Chern-Simons theory, Knot polynomials & Quivers

By
Vivek Kumar Singh

Based on : arXiv:1504.00364(JKTR), arXiv:1504.00371(JHEP), arXiv:1601.04199 (J.Phys.A), arXiv:1702.06316 (JHEP),arXiv:1805.03916(Annales Henri Poincaré(2019)), arXiv:2007.12532(Journal of Geometry and Physics),arXiv:2302.xxxx..
(P. Ramadevi, Satoshi Nawata, Andrei Mironov, Alexei Morozov, Andrey Morozov, Alexei Sleptsov and S.Dhara)

Quantum Colloquium Talk at NYUAD

CENTER FOR QUANTUM & TOPOLOGICAL SYSTEMS

جامعة نيويورك ابوظبي
NYU | ABU DHABI

Plan
of the talk

- Introduction
- Chern-Simons Theory
- Mutant Knots and Weaving knots
- Knot-Quiver Correspondence
- Summary and Discussion

What is Knot and Link ?

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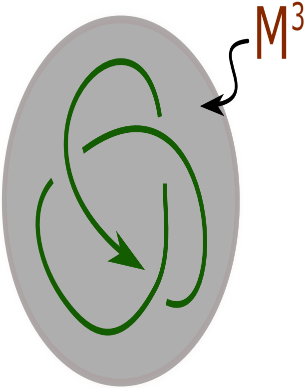
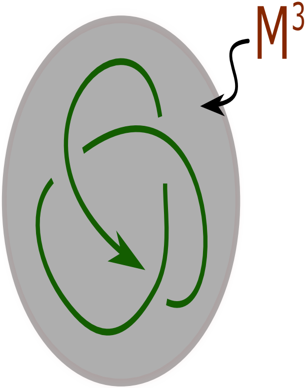
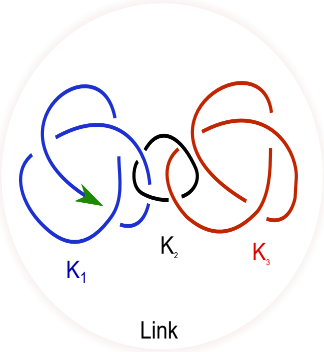


Introduction

What is Knot and Link ?





Periodic table of Knots

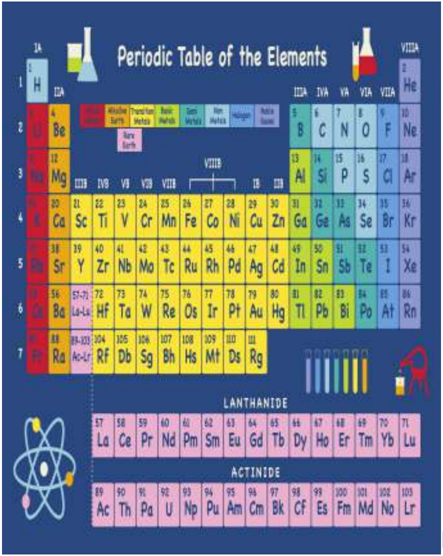


Figure: Classification of elements

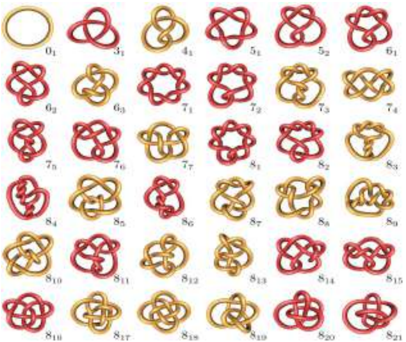


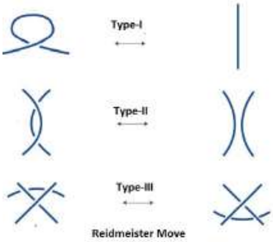
Figure: Classification of knots

Image source: Google

Classification of Knots

Classification Problem

Can we distinguish knots ?



Classification of Knots

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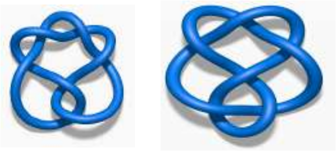

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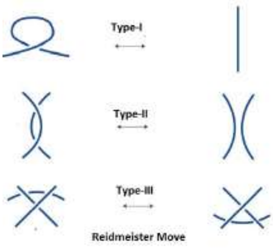


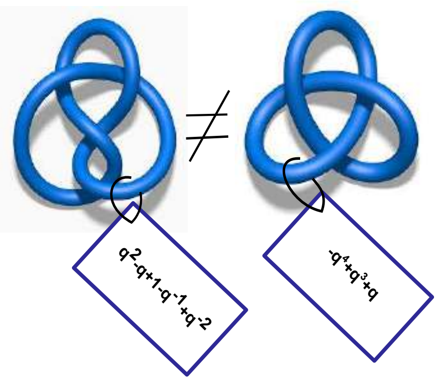
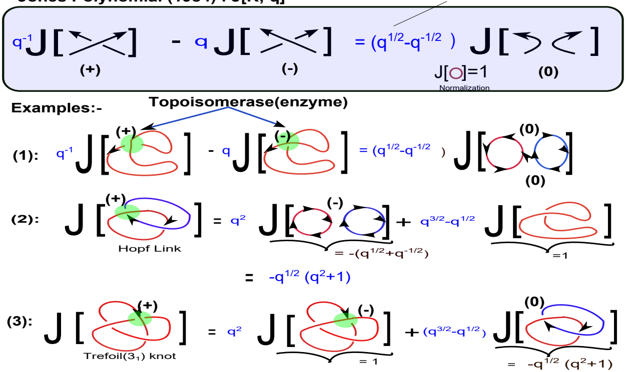


Figure: algebraic quantity associated with each knot

Image source: Google

Skein relation and Knot invariants

Jones Polynomial (1984) : J[K; q]



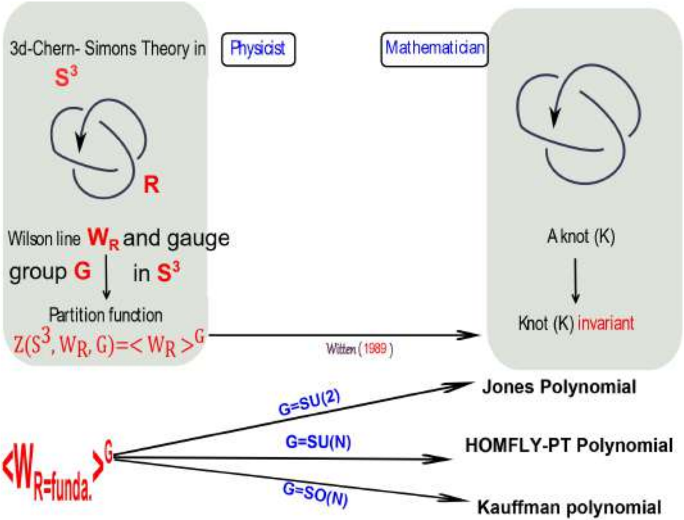
Examples: (5_1, 10_{132}), (8_8, 10_{129}), (10_{25}, 10_{56}), (10_{22}, 10_{35}), (10_{41}, 10_{94}) etc.

HOMFLY-PT Polynomial : H[K; A=q^(N/2),q]

Examples: (5_1, 10_{132}), (8_8, 10_{129}), (10_{25}, 10_{56}), (10_{22}, 10_{35}), (10_{40}, 10_{103}) etc.

Need Further Improvement !!!!!

Connection to Physics



Chern-Simons Theory

- Chern-Simons action $S_{CS}[A]$ on S^3 (metric independent)

$$S_{CS}[A] \;=\; \frac{k}{4\pi}\int_{S^3} Tr\left(A\wedge dA+\frac{2}{3}A\wedge A\wedge A\right)$$

k is the coupling constant, A's are the gauge connections.

Partition function
$\mathcal{Z}(M^3,k,G)$

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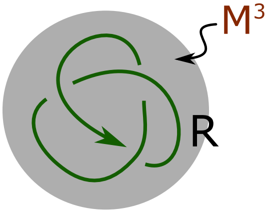
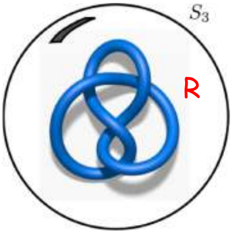


Chern-Simons Theory

Wilson loop operator

$$W_{\underline{R}}[K] = \mathrm{Tr}_{\mathrm{R}} exp \oint_K dx^u A_\mu^a \underline{T_R^a}$$



T_R^a=Generators for representation R of SU(N)

Knot invariant

Chern-Simons Theory

[Witten '89]

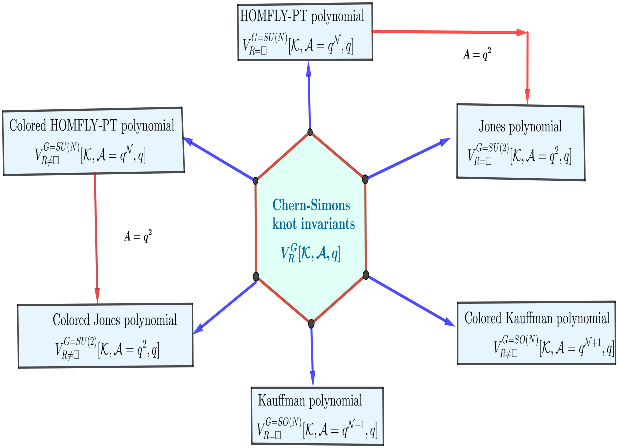
	R= □ (fundamental)	Higher rank representation
SU(2)	Jones Polynomial	Colored Jones
SU(N)	HOMFLY-PT Polynomial	Colored HOMFLY-PT

$J($  $, q) = q + q^3 - q^4$

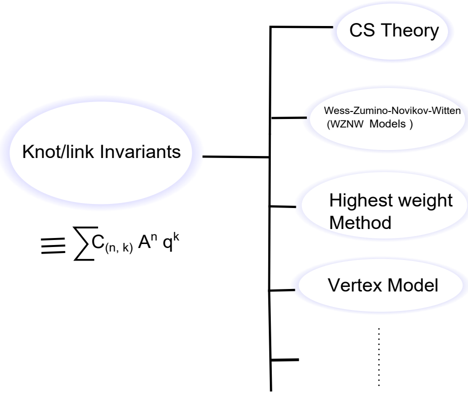
Variables: $q = e^{\frac{2\pi i}{k+N}}, \quad a = q^N$

Chern-Simons Invariants

Towards the solving classification problems of knot



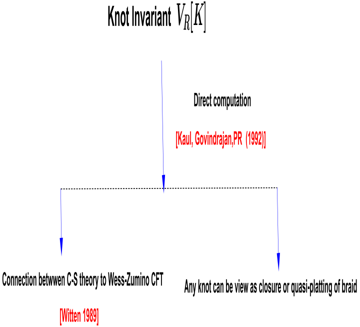
Method for computation of knot invariants

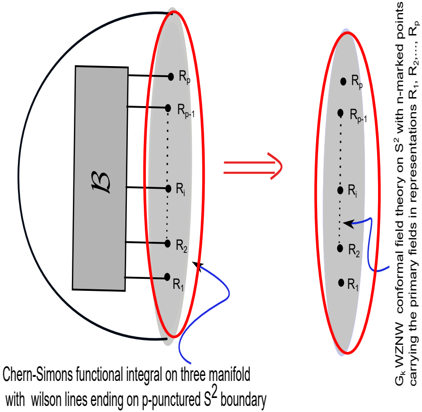


$$\equiv \sum C_{(n,\, k)} A^n q^k$$

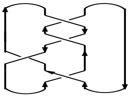
Direct Method to compute knot Invariant

The skein relation is too tedious for calculating higher crossing knots.

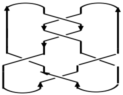




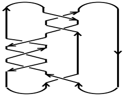
Examples



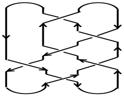
4_1



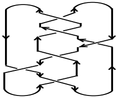
5_2



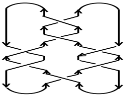
6_1



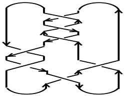
6_2



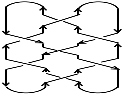
6_3



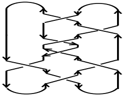
7_2



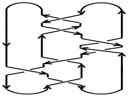
7_3



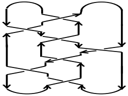
7_4



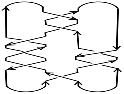
7_5



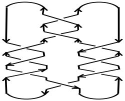
7_6



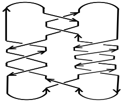
7_7



8_1



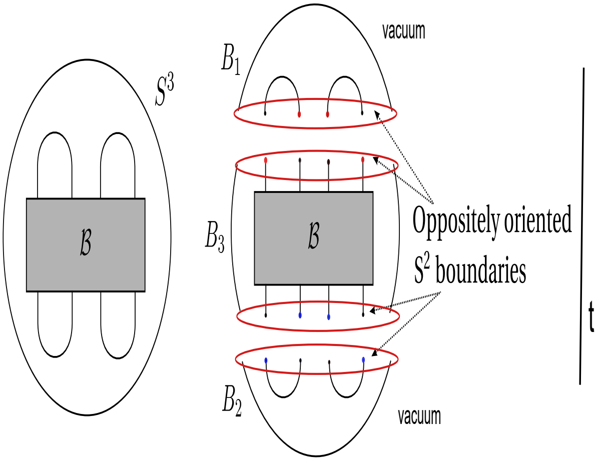
9_2



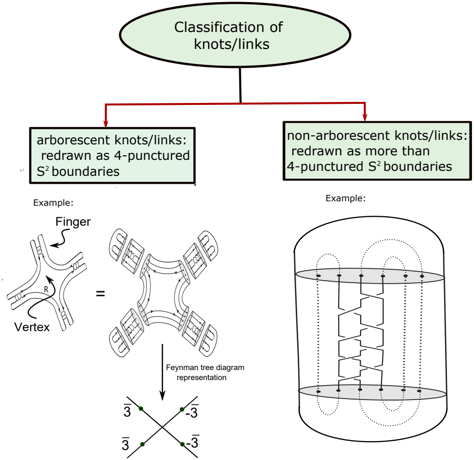
10_1

Direct Method to compute knot Invariant

The skein relation is too tedious for calculating higher crossing knots.
Witten's work (1989):



Knot classification

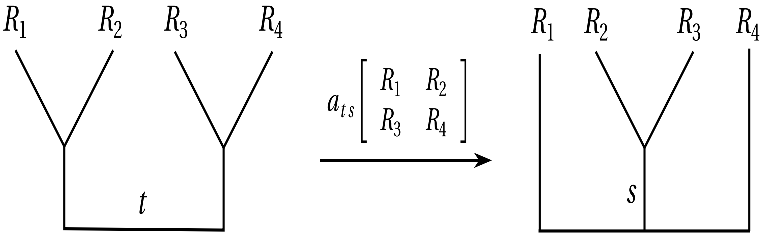


(a) $(\bar{3},\bar{3},-\bar{3},\bar{3})$ anti-parallel pretzel link

(b) 8_{18} knot

Figure: Classification of knot/link (a) arborescent (b) non- arborescent

Fusion matrix and braiding eigenvalue for 4-point conformal block



$$\lambda^{(+)}_{R_1,R_2;t} = \{R_1, R_2, t\}^{+} q^{(C_{R_1}+C_{R_2}-C_t/2)}$$

$$\lambda^{(-)}_{R_1,\bar{R}_2;t} = \{R_1, \bar{R}_2, t\}^{-} \ q^{(-C_t)/2}$$

[Moore,Seiberg '89]

where C_R denotes the quadratic Casimir of a representation R and intermediate states obey the fusion rule, *i.e.* $t \in (R_1 \otimes R_2) \cap (\bar{R}_3 \otimes \bar{R}_4)$ and $s \in (R_2 \otimes R_3) \cap (\bar{R}_1 \otimes \bar{R}_4)$

Fusion matrix

The quantum algebra $U_q(SU(2))$

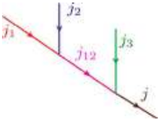
$$[J_z, J_\pm] = \pm J_\pm, \qquad [J_+, J_-] = \frac{q^{\frac{J_z}{2}} - q^{-\frac{J_z}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}} \equiv [J_z]$$

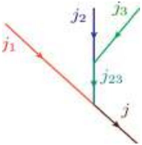
Representation: $|j, m\rangle$

$$J_\pm |j, m\rangle = \sqrt{[j \mp m][j \pm m + 1]}\ |j, m \pm 1\rangle \qquad J_z |j, m\rangle = [m]\ |j, m\rangle$$

$T_{j_1} \otimes T_{j_2} \otimes T_{j_3}$

$(T_{j_1} \otimes T_{j_2}) \otimes T_{j_3} = T_{j_1} \otimes (T_{j_2} \otimes T_{j_3})$



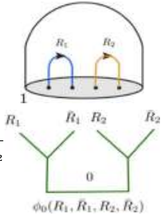


$$|(j_1, j_2)j_{12}, j_3; j, m\rangle = \sum_{j_{23}a} a_{j_{12}j_{23}} \begin{bmatrix} j_1 & j_2 \\ j_3 & j \end{bmatrix} |j_1, (j_2, j_3)j_{23}; j, m\rangle$$

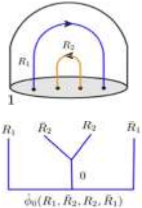
SU(2) quantum Racah coefficient

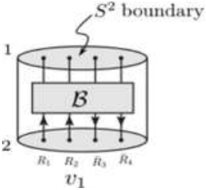
States

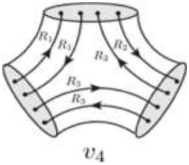
$\sqrt{\dim_q R_1 \dim_q R_2}$

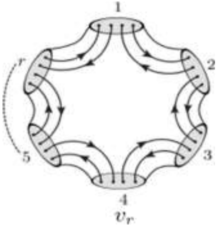


$\sqrt{\dim_q R_1 \dim_q R_2}$

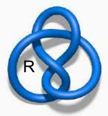






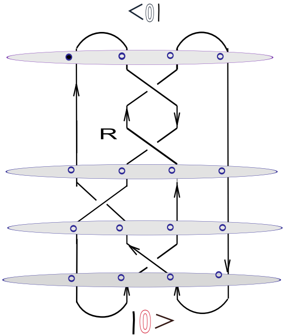


Example:- Figure Eight Knot



4_1 Knot

$$J[\ ]_R = \langle 0 | (b_2^-)^2 (b_1^-)^{-1} (b_2^+)^{-1} | 0 \rangle$$



J[1][SU(N)]=1 + A^(-2) + A^2 - q^(-2) - q^2

J[2][SU[N]=3 + q^(-6) - 1/(A^2*q^6) - q^(-4) + 1/(A^4*q^4) - 1/(A^2*q^4) + 1/(A^2*q^2) - A^2/q^2 - q^2/A^2 + A^2*q^2 - q^4 - A^2*q^4 + A^4*q^4 + q^6 - A^2*q^6

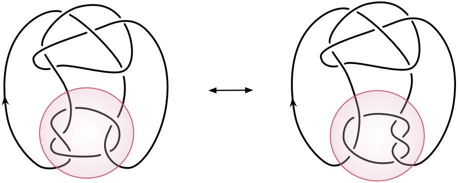
J[1][SO(N)]:=(-A + A^3 + q - 2 A^2 q + A^4 q + 2 A q^2 - 2 A^3 q^2 - q^3 + 3 A^2 q^3 - A^4 q^3 - 2 A q^4 + 2 A^3 q^4 + q^5 - 2 A^2 q^5 + A^4 q^5 + A q^6 - A^3 q^6)/(A^2 q^3)

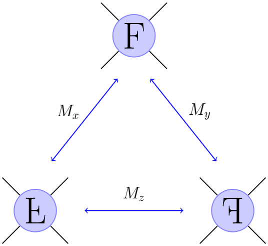
$$J[\ ]_R = \sum a_{0s}\begin{bmatrix} R & R \\ R & R \end{bmatrix} (\lambda_s^+)^{-1} a_{ts}\begin{bmatrix} R & R \\ R & R \end{bmatrix} (\lambda_t^-)^{-1} a_{tv}\begin{bmatrix} R & R \\ R & R \end{bmatrix} (\lambda_v^-)^{2} a_{0v}\begin{bmatrix} R & R \\ R & R \end{bmatrix}$$

Can Chern-Simon Knot invariants solve classification problem ?

- Mutant knots:

What is mutation ??





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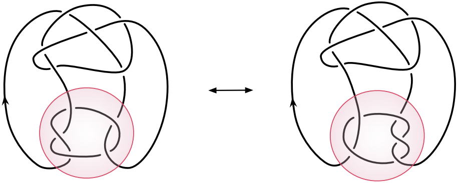

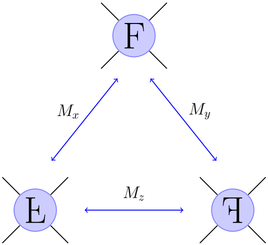

Can well-known polynomials like Jones, Homfly-PT, Kauffman polynomials distinguish ?

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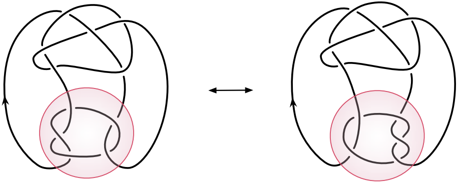

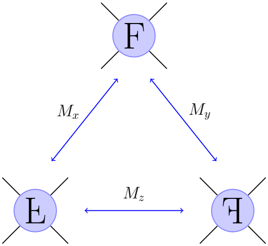

Can well-known polynomials like Jones, Homfly-PT, Kauffman polynomials distinguish ? NO!!!

Can Chern-Simons Knot invariant detect mutation?

• In arXiv:hep-th/9412084(1994), the results shows that mutation can not be studied in CS theory. Note that the explanation does not deal with the multiplicity issue properly.

• On the other hand (1996), Morton and Cromwell have shown that $[2,1]$-colored HOMFLY-PT polynomials can directly evaluating the difference of invariants of their satellites.

Moreover, the reason is explained in the view point of the cabling method by M. Ochiai and J. Murakami.

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Moreover, the reason is explained in the view point of the cabling method by M. Ochiai and J. Murakami.

- In fact, any symmetric or anti-symmetric rep. of SU(N) can not distinct (Identity operation). **Can CS theory detect mutation?** YES!! (Nawata, P.Ramadevi, V. K.Singh (JKTR, 2017))

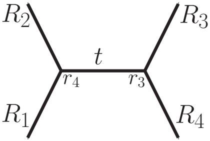
Crucial input: *multiplicity (denoted by red color)*

$$\yng(2,1) \otimes \overline{\yng(2,1)} = (0;0)_0 \oplus (1;1)_0 \oplus (1;1)_1 \oplus (2;2)_0 \oplus (2;1^2)_0 \oplus (1^2;2)_0 \oplus (1^2;1^2)_0 \oplus (21;21)_0 ,$$

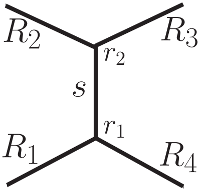
The two types of Wigner 6j has been recently determined for $\yng(2,1)$ (Gu, Jockers, 2014)

4- point conformal block basis

Now, the states in the four-point conformal blocks involve multiplicity

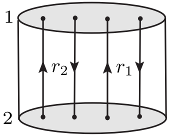


$$= |\phi^{(1)}_{t,r_3 r_4}(R_1, R_2, R_3, R_4)\rangle$$

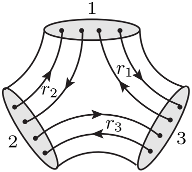


$$= |\phi^{(2)}_{s,r_1 r_2}(R_1, R_2, R_3, R_4)\rangle$$

Multi-boundary state



(a) two boundaries



(b) three boundaries

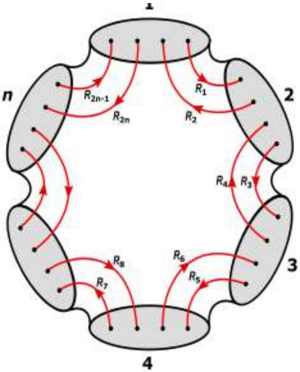
Figure: three-manifolds with boundaries

$$|\text{2-bdry}\rangle^{(a)} = \sum_{t,r_1 r_2} \{R, \bar{R}, \bar{t}, r_1\}\{R, \bar{R}, \bar{t}, r_2\}|\phi^{(1)}_{t,r_1 r_2}(\ldots)\rangle_{\mathbf{1}} \; |\phi^{(1)}_{t,r_2 r_1}(\ldots)\rangle_{\mathbf{2}}$$

$$|\text{3-bdry}\rangle^{(b)} = \sum_{t,r_1,r_2,r_3} \frac{\{R, \bar{R}, \bar{t}, r_1\}\{R, \bar{R}, \bar{t}, r_2\}\{R, \bar{R}, \bar{t}, r_3\}}{\sqrt{\dim_q t}} \,.|\phi^{(1)}_{t,r_1 r_2}(\ldots)\rangle_{\mathbf{1}} \; |\phi^{(1)}_{t,r_2 r_3}(\ldots)\rangle_{\mathbf{2}}|\phi^{(1)}_{t,r_3 r_1}(\ldots)\rangle_{\mathbf{3}} \,,$$

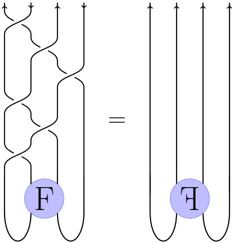
Multi-boundary state

Furthermore, it is straightforward to extrapolate it to multi-boundary states as



$$|n\text{-bdry}\rangle = \sum_{t,r_1,\cdots,r_n} \frac{\prod_{i=1}^{n}\{R,\bar{R},\bar{t},r_i\}}{\left(\sqrt{\dim_q t}\right)^{n-2}} \bigotimes_{i=1}^{n} |\phi^{(1)}_{t,r_i r_{i+1}}(\ldots)\rangle_{\mathbf{i}} \ .$$

M_y mutation operation on two -tangle

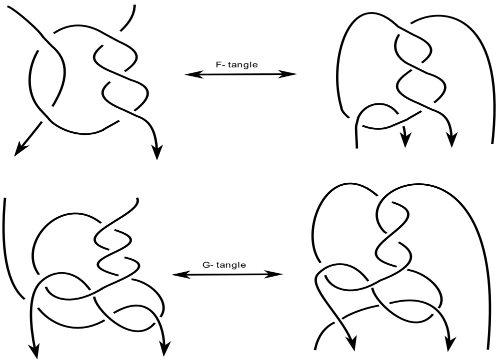


$$
\begin{aligned}
|\text{Ⅎ}\rangle &= \left([b_1^{(-)}]^{-1} b_2^{(+)} [b_1^{(-)}]^{-1}\right) b_1^{(-)} [b_3^{(-)}]^{-1} \left([b_1^{(-)}]^{-1} b_2^{(+)} [b_1^{(-)}]^{-1}\right) |\mathbf{F}\rangle \\
&= \sum_{t,r_1,r_2} \{R,\bar{R},\bar{t},r_1\}\{R,\bar{R},\bar{t},r_2\} |\phi^{(1)}_{t,r_2,r_1}(R,\bar{R},R,\bar{R})\rangle \langle \phi^{(1)}_{t,r_1,r_2}(R,\bar{R},R,\bar{R})|\mathbf{F}\rangle
\end{aligned}
$$

Note that $\{R,\bar{R},\bar{t},r_1\}$ indicated by signs ± 1 hence the amplitude of mutant tangle are related by sign when r1 $\neq$ r2.

Example :- Kinoshita-Terasaka knot and Conway knot

The F and G tangle for Kinoshita-Terasaka knot can be redrawn as follows:



Kinoshita-Terasaka knot and Conway knot

It is easy to see that

$$P_{\tiny\yng(2,1)}(K_{KT};a,q)-P_{\tiny\yng(2,1)}(K_{C};a,q)=a^{-5}q^{-18}(a-1)(a-q^2)(aq^2-1)(a-q^3)^2$$

$$(\mathrm{a}\,\mathrm{q}^3-1)^2(q-1)^2(q^3-1)^2(q^6-q^5+q^4-q^3+q^2-q+1)^2\ ,$$

so that the $SU(2)$ and $SU(3)$ quantum invariants cannot distinguish this mutant pair. The difference becomes apparent for $N > 3$ and especially, at $N = 4$, it factorizes as

$$J^{(4)}_{\tiny\yng(2,1)}(K_{KT};q)-J^{(4)}_{\tiny\yng(2,1)}(K_{C};q)=-q^{-30}(1-q)^6(1+q^2)(1-q^3)^2$$

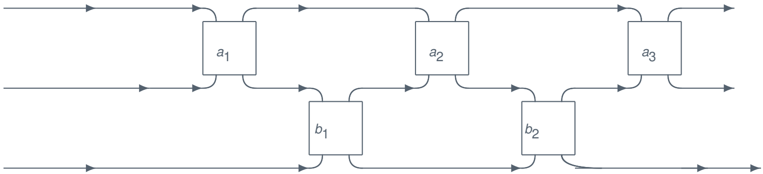
$(1 - \mathrm{q}^6)(1-q^{14})^2$, which is consistent with the result obtained by Ochiai with the computer software “Knot Theory By Computer” programmed based on the cabling method(Murakami (2000)).

Discussion

- This method is computationally efficient and it takes less than 15 minutes with a current desktop computer for the computation.
- More mutant pair discuss in arXiv:1601.04199(J.Phys. A 50 (2017)), arXiv:2007.12532(Journal of Geometry and Physics, 159(2021),)
- advanced new results of knot invariants-> knotebook.org website(DST-RFBR, P-162 funded ongoing project).
- Knot invariants:- useful to verify integrality structures predicted by U(N) and SO topological string duality conjectures (arXiv:1702.06316(JHEP08 (2017) 139)) and multi-boundary entanglement(arXiv:1711.06474(JHEP(2018)), arXiv:1906.11489(JHEP(2019),theory, arXiv:2007.07033(JHEP (2020)).

Computation Methods for non-arborescent knots

- For m=3 strand and each strand carrying representation R, parameterized by a sequence of integers (a_1, b_1, a_2, b_2)



- colored HOMFLY-PT using quantum $\mathcal{R}$ matrices will be

$$H_R = Tr\{(\mathcal{R} \otimes \mathcal{I})^{a_1} (\mathcal{I} \otimes \mathcal{R})^{b_1} (\mathcal{R} \otimes \mathcal{I})^{a_2} (\mathcal{I} \otimes \mathcal{R})^{b_2}\}$$

- Instead of working in tensor space $R^{\otimes 3}$, it is simpler to work using the irreducible representation

Examples

$$
\begin{aligned}
H_{[1]} &= \sum_{[111],[21],[3]} tr\{(\mathcal{R}_1^Q)^{a_1}(\mathcal{R}_2^Q)^{b_1}(\mathcal{R}_1^Q)^{a_2}(\mathcal{R}_2^Q)^{b_2}\} \\
&= q^{a_1+b_1+a_2+b_2} S^*_{[3]} + q^{-(a_1+b_1+a_2+b_2)} S^*_{[111]} + \\
&\quad tr\{(\mathcal{R}_1^{[21]})^{a_1}(U_{[21]}\mathcal{R}_1^{[21]}U_{[21]})^{b_1}(\mathcal{R}_1^{[21]})^{a_2}(U_{[21]}\mathcal{R}_1^{[21]}U_{[21]})^{b_2}\} S^*_{[21]}
\end{aligned}
$$

where S^*_Q are the quantum dimensions of the representation Q.

Examples

$$H_{[1]} = \sum_{[111],[21],[3]} tr\{(\mathcal{R}_1^Q)^{a_1}(\mathcal{R}_2^Q)^{b_1}(\mathcal{R}_1^Q)^{a_2}(\mathcal{R}_2^Q)^{b_2}\}$$
$$= q^{a_1+b_1+a_2+b_2} S^*_{[3]} + q^{-(a_1+b_1+a_2+b_2)} S^*_{[111]} +$$
$$tr\{(\mathcal{R}_1^{[21]})^{a_1}(U_{[21]}\mathcal{R}_1^{[21]}U_{[21]})^{b_1}(\mathcal{R}_1^{[21]})^{a_2}(U_{[21]}\mathcal{R}_1^{[21]}U_{[21]})^{b_2}\} S^*_{[21]}$$

where S^*_Q are the quantum dimensions of the representation Q.
• quantum $\mathcal{R}_1$ is diagonalisable and there is a unitary transformation U_Q to obtain $\mathcal{R}_2 = U_Q \mathcal{R}_1 U_Q$.

Examples

$$\begin{aligned} H_{[1]} &= \sum_{[111],[21],[3]} tr\{(\mathcal{R}_1^Q)^{a_1}(\mathcal{R}_2^Q)^{b_1}(\mathcal{R}_1^Q)^{a_2}(\mathcal{R}_2^Q)^{b_2}\} \\ &= q^{a_1+b_1+a_2+b_2} S^*_{[3]} + q^{-(a_1+b_1+a_2+b_2)} S^*_{[111]} + \\ & tr\{(\mathcal{R}_1^{[21]})^{a_1}(U_{[21]}\mathcal{R}_1^{[21]}U_{[21]})^{b_1}(\mathcal{R}_1^{[21]})^{a_2}(U_{[21]}\mathcal{R}_1^{[21]}U_{[21]})^{b_2}\} S^*_{[21]} \end{aligned}$$

where S^*_Q are the quantum dimensions of the representation Q.

- quantum $\mathcal{R}_1$ is diagonalisable and there is a unitary transformation U_Q to obtain $\mathcal{R}_2 = U_Q \mathcal{R}_1 U_Q$.
- U_Q is non-trivial when paths to obtain Q from $R^{\otimes 3}$ is two or more.

Examples

$$
\begin{aligned}
H_{[1]} &= \sum_{[111],[21],[3]} tr\{(\mathcal{R}_1^Q)^{a_1}(\mathcal{R}_2^Q)^{b_1}(\mathcal{R}_1^Q)^{a_2}(\mathcal{R}_2^Q)^{b_2}\} \\
&= q^{a_1+b_1+a_2+b_2} S^*_{[3]} + q^{-(a_1+b_1+a_2+b_2)} S^*_{[111]} + \\
&\quad tr\{(\mathcal{R}_1^{[21]})^{a_1}(U_{[21]}\mathcal{R}_1^{[21]}U_{[21]})^{b_1}(\mathcal{R}_1^{[21]})^{a_2}(U_{[21]}\mathcal{R}_1^{[21]}U_{[21]})^{b_2}\} S^*_{[21]}
\end{aligned}
$$

where S^*_Q are the quantum dimensions of the representation Q.

- quantum $\mathcal{R}_1$ is diagonalisable and there is a unitary transformation U_Q to obtain $\mathcal{R}_2 = U_Q \mathcal{R}_1 U_Q$.
- U_Q is non-trivial when paths to obtain Q from $R^{\otimes 3}$ is two or more.
- Highest weight method is one method which enables determining these U matrices.

Colored HOMFLY-PT for links carrying arbitrary symmetric Representations

arXiv:1805.03916(Ann. Henri Poincare (2019))

The braid word $\beta \in B_3$ for a link

There exist U matrix which relate two equivalent basis

$$|\left(([r_1]\otimes[r_2])_{X_\alpha}\otimes[r_3]\right)_{Q_\nu}\rangle \xrightarrow{U} |\left([r_1]\otimes([r_2]\otimes[r_3])_{Y_\beta}\right)_{Q_\nu}\rangle,$$

Conjecture :

$$\mathrm{U}\begin{bmatrix}[r_1] & [r_2] \\ [r_3] & \overline{[\ell_\nu, m_\nu, n_\nu]}\end{bmatrix} = \mathrm{U}_{U_q(sl_2)}\begin{bmatrix}(r_1-n_\nu)/2 & (r_2-n_\nu)/2 \\ (r_3-n_\nu)/2 & (\ell_\nu-m_\nu)/2\end{bmatrix}$$

Example $L7a3$ Link

$$\frac{H^{\mathrm{L7a3}}_{[r_1],[r_2]}}{s^*_{[r_1]}\cdot s^*_{[r_2]}} = T_{[r_2]}(q,A) + \sum_{k=1}^{min(r_1,r_2)} \frac{[r_1]![r_2]!}{[r_1-k]![r_2-k]!}\frac{\{q\}^{3k}}{A^{3r_2}}\frac{D_{-1}}{D_{r_2-1}} \times$$
$$\times \frac{\prod_{n=1}^{k} D_{r_1+n-1}\prod_{m=0}^{r_2-k-1} D_{2k+m}}{\prod_{i=0}^{r_2-k-1} D_{k+i-1}} \cdot G_{k,r_2}(q,A)\,,$$

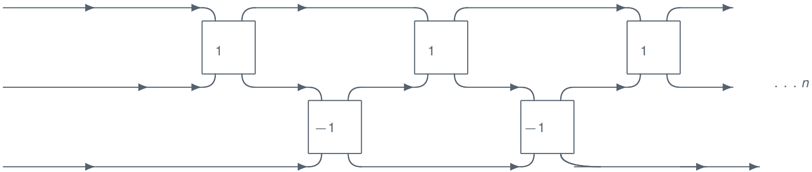
• The procedure is straightforward for $m=4$ or more strands but will involve new unitary matrices.

Weaving knot $W(p, n)$

Weaving knot obtained from closure of three-strand braid whose braid word is

$$(\sigma_1 \sigma_2^{-1} \sigma_3^{1} \ldots \sigma_{p-1}^{x})^n,$$

where $x = 1(-1)$ if p is even(odd). As example $p = 3$



(1)

Weaving knot $W(p, n)$

They attracted interest because it was conjectured that they possess maximum volume among all other knots of same crossing number. Exploring on this conjecture towards the volume, Champanerkar, Kofman and Purcell proved the following theorem.

Theorem (Theorem 1.1)

If $p \geq 3$ and $n \geq 7$, then

$$v_{\mathrm{oct}}(p-2)\, n \left(1 - \frac{(2\pi)^2}{n^2}\right)^{3/2} \leq \mathrm{vol}(W(p,n)) < \left(v_{\mathrm{oct}}(p-3) + 4\, v_{\mathrm{tet}}\right) n. \tag{2}$$

Here v_{oct} and v_{tet} denote the volumes of the ideal octahedron and ideal tetrahedron respectively.

weaving knot $W(p, n)$

The authors refer to these bounds as asymptotically sharp because their ratio approaches 1, as p and n tend to infinity. Since the crossing number of $W(p, n)$ is known to be $(p-1)n$, the volume bounds in the theorem imply

$$\lim_{p,n\to\infty} \frac{\mathrm{vol}(W(p,n))}{c(W(p,n))} = v_{\mathrm{oct}} \approx 3.66$$

Their study raises the general question of examining the asymptotic behaviour of other invariants of weaving knots.

Example $W(3, n)$

• In work of Mishra and R. Staffeldt(arXiv:1704.03982) attempted recursive method of relating the HOMFLY-PT of $W(3, n)$.
• Myself with Mishra, and Staffeldt, we have computed a closed formula for Jone's, Alexander, and Khovanov polynomials(arXiv:2302.XXXX to appear).
The Jones polynomial $\mathcal{J}^{W(3,n)}(t)$ of $W(3, n)$ is given by

$$\mathcal{J}^{W(3,n)}(t) = \sum_{k=-n}^{n} (-1)^k j[n, |k|] t^k, \text{ where, } j[n, k] = (-\delta_{(n-1, n-|k|)} + T[n, k]$$

$$T[n, k] = n \sum_{i=0}^{\frac{(n-k)}{2}} \frac{1}{n-i} \binom{n-i}{k+i} \binom{n-k-i-1}{i}$$

This gives us a neat description of Lucas number L_{2n} as

$$L_{2n} = \sum_{k=-n}^{n} |T[n, |k|],$$

Summary and Open problems

- We have explicitly worked out r=2 and r=3 colors for hybrid weaving knot $W_3(m, n)$ in the paper (arXiv:2103.10228) JHEP 06 (2021) 063 and Quasi-alternating knots arXiv:2202.09169(Nucl.Phys.B 980 (2022)).
- Quantum $\mathcal{R}$-matrices approach for higher colors is straightforward but no closed form expression
- closed form for r-colored HOMFLY-PT for hyperbolic weaving knots W(p,n)
- Will help to address volume conjecture?

KNOT-QUIVER Correspondence

Piotr Kucharski, Markus Reineke, Marko Stosic, Piotr Sułkowski–
arXiv:1707.04017(Adv. Theor. Math. Phys. 23 (2019))

Any knot one can assign a quiver, more precisely, defined as

$$P_r(A,q) = \sum_{d_1+\ldots+d_m=r} (-1)^{\sum \gamma_i d_i} \frac{q^{\sum_{i,j} C_{i,j} d_i d_j} (q;q)_r}{\prod_{i=1}^{m} (q;q)_{d_i}} q^{\sum \alpha_i d_i} A^{\sum \beta_i d_i}. \quad (3)$$

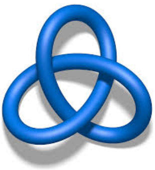
Here, $C_{i,j}$ is quiver charge matrix.

Example: [r] colored super polynomial for trefoil (3_1) :

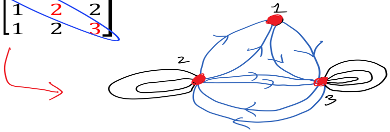
$$P_{[r]}(a,q,t) = \frac{a^{2r}}{q^{2r}} \sum_{k=0} \begin{bmatrix} r \\ k \end{bmatrix} q^{2k(r+1)} t^{2k} \left[\prod_{i=1}^{k} (1 + a^2 q^{2(i-2)} t)\right]$$

$$P_{[1]}(a,q,t) = t^0 \frac{a^2}{q^2} + a^2 q^2 t^2 + q^4 t^3$$

Quiver representation C:



$$C^{3_1} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$



Knot-Quiver Correspondence for double twist knot

arXiv:2302.XXXX to appear

$$J_r(K(p,-m);q) = \sum_{d_1,d_2\ldots d_{4mp+1}} (-1)^{\sum_i \gamma_i d_i} \frac{(q^2;q^2)_r}{\prod_{i=1}^{4mp+1}(q^2;q^2)_{d_i}} q^{\sum C^{K(p,m)} d_i d_j + \beta_i d_i}$$

The quiver charge matrix for an arbitrary p, m, takes the form

$$C^{K(-m,p)} = \left[\begin{array}{c|c|c|c|c|c|c|c}
F_0 & F_1 & \tilde{F}_1 & & \cdots & & F_p & \tilde{F}_p \\ \hline
F_1^T & U_1 & R_1 & \tilde{R}_1 & \cdots & & R_1 & \tilde{R}_1 \\ \hline
\tilde{F}_1^T & R_1^T & \tilde{U}_1 & T_1 & \tilde{T}_1 & \cdots & T_1 & \tilde{T}_1 \\ \hline
\tilde{F}_1^T & \tilde{R}_1^T & T_1 & U_2 & \tilde{R}_2 & \cdots & R_2 & \tilde{R}_2 \\ \hline
\vdots & \vdots & \ddots & \vdots & \cdots & & \vdots & \vdots \\ \hline
F_i^T & R_1^T & \cdots & U_i & \cdots & & R_i & \tilde{R}_p \\ \hline
\tilde{F}_i^T & \tilde{R}_1^T & \cdots & R_i^T & \tilde{U}_i & \cdots & T_i & \tilde{T}_i \\ \hline
\vdots & \vdots & \vdots & \vdots & \ddots & & \vdots & \vdots \\ \hline
F_p^T & R_1^T & T_1^T & R_2^T & & \cdots & U_p & R_p \\ \hline
\tilde{F}_p^T & \tilde{R}_1^T & \tilde{T}_1^T & \tilde{R}_2^T & & \cdots & R_p^T & \tilde{U}_p
\end{array}\right] \quad (4)$$

Future directions

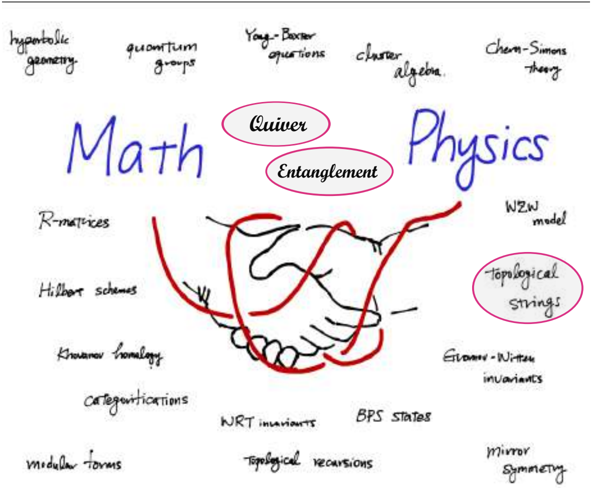


Image source: Satoshi Nawata webpage