A spatiotemporal extension of density matrices and time-reversal symmetry of measurements

Arthur Parzygnat (MIT)

based on joint work with James Fullwood (Hainan University)

2206.03607 2212.08088 2405.17555 2407.11123

Center for Quantum and Topological Systems at NYU Abu Dhabi

2024-11-11

Main idea

positive semidefinite $\rho \geq 0$ and $\mathrm{tr}[\rho] = 1$

- A density matrix ρ produces expectation values for observables A by the formula $\langle A \rangle_\rho = \mathrm{tr}[\rho A]$ $\quad A^\dagger = A$ (hermitian/self-adjoint)
- A state over time (to be defined) is a matrix $\mathcal{E} \star \rho$ involving an initial density matrix ρ and a quantum channel $\mathcal{E}: \mathcal{A} \to \mathcal{B}$ that gives the joint expectation values of measuring an observable A followed by B after evolution by $\mathcal{E}$.
- Motivating question: When does there exist a channel $\mathcal{F}: \mathcal{B} \to \mathcal{A}$ such that $\mathcal{F} \star \mathcal{E}(\rho)$ gives the joint expectation values of measuring B followed by A, i.e., when are the measurements time-reversal symmetric?

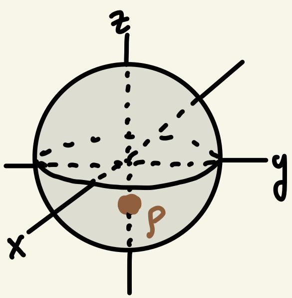
Static expectation values in quantum

Notation $\mathcal{A}, \mathcal{B}$ denote matrix algebras

Example $\mathcal{A} = \mathbb{M}_2$ is 2x2 complex matrices (algebra of a qubit)

Defn Let $\rho \in \mathcal{A}$ be a density matrix. Let $A \in \mathcal{A}$ be an observable. The **expectation value** of A with respect to ρ is the real number $\langle A \rangle_\rho := \mathrm{Tr}[\rho A]$.

Example $\rho = \begin{pmatrix} 1/3 & 0 \\ 0 & 2/3 \end{pmatrix}$ describes a state in which a qubit has 1/3 (2/3) chance of being spin up (down) along z. Hence,



$\langle \sigma_z \rangle_\rho = \mathrm{Tr}\left[\begin{pmatrix} 1/3 & 0 \\ 0 & 2/3 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right] = -1/3$ is the expected value of the spin in the z direction. Meanwhile,

$\langle \sigma_x \rangle_\rho = \mathrm{Tr}\left[\begin{pmatrix} 1/3 & 0 \\ 0 & 2/3 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right] = 0$ is the expected value of the spin in the x direction.

Dynamic expectation values in quantum

Defn Let $\rho \in \mathcal{A}$ be a density matrix and let $\mathcal{E}: \mathcal{A} \to \mathcal{B}$ be a quantum channel, i.e., a completely positive trace-preserving (CPTP) map. Let $A \in \mathcal{A}$, $B \in \mathcal{B}$ be observables with spectral decompositions

$$A = \sum_i \lambda_i P_i \qquad B = \sum_j \mu_j Q_j \qquad (\lambda_i, \mu_j \in \mathbb{R} \ \&\ P_i \in \mathcal{A},\ Q_j \in \mathcal{B} \text{ projectors}).$$

The __two-time expectation value__ of A and B with respect to the pair $(\rho, \mathcal{E})$ is the real number $\langle A, B \rangle_{(\rho,\mathcal{E})} := \sum_{i,j} \lambda_i \mu_j \operatorname{Tr}\left[\mathcal{E}(P_i \rho P_i) Q_j\right]$.

Remark The state after measuring A and getting outcome λ_i is $\dfrac{P_i \rho P_i}{\operatorname{Tr}[\rho P_i]}$ due to the state-update rule.

Example Suppose $\rho = \sum_i p_i |i\rangle\langle i|$, w/ $\{|i\rangle\}$ orthonormal basis, $\{p_i\}$ probabilities, $P_i = Q_i = |i\rangle\langle i|$, $\mathcal{E}(|i\rangle\langle j|) = \sum_k f_{ki} \delta_{ij} |k\rangle\langle k|$, $A = \sum_i \lambda_i P_i$, $B = \sum_j \mu_j Q_j$. $\{f_{ki}\}$ conditional probabilities

Then, $\langle A, B \rangle_{(\rho,\mathcal{E})} = \sum_{i,j} \lambda_i \mu_j f_{ji} p_i$ ($\{f_{ji} p_i\}$ is standard joint probability).

A no-go theorem for sequential measurements

Notation $\rho \in \mathcal{A}$ density matrix & $\mathcal{E}: \mathcal{A} \to \mathcal{B}$ channel $\Rightarrow$ $(\rho, \mathcal{E})$ process

Defn A process $(\rho, \mathcal{E})$ is representable iff there exists a matrix $X \in \mathcal{A} \otimes \mathcal{B}$ s.t. $\langle A, B \rangle_{(\rho,\mathcal{E})} = \mathrm{Tr}[X \, A \otimes B]$ for all observables $A \in \mathcal{A}$, $B \in \mathcal{B}$.

Theorem If $\mathcal{A}, \mathcal{B}$ are matrix algebras of dimension ≥ 2, then there exists a process $(\rho, \mathcal{E})$ that is not representable.

Example $\mathcal{E} = \mathrm{id}_{\mathbb{M}_n}$ and ρ any density matrix NOT equal to $\frac{\mathbb{1}_n}{n}$.

In what follows, we will bypass this no-go theorem by focussing on a subset of observables that are nevertheless quite robust.

The observables

Defn An observable $A \in \mathcal{A}$ is **light touch** iff its spectrum is either $\{-\lambda, \lambda\}$ for some $\lambda > 0$ or $\{\lambda\}$ for some $\lambda \in \mathbb{R}$.

Example $\mathcal{A} = M_2$ Pauli matrices $\mathbb{1}, \sigma_x, \sigma_y, \sigma_z$ are all light touch.
Also, if $\vec{n} \in S^2 \subseteq \mathbb{R}^3$, then $\vec{n} \cdot \vec{\sigma} = n_x \sigma_x + n_y \sigma_y + n_z \sigma_z$ is, too.

Theorem Let $(\rho, \mathcal{E})$ be a process. Then there exists a unique matrix $X \in \mathcal{A} \otimes \mathcal{B}$ such that $\langle A, B \rangle_{(\rho,\mathcal{E})} = \mathrm{Tr}[X\, A \otimes B]$ for all light touch observables $A \in \mathcal{A}$ and all observables $B \in \mathcal{B}$.

Example when $\mathcal{A} = M_2$, X is called a "pseudo-density matrix"
and it can be expressed as Ref: [Fitzsimons-Jones-Vedral]

$$X = \frac{1}{4} \sum_{\alpha,\beta} \langle \sigma_\alpha, \sigma_\beta \rangle \sigma_\alpha \otimes \sigma_\beta = \frac{1}{2} \{ \rho \otimes \mathbb{1}_2, \underbrace{(\mathrm{id}_{M_2} \otimes \mathcal{E})(\mathrm{SWAP})}_{\mathcal{J}[\mathcal{E}]} \}$$

$\sigma_0 = \mathbb{1}$, $\sigma_1 = \sigma_x$, $\sigma_2 = \sigma_y$, $\sigma_3 = \sigma_z$

$\{a, b\} = ab + ba$

$$\mathrm{SWAP} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

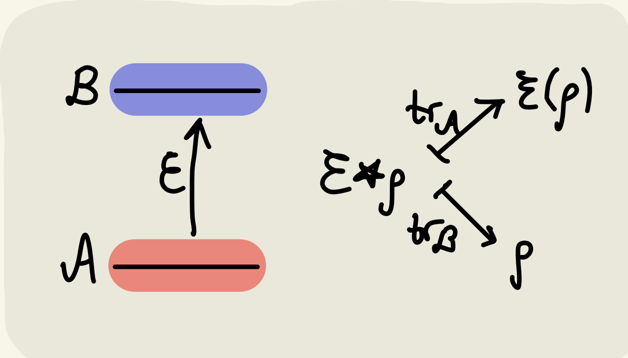
The state over time for sequential measurements

More generally, given a process $(\rho, \mathcal{E})$, the matrix $X \in \mathcal{A} \otimes \mathcal{B}$ s.t. $\langle A, B \rangle_{(\rho, \mathcal{E})} = \mathrm{Tr}[X \, A \otimes B]$ for all light touch $A \in \mathcal{A}$ and observables $B \in \mathcal{B}$ is given by the formula

$$X = \frac{1}{2}\Big\{\rho \otimes \mathbb{1}_{\mathcal{B}}, \overbrace{(\mathrm{id}_{\mathcal{A}} \otimes \mathcal{E})(\mathrm{SWAP}_{\mathcal{A}})}^{\mathcal{J}[\mathcal{E}]}\Big\}, \text{ where } \mathrm{SWAP}_{\mathcal{A}} = \sum_{i,j} |i\rangle\langle j| \otimes |j\rangle\langle i|.$$

This matrix is an example of a state over time.

Defn A **state over time function $\star$** is a function (for each $\mathcal{A}, \mathcal{B}$)



$$\mathrm{States}(\mathcal{A}) \times \mathrm{Channels}(\mathcal{A}, \mathcal{B}) \xrightarrow{\star} \mathcal{A} \otimes \mathcal{B}$$

sending $(\rho, \mathcal{E})$ to $\mathcal{E} \star \rho$ s.t. the two marginals of $\mathcal{E} \star \rho$ are ρ and $\mathcal{E}(\rho)$.

$\mathcal{E} \star \rho$ is called a **state over time**.

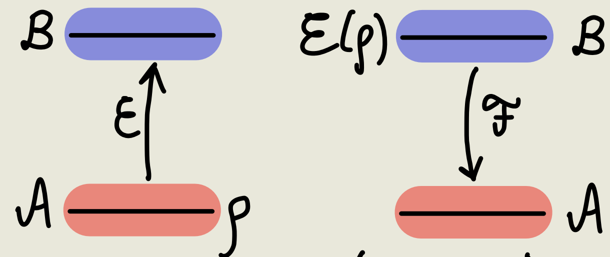
Note: Despite terminology, we do not demand $\mathcal{E} \star \rho$ to be a state!

Operational and Bayesian inverses

Defn A **Bayesian inverse** of $(\rho, \mathcal{E})$ is a channel $\mathcal{F} : \mathcal{B} \longrightarrow \mathcal{A}$ s.t.

$$\mathcal{E} \star \rho = S(\mathcal{F} \star \mathcal{E}(\rho)),$$

where $S(B \otimes A) = A \otimes B$ is the swap isomorphism.



$$\underbrace{\mathcal{E} \star \rho}_{\in \mathcal{A} \otimes \mathcal{B}} = S(\underbrace{\mathcal{F} \star \mathcal{E}(\rho)}_{\in \mathcal{B} \otimes \mathcal{A}})$$

Defn An **operational inverse** of $(\rho, \mathcal{E})$ is a channel $\mathcal{F} : \mathcal{B} \longrightarrow \mathcal{A}$ s.t. $\langle A, B \rangle_{(\rho, \mathcal{E})} = \langle B, A \rangle_{(\mathcal{E}(\rho), \mathcal{F})}$ for all light touch observables $A \in \mathcal{A}$, $B \in \mathcal{B}$.

Theorem Given $(\rho, \mathcal{E})$, a Bayesian inverse exists if and only if an operational inverse exists. Moreover, the two coincide.

Consequence Solve for $\mathcal{F}$ via Sylvester equation $\{\mathbb{1}_{\mathcal{B}} \otimes \rho, \mathscr{J}[\mathcal{E}^*]\} = \{\mathcal{E}(\rho) \otimes \mathbb{1}_{\mathcal{A}}, \mathscr{J}[\mathcal{F}]\}$

Examples with unitary dynamics and classical dynamics

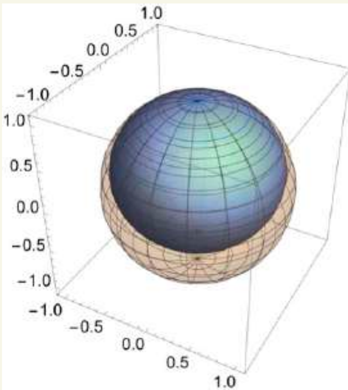
Theorem If $(\rho, \mathcal{E})$ is a process with $\mathcal{E} = \mathrm{Ad}_{\mathcal{U}}$, where $\mathcal{U}$ is a unitary operator, then the Bayesian inverse is $\mathcal{F} = \mathrm{Ad}_{\mathcal{U}^\dagger}$ regardless of the initial state ρ.

Remark When $\mathcal{A}, \mathcal{B}$ are finite dim'l commutative algebras, this definition reproduces Bayes' rule $\mathbb{P}(y|x)\,\mathbb{P}(x) = \mathbb{P}(x|y)\,\mathbb{P}(y)$. Alternatively, we can state this in the following way.

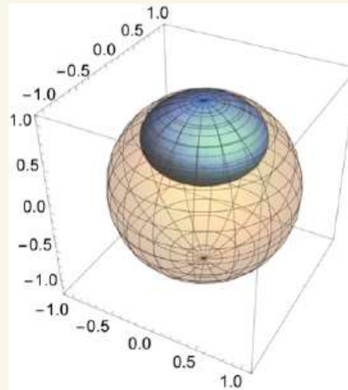
Theorem If $\rho = \sum_i p_i |i\rangle\langle i|$, w/ $\{|i\rangle\}$ orthonormal basis, $\{p_i\}$ probabilities, $\mathcal{E}(|i\rangle\langle j|) = \sum_k f_{ki}\,\delta_{ij}\,|k\rangle\langle k|$, where $\{f_{ki}\}$ define conditional probabilities, then the Bayesian inverse $\mathcal{F}$ satisfies $\mathcal{F}(|k\rangle\langle l|) = \sum_i g_{ik}\,\delta_{kl}\,|i\rangle\langle i|$, where $\{g_{ik}\}$ are the conditional probabilities satisfying the (classical) Bayes' rule $f_{ki}\,p_i = g_{ik}\,q_k$ $\forall i,k$, where $q_k = \sum_j f_{kj}\,p_j$.

Example with amplitude-damping channel I

Take $\mathcal{E} = \mathrm{Ad}_{E_0} + \mathrm{Ad}_{E_1}$ $\quad (\mathrm{Ad}_V(A) = VAV^\dagger)$ w/ $E_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{pmatrix}$, $E_1 = \begin{pmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{pmatrix}$



$\gamma = 0.2$



$\gamma = 0.6$

A qubit density matrix is of the form $\frac{1}{2}\begin{pmatrix} 1+z & x-iy \\ x+iy & 1-z \end{pmatrix}$ with $\|(x,y,z)\|^2 \leq 1$ and $\therefore$ identifies with a point in Bloch ball. The image of the Bloch ball is shown in blue for two values of $\gamma \in [0,1]$.

Now take input state $\rho = \frac{1}{2}\begin{pmatrix} 1+z & 0 \\ 0 & 1-z \end{pmatrix}$ $\Rightarrow \mathcal{E}(\rho) = \frac{1}{2}\begin{pmatrix} 1+z' & 0 \\ 0 & 1-z' \end{pmatrix}$,

where $z' = z + \gamma(1-z)$.

Fact With these parameters $z \in (-1,1)$, $\gamma \in (0,1)$, a Bayesian inverse exists if and only if $z \geq \frac{\gamma}{\gamma - 2}$

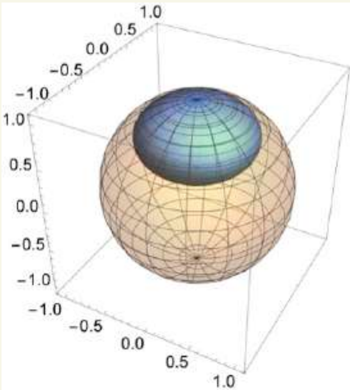
Example with amplitude-damping channel II

Fact (continued) The Bayesian inverse is given explicitly by

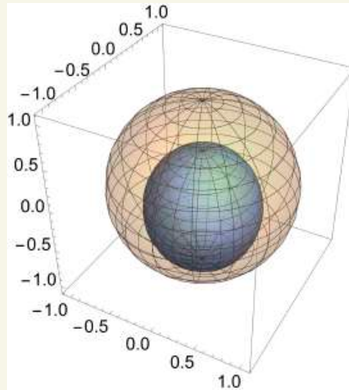
$$\mathcal{F} = \mathrm{Ad}_{F_0} + \mathrm{Ad}_{F_1} + \mathrm{Ad}_{F_2} \quad \text{where}$$

$$F_0 = \begin{pmatrix} \sqrt{\frac{1+z}{1+z'}} & 0 \\ 0 & \sqrt{\frac{(1-\gamma)(1+z')}{1+z}} \end{pmatrix},$$

$$F_1 = \begin{pmatrix} 0 & 0 \\ \sqrt{\frac{\gamma(1-z)}{1+z'}} & 0 \end{pmatrix}, \quad F_2 = \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{\frac{\gamma(z+z')}{1+z}} \end{pmatrix}.$$



original $\mathcal{E}$ w/ $\gamma = 0.6$



Bayesian inverse $\mathcal{F}$ w/ $\gamma = 0.6, z = 0.2$

This is a bit-flipped amplitude-damping channel with dephasing.

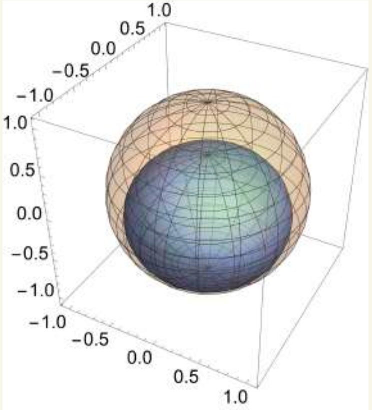
	σ_0	σ_1	σ_2	σ_3
σ_0	1	0	0	z'
σ_1	0	$\sqrt{1-\gamma}$	0	0
σ_2	0	0	$\sqrt{1-\gamma}$	0
σ_3	z	0	0	$1-\gamma(1-z)$

← Two-time expect. values $\langle \sigma_\alpha, \sigma_\beta \rangle$ w/ Pauli σ_α in top row measured first and then (after $\mathcal{E}$) Pauli σ_β in left column measured second.

Same but time-reversed $\langle \sigma_\beta, \sigma_\alpha \rangle$ using Bayesian inv. $\mathcal{F}$ →

	σ_0	σ_1	σ_2	σ_3
σ_0	1	0	0	z
σ_1	0	$\sqrt{1-\gamma}$	0	0
σ_2	0	0	$\sqrt{1-\gamma}$	0
σ_3	z'	0	0	$1-\gamma(1-z)$

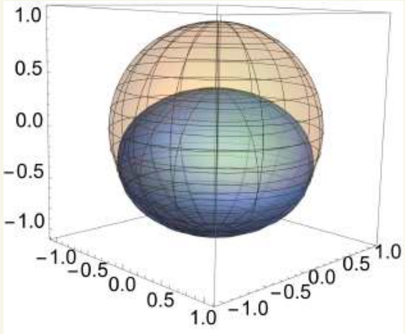
Example with amplitude-damping channel III



Remark The Bayesian inverse $\mathcal{F}$ is NOT equal to the Petz recovery map $\mathcal{R} = \mathrm{Ad}_{R_0} + \mathrm{Ad}_{R_1}$

w/ $R_0 = \begin{pmatrix} \sqrt{\frac{1+z}{1+z'}} & 0 \\ 0 & 1 \end{pmatrix}$ $R_1 = \begin{pmatrix} 0 & 0 \\ \sqrt{\frac{\gamma(1-z)}{1+z'}} & 0 \end{pmatrix}$

The Petz recovery map is depicted here →



Remark If $z < \frac{\gamma}{\gamma-2}$, no operational inverse exists! Although $\mathcal{F}$ is uniquely defined, it is not completely positive (an example when $z = -0.5$, $\gamma = 0.15$ is shown on right).

Summary & Key points

- Just like expectation values of observables at a fixed time are encoded in a density matrix, two-time expectation values of sequential measurements of light-touch observables are encoded in a state over time.
- Light touch observables are just as robust as projections in that their expectation values determine density matrices and states over time.
- Our theory of quantum Bayesian inverses has operational consequences for time-reversal symmetric multi-time expectation values.
- Immense plethora of open questions and low-hanging fruit!

  Eg. What state over time is characterized by two-time expectation values of Gell-mann matrices? Great problems for grad students!

Thank you!

All joint work with James Fullwood

"On quantum states over time"

2206.03607

Bypasses a no-go theorem of Horsman et al, proving existence of a state over time that satisfies many conditions

"Operator representation of spatiotemporal quantum correlations"

2405.17555

Provides an operational meaning to states over time and extends pseudo-densities

"From time-reversal symmetry to quantum Bayes' rules"

2212.08088

A modern reference on states over time and how they give rise to a quantum extension of Bayes' rule

"Time-symmetric correlations for open quantum systems"

2407.11123

What this talk was mostly based on (prediction using quantum Bayes' rule)