Atiyah algebroids for higher and groupoid gauge theories

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Motivation

Higher gauge theory is everywhere

- B-field in string theory / supergravity
- Everything related:
  - Generalized geometry
  - T-duality
  - $\vdots$
- RR-fields in string theory / supergravity
- 6d superconformal field theories
- $\vdots$

But also and here:

- Categorification, as deformation theory, to study math. objects
- Lessons learned: Groupoid gauge theory

Most importantly: Non-abelian gerbes exist and appear in physics!

Outline

I. What's subtle about higher gauge theory?
Usual connections on higher principal bundles do not match expectations from physics.

II. Adjusted connections in higher gauge theory
This can be fixed, and fixing this leads to interesting new mathematical structures.

III. Systematic construction: Atiyah algebroid
Atiyah algebroid perspective and generalization leads to a much better, systematic understanding.

IV. Adjusted connections in groupoid gauge theory
Adjustment also required in groupoid gauge theory, e.g. gauged sigma models.

I. What's subtle about higher gauge theory?

Homotopy Maurer–Cartan theory: L_∞-algebras

L_∞-algebra in "bracket picture":

- Graded vector space
  $\mathfrak{L} = \cdots \oplus \mathfrak{L}_{-2} \oplus \mathfrak{L}_{-1} \oplus \mathfrak{L}_0 \oplus \mathfrak{L}_1 \oplus \mathfrak{L}_2 \oplus \ldots$
- μ_1 is a differential, hence (cochain) complex:
  $$\ldots \xrightarrow{\mu_1} \mathfrak{L}_{-2} \xrightarrow{\mu_1} \mathfrak{L}_{-1} \xrightarrow{\mu_1} \mathfrak{L}_0 \xrightarrow{\mu_1} \mathfrak{L}_1 \xrightarrow{\mu_1} \mathfrak{L}_2 \xrightarrow{\mu_1} \ldots$$
- Graded totally antisymmetric multilinear products
  $$\mu_i : \wedge^i \mathfrak{L} \to \mathfrak{L}~, \quad |\mu_i| = 2 - i$$
- Satisfying higher/homotopy Jacobi identity:
  $$\sum_{i+j=n} \sum_{\sigma \in \mathrm{Sh}(i,n-i)} \pm \mu_{i+1}(\mu_j(\ell_{\sigma(1)}, \ldots, \ell_{\sigma(j)}), \ell_{\sigma(j+1)}, \ldots, \ell_{\sigma(n)}) = 0$$
- Metric/cyclic structure on L_∞-algebra $\mathfrak{L}$:
  $$\langle -, - \rangle : \mathfrak{L} \times \mathfrak{L} \to \mathbb{R}$$
  non-degenerate, graded symmetric, bilinear, cyclic.

L_∞-algebras come with their own gauge theory

Maurer–Cartan equation for differential graded Lie algebra, $(\mathfrak{g}, \mathrm{d})$:

$$\mathrm{d}a + \tfrac{1}{2}[a,a] = 0\ , \quad a \in \mathfrak{g}\ .$$

Homotopy Maurer–Cartan eqn: (a: gauge potential f: curvature)

$$f := \mu_1(a) + \tfrac{1}{2}\mu_2(a,a) + \tfrac{1}{3!}\mu_3(a,a,a) + \cdots = 0\ , \quad a \in \mathfrak{L}_1$$

(Higher) gauge transformations: homotopies.

Bianchi identity:

$$\mu_1(f) - \mu_2(f,a) + \tfrac{1}{2}\mu_3(f,a,a) - \tfrac{1}{3!}\mu_4(f,a,a,a) + \cdots = 0\ .$$

Homotopy Maurer–Cartan Action:

$$S_{\mathrm{MC}}[a] := \sum_{i \geq 1} \frac{1}{(i+1)!} \langle a, \mu_i(a,\ldots,a)\rangle_{\mathfrak{L}}\ .$$

BV: Any (...) field theory is a hMC theory, cf. Greg's/Luigi's talks.

Example: 4d Chern–Simons theory

Note: $\mathcal{C}om \otimes \mathcal{L}ie = \mathcal{L}ie$, therefore:

dg commutative algebra $\otimes$ L_∞-algebra $=$ L_∞-algebra

Let's consider $\Omega^\bullet(M) \otimes (\mathfrak{L}_{-1} \oplus \mathfrak{L}_0)$

- higher products are $\hat{\mu}_1 = \mathrm{d} + \mu_1$, μ_2, μ_3
- gauge potential

$$A + B \in \hat{\mathfrak{L}}_1 = \Omega^1(M) \otimes \mathfrak{L}_0 \quad \oplus \quad \Omega^2(M) \otimes \mathfrak{L}_{-1}$$

- Homotopy Maurer–Cartan equation:

$$\begin{aligned} F &= \mathrm{d}A + \tfrac{1}{2}\mu_2(A, A) + \mu_1(B) && = 0 \\ H &= \mathrm{d}B + \mu_2(A, B) + \tfrac{1}{3!}\mu_3(A, A, A) && = 0 \end{aligned}$$

- Bianchi identity:

$$\mathrm{d}H + \mu_2(A, H) = \mu_2(F, B) + \tfrac{1}{2}\mu_3(F, A, A)$$

But: SUGRA, etc: $\mathrm{d}H = \langle F \wedge F \rangle$

Alternative picture

Connection on bundle P: splitting of Atiyah algebroid sequence

$$0 \longrightarrow P \times_{\mathsf{G}} \mathsf{Lie}(\mathsf{G}) \longrightarrow \underbrace{TP/\mathsf{G}}_{\mathfrak{at}(P)} \longrightarrow TM \longrightarrow 0$$

Atiyah, 1957

Related approach: Cartan, Kotov/Strobl, Sati/Schreiber/Stasheff

- Locally, connection is map from $T_xM \to \mathfrak{g}$
- $\mathfrak{g}$ and TM have dual Chevalley–Eilenberg algebras
  - $\mathsf{CE}(\mathfrak{g})$ generated by $\xi^\alpha \in \mathfrak{g}[1]^*$ with $Q\xi^\alpha = -\frac{1}{2} f^\alpha_{\beta\gamma}\xi^\beta\xi^\gamma$
  - $\mathsf{CE}(TM) = \Omega^\bullet(M)$
- Gauge potential dually as morphism of graded com algebras:

$$a^* : \mathsf{CE}(\mathfrak{g}) \to \Omega^\bullet(M) \quad , \quad \xi^\alpha \mapsto A^a_\mu \mathrm{d}x^\mu := a^*(\xi^a)$$

- Curvature: failure of a to be morphism of dgcas:

$$F^a := (\mathrm{d} \circ a^* - a^* \circ Q)(\xi^a) = \mathrm{d}A^a + \tfrac{1}{2} f^a_{bc} A^b \wedge A^c$$

Non-flat connections

- Rather: work in category of dg manifold, dg morphisms
- Double CE algebra to Weil algebra $\mathsf{W}(\mathfrak{g}) := \mathsf{CE}(\mathfrak{inn}(\mathfrak{g}))$

$$\mathsf{W}(\mathfrak{g}) := C^\infty(\overset{\sigma\xi^\alpha}{T[1]}\overset{\xi^\alpha}{\mathfrak{g}[1]})\ ,\quad Q = Q_{\rm CE} + \sigma\ ,\quad \sigma Q_{\rm CE} = -Q_{\rm CE}\sigma$$

- Potentials/curvatures/Bianchi identities from dgca-morphisms

$$(A, F) : \mathsf{W}(\mathfrak{g}) \to \Omega^\bullet(M) = W(M)$$
$$\xi^\alpha \mapsto A^\alpha$$
$$(\sigma\xi^\alpha) = Q\xi^\alpha + \tfrac{1}{2}f^\alpha_{\beta\gamma}\xi^\beta\xi^\gamma \mapsto F^\alpha = (\mathrm{d}A + \tfrac{1}{2}[A, A])^\alpha$$
$$Q(\sigma\xi^\alpha) = -f^\alpha_{\beta\gamma}(\sigma\xi^\alpha)\xi^\beta \mapsto (\nabla F)^\alpha = 0$$

- Gauge transformations: homotopies between dgca-morphisms
- Topological invariants: invariant polynomials in $\mathsf{W}(\mathfrak{g})$
- Details: Sati/Schreiber/Stasheff (2008)

Chern–Simons terms

After all this, still Problem: wrong Bianchi identities, e.g. $\mathfrak{string}(n)$:

$$\mathrm{d}H = \langle F \wedge F \rangle \quad \text{vs} \quad \mathrm{d}H = -\tfrac{1}{2}(\mathrm{d}A, [A, A])$$

Solution: Sati/Schreiber/Stasheff (2008)
Can deform Weil algebra by Chern–Simons terms to correct.

- Weil algebra projects to Chevalley–Eilenberg
- Deform such that projection is preserved
- Q on deformation term produces invariant polynomial

Example: Higher gauge theory with $\mathfrak{string}(n)$

$$\mathfrak{string}(n) = (\mathbb{R} \xrightarrow{0} \mathfrak{spin}(n)) \,, \quad \mu_2 = [-,-] \,, \quad \mu_3 = (-,[-,-])$$

- Weil algebra and deformed Weil algebra (from Killing form):

$$\begin{aligned} Q_{\mathsf{W}} t^\alpha &= -\tfrac{1}{2} f^\alpha_{\beta\gamma} t^\beta t^\gamma + \hat{t}^\alpha & Q_{\tilde{\mathsf{W}}} t^\alpha &= -\tfrac{1}{2} f^\alpha_{\beta\gamma} t^\beta t^\gamma + \hat{t}^\alpha \\ Q_{\mathsf{W}} r &= \tfrac{1}{3!} f_{\alpha\beta\gamma} t^\alpha t^\beta t^\gamma + \hat{r} & Q_{\tilde{\mathsf{W}}} r &= \tfrac{1}{3!} f_{\alpha\beta\gamma} t^\alpha t^\beta t^\gamma - \kappa_{\alpha\beta} t^\alpha \hat{t}^\beta + \hat{r} \\ Q_{\mathsf{W}} \hat{t}^\alpha &= -f^\alpha_{\beta\gamma} t^\beta \hat{t}^\gamma & Q_{\tilde{\mathsf{W}}} \hat{t}^\alpha &= -f^\alpha_{\beta\gamma} t^\beta \hat{t}^\gamma \\ Q_{\mathsf{W}} \hat{r} &= -\tfrac{1}{2} f_{\alpha\beta\gamma} t^\alpha t^\beta \hat{t}^\gamma & Q_{\tilde{\mathsf{W}}} \hat{r} &= \kappa_{\alpha\beta} \hat{t}^\alpha \hat{t}^\beta \end{aligned}$$

- Gauge potentials: $(A, B) \;\in\; \Omega^1(U) \otimes \mathfrak{spin}(n) \;\oplus\; \Omega^2(U)$
- Curvatures:

$$F := \mathrm{d}A + \tfrac{1}{2}[A, A]$$
$$H := \mathrm{d}B + \underbrace{\langle A, \mathrm{d}A\rangle + \tfrac{1}{3}\langle A, [A, A]\rangle}_{\mathrm{cs}(A)}$$

- Correct Bianchi identity $\mathrm{d}H = \langle F \wedge F\rangle$

cf. also Mario's talk

II. Adjusted connections in higher gauge theory

Why did we have to deform?

The BRST complex for undeformed Weil algebra is open!

Explicitly: condition on 2-form "fake curvature" F:

$$Q^2 = 0 \quad \Leftrightarrow \quad f_{abc} c^a c^b F^c + f^\alpha_{a\beta} F^a \Lambda^\beta = 0$$

Without $F = 0$ condition:

- BRST complex open
- Higher parallel transport is not reparameterization invariant
  Schreiber, Baez (2005)
- 6d Self-duality equation $H = \star H$ is not gauge-covariant:

$$H \to \tilde{H} = g \rhd H - \mathcal{F} \rhd \Lambda$$

With this condition:

- Higher connections are locally abelian!
  Gastel (2019), CS, Schmidt (2020)

Why did we have to deform?

With adjustment: all condition/problems go away.

Much more generally thus:

A local adjustment is a deformation of the Weil algebra, such that the BRST complex closes. CS/Schmidt (2019)

Higher gauge theories with adjusted curvatures
are the ones that appear in physics!

Example: Gauged supergravities

- Theory with higher form potentials, constructed by hand
- Gauge structure encoded in a differential graded Lie algebra

Theorems: Borsten/Kim/CS (2021)

Any dg Lie algebra can be shift-truncated to $h\mathcal{L}ie$-algebra.

Such $h\mathcal{L}ie$-algebras contain adjustment data for an L_∞-algebras.

Example: 5d gauged supergravity, reps of $\mathfrak{e}_{6(6)}$

$$\begin{array}{ccccccccccccccc} V_{\mathfrak{e}_{6(6)}} & = & V_{-5} & \oplus & V_{-4} & \oplus & V_{-3} & \oplus & V_{-2} & \oplus & V_{-1} & \oplus & V_0 & \oplus & V_1 \\ & & \mathbf{27}\oplus\mathbf{1728} & & \mathbf{351}_c & & \mathbf{78} & & \mathbf{27} & & \mathbf{27}_c & & \mathbf{78} & & \mathbf{351} \\ \mathfrak{E}_{\mathfrak{e}_{6(6)}} & = & \mathfrak{E}_{-4} & \oplus & \mathfrak{E}_{-3} & \oplus & \mathfrak{E}_{-2} & \oplus & \mathfrak{E}_{-1} & \oplus & \mathfrak{E}_0 & & & & \end{array}$$

$$F^a = \mathrm{d}A^a + \tfrac{1}{2}X_{bc}{}^a A^b \wedge A^c + Z^{ab}B_b$$

$$H_a = \mathrm{d}B_a - \tfrac{1}{2}X_{ba}{}^c A^b \wedge B_c - \tfrac{1}{6}d_{abc}X_{de}{}^b A^c \wedge A^d \wedge A^e + d_{abc}A^b \wedge F^c + \Theta_a{}^\alpha C_\alpha$$

$$G_\alpha = \mathrm{d}C_\alpha - \tfrac{1}{2}X_{a\alpha}{}^\beta A^a \wedge C_\gamma + (\tfrac{1}{4}X_{a\alpha}{}^\beta t_{\beta b}{}^c + \tfrac{1}{3}t_{\alpha a}{}^d X_{(db)}{}^c)A^a \wedge A^b \wedge B_c$$
$$+ \tfrac{1}{2}t_{\alpha a}{}^b F^a \wedge B_b - \tfrac{1}{2}t_{\alpha a}{}^b H_b \wedge A^a - \tfrac{1}{6}t_{\alpha a}{}^b d_{bcd}A^a \wedge A^c \wedge F^d - Y_{a\alpha}{}^\beta D_\beta{}^a$$

Global picture: principal 2-bundles + adjusted connect.

- So far: higher connections only locally/infinitesimally
- But: T-duality, etc.: non-trivial topology $\Rightarrow$ principal bundles

Global picture Rist/CS/Wolf 2022

Adjusted crossed modules

- Crossed module $(\mathsf{H} \xrightarrow{\mathsf{t}} \mathsf{G}, \rhd)$ describing 2-group
- Additional map $\kappa : \mathsf{G} \times \mathsf{Lie}(\mathsf{G}) \to \mathsf{Lie}(\mathsf{H})$ with
  $\kappa(\mathsf{t}(h), V) = h(V \rhd h^{-1})$
  $\kappa(g_2 g_1, V) = g_2 \rhd \kappa(g_1, V) + \kappa\big(g_2, g_1 V g_1^{-1} - \mathsf{t}(\kappa(g_1, V))\big)$

Adjusted cocycles:

$$h_{ikl} h_{ijk} = h_{ijl}(g_{ij} \rhd h_{jkl}) , \qquad g_{ik} = \mathsf{t}(h_{ijk}) g_{ij} g_{jk} ,$$
$$\Lambda_{ik} = \Lambda_{jk} + g_{jk}^{-1} \rhd \Lambda_{ij} - g_{ik}^{-1} \rhd (h_{ijk} \nabla_i h_{ijk}^{-1}) ,$$
$$A_j = g_{ij}^{-1} A_i g_{ij} + g_{ij}^{-1} \mathrm{d} g_{ij} - \mathsf{t}(\Lambda_{ij}) ,$$
$$B_j = g_{ij}^{-1} \rhd B_i + \mathrm{d}\Lambda_{ij} + A_j \rhd \Lambda_{ij} + \tfrac{1}{2}[\Lambda_{ij}, \Lambda_{ij}] - \kappa\big(g_{ij}^{-1}, F_i\big) ,$$

Applications of principal 2-bundles with adjustment

Is this all really necessary/useful for physics? Yes!

- Heterotic/gauged supergravity
  kinematic data now on arbitrary manifolds.
- Geometric T-duality as principal 2-bundles

$$\check{\mathscr{P}} \xleftarrow{\;\check{\mathsf{p}}\;} \mathscr{P}_{\mathrm{C}} \xrightarrow{\;\hat{\mathsf{p}}\;} \hat{\mathscr{P}}$$

  topological picture
  with adjusted connections

  Nikolaus/Waldorf (2018)
  Kim/CS (2022)

- Categorified monopole/instanton

$$\mathsf{Spin}(5) \to \mathsf{Spin}(5)/\mathsf{Spin}(4)$$

  lifted to

$$\mathsf{String}(5) \to \mathsf{String}(5)/\mathsf{String}(4)$$

  Higher adjusted connections with $F \neq 0$

Summary so far

Widespread believe among theoretical physicists:

"Non-abelian gerbes do not exist/are useless"

- Both statements are wrong!
- Cause: much literature focusing on unadjusted connections.
- Adjustments and gerbes mathematically well-defined
- Adjustments solve usual physics issues, ready to be applied
- A number of sample applications working perfectly!

$\Rightarrow$ Explore adjusted connections more generally/systematically.

III. Systematic construction: Atiyah algebroid

Jalali Farahani, Kim, Saemann (2024)

Systematic construction

- Above construction very much "by hand"

  "Category theory is the subject where you can leave the definitions as exercises."

  John Baez

- Only partial insight into origin of adjustment:
  - Many infinitesimal adjustment from $h\mathcal{L}ie$-algebras
  - Sometimes adjustments exist, sometimes they don't
- What about general higher gauge theories?

Systematic construction

Total space: ✓

Higher space/groupoid with "principal" action of higher group.

Topological cocycles: ✓

Surjective submersion $\sigma : Y \twoheadrightarrow M$, e.g. $Y = \sqcup_a U_a$

$$\begin{array}{ccc} Y \times_M Y & \xrightarrow{\;g\;} & \mathsf{G} \\ \downdownarrows & & \downdownarrows \\ Y & \xrightarrow{\;*\;} & * \end{array}$$

Differential refinement/connection: $\mathcal{X}$

Split Atiyah-algebroid sequence

$$0 \longrightarrow P \times_{\mathsf{G}} \mathsf{Lie}(\mathsf{G}) \longrightarrow \underbrace{TP/\mathsf{G}}_{\mathfrak{at}(P)} \longrightarrow TM \longrightarrow 0$$

Atiyah algebroid as dg Lie groupoid

Recall from local description:

- Need to use dg manifold language
- Need to extend from Chevalley–Eilenberg to Weil algebra

Observations:

- To have local description, can replace (by Morita equivalence)
  - Manifold M by Čech groupoid $\check{\mathscr{C}}(\sigma) = (Y \times_M Y \rightrightarrows Y)$
  - Atiyah alg. $\mathfrak{at}(P)$ by dg groupoid $T[1](Y \times_M Y) \times \mathfrak{g} \rightrightarrows T[1]Y$
- There is a (dg) action of $T[1]\mathsf{G}$ on $\mathfrak{g}[1] = \mathsf{Lie}(\mathsf{G})[1]$:

$$X \triangleleft (g, \gamma) := g^{-1} X g + g^{-1} \gamma$$

- From this: action groupoid $\mathscr{A}(\mathsf{G}) = (\mathfrak{g}[1] \rtimes T[1]\mathsf{G} \rightrightarrows \mathfrak{g}[1])$
- Atiyah algebroid is pullback of dg Lie groupoids

$$\begin{array}{ccc} \mathfrak{at}(P) & \longrightarrow & \mathscr{A}(\mathsf{G}) \\ \downarrow & & \downarrow \\ T[1]\check{\mathscr{C}}(\sigma) & \xrightarrow{\mathrm{d}g} & T[1]\mathsf{BG} \end{array}$$

Construction of the action algebroid $\mathscr{A}(\mathscr{G})$

Lie functor as suggested by Ševera (2006)

- Principal $\mathscr{G}$-bundles over M subord. to $M \times \mathbb{R}^{0|1} \twoheadrightarrow M$
- Moduli: $\mathsf{Lie}(\mathscr{G}) = \underline{\mathsf{hom}}(\mathbb{R}^{0|2} \rightrightarrows \mathbb{R}^{0|1}, \mathsf{G} \rightrightarrows *)$
- Carries $\mathsf{Hom}(\mathbb{R}^{0|1}, \mathbb{R}^{0|1})$-action $\rightarrow$ Chevalley–Eilenberg algebra

Example: Differentiation of Lie group G.

- $g : M \times \mathbb{R}^{0|2} \rightarrow \mathsf{G}$ with $g(\theta_0, \theta_1, x)g(\theta_1, \theta_2, x) = g(\theta_0, \theta_2, x)$
- implies $g(\theta_0, \theta_1, x) = g(\theta_0, 0, x)(g(\theta_1, 0, x))^{-1}$ with

$$g(\theta_0, 0, x) = \mathbb{1} + \alpha\theta_0 \ , \quad \alpha \in \mathsf{Lie}(\mathsf{G})[1]$$

- compute $g(\theta_0, \theta_1) = \mathbb{1} + \alpha(\theta_0 - \theta_1) + \frac{1}{2}[\alpha, \alpha]\theta_0\theta_1$
- $Qg(\theta_0, \theta_1, x) := \frac{\mathrm{d}}{\mathrm{d}\varepsilon} g(\theta_0 + \varepsilon, \theta_1 + \varepsilon, x)$ with $Q\alpha = -\frac{1}{2}[\alpha, \alpha]$

Action groupoid $\mathscr{A}(\mathscr{G})$ is simply the inner hom groupoid in above.

Replacing the Atiyah algebroid by $\mathscr{A}(\mathsf{G})$

Recall: we constructed the Atiyah algebroid as pullback

$$\begin{array}{ccc} \mathfrak{at}(P) & \longrightarrow & \mathscr{A}(\mathsf{G}) \\ \downarrow & & \downarrow \\ T[1]\check{\mathscr{C}}(\sigma) & \xrightarrow{\mathrm{d}g} & T[1]\mathsf{BG} \end{array}$$

Note:

- Splitting $T[1]\check{\mathscr{C}}(\sigma) \to \mathfrak{at}(P)$ yields map $T[1]\check{\mathscr{C}}(\sigma) \to \mathscr{A}(\mathsf{G})$
- Such a dg map: principal G-bundle with a flat connection
- Flatness not surprising, as in local case.
- This description is in terms of the usual local cocycle: g_{ij}, A_i
- $\Rightarrow$ We can circumvent $\mathfrak{at}(P)$ completely, and use $\mathscr{A}(\mathsf{G})$
- Cocycles for princ. G-bundles + flat connections: dg-functors

Example: Ordinary principal bundles

Principal G-bundle with connection, G Lie group:

- Action groupoid: $\mathscr{A}(\mathsf{G}) = \mathfrak{g}[1] \ltimes T[1]\mathsf{G} \rightrightarrows \mathsf{G}$
- Bundle with connection from dg-functors $T[1]\check{\mathscr{C}}(\sigma) \to \mathscr{A}(\mathsf{G})$

$$\begin{array}{ccc} T[1](Y \times_M Y) & \xrightarrow{(A,\mathrm{d}g)} & \mathfrak{g}[1] \ltimes T[1]\mathsf{G} \\ \downarrow\downarrow & & \downarrow\downarrow \\ T[1]Y & \xrightarrow{\quad A \quad} & \mathfrak{g}[1] \end{array}$$

- Compatibility with groupoid structure:

$$g_{y_1 y_3} = g_{y_1 y_2} g_{y_2 y_3} \ , \quad A_{y_2} = g^{-1}_{y_1 y_2} A_{y_1} g_{y_1 y_2} + g^{-1}_{y_1 y_2} \mathrm{d} g_{y_1 y_2}$$

- Compatibility with differential:

$$\mathrm{d}A_y + \tfrac{1}{2}[A_y, A_y] = 0$$

Beyond flat connections

Recall from local picture:

- Need to switch from Chevalley–Eilenberg to Weil algebra

New construction:

- Ševera for $T[1]\mathscr{G}$ yields Weil algebra for $\mathfrak{g} = \mathsf{Lie}(\mathscr{G})$
- $\mathscr{A}(T[1]\mathscr{G})$ for some higher group $\mathscr{G}$ as inner hom groupoid.
- Bundles+connections: dg maps $T[1]\check{\mathscr{C}}(\sigma) \to \mathscr{A}(T[1]\mathscr{G})$?
- This produces too much!
- Need adjustment to cut down data appropriately.

On the origin of adjustments

Example: strict 2-group $\mathscr{G} = (\mathsf{H} \rtimes \mathsf{G} \rightrightarrows \mathsf{G})$

- Action groupoid $\mathscr{A}(\mathscr{G})$: here a 2-groupoid, looks as follows:

$$
\begin{array}{c}
(\dots) \times T[1]T[1]\mathfrak{h}[2] \times T[1]T[1]\mathsf{G} \times T[1]T[1]\mathsf{H} \\
\downdownarrows \\
(\dots) \times T[1]T[1]\mathfrak{h}[2] \times T[1]T[1]\mathsf{G} \\
\downdownarrows \\
\mathfrak{g}[1] \times \mathfrak{g}[2] \times \mathfrak{h}[2] \times \mathfrak{h}[3]
\end{array}
$$

- Considering pullback and then sections yields too much
- We want:

$$
\begin{array}{lll}
g_{ij} \in \Omega^0(Y^{[2]}, \mathsf{G}) \ , & & h_{ijk} \in \Omega^0(Y^{[3]}, \mathsf{H}) \ , \\
A_i \in \Omega^1(Y, \mathfrak{g}) \ , & F_i \in \Omega^2(Y, \mathfrak{g}) \ , & \Lambda_{ij} \in \Omega^1(Y^{[2]}, \mathfrak{h}) \ , \\
B_i \in \Omega^2(Y, \mathfrak{g}) \ , & H_i \in \Omega^3(Y, \mathfrak{h}) &
\end{array}
$$

On the origin of adjustments

Let's identify the relevant components:

$$\underbrace{T[1]\mathfrak{g}[1]}_{(A_i,F_i)}\times\underbrace{T[1]\mathfrak{h}[2]}_{(B_i,H_i)}\times\underbrace{T[1]T[1]\mathfrak{h}[2]}_{(\Lambda_{ij},\mathrm{d}\Lambda_{ij},\bar\Lambda_{ij},\mathrm{d}\bar\Lambda_{ij})}\times\underbrace{T[1]T[1]\mathsf{G}}_{(g_{ij},\mathrm{d}g_{ij},\bar g_{ij},\mathrm{d}\bar g_{ij})}\times\underbrace{T[1]T[1]\mathsf{H}}_{(h_{ijk},\mathrm{d}h_{ijk},\bar h_{ijk},\mathrm{d}\bar h_{ijk})}$$

$$\downdownarrows$$

$$\underbrace{T[1]\mathfrak{g}[1]}_{(A_i,F_i)}\times\underbrace{T[1]\mathfrak{h}[2]}_{(B_i,H_i)}\times\underbrace{T[1]T[1]\mathfrak{h}[2]}_{(\Lambda_{ij},\mathrm{d}\Lambda_{ij},\bar\Lambda_{ij},\mathrm{d}\bar\Lambda_{ij})}\times\underbrace{T[1]T[1]\mathsf{G}}_{(g_{ij},\mathrm{d}g_{ij},\bar g_{ij},\mathrm{d}\bar g_{ij})}$$

$$\downdownarrows$$

$$\underbrace{T[1]\mathfrak{g}[1]}_{(A_i,F_i)}\times\underbrace{T[1]\mathfrak{h}[2]}_{(B_i,H_i)}$$

We note:

- Differentials (e.g. $\mathrm{d}g_{ij}$) are fine, they are fixed.
- Second copies $(\bar g_{ij},\bar h_{ijk},\bar\Lambda_{ij})$ need to be constrained
- $(\bar g_{ij},\bar h_{ijk})$ by demanding: top. cocycles remain unchanged
- $\bar\Lambda_{ij}$ fixed by function $\bar\Lambda_{ij}=\kappa(g_{ij},F_i)$

On the origin of adjustments

$$\underbrace{T[1]\mathfrak{g}[1]}_{(A_i,F_i)} \times \underbrace{T[1]\mathfrak{h}[2]}_{(B_i,H_i)} \times \underbrace{T[1]T[1]\mathfrak{h}[2]}_{(\Lambda_{ij},\mathrm{d}\Lambda_{ij},\bar\Lambda_{ij},\mathrm{d}\bar\Lambda_{ij})} \times \underbrace{T[1]T[1]\mathsf{G}}_{(g_{ij},\mathrm{d}g_{ij},\bar g_{ij},\mathrm{d}\bar g_{ij})} \times \underbrace{T[1]T[1]\mathsf{H}}_{(h_{ijk},\mathrm{d}h_{ijk},\bar h_{ijk},\mathrm{d}\bar h_{ijk})}$$

$$\downarrow\downarrow$$

$$\underbrace{T[1]\mathfrak{g}[1]}_{(A_i,F_i)} \times \underbrace{T[1]\mathfrak{h}[2]}_{(B_i,H_i)} \times \underbrace{T[1]T[1]\mathfrak{h}[2]}_{(\Lambda_{ij},\mathrm{d}\Lambda_{ij},\bar\Lambda_{ij},\mathrm{d}\bar\Lambda_{ij})} \times \underbrace{T[1]T[1]\mathsf{G}}_{(g_{ij},\mathrm{d}g_{ij},\bar g_{ij},\mathrm{d}\bar g_{ij})}$$

$$\downarrow\downarrow$$

$$\underbrace{T[1]\mathfrak{g}[1]}_{(A_i,F_i)} \times \underbrace{T[1]\mathfrak{h}[2]}_{(B_i,H_i)}$$

With

$$\bar\Lambda_{ij} = \kappa(g_{ij}, F_i)$$

Theorem: Jalali Farahani/Kim/CS (2024)

Groupoid + dg structures: κ is an adjustment for a strict 2-group.

Note: Adjustment is data that restricts action groupoid as needed!

General systematic prescription

Let us summarize the construction

- Start from higher gauge group $\mathscr{G}$
- Double to the higher dg-group $T[1]\mathscr{G}$.
- Construct $\mathscr{A}(T[1]\mathscr{G})$ as inner hom dg-groupoid
- Restrict $\mathscr{A}(T[1]\mathscr{G})$ to $\widehat{\mathscr{A}(\mathscr{G})}$ by
  - Impose that topological cocycles are unchanged
  - Other doubled data fixed by adjustment maps
  - Derive conditions on maps from groupoid + dg relations
- Principal $\mathscr{G}$-bundle with adjusted connection:

  $$g : T[1]\check{\mathscr{C}}(\mathsf{G}) \longrightarrow \widehat{\mathscr{A}(\mathscr{G})}$$

  produces the local cocycles + relations.

Examples:

- This reproduces 1- and 2-connections.
- Concretely: $\mathscr{P} = \mathsf{String}(5) \to S^4$ is such a $\mathsf{String}(4)$-bundle

IV. Adjusted connections in groupoid gauge theory

Groupoid gauge theory

But what if I don't believe in string theory, supergravity, higher gauge theory?

You should still care about adjustments!

- Adjustment also appear in groupoid gauge theories.
- Recall: principal G-bundles:

$$\begin{array}{ccc} Y \times_M Y & \xrightarrow{\quad g \quad} & \mathsf{G} \\ \downarrow\downarrow & & \downarrow\downarrow \\ Y & \xrightarrow{\quad * \quad} & * \end{array}$$

- Principal groupoid bundles with groupoid $\mathscr{G} = (\mathscr{G}_1 \rightrightarrows \mathscr{G}_0)$

$$\begin{array}{ccc} Y \times_M Y & \xrightarrow{\quad g \quad} & \mathscr{G}_1 \\ \downarrow\downarrow & & \downarrow\downarrow \\ Y & \xrightarrow{\quad \phi \quad} & \mathscr{G}_0 \end{array}$$

Groupoid gauge theory

$$\begin{array}{ccc} Y \times_M Y & \xrightarrow{\;g\;} & \mathscr{G}_1 \\ \downdownarrows & & \downdownarrows \\ Y & \xrightarrow{\;\phi\;} & \mathscr{G}_0 \end{array}$$

- ϕ: scalar/Higgs field, $\mathscr{G}_0$-valued
- "gauged": local fields glued together by elements in $\mathscr{G}_1$

Examples:

- $\approx$ Yang–Mills–matter: assoc. vector bundles + group action
- Gauged sigma model: Action groupoid $M \ltimes \mathsf{G} \rightrightarrows M$
- But: more general groupoids possible.

Why adjustment?

- Two "levels" of connections/curvatures: $\nabla\phi$ and F_A
- similar to higher gauge theory: F and H

Summary of the situation

- First observed locally Strobl (2004) and Kotov/Strobl (2015)
  See also Fischer (2021)!
- (Fake-) Flat connections are readily defined.
- Non-fake-flat connections require adjustment: ∇ and ζ
- Deform. Weil algebra: Fischer, Jalali Farahani, Kim, CS (2024)

$$
\begin{aligned}
\mathrm{d}\mathtt{x} &= (\partial_i \mathtt{x})(\mathtt{p}^i + \rho^i_a \mathtt{a}^a)\\
\mathrm{d}\mathtt{p}^i &= -\rho^i_a \mathtt{f}^a + (\nabla_j \rho^i_a)\mathtt{a}^a \mathtt{p}^j + \frac{1}{2}\rho^i_a \zeta^a_{jk}\mathtt{p}^j\mathtt{p}^k\\
\mathrm{d}\mathtt{a}^a &= \mathtt{f}^a - \omega^a_{bi}\mathtt{p}^i\mathtt{a}^b - \frac{1}{2}C^a_{bc}\mathtt{a}^b\mathtt{a}^c - \frac{1}{2}\zeta^a_{ij}\mathtt{p}^i\mathtt{p}^j\\
\mathrm{d}\mathtt{f}^a &= -(C^a_{bc} + \rho^i_c \omega^a_{bi})\mathtt{a}^b\mathtt{f}^c - (\omega^a_{bi} - \zeta^a_{ij}\rho^j_b)\mathtt{p}^i\mathtt{f}^b + \left(\frac{1}{6}(\mathrm{d}^\nabla \zeta)^a_{ijk} - \frac{1}{2}\zeta^a_{il}\rho^l_b\zeta^b_{jk}\right)\mathtt{p}^i\mathtt{p}^j\mathtt{p}^k\\
&\quad + \frac{1}{2}\left(R^{\mathsf{bas}}_\nabla\right)^a_{bci}\mathtt{a}^b\mathtt{a}^c\mathtt{p}^i + \frac{1}{2}\left(R_\nabla + \mathrm{d}^{\nabla^{\mathsf{bas}}}\zeta\right)^a_{ijb}\mathtt{p}^i\mathtt{p}^j\mathtt{a}^b,
\end{aligned}
$$

- Adjustment conditions (BRST complex closes):

$$
R^{\mathsf{bas}}_\nabla = 0 \quad \text{and} \quad R_\nabla + \mathrm{d}^{\nabla^{\mathsf{bas}}}\zeta = 0
$$

- Global picture: recycle our procedure for groupoids!

Summary and Outlook

Summary:

- Non-abelian gerbes exist and are non-trivial!
- Fully systematic way of construction adjusted cocycles
  - Personally, now happy with adjustments
  - Framework ready to apply to any situation
- Adjustments also appear in groupoid gauge theories

Outlook:

- New examples of principal 3-bundles
- U-duality with principal 3-bundles
- Interesting groupoids for phenomenology
- Interesting new Higgs mechanism, etc.?