crossed complexes) for some aspects of homotopy types were discussed. Grothendieck wrote to Quillen:

> *At first sight, it seemed to me that the Bangor group had indeed come to work out (quite independently) one basic intuition of the program I had envisaged in those letters to Larry Breen – namely the study of n-truncated homotopy types (of semi-simplicial sets, or of topological spaces) was essentially equivalent to the study of so-called n-groupoids (where n is a natural integer). This is expected to be achieved by associating to any space (say) X its 'fundamental n-groupoid'* $\Pi_n(X)$*, generalizing the familiar Poincaré fundamental groupoid for* $n = 1$*. The obvious idea is that 0-objects of* $\Pi_n(X)$ *should be points of X, 1-objects should be 'homotopies' or paths between points, 2-objects should be homotopies between 1-objects, etc. This* $\Pi_n(X)$ *should embody the n-truncated homotopy type of X in much the same way as for* $n = 1$ *the usual fundamental groupoid embodies the 1-truncated homotopy type. For two spaces X, Y, the set of homotopy classes of maps* $X \to Y$ *(more correctly, for general X, Y, the maps of X into Y, in the homotopy category) should correspond to n-equivalence classes of n-functors from* $\Pi_n(X)$ *to* $\Pi_n(Y)$ *–, etc. There are very strong suggestions for a nice formalism including a notion of geometric realization of an n-groupoid, which should imply that any n-groupoid is n-equivalent to a* $\Pi_n(X)$*. Moreover*

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of $*$ and then a functor

$$\mathbb{B}(*,*) \times \mathbb{B}(*,*) \to \mathbb{B}(*,*),$$

which is then the tensor product, $\otimes$, of the corresponding monoidal category. This argument can be reversed, so any monoidal category can be thought of as a single object bicategory. The monoidal category is *strict* exactly when the bicategory, $\mathbb{B}$, is a 2-category.

[84]The process of passing from a strict ∞-groupoid to a crossed complex is a natural generalisation of the one we sketched earlier for 2-groupoids. One takes kernels of the source maps.

[85]It is worth remarking that, both here and later, Grothendieck refers to some ideas coming from algebraic geometry and non-Abelian cohomology, but our use of these texts will not require any real knowledge of that area. It just feels strange to cut up the quotation in an attempt to make a text without such mentions.

when the notion of n-groupoid (or more generally of an n-category) is relativised over an arbitrary topos to the notion of an n-gerbe (or more generally, an n-stack) these become the natural 'coefficients' for a formalism of non-commutative cohomological algebra, in the spirit of Giraud's thesis.

Later in the letter, he noted that these n-categories would have non-associative compositions including 'whiskerings' (as they are now called) of all types, but that the non-associativity would be up to a cell in the next higher dimension. This is one of the points of the idea of 'weak' as against 'strict' in these n-categories and groupoids. The composition would be associative 'up to higher cells', just as path composition is associative up to specified homotopies; see page 5. Likewise he points out that n-objects would be invertible up to an $(n+1)$-object.

4.6 Appendix: more technical comments.

We will pick up the main 'narrative' in the next full section, but before that will collect up, here, a few longer comments that may provide some more detail or insight into things that we have been looking at, but which can safely be skipped or skimmed on a first reading.

(a) *Relative homotopy groups:* We will give this in general as it can be useful later on, even though, for our main use, it was only the case $n = 2$ that is required[86].

The n^{th} homotopy group of a pointed space, (X, x_0), can be defined in another way from that we used above. In this alternative definition, it consists of homotopy classes of maps of pairs, $f : (I^n, \partial I^n) \to (X, \{x_0\})$, the homotopies being through maps of the same form. In other words, the maps, f, send the n-cube, I^n, to X in such a way as to send its boundary, ∂I^n, to the single point x_0, and the homotopies used must deform the way the interior is mapped whilst not changing the behaviour on the boundary.

For the relative homotopy groups of a pair, (X, A), with $A \subseteq X$, and a basepoint $x_0 \in A$, we again use the n-cube, I^n, and think of it as $I \times I^{n-1}$. Let $J^{n-1} = \{1\} \times I^{n-1} \cup I \times \partial I^{n-1}$, so consisting of all the faces of I^n except (the interior of) $\{0\} \times I^{n-1}$, then $\pi_n(X, A, x_0)$ is defined to be the set of homotopy classes of maps, $f : (I^n, \partial I^n, J^{n-1}) \to (X, A, \{x_0\})$ with homotopies being through maps of the same form[87].

Restricting such a map to $\{0\} \times I^{n-1}$, we get an element of $\pi_{n-1}(A, x_0)$, and this assignment gives a homomorphism,

$$\partial : \pi_n(X, A, x_0) \to \pi_{n-1}(A, x_0).$$

The above groups 'depend on the choice of base point' in the same way as we saw earlier. This means that, for a given n and a choice of basepoint, x_0, the groups, $\pi_n(X, A, x_0)$, do not depend on x_0 if A is pathwise connected, and

[86]Again, this will be a sketch, and for fuller details, the reader should 'consult the literature'. Fuller expositions can be found, for instance, in Hatcher's book, [35], p. 343, and, especially relevantly for us, in the book by Brown, Higgins and Sivera, [11], p. 35.

[87]We thus have all-but-one of the faces of the n-cube are sent to the base-point, whilst the last face is sent into A. We will see a similar situation later when looking at simplicial groups.

there is an action of the fundamental groupoid of A on the family of groups, $(\pi_n(X, A, x_0) \mid x_0 \in A)$, which can contain valuable information.

(b) *From chain complexes to crossed complexes:* We have met, above, the fundamental crossed complex of a CW-complex. Another type of crossed complex comes from chain complexes of modules over a group G. Suppose G is a group and (M, δ) is a (positively graded) chain complex of left G-modules. We can form a crossed complex by taking $C_1 = M_1 \rtimes G$, and $C_n = M_n$ for $n \geq 2$. We then take all the 'boundary' maps, ∂ to be δ, except for $\partial : M_2 \to M_1 \rtimes G$, which sends m to $(\partial m, 1_G)$. Finally we make $(m, g) \in C_1$ act on higher dimensions using just the action of g.

This construction, in fact, gives a functor from the category consisting of such pairs, (G, M), to the category, $CrsComp$, of crossed complexes. This functor has a left adjoint.

An especially important instance of this is when we apply that adjoint to the fundamental crossed complex of a CW-complex. X. This gives a chain complex of $\pi_1(X)$-modules, which is isomorphic to the complex of chains on the universal cover of X. There is, thus, a direct functorial construction going from the fundamental crossed complex, $\pi(X)$, of a CW-complex, X, to this crossed complex[88]. Applying this, *the crossed complex of a CW-complex must be at least as good at distinguishing spaces and homotopy equivalences as is the cellular homology of the universal covering because we can functorially derive the latter from the former.*

(c) *What kind of homotopy types are completely captured by crossed complexes?* This is a very natural question to ask. Each time we find a 'model' such as these, we have two related questions. The first is: how good a model is it? The other is: what kind of homotopy types are captured by the models, that is, are completely modelled by a model of that type? That the two questions are related is clear if we look at 1-types. The model of a 1-type is given by a group. Two spaces cannot be distinguished by their 'models' if they have the same fundamental group, so one might say the model is not that good (but in fact it is still very useful!). A space is completely modelled by its fundamental group if, and only if, all its other homotopy groups are trivial, but any space has the same 1-type as a space whose higher homotopy groups, π_k, for $k > 1$, are trivial[89], which shows how the two answers are connected.

That is quite easy and can, with a bit of work, be extended without much change to 2-types, and even general n-types. A bit more work is needed to describe the homotopy types that correspond to crossed complexes. It is relatively easy to show that there is a classifying space construction for crossed complexes and that there is a continuous map, $K \to B\pi(K)$, from a complex, K to the classifying space of its fundamental crossed complex. The question is thus of determining when this map is a weak equivalence. We know that a homotopy type leads to a crossed complex. What we hope for is to be able to say what properties a homotopy type has if this encoding in terms of crossed complexes is enough to give all the possible information, i.e., that the crossed complex 'characterises' the homotopy type.

There are higher order 'pairings' or 'actions' within the homotopy groups of a space given by the Whitehead products, for which see Hatcher, [35], for example.

[88]This construction relates to certain well known notions (Fox derivative, Reidemeister-Fox Jacobian of a presentation) from combinatorial group theory.

[89]X has the same 1-type as the classifying space of $\pi_1(X)$.

It is not too difficult to see that the classifying space of a crossed complex must have vanishing Whitehead products, but there are spaces which do have vanishing Whitehead products yet require more structure than that encoded in a crossed complex, in order to completely model their weak homotopy type. The key to answering the problem lies elsewhere, within Whitehead's theory[90].

In [68], Whitehead introduces the notion of a J_m-complex as follows:

Let K be a (connected) CW-complex (with a 0-cell chosen as basepoint for all the homotopy groups concerned). Let $\rho_n := \pi_n(K^{(n)}, K^{(n-1)})$ be the n^{th} relative homotopy group of the n-skeleton of K, relative to the $(n-1)$-skeleton[91]. There is a natural homomorphism,

$$j_n : \pi_n(K^{(n)}) \to \rho_n.$$

Definition: The complex, K, is said to be a J_m*-complex* if j_n is a monomorphism[92] for each $n = 2, \ldots, m$. The complex is said to be a J*-complex* if it is a J_m-complex for all $m > 2$.

He introduced the notation $\Gamma_n(K)$ for the kernel of j_n, so K is a J_m-complex if $\Gamma_n(K)$ is trivial for all $n = 2, \ldots, m$ and is a J-complex is $\Gamma_n(K)$ is trivial for all $n \geq 2$.

Theorem 1 *Let K be a connected CW-complex. The natural morphism,*

$$K \to B\pi(K),$$

is a weak equivalence if, and only if, K is a J-complex. ■

This result explicitly gives a characterisation of those homotopy types representable by crossed complexes. It has appeared in several places in the literature, e.g. in [15], Corollary 2.2.7, or more recently, in [11], but these have usually been slightly submerged in a mass of other results and so are not that well known. Carrasco and Cegarra's proof uses simplicial groups and is thus quite algebraic[93].

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[90]... but he did not, in fact, prove the characterisation needed for our question.

[91]N.B. This is the n^{th}-group, $\underline{\pi}(\underline{K})_n$, of the fundamental (reduced) crossed complex of the space, K, filtered by skeleta that was mentioned above.

[92]Whitehead actually uses the term 'isomorphism into' rather than 'monomorphism'.

[93]There is an important slightly technical point of interest here, relating to the fact that, from a crossed complex, one can build a *strict* ∞-groupoid by a construction generalising the way one constructs a 2-groupoid from a crossed module. We thus do get some infinity groupoids using the relative homotopy groups and classical constructions from Whitehead's 1950s papers, but they are *strict* and correspond to J-complexes. A question is: *how does one relate the J-complex condition* explicitly *to that categorical 'strictness'*? In low dimensions, weak versions of ∞-groupoids are well known, and relatively well understood, and some of them model, for instance, 3-types: see Joyal's letter to Grothendieck, [38]. How does the 'weakness' of these models, expressed in purely categorical terms, explicitly correspond to the non-vanishing of Whitehead's Γ functors?

5 Simplicial sets, higher combinatorics and ∞-groupoids

We have given quite a lot of historical background in the earlier sections on the lowest level 'classical' viewpoint. For the most part, that left us in the first half of the 20th century. We have mentioned, above, quite a few ideas that arose from work by Whitehead, in the 1940s and 50s[94]. Later on, these gave various algebraic structures equivalent to some forms of (strict) ∞-groupoid.

5.1 Kan complexes and ∞-groupoids

Grothendieck's sketch, which we saw earlier, is reminiscent of the construction of the singular complex of a space. The difference being that it was based on a globular intuition whilst the latter was simplicial, corresponding to the two sides of Grothendieck's 'yoga' from letters, [27–29], to Larry Breen, that will be revisited shortly. I mentioned this idea in a letter, [57], to Grothendieck in June 1983. Explicitly I said "*I believe, in fact, the ultimate in non-strict or lax ∞-groupoid structures is already essentially well known (even well loved) although not by that name. The objects to which I am referring are Kan complexes, [...]. Here composition is not even strictly defined – given $\alpha, \beta \in X_1$, X a Kan complex, one forms a composite by filling the horn,*

$$\begin{tikzcd} & {} \arrow[dr, "\beta"] & \\ {} \arrow[ur, "\alpha"] \arrow[rr, dotted, no head] & & {} \end{tikzcd}$$

in any way whatsoever. Two such fillers are homotopic, associativity is only defined up to homotopy and so on.'

Grothendieck's reply, a couple of weeks later, [31], raised several objections. He felt, amongst other things, that composition should be defined as a function, and that there should be just two boundary maps from each dimension giving the source and target of each n-object. (This then led on to a discussion of 'test categories' that will not be explored here, but does relate to the problem of the choice of 'test objects' that we looked at earlier, page 12 and also below.)

These objections are at the same time serious and not that difficult to counter, at least in part. (i) Firstly, the fact that 'filling' is not defined as a function could be met by making it one! We could then make composition 'algebraic', adding enough formal composites into the Kan complex, then 2-simplices between the formal composite and the corresponding horn, and so on. One way of making this idea explicit has been done by Thomas Nikolaus, [53]. He has introduced a form of Kan complex with given fillers, which he calls *algebraic Kan complexes*. There is also another way of doing this which links in with more classical constructions of simplicial homotopy, namely by generating composites freely in a certain sense, using a construction of Dwyer and Kan, in such a way as to be able to compare 'formal composites' with 'geometric composites'. We will look at this in the next section.

[94] These ideas have been developed a lot further by Brown and Higgins, and also by Baues in the work cited in the bibliography.

(ii) The second objection is partially handled by exploring Grothendieck's sketched construction in detail, as is done by Georges Maltsiniotis in [50, 51] and by Dimitri Ara, [1][95]. This approach uses n-globes as the test objects.

This raises the interesting question of which test objects should one use; cf. here page 12. Grothendieck preferred n-globes in some form. The difficulty is that although n-globes correspond to the intuition most neatly and to the occurrences of low dimensional versions in homological algebra and algebraic geometry, the idea of (weak) composition is much more difficult to handle for them. The use of n-cubes has a lot to recommend it (see the discussion in [11]) especially for composition, but the resulting theory of cubical sets is less well known and less well developed, so needs some development work to get to where we are going. In both cases one would also like nicely behaved globular or cubical nerves of categories, whilst in that context the simplicial case seems to be very neat as we will see in the coming pages.

Picking up the point about composition, another reason for claiming a strong link between Kan complexes and ∞-groupoids is the neat observation, usually attributed to Grothendieck himself and that we made earlier, that the nerve of a small category is a Kan complex if, and only if, the category is a groupoid. The simplicial filling properties of the nerve do correspond *precisely* to the algebraic structure of the composition in the category. The clearest and simplest example of this is that, given a $(2,1)$-horn, then the filler of that horn *is* the sequence formed by that pair of arrows, viewed as a 2-simplex of $Ner(\mathcal{C})$ and the missing 1-dimensional face *is* the composite of that pair. There is a *unique* filler for the horn, and this uniqueness is unusual (and significant). The related property, often known as the Segal condition, is also important, but will not be discussed here[96].

All this suggests that we *could* decide to take 'Kan complex' to be the idea of '∞-groupoid' that we would work with. In that case, the process of the passage from spaces to ∞-groupoids would just be the classical singular complex functor, whilst the functor going in the other direction would be the geometric realisation. Certainly this gives an equivalence of homotopy categories and, even better, that equivalence is given by an equivalence of the homotopy theoretic structure, interpreting that in the sense of Quillen's theory of model categories, either in its original form (from [60]) or in almost any of the refinements and variants made since his initial idea. This however is not the whole picture. It would not advance the study very much if that was all. The missing perspectives include the objections that Grothendieck raised and the result of the interaction of Grothendieck's ideas and those of Whitehead's algebraic / combinatorial homotopy. Examining this brings up the 'Homotopy Hypothesis' (that we will often call 'the (HH)'), that, in the case of Kan complexes, is a classical theorem of Milnor. This 'hypothesis' is really not one, but rather is a test for any putative definition of ∞-groupoid together with assorted 'baggage' of homotopies, and of higher homotopy structure, including fibrations, cofibrations and other similar concepts. The test is that *there should be, at very least, an equivalence of homotopy categories between that of some category of 'spaces' and that of that notion of ∞-groupoid being 'tested'* and this takes us right back to the questions

[95] More generally, useful references for higher category theory include [16] and [45].

[96] Segal 'spaces', which involve an abstraction of this condition, are another important model for ∞-categories and groupoids.

with which we started.

If a candidate notion of ∞-groupoid is to be tested against the (HH), then it must be able to model the various (relevant) structures of the category of 'spaces'. A key example of such structure would be the idea of fibrations, and if that works, then to translate some of the ideas of covering space, n-connected covering, etc. that we met earlier, to that ∞-groupoidal setting. The simplest context in which to try out this extra structural test is, of course, that of Kan complexes, and there everything works well[97]. There is a simple notion of (Kan) fibration, $p : E \to B$; see the footnote below[98]. For the case $B = \Delta[0]$, p is the unique map to that object[99], and is a Kan fibration exactly if E is a Kan complex[100]. If B is connected, and the fibre of p over any vertex of B is discrete[101], then p is the simplicial set analogue of a covering space. In general, if p is a Kan fibration, its fibres are Kan complexes (and so a Kan fibration is 'fibred in ∞-groupoids' if you are using Kan complexes as models for ∞-groupoids)[102].

5.2 Simplicially enriched categories and groupoids

(We will tacitly assume a bit more basic knowledge of simplicial objects in this section.) We next need to go towards understanding the Dwyer-Kan construction that we mentioned above. This takes a simplicial set and gives a *simplicially enriched groupoid*. To understand those objects it will not only help to meet *simplicially enriched categories*, but to see how these ideas fill in some of the blanks in a simplicially based version of Grothendieck's idea.

In the picture that we sketched above, there is a lot more structure around, namely that which we have loosely termed the 'assorted 'baggage' of homotopies and of higher homotopy structure'. For instance, the category of spaces can be 'enriched' over the category of simplicial sets. It has a structure that merits being called that of an ∞-category, as we will see shortly. Grothendieck's suggestion, implied by his comments in his letters to Larry Breen, was that any sensible category of ∞-groupoids should have an ∞-category structure and the

[97]This is a good point to remember that the best known meaning of 'works well' in this context, involves the idea of a Quillen model category structure in one of its forms. As we said earlier, for this we merely direct the reader to the extensive literature on that theory.

[98]Similarly defined to the topological notion, a morphism, $p : E \to B$, of simplicial sets is a *Kan fibration* if in any commutative square of the form

$$\begin{tikzcd} \Lambda[k,n] \arrow[r, "u"] \arrow[d, "i"'] & E \arrow[d, "p"] \\ \Delta[n] \arrow[r, "v"'] \arrow[ur, dashed, "f"] & B \end{tikzcd}$$

there exists a (dotted) arrow, as shown, so for any n-simplex, v, in B and a (n, k)-horn in E, which maps down to the correspond (n, k)-horn of v, the horn in E can be filled to an n-simplex mapping down to v.

[99]as $\Delta[0]$ is the terminal object of $\mathcal{S}$,

[100]In the language of model category theory, this says that the Kan complexes are the fibrant objects in the category, $\mathcal{S}$, of simplicial sets. In general, with a good notion of fibration, a *fibrant object* is one where the unique morphism from the object to the terminal object is a fibration.

[101]i.e., is a constant simplicial set,

[102]Given a lot more space, we could have described the fibrations in both cubical and globular ∞-groupoid models. In each case, the models correspond to the 'fibrant objects.

hypothetical correspondence

$$\textit{spaces} \leftrightarrow \infty\textit{-groupoids}$$

should be an equivalence of ∞-categories, in an appropriate sense. His sketch left a lot of ideas at the 'intuitive' level, i.e., the theory seems to need certain ideas so as to work well and, moreover, coming from the examples, the intuitions behind those ideas, about homotopies and higher homotopies, would seem to fit, but that still left a lot of detailed exploration, checking, etc., to be done.

Let us give a few more details. If our category of spaces has function space objects[103], then we can take their singular complexes to get simplicial sets of morphisms from X to Y. In fact, even if the particular category of spaces being considered does not have nice function spaces, we can still form up this simplicial set of maps by defining $\mathsf{Spaces}(X,Y)$ to be the simplicial set having $Spaces(X \times \Delta^n, Y)$ as its set of n-simplices[104]. There are 'composition maps',

$$\mathsf{Spaces}(X,Y) \times \mathsf{Spaces}(Y,Z) \to \mathsf{Spaces}(X,Z),$$

and a little routine checking shows that this gives *Spaces* the structure of a *simplicially enriched category*, a term often abbreviated to $\mathcal{S}$-category[105].

Remarks: (i) There are lots of other examples of $\mathcal{S}$-categories in the areas we have been discussing. The category, $\mathcal{S}$, itself is an $\mathcal{S}$-category with 'function spaces', $\mathsf{S}(K,L)$, given by $\mathsf{S}(K,L)_n = \mathcal{S}(K \times \Delta[n], L)$. Other examples include the category of simplicial Abelian groups, that of chain complexes of modules over a commutative ring, that of simplicial groups, and so on. All these are 'large' $\mathcal{S}$-categories. There are small ones as well. Any simplicial monoid is essentially a small $\mathcal{S}$-category with a single object, and conversely, for any $\mathcal{S}$-category, A, and any object x, the simplicial set, $\mathsf{A}(x,x)$, of endomorphisms of x is a simplicial monoid.

Another class of examples come from 2-categories. If $\mathbb{A}$ is a (strict) 2-category, then we can define a corresponding $\mathcal{S}$-category, A, by taking $\mathsf{A}(x,y) := Ner(\mathbb{A}(x,y))$ with the induced face and degeneracy maps. As any crossed module, C, gives a 2-group(oid), $\mathcal{X}(\mathsf{C})$, it also gives a small $\mathcal{S}$-category, $Ner\mathcal{X}(\mathsf{C})$, which is, in fact, an $\mathcal{S}$-groupoid. For that $\mathcal{S}$-groupoid, as its set of objects consists just of the single object, $*$, of C, we have that $Ner\mathcal{X}(\mathsf{C})$ is a simplicial

[103] so if we have X and Y, we also have a space, Y^X, of continuous functions from X to Y and there are nice isomorphisms such as $Spaces(X \times Y, Z) \cong Spaces(X, Z^Y)$.

[104] N.B. This is isomorphic to the singular complex of Y^X if that latter space exists, but this definition does not depend on the existence of that function space. There is a very slight condition on the category of spaces that is needed. The category of spaces being used must contain $X \times \Delta^n$ for all X and all $n \geq 0$.

[105] An $\mathcal{S}$-category, A, is a category enriched over the category of simplicial sets, so each 'hom' $\mathsf{A}(x,y)$ is a simplicial set and the composition,

$$\mathsf{A}(x,y) \times \mathsf{A}(y,z) \to \mathsf{A}(x,z),$$

is a simplicial map. Composition is associative, so given w,x,y,z, the square of simplicial maps,

$$\begin{tikzcd}
\mathsf{A}(w,x) \times \mathsf{A}(x,y) \times \mathsf{A}(y,z) \arrow[r] \arrow[d] & \mathsf{A}(w,x) \times \mathsf{A}(x,z) \arrow[d] \\
\mathsf{A}(w,y) \times \mathsf{A}(y,z) \arrow[r] & \mathsf{A}(w,z),
\end{tikzcd}$$

is commutative; similarly for the identities, which are zero simplices in the various $\mathsf{A}(x,x)$.

group[106]. Any simplicial group likewise gives a crossed module[107].

(ii) In the $\mathcal{S}$-category structure on the category of spaces, each simplicial function space, $\mathsf{Spaces}(X, Y)$, is a Kan complex[108]. Such 'Kan enriched' categories include the category of Kan complexes itself. If Kan complexes are taken to be a possible model for ∞-groupoids, then these Kan enriched categories are also enriched over ∞-groupoids, and would form a particular class of the structures that would warrant being called ∞-categories. In such an ∞-category, any k-arrow for $k > 1$ will be invertible. Of course, Grothendieck's criticisms would still apply to these and perhaps one needs something a bit more 'algebraic'.

The criticisms by Grothendieck of the idea of using Kan complexes as a model for ∞-groupoids included that, for that model, composition is not determined uniquely. It used a filler of a horn and that filler need not be unique, hence, for instance, a composable pair of 1-arrows might have a whole lot of 'composites'. Of course, that does mirror the topological context, but, in that setting, there is a reasonably natural choice of a filler. We have already mentioned one way to get around this namely by choosing one 'formal' filler for each horn in a Kan complex, K. Another way would be to freely combine simplices in a horn in some way to give a formal composite which, then, could be linked to the 'composites defined by the Kan condition' via some homotopies. This vague idea is somewhat of an analogue, in all dimensions, of the way in which Poincaré formed the edge-path groupoid of a simplicial complex. There[109] one formed a free groupoid on the 1-skeleton and then used the 2-simplices to link the formal composite of two faces of the 2-simplex to the generator in the third face. Technically the 'all dimensions' version of this is quite difficult to do and the formulae that result[110] are not 'self evident'. The result, however, is an $\mathcal{S}$-groupoid, $\mathcal{G}(K)$, which does work well. This construction is due to Dwyer and Kan in [25], (but beware of some silly typographic errors in that source, and look for the details elsewhere). Writing $\mathcal{S}$–*Grpd* for the category of $\mathcal{S}$-groupoids, the Dwyer-Kan functor, $\mathcal{G} : \mathcal{S} \to \mathcal{S}-Grpd$, has a right adjoint, $\overline{W}$, and, for any $\mathcal{S}$-groupoid, G, its '*classifying space*, $\overline{W}G$, is a Kan complex[111]. If we think of a groupoid as an $\mathcal{S}$-groupoid which is simplicially trivial in a fairly obvious sense, then its classifying space, as a simplicially enriched groupoid, is the same as the classifying space given by its nerve as a groupoid. As the adjoint pair, $\mathcal{G} \dashv \overline{W}$, induces an equivalence of homotopy categories between that of $\mathcal{S}$ and that of

[106]There are several simple ways to write down this simplicial group in terms of the data of C, which can be found in the literature, but will not be explored here. The ways back from simplicial groups to crossed modules using the Moore complex construction is also very neat.

[107]Simplicial groups represent all connected homotopy types, and the purely algebraic way from a simplicial group to the corresponding crossed module is an algebraic form of the Mac Lane - Whitehead model for the 2-type of a general (CW-)space.

[108]The proof of this follows the same route as that of proving that each $Sing(X)$ is a Kan complex. One uses that the horns of topological n-simplices are retracts of the simplex.

[109]cf. page 10

[110]One takes K_0 as the set of objects and then considers each $n+1$-simplex, σ, as being an n-dimensional edge going from the zeroth to the first vertex of σ. This gives a graph and one then takes the free groupoid on that graph to be $\mathcal{G}(K)_n$. There is then a way to define induced face and degeneracy maps.

[111]If we think of this Kan complex as an ∞-groupoid, we can analyse the composition *algebraically*. To see why, at least in the case of a simplicial group, G, note that the underlying simplicial set of a simplicial group is a Kan complex and there are algorithms giving fillers for horns. These, in turn, give algorithmic fillers for the 'classifying space', $\overline{W}G$, with a complete analysis of possible choices being feasible!

$\mathcal{S}-Grpd$, we could have taken $\mathcal{S}$-groupoids as our choice for ∞-groupoids.

These simplicial models are much more 'algebraic' than the simple Kan complex ones. Restricting to connected homotopy types, and thus to simplicial groups, we can gain some additional insight into the models of the special kinds of homotopy types that we mentioned earlier, e.g. those corresponding to strict infinity groupoids. They are related to special algebraic and combinatorial properties of the corresponding simplicial groups[112].

Here is a good point to mention the role simplicial groups play in the theory of simplicial fibrations, fibre bundles and related ideas in the theory of simplicial homotopy as it connects up not only with the adjunction, $\mathcal{G} \dashv \overline{W}$, but with another of Poincaré's ways of considering the fundamental group. We saw how he considered it as a group of deck transformation of the universal covering of the space. In other words, he looked at the group of automorphisms of the object $\tilde{X} \to X$ in the 'slice category' of spaces over X. When we are working in an $\mathcal{S}$-enriched category, $\mathbb{A}$, the automorphisms of an object, y, naturally form a simplicial group, $\mathsf{aut}(y)$, and if we have a specified map, $p : y \to x$ in $\mathbb{A}$, then the automorphisms of y *over* x form a simplicial subgroup of $\mathsf{aut}(y)$, generalising the group of deck transformations of a covering. This suggests that some, at least, of the rôle of '$\pi_1(X)$ as automorphisms of $\tilde{X} \to X$' might be transferred to the automorphisms of the fibrations that generalise the universal cover, and this *is* the case[113].

Another related link between simplicial groups and fibred things comes about in the adjunction, $\mathcal{G} \dashv \overline{W}$. This gives, for any (reduced) simplicial set, K, a unit morphism, $\eta_K : K \to \overline{W}\mathcal{G}K$. The next ingredient that we need is that for a simplicial group[114], G, its classifying space, $\overline{W}\mathsf{G}$, comes with a natural Kan fibration, $p_{\mathsf{G}} : W\mathsf{G} \to \overline{W}\mathsf{G}$, whose fibre is the underlying simplicial set, $U(\mathsf{G})$, of G, and whose simplicial group of automorphisms will be isomorphic to G. Now any morphism $f : K \to \overline{W}\mathsf{G}$ will induce a fibration over K by pullback of p_{G} along it. This induced fibration will have fibre $U(\mathsf{G})$[115]. We direct the reader to the literature for more on this. (A fairly brief treatment can be found in Curtis, [23].)

Some examples of this are of note for our earlier theme of factorising a homotopy type into simpler bits. If G is a group, when we can construct a constant simplicial group, $K(G, 0)$, with $K(G, 0)_n = G$ for all n and all face and degeneracy maps the identity isomorphism on G. Taking $G = \pi_1(K)$, there is a natural map, $K \to K(G, 0)$, related to the unit we mentioned above[116]. If we pull p_G along this, we get the universal (simplicial) cover of K. More generally, we could also use the unit composed with other quotients of $\mathcal{G}(K)$ and pullback

[112] A word of caution here is needed. The transition from simplicial sets to $\mathcal{S}$-groupoids involves a shift in the usual dimension of simplices, so a 1-simplex in K becomes a zero simplex in one of the 'hom-sets' of $\mathcal{G}(K)$. Because of this for a connected K, $\pi_n(K) \cong \pi_{n-1}(\mathcal{G}(K))$, which initially can lead to some confusion. The same goes for 'n-type'.

[113] Of particular interest would be the n-connected cover of a homotopy type considered as a fibration $X\langle n\rangle \to X$ in the simplicial category, Spaces, or in the analogous simplicial setting.

[114] We asked for K to be reduced ensure that the $\mathcal{G}K$ was a simplicial group, and so could apply this fact to it. This is mostly for the sake of the exposition.

[115] and is technically a *principal* G-*fibration* or G-*torsor*, and any such fibration will correspond to some f.

[116] As $\pi_1(K) = \pi_0(\mathcal{G}(K))$, there is a quotient map $\mathcal{G} \to K(\pi_1(K), 0)$, now apply $\overline{W}$ to that and compose with the unit η_K.

$p_{\mathcal{G}(K)}$ along that. In each case one is approximating to the information on the homotopy type of K, by means of a truncation or similar. Note that if we start with K being a Kan complex then this can be interpreted as being a series of operations on ∞-groupoids.

This rich structure suggests the extent that the classical Kan complex / Kan fibration set-up gives a useful first example of a situation satisfying the expectations of the (HH). It also sets up a family of results and sub-theories that operate as a model for how any good general abstract theory will look.

5.3 Chain complexes, globular and simplicial models

(Here we will look at globular models and how they relate via chain complexes to simplicial ones.)

Chain complexes (of Abelian groups) form very simple models of a class of homotopy types[117]. As mentioned earlier, they correspond to simplicial *Abelian* groups. We also mentioned that strict ∞-groupoids corresponded to crossed complexes, and that such objects are also chain complexes, but with not all the groups involved being Abelian, and there being some extra structure in the shape of some well behaved actions, etc. The functor from strict ∞-groupoids to crossed complexes extends that from 2-group(oid)s to crossed modules, and the idea there is very simple. Given a 2-group(oid), $\mathbb{G}$, you take those 2-arrows, x, which have their 1-source at an identity, so they look like

$$\begin{tikzcd} s_0(x) \arrow[rr, bend left, "id_{s_0(x)}"] \arrow[rr, bend right, "t_1(x)"'] \arrow[rr, phantom, "\Downarrow x"] & & t_0(x) \end{tikzcd} .$$

where $s_0(x)$ is the object that is the source of x, similarly $t_0(x)$ is its target[118].

In a strict ∞-groupoid, $\mathbb{G}$, based on a globular model, each n-arrow, x, would have sources, $s_k(x)$, and targets, $t_k(x)$, in all lower dimensions[119]. The group in the n^{th}-dimension of the corresponding crossed complex consists of those n-arrows, x, such that $s_{n-1}(x) = id_{s_{n-2}(x)}$, that is, the source trivial n-arrows.

If one goes to the more general form of ∞-groupoid, again in a globular form, then one can again consider the source trivial n-arrows in each dimension (but they may not form a group). The result will look somewhat like a chain complex as the target map will provide a boundary operator[120].

If one chooses Kan complexes as the model for ∞-groupoids, then handling the analogous construction seems hard as there is little or no algebra to help,

[117]... and also correspond to a fairly simple class of (strict) ∞-groupoids.

[118]so $x \in \mathbb{G}(s_0(x), t_0(x))$, whilst $s_1(x)$, (resp $t_1(x)$) denotes the 1-arrow that is the source (resp. target) of x *within* the groupoid $\mathbb{G}(s_0(x), t_0(x))$. In our particular situation, $s_1(x) = id_{s_0(x)}$, so $s_0(x) = t_0(x)$ and x actually is a vertex 2-loop of the 2-groupoid.

[119]Think of x as being a n-globe, with boundary consisting of two $(n-1)$-dimensional hemispheres, $s_{n-1}(x)$ and $t_{n-1}(x)$, which are thus in $\mathbb{G}_{n-1}$, meeting 'at the equator' in two $(n-2)$-dimensional hemispheres, $s_{n-2}(x)$ and $t_{n-2}(x)$, and so on.

[120]Chain complex models for these ∞-groupoidal structures have the advantage of the many years of experience in handling chain complexes in algebraic topology and homological algebra, although those latter chain complexes are usually of Abelian groups, which is not the case here. Some of the algebraic information encoded in them is thus easily accessible, although other parts of the structure may be less so.

but if we work with a $\mathcal{S}$-groupoid, not only does it work much better, after a bit of 'simplicial adjustment', but relates to known structures that we have seen already.

For ease of exposition, we will assume that the $\mathcal{S}$-groupoid is 'reduced', i.e., has a single object, which makes it essentially just a simplicial group. Suppose therefore that $\mathbb{G}$ is a simplicial group. We want, in some sense, to look at the group of 'source-trivial' n-simplices, but there is no clear sense of a single source for an n-simplex, at least for $n \geq 2$. Of course, if $g \in \mathbb{G}_1$, it is thought of as an arrow and the source will be $d_1 g$, so the group of source trivial 1-simplices is simply $Ker\, d_1$, but what about higher dimensions? One way to see a solution to this is to take an idea from the construction of the relative homotopy groups of a CW-complex, K, as used when constructing its fundamental crossed complex, $\pi(K)$. That construction used $\pi_n(K^{(n)}, K^{(n-1)}, x_0)$, so the elements can be represented by cellular singular n-cubes in K, all but one of whose faces is at the base-point. We adapt that to a simplicial group context and look at, for each n, the subgroup, $N\mathbb{G}_n$, of $\mathbb{G}_n$ consisting of those $g \in \mathbb{G}_n$ having all but their d_0-face at the identity. In other words, $N\mathbb{G}_n = \bigcap_{k=1}^{n} Ker\, d_k$. This subgroup forms the n-dimensional part of a chain complex as the remaining face, d_0, restricts to give a 'boundary' morphism, $\partial_n : N\mathbb{G}_n \to N\mathbb{G}_{n-1}$. It is easily checked that $\partial_{n-1}\partial_n$ is the trivial morphism as $d_0 d_0 = d_0 d_1$ is a consequence of the simplicial identities that encode how face maps and degeneracy maps interact. Those simplicial identities also easily show that the image of ∂_n is a normal subgroup of $N\mathbb{G}_{n-1}$, so $N\mathbb{G}$ is not just a chain complex of possibly non-Abelian groups, it has the extra property that boundaries form normal subgroups, and that means that we can form the homology, $H_*(N\mathbb{G})$, of such a complex, even though we are in a non-Abelian setting[121]. The complex, $(N\mathbb{G}, \partial)$, is well known from simplicial homotopy theory. It is often called the *Moore complex* as many of its properties were developed by John Moore in his seminar, [52], in the late 1950s. This complex relates to very many of the invariants of homotopy types represented by the simplicial group, $\mathbb{G}$. As the form of $\mathbb{G}$ is sufficiently general to represent *any* connected homotopy type, and it *is* evidently a form of ∞-groupoid, the way in which the more classical invariants of a homotopy type depend *algebraically* on the Moore complex of $\mathbb{G}$ provides some more intuitive interpretation of the model and of the way that well known homotopy invariants of a homotopy type correspond to properties of an ∞-groupoid.

5.4 Appendix: Some Moore complex properties

To help in this process, we will look at how properties of $(N\mathbb{G}, \partial)$ relate to structures that we have already met here.

(i) If $\mathbb{G}$ is a simplicial group, the homotopy groups of the underlying simplicial set of $\mathbb{G}$ are isomorphic to the homology groups of $(N\mathbb{G}, \partial)$. If $\mathbb{G} = \mathcal{G}(K)$ for a (reduced) simplicial set K, then $\pi_k(K) = \pi_{k-1}(\mathbb{G}) = H_{k-1}(N\mathbb{G}, \partial)$. (In particular we note the shift in dimension so that, for instance, $\pi_0\mathbb{G}$ is a group.) We thus have the homotopy groups of the connected homotopy type represented by K are very simply related to the Moore complex of $\mathcal{G}(K)$, and hence to the

[121] Although we have seen this just in the simplicial group setting, this fact encodes facts about 'whiskering' which is the general form of conjugation in ∞-groupoid models, so is an expected feature.

'source-trivial' part of a ∞-groupoid model for the homotopy type[122].

(ii) If $\mathbb{G}$ is a simplicial Abelian group, then $N\mathbb{G}$ is just a 'standard' chain complex and it is relatively easy, given an arbitrary chain complex, C, to build a simplicial Abelian group, whose Moore complex is isomorphic to C. This gives the classical Dold-Kan theorem[123].

(iii) If $\mathbb{G}$ is such that $N\mathbb{G}_n = 1$ for $n \geq 2$, then, of course, all the higher homotopy groups, $\pi_k(\mathbb{G})$, for $k \geq 1$, will be trivial[124]. In this case, the only non-trivial part of the Moore complex, $\partial_1 : N\mathbb{G}_1 \to N\mathbb{G}_0$, will be a crossed module. For a general $\mathbb{G}$, the 2-type model that it represents is given by the crossed module,

$$\partial : \frac{N\mathbb{G}_1}{\partial N\mathbb{G}_2} \to N\mathbb{G}_0.$$

(iv) If $\mathbb{G}$ is such that $N\mathbb{G}_n = 1$ for $n \geq 3$, then the three remaining terms of $N\mathbb{G}$ form what is called a *2-crossed module*[125]. We mention this more technical model because the corresponding (globular) ∞-groupoid is a weak 3-groupoid in which, in general, the interchange law fails to hold in the underlying 2-groupoid. This can be seen in the 2-crossed module as the Peiffer identity does not, in general, hold in the structure encoded by the bottom two terms[126].

(v) As a final example, we will look at what stops a general Moore complex, $N\mathbb{G}$, from being a crossed complex, as this is the simplicial version of what stops a general ∞-groupoid from being a strict ∞-groupoid[127].

Let D_n be the subgroup of $\mathbb{G}_n$ generated by the degenerate elements[128].

A Moore complex is a crossed complex if, and only if, for each $n \geq 2$, $N\mathbb{G}_n \cap D_n = 1$.

Turning this around, with a bit more work it also provides a functor from simplicial groups to crossed complexes, but we will not explore that here. It also allows one to give a description of the property of 'being a crossed complex' in terms of 'having a unique thin filler for each horn'; see Ashley, [3][129]. Note that, for comparison, in the nerve of a groupoid, there is a unique filler for each horn.

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[122] Of course, we should also look at the structure of homotopies through the eyes of this process. That is more tricky, ... and more revealing, but cannot be handled here.

[123] For an arbitrary simplicial group, $N\mathbb{G}$ has considerably more structure than being merely a chain complex of groups. (That structure is trivial in the Abelian case). That extra structure corresponds, in part, to the extra structure that an arbitrary homotopy type may have (Whitehead products, actions, etc.), and so corresponds to the *weak* ∞-groupoid structure, that is 'weak' as against 'strict'. The analogue of the Dold-Kan theorem with regard to this extra structure was given by Carrasco and Cegarra, [15].

[124] If $\mathbb{G} = \mathcal{G}(K)$, we have to remember that $\pi_0(\mathbb{G}) \cong \pi_1(K)$, and $\pi_1(\mathbb{G}) \cong \pi_2(K)$, so there is s shift in dimension and so, generally, if $N\mathbb{G}_n = 1$ for $n \geq 2$, then $\mathbb{G}$ is a simplicial groupoid model of a 2-type, *and not of a 1-type as one might think.*

[125] cf. Conduch́'s paper, [17], that was mentioned earlier.

[126] There is a pairing $N\mathbb{G}_1 \times N\mathbb{G}_1 \to N\mathbb{G}_2$, which 'lifts' the difference of the two sides of the Peiffer rule, so making it a boundary of a higher element. The weakening thus replaces an equality by a 'reason' why the two sides of the equation are to be thought equivalent. The corresponding structure for higher values of n is related to the notion of hypercrossed complex, found in Carrasco and Cegarra, [15].

[127] This is easily related to the simplicial group theoretic analogue of a J-complex, as mentioned on page 33.

[128] These elements are sometimes called 'thin' elements, in a sense that intuitively extends our earlier use of 'thin' to describe certain homotopies.

[129] In some way, the 'weakness' of a simplicial group modelling a given connected homotopy type, is related to the non-uniqueness of such thin fillers and the size of the various subgroups, $N\mathbb{G}_n \cap D_n$, of the Moore complex terms.

5.5 Higher dimensional combinatorial homotopy

Poincaré's combinatorial approach to the fundamental group of a polytope deserves to be explored up to higher dimensions. Let us put this as follows:

Pick a notion of ∞-groupoid, denoted $\Pi_\infty(X)$ for a 'space' X. We will work on the assumption here that $\Pi_\infty(X)$ is the singular complex of X, as that is a Kan complex and that we have been exploring the use of Kan complexes as models of ∞-groupoids.

Suppose, now, that X is a polytope / simplicial complex (or more generally a CW-complex, i.e., with explicit *CW-structure). Can we use the extra structure of a complex to produce an ∞-groupoid, $P_\infty(X)$,* presented *in some sense by the combinatorial structure, hopefully smaller than $\Pi_\infty(X)$, and an ∞-equivalence*

$$P_\infty(X) \leftrightarrow \Pi_\infty(X)?$$

As usual we could rewrite this question with ∞ replaced by n.

This is almost Whitehead's algebraic homotopy problem in disguise and a start on it is made in his papers, [68, 69]. There has been a lot of progress on it in the work that we have already mentioned by Brown and Higgins, see [11] and of Baues, [5], but *this work is using models that are not explicitly linked to ∞-groupoids*. There is a complication that the necessary weakness of the ∞-groupoids needs encoding in an *economical* way. It is not clear how to do this with some of the models[130] of ∞-groupoids. It would need a combinatorial ∞-groupoid theory to mirror combinatorial homotopy theory and to extend combinatorial group theory.

There are other aspects of that overall combinatorial approach that are worth mentioning. We have already pointed out the original link between combinatorial group theory and Whitehead's combinatorial homotopy. To some extent, working with simplicial sets or $\mathcal{S}$-groupoids can be thought of as an extension of that link, but, although it is relatively easy to define 'step-by-step' constructions of simplicial sets (or simplicial objects in algebraic categories) having desired properties, this is not the usual method used. Likewise, if one has a naturally occurring ∞-groupoid, the idea of working with a small 'presentation' of that object, perhaps reflecting some geometry of how it occurs 'in nature', is not one that has yet been explored to any great extent. Of course, 2-groups of *symmetries* have occurred in work on various properties in non-Abelian cohomology, but there are relatively few studies of explicit presentations of such, as yet, although the cohomology of 2-groups / crossed modules has begun to be applied to problems in algebraic and differential geometry and related areas of theoretical physics. We thus have less evidence of '2-groups as spaces', at least for that interpretation of our initial query. Crossed modules *do* yield interesting 'classifying spaces', but the models used in their construction usually give 'big' presentations, having a lot of cells, and are often obtained from a nerve construction followed by geometric realisation.

One block to constructing the 'combinatorial homotopy' of an algebraically defined ∞-groupoid is the second objection of Grothendieck that we mentioned. Algebraically occurring n- or ∞-categories are often 'globular' rather than simplicial. Likewise group presentations are more globular in their 'feeling', so what

[130] that we have not discussed here,

is needed is more a globular, abstract combinatorial homotopy rather than a simplicial one. In part, such a theory is given by the theory of computads / polygraphs introduced by Street, [64] for $n = 2$, and later by Burroni, [14]. These have wider application than merely presenting certain strict ∞-groupoids as they seem very useful for studying rewriting systems and for operads rather than merely spaces and then questions of finite presentation[131]. They usually are used to present strict ∞-categories, but skilled use allows some use of explicit 'weakness'. Their usefulness and power comes, in part, from being able to be *analogues* of 'spaces'. They lead to a 'folk' homotopy structure, see Lafont, Métayer, and Worytkiewicz, [43]. The 'cofibrant' objects are the free strict ∞-categories on polygraphs, so are in close analogy to CW-spaces, which are the cofibrant objects in one of the usual homotopy structures on the category of topological spaces. We are not, here, able to explore that line of development as much as it deserves.

5.6 ∞-categories?

We will briefly need to mention ∞-categories in the next section. Grothendieck, in his letters to Breen and to Quilllen to which we will be returning shortly, seems to make the assumption that, not only would certain features of spaces be modelled by n-groupoids, for any n including ∞, but that the right context would be to have some sort of n-equivalence of some corresponding n-categories. These n-categories, moreover, would be 'weak' rather than 'strict'.

There are many different models for what are new usually called $(\infty, 1)$-categories[132]. These are the ∞-categories in which all 2-arrows are invertible 'up to 3-arrows', similarly 3-arrows are invertible 'up to 4-arrows' and so on. They are, at present, the best understood class of ∞-categories. The different models correspond in part to the models for ∞-groupoids that we have met[133].

From the point of view that we have been exploring, the easiest potential approach to explaining what ∞-categories are is probably that via $\mathcal{S}$-categories, since we know that both the categories of spaces and of simplicial sets have $\mathcal{S}$-category structures. Of course, Grothendieck's objection regarding simplicial as against globular approaches still applies here, but if $\mathcal{S}$-groupoids are acceptable as ∞-groupoids, then probably $\mathcal{S}$-categories should be considered as, at least, a working alternative to a globular version – and the pre-existing extensive theory of homotopy in some of the key examples gives them some advantages. One *caveat* is that the really nice $\mathcal{S}$-categories seem to have $\mathbb{A}(x, y)$ being a Kan complex[134].

(A bit of 'terminology' may help here in bridging between natural, intuitive ∞-category terms and their analogues for $\mathcal{S}$-categories. If $\mathbb{A}$ is an $\mathcal{S}$-category which we are thinking of as an ∞-category, it is often useful to think of, and to speak of, the 0-simplices in $\mathbb{A}(x, y)$ as being the $(1-)$morphisms / 1-arrows of the infinity category from x to y, then this continues with the elements of $\mathbb{A}(x, y)_1$ being the 2-arrows or even 'homotopies', the 2-simplices as 3-arrows or

[131] Some idea of this application can be found in the work of Guiraud and Malbos, cf. [34].

[132] A useful survey of $(\infty, 1)$-categories is [8].

[133] More generally an (∞, r)-category is one in which the n-cells are invertible 'up to $(n+1)$-cells' for $n > r$, so in this terminology ∞-groupoids are $(\infty, 0)$-categories.

[134] There is a Quillen model category structure on the catgory $\mathcal{S}$–Cat, of (small) $\mathcal{S}$-categories in which these 'locally Kan' $\mathcal{S}$-categories are exactly the fibrant objects; see Bergner, [7].

'homotopies between homotopies', and, in general, of 'higher homotopies'. This terminology is in some ways not strictly accurate, but can be useful in helping the intuition.)

If the $\mathbb{A}(x, y)$ are Kan complexes, then the homotopies in $\mathbb{A}$ will be 'reversible up to higher homotopies[135]'.

There are directly simplicial models of $(\infty, 1)$-categories, namely the quasi-categories, that were mentioned in footnote 28. These are very like Kan complexes, but have a slightly, but significantly, weaker filling requirement on horns. As the nerve of a category is a quasi-category, there is a fairly clear intuition as to how to develop quasi-categorical analogues of many categorical properties. This gives one the most developed versions of ∞-category theory[136].

Globular approaches are also known, see Maltsiniotis, [51], and Ara's thesis, [1], but are to some extent less developed than the simplicial ones as that can draw on the classical theory of simplicial homotopy, which can be an advantage. There are also treatments that are 'model independent', i.e., they do not choose between simplicial, globular, operadic, ..., etc. models, but look at the structures from the point of view of Quillen model category theory.

6 Higher Galois theory and locally constant stacks.

When we started our discussion of spaces and groupoids, we mentioned three of the ways that Poincaré had of thinking of the fundamental group(oid). The first was as the algebraic structure of path classes in the space. The second was related to deck transformations and, via SGA1, to Galois theory. The last was a combinatorial group theoretic approach given a simplicial or CW-complex structure on the space, which we briefly revisited in section 5.5. We saw how Grothendieck's 'letter to Quillen' in *Pursuing Stacks* sketched a higher dimensional version of the path class idea, so what about deck transformations and Poincaré-Galois theory. How does this interpret for a Kan complex / simplicial model?

6.1 'Stacks'

We first need to explore briefly some notion of 'stack' and its relationship with covering spaces. (Other related notions of stack as occur in geometric contexts are discussed in other papers in this volume. A set of lectures giving a perspective linked to that taken here were given by Bertrand Toën, [65, 66], whilst there

[135] To see this, let $\alpha : f_0 \to f_1$ be in $\mathbb{A}(x, y)_1$, then you can use it to build horns:

$$\begin{tikzcd} & f_1 \arrow[dr, dashed, "\beta"] & \\ f_0 \arrow[ur, "\alpha"] \arrow[rr, "s_0(f_0)"'] & & f_0 \end{tikzcd} \qquad \begin{tikzcd} & f_0 \arrow[dr, "\alpha"] & \\ f_1 \arrow[ur, dashed, "\gamma"] \arrow[rr, "s_0(f_1)"'] & & f_1 \end{tikzcd}$$

in $\mathbb{A}(x, y)$, shown with 'full' arrows, which can be filled since $\mathbb{A}(x, y)$ is a Kan complex. This gives left and right inverses 'up to homotopy' for α, namely β and γ. You then show using higher horns that these are themselves homotopic, in a homotopy coherent variant of the classic argument for inverses in a group.

[136] The initial idea is due to Boardman and Vogt, [9]; the simplicial version was developed by Cordier, [18], and then with the author, [19–22], and this basic theory was then pushed much further by Joyal, [40], and Lurie, [47, 48].

are many other treatments available perhaps more optimised for applications of stacks in other areas of mathematics.)

A covering space, $q : Y \to X$, is equivalently a locally constant sheaf [137]. A reader who is 'new' to sheaves might initially think of them both as a continuously varying family of 'spaces', namely the fibres of q, indexed by the points of X, and also as a presheaf with nice gluing properties. A presheaf on X is just a functor from the opposite of the partially ordered set of open sets of the space, X, with inclusions as morphisms, to, in our case here, the category of sets. Given q, as above, we get for each open set, U, in X, the set of maps, $s : U \to Y$, such that $qs(x) = x$ for all $x \in U$. We will denote this $F(U)$. For any open $V \subset U$, such 'local sections', s, of course, restrict to local sections on V, so we get a presheaf, $F : Open(X)^{op} \to Sets$. That would work for any 'space over X', $f : X' \to X$, as local sections of f restrict to subsets giving a presheaf on X. If one has an arbitrary presheaf, $F : Open(X)^{op} \to Sets$, on X, it need not come from a space over X. Presheaves of local sections have a special 'gluing' property generalising that met in elementary calculus.

If we have a presheaf, F, of local sections of some map, $f : X' \to X$, then, for open sets, U_0, and U_1, in X and $U = U_0 \cup U_1$ and local sections s_i, $i = 0, 1$, each over the corresponding U_i, if $s_0(x) = s_1(x)$ for all $x \in U_0 \cap U_1$, then, clearly, we can define a function $s : U \to X'$ by $s(x) = s_0(x)$ if $x \in U_0$ and $s(x) = s_1(x)$ if $x \in U_1$, and this is a continuous section of f over U, because the two sets, U_0, and U_1, are open. To state the obvious, it is the unique local section over U that restricts to the given ones over the given two open sets.

This is the condition that the presheaf is a sheaf. It is also called the *descent condition*.

The local homeomorphism aspect of covering spaces means that, for small enough open sets, the restriction morphisms are, in fact, bijections, so the sets, $F(U)$, are 'really all the same' for small enough U, and F is, as we said, 'locally constant'. (Notice that $F(X)$ can be empty, yet $F(U)$ may be non empty for many open sets, U. 'Local sections' may not be restrictions of 'global' ones, but if a family of local sections is compatible over intersections of their domain, then it can be built up into a local sections on the union of their domains.) We thus have that covering spaces 'are' locally constant families of sets, and as sets 'are' homotopy 0-types, they are 'locally constant families of homotopy 0-types'.

The idea of a sheaf as a special form of presheaf on a space generalises in a useful way to general functors, $F : \mathcal{C}^{op} \to Sets$, and corresponds to giving an abstract analogue of 'open covering', or, more exactly, of a 'covering family of maps'. This leads on to the idea of a Grothendieck topology explored, here, in the chapter '*Sheaves and functors of points*' by Michel Vaquié. Categories of sheaves on such a 'site'[138] is called a *(Grothendieck) topos*.

It is easy to see how one can extend the idea of presheaves of sets to that of presheaves of other objects. You just replace *Sets* by the category of 'other objects'. To define stacks, of n-stacks, ∞-stacks and so on, one approach is to start with presheaves of, perhaps, categories, groupoids, n-groupoids or simplicial sets, depending on what 'flavour' of stacky objects you need. We will not give a detailed treatment as it would take too long, would be quite technical and is not really necessary for the limited use we have for it. We will thus be a

[137]We will not give a detailed introduction to sheaf theory here, but direct the reader to Michel Vaquié's chapter on '*Sheaves and functors of points*' in this volume.

[138] 'site' = 'category together with a Grothendieck topology'.

bit 'vague'! Perhaps the best way here is to take 'prestacks' to be presheaves of $(\infty, 1)$-categories on a site (which may be a topological space) or alternatively on $(\infty, 1)$-category objects internal to the corresponding topos[139]. We could also replace the basic category by an $\mathcal{S}$-category, or an ∞-category, if the potential application merited that extra structure[140]. The second stage is then to find a suitable $(\infty, 1)$-categorical analogue of the gluing condition. Not surprisingly that is often seen in terms of a notion of fibration with the ∞-stacks being the fibrant objects. If a (basic) simplicial model is taken for $(\infty, 1)$-categories, then the resulting fibrant objects, when considered as presheaves, take values in the subcategory consisting of Kan complexes[141]. The gluing condition then takes the form of local elements combining up to a form of coherent homotopy with higher homotopies linking the various levels. This leads to a simplicial set of 'descent data' (relative to a covering of an object) which is to be compared with the given value of the presheaf on that object. As this is quite a bit more technically challenging than earlier sections, we will leave this deliberately vague[142]. (For more precision, the reader may want to look at the nLab pages on stacks descent and presheaves of simplicial sets, with subsequent following up of the references given there.)

6.2 ... and their pursuit

This allows for a very useful generalisation, and leads us to a short quote from the letter of Grothendieck to Larry Breen, dated 17 February 1975 (in a translation, [27], annotated for use here) on the 'yoga of homotopy', which led on to the manuscript, 'Pursuing Stacks', [32]:

In other terms, the constructions on a topos[143] *X which one can make in terms of $(n-1)$-stacks which are* locally *constant, depend only of its 'n-truncated pro-homotopy*[144] *type' and define it. In the case where X is locally homotopically trivial in dim $\leq n$ and so defines an n-truncated ordinary homotopy type, one can interpret these last as an n-groupoid C_n, defined up to n-equivalence. In terms of these*[145].

The $(n-1)$-stacks on X should be able to be identified with the n-functors from the category C_n to the category $((n-1)-Cat)$ of all $(n-1)$-categories.

In the case $n = 1$, this is nothing other that the Poincaré theory of the classification of coverings of X in terms of the 'fundamental groupoid' C of X. By extension, C_n merits the name of fundamental n-groupoid *of X, which I propose to write $\Pi_n(X)$. Knowledge of this includes knowledge of the $\pi_i(X)$, $(0 \leq i \leq n$ and the Postnikoff invariants of all orders up to $H^{n+1}(\Pi_{n-1}(X), \pi_n)$.*

(17 February 1975)

139... in a sense that we leave you to work out or look up.

140We here are deviating from having simply a classical space on which things are happening.

141A detailed treatment of this would require the introduction of model category theory and the discussion of the model category structures on categories of simplicial presheaves.

142but hope that some of the intuition gets through the vagueness!

143For more on the idea of a topos, see other chapters in this collection. We think of this just as being the category of sheaves on a (topological) space or on a site. Remember that Grothendieck was mainly interested in applications to problems in Algebraic Geometry.

144see below in the text.

145There were two earlier forms of the idea explored in the letter, but which are more technically stated so will not be included here.

We will stop the quotation there as it does contain the point that we will be needing, but would also suggest that that letter and the following one, [28], of 17 July 1975, which continues some of the same themes, are well worth looking at. The main point for us is that Grothendieck's conception of an ∞-groupoidal model for 'spaces' includes not only the direct 'equivalence' between 'spaces' and some kind of weak ∞-groupoid, but also continues and extends the covering space formulation started by Poincaré and continued by Grothendieck himself in SGA1. The idea is that if we have a locally constant $(n-1)$-stack, then the fibre over any point should be an $(n-1)$-groupoid, and the automorphisms of that fibre should give an n-groupoid. The stack would then correspond to a morphism of n-groupoids from $\Pi_n(X)$ to that automorphism object, i.e., a representation of $\Pi_n(X)$. Doing a small reality check on this possibility for low values of n: for the case $n = 1$, a 0-groupoid is a set, and the automorphisms of a set form a group, thus a 1-groupoid (with one object); for $n = 2$, a 1-groupoid has an automorphism gadget that is a 2-groupoid, so that fits, and for $n = \infty$, the automorphism gadget of a Kan complex (considered as model for a ∞-groupoid) is a simplicial group, and that can also be considered to be an ∞-groupoid[146].

We need, however, to give some extra 'notes' on various points in the quotation. These by necessity are slightly more technical, but are only intended to make the quotation slightly easier to approach for the more general reader.

(i) The idea of a pro-homotopy type is that of a set of interrelated approximations to a general homotopy type. Such things are needed, for instance, for handling the more general objects found in algebraic geometry. Here Grothendieck is slightly simplifying things, but when he talks of 'locally homotopically trivial', *that* is the analogue of 'locally contractible', so then X looks a bit like a CW-space and the 'pro-homotopy type' simplifies to being a 'homotopy type'. This is relevant for the question: *What sort of spaces are we considering?* For spaces and concepts of spaces, (general topological spaces, schemes, toposes, etc.), that are more general than CW-spaces, probing the 'space' even via 'points' (i.e. singular 0-simplices) is very problematic. There may be very few points and very few singular simplices, hence the approach via paths and their generalisations will not be adequate. The pro-homotopy type can lead to a pro-∞-groupoid, in some sense, regardless of whether there are enough points or not.

(ii) An important point to note once again is that this sketch by Grothendieck also needs a working theory of ∞-categories, or rather of n-categories, n-functors, 'categorical' n-equivalences, etc., for all n including $n = \infty$. Such a theory existed for small values of n at the time the letter was written, but has involved much effort to formulate it precisely[147] for $n = \infty$.

Given that we can restate Grothendieck's idea as a form of Higher Poincaré-Galois Theory, namely: there should be an '∞-equivalence'

locally constant stacks of ∞-groupoids on X $\leftrightarrow$ representations of $\Pi_\infty(X)$, the

[146] In fact the classical theory of twisted cartesian products in simplicial homotopy theory can be thought of as being one simple form of the correspondence that Grothendieck is talking about.

[147] References for this include Maltsiniotis, [51], Ara's thesis, [1] and, for a simplicially based theory using *quasi-categories*, Joyal, [39, 40], Lurie, [47, 48] and below, here. Hoyois, [36], has now completed a detailed quasi-categorical attack on this aspect of Grothendieck's ideas.

fundamental ∞-groupoid of X.

Such a Galois theorem is, then, the ∞-version of a hierarchy of results for n-groupoids, which looks as follows:

- representations of the fundamental 1-groupoid classify coverings;
- representations of the fundamental 2-groupoid classify locally constant 1-stacks (in particular connected 1-stacks, usually called *gerbes*);
- ... (for subsequent ns);

and then representations of the fundamental ∞-groupoid classify locally constant ∞-stacks (in particular the higher gerbes as mentioned elsewhere in this collection).

7 Concluding discussion

It seems a good idea to try to bring together some of the themes we have been following in an attempt to answer some of the questions we started this chapter with. The title of the chapter suggested that some aspects of 'spaces' could be encoded by ∞-groupoids. We saw that probing a space firstly by paths and then by higher dimensional analogues, allowed an algebraic / combinatorial model of the space to be built. For both historical reasons and expositional purposes, we looked in detail only at the structures encoded this way using the simplices, Δ^n, but we could have used n-globes or n-cubes[148]. So 'what aspects?' initially has to be limited to those features which are available via probing with some test models. This works best with spaces that are sufficiently 'locally homotopically trivial', and thus 'spaces' ends up being limited in meaning to being 'topological spaces' and more often than not 'CW-spaces' as those are constructible from the usual test objects. Luckily many spaces[149] occurring in geometric and theoretical physical contexts are CW-spaces, so this gives a large degree of applicability to the resulting theory[150]. It also provides the simple answer to 'what spaces?'.

The relationship between (topological) spaces and ∞-groupoids is symbiotic. The aspects of a space that are coded up in its singular complex are '∞-groupoidal' and as such provide a (classical) somewhat combinatorial model for the space. Applying algebraic constructions allows one to mix the combinatorial structure with suitable algebra and extract useful and usable information. If no CW-structure is available then things are more tricky, but if there is one, then useful information of an ∞-categorical nature can be obtained with much smaller simpler models, but often at the expense of a loss of detailed structure.

[148]These two come with some advantages either from the intuitive viewpoint of by having a simpler compositional structure. Simplices, however, lead to simplicial sets and their very well understood homotopy theory makes them a natural first choice, at least at this point in time.

[149]... but not all,

[150]'Spaces' by themselves are unable to handle some important situations. Sometimes objets that one expects to be modelled by points of a space, have naturally symmetries, which does not accord well with just a directly spatial model. Such things as these 'orbifolds' although not 'directly spatial' do correspond to certain forms of stack, so fit into the overall ∞-groupoid model quite easily.

Mixing Whitehead's and Grothendieck's viewpoints, ∞-groupoids are a good first step to analysing more of the structure of a spatial homotopy type[151].

In many parts of mathematics, there are naturally occurring objects that look like ∞-groupoids or one of their avatars such as crossed complexes, crossed n-cubes, chain complexes, descending from the lofty heights of '∞-groupoidheim' to live among us by strictifying or nullifying some of the structure from the general case. Typically such structures arise because they enhance the notion of 'identity' (of sets) into that of 'identification'[152]. By this we mean that when there is some reason to 'identify' two situations in mathematics, it is often, nay *nearly always*, beneficial to record the collection of 'reasons' for so doing; cf. page 21 for some brief development of this idea[153]. This leads to a quite constructive form[154] of mathematics. As this often gives a type of ∞-groupoid, the 'spaces as ∞-groupoids' analogy can be turned around to provide a collection of spatial insights and tools in many other settings in mathematics.

Is the spatial intuition, thus being invoked, a good one to use or is it too constraining or, alternatively, too wide?

My own feeling on this is that only time will tell. Some of the ways of thought involved seem quite difficult to handle in some instances. Even in the theory of group presentations, it is fair to ask what the module of identities of a presentation tells one about the group itself, yet that is, perhaps, still not clear. Working out higher invariants of situations like that does tell one something, but it is not always clear how to use that.

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[151] Using n-stacks of such things allows interpretation of non-Abelian forms of cohomology and a transfer of the ∞-groupoid technology to other types of space.

[152] This situation is even more explicit In Homotopy Type Theory, since that theory is built around the idea of multiple identification.

[153] and the short article [59] for a fairly informal discussion of some of its consequences.

[154] In fact, something along these lines was pointed out by Mike Shulman in a discussion in the *n-cat café* about slogans in category theory. He mentioned that Errett Bishop *defined* a set as follows: *A set is defined by describing exactly what must be done in order to construct an element of the set and what must be done in order to show that two elements are equal.* In other words, 'equality' as a structure needs to be taken seriously!

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