The Definition of Conformal Field Theory

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The object of this work is to present a definition of a two-dimensional conformally invariant quantum field theory in mathematical language, and to describe the basic examples. I hope this will be helpful to mathematicians who are interested in physics; but apart from that there are several areas of pure mathematics where conformal field theories seem to play a fundamental but quite unexpected role. I shall give five examples.

(i) The "monster" group of Griess-Fischer is the group of automorphisms of a fairly simple and natural conformal field theory. The graded representation of the monster group whose Poincaré series is the modular function j is the basic Hilbert space of the field theory, and Griess's non-associative algebra is also part of its structure.

(ii) The representation theory of loop groups and of the group $\mathrm{Diff}(S^1)$ of diffeomorphisms of the circle is greatly illuminated by conformal field theory. In particular the modularity properties of the characters of the representations fall into place.

(iii) Field theory shows how the representations of $\mathrm{Diff}(S^1)$ are related to the geometry of the moduli spaces of Riemann surfaces. Thus the universal central extension of $\mathrm{Diff}(S^1)$ "is" the determinant line of the $\bar{\partial}$-operator on Riemann surfaces; and Mumford's classification of the holomorphic line bundles on moduli spaces can be simply proved.

(iv) Some, at least, of Vaughan Jones's new representations of braid groups arise from field theories, and his classification of subfactors in von Neumann algebras is reflected in the classification of field theories.

(v) The new "elliptic" cohomology theory of Landweber-Stong and Ochanine is undoubtedly connected with conformal field theory, though the connection is still mysterious.

This work is intended to be a coherent and self-contained exposition of material which is essentially well known. It contains no new results. The different sections are fairly independent, and aimed at slightly different readers: they are not meant to be read in order. The recent wave of interest in conformal field theory began with the well-known paper [BPZ] of Belavin, Polyakov, and Zamolodchikov, but I have not attempted the difficult task of indicating the history of the subject, or the provenance of particular ideas. I should like to point out, however, that two features of my exposition which I thought original when I wrote the first versions of this work, namely the emphasis on the semigroup $\mathcal{A}$ in §2, and the algebraic model of the fermion theory in §8, had been developed independently by Neretin [N1],[N2]. Apart from that I am greatly indebted to very many people who have taught me about the subject, especially Deligne, Frenkel, Friedan, Quillen, and Witten. Quillen was originally to be a joint author.

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§1. Introduction

I shall begin with a schematic description of the situation we want to axiomatize. Suppose that a string in the form of a circle S^1 is moving about in a manifold M. The configuration space of the system is then the loop space $\mathcal{L}M$, and the quantum states are the rays in a Hilbert space $\mathfrak{H}$ of complex-valued wave functions on $\mathcal{L}M$. The evolution of the system is described by a one-parameter group $\{e^{iHt}\}$ of unitary operators in $\mathfrak{H}$. If $T > 0$ the contraction operator e^{-HT} is an integral operator in $\mathfrak{H}$:

$$(e^{-HT}\psi)(\gamma) = \int_{\mathcal{L}M} K_T(\gamma,\gamma')\psi(\gamma')\mathcal{D}\gamma' \ , \tag{1.1}$$

where the kernel K_T is of the form

$$K_T(\gamma,\gamma') = \int e^{-S(\sigma)}\mathcal{D}\sigma \ , \tag{1.2}$$

the integral being over all paths $\sigma : [0,T] \rightarrow \mathcal{L}M$ from γ to γ', i.e. over all maps $\sigma : S^1 \times [0,T] \rightarrow M$ which restrict to γ,γ' at the ends of the cylinder. The crucial property of the functional S is that it depends only on the <u>conformal</u> structure of the surface $X = S^1 \times [0,T]$ the basic example is $S(\sigma) = \frac{1}{2}\int_\Sigma \|D\sigma\|^2$.

Needless to say, the preceding integrals have no precise sense. We extract from the discussion simply the Hilbert space $\mathfrak{H}$ and the idea of an operator in $\mathfrak{H}$ depending not so much on a number T as on the <u>Riemann surface</u> $X = S^1 \times [0,T]$. We could as well or as ill perform the integral (1.2) over maps defined on <u>any</u> Riemann surface X whose boundary consists of two circles, and so obtain an evolution operator $U_X : \mathfrak{H} \rightarrow \mathfrak{H}$. If two such surfaces X,X' are joined end-to-end to form a new one then we expect the semigroup property

$$U_{X \cup X'} = U_{X'} U_X . \tag{1.3}$$

It is now natural, given a Riemann surface X with a boundary consisting of m+n circles, to interpret the integral (1.2) over maps $X \to M$ as the kernel of an operator which transforms functions of m loops to functions of n loops, i.e. as the kernel of an operator

$$U_X : \mathcal{H}^{\otimes m} \to \mathcal{H}^{\otimes n} .$$

We might think of this operator as associated with a physical process in which m strings evolve into n strings. We still expect the composition rule (1.3) to hold when it makes sense.

The structure so far described is simply a functor from a certain category $\mathcal{C}$ to the category of Hilbert spaces: the objects of $\mathcal{C}$ are all compact one-dimensional manifolds (i.e. finite disjoint unions of circles), and a morphism from S_0 to S_1 is a Riemann surface X whose boundary ∂X is the disjoint union $S_0 \amalg S_1$. Composition of morphisms in $\mathcal{C}$ is defined by sewing the surfaces together along the common part of their boundaries. A functor assigns a Hilbert space $\mathcal{H}_S$ to each 1-manifold, and an operator $U_X : \mathcal{H}_{S_0} \to \mathcal{H}_{S_1}$ to each surface X with $\partial X = S_0 \amalg S_1$. A conformal field theory is no more and no less than such a functor.[1] It must satisfy a number of simple conditions motivated by the formulae (1.1) and (1.2). The most obvious is that

$$\mathcal{H}_{S_0 \amalg S_1} = \mathcal{H}_{S_0} \otimes \mathcal{H}_{S_1} .$$

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[1]Strictly speaking, a <u>projective</u> functor: one should allow a scalar multiplier in (1.3).

Another is that if X is a surface with exactly two boundary circles then the trace of the operator U_X depends only on the closed surface $\check{X}$ got by sewing the two ends of X together. The motivation for this is that integrating $K_X(\gamma,\gamma)$ over $\gamma \in \mathcal{L}M$ amounts to integrating $e^{-S(\sigma)}$ over all maps $\sigma : \check{X} \to M$.) This property implies the modularity of the <u>partition function</u> of the theory, which is defined as the trace of the operator e^{-HT} associated to the cylinder $X_T = S^1 \times [0,T]$. Because the tori $\check{X}_T$ and $\check{X}_{1/T}$ are conformally equivalent the partition function satisfies

$$\mathrm{tr}(E^{-HT}) = \mathrm{tr}(e^{-H/T}) \ .$$

There is another way to approach conformal invariance. The basic Hilbert space $\mathfrak{H} = \mathfrak{H}_{S^1}$ of the theory is thought of as the quantization of a classical system whose phase space is the tangent bundle $T\mathcal{L}M$. This can be identified with the space of solutions $\sigma : S^1 \times \mathbb{R} \to M$ of the classical equations of motion, which are conformally invariant, i.e. invariant under the group $\mathrm{Conf}(S^1 \times \mathbb{R})$ of diffeomorphisms of $S^1 \times \mathbb{R}$ which preserve $d\theta^2 - dt^2$ up to multiplication by a function of (θ,t). Thus $\mathrm{Conf}(S^1 \times \mathbb{R})$ acts on $T\mathcal{L}M$. We shall see that it follows from our definition of a field theory that $\mathrm{Conf}(S^1 \times \mathbb{R})$ acts projectively on $\mathfrak{H}$. One can think of a conformal field theory as a projective unitary representation of $\mathrm{Conf}(S^1 \times \mathbb{R})$ equipped with some additional structure. Speaking very roughly, the additional structure expresses the fact that is a representation of a disconnected "group" which has $\mathrm{Conf}(S^1 \times \mathbb{R})$ as its identity component.

The group $\text{Conf}(S^1 \times \mathbb{R})$ is a $\mathbb{Z}$-fold covering group of $\text{Diff}(S^1) \times \text{Diff}(S^1)$. For $S^1 \times \mathbb{R}$ possesses a circle S^1_R of right-moving light-paths $\{\theta = t-\alpha : \alpha \in S^1_R\}$ and a circle S^1_L of left-moving paths $\{\theta = -t+\alpha : \alpha \in S^1_L\}$; these two circles are permuted by any conformal diffeomorphism, and so we have a homomorphism

$$\text{Conf}(S^1 \times \mathbb{R}) \to \text{Diff}(S^1_L) \times \text{Diff}(S^1_R) ,$$

which is clearly surjective with kernel $\mathbb{Z}$. An irreducible projective representation $\mathfrak{H}$ of $\text{Conf}(S^1 \times \mathbb{R})$ decomposes canonically as a tensor product $\mathfrak{H} = \mathfrak{H}_L \otimes \mathfrak{H}_R$ of representations of $\text{Diff}(S^1_L)$ and $\text{Diff}(S^1_R)$. One of the interesting questions to ask about conformal field theories is how they decompose into left-handed and right-handed theories. These so-called <u>chiral</u> theories are to a mathematician – not to a physicist – the basic objects of study. They are rigid in the same sense as the representations of a compact group. Theories containing both chiralities, in contrast, are capable of continuous deformation. We shall consider the simplest example of this phenomenon in §10.

§2. $\mathrm{Diff}^+(S^1)$ and the semigroup of annuli

The group $\mathrm{Diff}^+(S^1)$ of orientation-preserving diffeomorphisms of the circle is an infinite dimensional Lie group which does not possess a complexification. In this section I shall describe a complex Lie semigroup $\mathcal{A}$ which can reasonably be regarded as a subsemigroup of the non-existent complexification. The relation between $\mathrm{Diff}^+(S^1)$ and $\mathcal{A}$ is exactly the same as that between the group $\mathbf{T} = \{z \in \mathbb{C} : |z| = 1\}$ and the semigroup $\mathbb{C}^\times_{<1} = \{z \in \mathbb{C} : 0 < |z| < 1\}$, or, better, between the subgroup $PSU_{1,1}$ of $\mathrm{Diff}^+(S^1)$ consisting of Mobius transformations and the sub-semigroup

$$PSL_2^<(\mathbb{C}) = \{g \in PSL_2(\mathbb{C}) : g(D) \subset \mathring{D}\}$$

of the complexification $PSL_2(\mathbb{C})$ of $PSU_{1,1}$. (Here D is the unit disc $\{z \in \mathbb{C} : |z| \leqslant 1\}$, and $\mathring{D}$ is its interior.) Another such pair consists of U_n and the semigroup of contraction operators $\{g \in GL_n(\mathbb{C}) : \|g\| < 1\}$.

The semigroup $\mathcal{A}$ is constructed by considering Riemann surfaces with boundaries. The surfaces we consider in this paper will always be compact smooth (i.e. C^∞) manifolds X with boundary ∂X, with a smooth almost complex structure defined everywhere in X. We shall usually consider surfaces <u>with parametrized boundaries</u>, i.e. with a given smooth identification of each boundary circle $S \subset \partial X$ with the standard circle $S^1 = \mathbb{R}/\mathbb{Z}$. If the parametrization of S agrees with the orientation induced by the complex structure of X we shall call the circle <u>outgoing</u>, otherwise <u>incoming</u>. Surfaces with parametrized boundaries can be sewn together by identifying incoming circles with outgoing ones. One can also sew together an incoming and an outgoing circle of the same surface. The sewing-together process is formally characterized as follows. If $\check{X}$ is obtained from a (possibly disconnected) surface X by sewing together some of the circles making up ∂X, and $\pi : X \to \check{X}$ is the identification map, then a function $f : U \to \mathbb{C}$ defined in an open set U of $\check{X}$ is holomorphic if and only if the composite $f \cdot \pi : \pi^{-1}(U) \to \mathbb{C}$ is holomorphic. It is true, though by no means obvious, that this does define a complex structure (and hence a smooth structure too) on the interior of $\check{X}$.

Let $\mathcal{A}$ denote the set of isomorphism classes of Riemann surfaces A which are topologically annuli (i.e. diffeomorphic to $\{z \in \mathbb{C} : a \leqslant |z| \leqslant b\}$) and are equipped with parametrizations of their boundary circles, one incoming and one outgoing. Such annuli form a semigroup in which the composite $A_2 \circ A_1$ is formed by sewing the outgoing end of A_1 to the incoming end of A_2.

If one forgets the parametrization of the ends then any annulus is isomorphic to $A_r = \{z \in \mathbb{C} : r \leqslant |z| \leqslant 1\}$ for a unique $r \in (0,1)$. The only holomorphic automorphisms of A_r are rigid rotations, so we have

Proposition (2.1). $\mathcal{A}$ is homeomorphic to

$$(0,1) \times (\mathrm{Diff}^+(S^1) \times \mathrm{Diff}^+(S^1))/\mathbf{T} .$$

Thus $\mathcal{A}$ has the right size to be a complexification of $\mathrm{Diff}^+(S^1)$. On the other hand $\mathcal{A}$ is a complex manifold in view of

Proposition (2.2). Any element A of $\mathcal{A}$ is uniquely representable as an annulus in $\mathbb{C}$ bounded by the circles

$$z \mapsto f_0(z) = a_1 z + a_2 z^2 + \ldots$$

$$z \mapsto f_\infty(z) = \{z^{-1} + b_2 z^{-2} + b_3 z^{-3} + \ldots\}^{-1} ,$$

where f_0 extends to a holomorphic embedding $f_0 : D \to \mathbb{C}$, and f_∞ extends to a holomorphic embedding of $D_\infty = \{z \in \mathbb{C} \cup \infty : |z| \geqslant 1\}$ in the Riemann sphere S^2. (We always identify $S^1 = \mathbb{R}/\mathbb{Z}$ with $\mathbf{T} \subset \mathbb{C}$ by $t \mapsto e^{2\pi it}$.)

Proof: Given an annulus A, let $\hat{A}$ be the closed surface got by sewing copies of D and D_∞ to its ends. Then $\hat{A}$ can be identified holomorphically with the standard S^2, and the identification is unique with the normaliztion prescribed in the proposition.

The space Hol(D) of holomorphic functions on D with smooth boundary values has a natural topology as a subspace of $C^\infty(S^1)$. Proposition (2.2) identifies $\mathcal{A}$ with an open set in the complex vector space $E = \mathbb{C} \oplus \mathrm{Hol}_1(D) \oplus \mathrm{Hol}_1(D)$ by $A \mapsto (a_1, a_1^{-1} f_0, f_\infty^{-1})$: here

$$\mathrm{Hol}_1(D) = \{f : f(0) = 0 \text{ and } f'(0) = 1\} .$$

In fact $\mathcal{A}$ is a bounded domain in E, because $|a_1| < 1$ and each coefficient $a_1^{-1} a_i$ or b_i in (2.2) is also uniformly bounded. (The area of the annulus, as a subset of $\mathbb{C}$, is $\pi\{1 - \Sigma k|a_k|^2 - \Sigma k|b_k|^2\}$.)

Proposition (2.3). The composition $\mathcal{A} \times \mathcal{A} \to \mathcal{A}$ is holomorphic.

To prove this we must consider the tangent spaces to $\mathcal{A}$. Because any annulus can be embedded holomorphically in $\mathbb{C}$ we have

Proposition (2.4). The tangent space to $\mathcal{A}$ at A is the space of complex tangent vector fields to A along ∂A, modulo those which extend holomorphically over A, i.e.

$$T_A = \{\mathrm{Vect}_{\mathbb{C}}(S^1) \oplus \mathrm{Vect}_{\mathbb{C}}(S^1)\}/\mathrm{Vect}(A).$$

Remark. As A shrinks to S^1, i.e. to the absent identity element of $\mathcal{A}$, the space T_A approaches $\mathrm{Vect}_{\mathbb{C}}(S^1)$, as one would expect if $\mathcal{A}$ is a complexification of $\mathrm{Diff}(S^1)$.

Proof of (2.3). We must show that the map of tangent spaces induced by $\mathcal{A} \times \mathcal{A} \to \mathcal{A}$ is complex-linear. Let A_1 and A_2 be annuli, and $A_3 = A_2 \circ A_1$. Write $\partial A_1 = S_0 \amalg S_1$, and $\partial A_2 = S_1 \amalg S_2$, so that $A_1 \cap A_2 = S_1$.

If $(\xi_0, \xi_1) \in \mathrm{Vect}_{\mathbb{C}}(S_0) \oplus \mathrm{Vect}_{\mathbb{C}}(S_1)$ represents a tangent vector to $\mathcal{A}$ at A_1, and (η_1, η_2) represents a tangent vector at A_2, then the composition law takes these vectors to (ζ_0, ζ_2), where

$$\zeta_0 = \xi_0 + \alpha_1|S_0 \qquad \zeta_2 = \eta_2 + \alpha_2|S_2$$

for some $\alpha_i \in \mathrm{Vect}(A_i)$ such that

$$\alpha_1|S_1 - \alpha_2|S_1 = \xi_1 - \eta_1 .$$

(It follows from Laurent's theorem that any vector field on S_1 is the difference of holomorphic vector fields on A_1 and A_2.) The map$((\xi_0, \xi_1), (\eta_1, \eta_2)) \mapsto (\zeta_0, \zeta_2)$ is clearly complex-linear.

There is an important holomorphic function $q : \mathcal{A} \to \mathbb{C}^\times$ whose value at A is the modulus of the torus $\check{A}$ obtained by sewing the ends of A together. (A torus with a preferred cycle is isomorphic to $\mathbb{C}^\times/\lambda$ for a unique λ with $0 < |\lambda| < 1$: I shall call λ the modulus.) More explicitly, if $A \subset \mathbb{C}$ is bounded by the curves f_0 and f_∞, then $q(A) = \lambda$ if there is a holomorphic map $F : A \to \mathbb{C}^\times$ such that $F(f_0(z)) = \lambda F(f_\infty(z))$. I shall omit the proof that $q : \mathcal{A} \to \mathbb{C}^\times$ is holomorphic, but the following result is almost obvious.

Proposition (2.5). We have $q(A) = q(B)$ if and only if A and B are conjugate in $\mathcal{A}$, i.e. related by the equivalence relation ~ generated by

$$A \sim B \quad \text{if} \quad A = C \circ D \quad \text{and} \quad B = D \circ C \text{ for some } C, D$$

When $\mathcal{A}$ is regarded as a bounded domain in the vector space E its boundary is made up of several different pieces. One piece lies in the hyperplane $a_1 = 0$. It is of complex codimension 1, and consists of "infinitely long" annuli. If it is adjoined to $\mathcal{A}$ we still have an open set of E. Another piece of the boundary consists of the points such that the embedded discs $f_0(D)$ and $f_\infty(D_\infty)$ in S^2 touch each other, i.e. those for which the "width" of the annulus collapses to zero at some point. This piece is of real codimension 1. It contains an extremal part $\Sigma(\mathcal{A})$ where $f_0(S^1) = f_\infty(S^1)$. This is a completion of $\mathrm{Diff}^+(S^1)$, in the sense that it contains a dense open subset $\mathring{\Sigma}(\mathcal{A})$ where $f_0|S^1$ and $f_\infty|S^1$ are injective, and $\mathring{\Sigma}(\mathcal{A})$ can be identified with $\mathrm{Diff}^+(S^1)$ by $(f_0,f_\infty) \mapsto f_\infty^{-1} \circ f_0$. There are also two other parts of the boundary consisting of points where $f_0|D$ or $f_\infty|D_\infty$ fail to be embeddings.

It is natural at this point to ask a number of questions to which I do not know the answers.

(i) Is $\Sigma(\mathcal{A})$ a Shilov boundary of $\mathcal{A}$?

(ii) Does the function q extend continuously from $\mathcal{A}$ to $\Sigma(\mathcal{A})$?

(iii) If so, what is the relation between $q|\mathrm{Diff}^+(S^1)$ and the rotation number in the sense of Poincaré?

I recall that a <u>Shilov boundary</u> of $\mathcal{A}$ as a subset of E is defined as a minimal closed subset $\mathcal{B}$ of the closure $\mathcal{A}^{cl}$ of $\mathcal{A}$ with the property that

$$\sup(f|\mathcal{A}) = \sup(f|\mathcal{B})$$

for every bounded holomorphic function $f : \mathcal{A} \to \mathbb{C}$ which extends continuously to $\mathcal{A}^{cl}$. If a Shilov boundary of $\mathcal{A}$ exists then it is certainly contained in $\Sigma(\mathcal{A})$, for any boundary point A of $\mathcal{A}$ which is not contained in $\Sigma(\mathcal{A})$ belongs to a holomorphic curve in $\mathcal{A}^{cl}$ got by deforming $\partial A \subset S^2$ by any vector field on S^2 which is holomorphic everywhere except for an essential singularity in the interior of A.

An optimist might hope that for diffeomorphisms of the circle the function q simultaneously measures the rotation number and how far the diffeomorphism is from being conjugate to a rotation, i.e.

<u>Conjecture</u>. The function q extends continuously from $\mathcal{A}$ to $\Sigma(\mathcal{A})$, and for a diffeomorphism f of S^1 one has $q(f) = \rho e^{i\alpha}$, where α is the rotation number of f, and $\rho = 1$ if and only if f is conjugate to a rotation.

To conclude this section I should mention that the semigroup $\mathcal{E}$ of holomorphic embeddings $f : D \to \mathring{D}$ is a sub-semigroup of $\mathcal{A}$: one identifies f with the annulus $A_f = D - f(\mathring{D})$. Heuristically, at least, $\mathcal{E}$

is "maximal parabolic" in $\mathcal{A}$, and contains the "minimal parabolic" $\mathcal{E}_0 = \{f \in \mathcal{E} : f(0) = 0\}$. In support of this terminology we notice that (cf. [BR])

$$\mathcal{A}/\mathcal{E}_0 \cong \mathrm{Diff}^+(S^1)/\mathbf{T} ,$$

$$\mathcal{A}/\mathcal{E} \cong \mathrm{Diff}^+(S^1)/PSU_{1,1} ,$$

and also that $\bar{\mathcal{E}}_0 \mathcal{E}_0$ is an open subset of $\mathcal{A}$.

It is easy to see that if $f \in \mathcal{E}$ then $q(f) = f'(\zeta)$, where ζ is the unique fixed point of f. Thus $q|\mathcal{E}_0$ is the homomorphism $f \mapsto f'(0)$, whose kernel is the commutator subgroup of $\mathcal{E}_0$.

§3. Wick-rotation, and representations of $\mathcal{A}$

In ordinary quantum field theory there is a Hilbert space $\mathfrak{H}$ of states on which the group $\mathbb{R}^4$ of translations of Minkowski space-time acts unitarily. It is well-known that the positivity of energy can be expressed by saying that the unitary action of $\mathbb{R}^4$ extends to an action of the semigroup

$$\mathbb{C}^4_+ \;=\; \{\ \xi \in \mathbb{C}^4 : \operatorname{Im}(\xi) \in P\ \}\ ,$$

where $P \subset \mathbb{R}^4$ is the positive light-cone. The action of $\mathbb{C}^4_+$ is by contraction operators, and is holomorphic. The "boundary" $\mathbb{R}^4$ is an open dense subset of the Shilov boundary of $\mathbb{C}^4_+$.

Now let us consider 2-dimensional Minkowskian space-time $\Sigma = \mathbb{R} \times S^1$, in which space is a circle. The group T of translations is $(\mathbb{R} \times \mathbb{R})/2\pi\mathbb{Z}$, where $(\xi,\eta) \in T$ acts on $(t,\theta) \in \Sigma$ by

$$(t,\theta) \mapsto (t + \xi + \eta,\ \theta + \xi - \eta)\ .$$

The positivity of energy is now expressed by saying that the unitary action of T is the boundary value of a holomorphic contraction representation of

$$T^+_{\mathbb{C}} = \{(\xi,\eta) \in (\mathbb{C} \times \mathbb{C})/2\pi\mathbb{Z} : \operatorname{Im}(\xi) > 0,\ \operatorname{Im}(\eta) > 0\}\ .$$

This is a covering group of $\mathbb{C}^\times_{<1} \times \mathbb{C}^\times_{<1}$, where $\mathbb{C}^\times_{<1} = \{q \in \mathbb{C}^\times : |q| < 1\}$. If one has a conformal theory one expects the group $\mathrm{Conf}(\Sigma)$ to act unitarily on $\mathfrak{H}$. I have already mentioned that

$$\mathrm{Conf}(\Sigma) = (\widetilde{\mathrm{Diff}}(S^1_L) \times \widetilde{\mathrm{Diff}}(S^1_R))/2\pi\mathbb{Z} ,$$

where $\widetilde{\mathrm{Diff}}(S^1)$ is the simply connected covering group of $\mathrm{Diff}(S^1)$, i.e. the group of diffeomorphisms $\varphi : \mathbb{R} \to \mathbb{R}$ such that $\varphi(\theta + 2\pi) = \varphi(\theta) + 2\pi$. The main idea of conformal field theory - in one interpretation - is that the positivity of energy is expressed by the fact that the action of $\mathrm{Conf}(\Sigma)$ on $\mathcal{H}$ extends to a holomorphic contraction representation of $(\tilde{\mathcal{A}} \times \tilde{\mathcal{A}})/2\pi\mathbb{Z}$, where $\tilde{\mathcal{A}}$ is the simply connected covering group of the semigroup $\mathcal{A}$ of annuli which was introduced in §2. In fact one wants the action to extend to a still larger semigroup (or rather category) which allows circles to split into two: that is described in §4.

The holomorphic action of $\mathbb{C}^4_+$ on a conventional state space is of course completely determined by its restriction to the cone iP. A contraction representation of iP can be extended holomorphically to $\mathbb{C}^4_+$ providing it satisfies the condition called "reflection-positivity", and then restricted to give a unitary representation of $\mathbb{R}^4$. In the two-dimensional case the sub-semigroup of $T^+_{\mathbb{C}}$ which corresponds to iP is the upper half-plane $\mathbb{C}_+$, the covering of $\mathbb{C}^\times_{<1}$. In the case of $\mathrm{Conf}(\Sigma)$ the corresponding semigroup is $\mathcal{A}$, embedded diagonally in $(\tilde{\mathcal{A}} \times \tilde{\mathcal{A}})/2\pi\mathbb{Z}$. We are therefore interested in two questions:

(i) when are unitary representations of $\mathrm{Diff}(S^1)$ the boundary values of holomorphic contraction representations of $\mathcal{A}$, and

(ii) when can contraction representations of $\mathcal{A}$ be continued analytically to holomorphic representations of $(\tilde{\mathcal{A}} \times \tilde{\mathcal{A}})/2\pi\mathbb{Z}$?

Concerning the first question I should mention that the corresponding finite dimensional situation - where a Lie group G is essentially the Shilov boundary of an open semigroup $G^+_{\mathbb{C}}$ contained in the complexification $G_{\mathbb{C}}$ - occurs frequently and has been much studied (cf. [ ]). For example when G is the subgroup $PSU_{1,1} \cong PSL_2(\mathbb{R})$ of $\mathrm{Diff}(S^1)$, and $G_{\mathbb{C}}^{+}$ is the sub-semigroup of $PSL_2(\mathbb{C})$ described at the beginning of §2, it is well-known (and obvious) that the irreducible unitary representations of G which extend to $G_{\mathbb{C}}^{+}$ are precisely the discrete series representations, i.e. the representations of G on the spaces of holomorphic forms on D, and also the trivial representation.

Returning to $\mathrm{Diff}^+(S^1)$ and $\mathcal{A}$, the representations of $\mathrm{Diff}^+(S^1)$ which have a chance of extending to $\mathcal{A}$ are the ones of positive energy ([S2], [PS]), i.e. those for which the subgroup **T** of rigid rotations acts by characters $\{e^{ik\theta}\}$ for which the values of k are bounded below These are all projective representations. In the following discussion we shall tacitly restrict our attention to representations for which the action of $\mathbb{C}^{\times}_{<1} \subset \mathcal{A}$ is diagonalizable and extends to an action of $\mathbb{T} \subset \mathrm{Diff}^+(S^1)$. I shall also not distinguish between representations which are "essentially equivalent" in the sense of [PS] Chapter 9.

Proposition (3.1). There is a 1-1 correspondence between positive energy projective representations of $\mathrm{Diff}^+(S^1)$ and holomorphic projective representations of $\mathcal{A}$. Unitary representations of $\mathrm{Diff}^+(S^1)$ correspond to representations of $\mathcal{A}$ which are reflection-positive in the sense that $U_A^* = U_{\bar{A}}$.

Proof: First suppose given a representation $A \mapsto U_A$ of $\mathcal{A}$ on a topological vector space E. Let A_q be the standard annulus with parameter $q \in \mathbb{C}^{\times}_{<1}$, and let $U_q = U_{A_q}$. The union of the subspaces $U_q.E$ for all q is a dense subspace $\check{E}$ of E. I shall prove that the group $\mathrm{Diff}^+_{an}(S^1)$ of real-analytic diffeomorphisms of S^1 acts on $\check{E}$. It is, however, well known that all positive energy representations of Diff^+_{an} extend to Diff^+.

If A is an annulus and φ is a diffeomorphism of S^1 I shall write φA (resp. $A\varphi^{-1}$) for the annulus obtained by changing the outgoing (resp. incoming) parametrization of A by φ. Let us call an annulus real-analytic if both its boundary parametrizations (in the sense of (2.2)) are real-analytic. If φ is a real-analytic diffeomorphism let U_φ denote the densely-defined operator $U_{\varphi A}U_A^{-1}$ in E, where A is a real-analytic annulus. This does not depend on A, for if A' is another choice then there is a standard annulus $B = A_q$ such that $A = B \circ C$ and $A' = B \circ C'$, and

$$U_{\varphi A}U_A^{-1} = U_{\varphi B}U_B^{-1} = U_{\varphi A'}U_{A'}^{-1} .$$

(We are suppressing a possible projective multiplier, which is immaterial.) Then U_φ maps $\check{E}$ to itself, and defines a representation of $\mathrm{Diff}^+_{an}(S^1)$, because

$$U_{\psi\varphi} = U_{\psi(\varphi A)}U_{\varphi A}^{-1}U_{\varphi A}U_A^{-1} = U_\psi U_\varphi .$$

Conversely, if E is a positive energy representation of $\mathrm{Diff}^+(S^1)$ then there is an obvious candidate for the operator U_q associated to A_q. But for any annulus A we can by (2.1) write $A = \varphi A_q \psi^{-1}$ in an essentially unique way, and then define $U_A = U_\varphi U_q U_\psi^{-1}$. We must show that U_A depends holomorphically on A, and that it defines a representation of $\mathcal{A}$. For the first, recall from (2.4) that the tangent space to $\mathcal{A}$ at A is $(\mathrm{Vect}_{\mathbb{C}}(S^1) \oplus \mathrm{Vect}_{\mathbb{C}}(S^1))/\mathrm{Vect}(A)$. Let $\xi \mapsto L_\xi$ be the derivative of $\varphi \mapsto U_\varphi$. Writing the derivative of $A \mapsto U_A$ as

$$\delta U_A \quad = \quad U_\varphi \ \{(U_\varphi^{-1}\delta U_\varphi)U_q - U_q(U_\psi^{-1}\delta U_\psi)\} \ U_\psi^{-1}$$

we see that $A \mapsto U_A$ is holomorphic if the map

$$(\xi, \eta) \mapsto L_\xi U_q - U_q L_\eta ,$$

defined on $\mathrm{Vect}_{\mathbb{C}}(S^1) \oplus \mathrm{Vect}_{\mathbb{C}}(S^1)$, vanishes on $\mathrm{Vect}(A_q)$. But that is obvious. (Holding q fixed in this calculation is permissible because q = constant defines a submanifold of $\mathcal{A}$ of <u>real</u> codimension one.) Finally, to show that $A \mapsto U_A$ is a homomorphism amounts to proving that two holomorphic maps $\mathcal{A} \times \mathcal{A} \to \mathrm{End}(E)$ coincide. But they coincide by definition at points of the form $(\varphi A_q, A_q, \psi^{-1})$, and as they are holomorphic that is enough.

The correspondence between unitarity and reflection-positivity needs no comment, except perhaps to point out that if $A = \varphi A_q \psi^{-1}$ then $\bar{A} = \psi A_{\bar{q}} \varphi^{-1}$.

I have little to say about question (ii) above, when a non-holomorphic representation of $\mathcal{A}$ can be continued to a holomorphic representation of the complexification $\mathcal{A}_{\mathbb{C}} = (\tilde{\mathcal{A}}_L \times \tilde{\mathcal{A}}_R)/2\pi\mathbb{Z}$. It is certainly true in the reflection-positive case. For any representation of $\mathcal{A}$ gives us a representation of the Lie algebra of $\mathcal{A}_{\mathbb{C}}$, which is the complexification of the Lie algebra of $\mathrm{Diff}(S^1_L) \times \mathrm{Diff}(S^1_R)$. But it is known that any unitary positive energy representation of this Lie algebra extends to a representation of the group, and then the representation of $\mathrm{Diff}(S^1_L) \times \mathrm{Diff}(S^1_R)$ gives rise to a holomorphic representation of $\mathcal{A}_{\mathbb{C}}$ as in the proof above. It would be interesting, however, to have a better treatment of this question.

§4. The category $\mathscr{C}$ and the definition of a field theory

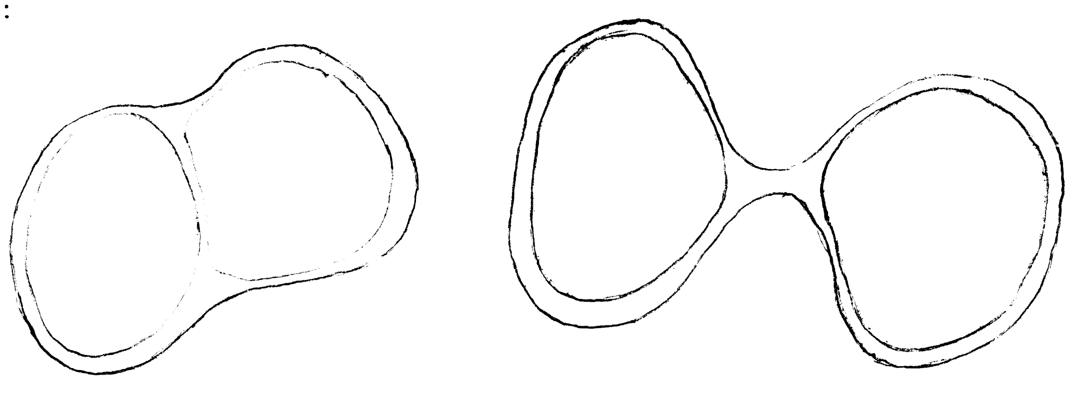
The category $\mathscr{C}$

The category $\mathscr{C}$ is defined as follows. There is a set of objects $\{C_n\}_{n\geqslant 0}$, where C_n is the disjoint union of a set of n parametrized circles. A morphism $C_m \to C_n$ is a Riemann surface X with boundary ∂X together with an orientation-preserving identification $C_n - C_m \to \partial X$. (Here $C_n - C_m$ means $C_n \amalg C_m$ with the orientation of C_m reversed.) We identify two surfaces if they are isomorphic by a map which respects the parametrization of the boundary. Composition of morphisms is defined by sewing surfaces together.[1]

The set $\mathscr{C}_{mn}$ of morphisms $C_m \to C_n$ is a topological space with one connected component $\mathscr{C}_\alpha$ for each topological type of surface. Thus when α is an annulus $\mathscr{C}_\alpha$ is the semigroup $\mathscr{A}$ of §2. Two other cases are worth mentioning.

(i) If α is a disc $\mathscr{C}_\alpha$ is $\mathrm{Diff}^+(S^1)/PSU_{1,1}$, for all discs are the same except for the parametrization of the boundary. This gives a description of the complex structure on $\mathrm{Diff}^+(S^1)/PSU_{1,1}$ (cf. [BR]). In terms of the semigroups of §2 we have $\mathscr{C}_\alpha \cong \mathscr{A}/\mathscr{E}$.

(ii) If α is a disc with two holes then $\mathscr{C}_\alpha$ has a Shilov boundary which consists of the space of ways in which a circle can split into two:



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[1]Purists will object that the category $\mathscr{C}$ has no identity morphisms, and will have their preferred remedies.

As a space $\mathcal{C}_\alpha$ is the quotient of the contractible space of complex structures on a fixed smooth surface Σ_α of type α by the group of all diffeomorphisms of Σ_α which are the identity on $\partial\Sigma_\alpha$. On the other hand it is well known that the moduli space of closed surfaces of a given topological type is a finite dimensional complex variety with some mild singularities. If α is a connected surface with $k > 0$ boundary components the complex structure of $\mathcal{C}_\alpha$ can be described by analogy with the description of $\mathcal{A}$ in §2, as follows. Let g be the genus of the closed surface got by adding k caps to α - we shall call g simply "the genus of α" - and let $\mathcal{M}'_{g,k}$ be the moduli space of closed surfaces of genus g with k marked points $\{x_i\}$ and prescribed tangent vectors $\{\xi_i\}$ at the points $\{x_i\}$. The space $\mathcal{M}'_{g,k}$ is a finite dimensional complex manifold with no singularities, and there is a tautological fibre bundle over it whose fibre at $\hat{X}$ is $\hat{X}$. The space $\mathcal{C}_\alpha$ is a fibration over $\mathcal{M}'_{g,k}$ whose fibre at $(\hat{X},\{x_i\},\{\xi_i\})$ is the space of k-tuples of disjointly embedded discs $f_i : D \to \hat{X}$ such that $f_i(0) = x_i$ and $f_i'(0) = \xi_i$. (This description needs adjustment when α is a disc: then $\mathcal{C}_\alpha$ is the space of embeddings $f: D \to S^2$ such that $f(0) = 0$, $f'(0) = 1$, and $f''(0) = 0$.

Composition of morphisms is a holomorphic map $\mathcal{C}_{km} \times \mathcal{C}_{mn} \to \mathcal{C}_{kn}$. It is enough to prove this when the composite surface has no closed components, and in that case it follows as in §2 from

<u>Proposition (4.1)</u>. If α has no closed components the tangent space to $\mathcal{C}_\alpha$ at X is $\mathrm{Vect}_{\mathbb{C}}(\partial X)/\mathrm{Vect}(X)$, the space of tangent vector fields to X along ∂X modulo those which extend holomorphically to X.

Remark. The dual cotangent space is therefore the space of holomorphic quadratic differentials on $\mathring{X}$ which have distributional boundary values on ∂X.

Proof of (4.1). The argument is the same as in §2, except that we need to know that if X is obtained from a closed surface $\hat{X}$ by removing discs, then any $Y \in \mathcal{C}_\alpha$ which is sufficiently close to X can be embedded holomorphically in $\hat{X}$. That is true because X is a Stein manifold.

Remark. We could put a finer topology on $\mathcal{C}_{mn}$ - without changing the topology on each $\mathcal{C}_\alpha$ - so that $\mathcal{C}_{mn}$ had just one connected component for each genus. For if β is disconnected $\mathcal{C}_\beta$ can be stuck on to the boundary of $\mathcal{C}_\alpha$ for an appropriate connected α. Thus the space $\{\mathrm{Diff}^+(S^1)/PSU_{1,1}\}^{\times 2}$ of pairs of discs can be attached to the boundary of $\mathcal{A}$ by collapsing the divisor $\{\mathrm{Diff}^+(S^1)/T\}^{\times 2}$ consisting of infinitely long cylinders. The resulting connected $\mathcal{C}_{mn}$ would be a complex variety with bad singularities. We shall not pursue this, however.

The definition: first version

We shall define a conformal field theory as a functor from $\mathcal{C}$ to complex topological vector spaces. We assume the vector spaces H are locally convex and complete, and equipped with a continuous hermitian form $\bar{H} \times H \to \mathbb{C}$. We shall not restrict ourselves to Hilbert spaces, as we want to allow indefinite inner products. We shall state the definition in terms of tensor products. These should be interpreted in the sense explained in Appendix A. But if H is a Hilbert space the tensor products can equally well be taken in the Hilbert space sense.

We shall make use of a number of elementary operations which can be performed on the morphisms of $\mathcal{C}$.

(a) The symmetric groups S_m and S_n act on $\mathcal{C}_{mn}$ by permuting the numbering of the boundary circles.

(b) If $X \in \mathcal{C}_{mn}$ then the complex conjugate surface $\bar{X}$ belongs to $\mathcal{C}_{nm}$, and $X \mapsto \bar{X}$ is an antiholomorphic map.

(c) By reversing the orientation of the incoming boundary circles we obtain the "crossing" isomorphism $\mathcal{C}_{mn} \to \mathcal{C}_{0,m+n}$, which I shall write $X \mapsto |X|$.

(d) By sewing k incoming to k outgoing circles we obtain a holomorphic map $\mathcal{C}_{mn} \to \mathcal{C}_{m-k,n-k}$.

We now give the provisional definition of a conformal field theory. We should warn the reader, however, that it is unsatisfactory because it does not allow for projective multipliers.

Definition (4.2). Let H be a topological vector space with a symmetric complex bilinear form and a given real structure (i.e. an anti-involution $H \to \bar{H}$). A conformal field theory based on H is a continuous functor U from $\mathcal{C}$ to topological vector spaces with the following properties.

(i) $U(C_n) = H \otimes \ldots \otimes H = H^{\otimes n}$.

(ii) The map $\mathcal{C}_{mn} \times H^{\otimes m} \to H^{\otimes n}$ is compatible with the action of the symmetric groups S_m and S_n.

(iii) "Crossing": for each $X \in \mathcal{C}_{mn}$ the operator $U(X) : H^{\otimes m} \to H^{\otimes n}$ is of trace class, and is defined by the element $U(|X|)$ of $H^{\otimes m} \otimes H^{\otimes n}$ together with the bilinear form on H.

(iv) "Sewing": the map $\mathcal{C}_{mn} \to \mathcal{C}_{m-k,n-k}$ of (d) above is compatible with the map

$$\mathrm{Hom}(H^{\otimes m};H^{\otimes n}) \to \mathrm{Hom}((H^{\otimes(m-k)};H^{\otimes(n-k)})$$

got by taking the trace over $H^{\otimes k}$. In particular, if $X \in \mathcal{C}_{11}$ and $\check{X}$ is the associated closed surface, then

$$\text{trace } U(X) = U(\check{X}) \ .$$

(v) "Reflection positivity": U is a *-functor in the sense that $U(\bar{X}) = U(X)^*$ for all morphisms X. Here the adjoint $U(X)^*$ refers to the hermitian structure on H got by combining the real structure with the complex bilinear form.

<u>Notes</u>. (i) In this definition we ought certainly to allow the space H to have a mod 2 grading. Then the permutations of $H^{\otimes n}$ should be performed with the usual sign conventions, and - most importantly - the trace in property (iv) should be replaced by the <u>supertrace</u>. We shall for the most part not bother to make this generalization explicit.

(ii) If we omit to give the real structure on H and the associated axiom (v) of reflection-positivity then we have a "non-unitary" field theory.

A conformal field theory is thus, among other things, a trace-class representation of the semigroup $\mathcal{A}$. As we saw in §3, this gives us a pseudo-unitary action on H of the Lie algebra of the conformal group $\mathrm{Conf}(S^1 \times \mathbb{R})$, i.e. of the Lie algebra of $\mathrm{Diff}^+(S^1_L) \times \mathrm{Diff}^+(S^1_R)$. Under the action of the rigid motions the space H breaks up as a discrete sum of finite dimensional pieces: $H = \oplus H_{a,b}$, where $(a,b) \in \mathbb{R}^2$, and $a-b \in \mathbb{Z}$.

The <u>partition function</u> Z_U of the theory is the function on the upper half-plane defined by

$$Z_U(\tau) = \text{trace } U(A_q) = \Sigma\, \bar{q}^a q^b \dim(H_{a,b}) \ ,$$

where $q = e^{2\pi i\tau}$ and A_q is the standard annulus $\{z : |q| < |z| < 1\}$ described in §2. Because the annuli A_q and $A_{\tilde{q}}$ produce isomorphic tori if $\tilde{q} = e^{-2\pi i/\tau}$ the partition function satisfies

$$Z_U(-\tau^{-1}) = Z_U(\tau) \qquad (4.3)$$

(but cf. Proposition (6.11)).The partition function completely determines H as a representation of $\text{Diff}^+(S^1_L) \times \text{Diff}^+(S^1_R)$, for the characters of the representations of $\text{Diff}^+(S^1)$ are all known, and are linearly independent.

Another aspect of the structure is seen by choosing, once for all, a disc with two holes Σ, regarded as an element of $\mathcal{C}_{21}$. For any theory, Σ gives us a map $H \otimes H \to H$ which makes H into a non-associative algebra. This composition law is called the <u>operator product expansion</u>. Together with the partition function the product in H determines the theory completely, for any Riemann surface can be obtained by sewing together discs, cylinders, and copies of Σ, by suitable diffeomorphisms. In the case of the theory whose group of automorphisms is the monster group, the algebra H contains Griess's non-associative algebra as a subalgebra.

Friedan has conjectured that a field theory U is completely determined by its restriction to <u>closed</u> surfaces, i.e. by the homomorphism $U : \mathcal{C}_{00} \to \mathbb{C}^\times$ defined on the commutative semigroup $\mathcal{C}_{00}$. This seems plausible, but I do not know a proof.

A field theory is called holomorphic if the operators U(X) depend holomorphically on X. That is the case if and only if $\mathrm{Diff}^{+}(S_L)$ acts trivially on H, and also if and only if the partition function is holomorphic. A theory is called chiral if it is either holomorphic or antiholomorphic.

The conformal anomaly

The preceding definition is too restrictive, and we must introduce slightly more general structures. In the usual terminology these are theories which have a "conformal anomaly". Mathematically this amounts to passing to projective representations of the category $\mathcal{C}$, i.e. the operator U(X) associated to a surface X is given only up to an indeterminate scalar multiplier. Physically one should think that U(X) is associated not to the surface X alone, but to the surface together with a chosen metric compatible with its conformal structure. The dependence on the metric is slight: if the volume element ω is multiplied by $e^{2\varphi}$, for some $\varphi : X \to \mathbb{R}$, then U(X) is multiplied by $e^{icS(\varphi)}$, where c is a constant depending on the theory (the "central charge") and $S(\varphi)$ is the Liouville action

$$S(\varphi) = \int_X \tfrac{1}{2}\{d\varphi \wedge *d\varphi + 4\varphi R\} .$$

Here R is the curvature 2-form of the metric.

To digress briefly, one can define a general notion of two dimensional field theory as a representation of a category $\mathcal{C}_{\mathrm{metric}}$ made from circles and surfaces equipped with metrics. The metrics must be piecewise twice differentiable, and the boundary circles must be geodesics. Circles of different lengths are, of course, non-isomorphic objects of $\mathcal{C}_{metric}$. It may be that the intriguing work of Zamolodchikov [Z] can be formulated in this language, but perhaps something more subtle is needed.

Returning to mathematics, just as a projective representation of a group G is a genuine representation of an extension $\tilde{G}$ of G by $\mathbb{C}^{\times}$, so a projective representation of a category $\mathcal{C}$ is an ordinary representation of an extension category $\tilde{\mathcal{C}}$ of $\mathcal{C}$ by $\mathbb{C}$. To give such an extension category is the same as giving a rule which assigns a complex line L_X to each morphism X of $\mathcal{C}$, and a map

$$\mu_{XY} : L_X \otimes L_Y \to L_{X \circ Y}$$

to each composable pair of morphisms. The maps μ_{XY} must be associative in the obvious sense. The objects of $\tilde{\mathcal{C}}$ are the same as the objects of $\mathcal{C}$, and a morphism in $\tilde{\mathcal{C}}$ is a pair (X,λ), where X is a morphism in $\mathcal{C}$ and $\lambda \in L_X$. In the next section we shall prove that there is essentially only one such extension of $\mathcal{C}$, got by assigning to X the determinant line Det_X of its $\bar{\partial}$-operator, in the sense of Quillen [Q] (cf. also Appendix B). More precisely, the most general extension is of the form[1] $L_X = (Det_X)^{\otimes p} \otimes (\overline{Det_X})^{\otimes q}$. If p = q = c one says that the theory has <u>central charge</u> c. (The determinant bundle will be discussed in detail in §6.)

The conditions of (4.2) make sense for a projective functor providing $X \mapsto L_X$ has the properties:

(i) $L_X = L_{\tilde{X}}$ if $\tilde{X}$ is obtained from X by reversing the parametrization of some boundary components;

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[1]See (5.18).

(ii) $L_{\overline{X}} = \overline{L}_X$;

(iii) there is a natural map $L_X \to L_{\check{X}}$ when $\check{X}$ is made from X by sewing boundary circles together.

When X is an annulus there is a preferred element $\epsilon_X \in L_X$, and so we can define the partition function as the trace of the operator $U(\epsilon_X)$. It has a modularity property analogous to (4.3), but we shall postpone discussion of that (and also the definition of ϵ_X) until §6.

An improved version of the definition

Definition (4.2) is cumbersome and unnatural, and the following reformulation is cleaner. I shall give it in the projective version.

We begin with a hermitian vector space H with a projective unitary action of $\mathrm{Diff}(S^1)$ in which the orientation-reversing diffeomorphisms act antilinearly. There is a unique way to associate to H a projective functor $S \mapsto H_S$ from compact oriented 1-manifolds (and orientation-preserving diffeomorphisms) to hermitian vector spaces (and $\mathbb{C}$-linear operators given up to an arbitrary scalar multiplier) with the two properties:

(a) $H_{\overline{S}} = \overline{H}_S$ if $\overline{S}$ is S with reversed orientation;

(b) $H_{S_1 \amalg S_2} = H_{S_1} \otimes H_{S_2}$.

Definition (4.4). A conformal field theory based on H is a continuous natural transformation which assigns to each Riemann surface X with (unparametrized) boundary a ray H_X in $H_{\partial X}$ satisfying

(i) $H_{\overline{X}} = \overline{H}_X$,

(ii) $H_{X \amalg Y} = H_X \otimes H_Y$,

(iii) $H_{\check{X}} = \text{trace } H_X$ if $X \to \check{X}$ is a sewing map.

Here a sewing map $X \to \check{X}$ is one which identifies two disjoint parts S_1 and S_2 of ∂X by an orientation-reversing diffeomorphism; and the trace map $H_{\partial X} \to H_{\partial \check{X}}$ is induced by the bilinear form $H(S_1) \otimes H(S_2) \to \mathbb{C}$.

In (4.4) it is important that we do not use Hilbert space tensor products, for then the Hermitian form $\bar{H} \times H \to \mathbb{C}$ would not extend to $\bar{H} \otimes H$.

The idea of a "projective" functor may seem unappealingly vague. The additional structure which an oriented 1-manifold S needs in order to define a vector space H_S rather than just a projective space can be described as follows.

We define a rigged 1-manifold as an oriented 1-manifold S together with a specific choice L of a determinant line bundle on the restricted Grassmannian $Gr(\Omega^0(S))$ of the space of smooth functions on S (see Appendix B). For given S the bundle L is canonically defined up to isomorphism, but the isomorphism is arbitrary up to an element of $\mathbb{C}^\times$. (A parametrization of S is more than enough to provide a canonical choice of L.) A morphism from (S_0, L_0) to (S_1, L_1) is a diffeomorphism $f : S_0 \to S_1$ together with an isomorphism $L_0 \cong f^*L_1$.

A surface X with $\partial X = S$ defines a point $Hol(X)$ in $Gr(\Omega^0(S))$. If S is rigged by L then we define the determinant line of X as the fibre L_X of L at $Hol(X)$. To obtain a vector in $H_{S,L}$ corresponding to Σ we must choose a point of L_X.

To describe chiral theories we shall need an even more general definition than (4.4), in which a surface X defines a subspace H_X of $H_{\partial X}$ which need not be one-dimensional. That is the subject of §5.

Minkowski space, ghosts, and BRS cohomology

Apart from the question of projective multipliers there are two other respects in which Definition (4.2) is not quite general enough for the needs of string theory. It is usual to study strings moving in a product $V \times M$, where V is Minkowski space of some dimension, and M is a compact Riemannian manifold. In that case the space of states of the string is a direct integral $H = \int H_p$, where H_p is the states of momentum p, and p runs through the dual space V* of V. A surface $X \in \mathcal{C}_{mn}$ defines an operator $U(X) : H^{\otimes m} \to H^{\otimes n}$ which is an integral of operators

$$U(X)_{\underline{\underline{p}},\underline{\underline{q}}} : H_{p_1} \otimes \ldots \otimes H_{p_m} \to H_{q_1} \otimes \ldots \otimes H_{q_n} ,$$

where $\Sigma\, p_i = \Sigma\, q_i$. Each operator $U(X)_{\underline{\underline{p}},\underline{\underline{q}}}$ is of trace class, but U(X) itself is not.

More importantly, strings are not supposed to be parametrized, while the spaces H we have been discussing describe parametrized strings. One would expect to replace H by the subspace which is invariant under $\mathrm{Conf}(S^1 \times \mathbb{R})$. In fact the spaces H which arise are projective representations of $\mathrm{Conf}(S^1 \times \mathbb{R})$ with a positive central charge c, and the invariant subspace would be 0. Instead of the invariant part of H one has recourse to its BRS cohomology H_{BRS}. The essential points about this are:

(i) it is defined only for a theory with $c = 26$,

(ii) it has a bi-grading (called the "ghost number"),

(iii) in good cases, at least, it has a positive definite metric,

(iv) instead of an operator $H_{BRS}^{\otimes m} \to H_{BRS}^{\otimes n}$ for each surface $X \in \mathcal{C}_{mn}$ one has a top-dimensional differential form ω_{mn} on the finite dimensional moduli space $\mathcal{M}_{m+n}$ of all closed surfaces with m+n marked points, with values in the space of operators $H_{BRS}^{\otimes m} \to H_{BRS}^{\otimes n}$ of bidegree (-m,-m).

In particular, m elements of H_{BRS} of bidegree (1,1) define a top-dimensional scalar-valued form on $\mathcal{M}_m$.

We shall say only a little about BRS cohomology in this paper. To define it one tensors the theory H with another theory H_{ghost} which has $c = -26$. The resulting theory $H \otimes H_{ghost}$ has a genuine (non-projective) action of $\mathcal{A}$. The space $H \otimes H_{ghost}$ has an operator $Q = Q_L + Q_R$ which satisfies $Q^2 = 0$, and Q_L and Q_R raise degree by (1,0) and (0, 1) respectively. The cohomology (ker Q)/(im Q) is the BRS cohomology. The theory H_{ghost} will be described in §8, and we shall return to Q and the property (iv) in §9.

§5. Modular functors

Definition and main properties

In studying chiral field theories and also the representations of loop groups one meets the concept of a modular functor. From one point of view this is a generalization of the idea of a central extension of $\mathrm{Diff}^+(S^1)$. On the other hand it can also be regarded as a coherent family of projective representations of the braid groups and mapping class groups.

We start with a finite set Φ of labels. Let $\mathcal{S}_\Phi$ be the category whose objects are Riemann surfaces with each boundary circle parametrized and equipped with a label from Φ. A morphism in $\mathcal{S}_\Phi$ is a holomorphic sewing map $X \to \check{X}$, i.e. one which sews together pairs of edges in accordance with the parametrization; we allow a pair of edges to be identified only if they have the same label. A morphism is allowed to permute the boundary circles, but it must preserve their parametrization.

Definition (5.1). A modular functor is a holomorphic functor E from $\mathcal{S}_\Phi$ to finite dimensional complex vector spaces with the following properties.

(i) $E(X \amalg Y) = E(X) \otimes E(Y)$.

(ii) If X_φ is obtained from X by cutting it along a simple closed curve and giving the label φ to the two new edges then the natural map

$$\bigoplus_{\varphi \in \Phi} E(X_\varphi) \to E(\check{X})$$

is an isomorphism.

(iii) For the Riemann sphere S^2 we have dim $E(S^2) = 1$.

Notes. (a) To say that E is holomorphic means that when $\{X_b\}_{b \in B}$ is a holomorphic family of surfaces parametrized by a complex manifold B the spaces $E(X_b)$ fit together to form a holomorphic vector bundle on B. In particular, E defines a holomorphic vector bundle E_α on the moduli space $\mathcal{C}_\alpha$ of surfaces of (labelled) topological type α, at least if α has no closed components. (Recall that $\mathcal{C}_\alpha$ was defined in §4. We exclude closed surfaces to avoid the singularities caused by their possible automorphisms.)

(b) The isomorphism of (i) above is supposed to be compatible with the maps interchanging the summands on each side. As in §4 we should certainly allow modular functors to be graded mod 2, and should use the graded tensor product in (i). The determinant line, for example, is a mod 2 graded modular functor for which $E(S^2)$ is in degree 1.

For any modular functor E we have a map $E(X) \otimes E(Y) \to E(X \circ Y)$ when X and Y are composable morphisms in $\mathcal{C}$ with their boundaries compatibly labelled. So E defines an extension $\mathcal{C}^E$ of the category $\mathcal{C}$. An object of $\mathcal{C}^E$ is a collection of circles each with a label from Φ, and a morphism is a pair (X,ϵ), where X is an morphism in $\mathcal{C}$ and $\epsilon \in E(X)$.

Definition (5.2). A weakly conformal field theory is a representation of $\mathcal{C}^E$ for some modular functor E, satisfying conditions as in (4.4).

Thus such a theory assigns a vector space H_S to each one-dimensional manifold and a vector space E_X to each surface, and there is a natural map $E_X \to H_{\partial X}$ for each X.

One may as well assume that the labelling set Φ of a modular functor contains no superfluous elements, i.e. no labels φ such that $E(X) = 0$ whenever X has an edge labelled by φ. We can then make the following elementary observations.

Proposition (5.3)

(i) There is a distinguished label $1 \in \Phi$ such that $\dim E(D) = 1$ when D is a disk with ∂D labelled 1, and $E(D) = 0$ if ∂D has any other label.

(ii) If $A_{\varphi\psi}$ is a annulus with ends labelled φ, ψ then $\dim E(A_{\varphi\psi}) = 1$ if $\varphi = \psi$ and $E(A_{\varphi\psi}) = 0$ otherwise. In particular, E defines a central extension $\mathcal{A}_\varphi$ of $\mathcal{A}$ by $\mathbb{C}^\times$ for each label φ.

(iii) There is an involution $\varphi \mapsto \bar{\varphi}$ of Φ such that if B is an annulus with both ends outgoing then $\dim E(B_{\varphi\psi}) = 1$ if $\psi = \bar{\varphi}$ and $E(B_{\varphi\psi}) = 0$ otherwise.

(iv) If $\tilde{X}$ is obtained from X by reversing the parametrization of an incoming boundary circle and changing its label from φ to $\bar{\varphi}$ then $E(\tilde{X}) = E(X) \otimes E(B_{\varphi\bar{\varphi}})$.

Proof: We first prove (ii) by observing that the $\Phi \times \Phi$ matrix $\dim E(A_{\varphi\psi})$ is idempotent with positive integer entries. The matrix $\dim E(B_{\varphi\psi})$ is then symmetric and invertible, so we obtain (iii). Assertion (iv) follows immediately, and finally we get (i) by considering the decomposition $S^2 = D \cup D$.

From now on we shall assume modular functors are normalized so that $E(D) = \mathbb{C}$ when ∂D is outgoing and labelled 1.

The sense in which a modular functor is a coherent family of projective representations of discrete groups is explained by

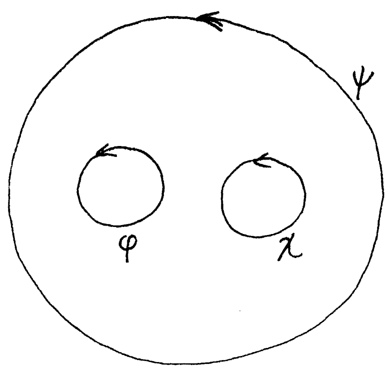
<u>Proposition (5.4)</u>. For any modular functor there is a canonical flat connection in the projective bundle of the bundle E_α on $\mathcal{C}_\alpha$, for every non-closed labelled surface α. These connections are compatible with the sewing-together of surfaces.

If the modular functor has central charge 0 (see below) then there is a canonical flat connection in the bundle E_α itself.

In other words, if X and X' are surfaces of type α there is an isomorphism $P(E(X)) \to P(E(X'))$ for each homotopy class of paths from X to X' in $\mathcal{C}_\alpha$. Thus for each α a modular functor gives a projective representation $\pi_1(\mathcal{C}_\alpha) \to PGL_{n_\alpha}(\mathbb{C})$. For example if α is a disc with k holes then $\pi_1(\mathcal{C}_\alpha) = \mathbb{Z}^k \times CBr_k$, where CBr_k is the coloured braid group on k strands. If α is a surface of genus g with one hole then $\pi_1(\mathcal{C}_\alpha)$ is the mapping class group of α.

<u>Verlinde's algebra</u>

An attractive way of looking at modular functors has been developed by Verlinde [V], following the "fusion-rule" approach of Belavin-Polyakov-Zamolodchikov [BPZ]. Let Σ be a disc with two holes, labelled



Let $n_{\varphi\chi\psi} = \dim E(\Sigma)$. Then the free abelian group $\mathbb{Z}[\Phi]$ is clearly a commutative ring under the multiplication

$$(\varphi,\chi) \mapsto \sum_{\psi} n_{\varphi\chi\psi}\, \psi \ .$$

The element $1 \in \Phi$ is the identity element of the ring. We shall say more about this ring later on. For the moment let us notice that the ring structure of $\mathbb{Z}[\Phi]$ is a very compact way of encoding the dimension of $E(X)$ for all labelled surfaces X. Thus if M_φ is the operator of multiplication by φ on $\mathbb{Z}[\varphi]$ then the dimension of $E(X)$ when X is a torus with an incoming and an outgoing hole labelled φ,ψ is

$$p_{\varphi\psi} = \operatorname{trace}(M_\varphi M_{\bar{\psi}}) \ ,$$

and if X_g is a closed surface of genus g then

$$\dim E(X_g) = \operatorname{trace}(P^{g-1}) \ ,$$

where P is the matrix $(p_{\varphi\psi})$.

<u>Loop groups</u>

The natural examples of modular functors arise from representations of loop groups in the following way. I shall suppose for simplicity that G is the complexification of a simply connected compact group. Let $\{E_\varphi\}_{\varphi\in\Phi}$ be the finite set of all irreducible projective positive energy representations of a certain level of the loop group LG. The indexing set Φ can be identified with a set of irreducible representations of G, for the zero-energy subspace of E_φ is an irreducible representation φ of G. The involution $\varphi \mapsto \bar{\varphi}$ takes a representation of G to its dual, and $1 \in \Phi$ is the trivial representation of G. Let X be a surface with k boundary components, all outgoing, labelled by $\varphi_1, \ldots, \varphi_k \in \Phi$. Then the group of holomorphic maps Hol(X;G) acts on $E_{\varphi_1} \otimes \ldots \otimes E_{\varphi_k}$ via restriction to ∂X, for the central extension of $(LG)^k$ is canonically split over Hol(X;G). We define E(X) as the part of $E_{\varphi_1} \otimes \ldots \otimes E_{\varphi_k}$ fixed under Hol(X;G). If some of the boundary circles are incoming we replace the corresponding factor E_φ by $t^*E_{\bar{\varphi}}$, where $t : S^1 \to S^1$ reverses the parametrization. Then $X \mapsto E(X)$ is a modular functor. This will be proved in §11. The point of the definition is that a surface X with p incoming and q outgoing circles labelled $\varphi_1, \ldots, \varphi_p$ and $\psi_1, \ldots, \psi_q$, together with an element ϵ of E(X) - i.e. a morphism (X, ϵ) in the extended category $\mathcal{C}^E$ - defines a trace-class operator

$$U_{X,\epsilon} : E_{\varphi_1} \otimes \ldots \otimes E_{\varphi_p} \to E_{\psi_1} \otimes \ldots \otimes E_{\psi_q} .$$

This is because for each φ there is a natural duality pairing

$$E_\varphi \otimes t^*E_{\bar{\varphi}} \to \mathbb{C} .$$

The concept of a modular functor is designed, among other things, to express the modularity properties of the characters of representations of loop groups. A representation E_φ decomposes under the action of the rigid rotations of S^1 as a sum $E_\varphi = \bigoplus_{k \geq 0} E_{\varphi,k}$ of finite dimensional pieces, where the rotation through the angle α acts as $e^{ik\alpha}$ on $E_{\varphi,k}$. Each piece $E_{\varphi,k}$ is a representation of the subgroup G of constant loops in LG. (Thus $E_{\varphi,0} = \varphi$.) The <u>partition function</u> and the <u>character</u> of E_φ are defined as the formal series

$$\chi_\varphi(q) = \Sigma\, q^k \dim(E_{\varphi,k})$$

and

$$\chi_\varphi(q,g) = \Sigma\, q^k \operatorname{trace}(g|E_{\varphi,k})$$

respectively. In fact these series converge when $|q| < 1$, and $\chi_\varphi(q)$ is best regarded as a function of an annulus with modulus q. More precisely,

$$\chi_\varphi(q) = \operatorname{trace}(U_{A,\epsilon} : E_\varphi \to E_\varphi) ,$$

where A is the standard annulus A_q, and ϵ is the standard element of the line $E(A_q)$, where the ends of A_q are labelled with φ. Then $\chi_\varphi(q)$ depends only on the image, say $\epsilon_{q,\varphi}$, of ϵ in E(X), where X is the torus got by sewing together the ends of A_q. We know from (5.1)(ii) that the elements $\epsilon_{q,\varphi}$ form a basis for E(X). On the other hand by (5.4) the modular group $SL_2(\mathbb{Z})$ acts projectively on E(X). This means that the partition function χ_φ is transformed by a modular transformation into a linear combination of characters of the same level.

The character $\chi_\varphi(q,g)$ should similarly be regarded as a function of a pair (A,P), where A is an annulus and P is a holomorphic principal G-bundle on A with a given trivialization of $P|\partial A$. Thus $\chi_\varphi(q,g) = \chi_\varphi(A_q,P_g)$, where P_g is $A_q \times G$ with the obvious trivialization over the incoming circle and g times the obvious trivialization over the outgoing end. The character depends only on $\epsilon_{q,\varphi} \in E(X)$ and the holomorphic G-bundle on X got by joining the ends of P_g. We shall explain this in detail in §11.

An example

The most basic modular functor is the determinant line, which is the subject of §6. We shall see (see (5.17)) that it and its powers are the only modular functors with only one label. A more typical example which can be described very explicitly is the following one, which corresponds to the level one representations of the loop group of U_n. (We shall meet other simple examples in §7.)

Let Φ be the set of characters of $\mathbb{Z}/n$. To a surface X with boundary we associate a Heisenberg group H_X which is an extension of $H_1(X;\mathbb{Z}/n)$ by $\mathbb{C}^\times$ with the commutator given by the intersection pairing. The centre of H_X is the image of $H_{\partial X} = \mathbb{C}^\times \oplus H_1(\partial X;\mathbb{Z}/n)$. A labelling $\varphi = (\varphi_1,\ldots,\varphi_k)$ of the boundary components defines a character χ_φ of $H_{\partial X}$ which is the identity on $\mathbb{C}^\times$. There is a unique irreducible representation E(X) of H_X in which $H_{\partial X}$ acts by χ_φ. It is zero unless $\partial\varphi = 0$ in $H^2(X,\partial X;\mathbb{C}^\times)$, i.e. unless $\Pi\ \varphi_i = 1$, in which case it has dimension n^g, where g is the genus of X. (It can be identified with the space of θ-functions of level n on the Jacobian of X.)

The ring $\mathbb{Z}[\Phi]$ in this case is simply the group ring of Φ.

Note. The preceding description is imprecise in two ways. First, H_X is defined only up to non-canonical isomorphism by the commutator pairing. Secondly, even when H_X is given, the representation E(X) is only uniquely defined as a projective space.

To clarify the definition of H_X we first introduce the extension $H_X^{\mathbb{Z}}$ of $H_1(X;\mathbb{Z})$ by $\mathbb{C}^{\times}$ defined by the cocycle

$$(\xi,\eta) \mapsto e^{2\pi i\langle\xi,\eta\rangle/2n}, \tag{5.5}$$

and then we define H_X as the quotient of $H_X^{\mathbb{Z}}$ by the central subgroup $H_1(X;n\mathbb{Z})$.

To deal with the second point we consider the extension $H_X^{\mathbb{R}}$ of $H_1(X;\mathbb{R})$ by $\mathbb{C}^{\times}$ defined by the same formula (5.5). The group $H_X^{\mathbb{R}}$ has a standard Heisenberg representation on the space F_X of holomorphic functions on $H_1(\hat{X};\mathbb{R})$, and we define E(X) as the part of F_X fixed by $H_1(X;n\mathbb{Z})$. (Here $\hat{X}$ is X with caps added to its boundary circles, and the complex structure of $H_1(\hat{X};\mathbb{R})$ comes from that of X.)

Extensions of $\mathcal{A}$

Modular functors give us extensions of $\mathcal{A}$ by $\mathbb{C}^{\times}$, and we shall now explain how these are classified.

Proposition (5.6). Holomorphic extensions of $\mathcal{A}$ by $\mathbb{C}^{\times}$ correspond precisely to extensions of $\mathrm{Diff}^+(S^1)$ by $\mathbb{C}^{\times}$.

Proof: We use the argument of (3.1). If A is an annulus we write φA (resp. $A\varphi^{-1}$) for the same annulus with its outgoing (resp. incoming) edge reparametrized by a diffeomorphism φ of S^1. Suppose that we are given a line L_A for each annulus A. Then we define L_φ for a real-analytic diffeomorphism φ by $L_\varphi = L_{\varphi A} \otimes L_A^*$, where A is a real-analytic annulus. The line L_φ does not depend on A, and it defines a central extension of the real-analytic diffeomorphism group, because

$$L_{\psi\varphi} = L_{\psi\varphi A} \otimes L_A^*$$

$$\cong L_{\psi\varphi A} \otimes L_{\varphi A}^* \otimes L_{\varphi A} \otimes L_A^*$$

$$\cong L_\psi \otimes L_\varphi ,$$

for we can choose A so that φA is also real-analytic. On the other hand it is known that the classification of extensions of $\mathrm{Diff}^+(S^1)$ is the same as that for the real-analytic diffeomorphisms.

Conversely, if we are given an extension $\varphi \mapsto L_\varphi$ of $\mathrm{Diff}^+(S^1)$ we define an extension of $\mathcal{A}$ by setting $L_{A_q} = \mathbb{C}$ for the standard annulus A_q, and $L_{\varphi A_q \psi} = L_\varphi \otimes L_\psi$. It is easy to see that $L_\varphi \otimes L_\psi$ depends only on the annulus $A = \varphi A_q \psi$. The dependence is holomorphic because any central extension of the Lie algebra $\mathrm{Vect}_{\mathbb{C}}(\partial A)$ is canonically split over Vect(A): see (6.7). We have therefore defined a correspondence between extensions of $\mathrm{Diff}^+(S^1)$ and extensions of $\mathcal{A}$.

Central extensions of $\mathrm{Diff}^+(S^1)$ were classified in [S2]. The universal central extension has kernel $\mathbb{R} \oplus \mathbb{Z}$, so extensions by $\mathbb{C}^\times$ correspond to homomorphisms $\mathbb{R} \oplus \mathbb{Z} \to \mathbb{C}^\times$, i.e. to elements of $\mathbb{C} \times \mathbb{C}^\times$. An extension can be completely described by its Lie algebra cocycle, in the following sense. The Lie algebra $\mathrm{Vect}(S^1)$ has the traditional basis $\{L_n = e^{in\theta} d/d\theta\}$. When one has a projective representation of $\mathrm{Diff}^+(S^1)$ one can choose the representatives of the L_n so that

$$[L_{-n}, L_n] = -2inL_0 + \frac{1}{12}\, cn(n^2-1) \quad . \tag{5.7}$$

Then the classification is given by

Proposition (5.8). A central extension of $\mathrm{Diff}^+(S^1)$ by $\mathbb{C}^\times$ is described by $(c,h) \in \mathbb{C} \times (\mathbb{C}/\mathbb{Z})$, where the "central charge" c is defined by (5.7), and h is any eigenvalue of L_0. (Thus h is detected by the restriction of the extension to the subgroup $PSL_2\mathbb{R}$.)

A modular functor gives us an extension of $\mathscr{A}$ for each label. We shall see in (6.9) that the extensions corresponding to the different labels all have the same central charge c, which will be called the central charge of the modular functor. The extension corresponding to the label 1 necessarily[1] has $h = 0$. The extension defined by the determinant line has $(c,h) = (-2,0)$.

The proof of (5.4)

A modular functor gives us an extension $\tilde{\mathscr{A}}_\varphi$ of $\mathscr{A}$, and hence an extension $\tilde{V}_\varphi$ of $\mathrm{Vect}_{\mathbb{C}}(S^1)$, for each label φ. Consider the bundle E_α on the moduli space $\mathscr{C}_\alpha$. There is an action of $\mathscr{A}$ on $\mathscr{C}_\alpha$ for each boundary circle, and it is covered by an action of the appropriate $\tilde{\mathscr{A}}_\varphi$, and hence of $\tilde{V}_\varphi$, on E_α. Putting these actions together gives us an action on E_α of a central extension $\tilde{V}_{\partial X}$ of $\mathrm{Vect}_{\mathbb{C}}(\partial X)$. At a point $X \in \mathscr{C}_\alpha$ the tangent space to $\mathscr{C}_\alpha$ is $\mathrm{Vect}_{\mathbb{C}}(\partial X)/\mathrm{Vect}(X)$, and so an extension $\tilde{V}_X$ of Vect(X) acts on the fibre E(X). But the Lie algebra Vect(X) has no finite dimensional projective representations (see Appendix *) [the Foreword and Postscript], so the

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[1]Because $D \circ A = D$ when A belongs to the subsemigroup $\mathscr{E}$ of $\mathscr{A}$ (see §2) the extension is split when restricted to $\mathscr{E}$, and hence when restricted to $PSL_2\mathbb{R}$.

extension $\tilde{V}_X$ is canonically split, and $\tilde{V}_X$ acts scalarly on E(X). Thus at X we have a differentiation operator D_ξ on sections of E_α for each $\xi \in \tilde{V}_{\partial X}/\mathrm{Vect}(X)$, i.e. D_ξ is defined up to an additive scalar for each tangent vector ξ to $\mathcal{C}_\alpha$ at X. This is a connection in the bundle of projective spaces of E_α, and it is flat because it comes from a Lie algebra action of $\tilde{V}_{\partial X}$. The nature of the definition of the connection makes it automatically compatible with sewing surfaces together.

Modular functors from a topological viewpoint

Immediately after the first version of this work was written the study of modular functors was transformed by Witten''s realization [W] that the vector spaces in question are in fact the state spaces of 2+1 dimensional "topological" field theories. To explain this it is best to look at modular functors in a slightly different way.

The main point is that for any modular functor E we know from Theorem (5.4) that the space E(X) is almost independent of the complex structure of X. For if X is a smooth surface the space $\mathcal{J}(X)$ of all complex structures on X (not identifying structures which are diffeomorphic) is contractible. The modular functor gives us a vector bundle on $\mathcal{J}(X)$: let $E_J(X)$ denote its fibre at J. By (5.4) the projective space of $E_J(X)$ is independent of J. But we can do better. There is a line bundle on $\mathcal{J}(X)$ whose fibre $\mathrm{Det}_J(X)$ at J is the determinant line of the Riemann surface (X,J). If the functor E has central charge c the bundle with fibres

$$E_J(X) = E_J(X) \otimes \mathrm{Det}_J(X)^{\otimes(\frac{1}{2}c)} \qquad (5.9)$$

has a flat connection, i.e. $E_J(X)$ is independent of J. To define $\mathrm{Det}^{\otimes(\frac{1}{2}c)}$, however, we must make a choice (unless $\frac{1}{2}c$ is an integer). This can be done universally for all $c \in \mathbb{C}$ by choosing a universal covering space $\tilde{\mathcal{P}}_X$ of the principal $\mathbb{C}^\times$-bundle $\mathcal{P}_X$ of the line bundle Det on $\mathcal{J}(X)$. The space $\tilde{\mathcal{P}}_X$ then has an action of $\mathbb{C}$, and $\mathrm{Det}^{\otimes(\frac{1}{2}c)}$ can be defined as $\tilde{\mathcal{P}}_X \times_{\mathbb{C}} \mathbb{C}$, where $\mathbb{C}$ acts on $\mathbb{C}$ by $(\lambda,\xi) \to e^{\pi i c\lambda}\xi$.

Definition (5.10). A rigged surface is a smooth surface X together with a choice of a universal covering space of $\mathcal{P}_X$.

Of course any two riggings of the same surface are isomorphic, but the group of automorphisms of a rigged surface $(X, \tilde{\mathcal{P}}_X)$ is a central extension by $\mathbb{Z}$ of the group of diffeomorphisms of X. In fact for a surface of genus >1 it is the universal central extension of the diffeomorphism group.

I have not been able to think of a less sophisticated definition of a rigged surface, although there are many possible variants. The essential idea is to associate functorially to a smooth surface a space - such as $\mathcal{P}_X$ - which has fundamental group $\mathbb{Z}$. Instead of $\mathcal{P}_X$ one can take $\mathcal{J}(H^1(\hat{X};\mathbb{R}))$, which is obtained by replacing the determinant line on $\mathcal{J}_X$ by the determinant line on the Siegel domain $\mathcal{J}(H^1(\hat{X};\mathbb{R}))$ of complex structures on the symplectic vector space $H^1(\hat{X};\mathbb{R})$. (Here $\hat{X}$ is X with discs attached to its boundary circles.) There is an obvious natural transformation $\mathcal{J}_X \to \mathcal{J}(H^1(X;\mathbb{R}))$. Another variant is to replace $\mathcal{P}_X$ by the Grassmannian of oriented Lagrangian subspaces of $H^1(X;\mathbb{R})$. (Let us notice that if Y is a 3-manifold with $\partial Y = X$ the image of $H^1(Y)$ in $H^1(X)$ is a Lagrangian subspace.) In [A2] Atiyah, following Witten, considers the space $\mathfrak{F}_X$ of 2-framings of X, i.e. trivializations of the sum of two copies of the tangent bundle of X. We have $\pi_1(\mathfrak{F}_X) = \mathbb{Z}$, but the natural map $\pi_1(\mathfrak{G}_X) \to \pi_1(\mathfrak{F}_X)$ is multiplication by 12. The 2-framings therefore lead to an extension of the mapping-class group by $\mathbb{Z}$ whose class is 12 times that of the extension considered here. (In particular, Atiyah's extension is trivial when X is a torus, whereas ours is the extension of $SL_2(\mathbb{Z})$ induced by the universal covering group of $SL_2(\mathbb{R})$, and is isomorphic to the braid group on three strings.) In any case, we can now reformulate (5.4) as follows.

Proposition (5.11). A modular functor defines a functor on the category of rigged smooth surfaces and isotopy classes of rigged diffeomorphisms.

The functor on the category of rigged smooth surfaces so obtained will be called a reduced modular functor. From what we have said so far it is defined for surfaces with parametrized boundaries (or, better, with rigged boundaries in the sense explained after Definition (4.4)) but it is clear that we could equally well regard it as a functor on the category of closed rigged smooth surfaces equipped with a finite number of labelled marked points with a preferred tangent direction at each. If the tangent directions are rotated there is a flat connection in the resulting vector bundle over the torus of tangent directions, and the holonomy of a rotation of 2π about a point labelled φ is $e^{2\pi i h_\varphi}$.

The central charge of a reduced modular functor is defined only modulo 1. (It is well-defined modulo 1 because $H^2(\Gamma;\mathbb{C}^\times) = \mathbb{C}^\times$ when Γ is the mapping-class group of a surface of large genus.) The original modular functor can be recovered from the reduced one up to tensoring with an arbitrary integral power of the determinant line.

When two rigged surfaces are sewn together the result is rigged, and a reduced modular functor inherits the composition properties of Definition (5.1). This follows from the simple behaviour of the determinant line, which we shall treat in §6. (The variant definitions of rigging mentioned above are less well adapted to sewing.)

It seems appropriate at this point to mention "Verlinde's conjecture", which gives a remarkable description of the algebra $\mathbb{Z}[\Phi]$ associated to a modular functor E, which we can assume to be reduced. (The statement and the idea of the proof are due to Verlinde [V]; a complete proof was first given by Moore and Seiberg [MS].) Let X be a torus, and let α,β be simple closed curves on X representing a basis for the homology $H_1(X)$. Because an annulus is canonically rigged we can identify E(X) with $\mathbb{C}[\Phi]$ by cutting X along α and using (5.1)(ii). The mapping-class group of X is $SL_2(\mathbb{Z})$. It acts projectively on E(X), and we transfer the action to $\mathbb{C}[\Phi]$. Let $S : \mathbb{C}[\Phi] \to \mathbb{C}[\Phi]$ be a representative of $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, and let M_φ denote as above the operation of multiplication by φ in the Verlinde algebra. Then we have

<u>Theorem (5.12)</u>. The matrix of $SM_\varphi S^{-1}$ is diagonal with respect to the natural basis of $\mathbb{C}[\Phi]$.

The theorem implies that the structure of the Verlinde algebra is completely determined by the matrix S. In fact

<u>Corollary (5.13)</u>. The structural constants $n_{\varphi\chi\psi}$ of $\mathbb{Z}[\Phi]$ are given by

$$n_{\varphi\chi\psi} = \sum_\theta (S^{-1})_{\chi\theta} S_{\theta\varphi} S_{\theta\psi} / S_{\theta 1} .$$

The corollary follows from the theorem because $n_{\varphi\chi\psi}$ is the (χ,ψ) matrix element of M_φ and the (θ,θ) entry of the diagonal matrix $SM_\varphi S^{-1}$ is $S_{\theta\varphi}/S_{\theta 1}$. That is proved by equating the $(\theta,1)$ entries of the matrices $SM_\varphi = (SM_\varphi S^{-1})S$, for $(M_\varphi)_{\chi 1} = \delta_{\varphi\chi}$.

We shall not give a proof of (5.12), but shall explain how it follows from the 2+1 dimensional description of modular functors to which we now turn.

Topological field theories

A topological field theory in d+1 dimensions can be defined, by analogy with (4.4), as a system comprising

(i) a functor $X \mapsto H(X)$ from closed oriented d-dimensional smooth manifolds to finite dimensional complex vector spaces,

(ii) a non-singular pairing $H(\bar{X}) \otimes H(X) \to \mathbb{C}$ for each X, where $\bar{X}$ denotes X with reversed orientation, and

(iii) a vector $\psi_Y \in H(\partial Y)$ for each smooth oriented (d+1)-dimensional manifold Y with boundary.

These data are required to obey the following two axioms.

(a) Multiplicativity: $H(X_1 \amalg X_2) = H(X_1) \otimes H(X_2)$ and

$$\psi_{Y_1 \amalg Y_2} = \psi_{Y_1} \otimes \psi_{Y_2} .$$

(b) Sewing: if $\partial Y = X_0 \amalg X_1 \amalg X_2$, and $\check{Y}$ is formed from Y by sewing X_2 to X_1 by an orientation-reversing diffeomorphism, then $\psi_Y \mapsto \psi_{\check{Y}}$ under the map $H(\partial Y) \to H(\partial\check{Y})$ induced by the pairing $H(X_1) \otimes H(X_2) \to \mathbb{C}$.

Witten realized that the modular functors coming from representations of loop groups are the state spaces of 2+1 dimensional theories, and subsequently Kontsevich [K] and others have given arguments - a little sketchy - to show that the two concepts are actually equivalent. Of course one must first widen the definition of a topological theory a little so that it is defined on the category of rigged surfaces and 3-manifolds. An oriented 3-manifold Y whose boundary ∂Y is rigged has itself a set of riggings which form a principal homogeneous set under the group $\mathbb{Z}$ which is the centre of the central extension of Diff(∂Y). I do not know an altogether straightforward way to define a rigging of a 3-manifold. One approach is to introduce the contractible space $\mathcal{M}_Y$ of metrics on Y. Each metric has an "η-invariant" (see [APS]) which is a non-zero element of the determinant line of ∂Y. (The invariant is essentially the phase of the determinant of the signature operator.) Thus we have a map

$$\eta : \mathcal{M}_Y \to \mathcal{P}_{\partial Y} .$$

A rigging of Y is a lift of this map to the covering space $\tilde{\mathcal{P}}_{\partial Y}$ which defines the rigging of ∂Y.

To relate modular functors to 2+1 dimensional theories it is helpful to introduce the intermediate idea of a relative 2+1 dimensional theory. Like a modular functor this has a set Φ of labels, and assigns a vector space H(X) to each rigged oriented surface with labelled boundary circles. It has the same sewing-together property as a modular functor. As for a field theory there is a vector $\psi_Y \in H(\partial Y)$ for each rigged 3-manifold with boundary, but it is required to satisfy a stronger sewing property than (b) above, for one must allow ∂Y to be decomposed $X_0 \cup X_1 \cup X_2$, where the X_i are surfaces with boundary which intersect along various boundary circles. An orientation-reversing diffeomorphism $f : X_1 \to X_2$ allows one to sew together X_1 and X_2 to form a 3-manifold $\check{Y}$ such that $\partial\check{Y} = \check{X}_0$ is obtained by sewing from X_0. To see that there is a natural map $H(\partial Y) \to H(\partial\check{Y})$, we write

$$H(\partial Y) = \bigoplus_{\varphi_{01},\varphi_{02},\varphi_{12}} H(X_0;\varphi_{01},\varphi_{02}) \otimes H(X_1;\varphi_{01},\varphi_{12}) \otimes H(X_2;\varphi_{12},\varphi_{02}),$$

where φ_{ij} is a multi-label for $X_i \cap X_j$. We project this sum to the sum of the terms where $\varphi_{01} = \varphi_{02}$. Then the last two factors in the tensor product are in duality under f, so the sum maps to

$$\bigoplus_{\varphi_{01} = \varphi_{02}} H(X_0;\varphi_{01},\varphi_{02}) = H(\check{X}_0) .$$

The axiom we require is that $H(\partial Y) \to H(\partial\check{Y})$ takes ψ_Y to $\psi_{\check{Y}}$.

When a reduced modular functor E is given it is obvious that there is at most one way to define the vectors ψ_Y corresponding to 3-manifolds Y. One begins with the standard 3-disc D and chooses ψ_D in the line $E(S^2)$. This can be done arbitrarily, because any modular functor has an automorphism which multiplies by $\lambda^{\chi(X)}$ on E(X). Any other 3-manifold Y can be obtained by sewing copies of D together, and its vector ψ_Y is determined by the sewing axiom. Kontsevich [K] has given a simple argument to show that the vector obtained is independent of the chosen decomposition of Y. I feel, however, that the matter is still far from well-understood.

I shall conclude this section with the proof of Verlinde's conjecture (5.12) for a 2+1 dimensional field theory. Let Σ be a disc with two holes, and let $Y = \Sigma \times [0,1]$. Then $\partial Y = \Sigma \cup \bar{\Sigma} \cup A \cup A \cup A$, where A is an annulus. We have

$$H(\partial Y) = \bigoplus_{\varphi,\chi,\psi} H(\Sigma_{\varphi\chi\psi}) \otimes H(\Sigma_{\varphi\chi\psi})^* \otimes H(A_{\varphi\varphi}) \otimes H(A_{\chi\chi}) \otimes H(A_{\psi\psi}) ,$$

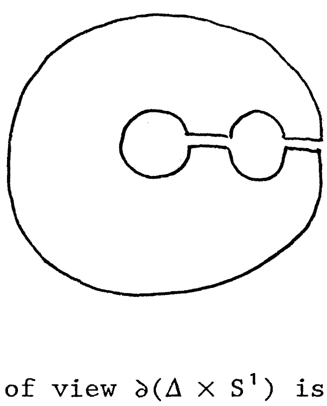
in what I hope is obvious notation. Let

$$\psi_Y = \sum_{\varphi,\chi,\psi} t_{\varphi\chi\psi} \otimes \epsilon_\varphi \otimes \epsilon_\chi \otimes \epsilon_\psi ,$$

where $t_{\varphi\chi\psi}$ is an endomorphism of $H(\Sigma_{\varphi\chi\psi})$ and ϵ_φ is the canonical element of $A_{\varphi\varphi}$. When two copies of the cylinder Y are sewn end-to-end we have $Y \cup Y \cong Y$, and hence $\psi_Y^2 = \psi_Y$ in the algebra $H(\partial Y)$. But $\epsilon_\varphi^2 = \epsilon_\varphi$, etc., so $t_{\varphi\chi\psi}$ is the identity map. Joining the ends of Y together to get $\check{Y} = \Sigma \times S^1$ we have

$$\psi_{\check{Y}} = \sum n_{\varphi\chi\psi} \; \epsilon_\varphi \otimes \epsilon_\chi \otimes \epsilon_\psi \tag{5.14}$$

On the other hand we can form $\check{Y}$ also from $\Delta \times S^1$, where Δ is the disc



From this point of view $\partial(\Delta \times S^1)$ is the union of eight annuli, and ψ_Y has to be of the form

$$\Sigma \lambda_\varphi \tilde{\epsilon}_\varphi \otimes \tilde{\epsilon}_\varphi \otimes \tilde{\epsilon}_\varphi \, , \qquad (5.15)$$

where $\tilde{\epsilon}_\varphi \in H(S^1 \times S^1)$ is formed in the same way as ϵ_φ, but with the axes of the torus interchanged. In terms of the modular transformation $S : H(S^1 \times S^1) \to H(S^1 \times S^1)$ we know that $\tilde{\epsilon}_\varphi$ is a multiple of $S\epsilon_\varphi$, and so the equality of (5.14) and (5.15) is exactly Verlinde's assertion (5.12).

Mumford's theorem

We can now easily prove the crucial theorem of Mumford which determines all one-dimensional modular functors. (I am greatly indebted to Deligne for showing me how to correct an earlier version of the following proof.)

Proposition (5.16). If a modular functor E satisfies dim E(X) = 1 for all X then it is determined by its restriction to $\mathcal{A}$.

Corollary (5.17). The only such modular functors are integral powers of the determinant line.

The same argument will prove

Proposition (5.18). The only central extensions of the category $\mathcal{C}$ by $\mathbb{C}^\times$ are those given by $X \mapsto \mathrm{Det}_X^{\otimes p} \otimes \overline{\mathrm{Det}}_X^{\otimes q}$ for $p, q \in \mathbb{C}$ such that $p - q \in \mathbb{Z}$.

Example. Let $E_m(X)$ denote the determinant line of the $\bar{\partial}$-operator acting on differentials of order m. Thus $E_m(X) \cong \Lambda^{(2m-1)(g-1)} \Omega_{\mathrm{hol}}^{\otimes m}(X)^*$

if m,g > 1 (cf. §6). Calculating the Lie algebra cocycles (see (8.14)) shows that

$$E_m(X) \cong E_0(X)^{\otimes(6m^2-6m+1)} \qquad (5.19)$$

when X is an annulus, and (5.8) shows that this isomorphism holds for all surfaces X.

Proof of (5.16). Let E_1 and E_2 be two functors with the same restriction to $\mathcal{A}$. Then $E = E_1^* \otimes E_2$ is a modular functor which is trivial on $\mathcal{A}$. The argument of (5.3) shows that for any α there is a connection in E_α which is flat - not just projectively flat - and compatible with sewing. This means that E_α is determined by a representation of $\pi_1(\mathcal{C}_\alpha)$. But it is a classical result that $\pi_1(\mathcal{C}_\alpha)$ is generated by "Dehn twists" along various curves γ in the surface α. In our language, if X is a point of $\mathcal{C}_\alpha$ one can write $X = Y \cup A$, where A is an annulus containing the curve γ. Holding Y fixed we have a map $\mathcal{A} \to \mathcal{C}_\alpha$, and the Dehn twist is the image of $\pi_1(\mathcal{A}) = \mathbb{Z}$. But $E|\mathcal{A}$ is trivial by hypothesis, so the action of $\pi_1(\mathcal{C}_\alpha)$ is trivial, and all the fibres of E_α can be canonically identified. This means that $X \mapsto E(X)$ is a functor on the category of smooth surfaces and diffeomorphisms, and also that the group of diffeomorphisms of X acts trivially on E(X). The isomorphisms $E(X) \otimes E(Y) \to E(X \amalg Y)$ and $E(X) \to E(\check{X})$ are still, of course, natural.

Let us write E_g for $E(X_g)$ when X_g is an arbitrary closed surface of genus g. If $X_g^{(k)}$ is got by removing k discs from X_g then $E(X_g^{(k)})$ can also be identified canonically with E_g. The complete data provided by the functor are then described by the sequence of lines E_g together with the maps

$$i : E_g \to E_{g+1}$$

$$m : E_{g_1} \otimes E_{g_2} \to E_{g_1+g_2} ,$$

where i is defined by sewing together the boundary circles of $X_g^{(2)}$, and m by sewing together $X_{g_1}^{(1)}$ and $X_{g_2}^{(1)}$. To prove the theorem we must show that one can choose isomorphisms $\epsilon_g : \mathbb{C} \to E_g$ which are compatible with the maps i and m. That is possible because the diagrams

$$\begin{array}{ccc} E_{g_1} \otimes E_{g_2} & \to & E_{g_1+g_2} \\ \downarrow & & \downarrow \\ E_{g_1+1} \otimes E_{g_2} & \to & E_{g_1+g_2+1} \end{array} \quad \text{and} \quad \begin{array}{ccc} E_{g_1} \otimes E_{g_2} & \to & E_{g_1+g_2} \\ \downarrow & & \downarrow \\ E_{g_1} \otimes E_{g_2+1} & \to & E_{g_1+g_2+1} \end{array}$$

commute.

Proofs of (5.17) and (5.18). We have seen in (5.7) that a holomorphic extension of $\mathcal{A}$ is determined by a pair (c,h), and that h = 0 for a modular functor with one label. In view of (5.16) it is therefore enough to show that c must be an even integer. That is true because the $(c/2)^{th}$ power of Det — which is defined for rigged surfaces — does not descend to unrigged surfaces unless c/2 is an integer. One reason is that the first Chern class of Det generates $H^2(\mathcal{C}_\alpha ; \mathbb{Z}) \cong \mathbb{Z}$ when α is a surface of large genus with one hole. I do not know if there is a simpler reason.

Unitarity

All the examples known to me of modular functors are unitary in the following sense.

Definition (5.20). A modular functor E is unitary if there is a positive non-degenerate transformation

$$\overline{E(X)} \otimes E(X) \to |\mathrm{Det}_X|^{c}$$

for each surface X with labelled boundary, such that, in the notation of (5.1), the diagram

$$\begin{array}{ccc} \bigoplus_{\varphi} \overline{E(X_\varphi)} \otimes E(X_\varphi) & \to & |\mathrm{Det}_X|^{c} \\ \downarrow & & \downarrow \\ \overline{E(X)} \otimes E(X) & \to & |\mathrm{Det}_X|^{c} \end{array}$$

commutes.

Thus a unitary modular functor provides unitary projective representations of the braid groups, etc. More importantly, the definition is designed to give us

Proposition (5.21). A pair of weakly conformal holomorphic field theories H and H' corresponding to the same unitary modular functor E with index set Φ gives rise to a conformal field theory based on the space $\bigoplus_{\varphi\in\Phi} \overline{H}_\varphi \otimes H'_\varphi$ and the central extension $|\mathrm{Det}|^{c}$ of $\mathscr{C}$.

§6. The determinant line

Definition and basic properties

The determinant line Det_X of a Riemann surface X with parametrized boundary[1] is the dual of the top exterior power of the space of holomorphic differentials on the closed surface $\hat{X}$ obtained by adding caps to the boundary circles of X, i.e.

$$\mathrm{Det}_X = \Lambda^g \,\Omega^1_{\mathrm{hol}}(\hat{X})^* \ . \tag{6.1}$$

This definition, however, does not lead one to expect the canonical isomorphism

$$\mathrm{Det}_X \otimes \mathrm{Det}_Y \cong \mathrm{Det}_{X \cup Y} \tag{6.2}$$

which exists when surfaces are sewn together.

An alternative definition of Det_X is as the determinant line of the $\bar{\partial}$-operator of X in the sense of Quillen [Q]. To define this, recall that on any Riemann surface there is a $\bar{\partial}$-operator

$$\bar{\partial}_X : \Omega^0(X) \rightarrow \Omega^{01}(X)$$

mapping smooth functions to (0,1)-forms. If X has a parametrized boundary then $\bar{\partial}_X$ has a natural boundary condition which makes it a Fredholm operator: one restricts it to the subspace $\Omega^0(X,\partial X)$ of functions which on each incoming boundary circle are of the form

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[1] Sophisticated readers should notice that to define Det_X we do not need the boundary of X to be parametrized, but only to be <u>rigged</u>, as was explained after Defn. (4.4).

$\sum_{n\geq 0} a_n e^{in\theta}$, and on each outgoing circle of the form $\sum_{n<0} b_n e^{in\theta}$. Any Fredholm operator $P : E \to F$ between topological vector spaces has a determinant line Det_P, which can be defined in various ways. A convenient definition for our purposes is given in Appendix B. For a single operator P we have

$$Det_P = Det\ (\ker P)^* \otimes Det\ (\mathrm{coker}\ P)\ ,$$

where on the right Det denotes the top exterior power. For the operator $\bar{\partial}_X$ this reduces to (6.1), but the important property of the definition is that the lines Det_P fit together to form a holomorphic line bundle on the space of Fredholm operators $E \to F$. More generally, if E and F are holomorphic bundles of topological vector spaces over some base space, and $P : E \to F$ is holomorphic and Fredholm (cf. Appendix B), then Det_P is a holomorphic line bundle on the base space.

It should be remembered that the determinant line of a Fredholm operator is a vector space with a mod 2 grading. The degree is the index of the operator. For a surface of genus g with m incoming and n outgoing circles the degree of the determinant line is $m + 1 - g$. Thus Det_{S^2} is canonically $\mathbb{C}$, but in degree 1. This means that $Det_{S^2 \amalg S^2 \amalg \cdots \amalg S^2}$ is also $\mathbb{C}$, but that the group of permutations of the spheres acts on it by the sign representation.

If the surface X has no closed components there is another description of Det_X. Let Hol(X) be the vector space of holomorphic functions on X. The space of smooth functions $\Omega^0(\partial X)$ has a splitting $\Omega^0_+(\partial X) \oplus \Omega^0_-(\partial X)$, where Ω^0_- denotes the functions which satisfy the boundary condition above.

Proposition (6.3). If X has no closed components then Det_X is canonically isomorphic to the determinant line of the operator $\pi_X : Hol(X) \to \Omega^0_+(\partial X)$ given by restriction to ∂X followed by projection on to Ω^0_+.[1]

Corollary (6.4). The lines Det_X form a holomorphic bundle on each moduli space $\mathcal{C}_\alpha$ of surfaces with parametrized boundaries.

We shall return to the case of closed surfaces below: see the remark after Proposition (6.5).

Proof of (6.3). We consider the diagram

$$\begin{array}{ccccccccc} 0 & \to & Hol(X) & \to & \Omega^0(X) & \xrightarrow{\bar\partial} & \Omega^{01}(X) & \to & 0 \\ & & \downarrow \pi_X & & \downarrow \bar\partial \oplus pr & & \downarrow id & & \\ 0 & \to & \Omega^0_+(\partial X) & \to & \Omega^{01}(X) \oplus \Omega^0_+(\partial X) & \to & \Omega^{01}(X) & \to & 0 . \end{array}$$

As the rows are exact this defines an isomorphism

$$Det_X \cong Det_{\bar\partial \oplus pr} \cong Det_{\pi_X} \otimes Det_{id} \cong Det_{\pi_X} .$$

The essential property of the determinant line is (6.2), which, as it is obvious that $Det_{X \amalg Y} \cong Det_X \otimes Det_Y$, is a particular case of

Proposition (6.4). A sewing map $X \to \check{X}$, i.e. one which sews outgoing edges of X to incoming ones, induces a canonical isomorphism $Det_X \cong Det_{\check{X}}$.

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[1] It is more convenient for the sequel if we change the definition of π_X by composing it with the automorphism of $\Omega^0_+(\partial X)$ which multiplies by -1 on the incoming circles. That does not affect the truth of (6.3).

Proof: First assume that $\check{X}$ is not closed, and that it is formed by sewing together the parts S_1 and S_2 of ∂X to form a curve S in $\check{X}$. We have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & Hol(\check{X}) & \to & Hol(X) & \xrightarrow{\Delta} & \Omega^0(S) & \to & 0 \\ & & \pi_{\check{X}} \downarrow & & \tilde{\pi} \downarrow & & id \downarrow & & \\ 0 & \to & \Omega_+(\partial\check{X}) & \to & \Omega_+(\partial\check{X}) \oplus \Omega^0(S) & \to & \Omega^0(S) & \to & 0 . \end{array}$$

Here Δ is defined by $\Delta(f) = f|S_1 - f|S_2$, and $\tilde{\pi}$ by

$$\tilde{\pi}(f) = ((f|\partial\check{X})_+, \Delta(f)) .$$

The rows are exact. (To see that Δ is surjective, let Y be the surface formed from two copies X_1 and X_2 of X by attaching $S_1 \subset X_1$ to $S_2 \subset X_2$. Because Y is a Stein manifold, any smooth function f on $S = X_1 \cap X_2$ can be written $f_1|S - f_2|S$, with $f_i \in Hol(X_i)$.) Thus $Det_X \cong Det(\tilde{\pi})$. But Det_X is the determinant of $\pi_X : Hol(X) \to \Omega^0_+(\partial X)$. We can identify $\Omega^0(S)$ with $\Omega^0_+(S_1) \oplus \Omega^0_-(S_2)$, and hence $\Omega^0_+(\partial X)$ with $\Omega^0_+(\partial\check{X}) \oplus \Omega^0(S)$. Then $\pi_X - \tilde{\pi}$ is the map $f \mapsto (f|S_1)_- - (f|S_2)_+$. This is of trace class by Lemma (6.6) below. But the determinant line does not change when the operator is changed by an operator of trace class (see Appendix B), so the result is proved.

The case when $\check{X}$ is closed can be dealt with by making a hole in $\check{X}$ so that it does have a boundary, and then using the following result.

Proposition (6.5). For any surface X the line Det_X does not change when the interiors of one or more holomorphically embedded discs are removed from X.

If X is not closed this is already implied by (6.4). It is therefore in keeping with the spirit of our approach to <u>define</u> Det_X when X is closed as $\mathrm{Det}_{X-\mathring{D}}$, where D is any disc in X. This definition does not depend on the disc chosen, for if D_1 and D_2 are disjoint discs in X then we know that

$$\mathrm{Det}_{X-\mathring{D}_1} \cong \mathrm{Det}_{X-\mathring{D}_1-\mathring{D}_2} \cong \mathrm{Det}_{X-\mathring{D}_2} .$$

We shall therefore leave the proof of (6.5) to Appendix B (B12).

In proving (6.4) we made use of

<u>Lemma (6.6)</u>. If S is a union of outgoing boundary circles of a Riemann surface X then the map $f \mapsto (f|S)_-$ from Hol(X) to $\Omega^0_-(S)$ is of trace class.

<u>Proof</u>: It is enough to prove this when X is an annulus and $S = S_1$ is its outgoing end. Indeed because diffeomorphisms preserve the decomposition $\Omega^0(S) = \Omega^0_+(S) \oplus \Omega^0_-(S)$ up to trace class operators (see [PS]( )) we can assume that X is $\{z \in \mathbb{C} : r \leq |z| \leq 1\}$ with the standard parametrization. But then $\mathrm{Hol}(X) \to \Omega^0_-(S_1)$ factorizes

$$\mathrm{Hol}(X) \to \Omega^0_+(S_0) \to \Omega^0_-(S_1) ,$$

where the second map is the diagonal operator taking z^k to $r^k z^k$. This is clearly of trace class.

The central extension of $\mathrm{Diff}^+(S^1)$

We can now see why - as was mentioned at the beginning of this paper - the determinant line gives rise to the basic central extension[1] of $\mathrm{Diff}^+(S^1)$. One way of formulating the result of [S2] §7(b) is that the basic central extension of $\mathrm{Diff}^+(S^1)$ consists of pairs (φ,λ) with $\varphi \in \mathrm{Diff}^+(S^1)$ and $\lambda \in L_\varphi$, where L_φ is the determinant line of the Toeplitz operator $T_\varphi : \Omega^0_+(S^1) \to \Omega^0_+(S^1)$ which is the (++) block of the action of φ on $\Omega^0(S^1)$. But L_φ is equivalently the determinant line of

$$P_\varphi : \Omega^0(S^1) \to \Omega^0_+(S^1) \oplus \Omega^0_-(S^1)$$

$$f \mapsto ((\varphi^* f)_+, f_-) .$$

If the diffeomorphism φ is regarded as the limit of a family of annuli A then P_φ is evidently the limit of the operators

$$\pi_A : \mathrm{Hol}(A) \to \Omega^0_+(\partial A) .$$

More precisely, in terms of the proof of (5.5), if A is the standard annulus $\{r \le |z| \le 1\}$, we have $\pi_{\varphi A} = P_\varphi \pi_A$, and hence

$$\det(P_\varphi) = \mathrm{Det}_{\varphi A} \otimes \mathrm{Det}^*_A .$$

This makes clear the sense in which the central extension of $\mathrm{Diff}^+(S^1)$ is the "boundary" of an extension of the semigroup $\mathcal{A}$.

To understand why the pairs $(\varphi, \lambda \in L_\varphi)$ form a group it is best to regard L_φ, in the notation of Appendix B, as $\mathrm{Det}(W;\varphi W)$, where W is an element of $\mathrm{Gr}(\Omega^0(S^1))$: this line does not depend on W.

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[1] The cleanest statement is simply that the extension is the group of automorphisms of a rigged circle.

An important property of the central extension of $\mathrm{Diff}(S^1)$ which is easy to see in terms of the determinant line - or rather in terms of the splitting of $\Omega^0(S^1)$ into positive and negative frequency - is the following "reciprocity law". When we have a surface X with boundary we can put together a copy of the standard extension of $\mathrm{Vect}_{\mathbb{C}}(S^1)$ for each boundary circle to obtain an extension $\widetilde{\mathrm{Vect}_{\mathbb{C}}}(\partial X)$ of $\mathrm{Vect}_{\mathbb{C}}(\partial X)$,

<u>Proposition (6.7)</u>. The restriction of the extension $\widetilde{\mathrm{Vect}_{\mathbb{C}}}(\partial X)$ to the subalgebra Vect(X) of holomorphic vector fields on X is canonically split.

<u>Proof</u>: The extension of $\mathrm{Vect}_{\mathbb{C}}(\partial X)$ measures the extent to which the vector fields fail to preserve the decomposition $\Omega^0(\partial X) = \Omega^0_+(\partial X) \oplus \Omega^0_-(\partial X)$. If we write T_ξ for the $\Omega_+ \to \Omega_+$ component of the action of a vector field ξ then we have the following explicit formula for the cocycle (cf. [PS](6.6.5)):

$$(\xi,\eta) \mapsto \mathrm{trace}([T_\xi, T_\eta] - T_{[\xi,\eta]}) \qquad (6.8)$$
$$= \mathrm{trace}(J[J,\xi][J,\eta])$$

Here J is the operator which defines the splitting $\Omega = \Omega_+ + \oplus\ \Omega_-$, i.e. $J|\Omega_\pm = \pm 1$. If the decomposition is changed by replacing J by another operator J_X such that $K = J_X - J$ is of trace class (and $J_X^2 = 1$) then the cocycle (6.8) changes by the coboundary

$$(\xi,\eta) \mapsto 2\ \mathrm{trace}([\xi,\eta]K)\ .$$

Let us choose J_X corresponding to the decomposition $\Omega^0(\partial X) = \mathrm{Hol}(X) \oplus \mathrm{Hol}'(Y)$, where Y is a collection of discs with $\partial Y = \partial X$, so that $X \cup Y$ is a closed surface, and Hol'(Y) means the functions which are holomorphic except for a pole (or zero) of an appropriate order at the centre of one of the discs. The difference $J_X - J$ is given by an integral operator on ∂X with a smooth kernel, so it is certainly of trace-class. On the other hand, the subspace Hol(X) is preserved by holomorphic vector fields on X, so the cocycle $\text{trace}(J_X[J_X,\xi][J_X,\eta])$ vanishes on Vect(X).

Another result which fits in naturally at this point is

Proposition (6.9). For any modular functor the extensions of $\text{Vect}(S^1)$ corresponding to its labels all have the same central charge.

Proof: Suppose that X is a disc with two holes with boundary circles S_0, S_1, S_2 labelled $\varphi_0, \varphi_1, \varphi_2$. Let c_i be the central charge corresponding to φ_i. We must show that $c_1 = c_2 = c_3$. Let $\xi_i \in H^2(\text{Vect}(X);\mathbb{C})$ be the class of the extension of Vect(X) pulled back from the determinant line extension of $\text{Vect}(S_i)$. Then $\xi_0 + \xi_1 + \xi_2 = 0$; but ξ_1 and ξ_2 are linearly independent, because by filling in, say, the second hole we embed X in an annulus A in such a way that $\xi_2 \mapsto 0$ but $\xi_1 \not\mapsto 0$ in $H^2(\text{Vect}(A);\mathbb{C}) = \mathbb{C}$. Now the modular functor gives us an extension of Vect(X) with class $\Sigma\, c_i\xi_i$. We know that it is split, and therefore $\Sigma\, c_i\xi_i = 0$, and hence $c_1 = c_2 = c_3$.

Modularity and the η-function

Our final task in this section is to give a completely explicit description of the isomorphism $\text{Det}_A \cong \text{Det}_X$ when A is an annulus and $X = \check{A}$ is the torus got by joining its ends. We need this to find the modularity properties of partition functions. Suppose, for example, that we have a holomorphic field theory with central charge c, so that an operator $U_{c,\alpha}$ in H is associated to the annulus A together with a choice of $\alpha \in \mathrm{Det}_A$, and $U_{A,\lambda\alpha} = \lambda^{\frac{1}{2}c} U_{A,\alpha}$ for $\lambda \in \mathbb{C}$. Then the trace of $U_{A,\alpha}$ depends only on the image $\check{\alpha}$ of α in the line Det_X. There is a canonical element $\epsilon_A \in \mathrm{Det}_A$, for the $\bar{\partial}$-operator is an isomorphism. The partition function Z of the theory is defined by

$$Z(\tau) = \text{trace } U_{A,\epsilon_A} ,$$

where A is the standard annulus determined by $q = e^{2\pi i\tau}$, with $\mathrm{Im}(\tau) > 0$. If τ is replaced by $\tau' = (a\tau + b)/(c\tau + d)$, for some $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in the modular group $\Gamma = SL_2(\mathbb{Z})$, then A changes to A', but the torus X does not change, so we have

$$Z(\tau') = \rho(\tau,g)^{\frac{1}{2}c}\, Z(\tau) , \tag{6.10}$$

where $\rho(\tau,g)$ is the ratio of the images of the elements $\epsilon_{A'}$ and ϵ_A in Det_X. (Note that $\rho(\tau,g)$ depends on g, and not just on τ and τ', because one must choose an isomorphism between the tori $\check{A}$ and $\check{A}'$.) The crucial result is

<u>Proposition (6.11)</u>. We have $\rho(\tau,g) = u(g)e^{2\pi i(\tau'-\tau)/12}$, where $u : \Gamma \to \boldsymbol{\mu}_{12}$ is a canonical homomorphism from Γ to the group $\boldsymbol{\mu}_{12}$ of 12^{th} roots of unity. In other words

$$(q')^{-c/12}Z(\tau') = u(g)^{\frac{1}{2}c}\, q^{-c/12}Z(\tau) .$$

The line Det_X attached to a torus X is the dual of the line of holomorphic differentials, so it contains a lattice $\Lambda = H_1(X,\mathbb{Z})$ given by the geometrical cycles. Let $\xi_\gamma \in \mathrm{Det}_X$ correspond to the cycle γ.

If X is formed from an annulus A this gives us a preferred element ξ_A. It is natural to expect that ξ_A should be related to the image of ϵ_A, and in fact it is very easy to prove

Proposition (6.12). The image of ϵ_A in Det_X is

$$\prod_{n>0} (1-q^n)^{-2}.\xi_A \ .$$

When A is changed to A' by $g \in \Gamma$ we have

$$\xi_{A'} = (c\tau + d)^{-1}.\xi_A \ , \tag{6.13}$$

and so, in the light of (6.12), the result (6.11) is equivalent to the modularity property

$$\eta(\tau')^2 = u(g)^{-1}(c\tau + d)^{-1}\eta(t)^2 \tag{6.14}$$

of the square of the Dedekind η-function, which is defined by

$$\eta(\tau) = e^{\pi i\tau/12} \prod_{n>0} (1-q^n) \ .$$

Indeed, (6.14) can be taken as a definition of the homomorphism u. The existence and modularity of the η-function amount to the following geometrical statement:

if L is a complex line equipped with a lattice Λ then there is a canonical isomorphism $L^{\otimes 12} \cong \mathbb{C}$, i.e. a canonical map $f : L \to \mathbb{C}$ which is homogeneous of degree 12; in particular L contains a distinguished set $\mu_{12}^L = f^{-1}(1)$ of 12 points.

Applying this to the case $(L,\Lambda) = (\mathbb{C}, \mathbb{Z} + \tau\mathbb{Z})$ for $\mathrm{Im}(\tau) > 0$ gives us a 12-sheeted covering of the upper half-plane with an action of the group $SL_2(\mathbb{Z})$ on it; and considering how the sheets are permuted gives the homomorphism $u : SL_2(\mathbb{Z}) \to \boldsymbol{\mu}_{12}$ which in fact describes the abelianization of $SL_2(\mathbb{Z})$.

At this point we could assume the properties of η and deduce (6.11) from (6.12). But it is obviously more satisfying to deduce the properties of η from general facts about the determinant line. I shall give an argument based on Mumford's theorem (5.9), but I should mention that Deligne has given a much more illuminating argument, which, however, it would require too long a digression to explain.

For my argument we consider alongside the line Det_X the determinant line $E_X^{(2)}$ of the $\bar{\partial}$-operator acting on forms of type (2,0). From (5.9) we know that $E_X^{(2)} \cong \mathrm{Det}_X^{\otimes 13}$, and, more precisely, that $\mathrm{Det}_X^{\otimes 13} \otimes (E_X^{(2)})^*$ has a canonical element μ_X which is multiplicative in the sense that $\mu_{X \circ Y} = \mu_X \cdot \mu_Y$. But the proof of (5.9) shows that for an annulus A one has $\mu_A = \epsilon_A^{13} \tilde{\epsilon}_A^{-1}$, where $\tilde{\epsilon}_A$ is the standard element of $E_A^{(2)}$. (The reason for this is that all of these standard elements are characterized by multiplicativity with respect to the relation $D_\infty \circ A \circ D = D_\infty \circ D$ in the category $\mathcal{C}$. Now the map $A \to X$ takes μ_A to μ_X, and so the image of $\epsilon_A^{13}\ \tilde{\epsilon}_A^{-1}$ depends only on X. But Proposition (6.12) can be generalized to a statement about the determinant line $E_X^{(m)}$ of the $\bar{\partial}$-operator acting on $\Omega^{\otimes m}$ for any value of m. An annulus $E_A^{(m)}$ has the usual canonical element $\epsilon_A^{(m)}$, and for a torus X there is an element $\xi_\psi^{(m)}$ for each cycle γ.

<u>Proposition (6.15)</u>. The image of $\epsilon_A^{(m)}$ in $E_X^{(m)}$ is

$$(-1)^m\, q^{\frac{1}{2}m(m-1)} \prod_{n>0} (1-q^n)^{-2}\, \xi^{(m)} \ .$$

On the other hand, when X is a torus $E_X^{(2)}$ can be identified with Det_X, for

$$\begin{aligned} E_X^{(2)} &\cong H^0(X;\Omega^{\otimes 2})^* \otimes H^1(X;\Omega^{\otimes 2}) \\ &\cong H^0(X;\Omega^{\otimes 2})^* \otimes H^0(X;\Omega^{\otimes(-1)})^* \\ &\cong H^0(X;\Omega^1)^* \\ &= \mathrm{Det}_X . \end{aligned}$$

(Here the second isomorphism comes from Serre duality, and the third from the product $\Omega^{\otimes 2} \otimes \Omega^{\otimes(-1)} \to \Omega^1$.) Under this identification $\xi_A^{(2)}$ corresponds to ξ_A.

Putting together the isomorphisms $E_X^{(2)} \cong \mathrm{Det}_X^{\otimes 13}$ and $E_X^{(2)} \cong \mathrm{Det}_X$ gives us $\mathrm{Det}_X^{\otimes 12} \cong \mathbb{C}$, by a map which takes

$$q^{-1} \Pi (1-q^n)^{-24} \xi_A^{12} = \eta(\tau)^{-24} \xi_A^{12}$$

to 1. This is precisely the statement that $\eta(\tau)^{24}$ is a modular form of weight 12.

It remains to prove (6.15), which, of course, includes (6.12). Using (6.4) we can identify $E_X^{(m)}$ with the determinant line of isomorphism

$$\pi_A : \Omega_{hol}^{\otimes m}(A) \to \Omega^{\otimes m}(S^1)$$

given by $\pi_A(f) = (\varphi_1^* f)_+ - (\varphi_0^* f)_-$, where $\varphi_0, \varphi_1 : S^1 \to A$ are the parametrizations of the ends of A. The operator π_A itself defines the element $\epsilon_A^{(m)}$, while $\xi_A^{(m)}$ is defined by the Fredholm operator $\tilde{\pi}$ such that $\tilde{\pi}(f) = \varphi_1^* f - \varphi_0^* f$, together with the obvious choice of isomorphism between its kernel and its cokernel. The ratio of $\xi_A^{(m)}$ to $\epsilon_A^{(m)}$ is therefore the determinant of $\tilde{\pi} \circ \pi_A^{-1}$ restricted to the subspace of $\Omega(S^1)$ spanned by all $z^k dz^{\,m}$ with $k \neq -m$. This operator is diagonal. It multiplies $z^k dz^{\,m}$ by $(1 - q^{k+m})$ if $k \geqslant 0$, and by $(1 - q^{-k-m})$ if $k < 0$. The determinant is therefore $(-1)^m q^{-\frac{1}{2}m(m-1)} \Pi (1-q^n)^2$, which proves (6.15).

§7. Spin structures: discrete coverings of $\mathcal{C}$

The categories $\mathcal{C}^{\mathrm{spin}}$ and $\mathcal{C}^{G}$

The kernel of the universal central extension of $\mathrm{Diff}^+(S^1)$ is $\mathbb{R} \oplus \mathbb{Z}$ [SΩ]. The coordinate functions on $\mathbb{R} \oplus \mathbb{Z}$, in the notation of (5.8) are (c,h). The determinant line accounts for the factor $\mathbb{R}$, while $\mathbb{Z}$ is the centre of the simply connected covering group of $\mathrm{Diff}^+(S^1)$, i.e. of the contractible group of diffeomorphisms $\varphi : \mathbb{R} \to \mathbb{R}$ satisfying $\varphi(\theta + 2\pi) = \varphi(\theta) + 2\pi$. Having discussed the determinant extension it is natural to ask about extensions connected with the fundamental group. The most important one is the spin covering.

A spin structure on a circle S is a real line bundle L on S together with an isomorphism $L \otimes L \cong TS$. There are two possible choices for L: trivial or Möbius. I shall write S_P and S_A for the respective pairs (S,L): the letters stand for "periodic" and "antiperiodic".

A spin structure on a Riemann surface X is a holomorphic line bundle L with an isomorphism $L \otimes L \cong TX$, where now TX denotes the complex tangent line bundle. If X has a boundary then a spin structure L on X induces one on the boundary, for the positively oriented tangent vectors to the boundary of X have square-roots which form a real line in $L|\partial X$. Every surface possesses a non-empty finite set of spin structures: they are acted on simply transitively by the group $H^1(X;\mathbb{Z}/2)$, for two spin structures on X differ by tensoring with a real line bundle.

The category $\mathcal{C}^{\mathrm{spin}}$ has objects $C_{n,n'}$ for $n,n' \geqslant 0$, where $C_{n,n'}$ is the union of n copies of S^1_A and n' of S^1_P. The morphisms $C_{m,m'} \to C_{n,n'}$ are Riemann surfaces (X,L) with a spin structure and a given isomorphism $(\partial X, L|\partial X) \cong C_{n,n'} - C_{m,m'}$. It is easy to see that there are no morphisms $C_{m,m'} \to C_{n,n'}$ unless $m' \equiv n' \pmod 2$, in which case the morphisms form a principal bundle over $\mathrm{Mor}_{\mathcal{C}}(C_{m+m'};C_{n+n'})$ with group $H^1(X,\partial X;\mathbb{Z}/2)$. In particular the identity component of the semigroup of endomorphisms of $C_{1,0}$ is a double covering of $\mathcal{A}$, and its Shilov boundary is a double covering of $\mathrm{Diff}^+(S^1)$.

There is no obvious generalization of $\mathcal{C}^{\mathrm{spin}}$. For any n one can construct an n-fold covering of the semigroup $\mathcal{A}$ by considering pairs (A,L) such that $L^{\otimes n} \cong TA$; but that does not work for general surfaces X because usually TX has no n^{th} root.

There is, nevertheless, an extension $\mathcal{C}^G$ of $\mathcal{C}$ associated to each finite group G. The objects of $\mathcal{C}^G$ are principal G-bundles over one-dimensional manifolds, and the morphisms are principal G-bundles over Riemann surfaces. Unlike $\mathcal{C}^{\mathrm{spin}}$, the extension $\mathcal{C}^G$ is <u>split</u>: $\mathcal{C}$ itself is a subcategory of $\mathcal{C}^G$.

The principal G-bundles over S^1 correspond to the conjugacy classes of elements of G: I shall write them S^1_g for $g \in G$. The endomorphisms of S^1_g in $\mathcal{C}^G$ which cover an annulus A correspond non-canonically to the elements of the centralizer Z_g of g in G. (Choose a path γ joining the base-points of the ends of A, and assign to a G-bundle on A the monodromy along γ.) More precisely, the endomorphisms of S^1_g over $\mathcal{A}$ form the extension of $\mathcal{A}$ by Z_g defined by the homomorphism $\mathbb{Z} = \pi_1(\mathcal{A}) \to Z_g$ which takes 1 to g.

<u>The associated modular functors</u>

The categories $\mathcal{C}^{\mathrm{spin}}$ and $\mathcal{C}^G$ give rise to modular functors in the sense of §5. Let us first consider $\mathcal{C}^G$. We shall denote the set of isomorphism classes of principal G-bundles on a 1-manifold S by $\mathcal{P}(S)$, and if P is a G-bundle on ∂X we shall denote by $\mathcal{P}(X;P)$ the set of isomorphism classes of pairs (Q,α), where Q is a G-bundle on X and α is an isomorphism $Q|\partial X \to P$. Thus if S is a union of components of ∂X the group $\mathrm{Aut}(P|S)$ of automorphisms of $P|S$ acts on $\mathcal{P}(X;P)$. The patching-together property of G-bundles is expressed as follows. If $X = X_1 \cup X_2$ is a union of surfaces attached along $S_0 = \partial X_1 \cap \partial X_2$, and $S_i = (\partial X_i) - S_0$, then

$$\mathcal{P}(X;P) = \coprod_{P_0 \in \mathcal{P}(S_0)} \mathcal{P}(X_1 ; P_1 \amalg P_0) \underset{\mathrm{Aut}(P_0)}{\times} \mathcal{P}(X_2 ; P_2 \amalg P_0) , \tag{7.1}$$

where $P_i = P|S_i$. There is a similar formula far attaching two edges of the same surface. We can now introduce a modular functor E whose set Φ of labels is the set of isomorphism classes of pairs (P,V), where $P \in \mathcal{P}(S^1)$ and V is a complex irreducible representation of Aut(P). For a surface X with boundary circles $S_1,\ldots,S_k$ labelled with (P_i,V_i) we define

$$E(X;P,V) = \mathrm{Map}_{\mathrm{Aut}(P)} (\mathcal{P}(X;P);V) , \tag{7.2}$$

where $P = P_1 \amalg \ldots \amalg P_k$ and $V = V_1 \otimes \ldots \otimes V_k$. The property (7.1) then translates into

$$E(X;P,V) = \bigoplus_{(P_0,V_0)} E(X_1;P_1 \amalg P_0,\ V_1 \otimes V_0^*) \otimes E(X_2;P_2 \amalg P_0,\ V_2 \otimes V_0) . \tag{7.3}$$

The Verlinde algebra A_G of this modular functor has arisen in another context in work of Lusztig [L]. Additively we have

$$A_G = \bigoplus_{[g]} R(C_g) ,$$

where $R(C_g)$ is the representation ring of the centralizer C_g of $g \in G$, and the sum is over the conjugacy classes of elements of G. Thus A_G can be identified with the equivariant K-group $K_G(G)$, where G acts on itself by conjugation. Analysis of (7.2) when X is a disc with two holes shows that the multiplication in A_G is given by the K_G direct image map induced by the multiplication $G \times G \to G$. (Thus if an element of $K_G(G)$ is regarded as a family of vector spaces $\{V_g\}_{g \in G}$ the product of $\{U_g\}$ and $\{V_g\}$ is $\{W_g\}$, where

$$W_g = \bigoplus_{g_1 g_2 = g} U_{g_1} \otimes V_{g_2} \quad .)$$

In particular, if G is abelian the ring A_G is the group ring of $G \times \hat{G}$, where $\hat{G} = \mathrm{Hom}(G, \mathbb{T})$.

To make everything as explicit as possible let us observe that when X is a torus E(X) is the vector space of functions on the set of conjugacy classes of pairs of commuting elements (g_1, g_2) of G. Constructing X from an annulus gives the standard isomorphism $A_G \otimes \mathbb{C} \to E(X)$ which maps $\chi \in R(C_g)$ to the function

$$(g_1, g_2) \longmapsto \begin{cases} \chi(g_2) & \text{if } g_1 = g \\ 0 & \text{if } g_1 \text{ is not conjugate to } g . \end{cases}$$

If A_G is regarded as a space of functions on a subset Γ of $G \times G$ by this map the multiplication is given by $(f_1, f_2) \mapsto f_1 * f_2$, where

$$(f_1 * f_2)(x,y) = \sum_{x_1 x_2 = x} f_1(x_1,y) f_2(x_2,y) \ .$$

The action of $SL_2(\mathbb{Z})$ in this description comes from its natural action on $\Gamma = \mathrm{Hom}(\mathbb{Z}^2;G)$, and one can easily check Verlinde's theorem (5.12).

Another aspect of the general theory which is easy to see in this example is the vector $\psi_Y \in E(X)$ associated to a 3-manifold Y such that $\partial Y = X$. For ψ_Y is simply the function on the set $\mathcal{P}(X)$ of classes of G-bundles on X whose value at a bundle P is given by

$$\psi_Y(P) = \sum_{(Q,\alpha) \in \mathcal{P}(Y,P)} \frac{1}{|\mathrm{Aut}(Q,\alpha)|} \ .$$

Little needs now to be said about the modular functor associated to $\mathcal{C}^{\,\mathrm{spin}}$. Instead of $\mathcal{P}(S)$ and $\mathcal{P}(X;P)$ we have the corresponding sets of spin structures $\mathcal{S}(S)$ and $\mathcal{S}(X;\sigma)$, where σ is a spin structure on ∂X. The group of automorphisms of σ is $H^0(\partial X;\mathbb{Z}/2)$. It acts on $\mathcal{S}(X;\sigma)$, and there is a sewing property just like (7.1). We obtain a modular functor with four labels $A^{\pm}$ and $P^{\pm}$, corresponding to the two spin structures on S^1 and the two representations of the group (± 1) of automorphisms.

The definition of a field theory based on $\mathcal{C}^{\,G}$ or $\mathcal{C}^{\,\mathrm{spin}}$ is clear. For $\mathcal{C}^{\,\mathrm{spin}}$, for example, we should have a projective functor $(S,\sigma) \mapsto H(S,\sigma)$ from oriented 1-manifolds with spin structure, and a ray in $H(S,\sigma)$ for each Riemann surface X with $\partial X = S$ equipped with an element of $\mathcal{S}(X,\sigma)$. I shall not repeat the conditions to be satisfied, but let us notice that $\mathrm{Aut}(\sigma)$ will act on $H(S,\sigma)$. If it were not for the projective nature of the functor we could say at once that such a theory defines a weakly conformal theory with respect to the modular functor described above. For each label (σ,V) for S we could define a space

$$H(S,(\sigma,V)) = \{H(S,\sigma) \otimes V\}^{\mathrm{Aut}(\sigma)} ,$$

and when $\partial X = S$ we should have an $\mathrm{Aut}(\sigma)$-equivariant map $\mathcal{S}(X,\sigma) \to H(S,\sigma)$ which would induce

$$\begin{aligned} E(X,(\sigma,V)) &= \mathrm{Map}_{\mathrm{Aut}(\sigma)}(\mathcal{S}(X,\sigma);V) \\ &= \{\mathbb{C}[\mathcal{S}(X,\sigma)] \otimes V\}^{\mathrm{Aut}(\sigma)} \\ &\to H(S,(\sigma,V)) . \end{aligned}$$

But not much needs to be altered to take the projectiveness into account. A central extension of $\mathcal{C}^{\mathrm{spin}}$ defines a line bundle L on $\mathcal{S}(X,\sigma)$ which is equivariant under $\mathrm{Aut}(\sigma)$, and we simply define a modified modular functor E for which $E(X,(\sigma,V))$ is the space of equivariant sections of $L \otimes V$.

The homological description of spin structures

Besides the direct geometrical description already given we shall need to make use of two other ways of characterizing spin structures. The first is cohomological. If γ is a smooth closed curve in a closed surface X then a spin structure L on X is either Möbius or trivial on γ. Define $\sigma_L(\gamma) = +1$ or -1 accordingly. It turns out that σ_L defines a quadratic form on $H_X = H_1(X;\mathbf{F}_2) \cong H^1(X,\mathbf{F}_2)$ which is associated with the cup-product (or intersection), i.e.

$$\sigma_L(\gamma_1 + \gamma_2) = \sigma_L(\gamma_1) + \sigma_L(\gamma_2) + \gamma_1\cdot\gamma_2 \ . \qquad (7.5)$$

Obviously the function σ_L describes the spin structure completely. This can alternatively be expressed by introducing the abelian group $\tilde{H}_X$ which is the extension of H_X by $\mathbf{F}_2$ got by using the cup-product as a cocycle. The formula (7.5) means that σ_L is a splitting of the extension $\tilde{H}_X \to H_X$. A theorem of Atiyah [A1] asserts

Proposition (7.6). Splittings of the extension $\tilde{H}_X$ correspond canonically to spin structures on X.

Remarks. The group $\tilde{H}_X$ is simply the group of units of the commutative ring $H^*(X;\mathbf{F}_2)$. Another description is that $\tilde{H}_X = \widetilde{KO}(X)$, and - because a spin structure is an orientation for KO-theory - the splitting associated to a spin structure is the corresponding Gysin map $\widetilde{KO}(X) \to KO^{-2}(\text{point}) = \mathbb{Z}/2$. Yet again, the set $\mathcal{S}_X$ of spin structures on X is an affine space of H_X, and there is a function $\sigma : \mathcal{S}_X \to \mathbf{F}_2$ which takes L to the parity of the dimension of the space of sections of L*. A choice of spin structure identifies $\mathcal{S}_X$ with H_X and makes σ into the quadratic form σ_L.

Proposition (7.6) can be generalized to surfaces with parametrized boundaries. (In fact all we shall use of the parametrization is a choice of base-point on each boundary circle; and all that is really needed is a choice of a double covering of the boundary.) If S is a 1-manifold with a base-point in each component, let S_0 be the set of base-points, and let $H_S = H_1(S,S_0;\mathbf{F}_2)$. Define $\tilde{H}_S = \mathbf{F}_2 \oplus H_S$. Then spin structures on S can be identified with splittings of $\tilde{H}_S \to H_S$. On the other hand a spin structure L on a surface X can be described by a function $\sigma_L : H_X \to \mathbf{F}_2$, where $H_X = H_1(X, \partial_0 X; \mathbf{F}_2)$ and $\partial_0 X$ is the set of base-points in ∂X. (Here a spin-structure on X means a choice of L together with an identification of the fibre L_x with a fixed choice for each $x \in \partial_0 X$.)

Proposition (7.7). For any surface X there is a canonical extension $\tilde{H}_X$ of H_X by $\mathbf{F}_2$, and a canonical homomorphism $\tilde{H}_{\partial X} \to \tilde{H}_X$, such that spin structures on X with a given restriction to ∂X correspond canonically to splittings of $\tilde{H}_X$ extending a given splitting of $\tilde{H}_{\partial X}$.

Proof: The main point is to define $\tilde{H}_X$. Suppose that ∂X has k components. Let Y be a standard sphere with k standard holes and a chosen tree y linking the base-points of its boundary components. Let $X^* = X \cup Y$. The inclusion $(X, \partial_0 X) \to (X^*, y)$ induces an isomorphism $H_X \to H_1(X^*, y) \cong H_{X^*}$. But we know how to define $\tilde{H}_{X^*}$ for the closed surface X^*, and it is now easy to deduce (7.7) from (7.6).

Spin structures and extensions of loop groups

The last way I shall mention of describing the spin structures on a surface X is in terms of the group $\mathbb{C}_X^\times$ of holomorphic maps $X \to \mathbb{C}^\times$.

Proposition (7.8). For every spin structure on ∂X (with an even number of periodic components) there is a central extension $\tilde{\mathbb{C}}_X^\times$ of $\mathbb{C}_X^\times$ by $\mathbf{F}_2$ such that splittings of $\tilde{\mathbb{C}}_X^\times$ correspond precisely to spin structures on X.

I shall give the proof in §12. (See also the end of §8.) At this point I shall make just two remarks.

(i) The group of components of $\mathbb{C}_X^\times$ is $H^1(X,\mathbb{Z})$, and if $\partial X = S^1$ then $\tilde{\mathbb{C}}_X^\times$ is induced from $\tilde{H}_X$ by the natural map $\mathbb{C}_X^\times \to H^1(X;\mathbf{F}_2) \cong H_X$. So the result is clear in this case.

(ii) Given a spin structure L on ∂X the group $\mathbb{C}_{\partial X}^\times$ of smooth maps $\partial X \to \mathbb{C}^\times$ acts on the polarized space $\Omega^{\frac{1}{2}}(\partial X;L)$ of $\frac{1}{2}$-forms on ∂X. This gives us a central extension $\tilde{\mathbb{C}}_{\partial X}^\times$ of $\mathbb{C}_{\partial X}^\times$ by $\mathbb{C}^\times$. The extension $\tilde{\mathbb{C}}_X^\times$ of (7.8) is a canonical subgroup of the restriction of $\tilde{\mathbb{C}}_{\partial X}^\times$ to $\mathbb{C}_X^\times$.

§8. The Grassmannian category: chiral fermions

Linear algebra

For clarity let us begin with finite dimensional vector spaces. A linear map $T : V_0 \to V_1$ is completely determined by its graph W_T, a subspace of $V_1 \oplus V_0$. It will be convenient in this section to define the graph of T by

$$W_T = \{(Tv,-v) : v \in V_0\} .$$

One may ask whether the category of finite dimensional vector spaces and linear maps is contained in a larger category in which the set of morphisms $V_0 \to V_1$ is the Grassmannian manifold $Gr(V_1 \oplus V_0)$ of <u>all</u> subspaces of $V_1 \oplus V_0$. If $W_{10} \in Tr(V_1 \oplus V_0)$, and $W_{21} \in Gr(V_2 \oplus V_1)$ one would try to define the composite $W_{21} * W_{10}$ by

$$W_{21} * W_{10} = \{(v_2,-v_0) \in V_2 \oplus V_0 : \exists\, v_1 \in V_1 \text{ such that } (v_2,-v_1) \in W_{21} \text{ and } (v_1,-v_0) \in W_{10}\}$$

$$= pr_{20}((W_{21} \oplus V_0) \cap (V_2 \oplus W_{10})) , \tag{8.1}$$

where $pr_{20} : V_2 \oplus V_1 \oplus V_0 \to V_2 \oplus V_0$ is the projection.

If $\dim(W_{10}) = n_0 + a$ and $\dim(W_{21}) = n_1 + b$, where $n_i = \dim(V_i)$, then generically $W_{21} * W_{10}$ has dimension $n_0 + a + b$. In fact, $\dim(W_{21} * W_{10}) = n_0 + a + b$ if the following conditions are satisfied:

$$\text{(i)} \quad W_{21} \oplus W_{10} \to V_1 \ \text{ is surjective, and}$$

$$\text{(ii)} \quad W_{21} \oplus W_{10} \to V_2 \oplus V_1 \oplus V_0 \ \text{ is injective.} \tag{8.2}$$

Where these conditions do not hold the composition law is obviously badly discontinuous.

But we can do better. A subspace W of $V_1 \oplus V_0$ defines a ray in the exterior algebra $\Lambda(V_1 \oplus V_0)$, and hence, up to a scalar multiple, a linear map $T_W : \Lambda(V_0) \to \Lambda(V_1)$. For

$$\begin{aligned} \Lambda(V_1 \oplus V_0) &\cong \Lambda(V_1) \otimes \Lambda(V_0) \\ &\cong \Lambda(V_1) \otimes \Lambda(V_0)^* \\ &\cong \mathrm{Hom}(\Lambda(V_0) \ ; \ \Lambda(V_1)) \ . \end{aligned} \tag{8.3}$$

The isomorphism $\Lambda^k(V_0) \to \Lambda^{n-k}((V_0)^*$ used in (8.3) depends on the choice of an element of $\det(V_0)^*$. Thus to get a specific map T_W we need not only W but also an element λ of $\det(W) \otimes \det(V_0)^* = \det(V_0;W)$; we shall therefore write it $T_{W,\lambda}$. One readily verifies

<u>Proposition (8.4)</u>. If $W_{10} \subset V_1 \oplus V_0$ and $W_{21} \subset V_2 \oplus V_1$ then

$$T_{W_{21},\mu} \circ T_{W_{10},\lambda} = T_{W_{21}*W_{10},\mu\otimes\lambda}$$

if the conditions (8.2) are satisfied, where $\mu \otimes \lambda$ refers to the isomorphism $\det(V_1;W_{21}) \otimes \det(V_0;W_{10}) \cong \det(V_0;W_{21}*W_{10})$ induced by the exact sequence

$$0 \to W_{21} * W_{10} \to W_{21} \oplus W_{10} \to V_1 \to 0 .$$

If the conditions (8.2) are not satisfied, then $T_{W_{21},\mu} \circ T_{W_{10},\lambda} = 0$.

<u>Corollary (8.5)</u>. (i) The category of finite dimensional vector spaces and linear maps is a subcategory of a category $\mathcal{V}$ which has the same objects, but in which a morphism from V_0 to V_1 is a pair (W,λ) with $W \in Gr(V_1 \oplus V_0)$ and $\lambda \in Det(V_0;W)$. (Here (W,λ) is regarded as independent of W if $\lambda = 0$, i.e. (W,λ) is really an element of $\Lambda(V_1 \oplus V_0) \otimes Det(V_0)^*$.)

(ii) The exterior algebra functor $V \mapsto \Lambda(V)$ extends to $\mathcal{V}$, though as a functor into vector spaces rather than algebras. The morphism associated to (W,λ) raises degrees by $\dim(W) - \dim(V_0)$.

An endomorphism T of $\Lambda(V)$ has a trace and, more importantly, a <u>supertrace</u> $tr_s(T) = \Sigma(-1)^p tr(T|\Lambda^p)$. It is elementary to check

<u>Proposition (8.6)</u>. If $(W,\lambda) : V \to V$ in $\mathcal{V}$ induces $T : \Lambda(V) \to \Lambda(V)$ then $tr_s(T)$ is the image of λ under

$$\Lambda(V \oplus V) \otimes Det(V)^* \to \Lambda(V) \otimes Det(V)^* \to \mathbb{C} ,$$

where the first map is induced by <u>subtraction</u> $V \oplus V \to V$.

<u>Remark</u>. If we used <u>addition</u> $V \oplus V \to V$ rather than subtraction we would obtain the <u>trace</u> instead of the supertrace.

More generally, a map $T : \Lambda(V_0 \oplus V) \to \Lambda(V_1 \oplus V)$ can be collapsed to a map $\check{T} : \Lambda(V_0) \to \Lambda(V_1)$ by taking the supertrace over $\Lambda(V)$.

Proposition (8.7). In this situation, if T is induced by (W,λ) in then $\check{T}$ is induced by $(\check{W},\check{\lambda})$, where

$$\check{W} = \{(v_1,-v_0) \in V_1 \oplus V_0 : (v_1,v,-v_0,-v) \in W \text{ for some } v \in V\} ,$$

and $\check{\lambda}$ is the image of λ under

$$\Lambda(V_1 \oplus V \oplus V_0 \oplus V) \otimes \mathrm{Det}(V_0 \oplus V)^* \cong \Lambda(V_1 \oplus V_0) \otimes \mathrm{Det}(V_0)^* \otimes \Lambda(V \oplus V) \otimes \mathrm{Det}(V)^*$$

$$\to \Lambda(V_1 \oplus V_0) \otimes \mathrm{Det}(V_0)^* .$$

Polarized vector spaces and Fock spaces

We are not really interested in finite dimensional vector spaces. Instead we want to consider the category of polarized topological vector spaces.

Definition (8.8). A polarization of a topological vector space E is a class of operators $J : E \to E$ such that $J^2 = 1$, any two differing by an operator of trace class.

Thus a polarized space E has a preferred class of decompositions $E = E^+ \oplus E^-$ into the ±1 eigenspaces of J. The typical example is the space $\Omega^0(S^1)$ of smooth functions on the circle, with E^+ and E^- spanned by $\{e^{in\theta}\}$ for $n < 0$ and $n \geqslant 0$ respectively.

A polarized space has a (restricted) Grassmannian Gr(E) which consists of the -1 eigenspaces E^- of all allowable J's. If W_0 and W_1 are two subspaces belonging to Gr(E) we can define a canonical relative determinant line $\mathrm{Det}(W_0;W_1)$. Holding W_0 fixed and letting W_1 vary we obtain a holomorphic line bundle Det_{W_0} on Gr(E). These bundles are all isomorphic, but not canonically: an isomorphism $\mathrm{Det}_{W_0} \to \mathrm{Det}_{W_1}$ is the same thing as an element of $\mathrm{Det}(W_0;W_1)$. For a discussion of all these facts I refer to Appendix B. Let us notice also that the line $\mathrm{Det}(W_0;W_1)$ is naturally a vector subspace of $\mathfrak{F}_{W_0}(E)$, i.e. the bundle Det_{W_0} on Gr(E) is a sub-bundle of the trivial bundle $\mathrm{Gr}(E) \times \mathfrak{F}_{W_0}(E)$.

We can now repeat the discussion in this section using the category of polarized topological vector spaces and systematically replacing the exterior algebra by its analogue for polarized spaces, which is the Fock space.

Definition (8.9). For $W \in \mathrm{Gr}(E)$ the Fock space $\mathfrak{F}_W(E)$ is the dual of the space of holomorphic sections of Det_W^*.

Thus the projective space of $\mathfrak{F}_W(E)$ is independent of W, and an isomorphism $\mathfrak{F}_{W_0}(E) \to \mathfrak{F}_{W_1}(E)$ is given by an element of $\mathrm{Det}(W_0;W_1)$. I recall from [PS] Chap. 10 that if $E = E^+ \oplus E^-$ is an allowable decomposition then we have a map

$$\Lambda((E^-)^* \oplus E^+) \to \mathfrak{F}_{E^-}(E)$$

which identifies the left-hand-side (interpreted algebraically) with a dense subspace of the Fock space.

The analogue of the finite dimensional isomorphism $\Lambda(V) \otimes \mathrm{Det}(V)^* \cong \Lambda(V)^*$ is a bilinear map

$$\mathfrak{F}_{E^+}(\tilde{E}) \times \mathfrak{F}_{E^-}(E) \to \mathbb{C}\ , \tag{8.10}$$

where $\tilde{E}$ denotes E with the reversed polarization-class, i.e. with J replaced by -J. (For the definition see Appendix B.)

We now have

Proposition (8.11). (i) There is a category $\mathcal{V}_{pol}$ whose objects are pairs (E,J) consisting of a polarized topological vector space and a choice of J, and whose morphisms $(E_0,J_0) \to (E_1,J_1)$ are pairs (W,λ) with $W \in Gr(\tilde{E}_0 \oplus E_1)$ and $\lambda \in Det(E_0^+ \oplus E_1^-;W)$.

(ii) The Fock space is a functor from $\mathcal{V}_{pol}$ to $\mathbb{Z}$-graded topological vector spaces and trace-class maps. It has exactly the same properties with respect to "sewing" and the supertrace as held in the finite dimensional case. In particular, a morphism (W,λ) raises dimension by the relative dimension $\dim(W : E_0^+ \oplus E_1^-)$.

The category $\mathcal{V}_{pol}$ is thus formally analogous to the category $\mathcal{C}$ made from circles and Riemann surfaces, and the Fock space functor is analogous to a field theory. To make the analogy complete we need the hermitian structure. If E has a hermitian inner product, and the polarization J is self-adjoint, then the Fock space $\mathfrak{F}(E)$ inherits an inner product. If E is positive-definite then so is $\mathfrak{F}(E)$. But there is another way to give $\mathfrak{F}(E)$ an inner product, using the canonical pairing (8.10). If E has a real structure, i.e. an operation of complex conjugation, which interchanges E^+ and E^-, then $\mathfrak{F}(\tilde{E}) \cong \overline{\mathfrak{F}(E)}$, and (8.10) becomes a hermitian form. If the conjugation exchanges E^+ and E^- only up to a finite dimensional discrepancy - more precisely, if it anticommutes with J modulo trace class operators - then the hermitian form is defined only up to a scalar multiple[1]. The inner product on $\mathfrak{F}(E)$ coming from a real structure on E is hyperbolic, i.e. as far as possible from being positive definite.

Chiral fermion theories

We can now give our first examples of conformal field theories. For any integer α one has the space $\Omega^{\alpha}(S^1)$ of differential forms of degree α on the circle - i.e. expressions of the form $f(\theta)(d\theta)^{\alpha}$. This is polarized by $\Omega^{\alpha} = \Omega^{\alpha}_{+} \oplus \Omega^{\alpha}_{-}$, where Ω^{α}_{+} (resp. Ω^{α}_{-}) is spanned by $e^{in\theta}(d\theta)^{\alpha}$ for $n < 0$ (resp. $n \geqslant 0$). The class of the polarization is independent of the parametrization ([PS] p.91), so $S \mapsto \Omega^{\alpha}(S)$ is a functor from oriented 1-manifolds to $\mathcal{V}_{\mathrm{pol}}$. Reversing the orientation reverses the polarization class, and so does complex conjugation in Ω^{α}. On the other hand for each Riemann surface X with boundary we have the space $\Omega^{\alpha}(X)$ of holomorphic α-forms $f(z)(dz)^{\alpha}$ on X, and (see [PS] (8.11.10)).

Proposition (8.12). The space $\Omega^{\alpha}(X)$ belongs to the restricted Grassmannian $Gr(\Omega^{\alpha}(\partial X))$. Its dimension relative to $\Omega^{\alpha}_{-}(\partial X)$ is

$$d_{\alpha}(X) = (2\alpha-1)(g+m-1) ,$$

where m is the number of outgoing boundary circles.

Here $\Omega^{\alpha}_{\pm}(\partial X)$ means the sum of a copy of $\Omega^{\alpha}_{+}(S^1)$ (resp. $\Omega^{\alpha}_{-}(S^1)$) for each incoming (resp. outgoing) circle.

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[1] (*) A better way to say this is: there is a hermitian form on $\mathfrak{F}_{E^-}(E)$ with values in $\mathrm{Det}(\bar{E}^-;E^+)$.

<u>Corollary (8.13)</u>. Ω^α defines a functor from $\mathcal{C}$ to $\mathcal{V}_{pol}$, and so $S \mapsto \mathfrak{F}(\Omega^\alpha(S))$ is a holomorphic conformal field theory. A surface X defines an operator which raises degrees by $d_\alpha(X)$.

The spaces Ω^α and $\Omega^{1-\alpha}$ are in duality, but the duality reverses the polarization, so the Fock spaces $\mathfrak{F}(\Omega^\alpha)$ and $\mathfrak{F}(\Omega^{1-\alpha})$ - and with them the field theories - are identical. By calculating the Lie algebra cocycle one finds that the theory $\mathfrak{F}(\Omega^\alpha)$ has central charge $c = 12\alpha(1-\alpha)-2$. A physicist does this calculation in the following way.

Let L_p denote $e^{-ip\theta}d/d\theta$ in $\mathrm{Vect}_{\mathbb{C}}(S^1)$, and let $\psi_q = e^{-iq\theta}(id\theta)^\alpha$ in Ω^α. We have

$$L_p \cdot \psi_q = -i(q + \alpha p)\psi_{p+q} \ . \tag{8.14}$$

Let $\omega = \psi_m \wedge \psi_{m-1} \wedge \psi_{m-2} \wedge \ldots$ in $\mathfrak{F}(\Omega^\alpha)$. Then

$$L_p\omega = -i \sum_{k=0}^{p-1} (m - k + \alpha p)\omega_k$$

if $p > 0$, where ω_k is obtained from ω by replacing ψ_{m-k} by ψ_{m-k+p}. If $p < 0$ then $L_p\omega = 0$. Hence

$$[L_{-p},L_p]\omega = \{(\alpha(1-\alpha) - 1/6)p^3 + (m(m+1) + 1/6)p\}\omega \ .$$

Comparing this with (5.6) we find

$$c = 12\alpha(1-\alpha) - 2 \quad \text{and} \quad h = \tfrac{1}{2}\alpha(1-\alpha) + \tfrac{1}{2}m(1+m) \ . \tag{8.15}$$

The most interesting of the theories $\mathfrak{F}(\Omega^{\alpha})$ is the one with $\alpha = -1$ or $\alpha = 2$, i.e. when $\Omega^{\alpha}(S^1)$ is the Lie algebra $\mathrm{Vect}(S^1)$ or its dual. This has $c = -26$. It is the theory of BRS ghosts which was mentioned at the end of §4. The grading of $\mathfrak{F}(\Omega^2(S))$ is the ghost number, and $d_2(X) = 3(g+m-1)$ is the ghost number anomaly. We shall return to this theory in §9.

If α is an integer the space $\Omega^{\alpha}(S^1)$ has no inner product. But if $\alpha = \frac{1}{2}$ it has a natural positive definite inner product, and we can use this instead of the real structure to define an inner product on $\mathfrak{F}(\Omega^{\frac{1}{2}})$. A new point arises, however, because to define $\Omega^{\frac{1}{2}}(S^1)$, and still more to define $\Omega^{\frac{1}{2}}(X)$ for a surface X, one must choose a square-root L of the tangent bundle TS^1 or TX, i.e. a spin structure. (See §7.) Thus we have

Proposition (8.16). $\Omega^{\frac{1}{2}}$ defines a functor from $\mathcal{C}^{\mathrm{spin}}$ to $\mathcal{V}_{\mathrm{pol}}$, and so $(S,L) \mapsto \mathfrak{F}(\Omega^{\frac{1}{2}}(S,L))$ is a positive-definite projective representation of $\mathcal{C}^{\mathrm{spin}}$, i.e. a weakly conformal field theory (cf. (5.2)), with central charge $c = 1$.

This theory is called the charged chiral fermion. It is the basic example of the structure we are studying. For surfaces X with spin structures which are Mobius (i.e. antiperiodic) on each boundary circle the dimension of $\Omega^{\frac{1}{2}}(X)$ relative to $\Omega^{\frac{1}{2}}_{+}(\partial X)$ is zero, and the associated operator preserves the grading. For the spin structure S^1_P, as there is no vacuum vector in $\mathfrak{F}(\Omega^{\frac{1}{2}}(S^1_P))$, it is not obvious how to grade the Fock space. The correct procedure is to grade it by $\mathbb{Z} + \frac{1}{2}$, so that $\psi_0 \wedge \psi_{-1} \wedge \psi_{-2} \wedge \ldots$ has degree $+\frac{1}{2}$. The operators of the theory then preserve the grading in all cases. From the formulae (8.15) we find that $\mathrm{Diff}^{+}(S^1)$ acts on $\mathfrak{F}(\Omega^{\frac{1}{2}}(S^1_A))$ with $(c,h) = (1,0)$, and on $\mathfrak{F}(\Omega^{\frac{1}{2}}(S^1_P))$ with $(c,h) = (1,1/8)$.

Field operators

We have seen that when X is a surface with $\partial X = \overline{S}_0 \amalg S_1$ the space $\Omega^\alpha(X)$, together with a point in its determinant line, defines an operator

$$U_X : \mathfrak{F}(\Omega^\alpha(S_0)) \to \mathfrak{F}(\Omega^\alpha(S_1)) .$$

If $\zeta = \Sigma\, n_i[z_i]$ is a divisor on X, i.e. a set $\{z_1,\ldots,z_k\}$ of points in the interior of X equipped with integral multiplicities $\{n_1,\ldots,n_k\}$, then it is natural to consider the space of holomorphic α-forms on X which vanish to order $-n_i$ at z_i. (If n_i is positive this means that the form is allowed to have a pole of order n_i at z_i). This space will be denoted $\Omega^\alpha(X;\zeta)$. Because it belongs to $Gr(\Omega^\alpha(\partial X))$ it too defines an operator $\mathfrak{F}(\Omega^\alpha(S_0)) \to \mathfrak{F}(\Omega^\alpha(S_1))$, at least up to a scalar multiple, which raises degree by $d_\alpha(X) + \Sigma\, n_i$. To get a precise operator we must choose an element of

$$\mathrm{Det}^\alpha(\zeta) = \mathrm{Det}(\Omega^\alpha(X);\Omega^\alpha(X;\zeta)) .$$

(I shall assume that an element of $\mathrm{Det}(\Omega^\alpha_+(\partial X);\Omega^\alpha(X))$ has already been chosen, and is kept fixed in what follows.) Now

$$\mathrm{Det}^\alpha(\zeta) = \bigotimes_i \mathrm{Det}(P^\alpha_{n_i}(z_i)) , \qquad (8.17)$$

where $P^\alpha_n(z)$ denotes the space of principal parts at z of meromorphic α-forms with n^{th} order poles if $n > 0$, while if $n < 0$ then it denotes the dual of the space of $(-n-1)$-jets of holomorphic α-forms at z. Thus

$$\mathrm{Det}(P_n^\alpha(z)) = (T_zX)^{\otimes(-\alpha n+\frac{1}{2}n(n+1))}$$

for all n. This means that as the divisor ζ varies we get not an operator-valued function on the complex manifold

$$\{(z_1,\ldots,z_k) \in X^k \ : \ z_i \neq z_j\}$$

but a holomorphic operator-valued form of multidegree $(d_1,\ldots,d_k)$, where $d_i = -\alpha n_i + \frac{1}{2}n_i(n_i + 1)$. We write this operator

$$\psi(\zeta)d\zeta^\alpha = \psi^{(n_1)}(z_1)\psi^{(n_2)}(z_2) \ \ldots \ \psi^{(n_k)}(z_k) \ . \ \Pi \ dz_i^{\otimes d_i} \ . \qquad (8.18)$$

Despite the notation it should not be regarded as a composition of operators associated to the different points z_i. It depends in an alternating way on the order of the points, providing z_i is assigned degree n_i: that follows from the graded nature of the isomorphism (8.17).

We have been assuming that the points z_i are distinct. As a point of the Grassmannian $\mathrm{Gr}(\Omega^\alpha(\partial X))$ the space $\Omega^\alpha(X;\zeta)$ behaves smoothly when the points come together (and their multiplicities are added appropriately), and so does the line $\mathrm{Det}^\alpha(\zeta)$. The isomorphism (8.17), however, breaks down and must be reconsidered. As an illustration let us consider the case of a divisor $\zeta = [z_2]-[z_1]$ contained in the annulus

$$X = X_{ab} = \{z \in \mathbb{C} : a \leqslant |z| \leqslant b\} \ .$$

For simplicity we shall assume that $\alpha = 0$. We can define a reference element of $\mathrm{Det}(\Omega^0_+(\partial X);\Omega^0(X))$ by means of the basis $\{f_k\}_{k\in}$ of $\Omega^0(X)$, where

$$f_k(z) = (z/b)^k \quad \text{for} \quad k \geqslant 0$$
$$= (z/a)^k \quad \text{for} \quad k < 0 \ .$$

A good basis for $\mathrm{Det}^0(\zeta)$ is then represented by the basis $\{\varphi_\zeta f_k\}_{k\in\mathbb{Z}}$ of $\Omega^0(X;\zeta)$, where $\varphi_\zeta(z) = (z-z_1)/(z-z_2)$. This behaves smoothly when $z_1 = z_2$. On the other hand, in terms of the isomorphism (8.17) the natural basis element $(dz_1)^0(dz_2)^{-1}$ of the right-hand side corresponds to the element of $\mathrm{Det}^0(\zeta)$ represented by the basis $\{g_k\}$ of $\Omega^0(X;\zeta)$, where

$$g_k(\zeta) = (z-z_2)^{-1} - (z_1-z_2)^{-1} \quad \text{if} \quad k = 0$$
$$= (z^k-z_1^k)/b^k \quad \text{if} \quad k > 0$$
$$= (z^k-z_1^k)/a^k \quad \text{if} \quad k < 0 \ .$$

It is easy to check that the determinant of $\{\varphi_\zeta f_k\}$ with respect to $\{g_k\}$ is z_1-z_2, and so we have

Proposition (8.19). The operator-valued form

$$\psi^*(z_1)\psi(z_2)(z_1-z_2)dz_2 \ : \ \mathfrak{F}(\Omega^0(S^1)) \to \mathfrak{F}(\Omega^0(S^1))$$

on the annulus X is holomorphic everywhere, and its value when $z_1 = z_2$ is simply the operator U_X associated to the annulus.

Here we have written, as is usual, $\psi(z)$ for $\psi^{(1)}(z)$, and $\psi^*(z)$ for $\psi^{(-1)}(z)$. The proposition is usually stated as an "operator product expansion"[1]

$$\psi^*(z_1)\psi(z_2) = \frac{1}{z_1 - z_2} + (\text{holomorphic}) .$$

It remains true when Ω^0 is replaced by Ω^α for any α, except that dz_2 becomes $(dz_1)^\alpha (dz_2)^{1-\alpha}$.

A similar, but easier, calculation can be performed for the positive divisor $\zeta = [z_1] + [z_2] + \ldots + [z_n]$ on the same annulus X. The operator $\psi(z_1)\psi(z_2) \ldots \psi(z_n)$ corresponds to the basis for $\Omega^\alpha(X;\zeta)$ which consists of $(z-z_i)^{-1}$ for $i = 1,\ldots,n$ together with $\{f_k\}_{k\in\mathbb{Z}}$ as above. On the other hand a basis for $\Omega^\alpha(X;\zeta)$ which is everywhere defined is given by $z^k h$ for $k = 0,1,\ldots,n-1$ together with the $\{f_k\}$, where $h = \Pi(z-z_i)^{-1}$. The determinant of the first basis in terms of the second is a Vandermonde determinant equal to $\prod_{i<j} (z_i - z_j)$. Thus we have

<u>Proposition (8.20)</u>. The operator $\psi(z_1)\psi(z_2) \ldots \psi(z_n)/\Pi(z_i - z_j)$ is holomorphic for all $z_1,\ldots,z_n$. Its value when $z_1 = z_2 = \ldots = z_n = z$ is usually denoted

$$C \ : \psi(z)\psi'(z)\psi''(z) \ \ldots \ \psi^{(n-1)}(z) \ : ,$$

where $C^{-1} = 1!2!3! \ldots (n-1)!$.

----

[1]For the omission of U_X from the following notation, see the remarks after (9.3).

It is an instructive exercise to translate the field operators just defined into their more usual description. The Fock space $\mathfrak{F}(\Omega^{\alpha}(S^1))$ is a module over the exterior algebra of $\Omega^{\alpha}(S^1)$. As above we write ψ_k for the basis element $z^{-k-\alpha}(dz)^{\alpha}$ of $\Omega^{\alpha}(S^1)$, and also for the corresponding multiplication operator on the Fock space. The formal series

$$\hat{\psi}(w) = \sum_{k\in\mathbb{Z}} w^{k+\alpha-1}\,\psi_k$$

does not define an operator in $\mathfrak{F}(\Omega^{0}(S^1))$, because it is unbounded and may not converge. Nevertheless, let U_{ab} denote the endomorphism of $\mathfrak{F}(\Omega^{\alpha}(S^1))$ defined by the annulus X_{ab}. This is a contraction operator which depends only on a/b, and satisfies $U_{ab}\psi_k = (a/b)^k\psi_k U_{ab}$. It is easy to see that if $a < |w| < b$ and $a \leqslant 1 \leqslant b$ the composite $U_{1b}\hat{\psi}(w)U_{a1}$ is a well-defined operator in $\mathfrak{F}(\Omega^{\alpha}(S^1))$. We have

<u>Proposition (8.21)</u>. The operator-valued $(1-\alpha)$-form $\psi(w)dw^{1-\alpha}$ on the annulus X_{ab} is given by

$$\psi(w) = U_{1b}\hat{\psi}(w)U_{a1} \ .$$

<u>Proof</u>: This is simply a matter of unravelling the definitions. The Fock space $\mathfrak{F} = \mathfrak{F}(\Omega^{\alpha}(S^1))$ has a natural basis $\{\omega_S\}$, where S runs through an appropriate class of subsequences of $\mathbb{Z}$. (Cf. [PS] Chap. 10.) The dual space $\mathfrak{F}^*$ has a dual basis $\{\omega_{\bar{S}}\}$, where $\bar{S} = \mathbb{Z}-S$. The operator U_{ab} multiplies ω_S by $(a/b)^{\ell(S)}$, and as an element of $\mathfrak{F}^* \otimes \mathfrak{F}$ we can write U_{ab} in the form

$$U_{ab} = \sum_S (a/b)^{\ell(S)}\ \omega_{\bar{S}} \otimes \omega_S \ . \qquad (8.22)$$

Now $\Omega^{\alpha}(X_{ab};[w])$ is spanned by $(z-w)^{-1}dz^{\alpha}$ modulo $\Omega^{\alpha}(X_{ab})$. When this form is restricted to the ends of the annulus it becomes

$$\varphi = w^{-1} \sum_{k<0} b^{k+\alpha} w^{-k} \psi_{-k-\alpha}$$

on one end, and

$$\theta = -w^{-1} \sum_{k \geqslant 0} a^{k+\alpha} w^{-k} \psi_{-k-\alpha}$$

on the other. The operator $\psi(w)$, as an element of $\mathfrak{F}^{*} \otimes \mathfrak{F}$, is therefore given by multiplying the expression (8.22) by $1 \otimes \varphi - \theta \otimes 1$, and this amounts to the assertion of (8.21).

The bosonic description of $\mathfrak{F}(\Omega^{\alpha})$

Each of the theories $\mathfrak{F}(\Omega^{\alpha})$ has an alternative "bosonic" description. We shall explain this very briefly now, and shall return to it in §9 and §12. For an oriented 1-manifold S the group $\mathbb{C}_S^{\times}$ of smooth maps $S \to \mathbb{C}^{\times}$ acts on $\Omega^{\alpha}(S)$ by multiplication, and preserves the polarization class ([PS](6.3.1)). It therefore acts projectively on $\mathfrak{F}(\Omega^{\alpha}(S))$. To give a bosonic description of the theory $\mathfrak{F}(\Omega^{\alpha})$ means, in one interpretation, to construct it purely in terms of the representation theory of the groups $\mathbb{C}_S^{\times}$ and $\mathbb{C}_X^{\times}$, the group of holomorphic maps from a surface X to $\mathbb{C}^{\times}$.

Of course all the spaces $\Omega^{\alpha}(S)$ and $\Omega^{\frac{1}{2}}(S_{\sigma})$, with $\alpha \in \mathbb{Z}$ and σ a spin structure, are isomorphic as representations of $\mathbb{C}_S^{\times}$, and $\mathfrak{F}(\Omega^{\alpha}(S^1))$ is the basic irreducible representation described in [PS] Chap. 10. Nevertheless one must beware of identifying the $\mathfrak{F}(\Omega^{\alpha}(S))$ for different α, even as projective spaces, as the isomorphisms involve a choice of parameter on S. (At first sight this seems to contradict Schur's lemma, but that lemma does not apply to projective representations.) Concretely, one has a fixed representation H of $\mathbb{C}_S^{\times}$, but a different action of $\mathrm{Diff}^{+}(S^1)$ on H for each α. These actions are described on page 208 of [PS].

The bosonic description begins by prescribing a definite projective multiplier on $\mathbb{C}_S^\times$, or equivalently an extension $\tilde{\mathbb{C}}_S^\times$ of $\mathbb{C}_S^\times$ by $\mathbb{C}^\times$ with a definite action of $\mathrm{Diff}^+(S^1)$ on it. Then $F_S = \mathfrak{F}(\Omega^\alpha(S))$ is constructed as the unique irreducible representation (of positive energy) of $\tilde{\mathbb{C}}_S^\times$. To complete the description one has only to give the ray F_X in $F_{\partial X}$ corresponding to each surface X. From the Fock space description we know that the ray is invariant under the subgroup $\mathbb{C}_X^\times$ of $\mathbb{C}_{\partial X}^\times$, so it defines a homomorphism $\tilde{\mathbb{C}}_X^\times \to \mathbb{C}^\times$, i.e. a splitting of the induced central extension of $\mathbb{C}_X^\times$. From the bosonic point of view one must give the splitting of the extension of $\mathbb{C}_X^\times$ directly. Then $\mathbb{C}_X^\times$ acts on $F_{\partial X}$. One proves that there is a unique ray in $F_{\partial X}$ which is pointwise fixed under $\mathbb{C}_X^\times$, and calls it F_X. This programme will be carried out in §12. In §9 I shall try to explain the relation of the representation theory to "bosonic fields".

Even spin structures and the real chiral fermion

There is an important variant of the linear algebra of this section. In finite dimensions the exterior algebra functor $V \mapsto \Lambda(V)$ has an analogue which takes a vector space V with an inner product to the spin representation of the orthogonal group O(V). To be precise, let us consider real vector spaces V with non-degenerate quadratic forms, not necessarily positive-definite. The spin representation of O(V) is a mod 2 graded complex projective irreducible representation Δ on which the orientation-reversing elements of O(V) act with degree 1. Alternatively, Δ is an irreducible graded module for the complexified Clifford algebra C(V). (See [ABS].) If dim(V) is odd then Δ is uniquely determined up to isomorphism, but if dim(V) is even there are two possibilities (which differ by reversing the grading), and a choice of Δ corresponds to choosing an orientation of V.

The spin representation is best understood in terms of the <u>isotropic Grassmannian</u> $\mathcal{J}(V)$ of all maximal isotropic subspaces of $V_{\mathbb{C}}$. (Cf. [PS] Chap. 12.) If dim(V) is odd then $\mathcal{J}(V)$ is connected, but if dim(V) is even $\mathcal{J}(V)$ has two connected components (for $W \in \mathcal{J}(V)$ defines a complex structure, and hence an orientation, on V). There is a holomorphic line bundle Pf on $\mathcal{J}(V)$ - the <u>Pfaffian</u> bundle - and in the even dimensional case the spin representation Δ is the space of holomorphic sections of Pf*, and can be graded in two ways. In particular, each $W \in \mathcal{J}(V)$ defines a ray Pf_W in W, which can also be characterized by the fact that it is annihilated by the subspace W of the Clifford algebra C(V). Indeed one can identify Δ with $Pf_W \otimes \Lambda(W^*)$, because $C(V)/\Lambda(W) \cong \Lambda(W^*)$. When dim(V) is odd, however, Δ is the sum of two copies of $\Gamma(Pf^*)$.

Let us also recall from [ABS] that if Δ_i is an irreducible $C(V_i)$-module for $i = 1,2$ then $\Delta_1 \otimes \Delta_2$ is an irreducible $C(V_1 \oplus V_2)$-module unless both V_1 and V_2 are odd dimensional, in wh ich case $\Delta_1 \otimes \Delta_2$ is the sum of the two distinct irreducible $C(V_1 \oplus V_2)$-modules.

We can now define a category $\mathcal{V}^{\text{orth}}$ whose objects are pairs (V,Δ), where Δ is an irreducible C(V)-module. A morphism $(V_0,\Delta_0) \to (V_1,\Delta_1)$ is a pair (W,λ), where W is a maximal isotropic subspace of $\bar{V}_0 \oplus V_1$ and $\lambda \in (\Delta_0^* \otimes \Delta_1)^{\text{even}}$ is annihilated by W. (Here $\bar{V}_0$ denotes V_0 with its quadratic form multiplied by -1.)

A morphism $(V_0,\Delta_0) \to (V_1,\Delta_1)$ in $\mathcal{V}^{\text{orth}}$ defines a linear map $\Delta_0 \to \Delta_1$ of degree 0, and the group of automorphisms of (V,Δ) is the complexification of $\text{Spin}^c(V)$. A general morphism $(V_0,\Delta_0) \to (V_1,\Delta_1)$ corresponds to a choice of isotropic subspaces P_0 and P_1 in $V_{0,\mathbb{C}}$ and $V_{1,\mathbb{C}}$ together with an isometry $P_0^{\perp}/P_0 \to P_1^{\perp}/P_1$.

Now let us turn to polarized infinite dimensional real vector spaces with quadratic forms. In this situation a <u>polarization</u> of V means a class of skew transformations $J : V \to V$ such that $J^2 = -1$ modulo trace-class operators, two members of the class differing as usual by trace-class operators. (Cf. (10.3).) The theory of irreducible modules for the Clifford algebra $C(V)$ proceeds just as in finite dimensions. There are two cases, according as dim(ker J) is even or odd. We shall refer to V as even or odd correspondingly. In either case there is an isotropic Grassmannian $\mathcal{J}(V)$ consisting of maximal isotropic subspaces W of $V_{\mathbb{C}}$ which belong to the polarization class (i.e. which are the (+i)-eigenspaces of allowable polarization operators J). It is connected if V is odd, and has two components if V is even. There are respectively one or two irreducible graded modules for C(V), and we can define a category $\mathcal{V}^{orth}_{pol}$ analogous to $\mathcal{V}^{orth}$.

Finally we come to Riemann surfaces and their boundary circles. We shall define a weakly conformal unitary field theory based on a modular functor with three labels which I shall call the <u>even spin</u> functor. The theory itself is called the <u>real chiral fermion</u>. It has central charge $c = \frac{1}{2}$, and is the simplest example of a theory with non-integral c.

For an oriented circle S with a spin structure L the space $\Omega^{\frac{1}{2}}(S;L)$ of sections of L* belongs to the category $\mathcal{V}^{orth}_{pol}$. It is even or odd according as L is Möbius or trivial. A label for the circle S consists of L together with an irreducible graded module for the Clifford algebra of $\Omega^{\frac{1}{2}}(S;L)$. To an oriented 1-manifold $S = S_1 \amalg \ldots \amalg S_k$, where the circle S_i is labelled (L_i, Δ_i), the theory assigns the Hilbert space $\Delta = \Delta_1 \otimes \ldots \otimes \Delta_k$. For a surface X with $\partial X = S$ we consider all spin structures L on X which reduce to $L = \amalg L_i$ on S. We define the modular functor by

$$E(X;L,\Delta) = \bigoplus_L E(X;L) \subset \Delta ,$$

where $E(X;L)$ is the subspace of the even part of Δ which is annihilated by $\Gamma(L)$, i.e. by the boundary values of holomorphic sections of L. If all of the L_i are Möbius then Δ is an irreducible representation of the Clifford algebra of $\Omega^{\frac{1}{2}}(S;L)$, and the maximal isotropic subspace $\Gamma(L)$ annihilates a unique ray in Δ. The spin structure L is called <u>even</u> or <u>odd</u> relative to Δ according as this ray belongs to the even or odd part of Δ. Thus when L is purely Möbius we have

$$\dim E(X;L,\Delta) = |\{\text{even spin structures on } X \text{ relative to } \Delta\}| .$$

If on the other hand L has 2q non-Möbius components then

$$\begin{aligned}\dim E(X;L,\Delta) &= 2^{q-1}|\{\text{spin structures on } X\}| \\ &= 2^{2g+q-1} ,\end{aligned}$$

where g is the genus of X. (In these statements the set of spin structures on X means not the set $\mathfrak{S}(X;L)$ of §7, but rather its quotient by Aut(L).)

In the Möbius case we observe that $E(X;L)$ is the Pfaffian line of L, which is the square-root of the determinant line of the $\bar{\partial}$-operator of L. This explains why the theory has $c = \frac{1}{2}$.

The Verlinde algebra generated by the three labels $A^{\pm}$,P is described by the multiplication rules

$$\begin{aligned} A^+ &= 1 \\ (A^-)^2 &= 1 \\ P^2 &= A^+ + A^- \\ A^{\pm}P &= P . \end{aligned}$$

§9. Field operators

Primary fields

We shall now describe how to reconstruct some of the usual formalism of quantum field theory from a functor of the type we are studying. Thus we begin from a vector space H, and have an operator $U_{X,\xi} : H^{\otimes m} \to H^{\otimes n}$ when X is a Riemann surface with $\partial X = C_n - C_m$, and $\xi \in \mathrm{Det}_X$. We suppose that

$$U_{X,\lambda\xi} = \bar{\lambda}^{-\frac{1}{2}c_L} \lambda^{-\frac{1}{2}c_R} U_{X,\xi} ,$$

whhere (c_L, c_R) is the central charge.[1]

First, the morphism $C_0 \to C_1$ defined by the standard unit disc D and the canonical element $\epsilon_D \in \mathrm{Det}_D$ gives us a map $\mathbb{C} \to H$, or equivalently a vector $\Omega \in H$. This is the vacuum vector.

The complex semigroup $\mathcal{A}$ acts projectively on H. The semigroup $\mathcal{E}_0$ of holomorphic embeddings $f : D \to \mathring{D}$ such that $f(0) = 0$ is a sub-semigroup of $\mathcal{A}$ over which the central extension is canonically split, for the standard element $\epsilon_f \in \mathrm{Det}_f$ is characterized by $\epsilon_D \epsilon_f = \epsilon_D$. It therefore makes sense to look for eigenvectors of $\mathcal{E}_0$ in H. The possible homomorphisms $\mathcal{E}_0 \to \mathbb{C}^\times$ are given by $f \mapsto f'(0)^p \overline{f'(0)}^q$ with $p,q \in \mathbb{R}$ such that $p-q \in \mathbb{Z}$.

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[1]The discussion applies with little change to a weakly conformal theory based on a modular functor with index set Φ rather than on the determinant line. The vacuum vector and the energy-momentum tensor lie in the space H_1 corresponding to $1 \in \Phi$.

Definition (9.1). A primary field of type (p,q) is a vector $\psi \in H$ such that

$$U(f)\psi = f'(0)^p \overline{f'(0)}^q \psi$$

for all $f \in \mathcal{E}_0$.

The reason for the terminology is the following. Suppose that X is a Riemann surface with m incoming and n outgoing boundary circles. We suppose an element $\xi \in \mathrm{Det}_X$ has been chosen, but we shall suppress it from the notation. Then for each primary field ψ of type (p,q) there is an operator-valued (p,q) form $\psi_X(z)dz^p d\bar{z}^q$ on X with values in the trace-class operators $H^{\otimes m} \to H^{\otimes n}$. To define ψ_X, choose a holomorphic embedding $f : D \to X$ with centre $z \in X$. Because $X_z = X - f(\mathring{D})$ is a morphism $C_{m+1} \to C_n$ it induces $U_{X_z} : H^{\otimes m} \otimes H \to H^{\otimes n}$, and $\psi_X(z)$ is defined by $\psi_X(z).\xi = U_{X_z}(\xi \otimes \psi)$. The condition of (9.1) implies that $\psi_X(z)dz^p d\bar{z}^q$ is a well-defined differential form on the interior of X. (It probably always has a distributional extension to the closed surface.)

One can also define multipoint fields. If $\psi_1,\ldots,\psi_k$ are primary fields of types (p_i,q_i) then there is a differential form $(\psi_1 \ldots \psi_k)_X$, usually written

$$\psi_1(z_1) \ldots \psi_k(z_k) dz_1^{p_1} \ldots dz_k^{p_k} d\bar{z}_1^{q_1} \ldots d\bar{z}_k^{q_k} ,$$

defined on the manifold

$$\{ (z_1,\ldots,z_k) \in \mathring{X}^k \ : \ z_i \neq z_j \quad \text{if } i \neq j \}$$

with values in the operators $H^{\otimes m} \to H^{\otimes n}$. It is defined by

$$(\psi_1 \ \ldots \ \psi_k)_X . \xi = U_{X'} (\xi \otimes \psi_1 \otimes \ldots \otimes \psi_k) \ ,$$

where now X' is obtained from X by removing k discs around the points $z_1, \ldots, z_k$. If $\psi_1 = \ldots = \psi_k$ then the operator is symmetric with respect to permuting $z_1, \ldots, z_k$ (or skew-symmetric if the theory is mod 2 graded).

By their construction the operator-valued differential forms ψ_X have the following naturality property, which is usually referred to as the Ward identity. If $Y = Z_1 \circ X \circ Z_2$ is a union of surfaces then

$$\psi_Y | X = U_{Z_1} \circ \psi_X \circ U_{Z_0} \ . \tag{9.2}$$

We also have, for example,

$$(\psi_1 \ \ldots \ \psi_k)_X \circ (\psi_{k+1} \ \ldots \ \psi_m)_Y = (\psi_1 \ \ldots \ \psi_m)_{X \circ Y}$$

when the surfaces X and Y are composable.

One can go on and define secondary, tertiary,... fields: the bigraded space H is filtered $H_0 \subset H_1 \subset H_2 \subset \ldots \subset H$, where H_0 is the primary fields, and $\mathcal{E}_0$ acts scalarly on H_k/H_{k-1} for each k. In fact any vector $\psi \in H$ gives rise to an operator-valued function on any surface; but when we use the notation $\psi_X(z)$ we must beware that if ψ is an r-ary field then $\psi_X(z)$ depends not only on z but also on the r^{th} order jet of a local coordinate at z. Thus for any surface X and for any $\psi \in H$ the formulae

$$i \frac{d}{dz} \psi_X(z) = (L_1\psi)_X(z) \ , \qquad i \frac{d}{d\bar{z}} \psi_X(z) = (\bar{L}_1\psi)(z) \ ,$$

make sense and are valid, where L_1 and $\bar{L}_1$ are the usual Virasoro operators on H representing the vector field $id/dz = e^{-i\theta}\, d/d\theta$ in the left- and right-hand actions of $Vect(S^1)$ on H. These formulae can be regarded as an infinitesimal version of (9.2). They hold because

$$idU_{X_z} = U_{X_z} \circ (L_1 dz + \bar{L}_1 d\bar{z}) \ .$$

In particular, in a holomorphic theory each primary field ψ gives rise to a <u>holomorphic</u> operator-valued differential form ψ_X, and even if ψ is not primary ψ_X is a holomorphic function in the domain of a given local parameter. If $X = X_{ab}$ is the annulus $\{z : a \leqslant |z| \leqslant b\}$ and ψ is of type (p,0) we can always write ψ_X as a Laurent series

$$\psi_X = U_{1b} \ \{\sum_{k \in \mathbb{Z}} \psi_k z^{k-p} dz^{\otimes p}\} \ U_{a1} \ , \tag{9.3}$$

where ψ_k is an unbounded operator in H and U_{a1} and U_{1b} are the operators associated to the annuli X_{a1} and X_{1b}. The advantage of this notation is that ψ_k depends only on ψ and not on a,b. The reader should be warned that in usual terminology it is the unbounded operator $\hat{\psi}(z) =$ $\Sigma\, \psi_k z^{k-n}$ which is called the field operator, rather than my ψ_X. In terms of the ψ_k the Ward identity (9.2) becomes

$$[L_n, \psi_k] = -i(k + n - pn)\psi_{k+n} \ . \tag{9.4}$$

In a holomorphic theory the operators ψ_k are completely characterized in terms of $\psi \in H$ by the property (9.4) together with

$$\psi_p \Omega = \psi \quad . \tag{9.5}$$

The energy-momentum tensor

The most important fields in any theory are the energy-momentum tensors T and $\overline{T}$. These are secondary fields of type (2,0) and (0,2) respectively: T transforms under $\mathcal{E}_0$ by

$$U_f.T = f'(0)^2 T + c_R S_f(0)\Omega \ , \tag{9.6}$$

where $S_f(0) = f'''(0)/f'(0) - 3/2(f''(0)/f'(0))^2$ is the Schwarzian derivative of f, and c_R is the central charge. It should be noticed that

$$f \longmapsto \begin{pmatrix} f'(0)^2 & c\, S_f(0) \\ 0 & 1 \end{pmatrix}$$

is the only two dimensional representation of $\mathcal{E}_0$ which combines the one dimensional representations $f \mapsto f'(0)^2$ and $f \mapsto 1$.

To define T and $\overline{T}$ we consider the variation of the vacuum map $U_D : \mathbb{C} \to H$ when the complex structure on the disc D is changed. An infinitesimal change of structure is an element of $V = \mathrm{Vect}_{\mathbb{C}}(S^1)/\mathrm{Vect}(D)$, so the variation is a map $V \to H$ which is $\mathbb{R}$-linear (but not $\mathbb{C}$-linear unless the theory is holomorphic). We can regard it as a $\mathbb{C}$-linear map $V \oplus \overline{V} \to H$. In V there is an eigenvector of $\mathcal{E}_0$ of type (2,0) represented by $z^{-1}d/dz$. The image of this in H is denoted by T, and the image of the corresponding element of $\overline{V}$ by $\overline{T}$. (I use the traditional notation $T,\overline{T}$ with misgivings, as $\overline{T}$ is not necessarily the complex conjugate of T.) In terms of the Virasoro generators we have $T = iL_2\Omega$ and $\overline{T} = -i\overline{L}_2\Omega$. To obtain the transformation properties of T and $\overline{T}$ under $\mathcal{E}_0$ we must notice that the map $V \to H$ is not $\mathcal{E}_0$-equivariant. Because $\mathrm{Vect}(S^1)$ acts projectively on H we actually have an $\mathcal{E}_0$-equivariant map $\mathbb{C} \oplus V \to H$, where $\mathbb{C}$ is that of the central extension of $\mathrm{Vect}(S^1)$. Taking this into account we find that T transforms by (9.6). (See [S1] ( ).)

On a Riemann surface X the vector T gives rise not to an operator-valued quadratic differential $T(z)dz^2$, but rather to a projective connection, i.e. when the local parameter is changed from z to ζ the operator $T(z)dz^2$ becomes

$$T(\zeta)d\zeta^2 + c_R S_\zeta U_X ,$$

where S_ζ is the Schwarzian derivative of the change of parameters.

The significance of the energy-momentum tensor is that it describes the variation of the operator $U_X : H^{\otimes m} \to H^{\otimes n}$ associated to a surface X when the complex structure of X is changed. An infinitesimal change of structure corresponds (see (4.1)) to an element of $\mathrm{Vect}_{\mathbb{C}}(\partial X)/\mathrm{Vect}(X)$.

Proposition (9.7). (i) The energy-momentum tensor $T(z)dz^2$ is a holomorphic projective connection on the interior of any surface. It possesses a distributional boundary value. Similarly, $\overline{T}(z)d\overline{z}^2$ is antiholomorphic.

(ii) For the infinitesimal deformation of X defined by a complex vector field ξ along ∂X we have

$$\delta U_X = \int_{\partial X} \xi(z).T(z)dz + \int_{\partial X} \overline{\xi(z)}.\overline{T}(z)d\overline{z} .$$

Remark. If $T(z)dz^2$ were a quadratic differential it would pair with the vector field ξ to give a 1-form which could be integrated around ∂X. But ξ is really an element of the central extension of $\mathrm{Vect}_{\mathbb{C}}(\partial X)$, for it represents a deformation of a surface equipped with a chosen point in its determinant line. Thus ξ pairs with a projective connection, for the projective connections are precisely the dual of the central extension of $\mathrm{Vect}(\partial X)$. ([S2](p.335).) also [S3] If ξ extends holomorphically over X the integrals above vanish by Cauchy's theorem because T (resp. $\overline{T}$) is holomorphic (resp. antiholomorphic). Conversely, the fact that $\delta U_X = 0$ when ξ extends holomorphically, i.e. the fact that the theory is conformally invariant, implies that T is holomorphic.

Proof of (9.7). Consider the variation of U_X in the space E of trace-class operators $H^{\otimes m} \to H^{\otimes n}$ when the structure of X is changed. This is expressed by a real-linear map

$$\mathrm{Vect}_{\mathbb{C}}(\partial X)/\mathrm{Vect}(X) \to E \tag{9.8}$$

when a section of the central extension of $\mathrm{Vect}_{\mathbb{C}}(\partial X)$ has been chosen. The dual of $\mathrm{Vect}_{\mathbb{C}}(\partial X)/\mathrm{Vect}(X)$ is the space Ω of holomorphic quadratic differentials on X with distributional boundary values. So (9.8) corresponds to an element of $E \otimes_{\mathbb{R}} \Omega = E \otimes_{\mathbb{C}} (\Omega \oplus \overline{\Omega})$. This means that we have a formula

$$\delta U_X = \int_{\partial X} (\xi t_X + \overline{\xi t}_X) \tag{9.9}$$

for <u>some</u> naturally defined operator-valued objects $t_X, \bar{t}_X$. Applying (9.9) when $X = D$ and $\xi = z^{-1}d/dz$ we find that $t_D(0) = T_D(0)$ and $\bar{t}_D(0) = \bar{T}_D(0)$. We then deduce that $t_X(z) = T_X(z)$ and $\bar{t}_X(z) = \bar{T}_X(z)$ in all cases by using the naturality property (9.2).

<u>Corollary (9.10)</u>. If X is an annulus $\{z : a \leqslant |z| \leqslant 1\}$ we can write the energy-momentum tensor in terms of the Virasoro generators:

$$T(z) = i\{\sum_{k\in\mathbb{Z}} z^{k-2}L_k\} \circ U_X .$$

<u>Infinitesimal automorphisms</u>

An important role of field operators is to describe the infinitesimal automorphisms and deformations of a theory. An automorphism of a theory based on a vector space H evidently means an invertible operator $A : H \to H$ which preserves the hermitian form and satisfies

$$A^{\otimes n} \circ U_X = U_X \circ A^{\otimes m}$$

for each morphism X from C_m to C_n in $\mathcal{C}$. An <u>infinitesimal automorphism</u> is accordingly a (densely defined) skew-hermitian operator δ in H such that

$$\{\sum_{i=1}^{n} 1 \otimes \ldots \otimes \delta \otimes \ldots \otimes 1\} \circ U_X = U_X \circ \{\sum_{j=1}^{m} 1 \otimes \ldots \otimes \delta \otimes \ldots \otimes 1\} \tag{9.11}$$

where the factors δ occur in the i^{th} and j^{th} places on the left and right.

Proposition (9.12). In a holomorphic theory a real primary field ψ of type (1,0) defines an infinitesimal automorphism δ_ψ whose domain of definition includes $U_X H$ for any annulus X, and which is characterized by

$$\delta_\psi \circ U_X = \int_\gamma \psi_X(z)dz , \qquad (9.13)$$

where γ is any simply closed curve going once around the annulus.

Proof: The right-hand side is independent of γ because $\psi_X(z)dz$ is a holomorphic 1-form, and the formula (9.2) shows that δ_ψ is independent of X and satisfies (9.11).

In a theory which is not chiral the 1-form $\psi_X(z)dz$ defined by a primary field of type (1,0) need not be closed, i.e. holomorphic. In that case an infinitesimal automorphism is defined by a pair of real primary fields (ψ_L, ψ_R) of types (0,1) and (1,0) such that $L_1\psi_L = \bar{L}_1\psi_R$. Then the 1-form $\psi_L(z)d\bar{z} + \psi_R(z)dz$ is closed, and its integral replaces the right-hand side of (9.13).

In the literature it is assumed that any infinitesimal automorphism of a field theory is given by a primary field in the way described. This may well follow from our axioms; if it does not then another axiom should probably be added to ensure it. The idea of such an axiom would be to express the fact that the space H associated to a circle is, in some sense which I do not know how to make precise, a continuous tensor product of spaces associated to the infinitesimal elements of the circle.

For the rest of the discussion of infinitesimal automorphisms I shall confine myself to holomorphic theories. The infinitesimal automorphisms evidently form a Lie algebra, and there is an induced Lie algebra structure on the finite dimensional space of primary fields of type (1,0), for if ψ_1 and ψ_2 are such fields then

$$[\psi_1, \psi_2] = \delta_{\psi_1} \psi_2 .$$

Proposition (9.14). Let $\mathfrak{g}$ be the Lie algebra of real primary fields of type (1,0) in a unitary holomorphic theory. Then the loop algebra $L\mathfrak{g}$ acts projectively on H, intertwining with the action of $\mathrm{Diff}^+(S^1)$. Conversely, if $L\mathfrak{g}$ acts in this way then $\mathfrak{g}$ is contained in the Lie algebra of primary fields of type (1,0).

Remark. Because we are assuming the hermitian form on H is positive-definite $\mathfrak{g}$ will be the Lie algebra of a compact group.

One consequence of Proposition (9.14) is that the field theories we shall construct in §11 from the loop groups of compact groups are genuinely different from one another. The proposition also gives a convenient criterion for deciding when the group of automorphisms of a theory is finite: there must be no primary fields of type (1,0). But the most important positive application of the proposition is the "vertex-operator" construction of the basic representation of the loop group LG when G is simply laced: we shall return to this in §12.

Proposition (9.14) is deduced by one of the very characteristic arguments of conformal field theory from the following "operator-product expansion".

Proposition (9.15). If ψ_1 and ψ_2 are primary fields of type (1,0) in a unitary holomorphic field theory then on any Riemann surface X we have

$$(\psi_1\psi_2)_X(z,\zeta) = \langle\psi_1,\psi_2\rangle U_X \frac{dz\,d\zeta}{(z-\zeta)^2} + \frac{1}{2\pi i}[\psi_1,\psi_2]_X \frac{d\zeta}{z-\zeta} + \varphi(z,\zeta) ,$$

where φ is holomorphic everywhere on $X \times X$, for some invariant inner product $\langle\ ,\ \rangle$ on the primary fields.

Proof of (9.14) using (9.15). Let $\psi_1,\ldots,\psi_n$ be a basis for $\mathfrak{g}$. We shall restrict ourselves to real analytic elements $\xi = \Sigma\, \xi_i\psi_i$ of the loop algebra $L\mathfrak{g}$. Then ξ is the boundary value of a holomorphic function ξ defined in an annulus X, and we can define the action δ_ξ of ξ on H by the formula

$$\delta_\xi \circ U_X = \int_\gamma \Sigma\, \xi_i\psi_{i,X}(z)dz ,$$

analogous to (9.13). Using the basic functoriality property (9.2) we can write the commutator $[\delta_\xi,\delta_\eta] \circ U_X$ in the form

$$\left\{\int_{\gamma_1} dz \int_{\gamma_2} d\zeta - \int_{\gamma_3} dz \int_{\gamma_2} d\zeta\right\}(\Sigma\, \xi_i(z)\eta_j(\zeta)(\psi_i\psi_j)_X(z,\zeta))$$

where $\gamma_1,\gamma_2,\gamma_3$ are three non-intersecting simple closed curves going once round the annulus, with γ_1 outside γ_2 and γ_2 outside γ_3. Let us first perform the integral over z, holding ζ fixed. Because the integrand is holomorphic for $z \neq \zeta$ and $\gamma_1 - \gamma_3$ is homologous to a small circle around ζ the result is the residue of the integrand at $z = \zeta$, which by (9.15) is

$$\Sigma\ \xi_i(\zeta)\eta_j(\zeta)[\psi_i,\psi_j]_X(\zeta) \quad + \quad \Sigma <\psi_i,\psi_j>\xi_i'(\zeta)\eta_j(\zeta)U_X\ .$$

Integrating this around γ_2 gives

$$[\delta_\xi,\delta_\eta] \quad = \quad \delta_{[\xi,\eta]} \quad + \quad c(\xi,\eta)1\ ,$$

where

$$c(\xi,\eta) \ = \ \int <\xi'(\zeta),\eta(\zeta)>d\zeta$$

is a cocycle defining a central extension of $L\mathfrak{g}$.

Proof of (9.15). First observe that the terms on the right-hand side of (9.16) make invariant sense, e.g. that the 2-form $dzd\zeta/(z-\zeta)^2$ on $X \times X$ is independent of the local parameter modulo holomorphic forms. Because of this we can assume that X is the standard disc D, and that $\zeta = 0$. The element $(\psi_1\psi_2)_D(z,0)$ of H can be expanded

$$(\psi_1\psi_2)_D(z,0) \ = \ \sum_{k\in\mathbb{Z}} A_k z^k \qquad (9.17)$$

with $A_k \in H$. Let R_α denote the automorphism of D which rotates it through α. Its action on $(\psi_1\psi_2)_D(z,0)$ is given by

$$R_\alpha\{(\psi_1\psi_2)_D(z,0)\} = e^{2i\alpha}(\psi_1\psi_2)_D(e^{-i\alpha}z,0) .$$

Applying this to (9.17) we find that $R_\alpha A_k = e^{(2-k)i\alpha}A_k$, i.e. that A_k is an element of H of degree 2-k. In a unitary theory there are no fields of negative degree, and only the vacuum vector has degree 0. So

$$(\psi_1\psi_2)_D(z,0) = \lambda\Omega z^{-2} + A_1 z^{-1} + (\text{holomorphic}) ,$$

for some $\lambda \in \mathbb{C}$. But by definition we have

$$\int_{S^1} (\psi_1\psi_2)_D(z,0) = \delta_{\psi_1}\psi_2$$

$$= [\psi_1,\psi_2] ,$$

so $2\pi i A_1 = [\psi_1,\psi_2]$. The proof is completed by observing that $(\psi_1,\psi_2)\mapsto\lambda$ is necessarily an invariant inner product on $\mathfrak{g}$.

Infinitesimal deformations

When one has a continuously varying family of conformal field theories one may as well assume that the hermitian vector space H is fixed and that all the variation takes place in the operators U_X associated to surfaces X. It is simplest to think in terms of Definition (4.4). Then an infinitesimal deformation will be a rule which associates to each surface X a vector $\Theta_X \in H_{\partial X}$ such that

(i) $\Theta_{X\sqcup Y} = \Theta_X \otimes \Omega_Y + \Omega_X \otimes \Theta_Y$,

and

(ii) $\Theta_X \mapsto \Theta_{\check{X}}$ for each sewing map $X \to \check{X}$.

In analogy with (9.12) we have

Proposition (9.18). In any theory a primary field θ of type (1,1) defines an infinitesimal deformation by the formula

$$\Theta_X = \int_X \theta_X(z)dz\wedge d\bar{z} \ .$$

As with automorphisms it is usually assumed that any infinitesimal deformation arises from a primary field in this way, but I do not know a proof. Thus a chiral theory should be automatically rigid. The space of deformations of the σ-model of a torus will be considered in §10.

Examples

We shall consider some fields in the holomorphic fermionic theories $\mathfrak{F}(\Omega^\alpha)$.

(i) The most obvious primary fields are the vectors

$$\psi^{(m)} = \psi_{m-\alpha} \wedge \psi_{m-\alpha-1} \wedge \psi_{m-\alpha-2} \wedge \ldots \in H \ .$$

Evidently $\psi^{(0)}$ is the vacuum vector Ω, with degree 0. In general $\psi^{(m)}$ has degree $\frac{1}{2}m(m-2\alpha+1)$, which is negative if m is between 0 and 2α-1. These are exactly the fields $\psi^{(m)}$ which were described in a different way in (8.20). That is obvious in the case of $\psi^{(1)} = \psi$ from the characterization of ψ_X by means of (9.4) and (9.5); we shall not pursue the general case here.

(ii) We saw in §8 that the theory $\mathfrak{F}(\Omega^{\frac{1}{2}})$ is $\mathbb{Z}$-graded[1]. This means that the circle group $\mathbb{T}$ acts on it as a group of automorphisms. The infinitesimal generator is the primary field

$$J = \psi_{\frac{1}{2}} \wedge \psi_{-3/2} \wedge \psi_{-5/2} \wedge \ldots$$

$$= \psi_{\frac{1}{2}} \, \psi^{*}_{-\frac{1}{2}} \, \Omega$$

in $\mathfrak{F}(\Omega^{\frac{1}{2}}(S^1_A))$, which is called the <u>current</u>. (In the second expression for J we write $\psi_{\frac{1}{2}}$ for the operator of multiplication by $\psi_{\frac{1}{2}}$ on $\mathfrak{F}(\Omega^{\frac{1}{2}})$, and $\psi^{*}_{-\frac{1}{2}}$ for the antiderivation corresponding to the dual basis element to $\psi_{-\frac{1}{2}}$.) The field J provides us with an action of the whole loop group $L\mathbb{T}$ on $\mathfrak{F}(\Omega^{\frac{1}{2}})$ extending the action of $\mathbb{T}$. (In physical language the grading or $\mathbb{T}$-action is the <u>charge</u>, and the action of the "current group" $L\mathbb{T}$ expresses the fact that charge is <u>local</u>.) To prove that the vector J really does generate the $\mathbb{T}$-action on $\mathfrak{F}(\Omega^{\frac{1}{2}})$ one can write the field $J(z)dz$ on an annulus in the form (9.3).

<u>Proposition (9.19)</u>. For J as above we have on the annulus $X = \{z : a \leqslant |z| \leqslant 1\}$ the relation

$$J(z)dz = \{\Sigma \, J_k z^{k-1} dz\} \circ U_X ,$$

where

$$J_k = \sum_{r \in \mathbb{Z}+\frac{1}{2}} \psi_{k+r} \, \psi^{*}_{r} \qquad \text{when} \quad k \neq 0 ,$$

---

[1]For simplicity I shall ignore the spin structure S^1_P for the moment, though actually the discussion applies to it just as well.

and

$$J_0 = \sum_{r>0} \psi_r \psi_r^* - \sum_{r<0} \psi_r^* \psi_r \ .$$

One must check that the expressions for the J_k given here are well-defined operators on vectors of finite energy in H. Granting that, the proposition is easily proved by checking the conditions (9.4) and (9.5), using $[L_n, \psi_m] = -i(m + \frac{1}{2}n)\psi_{m+n}$. The operator J_0 is by definition the infinitesimal automorphism δ_J of H corresponding to J, and it obviously multiplies each basis vector of H by its Fock space degree.

(iii) In $\mathfrak{F}(\Omega^\alpha)$ for any value of α there is a vector $J = \psi_{1-\alpha}\psi^*_{-\alpha}$ of degree 1 analogous to the current in $\mathfrak{F}(\Omega^{\frac{1}{2}})$. When $\alpha \neq \frac{1}{2}$ it is not a primary field, for

$$L_{-1}J = i(1-2\alpha)\Omega \ .$$

This means that under the action of a holomorphic map $f : D \to \mathring{D}$ such that $f(0) = 0$ the vector J transforms by

$$U_f.J = f'(0)J - \tfrac{1}{2}(1-2\alpha)f''(0)f'(0)^{-1}\Omega$$

(cf. (9.6)), or equivalently that we have an operator-valued 1-form $J(z)dz$ in each coordinate patch on any surface X, but when one changes coordinates by $z = z(\zeta)$ the form changes to

$$J(z(\zeta))z'(\zeta)d\zeta \;-\; \tfrac{1}{2}(1-2\alpha)\; U_X(z''(\zeta)/z'(\zeta))d\zeta \;.$$

We can still write $J(z)dz$ in the form (9.19), and the operator J_0 in H defines the Fock space degree, which is still called "charge". But the fact that $J(z)dz$ is not a 1-form corresponds to the failure of the operators U_X to preserve charge which was pointed out in (8.13). From the present viewpoint the charge anomaly can be calculated as follows. Let us suppose that all the boundary circles of X are outgoing. We can choose a holomorphic connection in the holomorphic tangent bundle to X. It is given by a 1-form $\gamma(z)dz$ in each coordinate patch, but a change of coordinates $z = z(\zeta)$ replaces $\gamma(z)dz$ by

$$\gamma(z(\zeta))z'(\zeta)d\zeta \;+\; (z''(\zeta)/z'(\zeta))d\zeta \;.$$

Then

$$(J(z) \;+\; \tfrac{1}{2}(1-2\alpha)\gamma(z)U_X)dz$$

is a global holomorphic 1-form on X with values in $H(\partial X)$. Applying the operator J_0 to $U_X \in H(\partial X)$ gives, by definition,

$$\frac{1}{2\pi i}\int_{\partial X} J(z)z^{-1}dz \;.$$

By Cauchy's theorem this equals

$$\{\frac{1}{4\pi}\;(2\alpha-1)\int_{\partial X}\gamma(z)dz\}U_X \;.$$

We can suppose the connection γ arose from a trivialization of TX. Then $\int_{\partial X} \gamma(z)dz$ is the angle by which the tangent vector d/dz to ∂X rotates relative to the trivialization when one travels around ∂X. This angle is $4\pi(g + m - 1)$, where g is the genus of X and m the number of boundary circles. So the charge anomaly is $(2\alpha-1)(g + m - 1)$, in agreement with (8.12).

BRST cohomology

For a holomorphic theory H with central charge 26 we define the BRST cohomology in the following way. First tensor the theory H with the "ghost" theory $\mathfrak{F}(\Omega^2)$. The Fock space grading of $\mathfrak{F}(\Omega^2)$, which was called the "charge" in the preceding example, induces a grading of $H = H \otimes \mathfrak{F}(\Omega^2)$ which is now called the "ghost number". We shall show that H contains a primary field Q of degree 1 whose associated infinitesimal automorphism δ_Q raises the ghost number by 1 and satisfies $\delta_Q^2 = 0$. The cohomology of H with respect to the differential δ_Q is the BRST cohomology H_{BRST}. Now the Fock space $\mathfrak{F}(\Omega^2(S^1))$ is a renormalized version of the exterior algebra on $\Omega^2(S^1)$, which is the dual of the Lie algebra of vector fields $\mathrm{Vect}(S^1) = \Omega^{-1}(S^1)$. The differential δ_Q is similarly, as we shall see, a renormalization of the standard differential of Lie algebra cohomology, and so $H_{BRST}(S^1)$ is a renormalized version of the cohomology of $\mathrm{Vect}(S^1)$ with coefficients in the module $H(S^1)$. In fact one can say more: for $\mathfrak{F}(\Omega^2(S^1))$ is a module over the exterior algebra on $\Omega^2(S^1)$, and this makes $H_{BRST}(S^1)$ a module over the ordinary cohomology of $\mathrm{Vect}(S^1)$ with coefficients in $\mathbb{C}$.

At the moment, however, I want just to point out the field-theoretic aspects of H_{BRST}. Because δ_Q is an infinitesimal automorphism the operator U_X associated to a surface X in the theory H commutes with δ_Q, i.e. it is a homomorphism of cochain complexes. It induces a map of the cohomology H_{BRST} which changes the degree by the ghost number anomaly. This is still not quite what we need. The surfaces X of a particular topological type α with m incoming and n outgoing circles form the moduli space $\mathcal{C}_\alpha$ of §4 whose tangent space at X is $\mathrm{Vect}_{\mathbb{C}}(\partial X)/\mathrm{Vect}(X)$. The operator U_X is really an element of $\hat{H}(\partial X) = H(\partial X) \otimes \mathfrak{J}(\Omega^2(\partial X))$, which is a module over the exterior algebra on $\mathrm{Vect}_{\mathbb{C}}(\partial X)$, the dual space of $\Omega^2(\partial X)$. The vector U_X is annihilated by the subspace $\mathrm{Vect}(X)$ of $\mathrm{Vect}_{\mathbb{C}}(\partial X)$. We can therefore define for each p a holomorphic differential form ω_p of degree p on $\mathcal{C}_\alpha$, with values in $\hat{H}(\partial X)$, by

$$\omega_p(X;\xi_1,\ldots,\xi_p) = \xi_1\xi_2 \ldots \xi_p U_X ,$$

where $\xi_i \in \mathrm{Vect}_{\mathbb{C}}(\partial X)$. The fact that $\delta_Q U_X = 0$ - because δ_Q is an infinitesimal automorphism - has the following generalization.

<u>Proposition (9.20)</u>. We have

$$d\omega_{p-1} = -\delta_Q \omega_p ,$$

where d denotes the exterior derivative of forms on $\mathcal{C}_\alpha$. Alternatively expressed, if the boundary circles of X are regarded as incoming rather than outgoing, the forms $\{\omega_p\}$ define a map of cochain complexes

$$\hat{H}(\partial X) \rightarrow \Omega_{hol}(\mathcal{C}_\alpha) \qquad (9.21)$$

Proof: An element $\xi \in \mathrm{Vect}_{\mathbb{C}}(\partial X)$ acts on H in two ways: by the exterior multiplication used above, and also by its action as an element of the Lie algebra of $\mathrm{Diff}(\partial X)$. I shall write L_ξ for the latter action, and (for this proof) i_ξ for the former. The two are related by the usual Cartan and naturality formulae

$$L_\xi = [\delta_Q, i_\xi] \ , \qquad i_{[\xi,\eta]} = [L_\xi, i_\eta]$$

where [,] is the graded commutator.

I shall give the proof of (9.20) when $p = 2$. We have

$$\begin{aligned} d\omega_1(X;\xi,\eta) &= L_\eta\omega_1(X;\xi) - L_\xi\omega_1(X;\eta) + \omega_1(X;[\xi,\eta]) \\ &= (L_\eta i_\xi - L_\xi i_\eta + i_{[\xi,\eta]})U_X \\ &= (L_\eta i_\xi - i_\eta L_\xi)U_X \\ &= \delta i_\eta i_\xi U_X \\ &= -\delta\omega_2(X;\xi,\eta) \ . \end{aligned}$$

The first line of this calculation is the definition of the exterior derivative, regarding ξ and η as vector fields on $\mathcal{C}_\alpha$. One can identify the Lie derivative L_ξ for forms on $\mathcal{C}_\alpha$ with the operator L_ξ on $\hat{H}$ because $X \mapsto U_X$ is equivariant with respect to the action of $\mathrm{Vect}_{\mathbb{C}}(\partial X)$.

Let us now suppose that X is a surface with m boundary circles, all incoming. We readily check that the cochain map (9.21) raises degree by 3g-3, where g is the genus of X. It is also compatible with the action of $\mathrm{Vect}_{\mathbb{C}}(\partial X)$ - by both kinds of operators L_ξ and i_ξ. The action of $Vect_{\mathbb{C}}(\partial X)$ is the infinitesimal version of the natural action of the semigroup $\mathcal{A}^m$. Inside $\mathcal{A}$ there is the subsemigroup $\mathcal{E}$ of holomorphic embeddings $f : D \to \mathring{D}$. The quotient space of $\mathcal{C}_\alpha$ by $\mathcal{E}^m$ is the moduli space $\mathcal{M}_g$ of closed surfaces of genus g, and has complex dimension 3g-3. Now the vacuum vector in $H(\partial X)$ is basic for $\mathcal{E}^m$ (i.e. annihilated by L_ξ and i_ξ for $\xi \in Vect(D)$). Its image under (9.21) is therefore a form on $\mathcal{C}_\alpha/\mathcal{E}^m = \mathcal{M}_g$ which is holomorphic, and of the top dimension 3g-3. It is natural to call this the <u>partition form</u> of the chiral theory H. A more physical theory in which both chiralities were present would have a partition form of bidegree (3g-3,3g-3), and this could be integrated over $\mathcal{M}_g$. That is the situation in string theory.

Besides the vacuum vector we can consider other classes in $H_{BRST}(\partial X) = H_{BRST}(S^1)^{\otimes m}$. For the theories which arise in practice all elements of $H_{BRST}(S^1)$ are represented by elements of $\hat{H}(S^1)$ which are basic for the action of the semigroup

$$\mathcal{E}_0 = \{f \in \mathcal{E} : f(0) = 0\} .$$

(This is true whenever $H(S^1)$ is a free module over the enveloping algebra $U(\mathfrak{a})$ of the Lie algebra $\mathfrak{a}$ of $\mathcal{E}_0$, which is spanned by the Virasoro generators $\{L_k\}_{k\geq 1}$. Cf. [FGZ].) The quotient space $\mathcal{C}_\alpha/\mathcal{E}_0^m$ is the moduli space $\mathcal{M}_{g,m}$ of surfaces with m marked points. It has complex dimension 3g-3+m. If we choose an element ψ_i of ghost number 1 in $H_{BRST}(S^1)$ for each boundary circle then the image

$$\langle\psi_1\psi_2 \ldots \psi_m\rangle$$

of $\psi_1 \otimes \ldots \otimes \psi_m$ under the map (9.21) is a top dimensional holomorphic form on $\mathcal{M}_{g,m}$, well defined up to the addition of an exact form. For this reason the elements of ghost number 1 in H_{BRST} are regarded as (chiral) "physical states".

We now return to the definition of the primary field Q and the verification of its properties. We can write it explicitly

$$Q \;=\; J \otimes \psi_{-1}\Omega \;+\; \Omega \otimes (\psi_0\psi_{-1}\psi^*_{-2} - 3\psi_1)\Omega \qquad (9.22)$$

in $\hat{H} = H \otimes \mathfrak{F}(\Omega^2)$. As the energies in $\mathfrak{F}(\Omega^2)$ are bounded below by -1 we need only check that $L_{-1}Q = L_{-2}Q = 0$ to see that Q is primary, and that is easily done by using the relation

$$[L_k, \psi_m] = -i(m+2k)\psi_{m+k}$$

of (8.14), together with the formula $L_{-2}T = 13\Omega$ in H which expresses that H has central charge c = 26. To understand where (9.22) comes from, however, it is best to recall the formula for the differential δ in the cochain complex $M \otimes \Lambda\mathfrak{g}^*$ of a Lie algebra $\mathfrak{g}$ with coefficients in a $\mathfrak{g}$-module M. Let $\{\xi_k\}$ be a basis for $\mathfrak{g}$, and $\{\alpha_k\}$ the dual basis of $\mathfrak{g}^*$. Then

$$\delta \;=\; \Sigma\, \xi_k \otimes e_k \;+\; \tfrac{1}{2}\,\Sigma\, 1 \otimes e_k\xi_k \;,$$

where e_k is the operation of multiplication by α_k on $\Lambda\mathfrak{g}^*$, and ξ_k denotes the action of ξ_k on either M or $\Lambda\mathfrak{g}^*$. The analogous operator on $\hat{H} = H \otimes \mathfrak{F}(\Omega^2)$, in the usual notation, is

$$\delta_Q = \sum_{k\in\mathbb{Z}} L_k \otimes \psi_{-k} + \tfrac{1}{2} \sum_{k\geqslant -1} 1 \otimes \psi_k L_{-k} + \tfrac{1}{2} \sum_{k<-1} 1 \otimes L_{-k}\psi_k \,, \tag{9.23}$$

where we have renormalized by "adding the infinite term"

$$\tfrac{1}{2} \sum_{k<-1} 1 \otimes [L_{-k},\psi_k] = (-\tfrac{1}{2} \sum_{k<-1} k).\psi_0$$

to ensure that $\delta_Q(\Omega \otimes \Omega) = 0$. In the light of (9.23) one finds, after a little experiment, that the sequence of operators

$$\begin{aligned} Q_m = \sum_{k\in\mathbb{Z}} L_{k+m} \otimes \psi_{-k} + \tfrac{1}{2} \sum_{k\geqslant -1} 1 \otimes \psi_k L_{-k+m} \\ + \tfrac{1}{2} \sum_{k<-1} 1 \otimes L_{-k+m}\psi_k + 3/2\ m(m+1)\psi_m \end{aligned}$$

satisfies the relations $[L_p,Q_m] = -im\ Q_{m+p}$ of (9.4) as well as $Q_0 = \delta_Q$. Finally, we obtain the expression (9.22) by setting

$$Q = Q_1(\Omega \otimes \Omega) \ .$$

To conclude we need to know that δ_Q raises the ghost number by 1, and also that $\delta_Q^2 = 0$. The first is obvious from (9.23). The second is equivalent to $\delta_Q Q = 0$, for δ_Q^2 is the graded commutator $[\delta_Q, \delta_Q]$. One knows a priori that δ_Q^2 is an infinitesimal automorphism of the theory, but I do not know a non-computational proof that it vanishes.

§10. The σ-model for a torus

The Hilbert space

We shall now construct the field theory corresponding to strings moving in a torus T. As was explained in the introduction, the essential point is to choose a vector space to play the role of the space of square-summable functions on LT. If A is a locally compact abelian group there is a simple group-theoretic characterization of the Hilbert space $H = L^2(A)$, and our strategy is to adopt this characterization as a definition in the case of LT.

In the finite dimensional case $L^2(A)$ possesses a unitary action of A by translations and a unitary action of the Pontrjagin dual group $\hat{A}$ by multiplication operators. These actions fit together to define an irreducible unitary representation of the Heisenberg group $(A \times \hat{A})^{\sim}$, which is the central extension of $A \times \hat{A}$ by $\mathbf{T}$ associated with the pairing $A \times \hat{A} \to \mathbf{T}$, i.e. the extension whose cocycle c is given by

$$c((a_1,\alpha_1),(a_2,\alpha_2)) = \langle\alpha_1, a_2\rangle \ . \tag{10.1}$$

The space $L^2(A)$ is characterized - up to a scalar multiplication - as the unique faithful irreducible representation of the Heisenberg group.

To generalize this, let us begin with a Riemannian torus $T = \mathfrak{t}/\Lambda$, where $\mathfrak{t}$ is a finite dimensional real vector space with a positive inner product, and Λ is a lattice in $\mathfrak{t}$. The dual lattice

$$\Lambda^* = \{\xi \in \mathfrak{t} : \langle\xi,\eta\rangle \in \mathbb{Z} \ \text{ for all } \ \eta \in \Lambda\}$$

gives rise to the dual torus $T^* = \mathfrak{t}/\Lambda^*$.

The loop groups LT and LT* are in duality under the bilinear pairing $\ll\ ,\ \gg$: $LT \times LT^* \to \mathbf{T}$ defined by

$$\ll f,g \gg = \tfrac{1}{2}\int_0^1 (\langle f,dg\rangle - \langle df,g\rangle) + \tfrac{1}{2}(\langle f(0),\Delta_g\rangle - \langle \Delta_f, g(0)\rangle), \quad (10.2)$$

where $\mathbf{T}$ is regarded as $\mathbb{R}/\mathbb{Z}$, and an element f of LT is regarded as a map $f : \mathbb{R} \to \mathfrak{t}$ satisfying

$$f(t + 1) = f(t) + \Delta_f$$

for some $\Delta_f \in \Lambda$. (Thus f is defined only modulo the addition of a constant element of Λ.) The pairing $\ll\ ,\ \gg$ is nondegenerate, and is invariant under $\mathrm{Diff}^+(S^1)$. The groups LT and LT* are thus essentially Pontrjagin duals of each other.

We can now define a Hilbert space H which is a faithful irreducible representation of the Heisenberg group $\tilde{\Pi}$ formed from $\Pi = LT \times LT^*$ by using the pairing $\ll\ ,\ \gg$. Because $\tilde{\Pi}$ is infinite dimensional it has many faithful irreducible representations. To single one out one must choose a positive polarization of Π.

Definition (10.3). (i) A polarization of a real vector space E with a skew form S is an equivalence class of operators $J : E \to E$ such that

(a) $S(J\xi, J\eta) = S(\xi,\eta)$, and

(b) $J^2 = -1$ modulo trace class operators.

Two such operators J are equivalent if their difference is of trace class.

(ii) A polarization J is positive if the quadratic form $\xi \mapsto S(\xi, J\xi)$ is positive-definite on a subspace of E of finite codimension.

(iii) If Π is an abelian Lie group with a skew pairing $c : \Pi \times \Pi \to \mathbf{T}$, and $\pi_0(\Pi)$ and $\pi_1(\Pi)$ are finitely generated, then a polarization of Π is a polarization of its Lie algebra Lie(Π).

In our case $\Pi = LT \times LT^*$, and

$$LT = T \times \Lambda \times V$$
$$LT^* = T^* \times \Lambda^* \times V ,$$

where V is the real vector space $(L\mathfrak{t})/\mathfrak{t}$. A polarization of Π is the same as a polarization of $V \oplus V$, and this space has a canonical polarization given by the decomposition $V_{\mathbb{C}} = V_{\mathbb{C}}^+ \oplus V_{\mathbb{C}}^-$ into positive and negative frequency. This gives us a definite irreducible unitary representation of $\tilde{\Pi}$ on the Hilbert space

$$H = L^2(T \times \Lambda) \otimes S(V_{\mathbb{C}}^+ \oplus V_{\mathbb{C}}^+) .$$

<u>Remark</u>. A positive polarization of $V \oplus V$ is the same thing as a quadratic form $q : V \to \mathbb{C}$ such that Im q is positive definite. In the present case $q(f) = i \sum n\langle a_{-n}, a_n\rangle$ if $f = \sum a_n e^{in\theta}$. One can identify $S(V_{\mathbb{C}}^+ \oplus V_{\mathbb{C}}^+)$ with a completion of the space of all functions on V of the form $f \mapsto p(f)e^{iq(f)}$, where $p : V \to \mathbb{C}$ is a polynomial, so it is a very natural candidate for $L^2(V)$.

Because $\mathrm{Diff}^+(S^1)$ acts on $\tilde{\Pi}$ by automorphisms, and preserves the polarization class, it acts on the irreducible representation H. (In principle the action could be projective, but in fact is not.) Orientation-reversing diffeomorphisms of S^1 reverse the polarization of

Π, so they give antiunitary maps $H \to \bar{H}$. We have therefore the data to assign a Hilbert space $H_{\partial X}$ to the boundary of any Riemann surface X. To complete the construction of a conformal field theory we must associate to X a ray Ω_X in $H_{\partial X}$, and check the conditions (4.4) - but in fact Ω_X will be defined in a way which makes the conditions manifest.

Before turning to these rays it is worth describing the action of the conformal group $\mathrm{Conf}(S^1 \times \mathbb{R})$ on H. We recall from §3 that this group is a covering group of $\mathrm{Diff}^+(S^1) \times \mathrm{Diff}^+(S^1)$, and that the natural geometric action of $\mathrm{Diff}^+(S^1)$ is diagonal with respect to this description. Let $\mathfrak{D}$ denote the simply connected covering group of $\mathrm{Diff}^+(S^1)$, i.e. the diffeomorphisms $\varphi : \mathbb{R} \to \mathbb{R}$ such that $\varphi(\theta + 2\pi) = \varphi(\theta) + 2\pi$. The group $\mathrm{Conf}(S^1 \times \mathbb{R})$ is a quotient of $\mathfrak{D} \times \mathfrak{D}$, and the latter acts on Π by

$$(\varphi_1, \varphi_2)^{-1}.(f,g) = \tfrac{1}{2}(\varphi_1^*(f+g) + \varphi_2^*(f-g),\ \varphi_1^*(f+g) - \varphi_2^*(f-g)) .$$

The representation H of Π is induced from the representation of its identity component Π_0 on $H_{0,0} = S(V_{\mathbb{C}}^+ \oplus V_{\mathbb{C}}^+)$. Under Π_0 it breaks up

$$H = \bigoplus H_{\lambda,\mu} ,$$

where (λ,μ) runs through the group of components $\Lambda \times \Lambda^*$, and $H_{\lambda,\mu} = H_{0,0}$ as a representation of $V \times V \subset \Pi_0$ but is acted on by $T \times T^*$ via the character (λ,μ). The summands $H_{\lambda,\mu}$ are acted on separately by $\mathfrak{D} \times \mathfrak{D}$: the action on $H_{\lambda,\mu}$ is obtained by twisting the action on $H_{0,0}$ by the crossed homomorphism $\epsilon_{\lambda,\mu} : \mathfrak{D} \times \mathfrak{D} \to \Pi_0$ associated to $(\lambda,\mu) \in \Pi$. To see the action of $\mathfrak{D} \times \mathfrak{D}$ on $H_{0,0}$ we rewrite $S(V_{\mathbb{C}}^+ \oplus V_{\mathbb{C}}^+)$ as

$S(W_L) \otimes S(W_R)$, where W_L and W_R are the diagonal and antidiagonal subspaces of $V_{\mathbb{C}}^{+} \oplus V_{\mathbb{C}}^{+}$. Then the left- and right-hand copies of $\mathfrak{D}$ act purely on the left- and right-hand factors $S(W_L)$ and $S(W_R)$. On each the representation is the standard metaplectic representation of $\mathfrak{D}$ described, for example, in [S2]. We thus obtain

Proposition (10.4). (i) The representation of $\mathrm{Conf}(S^1 \times \mathbb{R})$ on H has central charge (d,d), where d is the dimension of T.

(ii) The partition function of the theory, i.e. the trace of the action on H of the standard annulus A_q, is

$$|\varphi(q)|^{2d} \prod_{\lambda,\mu} q^{\frac{1}{2}||\lambda+\mu||^2} \;\; \bar{q}^{\frac{1}{2}||\lambda-\mu||^2} ,$$

where $\varphi(q) = \Pi(1-q^n)^{-1}$.

The ray associated to a surface

The general method of prescribing a ray in the Heisenberg representation of $\tilde{\Pi}$ is to give a maximal isotropic subgroup P of $\Pi_{\mathbb{C}}$ and a suitable character $\theta : \tilde{P} \to \mathbb{C}^{\times}$ which splits the central extension of P induced by $\tilde{\Pi}_{\mathbb{C}}$. The ray is then the eigenvector of $\tilde{P}$ corresponding to the character θ: it exists and is unique providing P is positive and compatible with the polarization of Π in the following sense.

Definition (10.5). (i) P is an isotropic subgroup of $\Pi_{\mathbb{C}}$ if the induced extension $\tilde{P}$ is abelian.

(ii) P is positive if $\mathrm{Im}\, c(\bar{p},p) \geqslant 0$ for all $p \in P$, and $\mathrm{Im}\, c(\bar{p},p) > 0$ except on a compact subgroup of P.

(iii) P is <u>compatible</u> with the polarization of Π if the endomorphism J_P of Lie $(\Pi_{\mathbb{C}})$ which is $+i$ (resp. $-i$) on Lie(P) (resp. Lie($\overline{P}$)) belongs to the polarization class of Π.

Remarks. (a) We write Im $c(\bar{p},p) \geqslant 0$ rather than $|c(\bar{p},p)| \leqslant 1$ because we are writing $\mathbb{C}^{\times}$ additively, i.e. as $\mathbb{C}/\mathbb{Z}$.

(b) A character $\theta : \tilde{P} \to \mathbb{C}^{\times} = \mathbb{C}/\mathbb{Z}$ is the same thing as a map $\theta : P \to \mathbb{C}/\mathbb{Z}$ satisfying

$$\theta(p_1 + p_2) = \theta(p_1) + \theta(p_2) + c(p_1,p_2) . \qquad (10.6)$$

We require it to satisfy $2\theta(p) = c(p,p)$.

We apply Definition (10.5) to the group $\Pi_{\partial X} = \mathrm{Map}(\partial X;T) \times \mathrm{Map}(\partial X;T^*)$. We associate to the surface X the group

$$P_X = \{(f,g) \in \mathrm{Map}(X;T_{\mathbb{C}}) \times \mathrm{Map}(X;T^*_{\mathbb{C}}) : dg = *idf\} .$$

Elements of P_X are determined by their restrictions to ∂X, so P_X is a subgroup of $\Pi_{\partial X,\mathbb{C}}$. In studying the cocycle on $\Pi_{\partial X}$ it is convenient to construct X from a plane polygon Y with $4g + 3k$ sides, where g is the genus of X and k is the number of boundary circles. We shall label the sides of Y cyclically

$$\alpha_1,\beta_1,\gamma_1,\delta_1,\alpha_2,\beta_2,\gamma_2,\delta_2,\ldots,\alpha_g,\beta_g,\gamma_g,\delta_g,\lambda_1,\sigma_1,\mu_1,\ldots,\lambda_k,\sigma_k,\mu_k, \qquad (10.7)$$

and identify γ_i with α_i^{-1}, δ_i with β_i^{-1}, and μ_i with λ_i^{-1}. Thus the sides σ_i become the boundary of X.

If (f_1, g_1) and (f_2, g_2) belong to P_X then we find

$$c((f_1,g_1),(f_2,g_2)) = -i \int_X \langle df_1, *df_2 \rangle \ .$$

This is symmetric in (f_1, f_2), so P_X is isotropic. We also see that there is a canonical choice for the character θ satisfying (10.6), namely

$$\theta_X(f,g) = -\frac{1}{2} i \int_X \langle df, *df \rangle \ .$$

Similarly, if $(f,g) \in P_X$ then

$$c((\bar{f},\bar{g}),(f,g)) = i \int_X \langle df, *d\bar{f} \rangle \ ,$$

which shows that P_X is positive. In fact we have

<u>Proposition (10.8)</u>. P_X is a positive maximal isotropic subgroup of $\Pi_{\partial X, \mathbb{C}}$, and is compatible with the polarization.

<u>Proof</u>: (a) To show that P_X is maximal isotropic we consider an element $(f,g) \in \Pi_{\partial X, \mathbb{C}}$ which is in the commutant of P_X, i.e. such that $\langle\langle f_1, g \rangle\rangle = \langle\langle g_1, f \rangle\rangle$ for all $(f_1, g_1) \in P_X$. In particular we can take $f_1 = \varphi$, $g_1 = -i\varphi$, where φ is an arbitrary holomorphic function $X \to \mathfrak{t}_{\mathbb{C}}$. We find

$$\int_{\partial X} \langle \varphi , dg - idf \rangle = 0 \ .$$

This shows that dg - idf is the boundary value of a holomorphic differential on X. Similarly, taking $f_1 = \bar{\varphi}$, $g_1 = i\bar{\varphi}$ we find that dg + idf is the boundary value of an antiholomorphic differential. Putting the two facts together, there is a harmonic 1-form ω on X such that $df = \omega|\partial X$, $dg = *i\omega|\partial X$. Let F and G be indefinite integrals of ω and $*i\omega$ defined on Y, and such that F agrees with f and G with g at one vertex of ∂Y. To complete the proof we must show that $\alpha_i(F), \beta_i(F)$, and $\sigma_i^*(F)$ belong to Λ, and $\alpha_i(G), \beta_i(G)$, and $\sigma_i^*(G)$ belong to Λ^*, where $\alpha_i(F)$ denotes $\int_{\alpha_i} dF$, etc., and $\sigma_i^*(F)$ denotes the constant difference between F and f along σ_i. But if $(f_1, g_1) \in P_X$ we calculate

$$<<f_1, g>> - <<g_1, f>> = \Sigma \{<\alpha_i(f_1), \beta_i(G)> - <\beta_i(f_1), \alpha_i(G)> + <\sigma_i(f_1), \sigma_i^*(G)>\}$$

$$- \Sigma \{<\alpha_i(g_1), \beta_i(F)> - <\beta_i(g_1), \alpha_i(F)> + <\sigma_i(g_1), \sigma_i^*(F)>\}$$

Now $\{\alpha_i(f_1), \beta_i(f_1), \sigma_i(f_1)\}$ describe the class of $f_1 : X \to T$ in $H^1(X;\Lambda)$. The proof is therefore completed by the observation that the group of components of P_X is $H^1(X;\Lambda) \oplus H^1(X;\Lambda^*)$, a fact which follows easily from the theorem that any element of $H^1(X;\mathfrak{t})$ can be represented by $*d\varphi$ for some harmonic map $\varphi : X \to \mathfrak{t}$.

(b) The operator J_{P_X} in $\text{Map}(\partial X;\mathfrak{t}) \oplus \text{Map}(\partial X;\mathfrak{t})$ which corresponds to P_X is $(f,g) \mapsto (j_X g, j_X f)$, where $j_X f$ is defined by $d(j_X f) = *dF$, where $F : X \to \mathfrak{t}$ is the unique harmonic extension of f. (Thus j_X is well-defined only up to the addition of a finite rank operator.) For the standard polarization j_X is replaced by the Hilbert transform. But it is easy to check that j_X differs from the Hilbert transform by a smoothing operator on ∂X, which is certainly of trace class.

Generalized toral theories and their parameter space

The theory we have just constructed is manifestly symmetric with respect to the tori T and T*, i.e. dual tori give rise to the same string theory. But we can say considerably more. The Hilbert space was constructed as a projective representation of the loop group Π of the torus $U = T \times T^*$. To define the cocycle (10.2) we did not need to identify the Lie algebra $\mathfrak{t}$ with its dual: we used only the vector space $\mathfrak{u} = \mathfrak{t} \oplus \mathfrak{t}^*$ with its natural indefinite inner product, and also the self-dual integral lattice $\Sigma = \Lambda \oplus \Lambda^*$ in it[1]. To define the polarization and the rays Ω_X, however, we did use the identification $\mathfrak{t} - \mathfrak{t}^*$: essentially we need an orthogonal transformation J of $\mathfrak{u}$ such that (i) $J^2 = 1$ and (ii) $\langle\xi, J\xi\rangle \geqslant 0$, for the definition of the subgroup P_X can be written

$$P_X = \{\varphi \in \mathrm{Map}(X;U_{\mathbb{C}}) \ : \ d\varphi = *iJd\varphi\} \ .$$

We do not even need the inner product on $\mathfrak{u}$ to have signature 0: it can be positive definite, in which case we must have $J = 1$, so that P_X is simply the group of holomorphic maps $X \to U_{\mathbb{C}}$. The fact that P_X is maximal isotropic is equivalent to the unimodularity of the lattice Σ, but to have a canonical splitting of its central extension we need Σ to be <u>even</u> (i.e. $\langle\lambda,\lambda\rangle \in 2\mathbb{Z}$) in addition. A lattice of the form $\Lambda \oplus \Lambda^*$ is automatically even. Let us recall, however, that an even unimodular lattice can exist only if the signature $p-q$ is divisible by 8. ([S7] Chapter 5, §2.2.)

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[1]Strictly it is the commutator pairing (and hence the isomorphism class of the extension) and not the cocycle which is defined by $(\mathfrak{u},\Sigma)$. The cocycle involves choosing an integral bilinear form B on $\mathfrak{u}$ such that $\langle\xi,\eta\rangle = B(\xi,\eta) + B(\eta,\xi)$. This exists only if the lattice is even. The general case is discussed in §12.

We now have a class of generalized toral theories parametrized by triples $(\mathcal{U},\Sigma,J)$, where Σ is an even unimodular lattice in the real inner-product space $\mathcal{U}$. If the inner product is indefinite the pair $(\mathcal{U},\Sigma)$ is determined ([S7] Chapter 5, Th. 6) up to isomorphism by the dimension and signature of $\mathcal{U}$, say $p + q$ and $p - q$. The automorphism group of $(\mathcal{U},\Sigma)$ is the discrete orthogonal group $\Gamma_{p,q} = O(\Sigma)$. The possible choices of J form the homogeneous space $O_{p,q}/O_p \times O_q$, so the parameter space of the theories is

$$M_{p,q} = \Gamma_{p,q} \backslash O_{p,q} / O_p \times O_q .$$

If $\mathcal{U}$ is positive definite, however, the parameter space is the discrete set of classes of even unimodular lattices.

The Hilbert space of the general toral theory breaks up $H = \bigoplus H_\sigma$, where σ runs through the lattice Σ. As before we have $H_0 = S(W_L) \otimes S(W_R)$ under $\mathfrak{D} \times \mathfrak{D}$, where now the left- and right-moving parts are associated to the splitting $\mathcal{U} = \mathcal{U}^+ \oplus \mathcal{U}^-$ into +1 and −1 eigenspaces of J. (Thus W_L (resp. W_R) is the positive- (resp. negative-) frequency part of $L\mathcal{U}_{\mathbb{C}} / \mathcal{U}_{\mathbb{C}}$.) In particular the theory has central charge (p,q), and is holomorphic if $\mathcal{U}$ is positive-definite. We shall discuss the positive-definite case in more detail in §12, without assuming that the lattice is unimodular. In general the splitting $\mathcal{U} = \mathcal{U}^+ \oplus \mathcal{U}^-$ is irrational with respect to the lattice Σ, and this prevents us factorizing the theory as a whole into left- and right-moving parts. When the splitting is rational the theory can be factorized as the product of a pair of weakly holomorphic theories based on a modular functor: this will be explained in §13.

It is interesting to consider the family of toral theories in the light of their infinitesimal automorphisms and deformations. From the formula (cf. (10.4))

$$\varphi(q)^p\varphi(\bar{q})^q \sum_{\sigma\epsilon\Sigma} q^{<\sigma_+,J\sigma_+>}\ \bar{q}^{<\sigma_-,J\sigma_->}$$

for the partition function we see that for a generic lattice (i.e. when $<\sigma\ ,J\sigma\ >$ is never integral) the fields of types (1,0),(0,1), and (1,1) are contained in H_0 and isomorphic to $\mathcal{U}^+_{\mathbb{C}}$, $\mathcal{U}^-_{\mathbb{C}}$ and $\mathcal{U}^+_{\mathbb{C}} \otimes \mathcal{U}^-_{\mathbb{C}}$ respectively, and are all primary. Now $\mathcal{U}^+_{\mathbb{C}} \oplus \mathcal{U}^-_{\mathbb{C}} = \mathcal{U}_{\mathbb{C}}$ is the complexified Lie algebra of the torus U, which is the obvious group of automorphisms of the theory. Similarly $\mathcal{U}^+_{\mathbb{C}} \otimes \mathcal{U}^-_{\mathbb{C}}$ is the complexified tangent space of the parameter space $M_{p,q}$, which suggests that the toral theories form a complete component of the moduli space of all theories.

Finally we should mention that when p = q = n the parameter space $M_{n,n}$ has as a covering space the moduli space of n-dimensional Riemannian tori T equipped with a translation-invariant 2-form ω. For if we write $U = T \times T^*$, as is always possible, and write $J : \mathfrak{t} \oplus \mathfrak{t}^* \to \mathfrak{t} \oplus \mathfrak{t}^*$ as a 2×2 matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

then $c : \mathfrak{t} \to \mathfrak{t}^*$ is a Riemannian metric on T and $\omega = ca : \mathfrak{t} \to \mathfrak{t}^*$ is a skew 2-form. One can easily check that c and ω can be prescribed arbitrarily, and determine b and d. In fact the moduli space of pairs (T,ω) is $GL_n(\mathbb{Z}) \backslash O_{n,n} / O_n \times O_n$, where $GL_n(\mathbb{Z})$ is the subgroup of $\Gamma_{n,n}$ which preserves the chosen decomposition $\Sigma = \Lambda \oplus \Lambda^*$.

From the point of view of string theory the theory corresponding to (T,ω) is that of strings moving in a constant background field ω: it is obtained by replacing the usual action functional for a map $f : X \to T$ from a surface to T by

$$S(f) = \frac{1}{2}\int_X \langle df, {*df}\rangle + \int_X f^*\omega \ .$$

The term involving ω depends only on the homotopy class of f, and so does not affect the classical equations of motion. From this standpoint, however, there are surprising equivalences between the theories for different (T,ω) coming from the fact that the true parameter space is a quotient by $\Gamma_{n,n}$ rather than $GL_n(\mathbb{Z})$.

§12. The loop group of a torus

The circle

For the loop group of a torus one can describe the representations and the associated modular functors very explicitly. Let us begin with the loops in $\mathbb{C}^\times$ - in fact, to keep the functoriality as clear as possible, let us begin with the group $\mathbb{C}^\times_S$ of smooth maps from an arbitrary compact oriented 1-manifold to $\mathbb{C}^\times$.

We have already pointed out in §8 that $\mathbb{C}^\times_S$ acts by pointwise multiplication on the polarized space $\Omega^{\frac{1}{2}}(S_\sigma)$ of $\frac{1}{2}$-forms on S. Here σ is a spin structure on S, i.e. a choice of a square-root of T*S, and $\Omega^{\frac{1}{2}}(S_\sigma)$ denotes the sections of the complexification of σ. This action gives us a central extension $\tilde{\mathbb{C}}^\times_{S,\sigma}$ of $\mathbb{C}^\times_S$ by $\mathbb{C}^\times$: an element of the extended group is a pair (γ,ϵ) with $\gamma \in \mathbb{C}^\times_S$ and ϵ an element of $\mathrm{Det}(W;\gamma W)$ for some subspace W of $\Omega^{\frac{1}{2}}(S_\sigma)$ belonging to the restricted Grassmannian. The extensions corresponding to different spin structures are isomorphic, but not canonically.

The Heisenberg group $\tilde{\mathbb{C}}^\times_{S,\sigma}$ has a canonical irreducible representation $H_{S,\sigma}$. This can be realized (cf. [PS]Ch.9) on the space of holomorphic $\mathbb{C}$-valued functions on $\mathbb{C}^\times_S/P$, where P is any suitable maximal isotropic subgroup of $\mathbb{C}^\times_S$. (The realization depends on the choice of a splitting of the induced extension $\tilde{P}$ of P: for $H_{S,\sigma}$ is the representation of $\tilde{\mathbb{C}}^\times_{S,\sigma}$ holomorphically induced from the character $\tilde{P} \to \mathbb{C}^\times$ given by the splitting.)

If X is a Riemann surface such that $\partial X = S$ then - as we shall prove - the group $\mathbb{C}^\times_X$ of holomorphic maps $X \to \mathbb{C}^\times$ is a suitable maximal isotropic subgroup of $\mathbb{C}^\times_S$. The natural way to define a splitting of the extension over $\mathbb{C}^\times_X$ is to choose an spin structure σ_X on X compatible with σ. Then the space W of holomorphic sections of σ_X belongs to the restricted Grassmannian of $\Omega^{\frac{1}{2}}(S_\sigma)$ and satisfies $\gamma W = W$ for all $\gamma \in \mathbb{C}_X^\times$; so $\gamma \mapsto (\gamma, \mathrm{id}_W)$ is a splitting.

At this point we have essentially defined a conformal field theory based on the category $\mathcal{C}^{\mathrm{spin}}$. For the realization of $H_{S,\sigma}$ as $\mathrm{Hol}(\mathbb{C}_S^\times/\mathbb{C}_X^\times)$ gives us a canonical ray in $H_{S,\sigma}$ - consisting of the constant functions - for each surface X with $\partial X = S$. And $H_{S,\sigma}$ has a hermitian form, characterized by the property that the action of $\mathbb{C}_S^\times$ is unitary (in the sense that $U(\gamma)^* = U(\bar{\gamma})$). The "sewing" property of the theory is obvious from this point of view.

This bosonic construction of the theory described by means of Fock spaces in §8 seems at first to have few advantages, for it entails proving that $\mathbb{C}_X^\times$ is a suitable maximal isotropic subgroup of $\mathbb{C}_{\partial X}^\times$; and in any case the theory of Heisenberg representations is not so elementary as that of Fock spaces. But the bosonic theory can be used in situations where there is no fermionic version, and it gives more information, as we shall see.

A central extension $\tilde{A}$ of an abelian group A by $\mathbb{C}^\times$ is determined up to non-canonical isomorphism by its commutator pairing

$$\langle\langle\ ,\ \rangle\rangle : A \times A \to \mathbb{C}^\times,$$

defined by $\langle\langle a_1, a_2\rangle\rangle = \tilde{a}_1\tilde{a}_2\tilde{a}_1^{-1}\tilde{a}_2^{-1}$, where $\tilde{a}_i \in \tilde{A}$ is a lift of a_i. The commutator pairing of the extension of $L\mathbb{C}^\times$ that we are interested in is

$$\langle\langle f,g\rangle\rangle = \tfrac{1}{2}\int_0^1 (f\,dg - g\,df) + \tfrac{1}{2}(f(0)\Delta_g - \Delta_f g(0)) \ . \qquad (12.1)$$

Here $\mathbb{C}^\times$ is regarded as $\mathbb{C}/\mathbb{Z}$, and elements of $L\mathbb{C}^\times$ as maps $f : \mathbb{R} \to \mathbb{C}$ such that $f(x + 1) = f(x) + \Delta_f$, where $\Delta_f \in \mathbb{Z}$ is the winding number. The pairing $\langle\langle\ ,\ \rangle\rangle$ is invariant under $\mathrm{Diff}^+(S^1)$.

We can obtain a definite central extension of $L\mathbb{C}^\times$ by giving a cocycle c, which must satisfy

$$c(f,g) - c(g,f) = \langle\langle f,g \rangle\rangle .$$

Unfortunately there is no choice of c which is invariant under $\mathrm{Diff}^+(S^1)$, but the formula

$$c(f,g) = \tfrac{1}{2}\int_0^1 f\,dg - \tfrac{1}{2}\Delta_f g(0) \qquad (12.2)$$

defines an extension $\tilde{A}$ of the group A of maps $f : \mathbb{R} \to \mathbb{C}$ such that $f(x + 1) = f(x) + \Delta_f$ which is invariant under the universal covering group of $\mathrm{Diff}^+(S^1)$, i.e. the diffeomorphisms φ of $\mathbb{R}$ satisfying $\varphi(x + 1) = \varphi(x) + 1$. Of course $L\mathbb{C}^\times = A/\mathbb{Z}$, so we get an extension of $L\mathbb{C}^\times$ by lifting the inclusion $\mathbb{Z} \to A$ to $\tilde{A}$. Because c vanishes on $\mathbb{Z} \times \mathbb{Z}$ the possible lifts simply correspond to homomorphisms $\mathbb{Z} \to \mathbb{C}^\times$, i.e. to elements $\sigma \in \mathbb{C}^\times$. It is easy to check that the double covering of $\mathrm{Diff}^+(S^1)$ acts on the extension corresponding to $\sigma = 1$, while $\mathrm{Diff}^+(S^1)$ itself acts when $\sigma = -1$. These are the two extensions which correspond to the two spin structures on the circle. Using them we can associate an extension $\tilde{\mathbb{C}}^\times_{S,\sigma}$ of $\mathbb{C}^\times_S$ to every oriented 1-manifold with a spin structure. In doing so it is important to notice the following point. From (12.1) we find that

$$\langle\langle f,g\rangle\rangle = \tfrac{1}{2}\,\Delta_f\Delta_g$$

if f and g have disjoint supports in S^1, in other words loops of winding number 1 <u>anticommute</u> if they have disjoint supports. For a disconnected 1-manifold it is therefore appropriate to look for extensions of $\mathbb{C}_S^\times$ whose commutator is given by

$$\langle\langle f,g\rangle\rangle = \sum_i \langle\langle f_i,g_i\rangle\rangle + \tfrac{1}{2}\sum_{i\neq j} \Delta_{f_i}\Delta_{g_i} ,$$

where f_i, g_i are the restrictions of f,g to the i^{th} component S_i of S. We therefore define $\tilde{\mathbb{C}}_{S,\sigma}^\times$ so that it contains each $\tilde{\mathbb{C}}_{S_i,\sigma_i}^\times$ as a subgroup, and satisfies

$$f.g = (-1)^{\Delta_f\Delta_g}\, g.f$$

when $f \in \tilde{\mathbb{C}}_{S_i,\sigma_i}$, $g \in \tilde{\mathbb{C}}_{S_j,\sigma_j}^\times$, and $i \neq j$. The natural way to achieve this is to use the cocycle

$$c(f,g) = \sum c_i(f,g) + \tfrac{1}{2}\sum_{i>j} \Delta_{f_i}\Delta_{g_j} , \tag{12.3}$$

where c_i is the cocycle defining $\tilde{\mathbb{C}}_{S_i}^\times$. Notice that this cocycle is defined only on a covering group A_S of $\mathbb{C}_S^\times$, and does not depend on the spin structure σ. The group $\tilde{\mathbb{C}}_S^\times$ is obtained as the quotient group of $\tilde{A}_S$ by the image of a lift of the inclusion $H^0(S;\mathbb{Z}) \to A_S$, and the lift <u>does</u> depend on the spin structure.

Proposition (12.4).

(i) For any Riemann surface X the subgroup $\mathbb{C}_X^\times$ is a positive maximal isotropic subgroup of $\mathbb{C}_{\partial X}^\times$, compatible with its polarization.

(ii) When restricted to $\mathbb{C}_X^\times \subset \mathbb{C}_{\partial X}^\times$ the cocycle (12.3) takes its values in $\{0,\frac{1}{2}\}$, so defines a canonical extension $\mathbb{C}_{X,\sigma}^\times$ of $\mathbb{C}_X^\times$ by $\mathbb{Z}/2$.

(iii) The extension $\mathbb{C}_{X,\sigma}^\times$ splits, and its splittings correspond canonically to the spin structures on X which restrict to σ on ∂X.

Proof: This is similar to Proposition (10.7), and we shall use the same notation. Suppose that two elements f,g of $\mathbb{C}_{\partial X}^\times$ are represented by smooth maps $f,g : Y \to \mathbb{C}$, where Y is the plane polygon with sides $\alpha_1,\ldots,\mu_k$. After some manipulation we find

$$c(f,g) = -\tfrac{1}{2}\, f \cup g \quad + \quad \tfrac{1}{2}\int_Y df \wedge dg \ , \tag{12.5}$$

where

$$f \cup g \quad = \quad \sum \{\alpha_i(f)\beta_i(g) \ - \ \beta_i(f)\alpha_i(g)\} \ .$$

If f and g are holomorphic then the integral over Y vanishes, and if f and g are well-defined on X then $c(f,g) \in \frac{1}{2}\mathbb{Z}$ and $c(f,g) - c(g,f) \in \mathbb{Z}$. So $\mathbb{C}_X^\times$ is isotropic. It is positive because if f is holomorphic then

$$\operatorname{Im} c(\bar{f},f) \quad = \quad \frac{1}{2i}\int_Y df \wedge d\bar{f} \geqslant 0 \ .$$

The proof that $\mathbb{C}_X^\times$ is maximal isotropic and compatible with the polarization is sufficiently like that of (10.7) to need no further comment.

Appendix B   Determinant lines

This appendix preserves the conventions of Appendix A, and all topological vector spaces are assumed to be allowable. An operator of determinant class means one of the form $1 + T$ where T is of trace class, and $\det(1 + T)$ is defined by (A.12).

Definition (B.1). An operator $T : E \to F$ between complete locally convex vector spaces is Fredholm if it is invertible modulo compact operators, or, equivalently, modulo operators of finite rank.

If T is Fredholm then it has closed range and finite dimensional kernel and cokernel. If E and F are Fréchet the converse is also true. (Cf. [S1]) The index of a Fredholm operator T is the integer $\dim(\ker T) - \dim(\operatorname{coker} T)$.

Definition (B.2).

(i) If $T : E \to F$ is Fredholm of index 0 then Det_T is the line whose points are pairs $[\theta,\lambda]$, where $\lambda \in \mathbb{C}$ and $\theta : E \to F$ differs from T by a trace-class operator, subject to the equivalence relation generated by

$$[\theta\varphi,\lambda] \sim [\theta,(\det\varphi).\lambda]$$

when $\varphi : E \to E$ is of determinant class.

(ii) If $T : E \to F$ is Fredholm of index n then $\mathrm{Det}_T = \mathrm{Det}_{\tilde{T}}$, where $\tilde{T} = T \oplus 0 : E \to F \oplus \mathbb{C}^n$ if $n > 0$, and $\tilde{T} = T \oplus 0 : E \oplus \mathbb{C}^{-n} \to F$ if $n < 0$.

Remarks. If T has index 0 one can always choose an invertible θ such that θ-T is of trace class. Then $\lambda \mapsto [\theta,\lambda]$ defines an isomorphism $\mathbb{C} \to \mathrm{Det}_T$. If θ is not invertible then $[\theta,\lambda] = 0$ for all λ.

Definition (B.4). If $T : E \to F$ is Fredholm of index 0 then det(T) is the element [T,1] of the line Det_T. If index(T) $\neq 0$ then det(T) $= 0 \in \mathrm{Det}_T$.

Corollary (B.5). T is invertible if and only if det(T) $\neq 0$.

Proposition (B.6). If dim(ker T) = p and dim(coker T) = q there is a canonical isomorphism

$$\mathrm{Det}_T \cong \Lambda^p(\ker T)^* \otimes \Lambda^q(\mathrm{coker}\, T) .$$

Proof: It is enough to prove this when p = q. Let $\alpha_1,\ldots,\alpha_p$ be a basis for (ker T)* and $\eta_1,\ldots,\eta_q$ be a basis for coker T. Let $\tilde{\alpha}_i : E \to \mathbb{C}$ be an extension of α_i. Then

$$[\, T + \Sigma\, \eta_i \otimes \tilde{\alpha}_i \,,\, 1 \,] \longleftrightarrow \alpha_1 \wedge \ldots \wedge \alpha_p \otimes \eta_1 \wedge \ldots \wedge \eta_p$$

defines the isomorphism.

We shall now show that the lines Det_T depend holomorphically on T, i.e.

Proposition (B.7). If $\{T_x : E_x \to F_x\}_{x \in X}$ is a holomorphic family of Fredholm operators then the lines Det_{T_x} form a holomorphic line bundle on X.

Here the meaning of 'holomorphic family' must be understood in the sense of [S1], i.e. $\{E_x\}$ and $\{F_x\}$ are holomorphic vector bundles on X in the weakest sense, but we assume that there exists a continuous parametrix $\{P_x : F_x \to E_x\}$ such that the <u>families</u> $\{T_xP_x - 1\}$ and $\{P_xT_x - 1\}$ are compact, i.e. are compact operators which depend continuously on x in the uniform topology.

<u>Proof of (B.7)</u>. We can assume the bundles $\{E_x\}$ and $\{F_x\}$ are trivial, and that the T_x have index 0. Then for each finite rank operator $t : E \to F$ the set

$$U_t = \{ x \in X : T_x + t \text{ is invertible} \}$$

is open in X. We trivialize the lines Det_{T_x} for $x \in U_t$ by $x \mapsto [T_x + t, 1]$, and in the intersection $U_{t_0} \cap U_{t_1}$ the transition function is

$$\begin{aligned} x \mapsto & \det((T_x + t_1)(T_x + t_0)^{-1}) \\ & = \det(1 + (t_1 - t_0)(T_x + t_0)^{-1}) , \end{aligned}$$

which is holomorphic.

The main general fact about determinant lines is

<u>Proposition (B.8)</u>. Let

$$\begin{array}{ccccccccc} 0 & \to & E' & \to & E & \to & E'' & \to & 0 \\ & & \downarrow T' & & \downarrow T & & \downarrow T'' & & \\ 0 & \to & F' & \to & F & \to & F'' & \to & 0 \end{array}$$

be a commutative diagram of topological vector spaces with exact rows and Fredholm columns. Then there is a canonical isomorphism

$$\mathrm{Det}_T \cong \mathrm{Det}_{T'} \otimes \mathrm{Det}_{T''}$$

which depends holomorphically on T',T,T".

"Exact rows" means of course that E' and F' are topological subspaces of E and F, and that E" and F" have the quotient topology.

Proof: If t',t" are finite rank operators such that T' + t' and T" + t" are invertible then one can find t of finite rank such that T + t forms a commutative diagram with T' + t' and T" + t". The desired isomorphism is

$$[T + t,\ 1] \longleftrightarrow [T' + t',\ 1] \otimes [T'' + t'',\ 1]\ .$$

The determinant line, the restricted Grassmannian, and the central extension of GL_{res}

A polarized topological vector space E (in the sense of Definition (8.8)) has a restricted Grassmannian Gr(E) which consists of the (+1)-eigenspaces of the preferred involutions J which define the polarization. If W_0 and W_1 are two points of Gr(E) there is a preferred class of Fredholm operators $T : W_0 \to W_1$, namely those which differ from the inclusion $W_0 \to H$ by trace class operators.

Definition (B.9). For W_0, W_1 in Gr(E) we write $\mathrm{Det}(W_0;W_1)$ for Det_T, where $T : W_0 \to W_1$ is such a Fredholm operator.

From (B.3) we know that $\mathrm{Det}(W_0;W_1)$ does not depend on the choice of T. If W_0 is held fixed then $\bigcup_W \mathrm{Det}(W_0;W)$ is a holomorphic line bundle on Gr(E). This is the determinant line bundle of [PS](Chap. 10): there the chosen W_0 was called H_+.

For three spaces W_0, W_1, W_2 we clearly have

$$\mathrm{Det}(W_0;W_2) \cong \mathrm{Det}(W_0;W_1) \otimes \mathrm{Det}(W_1;W_2) .$$

Now suppose that $g : E \to E$ belongs to the restricted general linear group of E, i.e. that $gJg^{-1} - J$ is of trace class for a preferred involution J.

Definition (B.10). For $g \in GL_{res}(E)$ we define

$$\mathrm{Det}_g = \mathrm{Det}(W;gW) ,$$

where W is an element of Gr(E).

The line Det_g is independent of W, for if W_0 and W_1 are two choices then g defines an isomorphism between $\mathrm{Det}(W_0;W_1)$ and $\mathrm{Det}(gW_0;gW_1)$, and hence between $\mathrm{Det}(W_0;gW_0)$ and

$$\mathrm{Det}(W_0;gW_0) \otimes \mathrm{Det}(W_0;W_1)^* \otimes \mathrm{Det}(gW_0;gW_1) = \mathrm{Det}(W_1;gW_1) .$$

Evidently $\mathrm{Det}_{g_1 g_2} \cong \mathrm{Det}_{g_1} \otimes \mathrm{Det}_{g_2}$, and so we can define a central extension of $GL_{res}(E)$ which consists of all pairs $(g.\lambda)$ with $\lambda \in \mathrm{Det}_g$. By its construction this group acts holomorphically on the line bundle $\bigcup_W \mathrm{Det}(W_0, W)$ for any choice of W_0.

Riemann surfaces

We conclude this appendix with a result on the determinant line of a Riemann surface.

Proposition (B.11). If a closed Riemann surface Z is the union of two surfaces X and Y which intersect in a 1-manifold S then Det_Z is canonically isomorphic to the determinant line of the map

$$\begin{array}{c} \mathrm{Hol}(X) \oplus \mathrm{Hol}(Y) \to \Omega^0(S) \\ (\ f\ ,\ g\ ) \longmapsto (f|S)-(g|S)\ . \end{array}$$

Proof: First consider the diagram

$$\begin{array}{ccccc} \Omega^0(Z) & \to & \Omega^0(X) \oplus \Omega^0(Y) & \to & J^0(S) \\ \downarrow \bar\partial & & \downarrow & & \downarrow \\ \Omega^{01}(Z) & \to & \Omega^{01}(X) \oplus \Omega^{01}(Y) \oplus \Omega^0(S) & \to & J^{01}(S) \oplus \Omega^0(S)\ , \end{array}$$

where $J^0(S)$ (resp. $J^{01}(S)$) is the space of infinite jets of functions (resp. (0,1)-forms) on Z along S, the middle vertical map is $(f,g) \mapsto (\bar{\partial}f, \bar{\partial}g, (f|S)-(g|S))$,

and the right-hand vertical map is $f \mapsto (\bar{\partial}f, f|S)$.

The horizontal maps are defined in the obvious way to give short exact sequences. Notice that an element f of $J^0(S)$ can be identified with the sequence $\{f_k\}$ of smooth functions on S such that $f = \Sigma f_k y^k$, where y is a coordinate on Z transverse to S. The right-hand vertical map is then

$$\{f_k\} \longmapsto \{ f_0 , f_0' + f_1 , f_1' + f_2 , \ldots \} ,$$

and is therefore an isomorphism. Thus by (B.8) we can identify Det_Z with the determinant of the middle map. Proposition (8.11) is then obtained from the diagram

$$\begin{array}{ccccc} Hol(X) \oplus Hol(Y) & \to & \Omega^0(X) \oplus \Omega^0(Y) & \to & \Omega^{01}(X) \oplus \Omega^{01}(Y) \\ \downarrow & & \downarrow & & \downarrow id \\ \Omega^0(S) & \to & \Omega^{01}(X) \oplus \Omega^{01}(Y) \oplus \Omega^0(S) & \to & \Omega^{01}(X) \oplus \Omega^{01}(Y) \end{array}$$

just as in the proof of (6.3).

Corollary (B.12). In the situation of (B.11) we have

$$Det_Z \cong Det_X \otimes Det_Y .$$

Proof: The right-hand side is the determinant of the map

$$\begin{aligned} \mathrm{Hol}(X) \oplus \mathrm{Hol}(Y) &\to \Omega^0(S) \\ (\ f\ ,\ g\ ) &\mapsto (f|S)_- - (g|S)_+ \ . \end{aligned}$$

This differs from the map of (B.9) by a trace-class operator (cf. the proof of (6.4)).

Notice that (B.12) provides the proof of Proposition (6.5), which was omitted earlier.