The Geometry of Pure Spinor Superfield Formalism

GTP Seminar - NYUAD

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Motivations & Premises

The construction of supersymmetric field theories faces essential difficulties, *e.g.* supersymmetry is only represented *on-shell* / up to gauge transformations.

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- **Quantization:** desirable for the symmetry to act on the full space of fields without regard to the dynamics;
- **Geometrization:** reasonable to think of supersymmetry as arising from the action of particular geometric symmetry on an appropriate (super)space;

$\rightsquigarrow$ **extending the space of fields / superfield formulation**

How to... Superfield Formulations

Harmonic Superspace (Galperin, Ivanov, Ogievetsky, Sokatchev...), Rheonomy (Castellani, D'Auria, Fre...), Pure Spinors (Nillson & Howe, Berkovits...)

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A View on Pure Spinors Superfield Formalism

Provide a view on pure spinor superfield formalism

- amenable to mathematicians;
- yields susy "multiplets" as understood by physicists.

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Structure of the talk

1. Definition of **multiplet;**
2. **Nilpotence variety** and **pure spinor superfield formalism**;
3. **Examples**;
4. (If time permits: general results and considerations).

- arXiv2404.07167 w/ R. Eager, R. Senghaas, J. Walcher;
- arXiv:2206.08388 w/ F. Hahner, I. Saberi, J. Walcher;
- see also arXiv:2111.01162

Multiplets - a first encounter in physics

A multiplet is a *representation of the supersymmetry algebra* $\mathfrak{g}$ of a physical theory.

Concretely, a multiplet is given by a collection of fields transforming one into another under the action of $\mathfrak{g}$: they are the *building blocks of actions of physical theories*.

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A minimal supersymmetric Lagrangian in $d = 4$ reads

$$\mathscr{L}_{chiral} = -\partial\bar{\varphi} \cdot \partial\varphi + i\bar{\psi}\partial\!\!\!/\psi + \bar{F}F$$

⇝ The triplet (φ, ψ, F) is a multiplet, called *chiral multiplet*.
⇝ *Supersymmetry transformations* of (φ, ψ, F) read

$$\delta_s\phi = \epsilon\psi, \quad \delta_s\psi = i\bar{\epsilon}\partial\!\!\!/\phi + \epsilon F, \quad \delta_s F = -i\epsilon\partial\!\!\!/\psi$$

Multiplets - toward a mathematical definition

Obviously, a multiplet is a *representation-theoretic* notion, though it is not obvious how to provide a rigorous - and encompassing - definition!

Working on a flat (possibly complexified) spacetime V, some pieces of data should be part of our definition:

- Bosonic and Fermionic fields are sections of vector bundles on the spacetime V (with parity / $\mathbb{Z}/2$-grading);
- Supersymmetry transformations are given by an action of a certain (super)algebra on these sections.

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On the other hand, attention must be paid...

In relevant examples, the representation of the supersymmetry algebra is *not strict*!

The $\mathcal{N} = 1$, $d = 4$ Vector Multiplet

This multiplet consists of the following collection of fields:

1. $A \in \Omega^1(\mathbb{R}^4)$ is a connection 1-form;
2. $(\lambda, \bar{\lambda}) \in C^\infty(\mathbb{R}^4, \Pi(S_+ \oplus S_-))$ are spinors of opposite chirality;
3. $D \in C^\infty(\mathbb{R}^4)$ is an auxiliary field;
4. $c \in C^\infty(\mathbb{R}^4)$ is a ghost field (of ghost degree $-1 \rightsquigarrow$ gauge).

- The ghost field has a *non-zero differential*:

$$c \xmapsto{d} dc \quad \leftrightsquigarrow \quad \delta_{brst} A_\mu = \partial_\mu c$$

- The ghost field has *higher-order supersymmetry transformation*:

$$Q \otimes \bar{Q} \otimes A \xmapsto{\rho^2} \iota_{\{Q,\bar{Q}\}} A \quad \leftrightsquigarrow \quad \delta_s c = (\epsilon \sigma^\mu \bar{\epsilon}) A_\mu.$$

The $\mathcal{N} = 1$, $d = 4$ Vector Multiplet

$\mathfrak{g}$-module structure on the vector multiplet

In physics lingo, the higher-order transformation of c is a *closure term* for the supersymmetry action: in this case, we say that "the supersymmetry action only closes up to gauge transformations".

Setting $\rho^i : \mathfrak{g}^{\otimes i} \longrightarrow End(\mathcal{E})[1-i]$, we have

$$\rho^1\text{-terms} \quad \leftrightsquigarrow \quad \begin{cases} \delta_s A_\mu = \epsilon\sigma_\mu\bar{\lambda} + \psi\sigma_\mu\bar{\epsilon}, \\ \delta\lambda_s = \epsilon D, \quad \delta_s\bar{\lambda} = -\bar{\epsilon}D, \\ \delta_s D = 0, \end{cases}$$

$$\rho^2\text{-terms} \quad \leftrightsquigarrow \quad \delta_s c = (\epsilon\sigma^\mu\bar{\epsilon})A_\mu.$$

The relation between ρ^1 and ρ^2 is given by

$$[\rho^1(x), \rho^1(y)] - \rho^1([x,y]) = -[d, \rho^2(x,y)].$$

In other words, ρ^2 provides a **homotopy** for the failure of ρ^1 to be a strict $\mathfrak{g}$-action $\rightsquigarrow$ we should consider weaker / L_∞-action!

Multiplet - a (tentative) mathematical definition

Definition ($\mathfrak{g}$-Multiplet)

Let (E, D) be an affine dgs vector bundle on $V = \mathbb{R}^d$, let $\mathfrak{g}$ be a super L_∞-algebra together with an injective map $\iota : \mathfrak{Aff}(V) \to \mathfrak{g}$.

A $\mathfrak{g}$-multiplet is a local $\mathfrak{g}$-module structure (E, D, ρ) on (E, D) such that the pullback of the module structure along $\iota : \mathfrak{Aff}(V) \to \mathfrak{g}$ agrees with the natural action on sections.

1. **Affine** : the total space of E carries an action of $\mathfrak{Aff}(V) = \mathbb{R}^d \rtimes \mathfrak{so}(d)$ such that its projection $\pi : E \to V$ is *equivariant* with respect to the action of $\mathfrak{Aff}(V)$ on V;
2. **dgs vector bundle** (E, D): $\mathbb{Z} \times \mathbb{Z}_2$-graded vector bundle $E = \bigoplus_k (E_+^k \oplus E_-^k)$ equipped with a collection of differential operators $D : \mathcal{E}_\pm^k \to \mathcal{E}_\pm^{k+1}$ such that $D \circ D = 0$, where $\mathcal{E}_\pm^k := \Gamma(\mathcal{X}, E_\pm^k)$ are the C^∞-sections of $E_\pm^k$.
3. **Local $\mathfrak{g}$-module structure** : super L_∞-map $\rho : \mathfrak{g} \to (\mathcal{D}(\mathcal{E}), [D, -])$ with $\mathcal{D}(\mathcal{E}) := \{x \in End(\mathcal{E}) : x \text{ is a differential operator}\} \subset End(\mathcal{E})$.

Multiplet - Examples

Multiplets lead to study (super)algebras that contain the affine algebra as a subalgebra.

We are interested in the case of the super Poincaré algebras $\mathfrak{p}$, but - as defined - the notion is broader...

1. Let $\mathfrak{h}$ be a Lie algebra and let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{Aff}(V)$. A $\mathfrak{g}$-multiplet contains a collection of fields transforming in a local representation of $\mathfrak{h} \rightsquigarrow$ "flavor symmetry" multiplets.

2. The Lie algebra $\mathfrak{Conf}(V)$ of (super)conformal transformations on V contains $\mathfrak{Aff}(V) \rightsquigarrow \mathfrak{Conf}(V)$-multiplets.

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Question : how to construct – and possibly "classify" – multiplets? (*i.e.* how to provide the building blocks for supersymmetric theories?)

Pure Spinor Superfield Formalism & Nilpotence Variety

The **Pure Spinor Superfield** formalism is a machinery that

$$\underbrace{\left\{ \begin{array}{c} \text{Bundles on} \\ Y_{\mathfrak{p}} \end{array} \right\}}_{\text{geometry}} \overset{\mathcal{PS}}{\rightsquigarrow} \underbrace{\left\{ \begin{array}{c} \text{Multiplets for} \\ \mathfrak{p} \end{array} \right\}}_{\text{representations}}$$

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The equations $Q^2 := \frac{1}{2}\{Q, Q\} = 0$, defines a set of quadrics, whose zero locus is called **nilpotence variety** $Y_{\mathfrak{g}} \subseteq \mathbb{A}^{\dim \mathfrak{g}_1}$.

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The equations are homogeneous, hence their space of solutions descends to a projective variety $\mathbb{P}Y_{\mathfrak{g}} \subseteq \mathbb{P}^{\dim \mathfrak{g}_1 - 1}$, the **projectivized nilpotence variety** of $\mathfrak{g}$.

Nilpotence Variety

Definition (Nilpotence Variety of $\mathfrak{g}$)

Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a super Lie algebra.

1. let R be the polynomial ring $Sym^\bullet(\mathfrak{g}_1^\vee[-1])$;
2. let I be the ideal defined by the set of equations $\{Q, Q\}$.

Then we call

- $Y_\mathfrak{g} := \mathrm{Spec}(R/I) \subset \mathrm{Spec}(R)$ is the **affine nilpotence variety**;
- $\mathbb{P}Y_\mathfrak{g} := \mathrm{Proj}(R/I) \subset \mathrm{Proj}(R)$ is the **projective nilpotence variety**.

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Very concretely, for super Poincaré algebras, expanding $Q = \lambda^a Q_a$ and identifying $R = \mathbb{C}[\lambda^a]$, if we denote Γ^μ_{ab} the structure constant of the bracket $\{Q_a, Q_b\} \sim \Gamma^\mu_{ab} p_\mu$, we have

$$R\,/\,I = \mathbb{C}[\lambda^a]\,/\,(\lambda^a \Gamma^\mu_{ab} \lambda^b).$$

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$\rightsquigarrow$ *Mathematically*, the nilpotence variety of $\mathfrak{g}$ can be seen as a "moduli space of cohomologies"...

$\rightsquigarrow$ *Physically*, these cohomologies are called **twists** of the related ($\mathfrak{g}$-invariant) physical theories.

Super Poincaré Algebra $\mathfrak{p}$ of V

As a super Lie algebra comes with a $\mathbb{Z}/2$-grading $\mathfrak{p} = \mathfrak{p}_0 \oplus \mathfrak{p}_1$:

1. The **fermionic part** $\mathfrak{p}_1$ is the tensor product of a spin representation S with an auxiliary vector space U

$$\mathfrak{p}_1 = S \otimes U,$$

   Recall that there are either one S or two $S_\pm$ representations of $Spin(V_{/\mathbb{C}})$.

   - Depending on the dimension, U can be equipped with a symmetric or antisymmetric bilinear form.
   - The "degree of supersymmetry" $\mathcal{N}$ is $\dim(U)$ as a multiple of its smallest possible dimension.

2. The **bosonic part** $\mathfrak{p}_0$ arises from *translations* V, *Lorentz transformations* $\mathfrak{so}(V)$ and *R-symmetry* $\mathfrak{r}$:

$$\mathfrak{p}_0 = (V \rtimes \mathfrak{so}(V)) \times \mathfrak{r},$$

   where $\mathfrak{r} = \{\mathfrak{gl}(U), \mathfrak{so}(U), \mathfrak{sp}(U)\}$, for U the auxiliary vector space.

   - The R-symmetry arises as automorphisms of U.

Supertranslations (aka Supersymmetry) Algebra $\mathfrak{t}$

It is a subalgebra of $\mathfrak{p}$. As a super Lie algebra it reads

$$\mathfrak{t} = \mathfrak{t}_0 \oplus \mathfrak{t}_1 = V \oplus \mathfrak{p}_1.$$

More precisely, it is a central extension of $\mathfrak{p}_1$ the form

$$0 \longrightarrow V \longrightarrow \mathfrak{t} \longrightarrow \mathfrak{p}_1 \longrightarrow 0,$$

the bracket on $\mathfrak{t}$ is given by the equivariant map

$$\Gamma : Sym^2(S) \to V$$

for S a spin representation.

It might be convenient to look at the super Poincaré algebra as graded algebra $\mathfrak{p} = \mathfrak{p}_0 \oplus \mathfrak{p}_1 \oplus \mathfrak{p}_2$, in a way such that supertranslations read $\mathfrak{t} := \mathfrak{p}_{>0}$ and $\{\cdot,\cdot\} : Sym^2(\mathfrak{p}_1) \to \mathfrak{p}_2$ is $\mathfrak{p}_0$-equivariant.

$d = 4$, $\mathcal{N} = 1$ Nilpotence Variety

- The super Poincaré algebra reads

$$\mathfrak{p} = \mathfrak{p}_0 \oplus \mathfrak{p}_1 = (V \rtimes \mathfrak{so}(V)) \oplus (S_+ \oplus S_-)$$

  where $S_\pm$ are chiral Weyl spinor representations of $Spin(V)$.

- Γ defines an isomorphism $\Gamma : S_+ \otimes S_- \xrightarrow{\cong} V$.
- This implies that $\{Q, Q\} = 0 \iff Q \in S_+$ or $Q \in S_-$.
- $Y(d = 4, \mathcal{N} = 1)$ consists in two $\mathbb{C}^2$-planes in $\mathbb{C}^4$ intersecting at the origin:

$$Y(4,1) = \mathbb{C}^2 \cup_{\{0\}} \mathbb{C}^2 = S_+ \cup_{\{0\}} S_-.$$

$d = 4$, $\mathcal{N} = 1$ Nilpotence Variety

The computation can be repeated in coordinates!

- A general supercharge can be written

$$Q = \lambda^\alpha Q_\beta + \bar{\lambda}^{\dot{\beta}} \bar{Q}_{\dot{\beta}}$$

as decomposed in its S_- and S_+ components.

- The equation $\{Q, Q\} = 0$ reduces to four quadratic equations

$$\lambda^\alpha \bar{\lambda}^{\dot{\beta}} \Gamma^\mu_{\alpha\dot{\beta}} = 0 \rightsquigarrow \begin{cases} \lambda^1\bar{\lambda}^1 + \lambda^2\bar{\lambda}^2 = 0, \\ \lambda^1\bar{\lambda}^1 - \lambda^2\bar{\lambda}^2 = 0, \\ \lambda^1\bar{\lambda}^2 + \lambda^2\bar{\lambda}^1 = 0, \\ \lambda^1\bar{\lambda}^2 - \lambda^2\bar{\lambda}^1 = 0. \end{cases}$$

- Adding and subtracting one finds

$$\lambda^1\bar{\lambda}^1 = \lambda^2\bar{\lambda}^2 = \lambda^1\bar{\lambda}^2 = \lambda^2\bar{\lambda}^1 = 0 \rightsquigarrow \lambda^\alpha = 0 \ \vee \ \bar{\lambda}^{\dot{\beta}} = 0.$$

$d = 3$, $\mathcal{N} = 1$ Nilpotence Scheme

- The super Poincaré algebra reads

$$\mathfrak{p} = \mathfrak{p}_0 \oplus \mathfrak{p}_1 = (V \rtimes \mathfrak{so}(V)) \oplus S$$

  where S is in the fundamental representation of $Spin(3)$.

- Γ defines an isomorphism $\Gamma : Sym^2(S) \xrightarrow{\cong} V$.
- This implies that $\{Q, Q\} = 0 \iff Q = 0$.
- $Y(3,1) = \{0\} \subset \mathbb{C}^2$... as an algebraic set!

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- This implies that $\{Q, Q\} = 0 \iff Q = 0$.
- $Y(3,1) = \{0\} \subset \mathbb{C}^2$... as an algebraic set!
- As a scheme, it is a **fat point**! Indeed expanding $\{Q, Q\} = 0$ one has

$$(\lambda^1)^2 = \lambda^1 \lambda^2 = (\lambda^2)^2 = 0.$$

- It follows that $Y(3,1) = \mathrm{Spec}(\mathbb{C}[\lambda^1, \lambda^2]/((\lambda^1)^2, \lambda^1\lambda^2, (\lambda^2)^2))$,

$d = 6$, $\mathcal{N} = (1,0)$ Projective Nilpotence Variety

- In $d = 6, \mathcal{N} = (1,0)$ we have symplectic spinors $\rightsquigarrow \mathfrak{t}_1 = S_+ \otimes U$, with (U, ω) a symplectic vector space.
- The nilpotence ideal $I = (\lambda_i^\alpha \Gamma^\mu_{\alpha\beta} \omega^{ij} \lambda_j^\beta)$ is a *determinantal ideal*

$$I = \left\{ (2 \times 2)\text{-minors of } [L] := \begin{pmatrix} \lambda_1^1 & \lambda_1^2 & \lambda_1^3 & \lambda_1^4 \\ \lambda_2^1 & \lambda_2^2 & \lambda_2^3 & \lambda_2^4 \end{pmatrix} \right\} \rightsquigarrow \text{“rank 1 locus” of } [L]$$

If follows that the nilpotence variety has a very nice a nice projective model, in fact the projective nilpotence variety $\mathbb{P}Y(6,(1,0))$ is a Segre 4-fourfold (sitting in $\mathbb{P}^7$):

$$Y(6;(1,0)) = \mathbb{P}^1 \times \mathbb{P}^3 \hookrightarrow \mathbb{P}^7$$

$\rightsquigarrow$ bundles are easily available on this (smooth!) variety...

Relations between Nilpotence Varieties

$$
\begin{array}{ccccccc}
\text{in } \mathbb{P}^3 & & \text{in } \mathbb{P}^7 & & \text{in } \mathbb{P}^{11} & & \text{in } \mathbb{P}^{15} \\
 & & & & & & Y(10;1) \\
 & & & & & & \downarrow \\
 & & Y(6;(1,0)) & \hookrightarrow & & \longrightarrow & Y(6;(2,0)) \\
 & & \downarrow & & & & \downarrow \\
 & & Y(5;1) & \hookrightarrow & & \longrightarrow & Y(5;2) \\
 & & \downarrow & & & & \downarrow \\
Y(4;1) & \hookrightarrow & Y(4;2) & \hookrightarrow & Y(4;3) & \hookrightarrow & Y(4;4) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
Y(2;(2,2)) & \hookrightarrow & Y(2,(4,4)) & \hookrightarrow & Y(2,(6,6)) & \hookrightarrow & Y(2,(8,8))
\end{array}
$$

Pure Spinor Superfield Formalism

In a nutshell, pure spinor superfield formalism constructs $\mathfrak{p}$-multiplets starting from the geometric data of modules on $Y_{\mathfrak{p}}$.

Pure Spinor Superfield Formalism

- Identifying the spacetime $V = \mathfrak{p}_2$ we consider the **supermanifold** $\mathcal{X}$

$$\mathcal{O}(\mathcal{X}) = C^\infty(\mathfrak{p}_{>0}) = C^\infty(V) \otimes_{\mathbb{C}} \wedge^\bullet(\mathfrak{p}_1^\vee) = C^\infty(\mathbb{C}^d) \otimes_{\mathbb{C}} \wedge^\bullet(\mathfrak{p}_1^\vee)$$

and call local coordinates $x^\mu|\theta^\alpha$ and $\mathcal{O}(\mathcal{X})$ the algebra of free superfields.

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  and call local coordinates $x^\mu|\theta^\alpha$ and $\mathcal{O}(\mathcal{X})$ the algebra of free superfields.

- There are two **commuting** action of the supersymmetry algebra, $(\ell, r) : \mathfrak{p}_1 \to End(\mathcal{X})$:

$$\ell(Q_\alpha) \equiv \widehat{Q}_\alpha := \partial_{\theta^\alpha} - i\Gamma^\mu_{\alpha\beta}\theta^\beta\partial_{x^\mu}$$
$$r(Q_\alpha) \equiv \widehat{\mathcal{D}}_\alpha := \partial_{\theta^\alpha} + i\Gamma^\mu_{\alpha\beta}\theta^\beta\partial_{x^\mu}$$

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- Take a (graded $\mathfrak{p}_0$-equivariant) module M on the nilpotence variety Y. This means that M is a graded $\mathfrak{p}_0$-equivariant R/I-module, for

$$R/I = \mathbb{C}[\lambda^\alpha]/I$$

  where I is the ideal cut out by $\{Q, Q\} = 0$.

Pure Spinor Superfield Formalism

- **Crucial step:** tensor the algebra of free superfields $\mathcal{O}(\mathcal{X})$ with the R/I-module M as to get a cochain complex

$$\mathcal{A}^\bullet(M) := (M \otimes_{\mathbb{C}} \mathcal{O}(\mathcal{X}),\ \mathcal{D}),$$

where $\mathcal{D} := \lambda^\alpha \otimes r(Q_\alpha) = \lambda^\alpha \widehat{\mathcal{D}}_\alpha$ and λ^α acts via the R/I-module structure.

$$\mathcal{D}^2 = \lambda^\alpha \lambda^\beta r(Q_\alpha) r(Q_\beta) = \frac{1}{2}\lambda^\alpha\lambda^\beta\{r(Q_\alpha), r(Q_\beta)\} =$$

$$= \frac{1}{2}\lambda^\alpha\lambda^\beta r(\{Q_\alpha, Q_\beta\}) = \frac{1}{2}\underbrace{\lambda^\alpha \Gamma^\mu_{\alpha\beta}\lambda^\beta}_{=0 \text{ on } Y_{\mathfrak{p}}} r(p_\mu) = 0.$$

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where $\mathcal{D} := \lambda^\alpha \otimes r(Q_\alpha) = \lambda^\alpha \widehat{\mathcal{D}}_\alpha$ and λ^α acts via the R/I-module structure.

$$\mathcal{D}^2 = \lambda^\alpha \lambda^\beta r(Q_\alpha) r(Q_\beta) = \frac{1}{2}\lambda^\alpha\lambda^\beta \{r(Q_\alpha), r(Q_\beta)\} = \\ = \frac{1}{2}\lambda^\alpha\lambda^\beta r(\{Q_\alpha, Q_\beta\}) = \frac{1}{2} \underbrace{\lambda^\alpha \Gamma^\mu_{\alpha\beta}\lambda^\beta}_{=0 \text{ on } Y_{\mathfrak{p}}} r(p_\mu) = 0.$$

- $\mathcal{A}^\bullet(M)$ has the structure of a dgs vector space ($\rightsquigarrow \mathbb{Z} \times \mathbb{Z}_2$-graded)

$$\deg(\lambda^\alpha) = (1, -) \quad \deg(x^\mu) = (0, +), \quad \deg(\theta^\alpha) = (0, -).$$

In fact, $\mathcal{A}^\bullet(M)$ can be viewed as the global sections of an **affine dgs vector bundle** $\pi : E \to V = \mathfrak{p}_2$, with typical fiber $E^k_x = (M)^k \otimes \wedge^\bullet \mathfrak{g}_1^\vee$ $\rightsquigarrow$ **multiplet!**

Pure Spinor Superfield Formalism

- We still have a *left* action ℓ! In particular one can argue that:
  1. $\ell(\mathfrak{p}_{>0})$ commutes with $\mathcal{D}$ $\Rightarrow$ it defines a $\mathfrak{p}_{>0}$-module structure on $\mathcal{A}^\bullet(M)$;
  2. it is equivariant with respect to $\mathfrak{p}_0$ $\Rightarrow$ can be extended to a full $\mathfrak{p}$-action

$$\tilde{\ell} : \mathfrak{p} \to \mathcal{A}^\bullet(M);$$

$\rightsquigarrow$ $\mathcal{A}^\bullet(M)$ **is endowed with the structure of a $\mathfrak{p}$-multiplet!**

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...from Superspace to Space(time)...

We would like to have the "ordinary" presentation of multiplet as collections of vector bundles on the spacetime V out of $\mathcal{A}^\bullet(M)$.

A *spectral sequence* argument allows for the connection:

$$\{\mathfrak{p}\text{-multiplet } \mathcal{A}^\bullet(M)\} \rightsquigarrow \{\text{vector bundles over spacetime}\}$$

Filtration and Associated Spectral Sequence

1. We consider the filtered complex $F^\bullet \mathcal{A}^\bullet(M)$ according to the filtered weights in the above table;

	homological deg	intrinsic parity	filtered weight
x	0	$+$	0
θ	0	-	1
λ	1	-	1

2. The differential does not respect the weight grading:

$$\mathcal{D} = \mathcal{D}_0 + \mathcal{D}_1 = \underbrace{\lambda^\alpha \partial_{\theta^\alpha}}_{w=0} + \underbrace{\lambda^\alpha \Gamma^\mu_{\alpha\beta} \theta^\beta \partial_{x^\mu}}_{w=2} .$$

3. The associated graded complex reads

$$\mathrm{Gr}\mathcal{A}^\bullet(M) = (C^\infty(V) \otimes_{\mathbb{C}} (M \otimes_{\mathbb{C}} \mathbb{C}[\theta^\alpha]), \mathcal{D}_0 = \lambda^\alpha \partial_{\theta^\alpha}) \cong C^\infty(V) \otimes_{\mathbb{C}} \mathcal{K}^\bullet(M)$$

where $\mathcal{K}^\bullet(M)$ is the **Koszul complex** of M:

$$\mathcal{K}^\bullet(M) := (M \otimes_{\mathbb{C}} \mathbb{C}[\theta^\alpha], \mathcal{D}_0 = \lambda^\alpha \partial_{\theta^\alpha}) .$$

Koszul Homology and Component Fields

In short, the Koszul homology of M ($\leftrightsquigarrow E_1^\bullet$) determines the component field description as known in the physics literature:

$$E_1^\bullet = H^\bullet(\mathrm{Gr}\mathcal{A}^\bullet(M)) \quad \leftrightsquigarrow \quad \left\{ \begin{array}{c} \text{Component Fields} \\ \text{in } \mathcal{A}^\bullet(M) \end{array} \right\}$$

- M is a graded $\mathfrak{p}_0$-equivariant module $\rightsquigarrow H^\bullet(\mathcal{K}^\bullet(M))$ gives finite dimensional representations of the Lorentz and R-symmetry algebra.
- $H^\bullet(\mathrm{Gr}\mathcal{A}^\bullet(M))$ determines a (graded) vector bundle over the spacetime V with fibers
$$(E'_x)^k = H^\bullet(\mathcal{K}(M))^{(k)}$$
- $\mathcal{D}_1$ induces a new differential $\mathcal{D}'$ and the $\mathfrak{p}$-module structure transfer as well.

$\rightsquigarrow$ this "page 1 multiplet" $(E', \mathcal{D}', \rho')$ determines a new multiplet defined over spacetime!

Pure Spinor Formalism, in a Nutshell

$$
\begin{array}{ccc}
(\mathcal{A}^\bullet(M), \mathcal{D}) & \cdots\cdots & \text{free over superspace} \\
\downarrow & & \downarrow \\
(H^\bullet(\mathrm{Gr}\mathcal{A}^\bullet(M)), \mathcal{D}') & \cdots\cdots & \text{free over spacetime} \\
\downarrow & & \downarrow \\
(H^\bullet(\mathcal{A}^\bullet(M)), 0) & \cdots\cdots & \text{not necessarily free}
\end{array}
$$

Mathematics	**Physics**
First page complex	Field content
First page differential	BRST / BV differential
Action of Q_α on representatives	SUSY transformations
Second page complex	gauge invariant (on-shell) fields

Properties of Modules and Properties of Multiplets

Module	Multiplet
$M = \mathcal{O}(S)$ for S hyperplane in Y	Exterior algebra in S (chiral / free superfields)
$M = \mathcal{O}_Y$ complete intersection of quadratic equations	Exterior algebra identified with $\Omega^\bullet(\mathbb{R}^d)$ ($\mathcal{O}_Y$ for $d = 4, \mathcal{N} = 4$)
M is Gorenstein	BV datum ($\mathcal{O}_Y$ for $d = 10$ SYM)
M is Cohen-Macaulay	BRST datum & antifield multiplet ($\mathcal{O}_Y$ for $d = 6$, $\mathcal{N} = (1, 0)$)
M is **not** Cohen-Macaulay	BRST datum & **no** antifield multiplet ($\mathcal{O}_Y$ for $d = 4$, $\mathcal{N} = 1$)

Geometry & Antifield Multiplets

Theorem (Antifield Multiplet and Dualizing Module)

Let the nilpotence variety Y be Cohen-Macaulay of dimension d, i.e. its ring of function R/I is a Cohen-Macaulay ring of (Krull) dimension d. Then the antifield multiplet $\mathcal{A}^\bullet(R/I)^\vee$ of $\mathcal{A}^\bullet(R/I)$ is given by

$$\mathcal{A}^\bullet(R/I)^\vee = \mathcal{A}^\bullet(\omega_{R/I})$$

where $\omega_{R/I} = Ext_R^{n-d}(R/I, R)$ is the **dualizing module** *of R/I and n is the (Krull) dimension of ambient ring R.*

Antifield multiplets $\mathcal{A}^\bullet(M) \leftrightsquigarrow$ Dualizing modules of M

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Warning: Dualizing Complexes & Pure Spinors

If Y is **not** Cohen-Macaulay, then one has a dualizing complex $\omega^\bullet_{R/I}$ instead of a single module, hence the PS formalism is not capable of producing the antifield multiplet of R/I.

$d = 4$, $\mathcal{N} = 1$ Chiral Multiplet via Pure Spinors

- Recall that the nilpotence variety is $Y = \mathbb{C}^2 \cup_{\{0\}} \mathbb{C}^2 = S_+ \cup_{\{0\}} S_-$.

$d = 4$, $\mathcal{N} = 1$ Chiral Multiplet via Pure Spinors

- Recall that the nilpotence variety is $Y = \mathbb{C}^2 \cup_{\{0\}} \mathbb{C}^2 = S_+ \cup_{\{0\}} S_-$.
- Choose $M = \mathbb{C}[\bar{\lambda}^{\dot{\alpha}}]$ and construct the PS complex

$$(\mathcal{A}^\bullet(M), \mathcal{D}) = \left(C^\infty(\mathfrak{t}) \otimes_{\mathbb{C}} \mathbb{C}[\bar{\lambda}^{\dot{\alpha}}], \mathcal{D} = \bar{\lambda}^{\dot{\alpha}} \partial_{\bar{\theta}^{\dot{\alpha}}} + \bar{\lambda}^{\dot{\alpha}} \Gamma^{\mu}_{\alpha\dot{\alpha}} \theta^\alpha \partial_\mu\right)$$

- Compute the relevant Koszul homology: using $\mathfrak{t}_1 = S_+ \oplus S_-$ one has

$$\mathcal{K}^\bullet(M) = \left(\wedge^\bullet S_+ \otimes \wedge^\bullet S_- \otimes \mathbb{C}[\bar{\lambda}^{\dot{\alpha}}], \mathcal{D}_0 = \bar{\lambda}^{\dot{\alpha}} \partial_{\bar{\theta}^{\dot{\alpha}}}\right)$$

with θ^α are coordinates for S_+ and $\bar{\theta}^{\dot{\alpha}}$ are coordinates for S_-.

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  with θ^α are coordinates for S_+ and $\bar{\theta}^{\dot{\alpha}}$ are coordinates for S_-.
- θ^α does not occur in $\mathcal{D}_0$, hence the cohomology reads

$$H^\bullet(\mathcal{K}^\bullet(M)) = \wedge^\bullet S_+ \otimes H^\bullet(\wedge^\bullet S_- \otimes \mathbb{C}[\bar{\lambda}^{\dot{\alpha}}]) \cong \wedge^\bullet S_+ \otimes \mathbb{C}.$$

- Reinstating the spacetime dependence one has that the $\mathcal{D}_0$-cohomology reads

$$C^\infty(\mathbb{C}^4) \otimes H^\bullet(\mathcal{K}^\bullet(M)) \cong C^\infty(\mathbb{C}^4) \otimes_{\mathbb{C}} \wedge^\bullet S_+.$$

$d = 4$, $\mathcal{N} = 1$ Chiral Multiplet via Pure Spinors

Field Content: Chiral Supermultiplet

Field	Representative in the $\mathcal{D}_0$-cohomology
ϕ	ϕ
ψ	$\psi\theta$
F	$F\theta_1\theta_2$

$d = 4$, $\mathcal{N} = 1$ Chiral Multiplet via Pure Spinors

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Supersymmetry Transformations of the Chiral Multiplet

The action on the supercharges in $\mathfrak{p}_1$ on the representatives in cohomology gives the supersymmetry transformations:

$$\begin{aligned}\rho(Q+\bar{Q})(\phi+\theta\psi+F\theta_1\theta_2) &= (\epsilon\partial_\theta + i(\theta\sigma^\mu\bar{\epsilon})\partial_\mu)(\phi+\theta\psi+F\theta_1\theta_2) \\ &= \underbrace{\epsilon\psi}_{\delta\phi} + \underbrace{(i\bar{\epsilon}\partial\!\!\!/\phi+\epsilon F)}_{\delta\psi}\theta + \underbrace{(-i\epsilon\partial\!\!\!/\psi)}_{\delta F}\theta_1\theta_2\end{aligned}$$

$d = 6$, $\mathcal{N} = (1,0)$ Multiplets via Pure Spinors

- Recall that $Y(6;(1,0)) = \mathbb{P}^1 \times \mathbb{P}^3 \hookrightarrow \mathbb{P}^7$
- All line bundles are of the form

$$\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^3}(n,m) = \pi_1^*\mathcal{O}_{\mathbb{P}^1}(n) \otimes_{\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^3}} \pi_3^*\mathcal{O}_{\mathbb{P}^3}(m) \quad (n,m) \in \mathbb{Z}^{\oplus 2}.$$

$\rightsquigarrow$ all multiplets $\mathcal{A}(m,n)$ coming from line bundles can be classified!

- For example, one finds:
  1. $\mathcal{O}_Y(0,0) \rightsquigarrow$ vector multiplet:

$$\mathcal{O}_Y(0,0) \rightsquigarrow \mathcal{A}^\bullet(0,0) = (\Omega^0, \quad \Omega^1, \quad S_- \otimes \mathbb{C}^2, \quad \Omega^0 \otimes \mathbb{C}^3)$$

  2. $\mathcal{O}_Y(1,0) \rightsquigarrow$ hypermultiplet:

$$\mathcal{O}_Y(1,0) \rightsquigarrow \mathcal{A}^\bullet(1,0) = (\Omega^0 \otimes \mathbb{C}^2, \quad S_+, \quad S_-, \quad \Omega^0 \otimes \mathbb{C}^2)$$

  3. $\mathcal{O}_Y(2,0) \rightsquigarrow$ antifield multiplet of the vector multiplet:

$$\mathcal{O}_Y(2,0) \rightsquigarrow \mathcal{A}^\bullet(2,0) = (\Omega^0 \otimes \mathbb{C}^3, \quad S_- \otimes \mathbb{C}^2, \quad \Omega^1, \quad \Omega^0)$$

$d = 6$, $\mathcal{N} = (1,0)$ Multiplets via Pure Spinors

- Recall that $Y(6;(1,0)) = \mathbb{P}^1 \times \mathbb{P}^3 \hookrightarrow \mathbb{P}^7$
- On the other hand, also higher-rank vector bundles can be considered, such as the *conormal bundle*

$$0 \longrightarrow \mathcal{N}_Y^\vee \longrightarrow \Omega^1_{\mathbb{P}^7|_Y} \longrightarrow \Omega^1_Y \longrightarrow 0.$$

- Remarkably, the conormal bundle is related to supergravity multiplet:

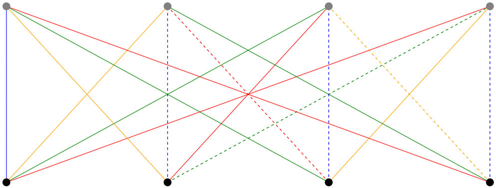
$$\mathcal{A}^\bullet(\mathcal{N}_Y^\vee) \ni (\ \ldots, \quad Sym^2_0(V), \quad (V \otimes S_-)_{\frac{3}{2}} \otimes \mathbb{C}^2, \quad \ldots \ )$$

The following is always true:

1. $\mathcal{O}_Y \rightsquigarrow$ vector (gauge) multiplet;
2. $\mathcal{N}_Y^\vee \rightsquigarrow$ supergravity multiplet;
3. $\pi_* \mathcal{O}_Y \rightsquigarrow$ chiral multiplet(s);

$d = 1$ Supersymmetry and Pure Spinors

- In $d = 1$ the nilpotence ideal is $I = \sum_{i=1}^{\mathcal{N}} \lambda_i^2$ for any amount of supersymmetry $\mathcal{N}$, hence the nilpotence variety $Y(1, \mathcal{N})$ is a *quadric hypersurface*.
- The most studied $d = 1$ multiplets arise from the graph technology of *Adinkras*: the following is an example of the most important class of Adinkras, the *valise* Adinkras:



- Via pure spinors formalism, valise Adinkras corresponds to characteristic bundle on the quadric $Y(1, \mathcal{N})$: the *spinor bundle*.

Outro - toward derived geometry

At this point, a reasonable and natural question is:

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As it turns out, the answer is **no**:

- A relevant example is the antifield multiplet of the $d = 4, \mathcal{N} = 1$ vector multiplet ($\leftrightsquigarrow \mathcal{O}_Y$).

Geometrically, the antifield multiplet of the vector multiplet is related to the dualizing module of $Y \rightsquigarrow$ if Y is singular, there is no dualizing module, but dualizing **complex** instead!

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  Geometrically, the antifield multiplet of the vector multiplet is related to the dualizing module of $Y \rightsquigarrow$ if Y is singular, there is no dualizing module, but dualizing **complex** instead!

- This point in the direction of a **derived** pure spinor formalism ($\rightsquigarrow$ input are not single modules, but *complexes* of modules)!

- Pure spinor superfield formalism as an instance of **Koszul duality**:

  d=0 susy : $D^\flat(\mathbb{P}^{\mathcal{N}-1}) \cong D^\flat(\Lambda^\bullet \mathfrak{t}\text{-}\mathbf{mod}) \rightsquigarrow$ *BGG correspondence*;

  d=1 susy : $D^\flat(Q_{\mathcal{N}-1}) \cong D^\flat(U(\mathfrak{t})\text{-}\mathbf{mod}) \rightsquigarrow$ *"deformed" BGG correspondence*

Pure Spinors in $d = 1$ and Geometry of Quadrics

1. The N-extended supersymmetry algebra $\mathfrak{t}_N$ in $d = 1$ is characterized by the relations $\{Q_i, Q_j\} = 2\delta_{ij} H$ for $i, j = 1, \ldots, N$;
2. The nilpotence variety of $\mathfrak{t}_N$ is a quadric hypersurface $Y_N := \mathrm{Spec}(\mathbf{k}[\lambda_1, \ldots, \lambda_N]/q_N)$ for $q_N := \sum_{i=1}^N \lambda_i^2$ the standard quadratic form;

Theorem ("Deformed" BGG correspondence & $d = 1$ SUSY)

Let R/I be the ring of functions on Y_N and let $U_{\mathbf{k}}(\mathfrak{t})$ be the universal enveloping algebra of $\mathfrak{t}_N$. Then

$$D^\flat(R/I\text{-}\mathbf{Mod}) \cong D^\flat(U_{\mathbf{k}}(\mathfrak{t})\text{-}\mathbf{Mod}). \tag{1}$$

In particular, the following (Abelian) categories are mapped into each other:

$$\mathbf{MCM}_{gr}(R/I) \rightleftarrows C\ell(q_N)\text{-}\mathbf{Mod}_{gr}. \tag{2}$$

Thank you very much!

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Multiplets and Pure Spinor Formalism

Definition (Multiplet)

A $\mathfrak{g}$-multiplet is a triple (E, D, ρ), where (E, D) is an affine dgs vector bundle E on V equipped with a (local) $\mathfrak{g}$-module structure $\rho : \mathfrak{g} \rightsquigarrow \mathcal{D}(E)$, such the following commute

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\rho} & \mathcal{D}(E) \\ {\scriptstyle \phi}\uparrow & \nearrow & \\ \mathfrak{aff}(V) & & \end{array}$$

A morphism of multiplet is map of cochain complexes $\psi : \Gamma(E) \to \Gamma(E')$ such that $\psi \circ \rho(x) = \rho'(x) \circ \psi$ for every $x \in \mathfrak{g}$.

Definition (Category of Multiplets)

The dg-category $\mathfrak{g}$-**Mult** of $\mathfrak{g}$-multiplets is the (full) subcategory of local $\mathfrak{g}$-modules whose object are $\mathfrak{g}$-multiplets.

Multiplets and Pure Spinor Formalism

Definition (Poincaré Superalgebra)

A superalgebra $\mathfrak{g}$ is of super Poincaré type if it can be written as an extension

$$0 \longrightarrow \mathfrak{t} \longrightarrow \mathfrak{g} \longrightarrow \mathfrak{g}_0 \longrightarrow 0,$$

where $\mathfrak{t}$ is the two-step nilpotent superalgebra of supertranslations.

Definition (Pure Spinor Functor)

$$\mathcal{PS} : C^{\bullet}_{\mathcal{CE}}(\mathfrak{t})\text{-}\mathbf{Mod}^{\mathfrak{g}_0} \longrightarrow \mathfrak{g}\text{-}\mathbf{Mult}.$$

Why Pure Spinors?

- Let V a vector space of dimension $2n$ or $2n+1$.
- Let S be a spin representation of $Spin(V)$, then S is a $Cl(V)$-module.
- Accordingly, there is an action $V \circlearrowright S$, with $(v, Q) \mapsto v \cdot Q$
- If $Q \in S$, we consider $\operatorname{Ann}(Q) := \{v \in V : v \cdot Q = 0\}$. Now, $\dim \operatorname{Ann}(Q) = m \leqslant n$.

Definition (Pure Spinor)

We say that Q is a *pure* spinor if $m = n$.
Alternatively, Q is pure if $\operatorname{Ann}(Q) \subset V$ is a maximal isotropic subspace.

In particular, for $\dim V = 2n$, considering $\mathbb{P}(S)$, we have that (projective) pure spinors are given by the homogeneous space $SO(2n)/U(n)$. The pure spinor space coincides - in some relevant cases - with the nilpotence variety of super Poincaré algebras.

CM condition

Let R be a commutative, Noetherian and local ring and let M be an R-module.

We say that M is CM if $\mathrm{depth}_R(M) = \dim_R(M)$.

There is also a homological useful characterization: namely we let R be polynomial a ring of Krull dimension n and $S \hookrightarrow R$ of Krull dimension d. Then we call $\omega_S^\bullet := Ext_R^\bullet(S, R)$ the dualizing complex of S (notice that this coincide with diff. forms of deg d if $S \hookrightarrow R$ is non-singular...). Now, S is CM if $Ext_R^i(S, R) = 0$ for every $i \neq n - d$, that is if the dualizing complex is a module.
In particular, if it is also free of rank 1, then we say that M is *Gorenstein*.

Typical example: plane curves with embedded points are not CM, *e.g.*

$$Spec\left(\mathbb{C}[X, Y]/(x^2, xy)\right).$$

Indeed $(x^2, xy) \cong (x) \cdot (xy)$: y-axis with embedded point $(0, 0)$.

Operators of a Theory

The *operators* of a theory consist of functionals of the fields of the theory are denoted with $\mathcal{O}(\mathcal{E})$.

For any point $x \in V$ we can define local operators via

$$\mathcal{O}_x(\mathcal{E}) := Sym^\bullet(J^\infty E|_x)^\vee,$$

where $J^\infty E$ denotes the jet bundles of E - in other words, the local operators at x evaluate polynomials in the fields and derivatives of fields at x.

Given a map $\rho : \mathfrak{g} \rightsquigarrow (\mathcal{D}(E), [D, -])$, the dual maps $(\rho^{(j)})^\vee$ define an action on the linear local operators, which extends to $\mathcal{O}_x(\mathcal{E})$ via Leibniz rule.
Fixing an element $Q \in \mathfrak{g}$, we can define a map

$$\delta_Q = \sum_j \rho^{(j)}(Q, \dots, Q)^\vee : \mathcal{O}_x(\mathcal{E}) \to \mathcal{O}_x(\mathcal{E}),$$

this defines the action of $Q \in \mathfrak{g}$ on the operators of the theory.

BRST Datum

A BRST datum on a multiplet (E, D, ρ) consists of:

- a local super L_∞ structure $\{\mu_k\}$ on $L \equiv E[-1]$ such that $\mu_1 = D$, and whose associated CE differential we denote by Q_{BRST};
- a local functional $S_0 \in \mathcal{O}(E)$ of bidegree $(0, +)$ called BRST action action, which is Q_{BRST}-closed and invariant for the L_∞ action ρ.

BV Datum

A BV datum on a multiplet (E, D, ρ) consists of:

- a graded antisymmetric map $\langle -, - \rangle : E \otimes E \to \omega_X$ of bidegree $(-1, +)$ which is fiberwise non-degenerate and invariant for the L_∞ action ρ;
- A $C^\bullet(\mathfrak{g})$-valued BV action of bidegree $(0, +)$ given by $S_{BV} = \sum_k S_B^k V$ where $S_B^k V \in C^k(\mathfrak{g}) \otimes \mathcal{O}(E)$, such that it satisfies the $\mathfrak{g}$-equivariant master equation

$$d_{\mathfrak{g}} S_B V + \frac{1}{2}\{S_B V, S_B V\} = 0.$$

Here

$$S_B^0 V(\Phi) = \int_X \langle \Phi, D\Phi \rangle + I_B V(\Phi)$$

where $I_B V$ is at least cubic in the fields and where

$$S_B^k V(x_1, \dots, x_k; \Phi) = \int_X \langle \Phi, \rho^k(x_1, \dots, x_k)\Phi \rangle$$

Frrom BRST to BV Datum

To move from a BRST datum to a BV datum one considers

$$L_B V = L \oplus L^\vee[-k],$$

which is equipped with a canonical evaluation pairing (of degree -k). The BRST action deforms the obvious L_∞ structure on the direct sum, thus giving rise to an L_∞ structure on $L_B V$, for which the evaluation pairing is invariant (after an application of the homological perturbation lemma).

1. If M is *Gorenstein*, its Koszul homology is naturally equipped with a perfect pairing, that equips the multiplet with a BV datum (in fact the minimal free resolution of M is self-dual if it is Gorenstein).
2. If M is Cohen-Macaulay, we can instead work as above: consider $L^\vee[-k]$ to be given by the dualizing module and look at $L \oplus L^\vee[-k]$ to define the BV datum.