

On the problem of gauge theories in locally covariant QFT

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Based on published and ongoing works with (various subsets of) the following people:

C. Becker, M. Benini, C. Dappiaggi, T.-P. Hack and R. J. Szabo

Outline

1. Locally covariant quantum field theory
2. Differential cohomology
3. Construction and analysis of the $U(1)$ Yang-Mills model
4. Towards new axioms for gauge QFTs

Locally covariant quantum field theory

Axiomatic QFT on curved spacetimes

- ◇ On the Minkowski spacetime \mathbb{R}^4 , a QFT is an assignment of $(C^*$ -)algebras $\mathcal{A}(U)$ to suitable open subsets $U \subseteq \mathbb{R}^4$ satisfying the **Haag-Kastler axioms**.
- ◇ Parts of the Haag-Kastler axioms can be rephrased by demanding that $\mathcal{A} : \mathcal{O}(\mathbb{R}^4) \rightarrow \text{Alg}$ is a **covariant functor** from a suitable category of open subsets in \mathbb{R}^4 to the category of $(C^*$ -)algebras.
- ◇ This part can be easily generalized to curved spacetimes; replace $\mathcal{O}(\mathbb{R}^4)$ by the category of **m -dimensional globally hyperbolic spacetimes** Loc^m .
- ◇ Hence, a reasonable starting point for axiomatic QFT on curved spacetimes seems to be the **functor category** $\text{Fun}(\text{Loc}^m, \text{Alg})$.
- ◇ The delicate issue is to impose suitable **extra axioms** characterizing those functors which “deserve to be called a QFT”. In other words, we would like to have a full subcategory QFT^m of $\text{Fun}(\text{Loc}^m, \text{Alg})$ such that
 - 1.) QFT^m is ‘big’ enough to contain the models relevant in physics.
 - 2.) QFT^m is ‘small’ enough to allow for model independent studies.

The axioms of locally covariant QFT

- ◇ The subcategory LCT^m of locally covariant QFTs is characterized by the **Brunetti-Fredenhagen-Verch (BFV) axioms** for functors $\mathcal{A} : \text{Loc}^m \rightarrow \text{Alg}$:

Locality: For any Loc^m -morphism $f : M \rightarrow N$ the Alg-morphism $f_* := \mathcal{A}(f) : \mathcal{A}(M) \rightarrow \mathcal{A}(N)$ is monic.

Causality: If $M_1 \xrightarrow{f_1} N \xleftarrow{f_2} M_2$ are Loc^m -morphisms with causally disjoint image, then $f_{1*} \mathcal{A}(M_1)$ and $f_{2*} \mathcal{A}(M_2)$ commute in $\mathcal{A}(N)$.

Time-slice: If $f : M \rightarrow N$ is Cauchy, then $f_* : \mathcal{A}(M) \rightarrow \mathcal{A}(N)$ is Alg-isomorphism.

- ◇ Model independent results include the spin-statistics theorem [Verch; see also Fewster's talk] and statements on pert. renormalization [Brunetti, Fredenhagen, ...].

- ◇ **Question:** Does LCT^m include physically relevant models?

😊 Klein-Gordon and Dirac are objects in LCT^m (Dirac uses spin manifolds)

☹ Yang-Mills is **not** an object in LCT^m

- ◇ **Goals of my talk:**

- 1.) Construct explicitly the full $U(1)$ Yang-Mills model (including all bundles).
- 2.) Use the results to explain which axioms are violated and why.
- 3.) Sketch some preliminary thoughts on different axioms which are motivated by the theory of stacks [see Schreiber's talk for more on stacks].

Differential cohomology

Why differential cohomology?

- ◇ Our construction of the quantum $U(1)$ Yang-Mills model is via quantization of the classical model \rightsquigarrow need to understand the configuration space first!
- ◇ The field configurations over a manifold M are pairs (P, ∇) consisting of a $U(1)$ -bundle P over M and a connection ∇ on P .

Rem: In previous studies, the bundle P has been fixed from the outside (typically to the trivial one). Here I want to take also bundles as parts of the dynamical degrees of freedom \rightsquigarrow much richer structure!

- ◇ **Gauge equivalence:** $(P, \nabla) \sim (P', \nabla')$ iff there exists a $U(1)$ -bundle isomorphism $\psi : P \rightarrow P'$ preserving the connections.
- ◇ Take **gauge orbit space** $\widehat{H}^2(M) = \{(P, \nabla)\} / \sim$ as configuration space.
- ◇ The role of differential cohomology is to provide an efficient description of the spaces $\widehat{H}^2(M)$, as well as their analogs $\widehat{H}^k(M)$ for all $k \in \mathbb{N}$.

NB: Physically, $k = 1$ describes the σ -model with target $U(1)$, $k = 2$ the $U(1)$ Yang-Mills model and $k > 2$ connections on higher $U(1)$ -bundles.

Differential cohomology theories

Def: A **differential cohomology theory** is a contravariant functor $\widehat{H}^* : \text{Man} \rightarrow \text{Ab}^{\mathbb{Z}}$ together with natural transformations (curv , char , ι , κ), such that the following diagram commutes and has exact rows and columns:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & \frac{H^{*-1}(M; \mathbb{R})}{H_{\text{free}}^{*-1}(M; \mathbb{Z})} & \longrightarrow & \frac{\Omega^{*-1}(M)}{\Omega_{\mathbb{Z}}^{*-1}(M)} & \xrightarrow{d} & d\Omega^{*-1}(M) \longrightarrow 0 \\
 & & \downarrow & & \downarrow \iota & & \downarrow \\
 0 & \longrightarrow & H^{*-1}(M; U(1)) & \xrightarrow{\kappa} & \widehat{H}^*(M) & \xrightarrow{\text{curv}} & \Omega_{\mathbb{Z}}^*(M) \longrightarrow 0 \\
 & & \downarrow & & \downarrow \text{char} & & \downarrow \\
 0 & \longrightarrow & H_{\text{tor}}^*(M; \mathbb{Z}) & \longrightarrow & H^*(M; \mathbb{Z}) & \longrightarrow & H_{\text{free}}^*(M; \mathbb{Z}) \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0
 \end{array}$$

Thm: [Simons, Sullivan; Bär, Becker] Differential cohomology theories **exist** (e.g. Cheeger-Simons theory) and are **unique** up to a unique natural isomorphism.

Construction and analysis of the $U(1)$ Yang-Mills model

Bird's-eye view on the construction

- ◇ The following construction is explained in my paper with Becker and Szabo.
 1. Take any differential cohomology theory $\widehat{H}^* : \text{Man} \rightarrow \text{Ab}^{\mathbb{Z}}$ and restrict it to the category Loc^m and some fixed degree $k \in \mathbb{N}$.
 2. Use Lagrangian $\mathcal{L}(h) = \frac{1}{2} \text{curv}(h) \wedge * \text{curv}(h)$ to construct subfunctor $\text{Sol}^k : \text{Loc}^m \rightarrow \text{Ab}$ of \widehat{H}^k describing solutions to EL-equations.
(NB: Sol^k fits into a restriction of the commutative diagram of exact sequences.)
 3. Realize that Sol^k is a contravariant functor to Abelian Fréchet-Lie groups with Poisson structure coming from Lagrangian \mathcal{L} (Peierls construction).
(NB: The isomorphism type of $\text{Sol}^k(M)$ is $U(1)^n \oplus H^k(M; \mathbb{Z}) \oplus F$, where $n \in \mathbb{N}^0$ and F is ∞ -dimensional Fréchet space.)
 4. Take smooth Pontryagin dual $\text{Hom}^\infty(\cdot, U(1))$ to get covariant functor $\text{PS}^k : \text{Loc}^m \rightarrow \text{PAb}$ valued in presymplectic Abelian groups.
(NB: $\text{PS}^k(M)$ describes cl. observables given by group characters $\text{Sol}^k(M) \rightarrow U(1)$.)
 5. Compose PS^k with the CCR-functor $\text{CCR} : \text{PAb} \rightarrow \text{Alg}$.

Thm: The construction above provides an object $\mathcal{A}^k := \text{CCR} \circ \text{PS}^k$ in the functor category $\text{Fun}(\text{Loc}^m, \text{Alg})$, for any $k \in \mathbb{N}$.

Properties of the $U(1)$ Yang-Mills model

Thm: \mathcal{A}^k satisfies the causality and time-slice axiom, but it violates the locality axiom (unless accidentally $(m, k) = (2, 1)$, i.e. σ -model in 2-dimensions).

- ◇ **Mathematical reason:** There exists subfunctor $\mathcal{A}_{\text{top}}^k$ of \mathcal{A}^k isomorphic to $\text{CCR} \circ (H^{m-k}(\cdot; \mathbb{R})^* \oplus H^k(\cdot; \mathbb{Z})^*)$. For $f: M \rightarrow N$ we have $\mathcal{A}(f)$ monic iff the push-forwards of f on the (dual) cohomology groups are monic. ⚡
- ◇ **Physical reason:** $\mathcal{A}_{\text{top}}^k$ is a topological theory measuring **top. invariants** of (higher) $U(1)$ -bundles and **electric fluxes**. Field configurations carrying top. charges in general do not extend from smaller to larger spacetimes.

(The same should hold true for **any** model with bundles or other topological degrees of freedom! In particular, the non-Abelian Yang-Mills model.)

Thm: For $k \geq 2$, \mathcal{A}^k does not satisfy the **additivity property**, i.e. there exists M and open cover by diamonds $\{\iota_i: D_i \rightarrow M\}$ such that $\text{id}_{\mathcal{A}^k(M)} \not\cong \bigvee_i \iota_{i*}$.

Sketch: Let M be such that $H^{k-1}(M, U(1)) \ni u \neq 0$ and take Weyl-operator $W(w) \in \mathcal{A}^k(M)$ such that $w(\kappa(u)) \neq 0$. Notice that for any Weyl-operator $W(w_i) \in \iota_{i*} \mathcal{A}^k(D_i)$ we have $w_i(\kappa(u)) = 0$, since $H^{k-1}(D_i, U(1)) = 0$. Hence, $W(w) \notin \bigvee_i \iota_{i*} \mathcal{A}^k(D_i)$.

- ◇ **Physical interpretation:** Flat fields can not be measured in diamonds.

Towards new axioms for gauge QFTs

Motivation

- ◇ The problems of $\mathcal{A}^k : \text{Loc}^m \rightarrow \text{Alg}$ can already be seen at the level of the configuration spaces $\widehat{H}^k : \text{Man} \rightarrow \text{Ab}$, morally:
 - \mathcal{A}^k is not additive, for $k \geq 2$ “ \iff ” \widehat{H}^k is not a sheaf, for $k \geq 2$
 - \mathcal{A}^k violates locality property “ \iff ” \widehat{H}^k is not flabby (or c -soft)
- ◇ Should we really use \widehat{H}^k for describing the configuration space?
- ◇ Modern point of view: [\[see Schreiber's talk for a proper explanation\]](#)
 - Instead of assigning to a manifold M the gauge orbit space $\widehat{H}^2(M) = \{(P, \nabla)\} / \sim$ we better should assign the **groupoid** $\mathcal{G}(M)$ of bundle-connection pairs with morphisms given by gauge equivalence.
 - This gives a contravariant pseudo-functor $\mathcal{G} : \text{Man} \rightarrow \text{Groupoids}$ to the category of groupoids (or equivalently a category fibered in groupoids over Man).
 - *Fact:* \mathcal{G} is a **stack** (called $BU(1)_{\text{con}}$), i.e. it satisfies “sheaf-like gluing conditions up to gauge transformation” for all open covers.
 - *Warning:* I will not discuss (i.e. I do not understand yet) the important concept of a **smooth stack**, which will eventually gives rise to smooth groupoids.
- ◇ **Goal:** Develop some axioms for theories of “observables on stacks”.

A working definition

- ◇ It seems reasonable to take as “functions on a groupoid” the **differential graded algebra** of functions on its simplicial set.

(NB: The van Est map gives the BRST-BV formalism of Fredenhagen-Rejzner when we work *infinitesimally/perturbatively* with the Lie algebroids of Lie groupoids.)

- ◇ This motivates me to study the covariant pseudo-functor category $\text{PsFun}(\text{Loc}^m, \text{dgAlg})$ for axiomatizing gauge QFTs.

Def: An object \mathcal{A} in $\text{PsFun}(\text{Loc}^m, \text{dgAlg})$ is called a **weak QFT** if it satisfies the diamond-locality (DL), causality (C), time-slice (T) and additivity (A) axioms:

- (DL):** For any diamond $\iota : D \rightarrow M$ the dgAlg -morphism $\iota_* := \mathcal{A}(\iota) : \mathcal{A}(D) \rightarrow \mathcal{A}(M)$ is cochain homotopic to a monic dgAlg -morphism.
- (C):** If $M_1 \xrightarrow{f_1} N \xleftarrow{f_2} M_2$ are Loc^m -morphisms with causally disjoint image, then the graded commutator $[\cdot, \cdot]_{\text{gr}} : f_{1*} \mathcal{A}(M_1) \otimes_{\text{gr}} f_{2*} \mathcal{A}(M_2) \rightarrow \mathcal{A}(N)$ is cochain homotopic to 0.
- (T):** If $f : M \rightarrow N$ is Cauchy, then $f_* : \mathcal{A}(M) \rightarrow \mathcal{A}(N)$ is cochain homotopic to a dgAlg -isomorphism.
- (A):** For any object M and any open cover $\{\iota_i : D_i \rightarrow M\}$ (by diamonds) we have $\text{id}_{\mathcal{A}(M)} \cong \bigvee_i \iota_{i*}$ up to cochain homotopy.

Remarks and open questions

◇ Remarks:

- Additive locally covariant QFTs are included in my axioms by considering algebras as degree-zero differential graded algebras.
- The axioms are designed to describe the *full content* of a gauge theory, i.e. gauge fields, ghosts, etc., and not only the gauge invariant content.
- For gauge invariant observables we have seen explicitly that additivity doesn't hold, while for the full content I expect this property to hold. This is related to the fact that gauge orbits don't satisfy the gluing conditions of a sheaf, while the full configuration groupoids glue in the sense of stacks.
- It seems reasonable to demand the locality axiom only for diamond embeddings $\iota : D \rightarrow M$, since the general locality axiom is dual to a flabbiness property for the configuration stack and we already know that bundles (even before taking gauge equivalence classes) in general do not extend.

◇ Open problems:

- 1.) Construct the quantum $U(1)$ Yang-Mills model via the (smooth?!?) stack $BU(1)_{\text{con}}$ and show that it fits into the new axioms.
- 2.) Can we prove some nice model independent results (e.g. spin statistics, perturbative renormalization, ...)?