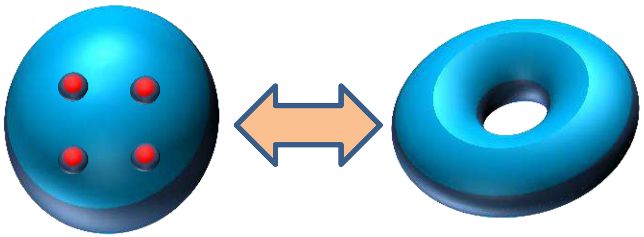

Defects in topologically ordered states

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Mag Lab, Tallahassee, 01/09/2014

- References

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Outline

Lecture 1

- Defects in conventional states
- Topologically ordered states, symmetries and twist defects
- The simplest twist defects---genons
- Properties of genons, “projective” non-Abelian statistics and parafermion zero modes
- Generalization to generic defects: A unified framework of defects in Abelian topological states

Lecture 2

- Realizations of genons and more general defects
    1. Bilayer FQH states with staircases
    2. Fractional Chern Insulators
    4. A spin model realization: generalized Kitaev model

Lecture 1

General properties of defects in topologically ordered states

Why are we interested in defects?

- Defects provide ways to probe the state of matter.
- New interesting properties may be carried by defects.
- **Example 1**: Superfluid vortex.
  Superfluid phase θ is not detectable.
  Global U(1) symmetry
  $\psi = \sqrt{\rho}e^{i\theta} \to \psi e^{i(\theta+\theta_0)}$,

$e^{i\theta}$

- The existence of vortex with quantized vorticity tells us that θ is a $U(1)$ phase periodic in 2π.
- Vortex is a *twist defect* obtained by twisting the $U(1)$ symmetry
- A superfluid vortex has $\log L$ divergent energy. It's an *extrinsic defect* that has to be introduced by external force.

Why are we interested in defects?

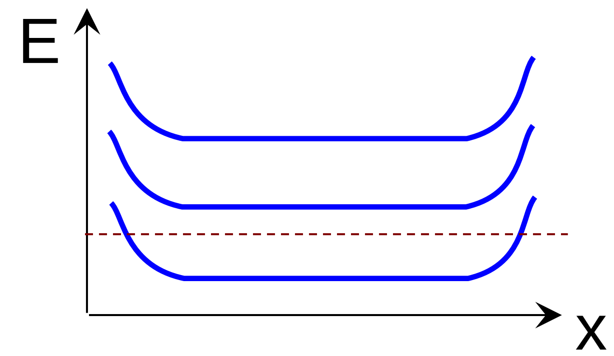
- **Example 2:** Superconductor vortex at a corner junction
- A superconductor vortex is not a defect, but an excitation with finite energy. A magnetic flux $\Phi = \int d^2xB = \frac{hc}{2e}$ localized in a vortex core with size ξ
- A corner junction between s-wave and d-wave superconductors traps a half vortex $\Phi = \frac{hc}{4e}$, which is an extrinsic defect.
- Order parameter phase θ changes by π.
- An example of a defect at interface between two phases.

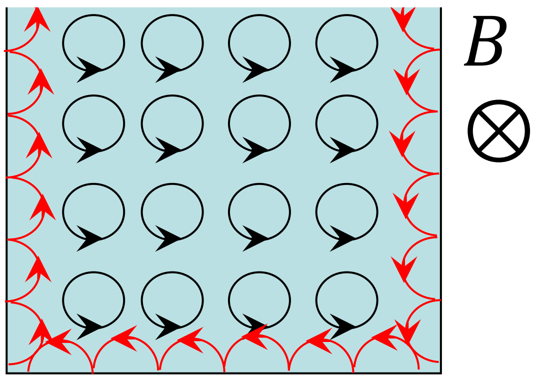
d-wave

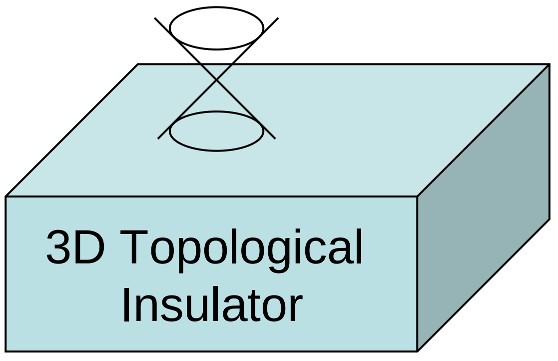
s-wave

Topological states of matter

- Quantum Hall effect occurs in 2d electron system with strong perpendicular magnetic field
- Integer quantum Hall (IQH) state: Filling integer number of Landau levels. Momentum space Chern number (Thouless et al 1982)
  Chiral edge states
- Quantum Hall state is an example of **topological states of matter**: New states that are classified by topological properties, such as robust edge states, topological response

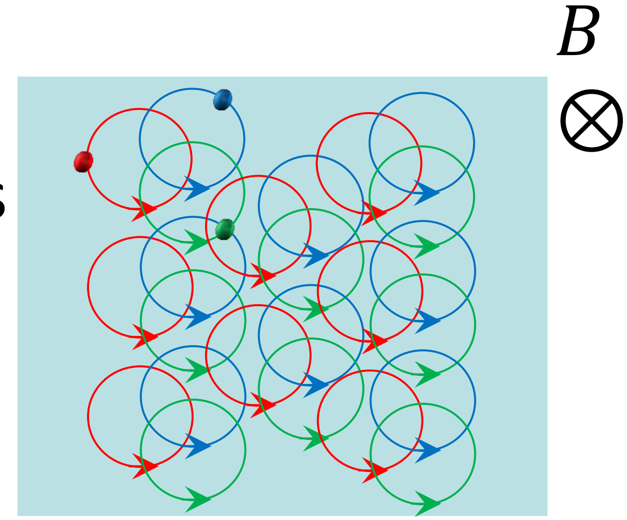


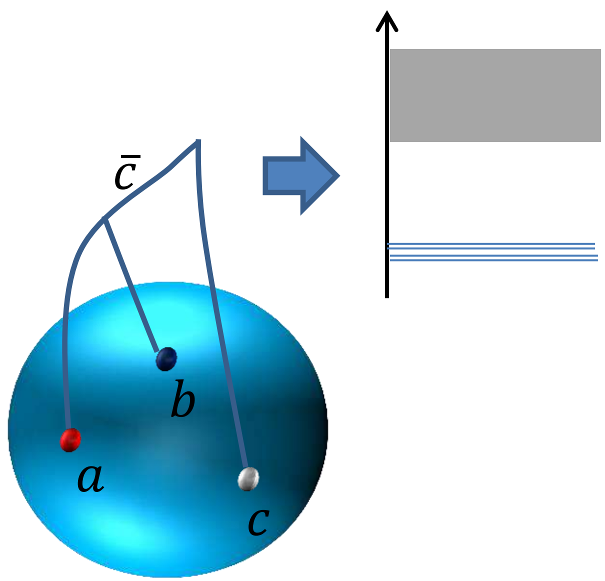


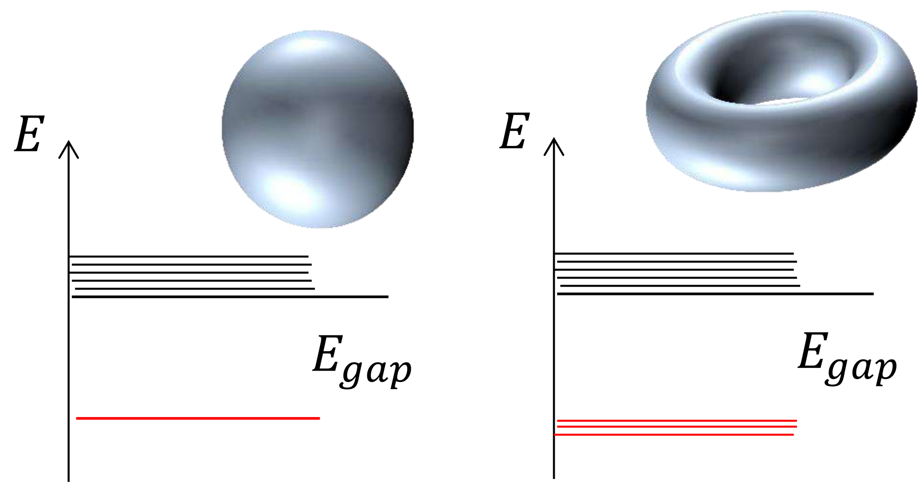


Topologically ordered states

- Topologically ordered states such as fractional quantum Hall (FQH) states have topological **ground state degeneracy**, and topological quasiparticles with fractional charge and fractional statistics.





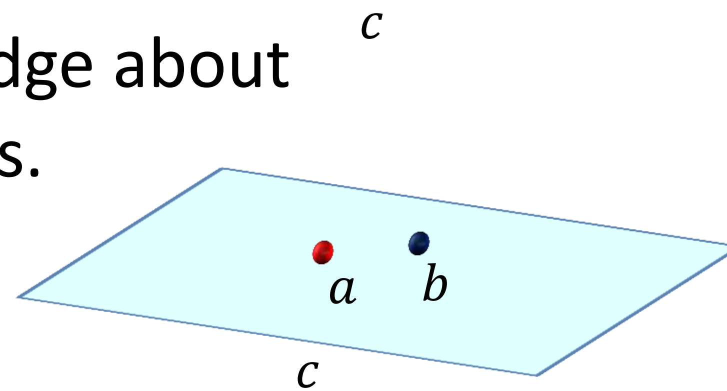


$g = 0$
1 ground state

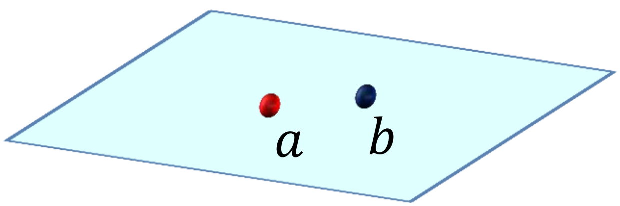
$g = 1$
m ground states

Key properties of topologically ordered states

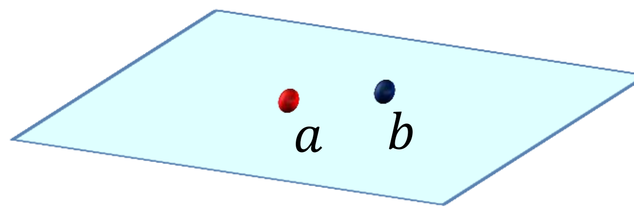
- Quasiparticles have no knowledge about distance. Only topology matters.



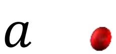
- **Fusion** $a \times b = N_{ab}^{c} c$
- **Braiding** c



$$= R_{ab}^{c} \cdot$$



- Braiding a, b and spinning a, b is equivalent to spinning c. **Topological spin** of particles h_a

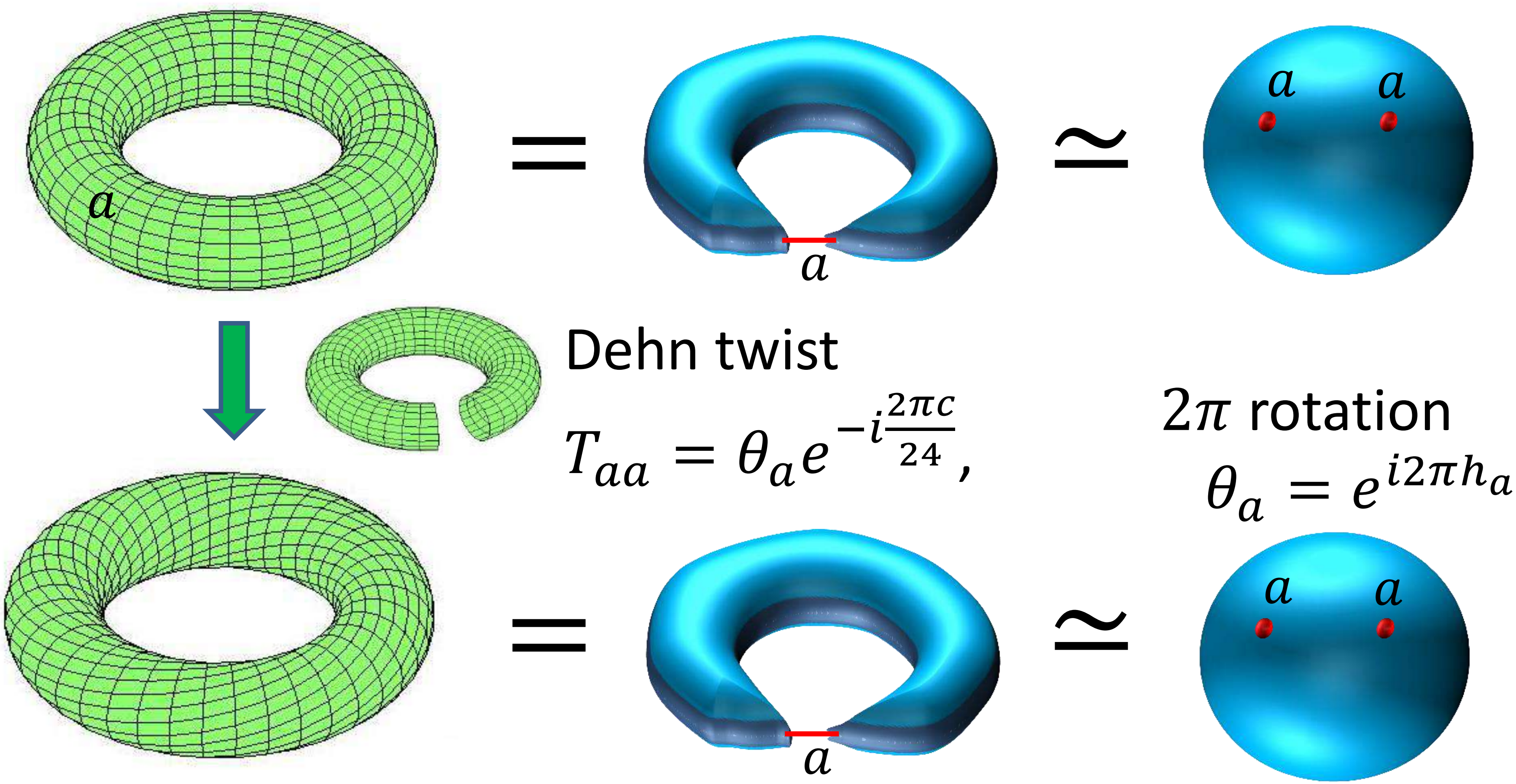


$$= e^{i2\pi h_a}$$

$$R_{ab}^{c} R_{ba}^{c} = e^{i2\pi(h_a + h_b - h_c)}$$

Key properties of topologically ordered states

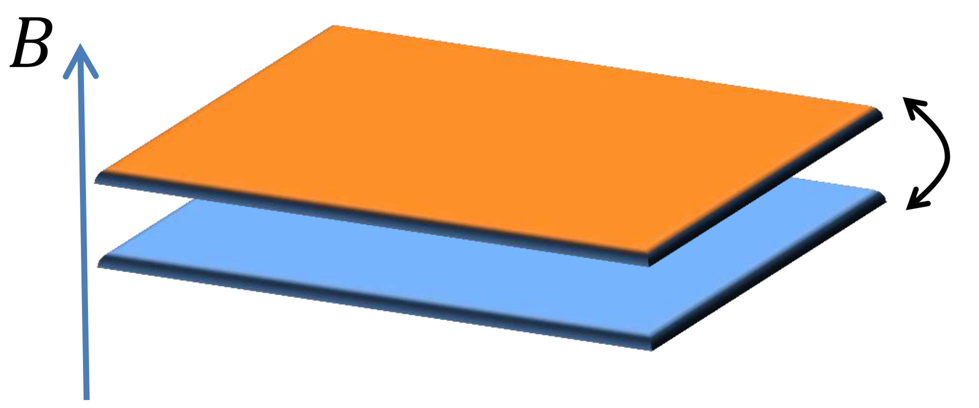
- Spin also determines the transformation of the torus ground states under **modular transformations**



Symmetries in topologically ordered states

- Topologically ordered states are robust without requiring any symmetry. However, symmetries may exist in topologically ordered states
- **Example 1**: Bilayer FQH states with the symmetry of exchanging two layers (Z_2 symmetry)
- E.g. (mnl) Halperin state with $m = n$:

$$\prod_{i<j}(z_i - z_j)^m \prod_{i<j}(w_i - w_j)^m \prod_{i,j}(z_i - w_j)^l \cdot e^{-\sum_i |z_i|^2 - \sum_i |w_i|^2}$$



Symmetries in topologically ordered states

- **Example 2**: particle-hole symmetry of Laughlin $1/m$ state.
- Quasiparticle charge $q = \frac{n}{m}, n = 0,1, \dots, m-1,$
- Braiding $\theta_{nn'} = 2\pi \frac{nn'}{m}$
- Fusion $a_n \times a_{n'} = a_{n+n' (\mathrm{mod}\, m)}$
- Braiding and fusion are invariant under the particle-hole transformation $n \to -n$
- In a generic state, the particle-hole symmetry is not an exact symmetry, but a **topological symmetry**:
- $|\Psi\rangle \to \ |\Psi'\rangle$ topologically equivalent to $|\Psi\rangle$

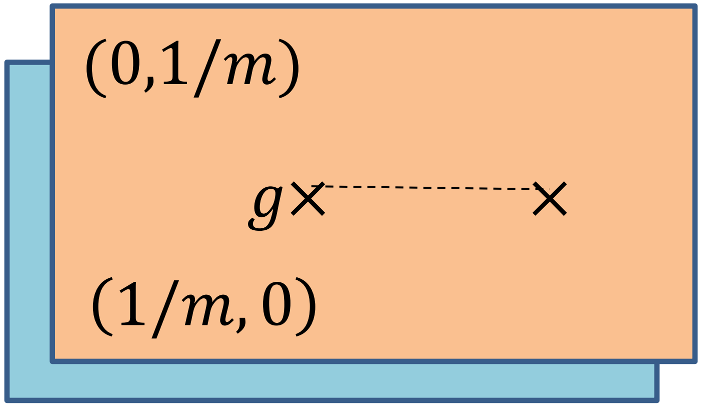
$n + n'$ $\qquad$ $n + n'$

$= e^{i\theta_{nn'}}$

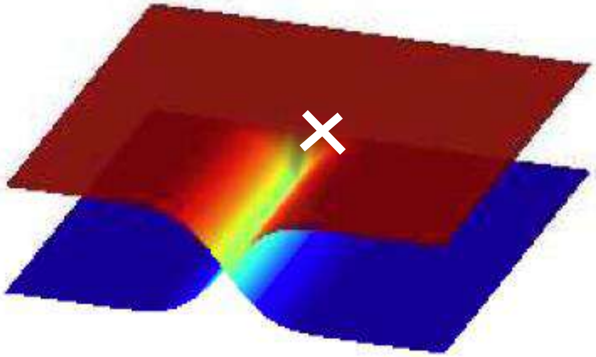
n $\quad$ n' $\qquad$ n $\qquad$ n'

Twist defects and genons

- A discrete global symmetry can be twisted.
- A topological particle
  is acted by the symmetry
  while crossing a "branch-cut"
  line.



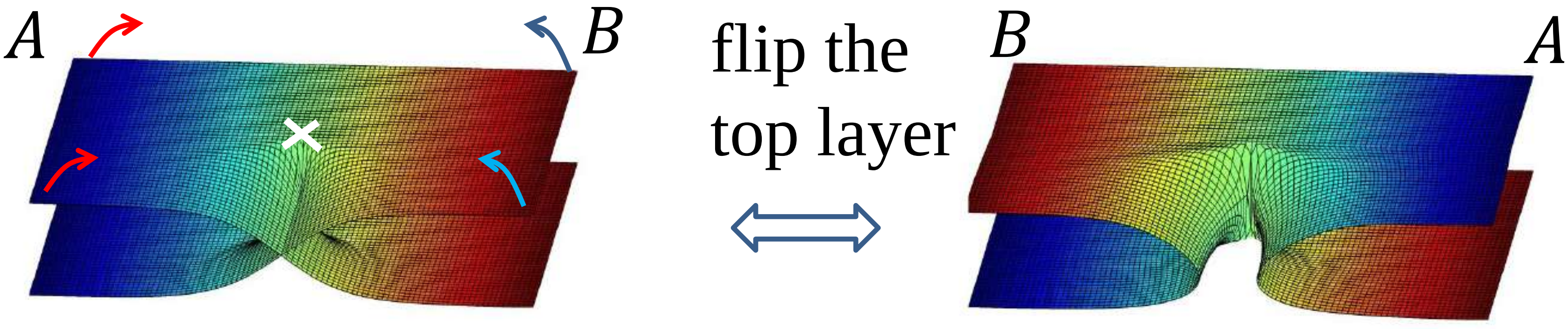
- The branch-cut line is in-visible
  and each end point of the branch-cut is a point-like
  twist defect. (Kitaev&Kong, '11)
- For bilayer FQH, the branch-cut line has a simple
  geometrical meaning
- The two layers are connected and
  become a Riemann surface

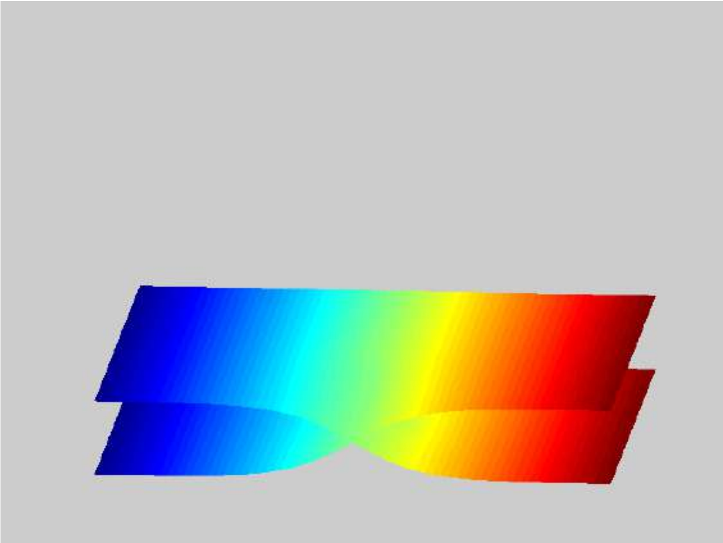


Barkeshli&XLQ PRX '12

Genons--genus generators

- In bilayer FQH states, a pair of defect creates a "worm hole" between the two layers. The defect is called a **genon**---genus generator





Quantum dimension of genons

- Every pair of defects add genus 1 to the manifold

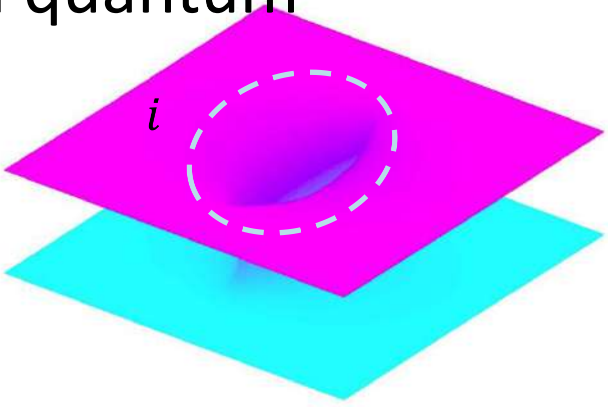
$2n$ defects on a sphere

genus $g = n - 1$ surface

- Ground state degeneracy (GSD) is $m^{n-1} = \frac{1}{m}(\sqrt{m})^{2n}$
- On comparison, $2n$ spins each with d local states have total degeneracy d^{2n}.
- A genon has the *quantum dimension* $d = \sqrt{m}$
- The non-integer quantum dimension indicates that genons are non-Abelian

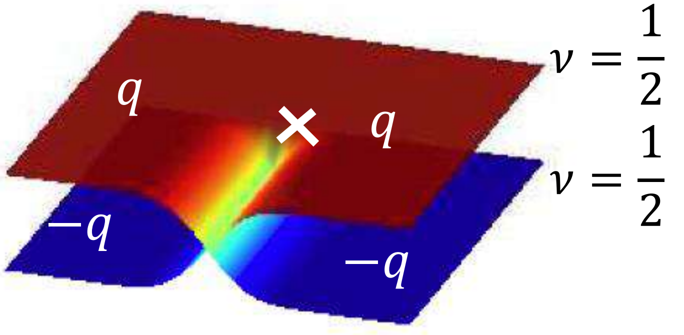
Quantum dimension of genons

- For the Halperin (mml) state, the genon quantum dimension is $d = \sqrt{m-l}$. For example (220) and (331) states both give $d = \sqrt{2}$ the same as Majorana fermion.
- For a more generic topological state at each layer with quasiparticles of quantum dimension d_i, the ground state degeneracy grows like $GSD \sim D^{2g}$ with $D = \sqrt{\sum_i d_i^2}$ the total quantum dimension of the single layer theory. Therefore genon quantum dimension is $d_X = D$.



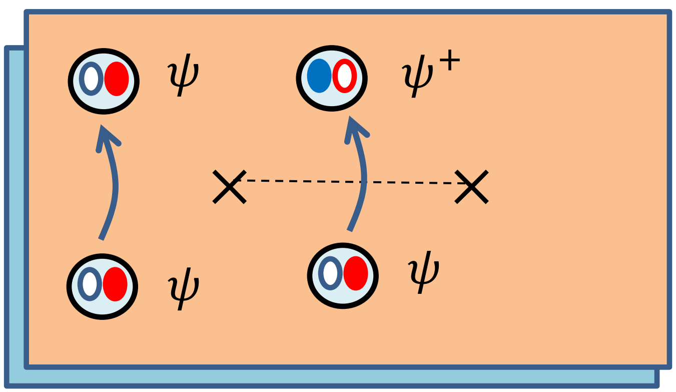
Understanding the genon quantum dimension: parafermion zero modes

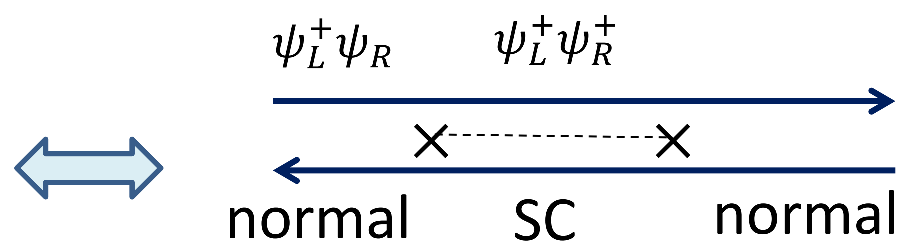
- First, Consider (220) state, i.e. two layers of Laughlin ½
- Each layer has a $q = 1/2$ semion (statistical angle $\theta_1 = \pi/2$)
- The ``exciton" $\left(\frac{1}{2}, -\frac{1}{2}\right)$ is a fermion



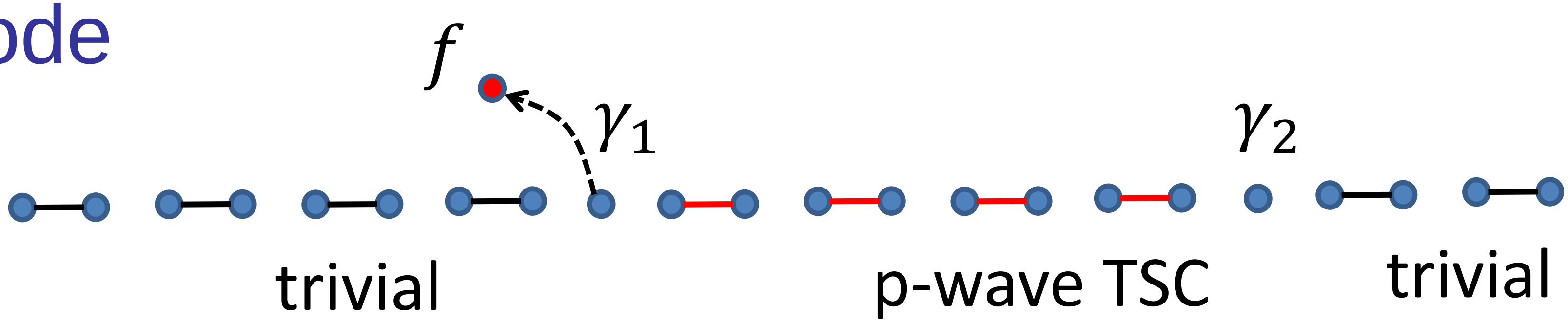
The branchcut line as an exciton superconductor

- Two excitons can annihilate at the branchcut. A "geometric" **exciton superconductor**
- A genon is the end of the branchcut line, which is the end of the 1d p-wave superconductor (Kitaev '01)
- Majorana zero mode at the genon, consistent with quantum dimension $\sqrt{2}$

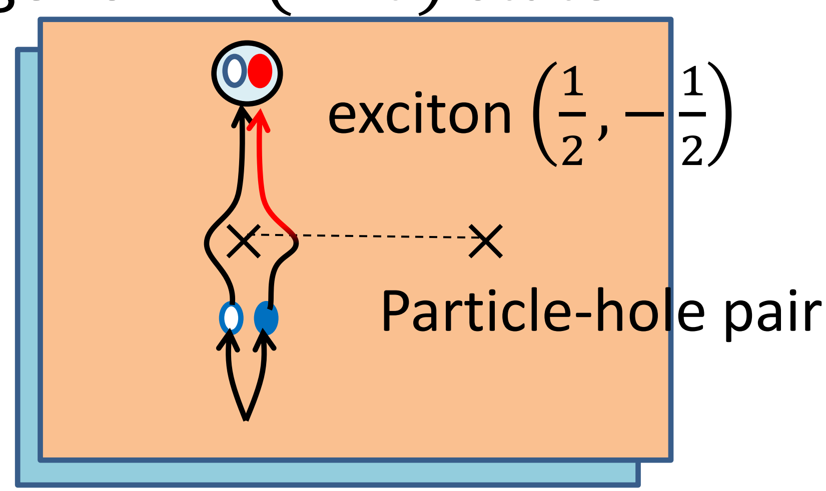




Geometrical understanding of the Majorana zero mode

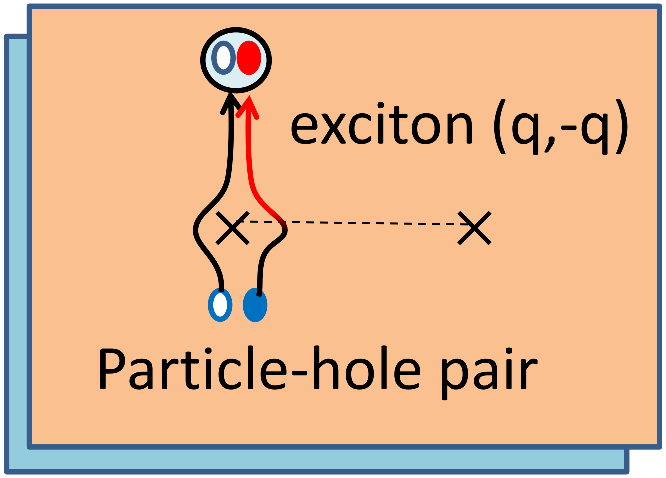


- The existence of Majorana zero mode means that a fermion can be emitted or absorbed by the zero mode, with no energy cost
- The same happens for the genon in (220) state: Emission or annihilation of the exciton fermion $\left(\frac{1}{2}, -\frac{1}{2}\right)$ can only occur at the genon



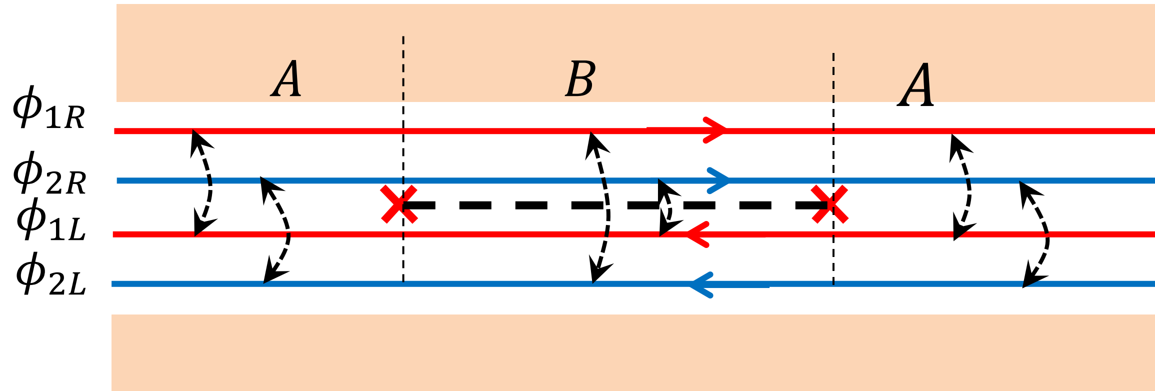
From Majorana zero modes to parafermion zero modes

- In more general (mml) states, the genon can emit exciton with charge $(1,-1)$, spin $\theta = \exp\left[i\,\frac{2\pi}{m-l}\right]$
- A Majorana zero mode is sqrt of a fermion. $d = \sqrt{2}$
- A parafermion zero mode is sqrt of an anyon. $d = \sqrt{\mathrm{m} - l}$
- $m - l$ states are shared by a pair of genons, which are not locally detectable.



Edge theory description of twist defects

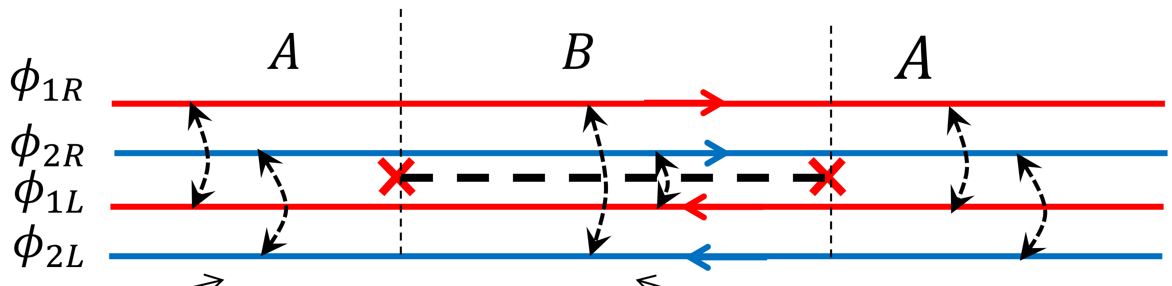
- "Cut and glue" scheme



- Genons are domain walls along 1D cuts in the system
- Chiral Luttinger liquid theory (Wen) with backscattering terms
- $\mathcal{L} = \mathcal{L}_L + \mathcal{L}_R + \mathcal{L}_{int}$
- $\mathcal{L}_L = \frac{m}{4\pi} K^{IJ} \partial_t \phi_{IL} \partial_x \phi_{JL} - V^{IJ} \partial_x \phi_{IL} \partial_x \phi_{JL}$
- $\mathcal{L}_R = -\frac{m}{4\pi} K^{IJ} \partial_t \phi_{IR} \partial_x \phi_{JR} - V^{IJ} \partial_x \phi_{IR} \partial_x \phi_{JR}$
- $\mathcal{L}_{int} = \begin{cases} \sum_I J \cos\left(K^{IJ}\left(\phi_{JR} - \phi_{JL}\right)\right), A \text{ region} \\ \sum_I J \cos\left(K^{IJ}\left(\phi_{JR} - \phi_{J'L}\right)\right), B \text{ region} \end{cases}$

Edge theory description of twist defects

- For example, for $l = 0$
- $\mathcal{L}_{int} =$
$$\begin{cases} J\cos\big(m(\phi_{1R}-\phi_{1L})\big)+J\cos\big(m(\phi_{2R}-\phi_{2L})\big), A \text{ region} \\ J\cos\big(m(\phi_{1R}-\phi_{2L})\big)+J\cos\big(m(\phi_{2R}-\phi_{1L})\big), B \text{ region} \end{cases}$$
- Decompose $\phi_{\pm IR} = \phi_{1R} \pm \phi_{2R}, \phi_{\pm IL} = \phi_{1L} \pm \phi_{2L}$, the mass term for ϕ_+ is the same in both regions. Only ϕ_- see the twist defect.
- Focus on the $\phi_- = \phi_1 - \phi_2$ sector

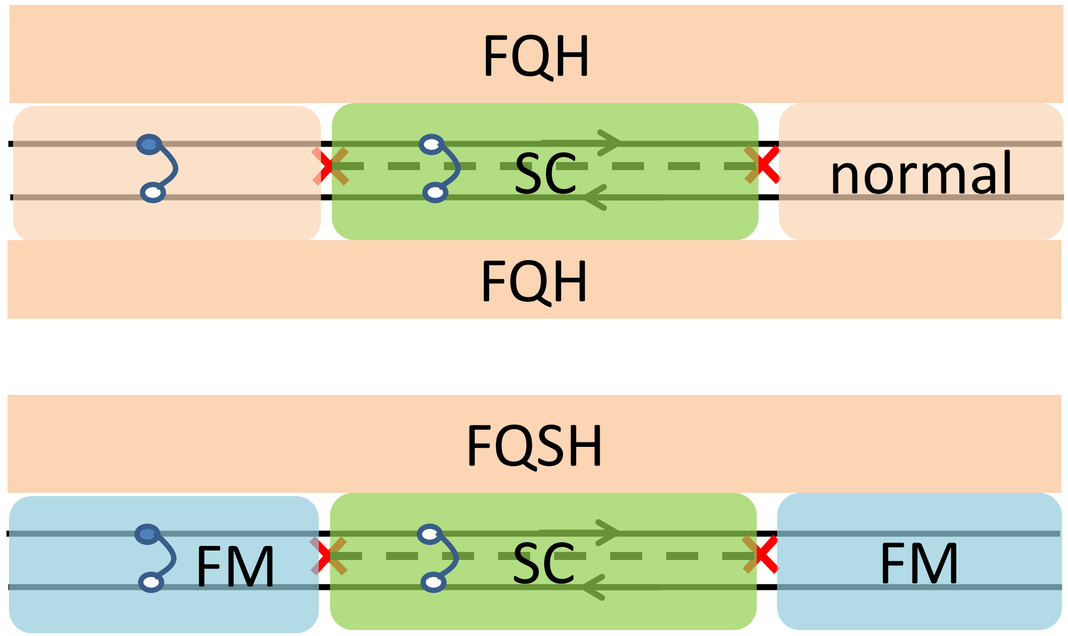


$$\cos\big((m-l)(\phi_{-L}-\phi_{-R})\big)$$

$$\cos\big((m-l)(\phi_{-L}+\phi_{-R})\big)$$

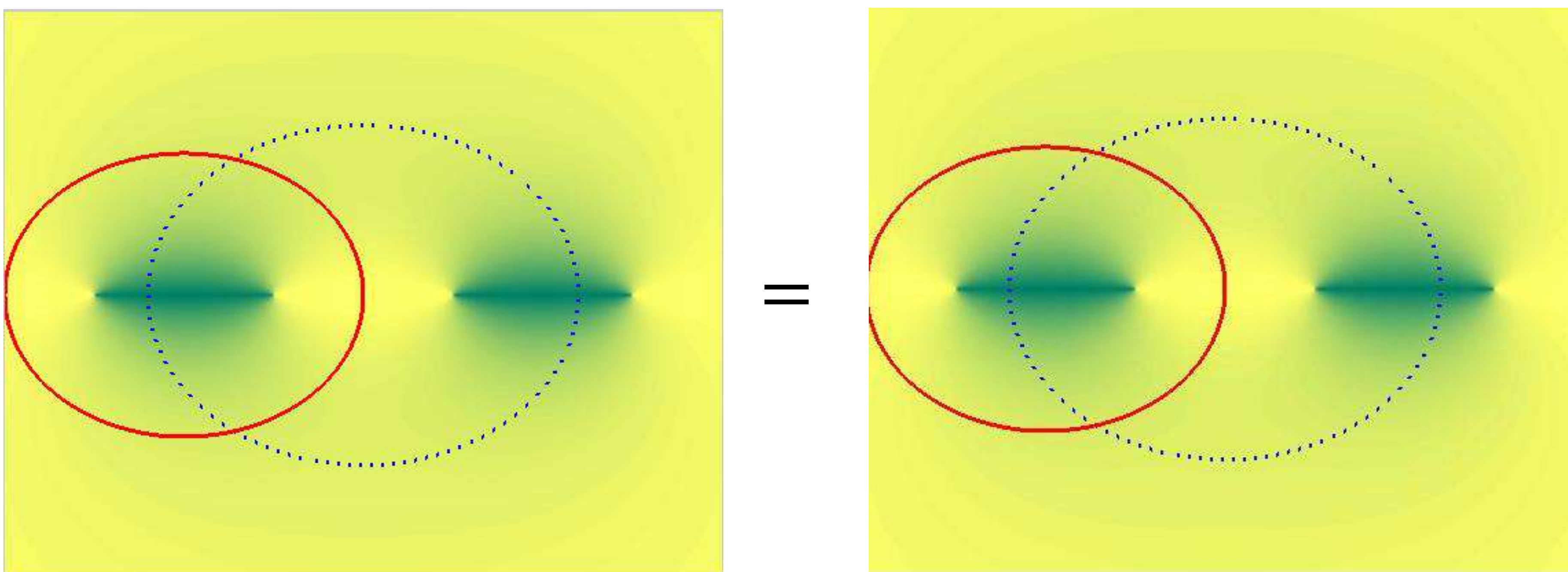
Relation with other twist defects

- The twist defect is a domain wall between particle-hole and particle-particle mass term for the exciton created by $e^{i(\phi_{-L}-\phi_{-R})}$
- The same defects can be realized in FQH or FQSH in proximity with SC (Linder et al, Clarke et al, Cheng, Vaezi, 2012)
- Quasiparticle ϕ replaces exciton ϕ_-



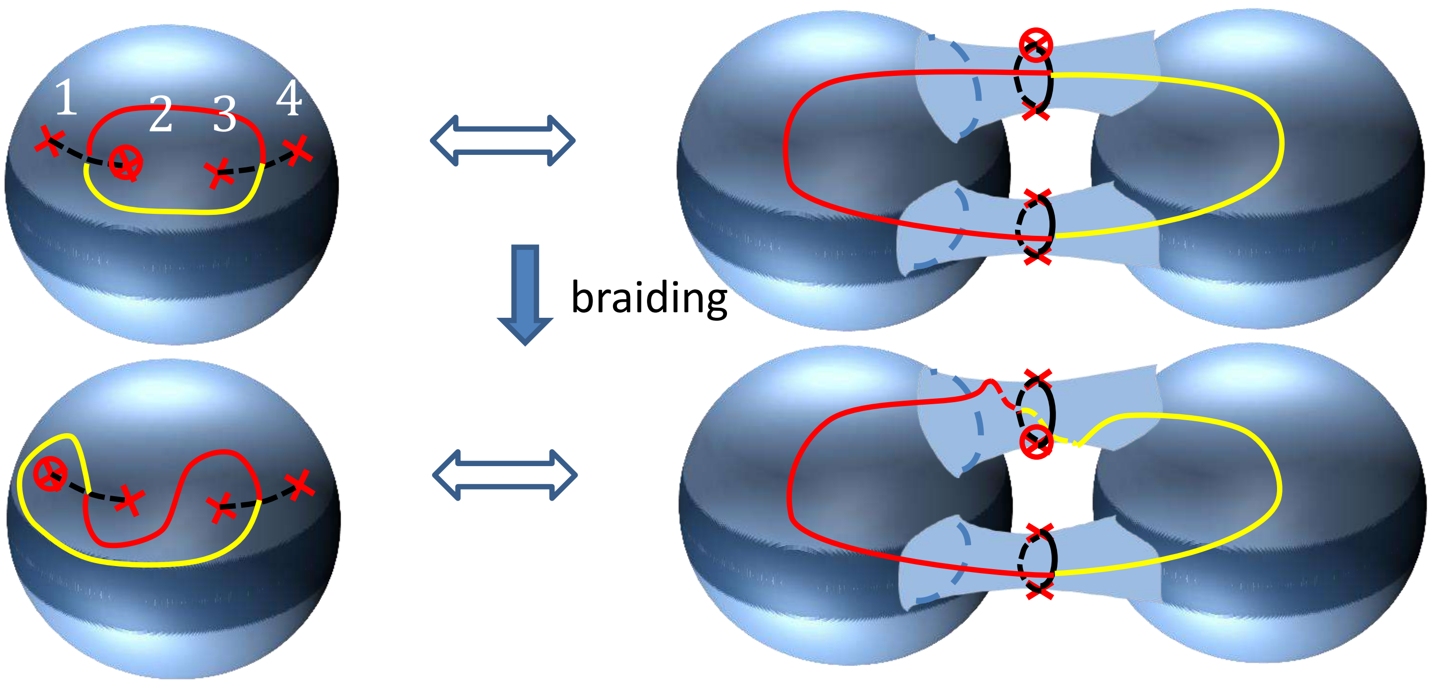
Braiding statistics of genons

- When two genons are braided, the corresponding genus $g = \mathrm{n} - 1$ surface carries a nontrivial large coordinate transformation---a Dehn twist
- This can be seen by tracking the change of nontrivial loops



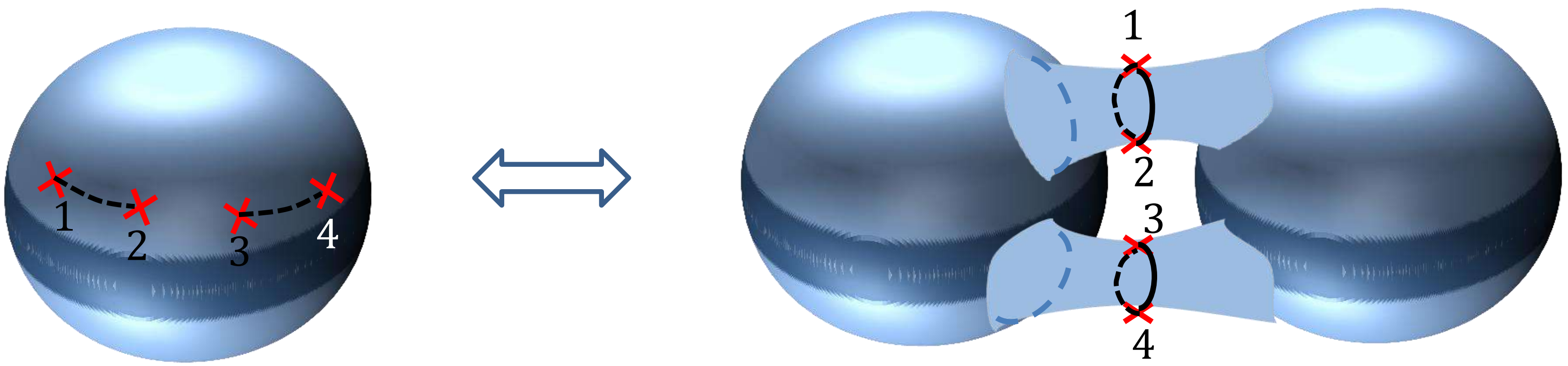
Braiding statistics of genons

- Braiding genons 1, 2➔Dehn twist around the orange loop T_x
- Braiding genons 2,3➔Dehn twist around the red-yellow loop T_y
- T_x, T_y non-commuting. Genons have non-Abelian statistics.



Properties of genons: braiding statistics

- Example: (220) state. 2 ground states for 4 defects. The braiding matrices are
- $U_{12} = e^{i\theta} \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}, U_{23} = e^{i\phi} \frac{1}{2} \begin{pmatrix} 1+i & 1-i \\ 1-i & 1+i \end{pmatrix}$
- Abelian phases are undetermined.
- The non-Abelian statistics is identical to Ising anyon! (Barkeshli, Jian & XLQ PRB '13)
- Related to intrinsic topological particles in orbifolding of topologically ordered states (Barkeshli-Wen '10)



Summary of general properties of genons

- Twist defects can be defined in a topological state with a global symmetry.
- In bilayer (or multilayer) states with the symmetry of permuting different layers, genons are defined around which particles move to different layers.
- Genons have non-Abelian projective statistics, and nontrivial quantum dimensions.
- Genons are different from topological quasiparticles: they have long range interaction. Statistics is only defined projectively.

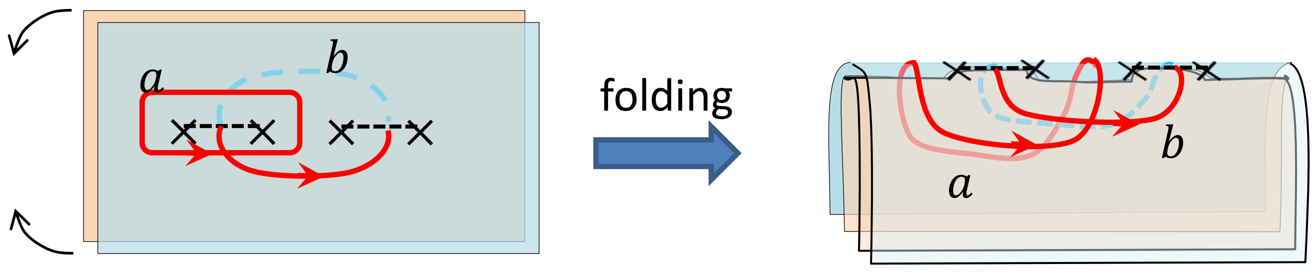
Other twist defects

- By twisting other symmetries, one can define other types of twist defects.
- Many twist defects can be understood as genons.

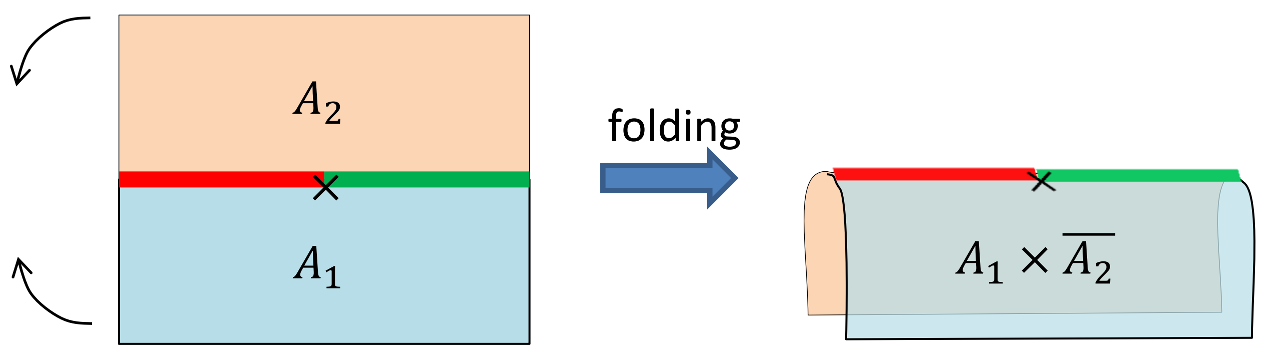
Topological order	Symmetries	Transformation of QP	References
Z_N toric code	Electromagnetic duality Z_2	$(e, m) \rightarrow (m, e)$	Bombin; You&Wen
	Particle-hole Z_2	$(e, m) \rightarrow (-e, -m)$	You&Jian&Wen
N-layer FQH	Layer permutation S_N	$(a_1, a_2, \dots, a_N) \rightarrow (a_{P_1}, a_{P_2}, \dots, a_{P_N})$	(Z_3 subgroup) Barkeshli&Jian&Qi
$1/k$ Laughlin	Particle-hole Z_2	$a \rightarrow -a$	Lindner et al; Clarke et al; Cheng, Vaezi
bilayer Z_k toric code	S_3 symmetry	"Hidden" S_3 permutation of QP	Teo&Roy&Chen
bilayer toric code (2d&3d)	Layer permutation Z_2	$(a_1, a_2) \rightarrow (a_2, a_1)$	Ran

A unified view to defects in Abelian states

- Different types of defects discussed here can all be mapped to gapped boundaries and domain wall between different boundary conditions
- Genons ➔ boundary in 4-layer systems

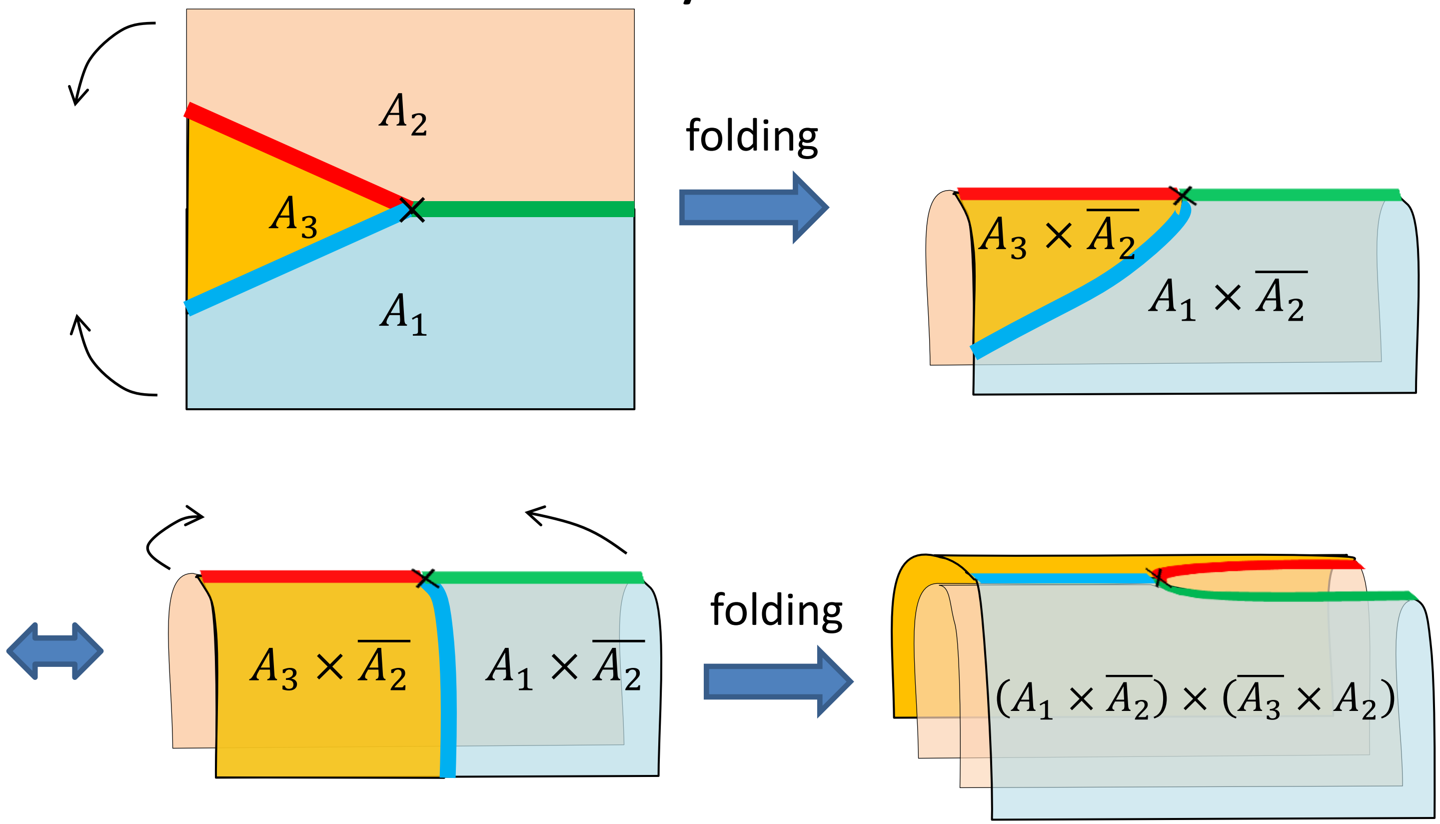


- Most general case, line defect between two phases ➔ boundary of bilayer system



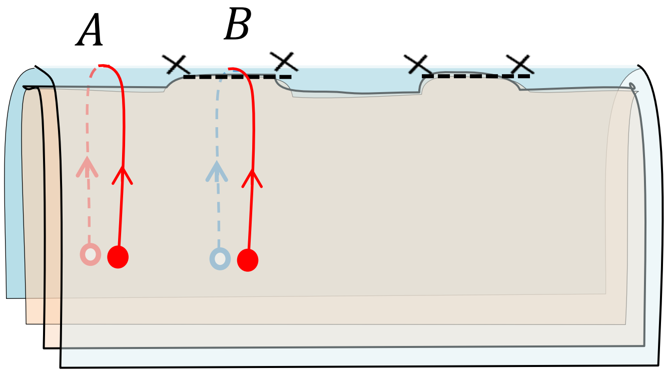
Generic line defects are equivalent to boundary defects

- Point defects are mapped to domain wall points between different line defects/boundary conditions
- Even the corner defect connecting three phases can still be viewed in this way



Boson condensation and Lagrangian subgroups

- In the folded picture, a boundary condition is determined by the particle that can annihilate (condense) at the boundary

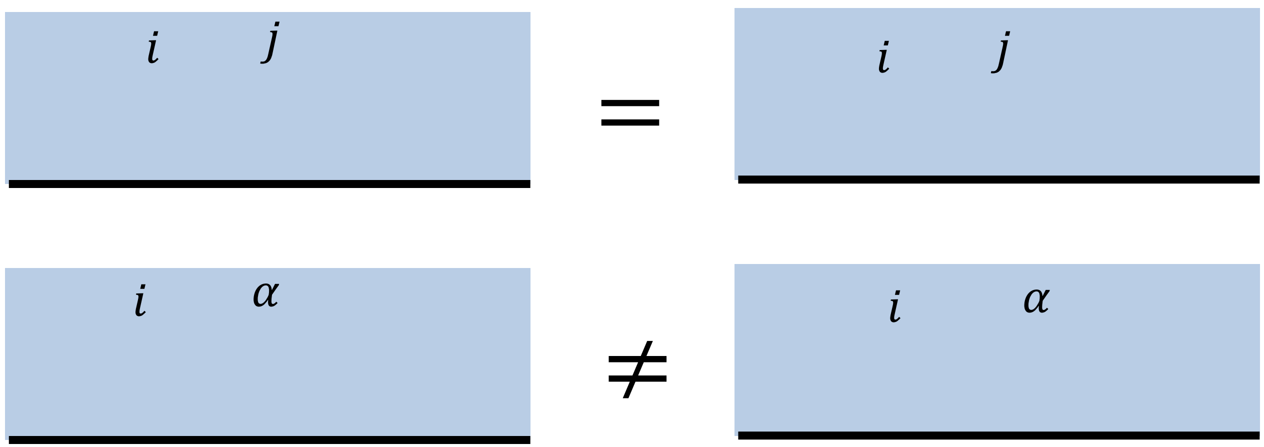


A region:
$(1,0,-1,0)$ and $(0,1,0,-1)$
B region:
$(1,0,0,-1)$ and $(0,1,-1,0)$

- The condensed particles must be boson, and mutually bosonic.
- Edge is completely gapped➔No particle is left after the boson condensation (Levin '13)

Boson condensation and Lagrangian subgroups

- A maximal set of condensed bosons form a **Lagrangian subgroup** (Levin '13, Barkeshli et al '13)
- Two conditions of Lagrangian subgroup M: i) $m_i^T K^{-1} m_j \in Z$. ii) No other particle α satisfies $\alpha^T K^{-1} m_j \in Z$.



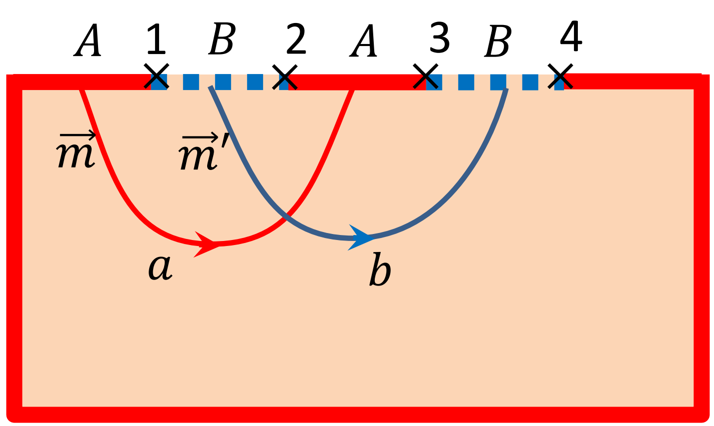
$, \forall\, i, j \in M$

$, \forall\, \alpha \notin M, \exists i$

- One-to-one correspondence between boundary conditions and Lagrangian subgroups (Barkeshli, et al '13, Levin '13)

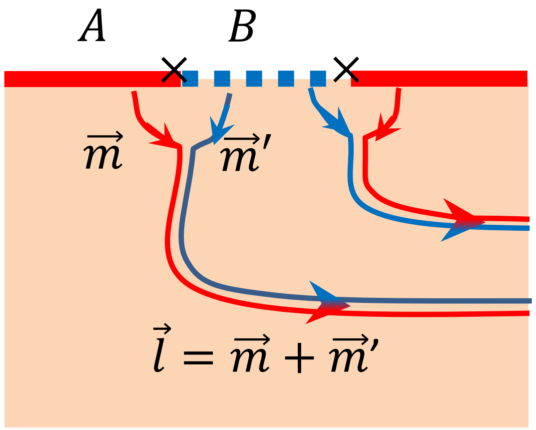
A unified view to defects in Abelian states

- Point defect carries nontrivial degeneracy, due to nontrivial braiding between different groups of bosons: $m_i \in M, m_i' \in M', m_i^T K^{-1} m_j' \notin Z$
- Wilson loop algebra
$$W_m(a) W_{m'}(b) = W_{m'}(b) W_m(a) e^{i2\pi m^T K^{-1} m'}$$
determines the (minimal) ground state degeneracy
- Defect quantum dimension $d = 1/\sqrt{\det |L|}$ with $L_{ij} = m_i^T K^{-1} m_j'$.
- m_i and m_i' are minimal basis sets for lattices M and M'.

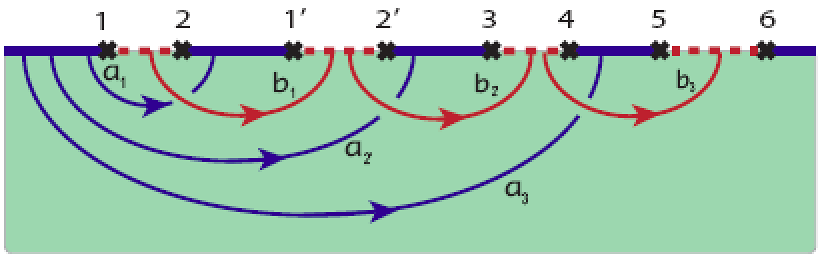


Parafermion zero modes and non-Abelian "statistics"

- Topological zero modes can be understood as pairs of particles $l = m + m'$
- Two bosons with nontrivial mutual statistics fuse into a nontrivial particle



- Generalization of the parafermion zero modes (Linder et al, Clarke et al, Fendley, '12)
- Effective "braiding" can be defined by coupling defects.



Summary of the first lecture

- Defects can be used to probe topologically ordered states
- Genons are simplest defects for multi-layer systems
- Non-Abelian defects can be realized in Abelian theories
- Generic point-like and line-like defects in Abelian theories can be understood based on the intuition from genons. General classification given by Lagrangian subgroups.
- Open question: Defects in non-Abelian theories?