Constraining and Un-constraining Supergravities

What you must know about supergravity but do not want to ask string theory?

- A randomly chosen supergravity theory, as it stands, is inconsistent
  - ▷ Restrictions are too few

(notably from anomalies $I_d^{(1)}$: $\quad I_{d+2} = \mathrm{d}I_{d+1}^{(0)}\,, \quad \delta I_{d+1}^{(0)} = \mathrm{d}I_d^{(1)}$)

  - ▷ imagination is even sparser (new ingredients ?)
  - ▷ Global surprises

Part I: Discrete anomalies in D=8

- Work with Bing Xin Lao

Part II: Quick D=6 review

- Work with Peng Cheng and Stefan Theisen

Part III: Exotic backgrounds

- Work with Peng Cheng and Ilarion Melnikov

D=10 Super-Poincaré Representations with 16 supercharges:

B.L.G.	representation	multiplet
$SO(8)$	$8_v \times 2^4 = 35 + 28 + 1 + 8_+ + 56_-$	gravity - anomalous
	$2^4 = 8_v + 8_-$	Yang-Mills - anomalous

Gravity + YM $\Leftrightarrow$ Anomaly cancelation possible if

- $\star$ $I_{12} = X_4(R,F) \wedge X_8(R,F)$
  - $\star$ Gauge group: $E_8 \times E_8$, $SO(32)$ ... but also $U(1)^{496}$, $E_8 \times U(1)^{248}$
  - $\triangleright$ Anomalous BI $\mathrm{d}H = X_4(R,F) \sim \text{tr}\, R \wedge R - \text{tr}\, F \wedge F$
  - $\triangleright$ GS couplings $\sim B_2 \wedge X_8(R,F)$

D=10 Super-Poincaré Representations with 32 supercharges:

B.L.G.	representation	multiplet
$SO(8)$	$2^8 = 35 + 28 + 1 + 56_v + 8_v + 8_+ + 56_- + 8_- + 56_+$	(1,1) - IIA
$SO(8) \times SO(2)$	$2^8 = 35_0 + 28_{-2} + 1_{-4} + (8_+)_{-3} + (56_-)_{-1}$	(0,2) - IIB
	$+ (35_-)_0 + 28_2 + 1_4 + (8_+)_3 + (56_-)_1$	

D=8/9 theories (Circle / T^2 reductions) with 16 suprecharges

For $\mathcal{L}_v g_{10} = 0 = \mathcal{L}_v H$

$$\begin{array}{ccc} S^1 & \hookrightarrow & X_{10} \\ & & \downarrow \pi \\ & & X_9 \end{array} \qquad \mathrm{d}e = \pi^* T \qquad (\mathcal{L}_v\, e = 0)$$

$$H = H_3 + H_2 \wedge e, \qquad \mathrm{d}H = 0 \quad \Rightarrow \quad \begin{cases} \bullet \quad \mathrm{d}H_2 = 0 \\ \bullet \quad \mathrm{d}H_3 = -H_2 \wedge T = -F_+^2 + F_-^2 \end{cases}$$

Supergravity in D=9/8 - theory with $SO(1, N, \mathbb{R})$ / $SO(2, N+1, \mathbb{R})$ symmetry

- ⋆ Global anomalies $\Rightarrow$ N - odd
- ⋆ Parity acts as an internal symmetry $\Rightarrow$ N = 1, 9, 17
  - ▷ non-orientable manifolds are consistent backgrounds!
- ⋆ String constructions known for N = 1, 9, 17
- ⋆ No parity anomalies

Discrete anomalies

- D=8 theory with maximal supersymmetry (32 supercharges)
  - ▷ moduli space: $\frac{SL(2,\mathbb{R})}{U(1)} \times \frac{SL(3,\mathbb{R})}{SO(3)}$
    - ∘ IIB: τ - complex structure of $\mathbb{T}^2$
    - ∘ M: $\tau = -2C_{8910} + Vol(\mathbb{T}^3)$
  - ▷ $S^{(8)}$ invariant under diffs, but .... not under $SL(2,\mathbb{Z})$
- Ungauged theory has composite $U(1)$ anomalies:
  - ◇ $I_{10} = \frac{F^Q}{2\pi}\left[2 \times \frac{1}{2} I_{3/2}^{d=8} - 4 \times \frac{1}{2} I_{1/2} + 2 \times \frac{3}{2} I_{1/2} + 2 \times I_{\rm SD}\right]_{8-\rm form} = \frac{F^Q}{2\pi} \wedge X_8$
    - ▷ $\delta\psi_\mu = \left[\nabla_\mu + \frac{i}{4} Q_\mu \gamma^9 + \frac{1}{4} Q_\mu^{ab} T^{ab}\right] \epsilon$
    - ▷ other fermions are also chiral
    - ▷ doublet of chiral three-forms (with +1 and -1 U(1) charges)
    - ▷ $F = \frac{d\tau \wedge d\bar{\tau}}{4i\tau_2^2}$ the curvature of the composite connection $Q = -d\tau_1/2\tau_2$
- Upon gauge fixing composite U(1) anomaly becomes $SL(2,\mathbb{Z})$ anomaly!
  - ◇ $X_8 = \frac{1}{48}\left(\frac{1}{4} p_1(TX)^2 - p_2(TX)\right)$ - present in D=10/11

Many lives of X_8

Euler density:

▷ $$\chi = \frac{1}{4!(4\pi)^2}\ \epsilon^{a_1\cdots a_8} R_{a_1a_2} R_{a_3a_4} R_{a_5a_6} R_{a_7a_8}$$

◇ Curvature two-from $\quad R_{ab} = \frac{1}{2} R_{abcd}\, e^c \wedge e^d$

Spinor density:

▷ $$\hat{\chi} = \frac{1}{4!(4\pi)^2} \cdot \frac{1}{2^4}$$
$$\epsilon^{i_1\cdots i_8} R_{a_1a_2} \left(\Gamma^{a_1a_2}\right)^{i_1i_2} R_{a_3a_4} \left(\Gamma^{a_3a_4}\right)^{i_3i_4} R_{a_5a_6} \left(\Gamma^{a_5a_6}\right)^{i_5i_6} R_{a_7a_8} \left(\Gamma^{a_7a_8}\right)^{i_7i_8}$$

Eight-forms:

▷ $$\hat{\chi} = \frac{1}{16}\left(8\chi + p_1(TX)^2 - 4p_2(TX)\right)$$

$$X_8 = \frac{1}{48}\left(\frac{1}{4}p_1(TX)^2 - p_2(TX)\right)$$
$$= \frac{1}{(2\pi)^4}\left(-\frac{1}{768}(\mathrm{tr}\, R^2)^2 + \frac{1}{192}\mathrm{tr}\, R^4\right)$$

M5 Anomaly & Inflow mechanism

11D supergravity: $S_{\text{SUGRA}} = \frac{1}{2\kappa^2}\int \mathcal{R} * 1 - \frac{1}{2}G_4 \wedge *G_4 - \frac{1}{6}C_3 \wedge G_4 \wedge G_4$

M5: $\mathrm{d}G_4 = Q\delta_5(M5)$

(2,0) tensor multiplet on M5:

- Woldvolume chiral 2-form
  - ◇ $I_\beta = \frac{1}{5760}\left(16p_1(TW)^2 - 112p_2(TW)\right) \sim L(TW)$
- Worldvolume fermions
  - ◇ $I_D = \frac{1}{2}\widehat{A}(TW)\text{ch}S(N)$
  - ◇ N-trivial: $I_D = 4 \times \frac{1}{2}\widehat{A}(TW) = \frac{1}{5760}\left(14p_1(TW)^2 - 8p_2(TW)\right)$
- Total anomaly: $I_{(2,0)} = \frac{1}{48}\left(\frac{1}{4}p_1(TX)^2 - p_2(TX)\right)$
- Cancelled via inflow from a bulk coupling $\sim C_3 \wedge X_8$

$$G_4\delta X_7^{(0)} \to \delta_5(M5)X_6^{(1)}(TX) \quad \leftrightarrow \quad \mathrm{d}^{-1}\delta\mathrm{d}^{-1}I_{(2,0)}$$

- For nontrivial normal bundle: $\delta(C \wedge G \wedge G)$ is needed

Consistency of the supergravity theory together with BPS objects

Anomalous variation in the D=8 path integral:

$$-12 \int \Sigma \; X_8(R)$$

with

$$e^{-i\Sigma(M,\tau)} = \left(\frac{c\tau + d}{c\bar{\tau} + d}\right)^{\frac{1}{2}} , \quad M \in \begin{pmatrix} a & b \\ c & d \end{pmatrix} , \quad M \in SL(2,\mathbb{R})\,, \quad \tau \in \mathbb{H}$$

can be canceled in quantum theory with $SL(2,\mathbb{Z}) \in SL(2,\mathbb{R})$

- Counterterm: $$\Delta S^{(8)} = 12i \int \arg(\eta^2(\tau)) X_8(R)$$
  - ▷ cancels the $SL(2,\mathbb{Z})$ anomaly
  - ▷ non-trivial phase (multiplier system)
- $$\int \left[X_8 + \tfrac{1}{2} G \wedge G\right] \in \mathbb{Z}$$
  - ▷ large volume limit: $\lim_{\mathrm{Im}\tau\to\infty} \mathcal{S} = 2\pi \int \tau_1 \left[X_8 + \frac{1}{2} G \wedge G\right]$
  - ▷ subtle contribution from the classical action

D=8 theories with 16 supercharges

Think of $\mathbb{T}^2$ reductions of Heterotic strings

- $\frac{SO(2,l)}{SO(2)\times SO(l)}$ coset
- Composite U(1) couplings to fermions are chiral
- Anomaly cancellation: $\Delta S^{(8)} = \int f(\mathbf{z},\bar{\mathbf{z}})\, Y_8^G$
  - ◇ for D=8 heterotic string with U(1)$^2 \times G$ (rank G = 16)
    - ▷ $f(\mathbf{z},\bar{\mathbf{z}})$ - modular function of T and U - scalars in $U(1)$ multiplets
    - ▷ $Y_8^G =$
      $\frac{1}{32(2\pi)^4}\left[(248+\mathrm{dim}G)\left[\frac{trR^4}{360}+\frac{(trR^2)^2}{288}\right]-(trR^2)^2+\frac{1}{6}trR^2TrF^2+\frac{2}{3}TrF^4\right]$
- $\Delta S^{(8)}$ agrees with the string amplitude only for
  - ▷ $G = SO(32)$
  - ▷ $G = E_8 \times E_8$

$$S_{\text{amp}} = \frac{1}{192(2\pi)^3}\int B_{89}X^{\text{GS}} + \frac{1}{4\times 192(2\pi)^4}\int\left[ln\left(\frac{\eta^{24}(U)}{\bar{\eta}^{24}(\bar{U})}\right)+ln\left(\frac{\eta^{24}(T)}{\bar{\eta}^{24}(\bar{T})}\right)\right]\times Y_8^G$$

For other G:

$$S_{\text{amp}} = \int \left[N_1 \mathcal{A}_{\text{trivial}} + N_2 \mathcal{A}_{\text{deg.}} + N_3 \mathcal{A}_{\text{non-deg.}}\right]$$

- ▷ N_1, N_2, N_3 - normalisations
- ▷ $\mathcal{A}_{\text{trivial}}$ - $\mathbb{T}^2$ reduction of D=10 GS term
- ▷ $\mathcal{A}_{\text{deg}}$ contains Y_8^G
- ▷ $\mathcal{A}_{\text{non-deg}}$ contains Y_8^G + *massive* VM contributions
  - ◇ $Y_8^{SO(8)^4} = \frac{1}{32(2\pi)^4}\left[\left(trR^4 - \frac{1}{4}(trR^2)^2\right) + \frac{1}{2}\sum_{i=1}^4 \left(\frac{1}{2}trR^2 + 2trF_i^2\right)^2\right]$

- Can be put on spaces with positive $c_1(TX)$, but ... $\mathbb{T}^2$ degenerates and massive states can become massless
- For $X = \mathbb{P}^1$ the resulting D=6 $(1,0)$ theories *without contribution from D=8 massive states* have chiral anomalies

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- Supergravity on non-spin, non-orientable, singular spaces

General story:

The compensating U(1) transformation for $SO(2,l;\mathbb{R})$ transformation M:

$$e^{-i\Sigma(M,Z)} = \frac{j(M,Z)}{|j(M,Z)|}$$

(Z - coordinates on the generalized upper half plane)

Conterterm:

$$\mathcal{S} = \frac{1}{r}\int \arg\Psi(Z) Y_8^G$$

with

$$\Psi(M\langle Z\rangle) = \chi(M) j(M,Z)^r \Psi(Z)$$

- $\Psi(Z)$ can be constructed as a Borcherds product:

$f(\tau) = \sum_{\gamma\in L'/L}\sum_{n\in\mathbb{Z}+q(\gamma)} c(\gamma,n)\mathfrak{e}_\gamma(n\tau)$ hol. modular form of weight $k = 1 - l/2$

$$\Rightarrow \qquad \Psi(Z) = \prod_{\beta\in L'/L}\prod_{\substack{m\in\mathbb{Z}+q(\beta)\\ m<0}} \Psi_{\beta,m}(Z)^{c(\beta,m)/2}$$

$\Psi(Z)$ - meromorphic function on $\mathbb{H}_l$ of (rational) weight $r = c(0,0)/2$ for the modular group $\Gamma(L)$ with character χ (if $c(0,0)\in 2\mathbb{Z}$)

Is the theory well-defined when the counterterm is not?

$$(\Psi) = \frac{1}{2} \sum_{\beta \in L'/L} \sum_{\substack{m \in \mathbb{Z}+q(\beta) \\ m<0}} c(\beta, m) H(\beta, m) \,.$$

where *rational quadratic divisors* (RQD):

$$H(\beta, m) = \sum_{\substack{\lambda \in \beta+L \\ q(\lambda)=m}} H_\lambda \quad \text{whith} \quad H_\lambda = \left\{ [Z_L] \in \mathcal{K}^+ \,|\, (Z_L, \lambda) = 0 \right\}$$

Reflective holomorphic modular form for the modular group $\Gamma(L)$ - zeroes are contained in the union of RQDs $\ell^\perp$ associated to roots of L:

The reflection $\sigma_\ell : \alpha \longmapsto \alpha - \frac{2(\alpha,\ell)}{(\ell,\ell)} \ell \,, \quad \alpha \in L$ belongs to $O^+(L)$

Reflection symmetry of $L \quad \Rightarrow \quad$ extra massless states (sym. enhancement)

Reflective lattices (admit reflective modular forms):

- ▷ $\chi = 1$
- ▷ **finite** number with $l \leq 26$ ... mostly classified
- ▷ further constrains (more restricitons? ... 2-reflective, simple?)

D=6 Super-Poincaré Representations with 8 supercharges:

B.L.G.	representation	multiplet
	$(2,3;1)\times 2^2 = (3,3;1) + (1,3;1) + (2,3;1)$	gravity
$SO(4)\times SU(2)$	$(2,1;1)\times 2^2 = (3,1;1) + (1,1;1) + (2,1;1)$	tensor
	$(1,2;1)\times 2^2 = (2,2:1) + (1,2;1)$	Yang-Mills
	$2^2 = (2,1;1) + (1,1;2)$	hyper

Chiral bosonis and fermionic fields $\Rightarrow$ Anomalies

Anomaly cancelation possible if

- $\star$ $I_8 = \frac{1}{2}\Omega_{\alpha\beta}\, X_4^\alpha X_4^\beta$
  - $\triangleright$ $\alpha, \beta = 0, 1, ...n_T$
  - $\triangleright$ $\Omega_{\alpha\beta}$ - symmetric inner product on the space of tensors with $(1, n_T)$ signature
- $\star$ GSS couplings $\sim \Omega_{\alpha\beta} B_2^\alpha X_4^\beta$
- $\star$ Anomalous BI $\mathrm{d}H^\alpha = X_4^\alpha$

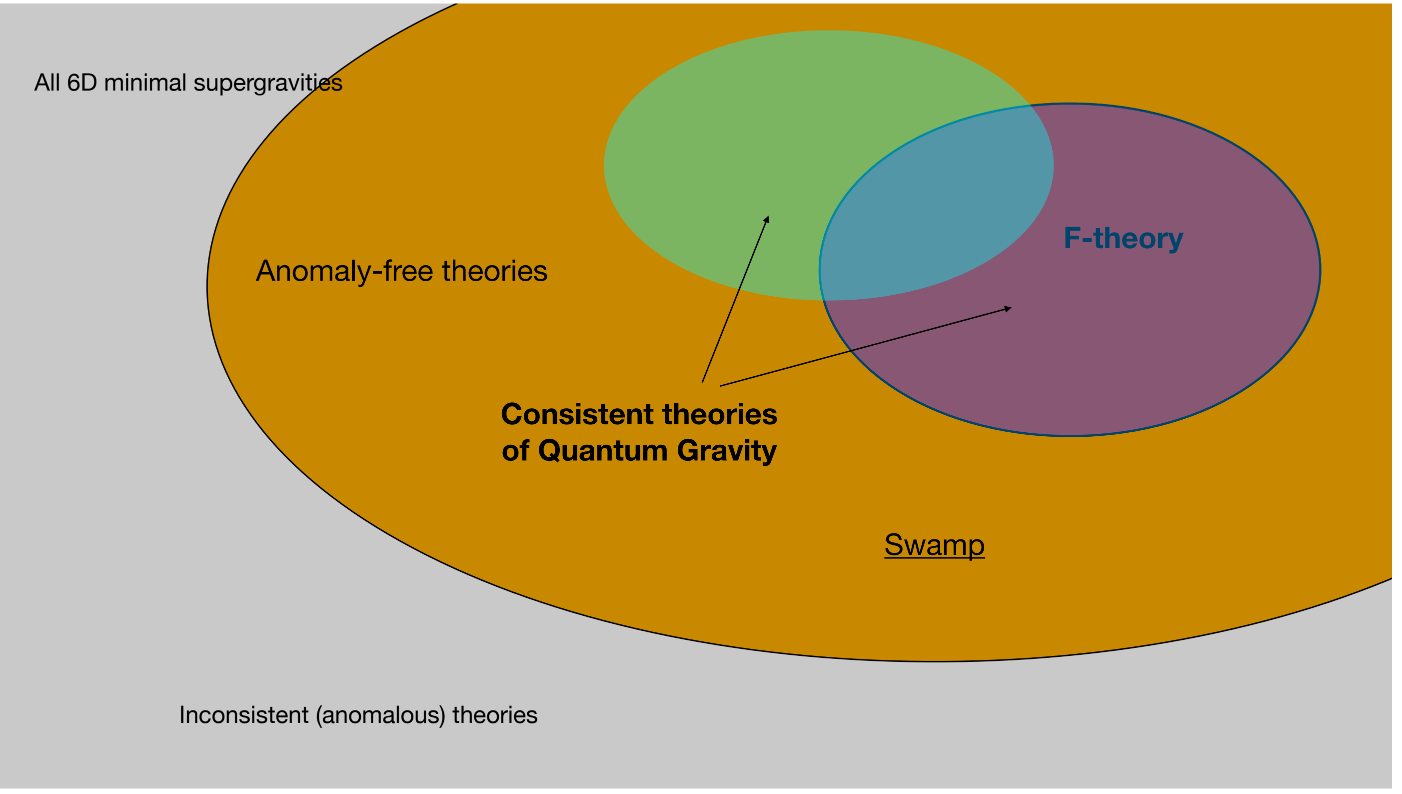
Anomaly-free theories:

- Heterotic strings on K3
  - perturbative: $n_T = 1 \ \ (c_2 = 24)$
  - non-perturbative: $n_T > 1 \ \ (c_2 + N_{\text{NS5}} = 24)$
- Perturbative IIB constructions (K3 orientifolds)
- F-theory
  - ▷ Geometrisation of the necessary conditions for the anomaly cancellation
  - ▷ Kodaira condition for elliptic CY3's $\Rightarrow$ bound of physical couplings
- Anomaly-free supergravity models (e.g. $n_T = 9 + 8k$ and $G = (E_8)^k$)

Questions:

- Extra consistency conditions?
  - ⋆ YES - unitarity of the worldsheet theory of the"supergraity strings" - according to H-C.Kim, G. Shiu and C. Vafa
- A (geometric) bound that all consistent theories should satisfy?
  - ⋆ The subject of this talk

A cartoon of the situation that can be imagined



The plan

- Review the unitarity argument in D=6 and ...
  - ⋆ Explain why we are we look answers to D=6 questions in D=5
- ... re-examine the unitary from D=5 point of view
- Establish a (geometric) bound for consistent theories

Supergravity strings in D=6

Consider an anomaly free D=6 theory with 8 spuercharges with

- n_T tensor multiplets
- Yang-Mills multiplets with a group $G = \prod_i G_i$
- hypermultiplets in different representations of the gauge group.

The anomaly polynomial:

⋆ $I_8 = \frac{1}{2}\Omega_{\alpha\beta}\, X_4^\alpha X_4^\beta$

▷ $\alpha, \beta = 0, 1, ...n_T$

▷ $\Omega_{\alpha\beta}$ - symmetric inner product on the space of tensors with $(1, n_T)$ signature

⋆ $X_4^\alpha = \frac{1}{8}a^\alpha \mathrm{tr} R^2 + \sum_i b_i^\alpha \frac{1}{4h_i^\vee}\mathrm{Tr}_{\mathrm{Adj}} F_i^2$

▷ $a, b_i \in \mathbb{R}^{1,n_T}$ – determined by the field content of the theory

Dyonic BPS strings with (0,4) worldsheet supersymmetry:

⋆ $\mathrm{d}H^\alpha = X_4^\alpha + Q^\alpha \prod_{a=1}^4 \delta\left(x^a\right) dx^a$

▷ Q^α – string charges

Anomaly inflow from $\Omega_{\alpha\beta} B_2^\alpha X_4^\beta$ to the BPS string $\quad \Rightarrow \quad$ (0,4) anomaly:

$$I_4 = -\Omega_{\alpha\beta}\, Q^\alpha \left(X_4^\beta(M_6)|_{W_2} + \frac{1}{2} Q^\beta \chi\,(N) \right)$$
$$= -\frac{1}{4}\, \Omega_{\alpha\beta} Q^\alpha \left(a^\beta p_1(TW_2) - 2\left(Q^\beta + a^\beta\right) c_2(SU(2)_1) + 2\left(Q^\beta - a^\beta\right) c_2(SU(2)_2) + \ldots\right)$$

Need to use:

- ▷ $\delta\,(x^a)\, dx^a$ – a particular representation of the Thom class Φ for $i : W_2 \hookrightarrow M_6$
  - * Thom isomorphism: $\quad i^* \Phi = \chi(N)$
- ▷ $\mathrm{tr} R^2|_{TW_2} = -2\, p_1(TW_2) - 2\, p_1(N)$
- ▷ $\chi(N) = c_2(SU(2)_1) - c_2(SU(2)_2)$ and $p_1(N) = -2(c_2(SU(2)_1) + c_2(SU(2)_2)$
- ▷ $SU(2)_2$ – $\mathcal{R}$-symmetry of the interacting part of the SCFT

Central charges (with c.o.m contribution):

$$c_L - c_R = -6\, \Omega_{\alpha\beta}\, a^\alpha\, Q^\beta \equiv -6 Q \cdot a$$
$$c_R = 3\, \Omega_{\alpha\beta}\, Q^\alpha\, Q^\beta - 6\, \Omega_{\alpha\beta}\, a^\alpha\, Q^\beta + 6 \equiv 3 Q \cdot Q - 6 Q \cdot a + 6$$

Constraints on charges Q

- Well-defined moduli space:
  - $j \cdot j > 0, \quad j \cdot b_i > 0, \quad j \cdot a < 0$
  - $j \in \mathbb{R}^{1,n_T}$ – a $(1, n_T)$ vector on the tensor branch ($SO(1,n_T)/SO(n_T)$ MS)
- Non-negative tension:
  - $j \cdot Q \geq 0$
- Non-negative levels for $SU(2)_1$ and G_i affine current algebras
  - $k = \frac{1}{2}(Q \cdot Q + Q \cdot a + 2) \geq 0$ and $k_i = Q \cdot b_i \geq 0$

Unitarity constraint on the worldsheet theory

▷ Left-moving current algebra for G is bounded by c_L

$$\sum_i \frac{k_i \cdot \dim G_i}{k_i + h_i^\vee} \leq c_L - 4 = 3\,Q \cdot Q - 9\,Q \cdot a + 2$$

▷ Allows to rule out anomaly-free supergravities without string-theoretic realisations

▷ Is not directly comparable with geometric bounds

Why is it worthwhile to re-examine the question in D=5?

- Different way of packing the (same) information
  - ◇ Consider e.g. reduction on a smooth elliptic CY3
    - ▷ D=6: $L_{\text{GS}} \sim b_{\alpha ij} B_2^\alpha \wedge F_2^i \wedge F^j \quad \Leftarrow \quad$ part of CY intersection form
    - ▷ D=5: - $-\frac{1}{6} C_{IJK} A^I \wedge F^I \wedge F^J \quad \Leftarrow \quad$ entire CY intersection form

    $(C_{IJK} = \int \omega_I \wedge \omega_J \wedge \omega_K; \qquad I = 1, ..., n_T + n_V + 1)$
  - ◇ In the S^1 reduction from D=6 to D=5, a one-loop computation should reveal new info and ... "hide" the anomaly
- Different scaling of central charges w.r.t string charge Q
  - ◇ D=6: $c_L, c_R \sim \# Q \cdot Q + \#' Q \cdot a + \#''$
  - ◇ D=5: $c_L, c_R \sim \tilde{\#} Q \cdot Q \cdot Q + \#' Q \cdot a + \tilde{\#}''$
- ○ General questions about which theories are liftable
  - ◇ reductions with Wilson lines
  - ◇ reductions with discrete holonomies
- ○ Unitarity constraints for generic minimal D=5 theories...

Anomaly cancellation in D=6 (2n) $\Leftrightarrow$ gauge/diff invariance in D=5 (2n-1)

- S^1 resuction of the GS terms:

$$\delta(\iota_v L_{\mathsf{GS}}) \neq 0 \quad !!!$$

- no D=5 anomaly to cancel it

Consider the simplest situation - $n_T = 1$ and $M_6 = M_5 \times S^1$ (no curvature):

- $I_8 = X_4 \wedge \tilde{X}_4; \qquad L_{\mathsf{GS}} = \hat{B}_2 \wedge \tilde{X}_4; \qquad \mathrm{d}H = X_4 \;\; (H = \mathrm{d}\hat{B} + X_3^{(0)})$
- reduction: $\hat{B}_2 \mapsto (B_2, A_1); \qquad X_4 \mapsto (x_4, x_3)$
  - $(\mathrm{d}x_4, \mathrm{d}x_3) = 0; \qquad (x_4 = \mathrm{d}x_3^{(0)}, x_3 = \mathrm{d}x_2^{(0)}); \qquad (\delta \mathrm{d}x_3^{(0)} = \mathrm{d}x_2^{(2)}, \delta x_2^{(0)} = 0)$
- $L_{\mathsf{GS}} \mapsto A_1 \wedge x_4 + B_2 \wedge x_3 \;\longrightarrow\; \mathrm{d}B_2 \wedge x_2^{(0)} \;\longrightarrow\; \tilde{F}_2 \llcorner x_2^{(0)} - x_3^{(0)} \wedge x_2^{(0)}$
  - CS-like terms with field dependent coefficients - not gauge/diff invariant
  - Can be cancelled by integrating out massive KK modes from chiral fields
  - Conditions for cancellation - the same as for the anomaly cancelation in D=6
  - many cases worked out by E. Poppitz, M, Unsal, F. Bonetti, T. Grimm, S. Hohenegger, P. Corvilain, D. Regalado ....

Another (scheme-independent) way to look at the problem

▷ Reduction of the anomaly

$$\int_{M_{2n-1}\times S^1} I^1_{2n}(\epsilon,\hat{\mathcal{A}},\hat{\mathcal{F}}) = \delta_\epsilon \int_{M_{2n-1}} \Phi\cdot X(\mathcal{A},\mathcal{F}) + \ldots$$

- ◇ $\hat{\mathcal{A}}\,/\,\mathcal{A}$ and $\hat{\mathcal{F}}\,/\,\mathcal{F}$ $-$ fields and curvatures in D=2n/2n-1
- ◇ ϵ $-$ the variation (gauge or diffeomorphism) parameter,
- ◇ Φ $-$ Wilson line along the circle (for gravity Φ - graviphoton curvature)
- ◇ $\cdot$ $-$ trace over group indices

▷ $X(\mathcal{A},\mathcal{F}) = \frac{\partial}{\partial\mathcal{F}} I^0_{2n+1}(\hat{\mathcal{A}},\hat{\mathcal{F}})$ - Bardeen-Zumino polynomial

- ◇ ... indicate correction terms when $G \longrightarrow G'$ or Diff(M_{2n}) $\longrightarrow$ Diff(M_{2n-1})

▷ Local counterterm $-\Phi\cdot X$ is *always* possible but can *never* be lifted to D=2n

▷ Liftability $\Rightarrow$ *different* counterterm

Obstruction to liftability

New CS couplings in D=5

- involve reduced D=6 YM fields, and the graviphoton

$$\mathcal{L}_{\mathrm{CS}} = -\frac{k_0}{6} A^{\mathrm{KK}} \wedge F^{\mathrm{KK}} \wedge F^{\mathrm{KK}} + \frac{k_R}{96} A^{\mathrm{KK}} \wedge \mathrm{tr} R^2$$

  - $k_0 = 2(9 - n_T)$ and $k_R = 8(12 - n_T)$

- Anomaly inflow

$$c_R = k_0 Q_{\mathrm{KK}}^3 + \frac{k_R}{2} Q_{\mathrm{KK}} \qquad \text{and} \qquad c_L = k_0\, Q_{\mathrm{KK}}^3 + k_R\, Q_{\mathrm{KK}}$$

  - The string source: $dF = d\rho(r) e_2 / 2$
    - $\mathrm{d}\rho(r) e_2/2$ – smooth representative of Thom class
    - e_2 – global angular form
    - $\int_{S^2} e_2 \wedge e_2 \wedge e_2 = 2 p_1(N)$
  - $\mathrm{tr} R^2|_{TW_2} = -2\, p_1(TW_2) - 2\, p_1(N)$

- In D=5 there are strings with *cubic* central charges (not quadratic!)
- All strings with cubic central charges carry some magnetic KK charge

Central charges for D=5 BPS strings

$$c_R = C_{IJK} Q^I Q^J Q^K + \frac{1}{2} a_I Q^I \qquad \text{and} \qquad c_L = C_{IJK} Q^I Q^J Q^K + a_I Q^I$$

◇ $I = 1, ..., n_T + n_V + 1$

▷ BPS strings in D=6 with transverse S^1 (normal bundle $\mathbb{R}^3 \times S^1$)

◇ Recall

$I_4 \sim \Omega_{\alpha\beta} Q^\alpha \left(a^\beta p_1(TW_2) - 2\left(Q^\beta + a^\beta\right) c_2(SU(2)_1) + 2\left(Q^\beta - a^\beta\right) c_2(SU(2)_2) + ...\right)$

◇ Take $c_2(SU(2)_1) = c_2(SU(2)_2) = c_2(N)$,

$$c_L = 2\, c_R = -12\, \Omega_{\alpha\beta}\, a^\alpha\, Q^\beta \equiv -12 Q \cdot a$$

▷ The interacting part of SCFT

$$c_R^{int} = -6\, Q \cdot a - 6 \qquad \text{and} \qquad c_L^{int} = -12\, Q \cdot a - 3$$

▷ The unitarity condition for linear strings

$$\sum_i \frac{(Q \cdot b_i) \cdot \dim G_i}{Q \cdot b_i + h_i^\vee} \leq c_L^{int} = -12\, Q \cdot a - 3$$

Kodaira positivity and F-theory models

- In all F-theory models the following bound holds:

$$j \cdot (-12a - \sum_i x_i b_i) \geq 0$$

  - $j \in \mathbb{R}^{1,n_T}$ – a $(1, n_T)$ vector on the tensor branch
  - $a, b_i \in \mathbb{R}^{1,n_T}$ – determined by the field content of the theory
  - x_i – number of D7 needed for G_i (multiplicity of respective singularity)

- Follows from the Kodaira condition - requirement that elliptic fibration over base B with singularities over divisors S_i is CY:

$$-12K = \sum_i x_i S_i + Y$$

  - Y – residual divisor which must be *effective*
  - For any nef divisor D : $D \cdot Y = D \cdot (-12K - \sum_i x_i S_i) \geq 0$

- KPC ($j \cdot (-12a - \sum_i x_i b_i) \geq 0$) is not expected to be satisfied in any consistent D=6 theory

The unitariry bound should hold in all consistent D=6 theories

▷ The strongest for of the constraint:

$$Q\cdot(-12a-\sum_i b_i(\frac{\dim G_i}{1+h_i^\vee}))\geq Q\cdot(-12a-\sum_i b_i(\frac{\dim G_i}{Q\cdot b_i+h_i^\vee}))\geq 3$$

- ◇ If the strong form is satisfied, it will hold also for $Q\cdot b_1>1$
- ◇ If it fails, need to check if $Q\cdot b_i=1$ is possible
- ◇ Impose: $Q\cdot Q+Q\cdot a+2\geq 0\,,\quad k_i=Q\cdot b_i\geq 0$ and $-Q\cdot a>0$

▷ (Assuming D=6 theory is F-theoretic) UC can be converted into geometric form:

$$D\cdot(-12K-\sum_i y_iS_i)\geq 3 \qquad \text{with} \quad y_i=\frac{\dim G^i}{1+h_i^\vee}$$

▷ x is always larger than y:

Type of gauge algebra	x_i-y_i	Gauge algebra
K_1	<2	$su(m)$, $sp(1)$, $sp(2)$, $sp(3)$ in Kodaira type I
K_2	≥ 2	All other groups

Comparing UC and KPC

$$D \cdot Y \geq 3 - \sum_i (x_i - y_i) D \cdot S_i$$

- In most of the cases the bound is automatic given KPC (KPC is stronger than UC)
  - ▷ At least 3 gauge group factors (gauge divisors $S_{1,2,3}$ ($D \cdot S_{1,2,3} > 0$ holds))
  - ▷ At least 2 gauge groups and at least 1 is type K_2 ($x_i - \frac{\dim G_i}{D \cdot S_i + h_i^\vee} \geq x_i - y_i \geq 2$)
- In other cases, KPC may be satisfied while UC is violated if

$$12\,n - 3 < \sum_i \mu_i D \cdot S_i \leq 12\,n - \sum_i (x_i - \mu_i) D \cdot S_i$$

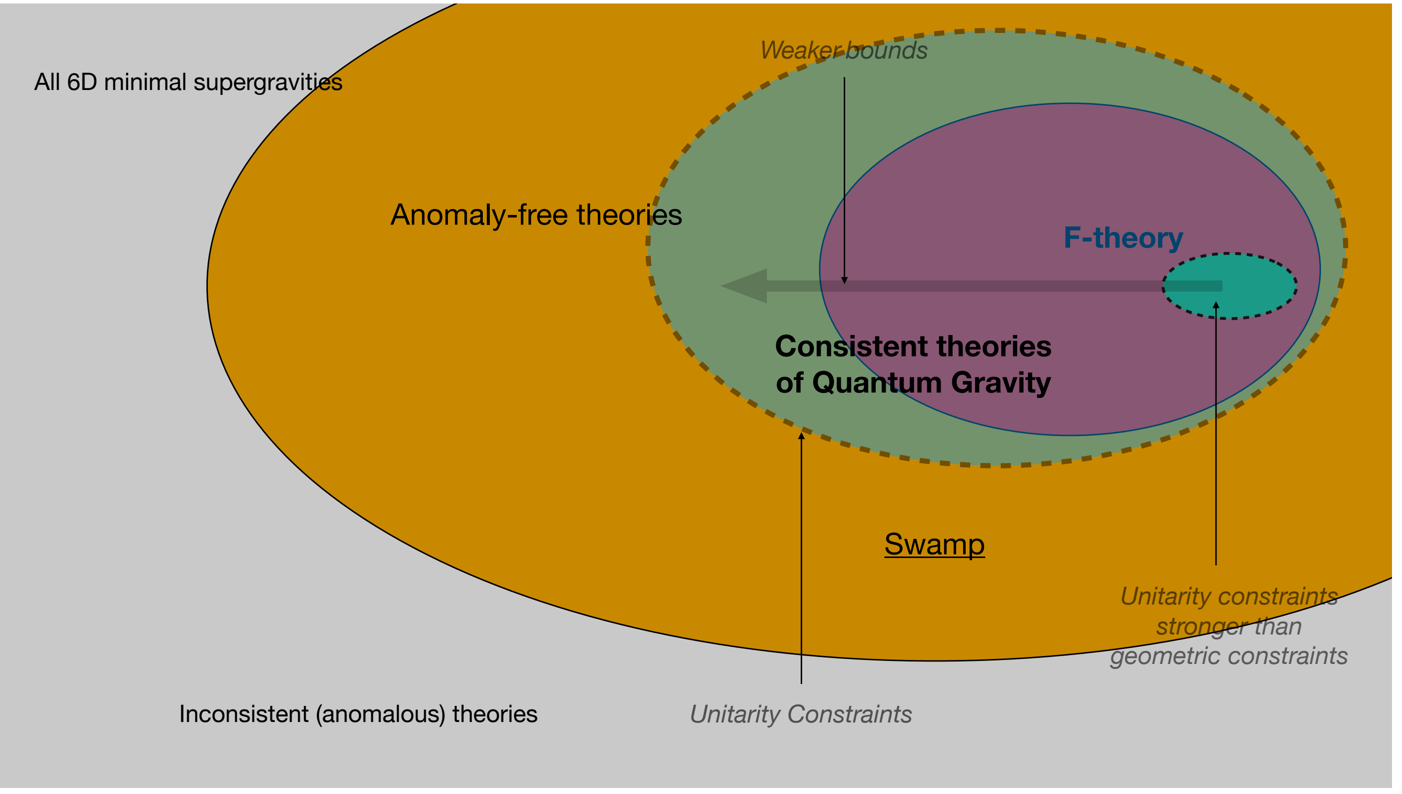
  - ▷ $Y = -12K - \sum_i x_i S_i$ – **NOT** numerically 0 (GDs S_i do not sweep $-12K$)
- 3 cases when UC imposes stronger constraints
  - ▷ $\exists S_1 \in \{S_i\}$ and nef D : $D \cdot S_1 = 1,\ D \cdot S_i = 0\,(i \neq 1)\ \&\ -D \cdot K \in \mathbb{Z}_+$
    $\Rightarrow D \cdot Y \geq 1$ for $SU(12n)$ and $D \cdot Y \geq 2$ for $SU(12n-1)$
  - ▷ $\exists S_1 \in \{S_i\}$ for $D \cdot Y \geq 1 \ \Rightarrow\ SO(24n-5)$, $SO(24n-4)$ or $Sp(6n)$ (I_{12n} type)
  - ▷ $\exists S_1, S_2 \in \{S_i\}$ for $D \cdot Y \geq 1\ \Rightarrow SU(a) \times SU(12n-a)$, $Sp(1) \times SU(12n-2)$, $Sp(2) \times SU(12n-4)$ or $SU(12n-6) \times Sp(3)$ (I_2, I_4 and I_6 type)

Example: $SU(N) \times SU(N)$, $n_H = 2$ (bifundamentals) and $n_T = 9$

$$\Omega = \text{diag}\left(+1, (-1)^9\right), \qquad a = \left(-3, (+1)^9\right)$$
$$b_1 = \left(1, -1, -1, -1, 0^6\right), \quad b_2 = \left(2, 0, 0, 0, (-1)^6\right)$$

- ▷ $Q = (1, 0, 0, 0, -1, 0.., 0)$
- ▷ $Q \cdot Q = 0$, $Q \cdot a = -2$ and $Q \cdot b_1 = Q \cdot b_2 = 1$
- ▷ UC: $2(N-1) \leq 24 - 3 \;\Rightarrow\; N \leq 11$ (stronger bound in D6 UC)
- ▷ KPC : $2N \leq 24 \rightarrow N \leq 12$
- ▷ Assuming F-theoretic realisation: $-12K = NS_1 + NS_2 + Y$
- ▷ For $N \geq 4$, the singular divisors are of type I_N
  - ◇ $S_1 \cdot K = S_2 \cdot K = 0$
  - ◇ 2 bifundamental hypers: $S_1 \cdot S_1 = -2 = S_2 \cdot S_2$ and $S_1 \cdot S_2 = 2$
  - ◇ $n_T = 9$ translates into $K \cdot K = 0$.
- ▷ Can verify that $Y = -12K - 12S_1 - 12S_2$ has to be numerically non-trivial ($-12K = 12S_1 + 12S_2$ cannot be realised on the base B of an elliptic Calabi-Yau threefold with the required singularity structure)

A refined cartoon of the space of D=6 theories



... and much left to be understood about consistency of quantum gravity

Exotic (sugra) backgrounds

- D=6 anomalies - [AG-W]

$$I_{3/2} - 21 I_{1/2} - 8 I_{\rm SD} = 0$$

  - ◇ $0 \times 2 + 5 I_{\rm SD} - 5 I_{\rm SD} \quad \Rightarrow$ anomaly-free $(0,2)$ theory with 21 TM
    - ▷ IIB compactified on $K3$
  - ◇ $0 \times 1 + I_{\rm SD} - I_{\rm SD} \quad \Rightarrow$ anomaly-free $(0,1)$ theory with 9 TM & 12 HM
    - ▷ The origin of this theory?

- ○ $(0,2)$ theory: $\mathcal{M} = O(5,21)/O(5) \times O(21)$

- ○ Freely-acting $\mathbb{Z}_2$ involution (Enriques) σ :
  - ◇ $USp(4) \longrightarrow USp(2) \times USp(2)$
    - ▷ gravitini: $4 \longrightarrow (2,1)^+ + (1,2)^-$
    - ▷ tensors in GM: $5 \longrightarrow (1,1)^+ + (2,2)^-$
    - ▷ σ has -1 eigenvalues acting on $H^{(2,0)}$ and $H^{(0,2)}$, +1 on $H^{(1,1)+}$
    - ▷ σ has $[(+1)^9, (-1)^{10}]$ eigenvalues acting on $H^{(1,1)-}$
  - ◇ $\mathcal{M}_{\rm T} \times \mathcal{M}_{\rm H} = \frac{O(1,9)}{O(9)} \times \frac{O(4,12)}{O(4) \times O(12)}$

- Enriques is not a spin manifold!

Duality action on fermions:

- (bosonic) duality symmetry $SL(2,\mathbb{Z}) \quad \Rightarrow \quad \text{pin}^+GL(2,\mathbb{Z})$
- Lorentz symmetry $SO(1,9)$ + duality group $\quad \Rightarrow \quad$ faithful action on the fermions by $\text{Spin}_\text{d}(1,9)$

$$1 \longrightarrow \mathbb{Z}_2 \xrightarrow{f} \text{Spin}_\text{d}(1,9) \longrightarrow SO(1,9)\times GL(2,\mathbb{Z}) \longrightarrow 1$$

- Duality group $\text{pin}^+GL(2,\mathbb{Z})$ includes the spacetime fermion number $e^{i\pi \boldsymbol{F}}$
- map f sends the $\mathbb{Z}_2$ generator to $(-\mathbb{I}_{16}, e^{i\pi\boldsymbol{F}}) \in Spin(1,9)\times \text{pin}^+GL(2,\mathbb{Z})$
- Generic $SL(2,\mathbb{Z}) \quad \Rightarrow \quad$ axion-dilaton dependent tranformation of fermions

Perturbative backgrounds (constant axion-dilaton) :

- Perturbative duality symmetries $\quad \mathcal{G}_\text{per} \subset \text{pin}^+GL(2,\mathbb{Z})$

$$\mathcal{G}_\text{per} \simeq D_8 = \langle r, s \mid r^4 = s^2 = (rs)^2 = 1\rangle$$

  - ◇ $r = e^{i\pi \boldsymbol{F}_\text{L}}$ and $s = \Pi$ worldsheet parity
  - ◇ $\mathbb{Z}_2 \times \mathbb{Z}_2$ analogue of bosonic $SL(2,\mathbb{Z})$

$$1 \longrightarrow \mathbb{Z}_2 \xrightarrow{f} \text{Spin'}_\text{d}(1,9) \longrightarrow SO(1,9)\times \mathbb{Z}_2\times\mathbb{Z}_2 \longrightarrow 1$$

- Fermions on 10D oriented spacetime X - sections of $\mathcal{F} \otimes T_X$ transition functions - in Spin'$_{\mathsf{d}}(1,9)$.
- For $X = \mathbb{R}^{1,5} \times Y$, $\quad \mathcal{F} \Rightarrow \mathcal{F}_+ \oplus \mathcal{F}_- \quad (16 \Rightarrow (4,1,2) \oplus (4',2,1))$
- Existence of fermions in $\mathcal{F}_+ \quad \Rightarrow \quad$ chiral Spin'$_{\mathsf{p}}(4)$ structure on Y
- $Y/G_E \quad \Rightarrow \quad$ Enriques surface .... **no** spinors (on sections of $\mathcal{F}_+$: $\; U_E \cdot \mathcal{S} = i\sigma_3\mathcal{S}$)
- $\mathbb{Z}_2$ symmetry, with generator

$$\tilde{U}_E = (g, e^{i\pi \boldsymbol{F}_{\mathsf{L}}}\Pi) \in SO(4) \times \mathcal{G}_{\mathsf{p}} \; .$$

- lift to Spin'$_{\mathsf{p}}(4)$: $\quad \tilde{U}_E \cdot \mathcal{S} = -\sigma_3 \mathcal{S} \sigma_2$
- Inv. fermions solving $\; \mathcal{S} = -\sigma_s \mathcal{S} \sigma_2 \quad \Rightarrow \quad$ (1/2) susy *on non-spin* background

Flat F-theory

- FHSV CY (holonomy $SU(2) \rtimes \mathbb{Z}_2 \subset SU(3)$)
- Other examples:
  - ◇ $O_2^3 \times S^1$ - $\quad U_E \cdot (x_1, x_2, x_3, x_4) = (x_1 + 1/2, -x_2, -x_3, x_4)$
  - ◇ Non-pertutrbative relatives O_k^3 for $\mathbb{Z}_3$, $\mathbb{Z}_4$ and $\mathbb{Z}_6$
  - ◇ Smooth non-spin 7 manifolds as Seifert bundles over singular CY_3/Γ

Non-orientable IIA backgrounds

- Circle T-duality on Enr $\times S^1$
  - no orientifold by WS parity
  - additional circle action on NS fields: T_{L} ($T_{\mathsf{L}}^2 =$ id) :

$$X^5_{\mathsf{L}} \mapsto -X^5_{\mathsf{L}} \qquad \& \qquad \mathcal{X}^5_{\mathsf{L}} \mapsto -\mathcal{X}^5_{\mathsf{L}}$$

- IIA symmetry: $\tilde{U}_{\mathsf{IIA}} = T_{\mathsf{L}}^{-1} \tilde{U}_E T_{\mathsf{L}} = T_{\mathsf{L}}^{-1} U_E e^{i\pi \boldsymbol{F}_{\mathsf{L}}} \,\Pi\, T_{\mathsf{L}}$
  - $\Pi\, T_{\mathsf{L}} = T_{\mathsf{R}}\, \Pi$
  - $R_5 = T_{\mathsf{L}} T_{\mathsf{R}}$ - spacetime reflection
- IIA symmetry: $\tilde{U}_{\mathsf{IIA}} = R_5 U_E e^{i\pi \boldsymbol{F}_{\mathsf{L}}} \,\Pi$
  - $R_5 U_E$ is free on Enr $\times S^1$
  - Volume projected out! $\tilde{X} = (Y \times S^1)/\mathbb{Z}_2$ with $\mathbb{Z}_2 = (\sigma, \rho)$
- Perturbative string description of nonoreinted background with a sugra limit
- massless spectrum agrees with IIB on Enr $\times S^1$
  - 11 vectors, 58 scalars
  - can be obtained from M-theory

M-theory lift, pinors & spacetime supersymmetry

- IIA on Enr $\times S^1$ lifts to M-theory on FHSV CY $Z = (Y \times S^1 \times S^1)/\mathbb{Z}_2$
  - ◇ $e^{i\pi \boldsymbol{F}_\mathrm{L}}$ in IIA $\Leftrightarrow$ Reflection on M-theory circle
  - ◇ IIA GSO - 10D spacetime reflection + WS parity
- IIA geometry:
  - ◇ $\tilde{X} = (Y \times S^1)/\mathbb{Z}_2$ - a circle bundle over Enr.
    - ◦ projection $p_1 : \tilde{X} \to X$ forgets the circle direction
    - ◦ section $i_1 : X \to \tilde{X}$ given by $i_1(x) = (x, 0)$ $(i_1 p_1 = \mathrm{id}_{\tilde{X}}$ and $p_1 i_1 = \mathrm{id}_X)$
  - ◇ Z - circle bundle over $\tilde{X}$
    - ◦ projection $p_2 : Z \to \tilde{X}$ & section $i_2 : \tilde{X} \to Z$
- Stiefel-Whitney classes:
  - ◦ $w_2(T_{\tilde{X}}) + w_1(T_{\tilde{X}}) \cup w_1(T_{\tilde{X}}) = 0$
  - ◦ Pin$^-$ structure
  - ◦ lift to the **pinor bundle** of Hol$(\tilde{X})$ $(SU(2) \rtimes \mathbb{Z}_2 \in O(5)$ (not $SO(5)$!))

$\Rightarrow$ non-zero covariantly constant sections

$\Rightarrow$ spacetime susy

Global future

- dual descriptions (e.g. heterotic strings)
- M-theory on **non-spin** manifolds (cf. m_c structure based on sugra description)
- situation when the WS description is not possible
- (lower-dimensional) spacetime physics with discrete (gauge) symmetries
- ...