RINGS OF SEPARATED POWER SERIES

by

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Contents

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1. Introduction

Let K be a field, complete with respect to the non-trivial ultrametric absolute value $|\cdot|: K \to \mathbb{R}_+$. By K° denote the valuation ring, by $K^{\circ\circ}$ its maximal ideal, and by $\widetilde{K}$ the residue field $K^\circ/K^{\circ\circ}$. Let K' be an algebraically closed field containing K and consider the polydisc

$$\Delta_{m,n} := \left((K')^\circ\right)^m \times \left((K')^{\circ\circ}\right)^n.$$

In 1961, Tate [**39**] introduced rings T_m of analytic functions on the closed polydiscs $\Delta_{m,0}$. These rings lift the affine algebraic geometry of the field $\widetilde{K}$. In particular, the Euclidean Division Theorem for $\widetilde{K}[\xi]$ lifts to a global Weierstrass Division Theorem for T_m. The basic properties of T_m that follow from Weierstrass Division include Noetherianness, Noether Normalization, unique factorization, and a Nullstellensatz. These results pave the way for the development of rigid analytic geometry (see [**6**] and [**10**]).

Because in its metric topology K' is totally disconnected and not locally compact, to construct rigid analytic spaces one relies on a Grothendieck topology to provide a suitable framework for sheaf theory. For example, the basic admissible open affinoids of rigid analytic geometry are obtained by an analytic process analogous to localization in algebraic geometry (see [**6**, Section 7.2.3]). The resulting domains, rational domains, satisfy a certain universal property (see [**6**, Section 7.2.2]) and therefore give a local theory of rigid analytic spaces. The local data are linked together with a notion of admissible open cover and Tate's Acyclicity Theorem. This makes it possible, for example, to endow every algebraic variety over K with an analytic structure, that of a rigid analytic variety.

The representation

$$\Delta_{m,n} = \varinjlim_{\varepsilon} ((K')^\circ)^m \times (\varepsilon (K')^\circ)^n,$$

where $\varepsilon \in (K')^{\circ\circ}$, yields a ring of analytic functions on $\Delta_{m,n}$ by taking a corresponding inverse limit of Tate rings. This gives the polydisc $\Delta_{m,n}$ the structure of a rigid analytic variety. But its global functions are, in general, unbounded. Even if one restricts attention to those functions with finite supremum norm, the geometric behavior can be pathological. For example, let $\{a_i\}_{i\in\mathbb{N}} \subset (K')^{\circ\circ}$ be a sequence such that $\lim_{i\to\infty} |a_i| = 1$. Put

$$f(\rho) := \sum a_i \rho^i.$$

Then f converges and has infinitely many zeros on $\Delta_{0,1}$. This follows by restricting to the closed subdiscs $\varepsilon \cdot \Delta_{1,0}$ and applying Weierstrass Preparation.

The rings $S_{m,n}$, defined below, represent Noetherian rings (often, K-Banach algebras) of bounded analytic functions on $\Delta_{m,n}$ with a tractable algebraic and geometric behavior. We address the issue of the corresponding sheaf theory in [**22**].

These rings have been used in various contexts. In [**16**], where the $S_{m,n}$ were first defined, they were used to obtain a uniform bound on the number of isolated points in fibers of affinoid maps. This result was strengthened in [**2**] to give a uniform bound on the piece numbers of such fibers. In [**11**], rings $S_{0,n}$ were used to lift the rings $\widetilde{K}[\![\rho]\!]$ in order to obtain analytic information about local rings of algebraic varieties over $\widetilde{K}$. In [**17**] (and later in [**21**]), the $S_{m,n}$ were used to provide the basis for a theory of rigid subanalytic sets; i.e., images of K-analytic maps. This theory of rigid subanalytic sets was developed considerably further in [**21**], [**19**], [**18**], [**20**]. The manuscript [**21**] (unpublished) contains a quantifier simplification theorem suitable for the development of a theory of subanalytic sets based on the Tate rings. That manuscript was produced in 1995, well before the completion of this paper, and hence it was written to be self contained. As a result the proofs were rather ad hoc. In the paper [**23**] we give a smoother and more general treatment of that quantifier simplification theorem, based on some of the machinery developed in this paper, specifically the Weierstrass Division and Preparation Theorems (Theorem 2.3.8 and Corollary 2.3.9) and the concept of "generalized ring of fractions" developed in Section 5.

(The theory of the images of semianalytic sets under proper K-analytic maps was developed by Schoutens in [**32**]–[**36**]. Recently in [**12**], [**37**] and [**13**] Gardener and Schoutens have given a quantifier elimination in the language of Denef and van den Dries [**9**] over the Tate rings T_m, using the results of Raynaud–Mehlmann [**27**], Berkovich [**3**], and Hironaka, [**15**]. The proof of their elimination theorem also depends on the model completeness result of [**21**], see [**23**, Section 4].)

The theory of the rings $S_{m,n}$ was not developed systematically in papers [**16**], [**17**], [**18**], [**19**], [**20**] and [**21**]. Instead, partial results were proved as needed. The accumulation of these partial results convinced us that a systematic theory of the rings $S_{m,n}$ would be possible and would provide a natural basis for rigid analytic geometry on the polydiscs $\Delta_{m,n}$. The theory developed in this paper has been applied in [**23**] to prove a quantifier elimination theorem which provides the basis for the theory of rigid subanalytic sets based on the Tate rings, and in [**22**] which treats the basic sheaf theory of quasi-affinoid varieties and proves the quasi-affinoid acyclicity theorem. The theory has also been applied in [**31**] to yield a global Artin Approximation Theorem for the pair of rings $H_{m,n} \hookrightarrow S_{m,n}$, where $H_{m,n}$ is the algebraic closure of T_{m+n} in $S_{m,n}$. Here the $S_{m,n}$ play the role of a kind of completion of Tate rings.

The goals of this paper are (i) to develop the commutative algebra of the power series rings $S_{m,n}$ (Section 4) and (ii) to develop the ingredients of sheaf theory for $S_{m,n}$-analytic varieties; in particular to show that rational domains in this setting (which we term quasi-affinoid) satisfy the same universal property as affinoid rational domains. This provides a foundation for a relative rigid analytic geometry over open polydiscs.

In the next few paragraphs we outline the contents of this paper.

In Section 2, we define the rings $S_{m,n}$ of separated power series, prove that they are Noetherian and prove two Weierstrass Preparation Theorems as in [**16**], [**17**] and [**2**], one relative to the variables ranging over closed discs, the other relative to the variables ranging over open discs. These Weierstrass Preparation Theorems were crucial in the applications mentioned above. But, because there are two types of variables, a suitably large collection of Weierstrass automorphisms does not exist. Thus these Weierstrass Preparation Theorems do not yield Noether Normalization for quotient rings of the $S_{m,n}$ (see Example 2.3.5), making the basic theory considerably more difficult to establish than in the affinoid case.

We are interested in studying properties of quotient rings $S_{m,n}/I$. In affinoid geometry, the key technique is Noether Normalization. The difficulties stemming from the failure of Noether Normalization for $S_{m,n}$ are overcome in Section 3 by a careful analysis of the behavior of restriction maps from $\Delta_{m,n}$ to closed subpolydiscs and to certain disjoint unions of open subpolydiscs.

Section 4 contains the Nullstellensatz and results on flatness, excellence, and unique factorization. The Nullstellensatz yields a supremum seminorm on the maximal ideal space of a quasi-affinoid algebra (i.e., a quotient ring of $S_{m,n}$).

In Section 5, we relate the behavior of the supremum seminorm to the residue norm derived from the Gauss norm on $S_{m,n}$, patching together uniform data that hold on affinoid algebras induced by restriction maps. The results are used to show that K-algebra homomorphisms of quasi-affinoid algebras are continuous, that all residue norms on a quasi-affinoid algebra are equivalent (i.e., the topology of a quasi-affinoid algebra is independent of presentation), and that quasi-affinoid rational domains satisfy an appropriate Universal Mapping Property. We prove when $\operatorname{Char} K = 0$, and in many cases also when $\operatorname{Char} K = p$, that on a reduced quasi-affinoid algebra the supremum norm and the residue norms are equivalent.

Section 6 contains some finiteness theorems, in particular it contains a weak analogue of Zariski's Main Theorem for quasi-finite maps, which is applied to show that quasi-affinoid subdomains are finite unions of R-subdomains.

We employ three different sorts of argument in this paper. The first sort of argument, "slicing", combines a generalization of the notion of discrete valuation ring (DVR) and a generalization of the notion of orthonormal basis. Each "level" of a formal power series ring over a DVR projects to a formal power series ring over a field, whose algebraic properties can often be lifted. Similar arguments were employed in [**14**] and in [**4**]. The second sort of argument exploits the relation between residue order and restrictions to closed polydiscs. A special case of this type of argument was used in [**5**]. To treat the case of a discretely valued ground field we must understand how generating systems of modules behave under ground field extension. Here we use the notion of stable fields (see [**6**]). The third sort of argument uses techniques of commutative algebra to extract information from completions at maximal ideals.

Following is a telegraphic summary of the principal results of this paper.

Theorem 2.1.3. *If $\widetilde{K}$ is algebraic over $\widetilde{E}$ then*

$$S_{m,n}(E,K) = K\widehat{\otimes}_E E\langle\xi\rangle[\![\rho]\!].$$

Corollary 2.2.4. *$S_{m,n}$ is Noetherian.*

Theorem 2.3.2 and Corollary 2.3.3. *Weierstrass Division and Preparation Theorems for $S_{m,n}$.*

Theorem 2.3.8, and Corollary 2.3.9. *Weierstrass Division and Preparation Theorems for $A\langle\xi\rangle[\![\rho]\!]_s$.*

Theorem 3.1.3. *Submodules of $(S_{m,n})^\ell$ are v-strict. In particular, ideals of $S_{m,n}$ are strictly closed.*

Theorem 3.2.3. *Strictness of a generating system is preserved under restriction to suitably large rational polydiscs.*

Corollary 3.3.2. *For a submodule $M \subset (S_{m,n})^\ell$, and ε large enough*

$$\iota_\varepsilon^{-1}(\iota_\varepsilon(M)\cdot T_{m,n}(\varepsilon)) = M.$$

Theorems 3.4.3, 3.4.6. *The restriction of a quasi-affinoid algebra to a suitably chosen finite union of open polydiscs is an isometry in residue norms.*

Theorem 4.1.1. *The Nullstellensatz for $S_{m,n}$.*

Corollary 4.2.2. *$S_{m,n}$ is a regular ring of dimension $m+n$.*

Proposition 4.2.3. *If $\operatorname{Char} K = 0$, $S_{m,n}$ is excellent.*

Proposition 4.2.5. *$S_{m,n}$ is often excellent when $\operatorname{Char} K = p \neq 0$.*

Theorem 4.2.7. *$S_{m,n}$ is a UFD.*

Theorem 5.1.5. *For a quasi-affinoid algebra, the ring of power-bounded elements is integral over the ring of elements of residue norm ≤ 1.*

Corollary 5.1.8. *Characterization of power-boundedness, topological nilpotence and quasi-nilpotence in terms of the supremum seminorm.*

Theorem 5.2.3, Corollary 5.2.4. *Quasi-affinoid morphisms are continuous. In particular all residue norms on a quasi-affinoid algebra are equivalent.*

Theorem 5.2.6. *Homomorphism Extension Lemma.*

Proposition 5.3.2. *Generalized rings of fractions are well-defined.*

Theorem 5.3.5. *Quasi-rational domains satisfy the appropriate universal mapping property.*

Proposition 5.4.3. *Tensor products exist in the category of quasi-affinoid algebras.*

Theorems 5.5.3, 5.5.4. *In characteristic zero, and often in characteristic* p*, the residue norm and the supremum norm of a reduced quasi-affinoid algebra are equivalent.*

Theorem 6.1.2. *A quasi-affinoid map that is finite-to-one is piecewise finite.*

Theorem 6.2.2. *A quasi-affinoid subdomain is a finite union of* R*-subdomains.*

Corollary 6.2.3. *Quasi-affinoid subdomains are open.*

2. Rings of Separated Power Series

In this section, we define the rings $S_{m,n} = S_{m,n}(E,K)$ of separated power series, prove that these rings are Noetherian (Corollary 2.2.4) and that they satisfy Weierstrass Preparation and Division theorems (Corollary 2.3.3 and Theorem 2.3.2), but not (Example 2.3.5) Noether Normalization.

2.1. Definitions. — Let K be a field, complete with respect to a non-trivial ultrametric absolute value $|\cdot|\colon K \to \mathbb{R}_+$, let K° denote the valuation ring of K, let $K^{\circ\circ}$ denote its maximal ideal and let $\sim\colon K^\circ \to \widetilde{K} := K^\circ/K^{\circ\circ}$ denote the canonical residue epimorphism. Throughout this paper, we will be concerned with power series whose coefficients lie in certain subrings B of K° called quasi-Noetherian rings.

Let B be a valued subring of K° such that each $x \in B$ with $|x| = 1$ is a unit of B (such rings are called **B-rings**.) It follows from the ultrametric inequality that B is a local ring. The ring B is called **quasi-Noetherian** iff for each ideal $\mathfrak{a}$ of B there is a zero-sequence $\{x_i\}_{i\in\mathbb{N}} \subset \mathfrak{a}$ (called a **quasi-finite generating system**) such that each $a \in \mathfrak{a}$ can be written in the form $a = \sum_{i\geq 0} b_i x_i$ for some elements $b_i \in B$. However, not all such sums need belong to $\mathfrak{a}$. (See [**6**, Section 1.8] and [**14**].)

We will make use of the following properties of quasi-Noetherian rings without further reference. Clearly, any subring $B \subset K^\circ$ which is a DVR is quasi-Noetherian, since it is Noetherian. Let $B \subset K^\circ$ be quasi-Noetherian. For any zero sequence $\{a_i\}_{i\in\mathbb{N}} \subset K^\circ$, the local ring

$$A := B[a_0, a_1, \dots]_{\{a\in B[a_0,a_1,\dots]:|a|=1\}}$$

is quasi-Noetherian ([**6**, Proposition 1.8.2.4]). The completion of B is itself quasi-Noetherian ([**6**, Proposition 1.8.2.2]). The value semigroup $|B\setminus\{0\}| \subset \mathbb{R}_+ \setminus \{0\}$ is discrete ([**6**, Corollary 1.8.1.3]). Therefore, there is a sequence $\{b_i\}_{i\in\mathbb{N}} \subset B\setminus\{0\}$ with $|B\setminus\{0\}| = \{|b_i|\}_{i\in\mathbb{N}}$ and $1 = |b_0| > |b_1| > \cdots$. The sequence of ideals

$$B_i := \{b \in B : |b| \leq |b_i|\},\ i \in \mathbb{N}$$

is called the **natural filtration** of B. Note that B_1 is the unique maximal ideal of B. By $\widetilde{B}$ denote the residue field B/B_1 of B. For $i \in \mathbb{N}$, put $\widetilde{B}_i := B_i/B_{i+1}$; then $\widetilde{B} = \widetilde{B}_0 \subset \widetilde{K}$. Since $B_1 \cdot B_i \subset B_{i+1}$, the B-modules $\widetilde{B}_i$ can be viewed in a canonical way as $\widetilde{B}$-vector spaces. Each $\widetilde{B}$ vector space $\widetilde{B}_i$ is finite-dimensional; in fact, this property characterizes the class of quasi-Noetherian rings ([**6**, Theorem 1.8.1.2]). For $i \in \mathbb{N}$ we may identify the $\widetilde{B}$-vector space $\widetilde{B}_i$ with the $\widetilde{B}$-vector subspace $\left(b_i^{-1}B_i\right)^\sim$ of $\widetilde{K}$ via the map

$$\pi_i : (a + B_{i+1}) \mapsto \left(b_i^{-1}a\right)^\sim .$$

When $i > 0$, this identification of $\widetilde{B}_i$ with a $\widetilde{B}$-vector subspace of $\widetilde{K}$ is not canonical; it will, however, be used frequently.

Let R be a ring and let $\{\mathfrak{a}_\lambda\}_{\lambda\in I}$ be an inverse system of ideals of R. When we endow R with the topology induced by taking $\{\mathfrak{a}_\lambda\}_{\lambda\in I}$ to be a system of neighborhoods of 0, R is said to be a ring with a **linear topology**. In this subsection, we will assume that R is complete and Hausdorff in this linear topology. For example, let R be a subring of K°; then the topology induced on R by the absolute value $|\cdot|$ is a Hausdorff linear topology.

Let $\xi = (\xi_1, \dots, \xi_m)$ be variables. A formal power series $\sum a_\mu \xi^\mu$ with coefficients in R is called **strictly convergent** iff $\{a_\mu\}_{\mu\in\mathbb{N}^m}$ is a zero-sequence in R. By $R\langle\xi\rangle$, we denote the collection of all strictly convergent power series; it is a subring of the formal power series ring $R[[\xi]]$. The ring $R\langle\xi\rangle$ is complete and Hausdorff in the **uniform topology**; i.e., in the linear topology given by the system of ideals $\{\mathfrak{a}_\lambda \cdot R\langle\xi\rangle\}_{\lambda\in I}$. In case $R = K^\circ$, by $K\langle\xi\rangle$, we denote the K-algebra $K \otimes_R R\langle\xi\rangle$ of strictly convergent power series over K.

Let $\rho = (\rho_1, \dots, \rho_n)$ be variables. Then
$$R\langle\xi\rangle[[\rho]] = R[[\rho]]\langle\xi\rangle$$
when we endow $R[[\rho]]$ with the **product topology**; i.e., the topology induced by the inverse system of ideals
$$\left\{(\rho)^d + \sum_{|\nu|<d} \rho^\nu \cdot \mathfrak{a}_{\lambda_\nu}[[\rho]]\right\}_{d\in\mathbb{N}}.$$
In case R carries the discrete topology, $R\langle\xi\rangle = R[\xi]$ and
$$R[\xi][[\rho]] = R[[\rho]]\langle\xi\rangle\,,$$
where $R[[\rho]]$ carries the (ρ)-adic topology. If $R \subset K^\circ$ then the absolute value $|\cdot|$ on R induces a linear topology, and $|\cdot|$ extends to an R-module norm on $R\langle\xi\rangle[[\rho]]$ called the **Gauss norm**, given by
$$\left\|\sum_{\mu\nu} f_{\mu\nu}\xi^\mu\rho^\nu\right\| := \sup_{\mu\nu} |f_{\mu\nu}|\,.$$
These definitions will be used in Subsection 2.3 where we discuss Weierstrass Division Theorems.

Definition 2.1.1. — Fix a complete, quasi-Noetherian subring $E \subset K^\circ$ and, if $\operatorname{Char} K = p > 0$, assume in addition that E is a DVR. Let $\xi = (\xi_1, \dots, \xi_m)$ and $\rho = (\rho_1, \dots, \rho_n)$ be variables. We define a K-subalgebra $S_{m,n}(E,K)$ of $K[[\xi,\rho]]$, called a **ring of separated power series**.

Let $\mathfrak{B}$ be the family of quasi-Noetherian subrings of K° which consists of all local rings of the form
$$\left(E[a_0, a_1, \dots]_{\{a\in E[a_0,a_1,\dots]:|a|=1\}}\right)^\frown,$$

where $\widehat{\ }$ denotes completion in $|\cdot|$, and where $\{a_i\}_{i\in\mathbb{N}} \subset K^\circ$ is a zero-sequence. Then put

$$\begin{aligned}
S_{m,n} = S_{m,n}(E,K) &:= K \otimes_{K^\circ} \left(\varinjlim_{B\in\mathfrak{B}} B\langle\xi\rangle [\![\rho]\!] \right), \\
S_{m,n}^\circ &:= \varinjlim_{B\in\mathfrak{B}} B\langle\xi\rangle[\![\rho]\!], \\
S_{m,n}^{\circ\circ} &:= K^{\circ\circ} \cdot S_{m,n}^\circ, \\
\widetilde{S}_{m,n} &:= \varinjlim_{B\in\mathfrak{B}} \widetilde{B}[\xi][\![\rho]\!].
\end{aligned}$$

For $f = \sum a_{\mu\nu}\xi^\mu\rho^\nu \in S_{m,n}$ we define the **Gauss norm** of f by

$$\|f\| := \sup_{\mu,\nu} |a_{\mu,\nu}|.$$

Note that $S_{m,n}$ contains the **Tate ring** $T_{m+n}(K) = K\langle\xi,\rho\rangle$ and $S_{m,0}$ coincides with T_m. In case $K = \mathbb{Q}_p$, the field of p-adic numbers, we have $S_{m,n} = \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \mathbb{Z}_p\langle\xi\rangle [\![\rho]\!]$, where $\mathbb{Z}_p$ denotes the ring of p-adic integers. When K is algebraically closed and E is a DVR with $E \subset K^\circ$ and $\widetilde{E} = \widetilde{K}$, the rings $S_{m,n}(E,K)$ are the rings defined in [**17**]. Following the usage in [**17**], when E is understood, we may write

$$K\langle\xi\rangle [\![\rho]\!]_s := S_{m,n}(E,K).$$

(The subscript s stands for "separated".) In the case that $\widetilde{E} = \widetilde{K}$ the rings $S_{0,n}$ and their quotient rings are the formal completions considered in [**11**, Section 2.3.2,], and used to derive properties of the formal localizations. The description of these rings given in Definition 2.1.1 is due to Bartenwerfer [**2**].

The family $\mathfrak{B}$, described in Definition 2.1.1, satisfies the following properties, which we use without further reference.

(a) $\mathfrak{B}$ forms a direct system under inclusion,
(b) $\varinjlim_{B\in\mathfrak{B}} B = K^\circ$,
(c) for each $B \in \mathfrak{B}$ and $b \in B$ there is some $B' \in \mathfrak{B}$ with $(b^{-1}B \cap K^\circ) \subset B'$,

and

(d) for any $B \in \mathfrak{B}$ and any zero-sequence $\{a_i\}_{i\in\mathbb{N}} \subset K^\circ$,

$$\big(B[a_0,a_1,\ldots]_{\{a\in B[a_0,a_1,\ldots]:|a|=1\}}\big)^\wedge \in \mathfrak{B}.$$

If $E \subset E'$ and $K \subset K'$ then

$$S_{m,n}(E,K) \subset S_{m,n}(E',K').$$

If K' is a finite algebraic extension of K then $S_{m,n}(E,K') = K' \otimes_K S_{m,n}(E,K)$.

Remark 2.1.2. — The following are easy consequences of the properties of B-rings (cf. [**2**] and [**17**]).

(i) If $f = \sum a_{\mu,\nu}\xi^\mu\rho^\nu \in S_{m,n}$ then
$$\|f\| = \sup_{\mu,\nu} |a_{\mu\nu}| = \max_{\mu,\nu} |a_{\mu\nu}|,$$
i.e., the supremum is attained.

(ii) We have the following characterizations of the subring $S^\circ_{m,n}$, the ideal $S^{\circ\circ}_{m,n}$, and the residue ring $\widetilde{S}_{m,n}$:
$$\begin{array}{rcl} S^\circ_{m,n} & = & \{f \in S_{m,n} : \|f\| \leq 1\}, \\ S^{\circ\circ}_{m,n} & = & \{f \in S_{m,n} : \|f\| < 1\} \text{ and} \\ \widetilde{S}_{m,n} & = & S^\circ_{m,n}/S^{\circ\circ}_{m,n}. \end{array}$$

As in [**6**, Corollary 1.5.3.2], the Gauss norm $\|\cdot\|$ is an absolute value on $S_{m,n}$ extending that on K.

The **canonical residue epimorphism** $\sim: K^\circ \to \widetilde{K}$ extends to the residue epimorphism $\sim: S^\circ_{m,n} \to \widetilde{S}_{m,n} : \sum a_{\mu\nu}\xi^\mu\rho^\nu \mapsto \sum \widetilde{a}_{\mu\nu}\xi^\mu\rho^\nu$. Let I be an ideal of $S_{m,n}$, and put $I^\circ := S^\circ_{m,n} \cap I$. Since $\sim: S^\circ_{m,n} \to \widetilde{S}_{m,n}$ is surjective, the image of I° under $\sim$ is an ideal of $\widetilde{S}_{m,n}$, which we denote by $\widetilde{I}$.

In general the $S_{m,n}(E,K)$ are not complete in $\|\cdot\|$. However, for many choices of $E \subset K$ they are. When $\widetilde{K}$ is algebraic over $\widetilde{E}$, we will show that
$$S_{m,n}(E,K) = K\widehat{\otimes}_E E\langle\xi\rangle[\![\rho]\!],$$
where $\widehat{\otimes}_E$ denotes the complete tensor product of normed E-modules (see [**6**, Section 2.1.7]). This situation is clarified in the next theorem. Observe that the natural map
$$\sigma : K \otimes_E E\langle\xi\rangle[\![\rho]\!] \to K[\![\xi,\rho]\!] : \sum \alpha_i \otimes f_i \mapsto \sum a_i f_i$$
is injective. Indeed, it is easy to see that the field of fractions $Q(E)$ of E is a flat E-algebra. Hence, $K[\![\xi,\rho]\!]$, being a $Q(E)$-vector space, is also a flat E-algebra. It now follows from [**25**, Theorem 7.6], that $\operatorname{Ker}\sigma = (0)$. The image of σ is contained in $S_{m,n}(E,K)$. Moreover, since σ is contractive, it extends to a map
$$\widehat{\sigma} : K\widehat{\otimes}_E E\langle\xi\rangle[\![\rho]\!] \to K[\![\xi,\rho]\!].$$
It is not hard to see that the image of $\widehat{\sigma}$ is contained in $S_{m,n}(E,K)$, when $S_{m,n}(E,K)$ is complete (see below).

Theorem 2.1.3. — *Let $n > 0$.* (i) *$S_{m,n}(E,K)$ is $\|\cdot\|$-complete if, and only if, $\widetilde{K}$ has finite transcendence degree over $\widetilde{E}$. In that case let $E' \subset K^\circ$ be a finitely generated extension of E such that $\widetilde{K}$ is algebraic over $\widetilde{E}'$. Then*

$$S_{m,n}(E,K) = S_{m,n}(E',K) = K\widehat{\otimes}_{E'}E'\langle\xi\rangle[\![\rho]\!],$$

where $\widehat{\otimes}_{E'}$ denotes complete tensor product of normed E'-modules (see [**6**, Section 2.1.7]*).*

(ii) *There is a quasi-Noetherian ring E', $E \subset E' \subset K^\circ$, such that $S_{m,n}(E',K)$ is $\|\cdot\|$-complete (and contains $S_{m,n}(E,K)$).*

(iii) *$\widetilde{S}_{m,n}(E,K)$ is (ρ)-adically complete if, and only if, $\widetilde{K}$ is a finitely generated field extension of $\widetilde{E}$.*

(iv) *$S^\circ_{m,n}(E,K)$ is (ρ)-adically complete if, and only if, $\widetilde{K}$ is a finitely generated field extension of $\widetilde{E}$ and K is discretely valued. (In which case we may take $E = K^\circ$.)*

Proof. — (i) Suppose that $\widetilde{K}$ has infinite transcendence degree over $\widetilde{E}$. Let $t_i \in K^\circ$, $i \in \mathbb{N}$, be such that the $\widetilde{t}_i$ are algebraically independent over $\widetilde{E}$. Let $f_i = \sum_{j=1}^\infty t_i^j \rho_1^j \in S_{0,1} \subset S_{m,n}$ $(n > 0)$. Choose $a \in K^{\circ\circ}$ (i.e., $|a| < 1$). The series $f = \sum_{i=1}^\infty a^i f_i$ is Cauchy in $\|\cdot\|$ but does not belong to $S_{m,n}$. Indeed for any $B \in \mathfrak{B}$, $\widetilde{B}$ is a finitely generated field extension of $\widetilde{E}$ and for $i \geq 1$, $\widetilde{B}_i$ is a finite dimensional vector space over $\widetilde{B}$. Hence $f \notin B[\![\rho_1]\!]$.

For the converse, assume that $\widetilde{K}$ is of finite transcendence degree over $\widetilde{E}$. Note that if $E' \in \mathfrak{B}$ then $S_{m,n}(E,K) = S_{m,n}(E',K)$. Hence we may assume that $\widetilde{K}$ is algebraic over $\widetilde{E}$. Let $f_i \in S^\circ_{m,n}$ with $\|f_i\| \to 0$. There are $a_i \in K^\circ$ with $|a_i| = \|f_i\|$ and $B^{(i)} \in \mathfrak{B}$ such that $\frac{1}{a_i} f_i \in B^{(i)}\langle\xi\rangle[\![\rho]\!]$, i.e., $f_i \in a_i B^{(i)}\langle\xi\rangle[\![\rho]\!]$. Let

$$B^{(i)} = B^{(i)}_0 \supset B^{(i)}_1 \supset \dots$$

be the natural filtration of $B^{(i)}$. Since $\widetilde{K}$ is algebraic over $\widetilde{E}$, each field $\widetilde{B}^{(i)}_0 = \widetilde{B}^{(i)}$, and hence each $\widetilde{B}^{(i)}_j$, is a finite-dimensional $\widetilde{E}$-vector space. Let $\widetilde{B}^{(i)}_j$ be generated over $\widetilde{E}$ by the residues modulo $B^{(i)}_{j+1}$ of $b_{ijk} \in B^{(i)}_j$, $k = 1, \dots, \dim \widetilde{B}^{(i)}_j$. Let $\{c_i\}_{i\in\mathbb{N}}$ be a rearrangement of $\{a_i b_{ijk} : i \in \mathbb{N}, j \in \mathbb{N}, k = 1, \dots, \dim \widetilde{B}^{(i)}_j\}$ in non-increasing size. (Recall that $a_i \to 0$.) Putting

$$B := \left(E[c_0, c_1, \dots]_{\{a \in E[c_0,c_1,\dots] : |a|=1\}}\right)^{\widehat{}} \in \mathfrak{B}$$

yields $a_i B^{(i)} \subset B$ for all i and $\sum_i f_i \in B\langle\xi\rangle[\![\rho]\!]$. Hence $S_{m,n}(E,K)$ is complete.

As we observed above, there is a map $\widehat{\sigma} : K\widehat{\otimes}_E E\langle\xi\rangle[\![\rho]\!] \to S_{m,n}$. If $\widetilde{K}$ is algebraic over $\widetilde{E}$ then for every $B \in \mathfrak{B}$, $\widetilde{B}$ and the $\widetilde{B}_i$ are all finite-dimensional

$\widetilde{E}$ vector spaces. Hence for each $B \in \mathfrak{B}$, there is a map

$$\tau : B\langle\xi\rangle[\![\rho]\!] \to K\widehat{\otimes}_E E\langle\xi\rangle[\![\rho]\!],$$

which is a left inverse of $\widehat{\sigma}$.

(ii) Repeated use of [**6**, Proposition 1.8.2.3 and Theorem 1.8.1.2], shows that there is a quasi-Noetherian ring E', $E \subset E' \subset K^\circ$, such that $\widetilde{K}$ is an algebraic extension of $\widetilde{E'}$. Hence $S_{m,n}(E,K) \subset S_{m,n}(E',K)$ and by (i) $S_{m,n}(E',K)$ is complete.

(iii) If $\widetilde{K}$ is a finitely generated field extension of $\widetilde{E}$ then replacing E by a suitable finitely generated extension we may assume that $\widetilde{E} = \widetilde{K}$. But then

$$\widetilde{S}_{m,n} = \widetilde{E}[\xi][\![\rho]\!],$$

which is (ρ)-adically complete.

If, on the other hand, there are $\widetilde{t}_i \in \widetilde{K}$ such that $\widetilde{t}_{i+1} \notin \widetilde{E}(\widetilde{t}_1, \ldots, \widetilde{t}_i)$ then $f := \sum \widetilde{t}_i \rho_1^i \notin \widetilde{S}_{m,n}$, since for every $B \in \mathfrak{B}$, $\widetilde{B}$ is a finitely generated field extension of $\widetilde{E}$.

(iv) If K is not discretely valued there are $a_i \in K^\circ$ with $|a_i| < |a_{i+1}| < 1$ for $i = 0, 1, 2, \ldots$. Then $\sum_i a_i \rho_1^i \notin S_{m,n}$. On the other hand, if $\widetilde{K}$ is a finitely generated extension of $\widetilde{E}$ and K is discretely valued, then $K^\circ \in \mathfrak{B}$. $\square$

Remark 2.1.4. — (i) Suppose $\operatorname{Char} K = p \neq 0$. In this case we require E to be a complete DVR. By the Cohen Structure Theorem ([**25**, Theorem 29.4]), E has a coefficient field (i.e., an isomorphic copy of $\widetilde{E} \subset E$) which we also denote by $\widetilde{E}$). If π is a prime of E then $\widetilde{E} \subset E = \widetilde{E}[\pi]\widehat{\ }$. Thus $S_{m,n}(E,K) = S_{m,n}(\widetilde{E},K)$. Hence we could have required in the equicharacteristic p case that $E \subset K$ be a field, without loss of generality.

(ii) Let K be a perfect field of characteristic p, and let $E \subset K^\circ$ be a subfield. Then there is a field E', $E \subset E' \subset K^\circ$, with E' perfect and $\widetilde{K}$ algebraic over $\widetilde{E'}$. Hence, using (i) above, for any DVR $E \subset K^\circ$ there is a field $E' \subset K^\circ$ such that $S_{m,n}(E,K) \subset S_{m,n}(E',K)$, $S_{m,n}(E',K)$ is complete in $\|\cdot\|$ and $S_{m,n}(E',K)$ is a finite $S_{m,n}(E',K)^p$–module. (The monomials $\xi^\mu\rho^\nu$ with $0 \le \mu_i < p$, $0 \le \nu_j < p$, form a basis.)

By definition, $S_{m,n}^\circ$ is the direct limit of complete rings (the $B\langle\xi\rangle[\![\rho]\!]$). Next we show that while $S_{m,n}$ may not be a complete K-algebra it is the direct limit of complete F-algebras for some complete, nontrivially valued subfield F of K. This decomposition will be used in Subsection 5.2.

Let F be a complete subfield of K such that F° is a DVR and $\widetilde{F}$ is finitely generated as a field. (For example, in the mixed characteristic case let $F = \mathbb{Q}_p$, the field of p-adic numbers, and in the equicharacteristic case let F be the fraction field of $\mathbb{Q}[\![t]\!]$ or $\mathbb{F}_p[\![t]\!]$, depending on the characteristic of K, where

$t \in K^{\circ\circ}$.) Let $B' \in \mathfrak{B}$. There is a $B \in \mathfrak{B}$ such that $B' \cup F^\circ \subset B$. Consider the F-algebra

$$F\widehat{\otimes}_{F^\circ} B\langle\xi\rangle[\![\rho]\!].$$

By the definition of the complete tensor product $\widehat{\otimes}$ this is an F-Banach algebra (i.e., is complete in $\|\cdot\|$). In general there is no $B'' \in \mathfrak{B}$ such that $(F\widehat{\otimes}_{F^\circ} B\langle\xi\rangle[\![\rho]\!])^\circ \subset B''\langle\xi\rangle[\![\rho]\!]$. However, the natural map

$$\sigma : F \otimes_{F^\circ} B\langle\xi\rangle[\![\rho]\!] \to S_{m,n} : \sum \alpha_i \otimes f_i \mapsto \sum \alpha_i f_i$$

is an isometry because $F^\circ \otimes_{F^\circ} B\langle\xi\rangle[\![\rho]\!] = B\langle\xi\rangle[\![\rho]\!]$. The next proposition shows that σ extends to $F\widehat{\otimes}_{F^\circ} B\langle\xi\rangle[\![\rho]\!]$.

Proposition 2.1.5. — *With the above notation,*

$$F\widehat{\otimes}_{F^\circ} B\langle\xi\rangle[\![\rho]\!] \subset S_{m,n}(E,K).$$

Indeed

$$S_{m,n} = \varinjlim_{F^\circ \subset B \in \mathfrak{B}} F\widehat{\otimes}_{F^\circ} B\langle\xi\rangle[\![\rho]\!].$$

Proof. — It is sufficient to show that if $f \in F\widehat{\otimes}_{F^\circ} B\langle\xi\rangle[\![\rho]\!]$ and $\|f\| \le 1$ then there is a $B'' \in \mathfrak{B}$ such that $f \in B''\langle\xi\rangle[\![\rho]\!]$. Let $f \in F\widehat{\otimes}_{F^\circ} B\langle\xi\rangle[\![\rho]\!]$ with $\|f\| \le 1$. Then there are $f_i \in B\langle\xi\rangle[\![\rho]\!]$ and $m_i \in \mathbb{N}$ such that $f = \sum \pi^{-m_i} f_i$, where π is a prime of F°, and $\|\pi^{-m_i} f_i\| \to 0$. Hence for each i there is a nullsequence $\{a_{ij}\}_{j\in\mathbb{N}}$ with $\pi^{-m_i} f_i \in B'\langle\xi\rangle[\![\rho]\!]$, where

$$B' := (B[a_{ij} : j \in \mathbb{N}]_{\{a \in B[a_{ij} : j\in\mathbb{N}] : |a|=1\}})^{\widehat{}}$$

and $|a_{ij}| \le \|\pi^{-m_i} f_i\|$ for all i and j. Since $\|\pi^{-m_i} f_i\| \to 0$, any rearrangement of the double sequence $\{a_{ij}\}_{i,j\in\mathbb{N}}$ as a sequence will be a null-sequence. Let $\{c_i\}_{i\in\mathbb{N}}$ be such a rearrangement. Then if

$$B'' := \left(B[c_0, c_1, \dots,]_{\{a \in B[c_0,c_1,\dots] : |a|=1\}}\right)^{\widehat{}},$$

$f \in B''\langle\xi\rangle[\![\rho]\!]$. □

In general the F-Banach Algebras $F\widehat{\otimes}_{F^\circ} B\langle\xi\rangle[\![\rho]\!] \subset S_{m,n}$ constructed above are not Noetherian and the Weierstrass Preparation and Division Theorems need not hold in them. An argument similar to the proof of Proposition 2.1.5 shows that we can write

$$S_{m,n}(E,K) = \varinjlim_{B\in\mathfrak{B}} (K \cdot B\langle\xi\rangle[\![\rho]\!])^{\widehat{}}$$

as the direct limit of K-Banach Algebras. These K-Banach algebras likewise may fail to satisfy the Weierstrass Preparation and Division Theorems of Subsection 2.3.

Remark 2.1.6. — (i) The rings $S_{m,n} = S_{m,n}(E,K)$ can have quite different properties depending on the choice of E. As we saw in Theorem 2.1.3, if $\widetilde{E}$ is large enough the $S_{m,n}(E,K)$ will be complete and the $\widetilde{S}_{m,n}$ may even be (ρ)-adically complete. On the other hand if $\widetilde{E} \subset \widetilde{K}$ is small, the $S_{m,n}$ will be far from complete and the $\widetilde{S}_{m,n}$ far from (ρ)-adically complete. Nevertheless, for all choices of E, $S^\circ_{m,n}$ is, by definition, the direct limit of the $\|\cdot\|$-complete and (ρ)-adically complete rings $B\langle\xi\rangle[\![\rho]\!]$, and this key property allows the development of the theory.

(ii) There is a larger class of power series rings in which many of the results and proofs of this paper remain valid. This larger class is defined as follows. Fix a family $\mathfrak{B}$ of complete, quasi-Noetherian subrings $B \subset K^\circ$ that satisfy the properties (a), (b), (c) and (d) listed after Definition 2.1.1, and put

$$S_{m,n} = S_{m,n}(\mathfrak{B},K) := K \otimes_{K^\circ} \varinjlim_{B\in\mathfrak{B}} B\langle\xi\rangle[\![\rho]\!].$$

Example 2.1.7 shows that this definition is more general.

(iii) If we wished to work over complete rings we could also have proceeded as follows: Form the rings $S_{m,n}(E,K)$ as in Definition 2.1.1, or the rings $S_{m,n}(\mathfrak{B},K)$ defined above, and then take their completions $\widehat{S_{m,n}} = S_{m,n}(E,K)\widehat{\ }$ or $S_{m,n}(\mathfrak{B},K)\widehat{\ }$. In general the rings $S_{m,n}(E,K)\widehat{\ }$ would be different from the rings $S_{m,n}(E',K)$ for any E'. However all the results of the paper are true for these rings $\widehat{S_{m,n}}$. The proofs that use "slicing" arguments may be modified as follows. Though an arbitrary $f \in (\widehat{S_{m,n}})^\circ$ need not belong to $B\langle\xi\rangle[\![\rho]\!]$ for any $B \in \mathfrak{B}$, there is an increasing sequence $B^{(0)} \subset B^{(1)} \subset \ldots$ from $\mathfrak{B}$ and $f^{(i)} \in B^{(i)}\langle\xi\rangle[\![\rho]\!]$ such that $\|f - f^{(i)}\| \to 0$.

Example 2.1.7. — We give an example of a $\mathfrak{B}$, as in Remark 2.1.6(ii), such that there is no E with $S_{m,n}(\mathfrak{B},K) \subset S_{m,n}(E,K)$. Consider $F = \mathbb{F}_p(t_1,t_2,\ldots)(z)$ with absolute value derived from the (z)-adic valuation and let K be the completion of the algebraic closure of F. Let $\{\alpha_i\}$ be a sequence of positive rationals converging to zero, and define inductively

$$\begin{aligned} E_0 &:= \mathbb{F}_p(t_i + z^{\alpha_i}, i \in \mathbb{N}) \\ E_i &:= \left(E_{i-1}[t_i^{p^{-n}} : n \in \mathbb{N}]_{\{a\in E_{i-1}[t_i^{p^{-n}}\,:\,n\in\mathbb{N}]:\,|a|=1\}}\right)\widehat{\ } \ . \end{aligned}$$

Let $\mathfrak{B}_i$ be the family of all quasi-Noetherian rings of the form

$$\left(E_i[a_0,a_1,\ldots]_{\{a\in E_i[a_0,a_1\ldots]:|a|=1\}}\right)\widehat{\ }$$

where $\{a_i\}_{i\in\mathbb{N}}$ is a null sequence from K°, and let

$$\mathfrak{B} := \cup_i \mathfrak{B}_i.$$

We will show that for $n > 0$ there is no complete DVR $E \subset K^\circ$ such that $S_{m,n}(\mathfrak{B}, K) \subset S_{m,n}(E, K)$. Suppose that $S_{m,n}(\mathfrak{B}, K) \subset S_{m,n}(E, K)$. Since K is algebraically closed, by Remark 2.1.4 we may assume that $E \subset K^\circ$ is a field and that $\widetilde{E} = \widetilde{K}$.

Note that E_i has a countable dense subset $\{c_0, c_1, \dots\}$. Hence

$$\sum c_i \rho_1^i \in S_{m,n}(\mathfrak{B}, K) \subset S_{m,n}(E, K).$$

Therefore for each $i \in \mathbb{N}$ there is a zero sequence $\{a_1, a_2, \dots\}$ from K° such that

$$E_i \subset \left(E[a_1, a_2 \dots]_{\{a \in E[a_1, a_2, \dots] : |a| = 1\}}\right)^\frown =: E_i'.$$

Since $\widetilde{E} = \widetilde{K}$ we may assume that $|a_j| < 1$ for all j. Since $t_i^{p^{-n}} \in E_i$, there are $e_{nj} \in E_i'$ with $e_{n0} \in E$ such that

$$t_i^{p^{-n}} = e_{n0} + \sum_{j=1}^{\infty} e_{nj} a_j.$$

Then

$$t_i = e_{n0}^{p^n} + \sum_{j=1}^{\infty} e_{nj}^{p^n} a_j^{p^n}.$$

Since $|a_j| < 1$ for all j, we see that the sequence $e_{n0}^{p^n}$ converges to t_i. Since $E \subset K^\circ$ is a field the absolute value is trivial on E and hence $t_i \in E$. The quasi-Noetherian ring E_0' contains both E and E_0. Thus it contains the elements z^{α_i}, $i \in \mathbb{N}$. Since $|z^{\alpha_i}| = p^{-\alpha_i}$ this contradicts the discreteness of the value semigroup of E_0'. One can construct a similar counterexample in characteristic zero.

Remark 2.1.8. — We will use the term **affinoid** to refer to objects defined over the Tate rings and the term **quasi-affinoid** to refer to objects defined over rings of separated power series. Hence, for example, an **affinoid algebra** is a quotient of a T_m and a **quasi-affinoid algebra** is a quotient of an $S_{m,n}$.

2.2. Noetherianness. — In this subsection, we lift the Noetherian property of the residue rings $\widetilde{S}_{m,n}$ to the $S_{m,n}$ by lifting generators of ideals. This also yields the property that ideals of $S_{m,n}$ are strictly closed in $\|\cdot\|$, a property that will be further analyzed in Subsection 3.1.

Lemma 2.2.1. — *Suppose $A = \varinjlim A_\lambda$ is a Noetherian ring which is the direct limit of the rings A_λ. Put $\mathcal{A} := \varinjlim A_\lambda[[\rho]] \subset A[[\rho]]$. The following are equivalent:*

(i) *$\mathcal{A}$ is Noetherian.*
(ii) *$A[[\rho]]$ is a flat $\mathcal{A}$-algebra.*

(iii) *$A[[\rho]]$ is a faithfully flat $\mathcal{A}$-algebra.*
(iv) *Each ideal of $\mathcal{A}$ is closed in the (ρ)-adic topology.*

If each A_λ is Noetherian and if for every λ there is some $\mu \geq \lambda$ such that A is a flat A_μ-algebra, then $A[[\rho]]$ is a flat $\mathcal{A}$-algebra.

Proof. — It is no loss of generality to assume that each $A_\lambda \subset A$. We first show (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (i).

(i) $\Rightarrow$ (ii). Let I be the ideal of $\mathcal{A}$ generated by the variables $\rho_1, \ldots, \rho_n$. Since $A[[\rho]]$ is Noetherian and since $I \cdot A[[\rho]]$ is contained in the Jacobson radical of $A[[\rho]]$, $A[[\rho]]$ is I-adically ideal separated as an $\mathcal{A}$-module. Since for every $\ell \in \mathbb{N}$

$$\mathcal{A}/I^\ell = A[[\rho]]/(\rho)^\ell,$$

(ii) follows from (i) by the Local Flatness Criterion ([**25**, Theorem 22.3]).

(ii) $\Rightarrow$ (iii). Let I be any ideal of $\mathcal{A}$; then $I \cdot A[[\rho]]$ is the unit ideal if, and only if, for some $f_1, \ldots, f_\ell \in I$ and $\alpha_1, \ldots, \alpha_\ell \in A$, the constant term of $\sum \alpha_i f_i$ is a unit. The latter condition holds if, and only if, I generates the unit ideal of $\mathcal{A}$. Therefore (iii) follows from (ii) by [**25**, Theorem 7.2].

(iii) $\Rightarrow$ (iv). Since $A[[\rho]]$ is Noetherian and since $(\rho) \cdot A[[\rho]]$ is contained in the Jacobson radical, each ideal of $A[[\rho]]$ is closed in the (ρ)-adic topology by the Krull Intersection Theorem ([**25**, Theorem 8.10 (i)]). Let I be any ideal of $\mathcal{A}$; then the (ρ)-adic closure of I in $\mathcal{A}$ is equal to $I \cdot A[[\rho]] \cap \mathcal{A}$. Hence to prove (iv), we must show that $I = I \cdot A[[\rho]] \cap \mathcal{A}$. If $A[[\rho]]$ is faithfully flat over $\mathcal{A}$, this follows from [**25**, Theorem 7.5].

(iv) $\Rightarrow$ (i). Let I be an ideal of $\mathcal{A}$. Since $A[[\rho]]$ is Noetherian, there are finitely many elements $f_1 \ldots, f_\ell$ of I which generate the ideal $I \cdot A[[\rho]]$. Let J be the ideal of $\mathcal{A}$ generated by $f_1, \ldots, f_\ell$. To prove (i), we show that $J = I$. If each ideal of $\mathcal{A}$ is closed in the (ρ)-adic topology, then, as above,

$$\begin{aligned} I &= \mathcal{A} \cap I \cdot A[[\rho]] \\ &= \mathcal{A} \cap J \cdot A[[\rho]] \\ &= J, \end{aligned}$$

proving (i).

Now suppose that each A_λ is Noetherian and that for every λ there is some $\mu \geq \lambda$ such that A is a flat A_μ-algebra. We show that $A[[\rho]]$ is a flat $\mathcal{A}$-algebra.

If A is a flat A_μ-algebra then

$$A[\rho] = A \otimes_{A_\mu} A_\mu[\rho]$$

is a flat $A_\mu[\rho]$-algebra. Since, in addition, A is Noetherian, by the Artin-Rees Lemma ([**25**, Theorem 8.6]), the $A_\mu[\rho]$-module $A[\rho]$ is (ρ)-adically ideal-separated. Since $A_\mu[\rho]$ is Noetherian, by the Local Flatness Criterion ([**25**], Theorem 22.3), for every $\ell \in \mathbb{N}$, $A[\rho]/(\rho)^\ell$ is a flat $A_\mu[\rho]/(\rho)^\ell$-algebra. Since

$A[[\rho]]/(\rho)^\ell = A[\rho]/(\rho)^\ell$ and $A_\mu[[\rho]]/(\rho)^\ell = A_\mu[\rho]/(\rho)^\ell$, and since $\rho_1,\dots,\rho_n$ are contained in the Jacobson radical of $A[[\rho]]$, by another application of the local flatness criterion, $A[[\rho]]$ is a flat $A_\mu[[\rho]]$-algebra. To show that $A[[\rho]]$ is a flat $\mathcal{A}$-algebra, we use [**25**], Theorem 7.6. Suppose $f_1,\dots,f_\ell \in \mathcal{A}$; then for some μ such that $A[[\rho]]$ is a flat $A_\mu[[\rho]]$-algebra, $f_1,\dots,f_\ell \in A_\mu[[\rho]]$. Suppose, furthermore, for some $g_1,\dots,g_\ell \in A[[\rho]]$ that $\sum g_i f_i = 0$. Since $A[[\rho]]$ is a flat $A_\mu[[\rho]]$-algebra, there are $r \in \mathbb{N}$, $\varphi_{ij} \in A_\mu[[\rho]]$ and $\gamma_j \in A[[\rho]]$, $1 \le i \le \ell$, $1 \le j \le r$, such that

$$\sum_i f_i\varphi_{ij} = 0 \text{ for all } j, \text{ and } g_i = \sum_j \varphi_{ij}\gamma_j \text{ for all } i.$$

Since $A_\mu[[\rho]] \subset \mathcal{A}$, it follows immediately that $A[[\rho]]$ is a flat $\mathcal{A}$-algebra. □

The following is an immediate consequence of Lemma 2.2.1, taking the A_λ to be the $\widetilde{B}[\xi]$, $B \in \mathfrak{B}$.

Corollary 2.2.2. — *The residue rings $\widetilde{S}_{m,n}$ are Noetherian; each ideal of $\widetilde{S}_{m,n}$ is closed in the (ρ)-adic topology.*

The next lemma allows us to lift generators of an ideal $\widetilde{I}$ of $\widetilde{S}_{m,n}$ to generators of the ideal I of $S_{m,n}$.

Lemma 2.2.3. — *Let $I \subset S_{m,n}$ be an ideal and let $g_1,\dots,g_r \in I^\circ$ be such that $\{\widetilde{g}_1,\dots,\widetilde{g}_r\}$ generates $\widetilde{I}$. Let $f \in S^\circ_{m,n}$ and choose $B \in \mathfrak{B}$ such that $f, g_1,\dots,g_r \in B\langle\xi\rangle[[\rho]]$. Suppose that $\|f-h\| < \|f\|$ for some $h \in I$. Then there are $f_1,\dots,f_r \in B\langle\xi\rangle[[\rho]]$ with*

$$\left\| f - \sum_{i=1}^r f_i g_i \right\| < \|f\| \tag{2.2.1}$$

and $\|f\| = \max_{1\le i\le r} \|f_i\|$.

Proof. — Let $B = B_0 \supset B_1 \supset \cdots$ be the natural filtration of B, and suppose

$$f \in B_p\langle\xi\rangle[[\rho]] \setminus B_{p+1}\langle\xi\rangle[[\rho]].$$

Find $b_p \in B$ with $B_p = \{b \in B : |b| \le |b_p|\}$, let $\pi_p : B_p \to \widetilde{B}_p \subset \widetilde{K}$ be the B-module residue epimorphism $a \mapsto (b_p^{-1}a)^\sim$, and write

$$\widetilde{K} = \widetilde{B}_p \oplus V$$

for some $\widetilde{B}$-vector space V. This implies that

$$\widetilde{K}[\xi][[\rho]] = \widetilde{B}_p[\xi][[\rho]] \oplus V[\xi][[\rho]] \tag{2.2.2}$$

as $\widetilde{B}[\xi][[\rho]]$-modules. (This useful decomposition can be found in [**14**].)

Since $\|f - h\| < \|f\|$ for some $h \in I$, we have $\pi_p(f) \in \widetilde{I}$. Since $\widetilde{g}_1, \ldots, \widetilde{g}_r$ generate $\widetilde{I}$, we have

$$\pi_p(f) = \sum_{i=1}^{r} \widetilde{f}_i \widetilde{g}_i \in \widetilde{B}_p[\xi][\![\rho]\!]$$

for some $\widetilde{f}_1, \ldots, \widetilde{f}_r \in \widetilde{K}[\xi][\![\rho]\!]$. By (2.2.2), we may assume $\widetilde{f}_1, \ldots, \widetilde{f}_r \in \widetilde{B}_p[\xi][\![\rho]\!]$. Thus there are $f_1, \ldots, f_r \in B_p\langle\xi\rangle[\![\rho]\!]$ corresponding to $\widetilde{f}_1, \ldots, \widetilde{f}_r$ under the residue map π_p. Clearly, $\|f - \sum f_j g_j\| < \|f\|$. □

Since each $B \in \mathfrak{B}$ has discrete value semigroup and since $B\langle\xi\rangle[\![\rho]\!]$ is complete in $\|\cdot\|$, Lemma 2.2.3 implies that the separated power series rings are Noetherian.

Corollary 2.2.4. — (cf. [**17**], Proposition 2.6.2.) *The rings $S_{m,n}$ are Noetherian. Indeed, let $I \subset S_{m,n}$ be an ideal and suppose the residues of $g_1, \ldots, g_r \in I^\circ$ generate $\widetilde{I}$ in $\widetilde{S}_{m,n}$. Then for every $f \in I$ there are $f_1, \ldots, f_r \in S_{m,n}$ with*

$$f = \sum_{i=1}^{r} f_i g_i,$$

and $\|f\| = \max_{1 \le i \le r} \|f_i\|$. Moreover, if for some $B \in \mathfrak{B}$, $f, g_1, \ldots, g_r \in B\langle\xi\rangle[\![\rho]\!]$, then $f_1, \ldots, f_r$ may also be taken to lie in $B\langle\xi\rangle[\![\rho]\!]$.

In fact, Lemma 2.2.3 yields the slightly stronger result, Corollary 2.2.6.

Definition 2.2.5. — (cf. [**6**, Definition 1.1.5.1].) Let (A, v) be a multiplicatively valued ring. An ideal I of A is called **strictly closed** in v iff for every $f \in A$ there is some $g \in I$ such that $v(f - g) \le v(f - h)$ for every $h \in I$.

Corollary 2.2.6. — *Ideals of $S_{m,n}$ are strictly closed in $\|\cdot\|$. Indeed, let $I \subset S_{m,n}$ be an ideal and suppose the residues of $g_1, \ldots, g_r \in I^\circ$ generate $\widetilde{I}$ in $\widetilde{S}_{m,n}$. Then for every $f \in S_{m,n}$ there are $f_1, \ldots, f_r \in S_{m,n}$ with*

$$\left\| f - \sum_{i=1}^{r} f_i g_i \right\| \le \|f - h\|$$

for every $h \in I$, and $\|f\| \ge \max_{1 \le i \le r} \|f_i\|$. Moreover, if for some $B \in \mathfrak{B}$, $f, g_1, \ldots, g_r \in B\langle\xi\rangle[\![\rho]\!]$, then $f_1, \ldots, f_r$ may be taken to lie in $B\langle\xi\rangle[\![\rho]\!]$.

Taking $n = 0$ in the above, we obtain [**6**, Corollary 5.2.7.8].

In Subsection 3.1, we will be interested in some refinements of Corollary 2.2.6.

Definition 2.2.7. — Let I be an ideal of $S_{m,n}$. For $f \in S_{m,n}$, we define the **residue norm**

$$\|f\|_I := \inf\{\|f-h\| : h \in I\}.$$

From Corollary 2.2.6, it follows that there is some $h \in I$ such that $\|f\|_I = \|f-h\|$.

The direct sum $(S_{m,n}/I)^\ell$ is a normed $(S_{m,n}/I)$-module via

$$\|\cdot\|_I \colon (S_{m,n}/I)^\ell \to \mathbb{R}_+ : (f_1, \dots, f_\ell) \mapsto \max_{1 \le i \le \ell} \|f_i\|_I.$$

We will be concerned with submodules M of $(S_{m,n}/I)^\ell$, which will be endowed with the norm $\|\cdot\|_I$. Residue modules play an important role.

Definition 2.2.8. — Let $(M, |\cdot|)$ be a normed K-module. By M° and $M^{\circ\circ}$ denote, respectively, the K°-modules

$$M^\circ : = \{f \in M \colon |f| \le 1\} \quad \text{and}$$
$$M^{\circ\circ} := \{f \in M \colon |f| < 1\}.$$

We define the **residue module** $\widetilde{M}$ by

$$\widetilde{M} := M^\circ / M^{\circ\circ}.$$

It is a $\widetilde{K}$-module.

From Corollary 2.2.6, it follows that

$$(S_{m,n}/I)^\circ = S^\circ_{m,n}/I^\circ \text{ and } (S_{m,n}/I)^\sim = \widetilde{S}_{m,n}/\widetilde{I}.$$

2.3. Weierstrass Division Theorems. — We recall in Theorem 2.3.2 the Weierstrass Division Theorems for the rings $S_{m,n}$ (see [**16**] and [**17**]) in the form given in [**2**, Section 1.2]. These will be used in Section 4 and extensively in Section 5. In Theorem 2.3.8, we prove an extension of these division theorems to handle Weierstrass divisors with coefficients in a quasi-affinoid algebra. The statement and proof of Theorem 2.3.8 rely on results of Sections 4 and 5, but the theorem itself is only used in Section 6 and in [**23**].

Definition 2.3.1. — (cf. [**17**, Sections 2.3 and 2.4].) An element $f \in \widetilde{S}_{m,n}$ is **regular in ξ_m of degree** s iff for some $c \in \widetilde{K}$, cf is congruent modulo $(\rho) \cdot \widetilde{S}_{m,n}$ to a monic polynomial in ξ_m of degree s. An element $f \in \widetilde{S}_{m,n}$ is **regular in ρ_n of degree** s iff $f(\xi, 0, \dots, 0, \rho_n) = \rho_n^s \cdot g(\xi, \rho_n)$ for some unit $g \in \widetilde{K}[\xi][\![\rho_n]\!]$. An element $f \in S_{m,n} \setminus \{0\}$ is **regular of degree** s **in** ξ_m (respectively, ρ_n) iff for some $c \in K$, $(cf)^\sim \in \widetilde{S}_{m,n}$ is regular of degree s in ξ_m (respectively, ρ_n).

The formal power series ring $\widetilde{B}[\xi][\![\rho]\!]$, whence $\widetilde{S}_{m,n}$, has the usual local Weierstrass Division Theorem for elements regular in ρ, as in [**41**, Theorem VI.1.5]. As in [**1**, Section 2.2] or in [**17**, Proposition 2.4.1], this lifts to the complete, linearly topologized ring $B\langle\xi\rangle[\![\rho]\!]$. As explained in Subsection 2.1, $\widetilde{B}[\xi][\![\rho]\!]$ is equal to the strictly convergent power series ring $\widetilde{B}[\![\rho]\!]\langle\xi\rangle$. The Euclidean Division Theorem for $\widetilde{B}[\xi]$ lifts to a Weierstrass Division Theorem in $\widetilde{B}[\![\rho]\!]\langle\xi\rangle$ for elements regular in ξ, as in [**6**, Theorem 5.2.1.2]. This may be lifted to $B\langle\xi\rangle[\![\rho]\!]$ as in [**17**, Proposition 2.3.1], or as in [**2**, Section 1.2], using the Hensel's Lemma of [**8**, Section 4]. This yields the following theorem.

Theorem 2.3.2. — **(Weierstrass Division Theorem,** cf. [**17**, Propositions 2.3.1 and 2.4.1].**)** *Let $f, g \in S^\circ_{m,n}$ with $\|f\| = 1$.*

(i) *If f is regular in ξ_m of degree s, then there exist unique $q \in S^\circ_{m,n}$ and $r \in S^\circ_{m-1,n}[\xi_m]$ of degree at most $s-1$ such that $g = qf + r$. If $g \in I \cdot S^\circ_{m,n}$ for some (closed) ideal I of $S^\circ_{m-1,n}$, then $q, r \in I \cdot S^\circ_{m,n}$.*

(ii) *If f is regular in ρ_n of degree s, then there exist unique $q \in S^\circ_{m,n}$ and $r \in S^\circ_{m,n-1}[\rho_n]$ of degree at most $s-1$ such that $g = qf + r$. If $g \in I \cdot S^\circ_{m,n}$ for some (closed) ideal I of $S^\circ_{m,n-1}$, then $q, r \in I \cdot S^\circ_{m,n}$.*

Moreover, if $f, g \in B\langle\xi\rangle[\![\rho]\!]$ for some $B \in \mathfrak{B}$, also, $q, r \in B\langle\xi\rangle[\![\rho]\!]$.

Dividing ξ_m^s (or ρ_n^s) by an element $f \in S_{m,n}$ regular in ξ_m (or ρ_n) of degree s, we obtain the following corollary.

Corollary 2.3.3. — **(Weierstrass Preparation Theorem)** *Let $f \in S^\circ_{m,n}$ with $\|f\| = 1$.*

(i) *If f is regular in ξ_m of degree s, then there exist a unique unit u of $S^\circ_{m,n}$ and a unique monic polynomial $P \in S^\circ_{m-1,n}[\xi_m]$ of degree s such that $f = u \cdot P$; in addition, P is regular in ξ_n of degree s.*

(ii) *If f is regular in ρ_n of degree s, then there exist a unique unit u of $S^\circ_{m,n}$ and a unique monic polynomial $P \in S^\circ_{m,n-1}[\rho_n]$ of degree s such that $f = u \cdot P$; in addition, P is regular in ρ_n of degree s.*

Moreover, if $f \in B\langle\xi\rangle[\![\rho]\!]$ for some $B \in \mathfrak{B}$, also $u, P \in B\langle\xi\rangle[\![\rho]\!]$.

Unlike the rings $B[\![\xi, \rho]\!]$ and $B\langle\xi, \rho\rangle$, there may be no automorphism of $S_{m,n}$ under which a given element f with $\|f\| = 1$ becomes regular (see Example 2.3.5).

Definition 2.3.4. — (cf. [**17**, Section 3.12].) An element $f = \sum f_\mu(\rho)\xi^\mu \in \widetilde{S}_{m,n}$ is **preregular in ξ of degree μ_0** iff $f_{\mu_0} \not\equiv 0$ modulo $(\rho) \cdot \widetilde{S}_{m,n}$ and $f_\mu \equiv 0$ modulo $(\rho) \cdot \widetilde{S}_{m,n}$ for all μ lexicographically larger than μ_0. An element $f = \sum f_\nu(\xi)\rho^\nu \in \widetilde{S}_{m,n}$ is **preregular in ρ of degree ν_0** iff $f_{\nu_0} \in \widetilde{K} \setminus \{0\}$ and for all lexicographically smaller indices ν, $f_\nu = 0$. An element $f \in S_{m,n} \setminus \{0\}$

is **preregular in** ξ **of degree** μ_0 (respectively, **in** ρ **of degree** ν_0)) iff for some $c \in K$, $(cf)^\sim \in \widetilde{S}_{m,n}$ is preregular of the same degree.

If f is preregular in ξ (respectively, ρ) then after an automorphism of the form $\rho \mapsto \rho$, $\xi_m \mapsto \xi_m$, $\xi_i \mapsto \xi_i + \xi_m^{c_i}$ (respectively, $\xi \mapsto \xi$, $\rho_n \mapsto \rho_n$, $\rho_j \mapsto \rho_j + \rho_n^{c_j}$) f becomes regular in ξ_m (respectively, ρ_n) of some degree s. Such automorphisms are called **Weierstrass automorphisms.**

Example 2.3.5. — The element $\xi \cdot \rho \in S_{1,1}$ is not preregular. Indeed, there is no finite monomorphism $S_{m,n} \to S_{1,1}/(\xi\rho)$ for any $m, n \in \mathbb{N}$. Since the map

$$S_{1,0} \oplus S_{0,1} \to S_{1,1}/(\xi\rho) \colon (f,g) \mapsto f+g$$

is surjective and $\dim S_{1,0} = \dim S_{0,1} = 1$ (see Corollary 4.2.2), we must have $\dim(S_{1,1}/(\xi\rho)) = 1$. Thus, if there were a finite monomorphism

$$\varphi : S_{m,n} \to S_{1,1}/(\xi\rho),$$

either $m = 1$ and $n = 0$, or $m = 0$ and $n = 1$. We treat the case $m = 1$ and $n = 0$. Let

$$\alpha : S_{1,1}/(\xi\rho) \to S_{0,1} = S_{1,1}/(\xi\rho, \xi)$$

be the canonical projection. Since α is surjective,

$$\alpha \circ \varphi : S_{1,0} \to S_{0,1}$$

is finite. Since $\dim S_{0,1} = 1$, $\alpha \circ \varphi$ must be injective. By [**6**, Proposition 3.8.1.7], we can reduce modulo $K^{\circ\circ}$ to obtain a finite $\widetilde{K}$-algebra homomorphism

$$(\alpha \circ \varphi)^\sim : \widetilde{K}[\xi] \to \widetilde{S}_{0,1}.$$

But such a map cannot exist, since the transcendence degree of $\widetilde{S}_{0,1}$ over $\widetilde{K}$ is infinite.

Remark 2.3.6. — For every nonzero $f \in S_{0,n}$, there is a Weierstrass automorphism of $S_{0,n}$ under which f becomes regular in ρ_n of some degree. Therefore, arguing as in [**6**, Theorem 6.1.2.1], one proves the following version of Noether Normalization: Let d be the Krull dimension of $S_{0,n}/I$; then there is a finite K-algebra monomorphism $\varphi : S_{0,d} \to S_{0,n}/I$.

In Definition 5.2.7, we will define the ring $A\langle\xi\rangle[\![\rho]\!]_s \subset A\,[\![\xi, \rho]\!]$ of separated power series with coefficients in a quasi-affinoid algebra A. Using the results of Subsection 5.2, we state and prove here relative Weierstrass Division Theorems for such rings. These theorems will be used only in Section 6 and in [**23**].

Definition 2.3.7. — Let A be a quasi-affinoid algebra. By the Extension Lemma, Theorem 5.2.6, for each $x \in \operatorname{Max} A$, there is a unique homomorphism

$$\varepsilon_x : A\langle\xi_1, \ldots, \xi_m\rangle[\![\rho_1, \ldots \rho_n]\!]_s \to S_{m,n}(E, A/x)$$

extending the map $A \to A/x$ and preserving the variables ξ and ρ. An element $f \in A\langle\xi\rangle[\![\rho]\!]_s$ is **regular in** ξ_m (respectively, ρ_m) **of degree** s iff for each $x \in \operatorname{Max} A$, $\varepsilon_x(f) \in S_{m,n}(E, A/x)$ is regular in ξ_m (respectively, ρ_n) of degree s. **Preregular** elements are defined similarly.

Theorem 2.3.8. — **(Weierstrass Division Theorem)** *Let A be a quasi-affinoid algebra, and let $f, g \in A\langle\xi\rangle[\![\rho]\!]_s$.*

(i) *If f is regular in ξ_m of degree s, then there exist unique $q \in A\langle\xi\rangle[\![\rho]\!]_s$ and $r \in A\langle\xi'\rangle[\![\rho]\!]_s[\xi_m]$ of degree at most $s-1$ such that $g = qf + r$ (where $\xi' = (\xi_1, \ldots, \xi_{m-1})$.)*

(ii) *If f is regular in ρ_n of degree s, then there exist unique $q \in A\langle\xi\rangle[\![\rho]\!]_s$ and $r \in A\langle\xi\rangle[\![\rho']\!]_s[\rho_n]$ of degree at most $s-1$ such that $g = qf + r$ (where $\rho' = (\rho_1, \ldots, \rho_{n-1})$.)*

Proof. — (i) *Existence.* Write

$$f = \sum_{\mu,\nu} a_{\mu\nu}\xi^\mu\rho^\nu = \sum_{i\geq 0} f_i\xi_m^i.$$

Since f is regular in ξ_m of degree s, for each $x \in \operatorname{Max} A$, $\varepsilon_x(f_s)$ is a unit of $S_{m-1,n}(E, A/x)$. It follows by the Nullstellensatz, Theorem 4.1.1, that f_s is a unit of $A\langle\xi\rangle[\![\rho]\!]_s$. Since $\varepsilon_x(f_s^{-1}) \cdot \varepsilon_x(f)$ is regular in ξ_m of degree s for each $x \in \operatorname{Max} A$, we may therefore take $f_s = 1$. It follows that

$$\varepsilon_x(f_i) \in S^\circ_{m-1,n}(E, A/x), \quad i < s,$$

and

$$\varepsilon_x(f_i) \in (\rho)S^\circ_{m-1,n}(E, A/x) + S^{\circ\circ}_{m-1,n}(E, A/x), \quad i > s,$$

for every $x \in \operatorname{Max} A$. By Corollary 5.1.8, f_i is power-bounded for $i < s$ and f_i is quasi-nilpotent for $i > s$.

Write $A = S_{m',n'}/I$ and consider the canonical projection

$$\varphi : S_{m+m',n+n'} \to A\langle\xi\rangle[\![\rho]\!]_s$$

modulo $I \cdot S_{m+m',n+n'}$. Let

$$F = \sum F_i\xi_m^i$$

be a preimage of f, where each $F_i \in S_{m-1+m',n+n'}$. By Lemma 3.1.6, there is an r so that for $i > s$,

$$F_i = \sum_{j=1}^{r} H_{ij}F_{s+j},$$

where $\|H_{i1}\|, \ldots, \|H_{ir}\| \leq 1$.

By the Extension Lemma, Theorem 5.2.6, there is a K-algebra homomorphism ψ such that

$$\begin{tikzcd} S_{m+m',n+n'} \arrow[r, hook] \arrow[d, "\varphi"'] & S_{m+m'+s,n+n'+r} \arrow[dl, "\psi"] \\ A\langle\xi\rangle[\![\rho]\!]_s & \end{tikzcd}$$

commutes, and

$$\psi(\xi_i) = \xi_i, \quad 1 \le i \le m; \qquad \psi(\xi_{m+i}) = \varphi(\xi_{m+i}), \quad 1 \le i \le m';$$
$$\psi(\xi_{m+m'+i}) = f_{i-1}, \quad 1 \le i \le s,$$

and

$$\psi(\rho_i) = \rho_i, \quad 1 \le i \le n; \qquad \psi(\rho_{n+i}) = \varphi(\rho_{n+i}), \quad 1 \le i \le n';$$
$$\psi(\rho_{n+n'+i}) = f_{s+i}, \quad 1 \le i \le r.$$

Note that f is the image under ψ of

$$f^* := \sum_{i=0}^{s-1} \xi_{m+m'+i+1}\xi_m^i + \xi_m^s + \sum_{i>s} \xi_m^i \left(\sum_{j=1}^{r} H_{ij}\rho_{n+n'+j} \right)$$

and $f^* \in S_{m+m'+s,n+n'+r}$ is regular in ξ_m of degree s.

Let $G \in S_{m+m'+s,n+n'+r}$ be a preimage of g under ψ. By Theorem 2.3.2, there are unique $Q \in S_{m+m'+s,n+n'+r}$ and $R \in S_{m-1+m'+s,n+n'+r}[\xi_m]$ of degree at most $s-1$ with

$$G = Qf^* + R.$$

Putting $q = \psi(Q)$ and $r = \psi(R)$ satisfies the existence assertion of part (i).

Uniqueness. Let $q \in A\langle\xi\rangle[\![\rho]\!]_s$ and let $r \in A\langle\xi'\rangle[\![\rho]\!]_s[\xi_m]$ be of degree at most $s-1$. Suppose

$$0 = qf + r;$$

we must show that $q = r = 0$. Let $Q \in S_{m+m'+s,n+n'+r}$ and $R \in S_{m-1+m'+s,n+n'+r}[\xi_m]$ with $\deg R \le s-1$ be preimages under ψ of q and r, respectively. Then

$$G := Qf^* + R \in \operatorname{Ker}\psi = I \cdot S_{m+m'+s,n+n'+r}.$$

The ideal I is closed by Corollary 2.2.6; hence by Theorem 2.3.2 (i), $Q, R \in \operatorname{Ker}\psi$, as desired.

(ii) The proof of this part is entirely analogous to the above. □

The corresponding Weierstrass Preparation Theorem follows in the usual way.

Corollary 2.3.9. — **(Weierstrass Preparation Theorem)** *Let A be a quasi-affinoid algebra, and let $f \in A\langle\xi\rangle[\![\rho]\!]_s$.*

(i) *If f is regular in ξ_m of degree s, then there exist unique unit $u \in A\langle\xi\rangle[\![\rho]\!]_s$ and monic polynomial $P \in A\langle\xi'\rangle[\![\rho]\!][\xi_m]$ of degree s such that $f = uP$. Furthermore P is regular in ξ_m of degree s.*

(ii) *If f is regular in ρ_n of degree s, then there exist unique unit $u \in A\langle\xi\rangle[\![\rho]\!]_s$ and monic polynomial $P \in A\langle\xi\rangle[\![\rho']\!]_s[\rho_n]$ of degree s such that $f \in uP$. Furthermore P is regular in ρ_n of degree s.*

3. Restrictions to Polydiscs

In this section, we study the restriction maps from $\Delta_{m,n}$ (see Introduction) to "closed" (and to "open") sub-polydiscs, and show how to transfer information from their (quasi-)affinoid function algebras back to $S_{m,n}$.

The closed subpolydiscs with which we are concerned in this section are Cartesian products where the first m factors are closed unit discs and the next n factors are closed discs of radius $\varepsilon \in \sqrt{|K \setminus \{0\}|}$. Such products are K-affinoid varieties, and we denote their corresponding rings of K-affinoid functions by $T_{m,n}(\varepsilon, K)$.

To transfer algebraic information from the affinoid algebras $T_{m,n}(\varepsilon)$ to $S_{m,n}$, we analyze the metric behavior of the inclusions $\iota_\varepsilon : S_{m,n} \hookrightarrow T_{m,n}(\varepsilon)$ as $\varepsilon \to 1$. We carry out our computations by reducing to the case that $\varepsilon \in |K \setminus \{0\}|$. In the case that K is discretely valued, this entails working with certain algebraic extensions K' of K and understanding the inclusion $S_{m,n}(E, K) \hookrightarrow S_{m,n}(E, K')$. The reader interested only in the case that K is algebraically closed may omit the complications arising from field extensions.

We are interested in studying properties of quotient rings $S_{m,n}/I$. We study such quotient rings by studying metric properties (e.g., pseudo-Cartesian and strict) of generating systems of submodules of $(S_{m,n})^\ell$, and how they transform under restriction maps to rational sub-polydiscs.

In Subsection 3.1, we introduce metric properties of generating systems of submodules of $(S_{m,n})^\ell$ and of $(\widetilde{S}_{m,n})^\ell$. In particular we introduce a valuation, the total value v, on $S_{m,n}$ which lifts the (ρ)-adic valuation on $\widetilde{S}_{m,n}$ and refines the Gauss norm on $S_{m,n}$. This allows us to formulate the "slicing" arguments whereby (ρ)-adic properties of $\widetilde{S}_{m,n}$ are seen to lift to $S_{m,n}$. The valuations $\|\cdot\|$ and v induce norms $\|\cdot\|_M$ and v_M on a quotient module $(S_{m,n})^\ell/M$. We prove a number of estimates.

In Subsection 3.2, we study restrictions to closed subpolydiscs. The main result is Theorem 3.2.3, which says that if ε is suitably large, then a strict generating system remains strict under restriction.

In Subsection 3.3, we transfer information from $T_{m,n}(\varepsilon)$ back to $S_{m,n}$. The main results are Theorem 3.3.1 and its corollaries, which show, roughly speaking, how to replace powers of ε with powers of (ρ) for ε near 1. More precisely, they establish a key relation between v_M and $\|\cdot\|_{\iota_\varepsilon(M)\cdot T_{m,n}(\varepsilon)}$ uniformly in ε for ε suitably large, which is used extensively in the rest of this paper. This is how we overcome the difficulties stemming from the failure of Noether normalization for $S_{m,n}$.

In Subsection 3.4 we study restrictions from $\Delta_{m,n}$ to certain disjoint unions of open subpolydiscs. When the centers of the polydiscs are K-rational, these maps have the form $\varphi : S_{m,n} \to \oplus_{j=0}^{r} S_{0,n+m}$. In the case of non-$K$-rational centers, the restriction maps are only slightly more complicated. We show in Theorems 3.4.3 and 3.4.6 that such restrictions are isometries in the residue norms derived from $\|\cdot\|$ and respectively I and $\varphi(I)$, provided the finite collection of open polydiscs is chosen appropriately. Theorems 3.4.3 and 3.4.6 will be used in Subsection 5.5 to derive the fact that on certain reduced quotients $S_{m,n}/I$, the residue and supremum norms are equivalent from the simpler case of reduced quotients $S_{0,n+m}/I$.

3.1. Strict and Pseudo-Cartesian Modules. — We introduce metric properties of generating systems of submodules of $(S_{m,n})^\ell$ and $(\widetilde{S}_{m,n})^\ell$ and their quotients. We introduce a valuation, the total value v, on $S_{m,n}$ which lifts the (ρ)-adic valuation on $\widetilde{S}_{m,n}$ and refines the Gauss norm on $S_{m,n}$. The lemmas of this subsection show how certain metric properties of generating systems of modules lift from residue modules and transform under maps and ground field extension.

Let (A, v) be a multiplicatively valued ring, and let (N, w) be a normed A-module; i.e.,

$$w(an) \leq v(a)w(n)$$

for all $a \in A$, $n \in N$. Let M be an A-submodule of N. A finite generating system $\{g_1, \dots, g_r\}$ of M is called w-**strict** iff for all $f \in N$ there exist $a_1, \dots, a_r \in A$ such that

$$\begin{aligned} & w(f) \geq \max_{1\leq i\leq r} v(a_i)w(g_i), \text{ and} \\ & w\left(f - \sum_{i=1}^{r} a_i g_i\right) \leq w(f-h) \text{ for all } h \in M. \end{aligned} \tag{3.1.1}$$

The generating system $\{g_1, \dots, g_r\}$ is called w-**pseudo-Cartesian** iff (3.1.1) is only assumed to hold for all $f \in M$; i.e., iff for all $f \in M$ there exist $a_1, \dots, a_r \in A$ such that

$$w(f) \geq \max_{1\leq i\leq r} v(a_i)w(g_i), \text{ and}$$

$$f = \sum_{i=1}^{r} a_i g_i.$$

An A-module $M \subset N$ is called w-**strict** (w-**pseudo-Cartesian**) iff it has a w-strict (w-pseudo-Cartesian) generating system. Usually, N will be a quotient of the ℓ-fold norm-direct sum of $S_{m,n}$.

Along with the Gauss norm, we will be interested primarily in two other valuations. One, the residue order, is a rank-one additive valuation on $\widetilde{S}_{m,n}$. The other, the total value, is a rank-two multiplicative valuation on $S_{m,n}$. These valuations are defined below.

Assume $n \geq 1$, and define the map $\widetilde{\mathrm{o}} : \widetilde{S}_{m,n} \to \mathbb{Z} \cup \{\infty\}$ as follows. Put $\widetilde{\mathrm{o}}(0) := \infty$, and for $f \in \widetilde{S}_{m,n} \setminus \{0\}$, put $\widetilde{\mathrm{o}}(f) := \ell$, where $f \in (\rho)^\ell \setminus (\rho)^{\ell+1}$. It will not lead to confusion if we also define the map $\widetilde{\mathrm{o}} : S_{m,n} \to \mathbb{Z} \cup \{\infty\}$ by $\widetilde{\mathrm{o}}(0) := \infty$, and for $f \in S_{m,n} \setminus \{0\}$, $\widetilde{\mathrm{o}}(f) := \widetilde{\mathrm{o}}((cf)^\sim)$, where $c \in K$ satisfies $\|cf\| = 1$. The map $\widetilde{\mathrm{o}}$ is called the **residue order.** The residue order is an additive valuation on $\widetilde{S}_{m,n}$.

Consider $(\mathbb{R}_+ \setminus \{0\})^2$ as an ordered group with coordinatewise multiplication and lexicographic order. Define a map $v : S_{m,n} \to (\mathbb{R}_+ \setminus \{0\})^2 \cup \{(0,0)\}$ as follows. Put $v(0) := (0,0)$, and for $f \in S_{m,n} \setminus \{0\}$, put

$$v(f) := \left(\|f\| , 2^{-\widetilde{\mathrm{o}}(f)} \right).$$

Then v is a multiplicative valuation on $S_{m,n}$, called the **total value**. Note that v extends the absolute value on K in an obvious sense.

The total value yields information on elements $f(\xi,\rho) \in S_{m,n}$ as $|\rho| \to 1$, in a sense to be made precise in Subsections 3.2 and 3.3. Our aim in this subsection is to establish an analogue of Corollary 2.2.6 for the total value. This analogue will be established by lifting a similar result for the residue order from the residue ring $\widetilde{S}_{m,n}$.

Let $M \subset (S_{m,n})^\ell$ be a submodule. Put $M^\circ := (S^\circ_{m,n})^\ell \cap M$ and let $\widetilde{M}$ be the image of M° under the canonical residue epimorphism $\sim: (S^\circ_{m,n})^\ell \to (\widetilde{S}_{m,n})^\ell$.

The next lemma establishes a basic lifting property of $\widetilde{\mathrm{o}}$-strict generating systems. The lemma ensures that the lifting behaves well with respect to restrictions. More precisely,

$$\|a_i(\xi, c\cdot\rho)\| = |c|^{\widetilde{\mathrm{o}}(a_i)} \|a_i\|$$

for any $c \in K^\circ \setminus \{0\}$ and any $a_i \in S_{m,n}$ that satisfies condition (i). Condition (ii) stems from the definition of strictness. And condition (iii) says that we've done the whole slice.

Lemma 3.1.1. — *Let M be a submodule of $(S_{m,n})^\ell$. Let $B \in \mathfrak{B}$ and let $\{g_1,\dots,g_r\} \subset (B\langle\xi\rangle[[\rho]])^\ell \cap M$ satisfy $\|g_i\| = 1$ for $i = 1,\dots,r$. Suppose $\{\widetilde{g}_1,\dots,\widetilde{g}_r\}$ is an $\widetilde{\mathrm{o}}$-strict generating system of $\widetilde{M}$. Let $B = B_0 \supset B_1 \supset \dots$ be the natural filtration of B and suppose $f \in (B_p\langle\xi\rangle[[\rho]])^\ell \setminus (B_{p+1}\langle\xi\rangle[[\rho]])^\ell$. Then there are $a_1,\dots,a_r \in B_p\langle\xi\rangle[[\rho]]$ such that*

(i) *for $i = 1,\dots,r$ if $a_i \neq 0$ then $a_i \in (\rho)^{\widetilde{\mathrm{o}}(a_i)} B_p\langle\xi\rangle[[\rho]] \setminus B_{p+1}\langle\xi\rangle[[\rho]]$,*

(ii) *$v(f) \geq \max_{1\leq i\leq r} v(a_i g_i)$, and*

(iii) *if $v(f-h) < v(f - \sum_{i=1}^r a_i g_i)$ for some $h \in M$, then $\|f - \sum_{i=1}^r a_i g_i\| < \|f\|$.*

(When condition (i) *holds, to verify* (ii)*, it suffices to verify*

(ii)′ $\widetilde{\mathrm{o}}(f) \leq \min_{1\leq i\leq r} \widetilde{\mathrm{o}}(a_i g_i)$, *since* $a_1, \dots, a_r \in B_p\langle\xi\rangle[\![\rho]\!]$.*)*

Proof. — Let $\pi_p : B_p \to \widetilde{B}_p \subset \widetilde{K}$ be the B-module residue epimorphism $a \mapsto (b_p^{-1}a)^\sim$ and write $\widetilde{K} = \widetilde{B}_p \oplus V$ for some $\widetilde{B}$-vector space V. Then

$$\widetilde{K}[\xi][[\rho]] = \widetilde{B}_p[\xi][[\rho]] \oplus V[\xi][[\rho]] \tag{3.1.2}$$

as $\widetilde{B}[\xi][[\rho]]$-modules, and $\widetilde{\mathrm{o}}(a+b) = \min\{\widetilde{\mathrm{o}}(a), \widetilde{\mathrm{o}}(b)\}$ when $a \in \widetilde{B}_p[\xi][[\rho]]$ and $b \in V[\xi][[\rho]]$. Since $\{\widetilde{g}_1, \dots, \widetilde{g}_r\}$ is $\widetilde{\mathrm{o}}$-strict, there are $\widetilde{c}_1, \dots, \widetilde{c}_r \in \widetilde{S}_{m,n}$ so that

$$\widetilde{\mathrm{o}}(\pi_p(f)) \le \min_{1\le i\le r} \widetilde{\mathrm{o}}(\widetilde{c}_i\widetilde{g}_i) \quad \text{and} \quad \widetilde{\mathrm{o}}\left(\pi_p(f) - \sum_{i=1}^r \widetilde{c}_i\widetilde{g}_i\right) \ge \widetilde{\mathrm{o}}(f-h) \tag{3.1.3}$$

for all $h \in \widetilde{M}$.

By (3.1.2), we may write $\widetilde{c}_i = \widetilde{a}_i + \widetilde{b}_i$ where $\widetilde{a}_i \in \widetilde{B}_p[\xi][[\rho]]$ and $\widetilde{b}_i \in V[\xi][[\rho]]$, $1 \le i \le r$. Since $\widetilde{g}_1, \dots, \widetilde{g}_r \in (\widetilde{B}[\xi][[\rho]])^\ell$, by (3.1.2)

$$\widetilde{\mathrm{o}}\left(\pi_p(f) - \sum \widetilde{a}_i\widetilde{g}_i\right) \ge \widetilde{\mathrm{o}}\left(\pi_p(f) - \sum \widetilde{c}_i\widetilde{g}_i\right) \text{ and}$$
$$\min_{1\le i\le r} \widetilde{\mathrm{o}}(\widetilde{a}_i\widetilde{g}_i) \ge \min_{1\le i\le r} \widetilde{\mathrm{o}}(\widetilde{c}_i\widetilde{g}_i).$$

Thus, (3.1.3) holds with $\widetilde{a}_i$ in place of $\widetilde{c}_i$. Now for any $\widetilde{a} \in \widetilde{B}_p[\xi][[\rho]]$, if $\widetilde{a} \ne 0$ then $\widetilde{a} \in (\rho)^{\widetilde{\mathrm{o}}(\widetilde{a})}\widetilde{B}_p[\xi][[\rho]]$. Hence there are $a_1, \dots, a_r \in B_p\langle\xi\rangle[[\rho]]$ such that for $1 \le i \le r$, $\pi_p(a_i) = \widetilde{a}_i$, $a_i = 0$ if $\widetilde{a}_i = 0$ and $a_i \in (\rho)^{\widetilde{\mathrm{o}}(\widetilde{a}_i)}B_p\langle\xi\rangle[[\rho]]$ if $\widetilde{a}_i \ne 0$. It is clear that $a_1, \dots, a_r$ satisfy the lemma. □

We show in Theorem 3.1.3 that every submodule of $(S_{m,n})^\ell$ is v-strict. In light of Lemma 3.1.1, the next lemma reduces this to showing that every submodule of $(\widetilde{S}_{m,n})^\ell$ is $\widetilde{\mathrm{o}}$-strict.

Lemma 3.1.2. — *Let M be a submodule of $(S_{m,n})^\ell$ and suppose $\{g_1, \dots, g_r\} \subset M^\circ$ satisfies $\widetilde{g}_1, \dots, \widetilde{g}_r \ne 0$. Then $\{g_1, \dots, g_r\}$ is a v-strict generating system of M if and only if $\{\widetilde{g}_1, \dots, \widetilde{g}_r\}$ is an $\widetilde{\mathrm{o}}$-strict generating system of $\widetilde{M}$. Moreover*

(i) *if $\{g_1, \dots, g_r\}$ is v-strict and $f, g_1, \dots, g_r \in (B\langle\xi\rangle[\![\rho]\!])^\ell$ then there are $h_1, \dots, h_r \in B\langle\xi\rangle[\![\rho]\!]$ such that*

$$v\left(f - \sum_{i=1}^r h_i g_i\right) \le v(f-h)$$

for all $h \in M$ and

$$v(f) \ge \max_{1\le i\le r} v(h_i g_i),$$

and

(ii) *if* $\{\widetilde{g}_1, \dots, \widetilde{g}_r\}$ *is* $\widetilde{\mathrm{o}}$*-strict and* $\widetilde{f}, \widetilde{g}_1, \dots, \widetilde{g}_r \in (\widetilde{B}[\xi][\![\rho]\!])^\ell$ *then there are* $\widetilde{h}_1, \dots, \widetilde{h}_r \in \widetilde{B}[\xi][\![\rho]\!]$ *such that*

$$\widetilde{\mathrm{o}}\left(\widetilde{f} - \sum_{i=1}^{r} \widetilde{h}_i \widetilde{g}_i\right) \geq \widetilde{\mathrm{o}}(\widetilde{f} - \widetilde{h})$$

for all $\widetilde{h} \in \widetilde{M}$ *and*

$$\widetilde{\mathrm{o}}(\widetilde{f}) \leq \min_{1 \leq i \leq r} \widetilde{\mathrm{o}}(\widetilde{h}_i \widetilde{g}_i).$$

Proof. — ($\Rightarrow$) Let $\widetilde{f} \in (\widetilde{S}_{m,n})^\ell \setminus \{0\}$ and lift $\widetilde{f}$ to an element $f \in (S^\circ_{m,n})^\ell$. Find $a_1, \dots, a_r \in S^\circ_{m,n}$ such that $v(f) \geq \max_{1 \leq i \leq r} v(a_i g_i)$ and

$$v\left(f - \sum_{i=1}^{r} a_i g_i\right) \leq v(f - h) \tag{3.1.4}$$

for every $h \in M$. Since $\|f\| = 1$, we must have that

$$\widetilde{\mathrm{o}}(f) \leq \min\left\{\widetilde{\mathrm{o}}(a_i g_i) : \|a_i g_i\| = 1\right\}.$$

Thus $\widetilde{\mathrm{o}}(f) \leq \min_{1 \leq i \leq r} \widetilde{\mathrm{o}}(\widetilde{a}_i \widetilde{g}_i)$. If $\|f - \sum_{i=1}^{r} a_i g_i\| < 1$ then $\widetilde{f} = \sum_{i=1}^{r} \widetilde{a}_i \widetilde{g}_i \in \widetilde{M}$ and we are done. Otherwise, assume $\|f - \sum_{i=1}^{r} a_i g_i\| = 1$. Let $\widetilde{h} \in \widetilde{M}$ and lift $\widetilde{h}$ to $h \in M^\circ$. Hence, by (3.1.4), $\|f - h\| = 1$ and

$$\widetilde{\mathrm{o}}\left(\widetilde{f} - \sum_{i=1}^{r} \widetilde{a}_i \widetilde{g}_i\right) = \widetilde{\mathrm{o}}\left(f - \sum_{i=1}^{r} a_i g_i\right) \geq \widetilde{\mathrm{o}}(f - h) = \widetilde{\mathrm{o}}(\widetilde{f} - \widetilde{h}),$$

and we have proved that $\{\widetilde{g}_1, \dots, \widetilde{g}_r\}$ is $\widetilde{\mathrm{o}}$-strict.

($\Leftarrow$) Parts (i) and (ii), as well as ($\Leftarrow$) follow immediately from Lemma 3.1.1 using the facts that $\|S_{m,n}\| = |K|$, $|B \setminus \{0\}| \subset \mathbb{R}_+ \setminus \{0\}$ is discrete and $B\langle\xi\rangle[[\rho]]$ is complete in $\|\cdot\|$ for every $B \in \mathfrak{B}$. $\square$

Now the proof of Theorem 3.1.3 reduces to a computation involving the Artin-Rees Lemma for the (ρ)-adic topology on $(\widetilde{S}_{m,n})^\ell$.

Theorem 3.1.3. — *Each submodule of* $(S_{m,n})^\ell$ *is* v*-strict. Each submodule of* $(\widetilde{S}_{m,n})^\ell$ *is* $\widetilde{\mathrm{o}}$*-strict.*

Proof. — By Lemma 3.1.2, we need only prove the last assertion. Let $M \subset (\widetilde{S}_{m,n})^\ell$ be a submodule.

***Claim* (A).** — *If* $\{g_1, \dots, g_r\}$ *is an* $\widetilde{\mathrm{o}}$*-pseudo-Cartesian generating system of* M *then it is* $\widetilde{\mathrm{o}}$*-strict.*

The ideal (ρ) is contained in the Jacobson radical of $\widetilde{S}_{m,n} = \varinjlim \widetilde{B}[\xi][[\rho]]$. Hence by the Krull Intersection Theorem ([**25**, Theorem 8.10]), the $\widetilde{\text{o}}$-topology on $(\widetilde{S}_{m,n})^\ell$ is separated and M is a closed set.

Let $f \in (\widetilde{S}_{m,n})^\ell$. Since M is closed and since $\widetilde{\text{o}}((\widetilde{S}_{m,n})^\ell) = \mathbb{N} \cup \{\infty\}$ there is some $f_0 \in M$ such that

$$\widetilde{\text{o}}(f - f_0) \geq \widetilde{\text{o}}(f - h)$$

for all $h \in M$. Putting $h = 0$ in the above we have $\widetilde{\text{o}}(f_0) \geq \widetilde{\text{o}}(f)$ by the ultrametric inequality. There are $a_1, \ldots, a_r \in \widetilde{S}_{m,n}$ such that

$$\widetilde{\text{o}}(f_0) = \min_{1 \leq i \leq r} \widetilde{\text{o}}(a_i g_i) \quad \text{and} \quad f_0 = \sum_{i=1}^{r} a_i g_i.$$

Thus, we have that $\widetilde{\text{o}}(f) \leq \widetilde{\text{o}}(f_0) = \min_{1 \leq i \leq r} \widetilde{\text{o}}(a_i g_i)$ and

$$\widetilde{\text{o}}\left(f - \sum_{i=1}^{r} a_i g_i\right) \geq \widetilde{\text{o}}(f - h)$$

for all $h \in M$. This proves the claim.

For $i \in \mathbb{N}$, put

$$M_i := \{f \in M \colon \widetilde{\text{o}}(f) \geq i\}.$$

We have $M = M_0 \supset M_1 \supset \ldots$. By the Artin-Rees Lemma ([**25**, Theorem 8.5]) there is some $c \in \mathbb{N}$ such that for all $i > c$

$$M_i = (\rho)^{i-c} M_c. \tag{3.1.5}$$

Each quotient M_i/M_{i+1} is a finite module over $\widetilde{S}_{m,n}/(\rho) = \widetilde{T}_m$. Find $r \in \mathbb{N}$ sufficiently large so that each M_i/M_{i+1} can be generated by r elements for $0 \leq i \leq c$. By $\pi_i : M_i \to M_i/M_{i+1}$, denote the canonical projection. For each $1 \leq i \leq c$, choose $g_{ij} \in M_i \setminus M_{i+1}$, $1 \leq j \leq r$, so that $\pi_i(g_{i1}), \ldots, \pi(g_{ir})$ generate the $\widetilde{T}_m$-module M_i/M_{i+1}.

***Claim* (B).** — *$\{g_{ij}\}$ is an $\widetilde{\text{o}}$-strict generating system of M.*

By Claim A, it suffices to show that $\{g_{ij}\}$ is an $\widetilde{\text{o}}$-pseudo-Cartesian generating system.

Let $f \in M$, and let $B \in \mathfrak{B}$ be such that $\{f\} \cup \{g_{ij}\} \subset (\widetilde{B}[\xi][[\rho]])^\ell$. Write $\widetilde{K} = \widetilde{B} \oplus V$ for some $\widetilde{B}$-vector space V. Then

$$\widetilde{K}[\xi][[\rho]] = \widetilde{B}[\xi][[\rho]] \oplus V[\xi][[\rho]] \tag{3.1.6}$$

as $\widetilde{B}[\xi][\![\rho]\!]$-modules, and $\widetilde{\text{o}}(a + b) = \min\{\widetilde{\text{o}}(a), \widetilde{\text{o}}(b)\}$ when $a \in \widetilde{B}[\xi][\![\rho]\!]$ and $b \in V[\xi][[\rho]]$. Put $N := (\widetilde{B}[\xi][[\rho]])^\ell \cap M$; and for $i \in \mathbb{N}$, put

$$N_i := \{h \in N : \widetilde{\text{o}}(h) \geq i\} = (\widetilde{B}[\xi][[\rho]])^\ell \cap M_i.$$

It follows from (3.1.6) that $\pi_i(g_{i1}),\dots,\pi_i(g_{ir})$ generate the $\widetilde{B}[\xi]$-module N_i/N_{i+1} for $0 \le i \le c$. Furthermore, by (3.1.5), $\{\pi_i(\rho^\nu g_{cj})\}_{1\le j\le r, |\nu|=i-c}$ generates the $\widetilde{B}[\xi]$-module N_i/N_{i+1} for $i > c$. Since $\widetilde{\mathrm{o}}(g_{ij}) = i$ and since $\widetilde{B}[\xi][[\rho]]$ is complete in $\widetilde{\mathrm{o}}$, the claim follows. □

Lemma 3.1.4. — *Let M be a submodule of $(S_{m,n})^\ell$ and suppose that $\{g_1,\dots,g_r\} \subset M^\circ$ satisfy $\widetilde{g}_1,\dots,\widetilde{g}_r \ne 0$. Then $\{g_1,\dots,g_r\}$ is a $\|\cdot\|$-strict generating system of M if, and only if, $\{\widetilde{g}_1,\dots,\widetilde{g}_r\}$ generate $\widetilde{M}$. In particular, since $\widetilde{S}_{m,n}$ is Noetherian, each submodule of $(S_{m,n})^\ell$ is $\|\cdot\|$-strict.*

Proof. — As in Lemmas 3.1.1 and 3.1.2. □

It follows from Theorem 3.1.3 and Lemma 3.1.4 that we may make the following definitions.

Definition 3.1.5. — (cf. Definition 2.2.7.) Let M be a submodule of $(S_{m,n})^\ell$. For $f \in (S_{m,n})^\ell$ we define the **residue norms**

$$\begin{aligned} v_M(f) &:= \inf\{v(f-h) \colon h \in M\}, \text{ and} \\ \|f\|_M &:= \inf\{\|f-h\| \colon h \in M\}. \end{aligned}$$

There is some $h \in M$ such that $v_M(f) = v(f-h)$ and $\|f\|_M = \|f-h\|$. Let M be a submodule of $(\widetilde{S}_{m,n})^\ell$. For $f \in (\widetilde{S}_{m,n})^\ell$ we define

$$\widetilde{\mathrm{o}}_M(f) := \sup\{\widetilde{\mathrm{o}}(f-h) \colon h \in M\}.$$

There is some $h \in M$ such that $\widetilde{\mathrm{o}}_M(f) = \widetilde{\mathrm{o}}(f-h)$.

It follows from Lemma 3.1.4 that $\|\cdot\|_M$ is a norm on $(S_{m,n})^\ell/M$. If E is such that $S_{m,n} = S_{m,n}(E,K)$ is complete in $\|\cdot\|$ (see Theorem 2.1.3) then $(S_{m,n})^\ell/M$ is complete in $\|\cdot\|_M$.

The following lemma is an application of Theorem 3.1.3. It is used in Theorem 2.3.8. In the statement of the lemma, the set A will usually consist of the coefficients f_i of a power series

$$F = \sum_{i\ge 0} f_i(\xi,\rho)\lambda^i \in B\langle\xi,\lambda\rangle[\![\rho]\!] \text{ (respectively, } B\langle\xi\rangle[\![\rho,\lambda]\!]).$$

The lemma allows us to write all the coefficients of F as linear combinations of the first few:

$$F = \sum_{i\ge 0}\sum_{j=1}^{r} h_{ij} f_j \lambda^i$$

in such a way that each power series

$$F_j := \sum_{i\ge 0} h_{ij}\lambda^i \in B'\langle\xi,\lambda\rangle[\![\rho]\!] \text{ (respectively, } B'\langle\xi\rangle[\![\rho,\lambda]\!]),$$

for some $B \subset B' \in \mathfrak{B}$. Although $B\langle\xi\rangle[\![\rho]\!]$ is not in general Noetherian, we are still able to do this. The estimate in the lemma is sufficient to guarantee convergence of F_j in the $(B_1 + (\rho))$-adic (respectively, $(B_1 + (\rho, \lambda))$-adic) topology.

Lemma 3.1.6. — *Let $B \in \mathfrak{B}$ and $A \subset B\langle\xi\rangle[\![\rho]\!]$. Then there are*

$$f_1, \dots, f_r \in A, \quad \ell_0, c, e \in \mathbb{N}, \text{ and } B \subset B' \in \mathfrak{B}$$

with the following property. Let $B' = B'_0 \supset B'_1 \supset \dots$ be the natural filtration of B'. For each $f \in A$ there are $h_1, \dots, h_r \in B'\langle\xi\rangle[\![\rho]\!]$ such that

$$f = \sum_{i=1}^{r} h_i f_i.$$

If, in addition,

$$f \in B'_\ell\langle\xi\rangle[\![\rho]\!] + (\rho)^{2\ell e + c} B'\langle\xi\rangle[\![\rho]\!]$$

for some $\ell > \ell_0$, then we may choose $h_1, \dots, h_r$ such that

$$h_1, \dots, h_r \in B'_\ell\langle\xi\rangle[\![\rho]\!] + (\rho)^{\ell e} B'\langle\xi\rangle[\![\rho]\!].$$

Proof. — Put $I := A \cdot S_{m,n}$, and let $\{g_1, \dots, g_d\} \subset S_{m,n} \setminus \{0\}$ be a v-strict generating system of I. Since $S_{m,n}$ is Noetherian, there are $f_1, \dots, f_s \in A$ and $h_{ij} \in S_{m,n}$ such that

$$g_i = \sum_{j=1}^{s} h_{ij} f_j, \quad 1 \le i \le d.$$

Without loss of generality, we may assume that all $g_i, h_{ij} \in S^\circ_{m,n}$ and

$$\|g_1\| = \dots = \|g_d\| = |\alpha|,$$

for some $\alpha \in K^\circ \setminus \{0\}$. Find $B \subset B' \in \mathfrak{B}$ such that

$$\frac{1}{\alpha} g_1, \dots, \frac{1}{\alpha} g_d \in B'\langle\xi\rangle[\![\rho]\!].$$

Let $B' = B'_0 \supset B'_1, \dots$ be the natural filtration of B' and find ℓ_0 so that

$$\alpha \in B'_{\ell_0} \setminus B'_{\ell_0+1}.$$

Put

$$e := \max_{1 \le i \le d} \widetilde{\mathrm{o}}(g_i).$$

To find a suitable $c \in \mathbb{N}$, consider the ideal

$$J := A \cdot (B'/B'_{\ell_0})[\xi][\![\rho]\!].$$

The ring $(B'/B'_{\ell_0})[\xi][\![\rho]\!]$ is Noetherian, so the Artin–Rees Lemma, [**25**, Theorem 8.5], yields a $c \in \mathbb{N}$ such that for all $q \geq c$,

$$J \cap (\rho)^q \subset (\rho)^{q-c} \cdot J.$$

Find $f_{s+1}, \ldots, f_r \in A$ so that the images of $f_1, \ldots, f_r$ in $(B'/B'_{\ell_0})[\xi][\![\rho]\!]$ generate J.

Let $f \in A$ with

$$f \in B'_\ell\langle\xi\rangle[\![\rho]\!] + (\rho)^{2\ell e+c} B'\langle\xi\rangle[\![\rho]\!].$$

There are $H_1, \ldots, H_r \in B'\langle\xi\rangle[\![\rho]\!]$ such that

$$f - \sum_{i=1}^{r} H_i f_i =: f' \in B'_{\ell_0}\langle\xi\rangle[\![\rho]\!].$$

By choice of c, if $\ell > \ell_0$, we may assume that

$$H_1, \ldots, H_r \in (\rho)^{2\ell e} \cdot B'\langle\xi\rangle[\![\rho]\!].$$

We have

$$f' \in B'_{\ell_0}\langle\xi\rangle[\![\rho]\!]$$

and if $\ell > \ell_0$, we have moreover that

$$f' \in B'_\ell\langle\xi\rangle[\![\rho]\!] + (\rho)^{2\ell e} B'_{\ell_0}\langle\xi\rangle[\![\rho]\!].$$

Let

$$\tau_{\ell_0} : B'_{\ell_0} \to \widetilde{B}'_{\ell_0} \subset \widetilde{K}$$

be any residue epimorphism. Note, by choice of B', that

$$\left\{(\alpha^{-1}g_1)^{\sim}, \ldots, (\alpha^{-1}g_d)^{\sim}\right\} \subset \widetilde{B}'[\xi][\![\rho]\!]$$

is an $\widetilde{\mathrm{o}}$–strict generating system of $\widetilde{I}$. Thus by Lemma 3.1.2, there are $\widetilde{H}'_{11}, \widetilde{H}'_{21}, \ldots, \widetilde{H}'_{d1} \in \widetilde{B}'_{\ell_0}[\xi][\![\rho]\!]$ such that

$$\pi_{\ell_0}(f') = \sum_{i=1}^{d} \widetilde{H}'_{i1}(\alpha^{-1}g_i)^{\sim}.$$

If $\ell > \ell_0$, we have, moveover, that

$$\widetilde{H}'_{11}, \ldots, \widetilde{H}'_{d1} \in (\rho)^{2\ell e-c} \widetilde{B}'_{\ell_0}[\xi][\![\rho]\!].$$

Lift $\widetilde{H}'_{11}, \ldots, \widetilde{H}'_{d1}$ to elements $H'_{11}, \ldots, H'_{d1} \in B'_{\ell_0}\langle\xi\rangle[\![\rho]\!]$ such that for each i,

$$H'_{i1} \in (\rho)^{\widetilde{\mathrm{o}}(\widetilde{H}'_{i1})} B'_{\ell_0}\langle\xi\rangle[\![\rho]\!].$$

Put

$$f'' := f' - \sum_{i=1}^{d} H'_{i1} g_i \in B'_{\ell_0+1}\langle\xi\rangle[\![\rho]\!],$$

and observe that if $\ell > \ell_0$

$$f'' \in B'_\ell\langle\xi\rangle[\![\rho]\!] + (\rho)^{2\ell e - c} B'_{\ell_0+1}\langle\xi\rangle[\![\rho]\!].$$

Iterating this procedure $\ell - \ell_0$ times, we obtain sequences

$$H'_{ij} \in (\rho)^{2\ell e - jc} B'_{\ell_0+j}\langle\xi\rangle[\![\rho]\!]$$

such that

$$f''' := f' - \sum_{i=1}^{d}\sum_{j=1}^{\ell-\ell_0} H'_{ij} g_i \in B'_\ell\langle\xi\rangle[\![\rho]\!].$$

Finally, since $\{g_1, \ldots, g_d\}$ is a $\|\cdot\|$-strict generating system for I, by Lemma 3.1.2, there are $H''_1, \ldots, H''_d \in B'_\ell\langle\xi\rangle[\![\rho]\!]$ such that

$$f''' = \sum H''_i g_i.$$

Put

$$h_i := H_i + \sum_{p,j} (H'_{pj} + H''_p) h_{pi}.$$

□

The next five lemmas give criteria under which a generating system of a module is strict and under which strictness is preserved by contractive homomorphisms and field extensions. For technical reasons, we work over a quotient ring $S_{m,n}/I$. The modules M we consider will carry the residue norm $\|\cdot\|_I$. We will also consider residue modules $\widetilde{M}$ (see Definition 2.2.8).

Lemma 3.1.7. — *Let M be a submodule of $(S_{m,n})^\ell/N$ and suppose that $g_1, \ldots, g_r \in M^\circ$ satisfy $\widetilde{g}_1, \ldots, \widetilde{g}_r \neq 0$. Then:*

(i) *$\{g_1, \ldots, g_r\}$ is a $\|\cdot\|_N$-strict generating system of M if, and only if, $\{\widetilde{g}_1, \ldots, \widetilde{g}_r\}$ generates $\widetilde{M}$.*

(ii) *$\{g_1, \ldots, g_r\}$ is a v_N-strict generating system of M if, and only if, $\{\widetilde{g}_1, \ldots, \widetilde{g}_r\}$ is an $\widetilde{\mathrm{o}}_N$-strict generating system of $\widetilde{M}$.*

Hence each submodule of $(S_{m,n})^\ell/N$ is $\|\cdot\|_N$-strict and v_N-strict. Each submodule of $(\widetilde{S}_{m,n})^\ell/\widetilde{N}$ is $\widetilde{\mathrm{o}}_{\widetilde{N}}$-strict.

Proof. — (i) ($\Rightarrow$) Lift an element $\widetilde{f} \in \widetilde{M} \setminus \{0\}$ to an element $f \in M$ with $\|f\|_N = 1$. Since $\{g_1, \ldots, g_r\}$ is $\|\cdot\|_N$-strict, there are $h_1, \ldots, h_r \in S_{m,n}$ with

$$f = \textstyle\sum_{i=1}^r g_i h_i \quad \text{and}$$
$$1 = \|f\|_N = \max_{1\le i\le r} \|g_i\|_N \|h_i\| = \max_{1\le i\le r} \|h_i\|.$$

Hence $\widetilde{f} = \sum_{i=1}^r \widetilde{g}_i \widetilde{h}_i$; i.e., $\{\widetilde{g}_1, \ldots, \widetilde{g}_r\}$ generates $\widetilde{M}$.

($\Leftarrow$) Put

$$\mathcal{M} := \{f \in (S_{m,n})^\ell \colon f + N \in M\}.$$

Find $A_1, \dots, A_s \in N^\circ$ and $G_1, \dots, G_r \in \mathcal{M}$ such that $\{\widetilde{A}_1, \dots, \widetilde{A}_s\}$ generates $\widetilde{N}$ and $g_i = G_i + N$, $1 \le i \le r$. By Lemma 3.1.4, we may assume that $\|G_i\| = \|g_i\|_N = 1$, $1 \le i \le r$. It follows that $\{\widetilde{A}_1, \dots, \widetilde{A}_s, \widetilde{G}_1, \dots, \widetilde{G}_r\}$ generates $\widetilde{\mathcal{M}}$; hence by Lemma 3.1.4, $\{A_1, \dots, A_s, G_1, \dots, G_r\}$ is a $\|\cdot\|$-strict generating system of $\mathcal{M}$.

Let $f \in M$. By Lemma 3.1.4, there is a $F \in \mathcal{M}$ such that $f = F + N$ and $\|F\| = \|f\|_N$. We may write

$$F = \sum_{i=1}^r G_i h_i + \sum_{i=1}^s A_i h_{r+i}$$

for some $h_1, \dots, h_{r+s} \in S_{m,n}$ with

$$\|F\| = \|f\|_N = \max_{1 \le i \le r+s} \|h_i\|.$$

Hence

$$f = \sum_{i=1}^r g_i h_i \quad \text{and} \quad \|f\|_N = \max_{1 \le i \le r} \|g_i\|_N \|h_i\|,$$

as desired.

(ii) ($\Rightarrow$) Lift an element $\widetilde{f} \in \widetilde{M} \setminus \{0\}$ to an element $f \in M$ with $\|f\|_N = 1$. Since $\{g_1, \dots, g_r\}$ is v_N-strict, there are $h_1, \dots, h_r \in S^\circ_{m,n}$ such that

$$\begin{aligned} v_N(f) &\ge \max_{1 \le i \le r} v_N(g_i) \cdot v(h_i) \quad \text{and} \\ v_N\left(f - \sum_{i=1}^r g_i h_i\right) &\le v_N(f - h) \end{aligned} \tag{3.1.7}$$

for every $h \in M$. Since $v_N(f) = (\|f\|_N, 2^{-\widetilde{\mathrm{o}}_{\widetilde{N}}(f)})$ and $\|f\|_N = 1$, we have

$$\widetilde{\mathrm{o}}_{\widetilde{N}}(f) \le \min\{\widetilde{\mathrm{o}}_{\widetilde{N}}(g_i) + \widetilde{\mathrm{o}}(h_i) \colon \|h_i\| = 1\}.$$

Thus, $\widetilde{\mathrm{o}}_{\widetilde{N}}(\widetilde{f}) \le \min_{1 \le i \le r}(\widetilde{\mathrm{o}}_{\widetilde{N}}(\widetilde{g}_i) + \widetilde{\mathrm{o}}(\widetilde{h}_i))$. If $\|f - \sum_{i=1}^r g_i h_i\|_N < 1$ then $\widetilde{f} = \sum_{i=1}^r \widetilde{g}_i \widetilde{h}_i \in \widetilde{M}$, and we are done. Otherwise, $\|f - \sum_{i=1}^r g_i h_i\| = 1$. Let $\widetilde{h} \in \widetilde{M}$ and lift $\widetilde{h}$ to an element $h \in M^\circ$ with $\|h\|_N = 1$. By (3.1.7), $\|f - h\|_N = 1$ and

$$\widetilde{\mathrm{o}}_{\widetilde{N}}\left(\widetilde{f} - \sum_{i=1}^r \widetilde{g}_i \widetilde{h}_i\right) = \widetilde{\mathrm{o}}_N\left(f - \sum_{i=1}^r g_i h_i\right) \ge \widetilde{\mathrm{o}}_{\widetilde{N}}(f - h) = \widetilde{\mathrm{o}}_{\widetilde{N}}(\widetilde{f} - \widetilde{h}),$$

and we are done.

($\Leftarrow$) Put
$$\mathcal{M} := \{f \in (S_{m,n})^\ell \colon f + N \in M\}.$$
Find $A_1, \ldots, A_s \in N^\circ$ and $G_1, \ldots, G_r \in \mathcal{M}$ such that $\{\widetilde{A}_1, \ldots, \widetilde{A}_s\}$ is an $\widetilde{\text{o}}$-strict generating system of $\widetilde{N}$ and $g_i = G_i + N$, $1 \le i \le r$. By Theorem 3.1.3, we may assume that $v(G_i) = v_N(g_i)$, $1 \le i \le r$.

As in part (i), it suffices to show that $\{A_1, \ldots, A_s, G_1, \ldots, G_r\}$ is a v-strict generating system of $\mathcal{M}$. By Lemma 3.1.2, this reduces to showing that $\{\widetilde{A}_1, \ldots, \widetilde{A}_s, \widetilde{G}_1, \ldots, \widetilde{G}_r\}$ is an $\widetilde{\text{o}}$-strict generating system of $\widetilde{\mathcal{M}}$. Let $F \in (\widetilde{S}_{m,n})^\ell$ and put $f := F + \widetilde{N}$. Since $\{\widetilde{g}_1, \ldots, \widetilde{g}_r\}$ is an $\widetilde{\text{o}}_{\widetilde{N}}$-strict generating system of $\widetilde{M}$, there are $h_1, \ldots, h_r \in \widetilde{S}_{m,n}$ such that
$$\begin{aligned} \widetilde{\text{o}}_{\widetilde{N}}(f) &\le \min_{1 \le i \le r} (\widetilde{\text{o}}_{\widetilde{N}}(g_i) + \widetilde{\text{o}}(h_i)) \quad \text{and} \\ \widetilde{\text{o}}_{\widetilde{N}}\left(f - \sum_{i=1}^r g_i h_i\right) &\ge \widetilde{\text{o}}_{\widetilde{N}}(f - h) \end{aligned}$$
for every $h \in \widetilde{M}$. Since $\{\widetilde{A}_1, \ldots, \widetilde{A}_s\}$ is an $\widetilde{\text{o}}$-strict generating system, there are $h_{r+1}, \ldots, h_{r+s} \in \widetilde{S}_{m,n}$ such that
$$\begin{aligned} \widetilde{\text{o}}\left(F - \sum_{i=1}^r \widetilde{G}_i h_i\right) &\le \min_{1 \le i \le s} \widetilde{\text{o}}(\widetilde{A}_i h_{r+i}) \quad \text{and} \\ \widetilde{\text{o}}_{\widetilde{N}}\left(f - \sum_{i=1}^r g_i h_i\right) &= \widetilde{\text{o}}\left(F - \sum_{i=1}^r \widetilde{G}_i h_i - \sum_{i=1}^s \widetilde{A}_i h_{r+i}\right). \end{aligned}$$
Let $H \in \widetilde{\mathcal{M}}$, and put $h := H + \widetilde{N}$. We have
$$\begin{aligned} \widetilde{\text{o}}\left(F - \sum_{i=1}^r \widetilde{G}_i h_i - \sum_{i=1}^s \widetilde{A}_i h_{r+i}\right) &= \widetilde{\text{o}}_{\widetilde{N}}\left(f - \sum_{i=1}^r g_i h_i\right) \\ &\ge \widetilde{\text{o}}_{\widetilde{N}}(f - h) \\ &\ge \widetilde{\text{o}}(F - H), \end{aligned}$$
as desired.

To prove the last assertions of the Lemma, observe that by part (i), each submodule of $(S_{m,n})^\ell/N$ is $\|\cdot\|_N$-strict because $(\widetilde{S}_{m,n})^\ell/\widetilde{N}$ is Noetherian (Corollary 2.2.2). The fact that each submodule M of $(\widetilde{S}_{m,n})^\ell/\widetilde{N}$ is $\widetilde{\text{o}}_{\widetilde{N}}$-strict follows from the fact that we may include in an $\widetilde{\text{o}}$-strict generating system of the inverse image submodule $\mathcal{M}$ of $(\widetilde{S}_{m,n})^\ell$ an $\widetilde{\text{o}}$-strict generating system of $\widetilde{N}$ (use Theorem 3.1.3). Finally, to see that each submodule of $(S_{m,n})^\ell/N$ is v_N-strict, we apply part (ii). $\square$

Lemma 3.1.8. — *Let M be a submodule of $(S_{m,n})^\ell/N$ and let $g_1,\dots,g_r$ be generators with $\|g_1\|_N=\cdots=\|g_r\|_N=1$. Put*

$$\Phi := \{(h_1,\dots,h_r)\in(S_{m,n})^r : \sum_{i=1}^r g_ih_i=0\} \text{ and}$$
$$\Psi := \{(h_1,\dots,h_r)\in(\widetilde{S}_{m,n})^r : \sum_{i=1}^r \widetilde{g}_ih_i=0\}.$$

Then $\{g_1,\dots,g_r\}$ is a $\|\cdot\|_N$-strict generating system of M if, and only if, $\widetilde{\Phi}=\Psi$.

Proof. — ($\Rightarrow$) Assume $\{g_1,\dots,g_r\}$ is a $\|\cdot\|_N$-strict generating system of M. Let $\widetilde{h}=(\widetilde{h}_1,\dots,\widetilde{h}_r)\in\Psi\setminus\{0\}$ and find $h\in(S_{m,n})^r$ that lifts $\widetilde{h}$. We have:

$$\left\|\sum_{i=1}^r g_ih_i\right\|_N<\max_{1\le i\le r}\|h_i\|.$$

Since $\{g_1,\dots,g_r\}$ is $\|\cdot\|_N$-strict, there is an $h'=(h'_1,\dots,h'_r)\in(S_{m,n})^r$ such that

$$\sum_{i=1}^r g_ih_i=\sum_{i=1}^r g_ih'_i \text{ and } \max_{1\le i\le r}\|h'_i\|=\left\|\sum_{i=1}^r g_ih_i\right\|_N<\max_{1\le i\le r}\|h_i\|=1.$$

Put $H:=h-h'\in\Phi$, and note that $\widetilde{H}=\widetilde{h}$. This proves $\widetilde{\Phi}=\Psi$.

($\Leftarrow$) By Lemma 3.1.4, there are $G_1,\dots,G_r\in(S_{m,n})^\ell$ with $\|G_i\|=1$ and $g_i=G_i+N$, $1\le i\le r$. Put

$$\mathcal{M}:=\{f\in(S_{m,n})^\ell : f+N\in M\}.$$

Let $\{A_1,\dots,A_s\}$ be a $\|\cdot\|$-strict generating system of N with $\|A_1\|=\cdots=\|A_s\|=1$. Since M has a $\|\cdot\|_N$-strict generating system by Lemma 3.1.7, it suffices to show that $\{g_1,\dots,g_r\}$ is $\|\cdot\|_N$-pseudo-Cartesian. Indeed, since for any $f\in M$ there is an $F\in\mathcal{M}$ with $f=F+N$ and $\|F\|=\|f\|_N$, it suffices to show that $\{G_1,\dots,G_r,A_1,\dots,A_s\}$ is a $\|\cdot\|$-pseudo-Cartesian generating system of $\mathcal{M}$.

Let $F\in\mathcal{M}$ and write

$$F=\sum_{i=1}^r G_ih_i+\sum_{i=1}^s A_ih_{r+i} \tag{3.1.8}$$

for some $h_1,\dots,h_{r+s}\in S_{m,n}$. Since $\{A_1,\dots,A_s\}$ is $\|\cdot\|$-strict, we may always assume that

$$\max_{r+1\le i\le r+s}\|h_i\|\le\max\{\|F\|,\|h_1\|,\dots,\|h_r\|\}. \tag{3.1.9}$$

If $\|F\| \geq \max_{1\leq i\leq r} \|h_i\|$, then by (3.1.9) we are done. Therefore, assume that

$$0 \neq \|F\| < \max_{1\leq i\leq r} \|h_i\| \leq 1. \tag{3.1.10}$$

Let $\{C_1, \ldots, C_t\}$ be a $\|\cdot\|$-strict generating system of Φ with $\|C_1\| = \cdots = \|C_t\| = 1$. Find $B \in \mathfrak{B}$ such that

$$\begin{gathered} h_1, \ldots, h_{r+s} \in B\langle\xi\rangle[\![\rho]\!], \\ G_1, \ldots, G_r, A_1, \ldots, A_s \in (B\langle\xi\rangle[\![\rho]\!])^{\ell}, \\ C_1, \ldots, C_t \in (B\langle\xi\rangle[\![\rho]\!])^{r}. \end{gathered}$$

Using (3.1.9) and the fact that $|B \setminus \{0\}|$ is discrete, it suffices to find $h_i' \in B\langle\xi\rangle[\![\rho]\!]$ with

$$\begin{gathered} F = \sum_{i=1}^{r} G_i h_i' + \sum_{i=1}^{s} A_i h_{r+i}' \quad \text{and} \\ \max_{1\leq i\leq r} \|h_i'\| < \max_{1\leq i\leq r} \|h_i\|. \end{gathered} \tag{3.1.11}$$

Let $B = B_0 \supset B_1 \supset \ldots$ be the natural filtration of B, and suppose

$$(h_1, \ldots, h_r) \in (B_p\langle\xi\rangle[\![\rho]\!])^r \setminus (B_{p+1}\langle\xi\rangle[\![\rho]\!])^r.$$

By (3.1.9),

$$h_1, \ldots, h_{r+s} \in B_p\langle\xi\rangle[\![\rho]\!].$$

Let

$$\pi_p \colon B_p \to \widetilde{B}_p = (b_p^{-1} B_p)^{\sim} \subset \widetilde{K}$$

be the projection.

Write $\widetilde{K} = \widetilde{B}_p \oplus V$ for some $\widetilde{B}$-vector space V. Then

$$\widetilde{K}[\xi][\![\rho]\!] = \widetilde{B}_p[\xi][\![\rho]\!] \oplus V[\xi][\![\rho]\!] \tag{3.1.12}$$

as $\widetilde{B}[\xi][\![\rho]\!]$-modules. By (3.1.8) and (3.1.10),

$$\pi_p((h_1, \ldots, h_r)) \in \Psi = \widetilde{\Phi}.$$

Thus for some $\widetilde{e}_1, \ldots, \widetilde{e}_t \in \widetilde{K}[\xi][\![\rho]\!]$,

$$\pi_p((h_1, \ldots, h_r)) = \sum_{i=1}^{t} \widetilde{C}_i \widetilde{e}_i.$$

By (3.1.12), we may assume $\widetilde{e}_1, \ldots, \widetilde{e}_t \in \widetilde{B}_p[\xi][\![\rho]\!]$. Find $e_1, \ldots, e_t \in B_p\langle\xi\rangle[\![\rho]\!]$ with $\pi_p(e_i) = \widetilde{e}_i$, $1 \leq i \leq t$. Put

$$e := \sum_{i=1}^{t} C_i e_i \in \Phi,$$

and

$$(h'_1, \ldots, h'_r) := (h_1, \ldots, h_r) - e.$$

Note that (3.1.11) is satisfied because $\pi_p(e) = \pi_p((h_1, \ldots, h_r))$. □

Lemma 3.1.9. — *Let M be a submodule of $(\widetilde{S}_{m,n})^\ell/N$ and suppose $g_1, \ldots, g_r$ generate M. Put*

$$\Psi := \left\{ (h_1, \ldots, h_r) \in (\widetilde{S}_{m,n})^r : \sum_{i=1}^{r} g_i h_i = 0 \right\},$$

and for each $i \in \mathbb{N}$, put

$$\begin{aligned} M_i &:= \{f \in M : \widetilde{\mathrm{o}}_N(f) \geq i\} \text{ and} \\ \Psi_i &:= \left\{ (h_1, \ldots, h_r) \in (\widetilde{S}_{m,n})^r : \widetilde{\mathrm{o}}_N\left(\sum_{i=1}^{r} g_i h_i\right) \geq e + i \right\} \end{aligned}$$

where

$$e := \max_{1 \leq i \leq r} \widetilde{\mathrm{o}}_N(g_i).$$

Then:

(i) *If $\{g_1, \ldots, g_r\}$ is an $\widetilde{\mathrm{o}}_N$-strict generating system of M, then*

$$\Psi_i = \Psi + \bigoplus_{j=1}^{r} (\rho)^{i+e-\widetilde{\mathrm{o}}_N(g_j)} \widetilde{S}_{m,n}$$

for all i.

Conversely:

(ii) *By the Artin-Rees Lemma (*[**25**, Theorem 8.5]*) there is some $c \in \mathbb{N}$ such that for all $i > c$,*

$$M_i = (\rho)^{i-c} M_c.$$

If

$$\Psi_i = \Psi + \bigoplus_{j=1}^{r} (\rho)^{i+e-\widetilde{\mathrm{o}}_N(g_j)} \widetilde{S}_{m,n}$$

for $1 \leq i \leq c - e$, then $\{g_1, \ldots, g_r\}$ is an $\widetilde{\mathrm{o}}_N$-strict generating system of M.

Proof. — (i) Assume $\{g_1, \ldots, g_r\}$ is an $\widetilde{\mathrm{o}}_N$-strict generating system of M. Clearly, $\Psi + \oplus_{j=1}^{r} (\rho)^{i+e-\widetilde{\mathrm{o}}_N(g_j)} \widetilde{S}_{m,n} \subset \Psi_i$. Let $h = (h_1, \ldots, h_r) \in \Psi_i$; we wish to find $H \in \Psi$ and $h' \in \oplus_{j=1}^{r} (\rho)^{i+e-\widetilde{\mathrm{o}}_N(g_j)} \widetilde{S}_{m,n}$ such that

$$h = H + h'. \tag{3.1.13}$$

Since $h \in \Psi_i$, we have

$$\widetilde{\mathrm{o}}_N\left(\sum_{j=1}^{r} g_j h_j\right) \geq e + i.$$

Since $\{g_1, \ldots, g_r\}$ is $\widetilde{\mathrm{o}}_N$-strict, there is an $h' = (h'_1, \ldots, h'_r) \in (\widetilde{S}_{m,n})^r$ such that

$$\sum_{j=1}^{r} g_j h'_j = \sum_{j=1}^{r} g_j h_j \text{ and}$$

$$\min_{1 \leq j \leq r} (\widetilde{\mathrm{o}}_N(g_j) + \widetilde{\mathrm{o}}(h'_j)) = \widetilde{\mathrm{o}}_N\left(\sum_{j=1}^{r} g_j h_j\right) \geq e + i.$$

Thus $h'_j \in (\rho)^{i+e-\widetilde{\mathrm{o}}_N(g_j)}\widetilde{S}_{m,n}$. Put $H := h - h' \in \Psi$. We have

$$h = H + h' \in \Psi + \bigoplus_{j=1}^{r} (\rho)^{i+e-\widetilde{\mathrm{o}}_N(g_j)}\widetilde{S}_{m,n},$$

satisfying (3.1.13).

(ii) Since M is $\widetilde{\mathrm{o}}_N$-strict by Lemma 3.1.7, it suffices to show that $\{g_1, \ldots, g_r\}$ is $\widetilde{\mathrm{o}}_N$-pseudo-Cartesian. Let

$$f = \sum_{i=1}^{r} g_i h_i \in M.$$

***Case* (A).** — $\widetilde{\mathrm{o}}_N(f) \leq c$.

By assumption,

$$(h_1, \ldots, h_r) \in \Psi_{\widetilde{\mathrm{o}}_N(f)-e} = \Psi + \bigoplus_{j=1}^{r} (\rho)^{\widetilde{\mathrm{o}}_N(f)-\widetilde{\mathrm{o}}_N(g_j)}\widetilde{S}_{m,n};$$

i.e.,

$$(h_1, \ldots, h_r) = H + h'$$

for some $H \in \Psi$ and $h' \in \oplus_{j=1}^{r}(\rho)^{\widetilde{\mathrm{o}}_N(f)-\widetilde{\mathrm{o}}_N(g_j)}\widetilde{S}_{m,n}$. Write $h' = (h'_1, \ldots, h'_r)$. Since $H \in \Psi$,

$$f = \sum_{i=1}^{r} g_i h'_i \text{ and } \min_{1 \leq i \leq r} (\widetilde{\mathrm{o}}_N(g_i) + \widetilde{\mathrm{o}}(h'_i)) \geq \widetilde{\mathrm{o}}_N(f),$$

as desired.

***Case* (B).** — $\widetilde{\mathrm{o}}_N(f) > c$.

By choice of c,

$$f \in M_{\widetilde{\mathrm{o}}_N(f)} = (\rho)^{\widetilde{\mathrm{o}}_N(f)-c} M_c \ ;$$

i.e.,

$$f = \sum_{|\nu| = \widetilde{\mathrm{o}}_N(f)-c} \rho^\nu f_\nu, \quad f_\nu \in M_c.$$

Now apply Case A to the f_ν. □

Let K' be a complete, valued field extension of K, write $S_{m,n} := S_{m,n}(E,K)$ and $S'_{m',n'} := S_{m',n'}(E',K')$, and suppose I is an ideal of $S_{m,n}$ and J is an ideal of $S'_{m',n'}$. Put

$$A := S_{m,n}/I \quad \text{and} \quad B := S'_{m',n'}/J,$$

and by $\|\cdot\|_I$ and $\|\cdot\|_J$ denote the respective residue norms on A and B, as in Definition 3.1.5. Suppose

$$\varphi : A \to B$$

is a K-algebra homomorphism such that

$$\|\varphi(f)\|_J \leq \|f\|_I$$

for all $f \in A$. Then φ induces a K°-algebra homomorphism

$$\varphi^\circ : A^\circ \to B^\circ,$$

where

$$A^\circ = S^\circ_{m,n}/I^\circ \quad \text{and} \quad B^\circ = (S'_{m',n'})^\circ/J^\circ.$$

In addition, φ induces a $\widetilde{K}$-algebra homomorphism

$$\widetilde{\varphi} : \widetilde{A} \to \widetilde{B},$$

where

$$\widetilde{A} = \widetilde{S}_{m,n}/\widetilde{I} \quad \text{and} \quad \widetilde{B} = \widetilde{S}'_{m',n'}/\widetilde{J}.$$

Lemma 3.1.10. — *With notation as above, let M be a submodule of A^ℓ and put $N := \varphi(M) \cdot B \subset B^\ell$. Suppose $\widetilde{\varphi}$ is flat. Then:*

(i) *If $\{g_1, \ldots, g_r\}$ is a $\|\cdot\|_I$-strict generating system of M, then $\{\varphi(g_1), \ldots, \varphi(g_r)\}$ is a $\|\cdot\|_J$-strict generating system of N.*
(ii) *φ is flat.*
(iii) *φ° is flat.*

Proof. — (i) We may assume that $\|g_1\|_I = \cdots = \|g_r\|_I = 1$. Put

$$\Phi_A := \left\{ (h_1, \ldots, h_r) \in A^r : \sum_{i=1}^r g_i h_i = 0 \right\},$$

$$\Phi_B := \left\{ (h_1, \ldots, h_r) \in B^r : \sum_{i=1}^r \varphi(g_i) h_i = 0 \right\},$$

$$\Psi_A := \left\{ (h_1, \ldots, h_r) \in \widetilde{A}^r : \sum_{i=1}^r \widetilde{g}_i h_i = 0 \right\},$$

$$\Psi_B := \left\{ (h_1, \ldots, h_r) \in \widetilde{B}^r = \sum_{i=1}^r \widetilde{\varphi}(\widetilde{g}_i) h_i = 0 \right\}.$$

By Lemma 3.1.8, $\widetilde{\Phi}_A = \Psi_A$. Since $\widetilde{\varphi}$ is flat, by [**25**, Theorem 7.6], $\Psi_B = \widetilde{B} \cdot \widetilde{\varphi}(\Psi_A)$. We have:

$$\Psi_B = \widetilde{B} \cdot \widetilde{\varphi}(\Psi_A) = \widetilde{B} \cdot \widetilde{\varphi}(\widetilde{\Phi}_A) \subset \widetilde{\Phi}_B \subset \Psi_B;$$

i.e., $\widetilde{\Phi}_B = \Psi_B$. Part (i) now follows from Lemma 3.1.8.

(ii) Let $\mathfrak{a}$ be an ideal of A. By [**25**, Theorem 7.6], we must show that the canonical map

$$\mathfrak{a} \otimes_A B \to A \otimes_A B \tag{3.1.14}$$

is injective.

Let $\{g_1, \ldots, g_r\}$ be a $\|\cdot\|_I$-strict generating system of $\mathfrak{a}$ with $\|g_1\| = \cdots = \|g_r\| = 1$. Define Φ_A, Φ_B, Ψ_A, Ψ_B as in part (i). To prove that (3.1.14) is injective, it suffices to show that $\Phi_B = B \cdot \varphi(\Phi_A)$. By Lemma 3.1.7, it is enough to show that $\widetilde{\Phi}_B$ is generated by $\widetilde{\varphi}(\widetilde{\Phi}_A)$. By part (i), and Lemma 3.1.8 $\widetilde{\Phi}_B = \Psi_B$. Since $\widetilde{\varphi}$ is flat, $\Psi_B = \widetilde{B} \cdot \widetilde{\varphi}(\Psi_A)$. Finally, by Lemma 3.1.8, $\Psi_A = \widetilde{\Phi}_A$. This proves part (ii).

(iii) Let $g_1, \ldots, g_r \in A^\circ$ and define Φ_A and Φ_B as in part (i). By [**25**, Theorem 7.6], we must show that

$$\Phi_B^\circ = B^\circ \cdot \varphi^\circ(\Phi_A^\circ).$$

This follows immediately from parts (i) and (ii) since there is a $\|\cdot\|_I$-strict generating system of the A°-module Φ_A°. $\square$

It is often convenient to work over an extension field of K. The next lemma shows that $S_{m,n}$ and the total value v behave well with respect to ground field extension.

Lemma 3.1.11. — *Let K' be a complete, valued field extension of K, let $E' \subset (K')^\circ$ be a complete, quasi-Noetherian ring, and put $S_{m,n} := S_{m,n}(E, K)$,*

$S'_{m,n} := S_{m,n}(E', K')$. Assume $S'_{m,n} \supset S_{m,n}$; e.g., take $E' \supset E$. Let M be a submodule of $(S_{m,n})^\ell$ and put $M' := M \cdot S'_{m,n}$.

(i) *$\widetilde{S}'_{m,n}$ is a faithfully flat $\widetilde{S}_{m,n}$-algebra.*
(ii) *Suppose $\{g_1, \ldots, g_r\} \subset M$ is a v-strict generating system of M, then $\{g_1, \ldots, g_r\}$ is also a v-strict generating system of M', and for every $f \in (S_{m,n})^\ell$, $v_M(f) = v_{M'}(f)$. In particular $\|f\|_M = \|f\|_{M'}$.*
(iii) *$S_{m,n}(E', K')$ is a faithfully flat $S_{m,n}(E, K)$-algebra.*
(iv) *$S_{m,n}(E', K')^\circ$ is a faithfully flat $S_{m,n}(E, K)^\circ$-algebra.*

Proof. — (i) By Corollary 2.2.2, both $\widetilde{S}_{m,n}$ and $\widetilde{S}'_{m,n}$ are Noetherian. Since $(\rho) \subset \operatorname{rad} \widetilde{S}'_{m,n}$, $\widetilde{S}'_{m,n}$ is (ρ)-adically ideal-separated. For each $\ell \in \mathbb{N}$,

$$\widetilde{S}_{m,n}/(\rho)^\ell = \widetilde{K}[\xi, \rho]/(\rho)^\ell \to \widetilde{K}'[\xi, \rho]/(\rho)^\ell = \widetilde{S}'_{m,n}/(\rho)^\ell$$

is flat. Hence by the Local Flatness Criterion [**25**, Theorem 22.3], $\widetilde{S}'_{m,n}$ is a flat $\widetilde{S}_{m,n}$-algebra. Let $\mathfrak{m}$ be a maximal ideal of $\widetilde{S}_{m,n}$. By [**25**, Theorem 7.2], to prove that $\widetilde{S}'_{m,n}$ is faithfully flat over $\widetilde{S}_{m,n}$, we must show that $\mathfrak{m} \cdot \widetilde{S}'_{m,n} \neq \widetilde{S}'_{m,n}$. Since $(\rho) \subset \mathfrak{m}$, this follows from the faithful flatness of $\widetilde{K}'[\xi]$ over $\widetilde{K}[\xi]$.

(ii) We may assume that $\|g_i\| = 1$, $1 \le i \le r$. Put

$$N := \left\{ (h_1, \ldots, h_r) \in (\widetilde{S}_{m,n})^r : \sum_{i=1}^r \widetilde{g}_i h_i = 0 \right\},$$

$$N' := \left\{ (h_1, \ldots, h_r) \in (\widetilde{S}'_{m,n})^r : \sum_{i=1}^r \widetilde{g}_i h_i = 0 \right\},$$

and for each $i \in \mathbb{N}$, put

$$N_i := \left\{ (h_1, \ldots, h_r) \in (\widetilde{S}_{m,n})^r : \widetilde{\mathrm{o}} \left(\sum_{i=1}^r \widetilde{g}_i h_i \right) \ge e + i \right\},$$

$$N'_i := \left\{ (h_1, \ldots, h_r) \in (\widetilde{S}'_{m,n})^r : \widetilde{\mathrm{o}} \left(\sum_{i=1}^r \widetilde{g}_i h_i \right) \ge e + i \right\},$$

where $e := \max_{1 \le i \le r} \widetilde{\mathrm{o}}(g_i)$. By Lemma 3.1.2, $\{\widetilde{g}_1, \ldots, \widetilde{g}_r\}$ is an $\widetilde{\mathrm{o}}$-strict generating system of $\widetilde{M}$. Hence by Lemma 3.1.9(i),

$$N_i = N + \bigoplus_{j=1}^r (\rho)^{i+e-\widetilde{\mathrm{o}}_N(g_j)} \widetilde{S}_{m,n}$$

for all $i \in \mathbb{N}$. By part (i),

$$N'_i = \widetilde{S}'_{m,n} \otimes_{\widetilde{S}_{m,n}} N_i \text{ and } N' = \widetilde{S}'_{m,n} \otimes_{\widetilde{S}_{m,n}} N.$$

Hence,

$$N_i' = N' + \bigoplus_{j=1}^{r} (\rho)^{i+e-\widetilde{o}_N(g_j)} \widetilde{S}'_{m,n}$$

for all $i \in \mathbb{N}$. Finally, by applying Lemmas 3.1.9 and 3.1.2 again, we see that $\{g_1, \ldots, g_r\}$ is a v-strict generating system of M'. The last assertions of part (ii) follow from Lemma 3.1.1 as in the proof of Lemma 3.1.2.

(iii) First we prove that $S'_{m,n}$ is a flat $S_{m,n}$-algebra. The faithful flatness will follow from part (iv) by faithfully flat base change; i.e., $S'_{m,n} = (S'_{m,n})^\circ \otimes_{S^\circ_{m,n}} S_{m,n}$. Of course, the proof of part (iv) makes use only of the assertion that $S'_{m,n}$ is flat over $S_{m,n}$.

Let I be an ideal of $S_{m,n}$. By [**25**, Theorem 7.7], we must show that the canonical map

$$I \otimes_{S_{m,n}} S'_{m,n} \to S_{m,n} \otimes_{S_{m,n}} S'_{m,n} \tag{3.1.15}$$

is injective.

Let $g_1, \ldots, g_r \in S_{m,n}$ be a v-strict generating system of I with $\|g_1\| = \cdots = \|g_r\| = 1$. Put

$$\begin{aligned}
N &:= \left\{ (h_1, \ldots, h_r) \in (S_{m,n})^r : \sum_{i=1}^{r} g_i h_i = 0 \right\} \\
N' &:= \left\{ (h_1, \ldots, h_r) \in (S'_{m,n})^r : \sum_{i=1}^{r} g_i h_i = 0 \right\} \\
P &:= \left\{ (h_1, \ldots, h_r) \in (\widetilde{S}_{m,n})^r : \sum_{i=1}^{r} \widetilde{g}_i h_i = 0 \right\} \\
P' &:= \left\{ (h_1, \ldots, h_r) \in (\widetilde{S}'_{m,n})^r : \sum_{i=1}^{r} \widetilde{g}_i h_i = 0 \right\}.
\end{aligned}$$

To prove that (3.1.15) is injective, it suffices to show that $N' = S'_{m,n} \cdot N$. By Lemma 3.1.8, $\widetilde{N} = P$, by part (ii) and Lemma 3.1.8, $(N')^\sim = P'$, and by part (i), $P' = \widetilde{S}'_{m,n} \cdot P$. Hence

$$P' = \widetilde{S}'_{m,n} \cdot P = \widetilde{S}'_{m,n} \cdot \widetilde{N} = (N')^\sim.$$

After an application of Lemma 3.1.4, one sees that $N' = S'_{m,n} \cdot N$, as desired.

(iv) Let $g_1, \ldots, g_r \in S^\circ_{m,n}$ and define N, N' as in the proof of part (iii), above. We must show that

$$(N')^\circ = (S'_{m,n})^\circ \cdot N^\circ.$$

This follows immediately from the existence of a v-strict generating system for N, from part (ii) and from the fact that $S'_{m,n}$ is flat over $S_{m,n}$. Since $K^{\circ\circ}$, $(\rho) \subset \operatorname{rad} S^{\circ}_{m,n}$, the faithfulness follows from that of

$$\widetilde{K}[\xi] \to ((K')^{\circ}/K^{\circ\circ} \cdot (K')^{\circ})[\xi].$$

□

3.2. Restrictions to Rational Polydiscs. — Let $\varepsilon \in \sqrt{|K \setminus \{0\}|}$ with $1 > \varepsilon > 0$. Put

$$T_{m,n}(\varepsilon) = T_{m,n}(\varepsilon, K) := \left\{ \sum a_{\mu\nu} \xi^{\mu} \rho^{\nu} \in K[\![\xi, \rho]\!] : \lim_{|\mu|+|\nu| \to \infty} \varepsilon^{|\nu|} |a_{\mu\nu}| = 0 \right\}.$$

By [**6**, Theorem 6.1.5.4], $T_{m,n}(\varepsilon)$ is K-affinoid. Define a modified Gauss norm $\|\cdot\|_{\varepsilon}$ on $T_{m,n}(\varepsilon)$ by

$$\left\| \sum a_{\mu\nu} \xi^{\mu} \rho^{\nu} \right\|_{\varepsilon} := \max_{\mu,\nu} \varepsilon^{|\nu|} \, |a_{\mu\nu}|$$

(see [**6**, Proposition 6.1.5.2]). By [**6**, Proposition 6.1.5.5], $\|\cdot\|_{\varepsilon} = \|\cdot\|_{\sup}$ on $T_{m,n}(\varepsilon)$. In this subsection we make extensive use of $\| \cdot \|_{\sup}$ on affinoid algebras. Quasi-affinoid algebras also possess supremum seminorms, but we will not make use of them until after we prove the quasi-affinoid Nullstellensatz, Theorem 4.1.1.

By ι_{ε} denote the natural inclusion

$$\iota_{\varepsilon} : S_{m,n} \hookrightarrow T_{m,n}(\varepsilon),$$

which corresponds to the **restriction to the rational polydisc** $\operatorname{Max} T_{m,n}(\varepsilon)$. In the case that $\varepsilon \in |K|$ with $1 > \varepsilon > 0$, fix $c \in K$ with $|c| = \varepsilon$. Then the K-affinoid map

$$\varphi_{\varepsilon} : T_{m,n}(\varepsilon) \to T_{m+n}$$

given by $\xi \mapsto \xi$ and $\rho \mapsto c \cdot \rho$ identifies $T_{m,n}(\varepsilon)$ with T_{m+n}, and for $f \in T_{m,n}(\varepsilon)$, we have $\|f\|_{\sup} = \|\varphi_{\varepsilon}(f)\|$. By ι'_{ε} we denote the inclusion

$$\iota'_{\varepsilon} := \varphi_{\varepsilon} \circ \iota_{\varepsilon} : S_{m,n} \hookrightarrow T_{m+n};$$

thus $\iota'_{\varepsilon}(f) = f(\xi, c \cdot \rho)$ for $f \in S_{m,n}$. Note that the morphisms φ_{ε} and ι'_{ε} depend on the choice of c.

We are interested in the uniform behavior of the inclusions ι_{ε} as $\varepsilon \to 1$. In particular, we show in Theorem 3.2.3 that the image under ι_{ε} of a strict generating system remains strict for ε sufficiently large.

For this purpose we define a map $\sigma : S_{m,n} \to \mathbb{R}_{+}$ as follows (assuming that $n \geq 1$). Let $f = \sum f_{\nu}(\xi) \rho^{\nu} \in S_{m,n}$ and put $i := \widetilde{\operatorname{o}}(f)$. If $i = 0, \infty$ put

$\sigma(f) := 0$. Otherwise, put

$$\sigma(f) := \max_{|\nu|<i} \left(\frac{\|f_\nu\|}{\|f\|} \right)^{\frac{1}{i-|\nu|}}.$$

Note that $0 \leq \sigma(f) < 1$. The number $\sigma(f)$ is called the **spectral radius** of f.

The following observations are useful in computations involving the spectral radius:

$$\|\iota_\varepsilon(f)\|_{\sup} \geq \varepsilon^{\widetilde{\mathrm{o}}(f)} \|f\|,$$

with equality when $1 > \varepsilon \geq \sigma(f)$, and

$$\sigma(f) = \inf\{|c| : c \in (K')^\circ \text{ and } \widetilde{\mathrm{o}}(f(\xi, c \cdot \rho)) = \widetilde{\mathrm{o}}(f)\},$$

where $K' \supset K$ is algebraically closed. Hence if $f \cdot g \neq 0$,

$$\sigma(f \cdot g) = \max\{\sigma(f), \sigma(g)\}.$$

It is suggestive to compare the spectral radius with the *spectral value* of a monic polynomial defined in [**6**, Section 1.5.4].

We define the **spectral radius** of a submodule M of $(S_{m,n})^\ell$, $n \geq 1$ by

$$\sigma(M) := \inf_{\{g_1,\dots,g_r\} \in \mathcal{M}} \max\{\sigma(g_1), \dots, \sigma(g_r)\},$$

where $\mathcal{M}$ is the collection of all v-strict generating systems $\{g_1, \dots, g_r\}$ of M.

Remark 3.2.1. — (i) Let $\varepsilon \in |K|$ with $1 > \varepsilon > 0$. We have the following commutative diagram

$$\begin{tikzcd} & T_{m,n}(\varepsilon) \arrow[dr, "\varphi_\varepsilon"] & \\ S_{m,n} \arrow[ur, "\iota_\varepsilon"] \arrow[rr, "\iota'_\varepsilon"'] & & T_{m+n} \end{tikzcd}$$

and φ_ε is an isometric isomorphism. Since φ_ε is an isometry, this yields an identification of $\widetilde{T}_{m,n}(\varepsilon)$ with $\widetilde{K}[\xi, \rho] = \widetilde{T}_{m+n}$, where $\widetilde{T}_{m,n}(\varepsilon)$ is the quotient of the subring of power-bounded elements of $T_{m,n}(\varepsilon)$ modulo its ideal of topologically nilpotent elements (see [**6**, Section 6.3]).

(ii) Let $\varepsilon \in \sqrt{|K \setminus \{0\}|}$ with $1 > \varepsilon > 0$. Let K' be a finite algebraic extension of K and suppose $\{c_1, \dots, c_s\}$ is a K-Cartesian basis of K' (see [**6**, Definition 2.4.1.1]). Then $\{c_1, \dots, c_s\}$ is also a $\|\cdot\|_{\sup}$-Cartesian basis for the $T_{m,n}(\varepsilon)$-module $T'_{m,n}(\varepsilon) := T_{m,n}(\varepsilon, K')$. This is easily seen using the modified

Gauss norm $\|\cdot\|_\varepsilon$, as follows. Let $f \in T'_{m,n}(\varepsilon)$; then $f = \sum_{i=1}^{s} c_i f_i$ with each $f_i \in T_{m,n}(\varepsilon)$ and $\|f\|_\varepsilon = \max_{1\le i\le s} |c_i|\,\|f_i\|_\varepsilon$.

(iii) Using the notation of part (ii), observe that $\{\frac{c_1}{c_1}, \ldots, \frac{c_s}{c_1}\}$ is a K-Cartesian basis of K'; hence we may assume that $c_1 = 1$. Let M be a submodule of $(T_{m,n}(\varepsilon))^\ell$ and put $M' := T'_{m,n}(\varepsilon)\cdot M$. Let $f \in (T_{m,n}(\varepsilon))^\ell$; then $\|f\|_M = \|f\|_{M'}$ (see Definition 3.1.5). This is proved as follows. By Lemma 3.1.4, there is a $g \in M'$ such that $\|f-g\| = \|f\|_{M'}$. We may write $g = \sum_{i=1}^{s} c_i g_i$ with each $g_i \in M$. By part (ii),

$$\begin{aligned}\|f-g\|_\varepsilon &= \max\{\|f-g_1\|_\varepsilon, |c_2|\,\|g_2\|_\varepsilon, \ldots, |c_s|\,\|g_s\|_\varepsilon\}\\ &\ge \|f-g_1\|_\varepsilon\\ &\ge \|f\|_M.\end{aligned}$$

Since $\|f\|_{M'} \le \|f\|_M$, we have $\|f\|_M = \|f\|_{M'}$.

Our immediate goal, Theorem 3.2.3, is to show that a strict generating system remains strict upon restriction to a suitably large rational polydisc. Lemma 3.2.2 is the inductive step of the slicing argument involved. It makes special use of condition (i) of Lemma 3.1.1.

Lemma 3.2.2. — *Let M be a submodule of $(S_{m,n})^\ell$, let $g_1, \ldots, g_r \in M$ with $\|g_1\| = \cdots = \|g_r\| = 1$, and suppose that $\{\widetilde{g}_1, \ldots, \widetilde{g}_r\}$ is an $\widetilde{\mathrm{o}}$-strict generating system of $\widetilde{M}$. Suppose $B \in \mathfrak{B}$ satisfies $\{g_1, \ldots, g_r\} \subset (B\langle\xi\rangle[[\rho]])^\ell \cap M$, and let $B = B_0 \supset B_1 \supset \ldots$ be the natural filtration of B. Let $\varepsilon \in \sqrt{|K\setminus\{0\}|}$ be such that $1 > \varepsilon \ge \max\{\sigma(g_1), \ldots, \sigma(g_r)\}$. Suppose*

$$f \in M \cap \left((B_p\langle\xi\rangle[[\rho]])^\ell \setminus (B_{p+1}\langle\xi\rangle[[\rho]]^\ell\right).$$

Then there are $a_1, \ldots, a_r \in \{0\} \cup (B_p\langle\xi\rangle[[\rho]] \setminus B_{p+1}\langle\xi\rangle[[\rho]])$ such that

(i) $\|\iota_\varepsilon(f)\|_{\sup} \ge \max_{1\le i\le r} \|\iota_\varepsilon(a_i g_i)\|_{\sup}$ *(recall $\|\cdot\|_{\sup} = \|\cdot\|_\varepsilon$ on $T_{m,n}(\varepsilon)$) and*

(ii) $\|f - \sum_{i=1}^r a_i g_i\| < \|f\|$.

Proof. — Choose $a_1, \ldots, a_r \in \{0\} \cup (B_p\langle\xi\rangle[[\rho]] \setminus B_{p+1}\langle\xi\rangle[[\rho]])$ as in Lemma 3.1.1. By Lemma 3.1.1 (i), $\sigma(a_i g_i) \le \varepsilon$, so

$$\|\iota_\varepsilon(a_i g_i)\|_{\sup} = \varepsilon^{\widetilde{\mathrm{o}}(a_i g_i)}\|a_i g_i\| \le \varepsilon^{\widetilde{\mathrm{o}}(a_i g_i)}\|f\|.$$

By Lemma 3.1.1 (ii)$'$, we get

$$\|\iota_\varepsilon(a_i g_i)\|_{\sup} \le \varepsilon^{\widetilde{\mathrm{o}}(a_i g_i)}\|f\| \le \varepsilon^{\widetilde{\mathrm{o}}(f)}\|f\| \le \|\iota_\varepsilon(f)\|_{\sup},$$

which yields (i). Since $f \in M$, (ii) follows from Lemma 3.1.1 (iii). □

Theorem 3.2.3. — *Let M be a submodule of $(S_{m,n})^\ell$, $n \geq 1$, with $\mathfrak{v}$-strict generating system $\{g_1, \dots, g_r\} \subset M^\circ$. Let $\varepsilon \in \sqrt{|K \setminus \{0\}|}$ with $1 > \varepsilon \geq \max_{1 \leq i \leq r} \sigma(g_i)$, and assume either that K is a stable field (see* [**6**, Definition 3.6.1.1]*) or that $\varepsilon \in |K|$. Then $\{\iota_\varepsilon(g_1), \dots, \iota_\varepsilon(g_r)\}$ is a $\|\cdot\|_{\sup}$-strict generating system of the $T_{m,n}(\varepsilon)$-module $\iota_\varepsilon(M) \cdot T_{m,n}(\varepsilon) \subset (T_{m,n}(\varepsilon))^\ell$.*

Proof. — Suppose first that $\varepsilon \in |K|$. Then by Remark 3.2.1 (i), we have the following commutative diagram,

$$\begin{tikzcd} & T_{m,n}(\varepsilon) \arrow[dr, "\varphi_\varepsilon"] & \\ S_{m,n} \arrow[ur, "\iota_\varepsilon"] \arrow[rr, "\iota'_\varepsilon"'] & & T_{m+n} \end{tikzcd}$$

where φ_ε is an isometric isomorphism. We will therefore show that $\{\iota'_\varepsilon(g_1), \dots, \iota'_\varepsilon(g_r)\}$ is a $\|\cdot\|$-strict generating system of the T_{m+n}-module $\iota'_\varepsilon(M) \cdot T_{m+n} \subset (T_{m+n})^\ell$.

By Lemma 3.1.4 (applied to $T_{m+n} = S_{m+n,0}$), it suffices to show for each $f \in \iota'_\varepsilon(M) \cdot T_{m+n} \setminus \{0\}$ that there are $a_1, \dots, a_r \in T_{m+n}$ such that

$$\|f\| = \max_{1 \leq i \leq r} \left\|a_i \iota'_\varepsilon(g_i)\right\| \text{ and } \left\|f - \sum_{i=1}^r a_i \iota'_\varepsilon(g_i)\right\| < \|f\|. \tag{3.2.1}$$

Write $f = \sum_{i=1}^r f_i \iota'_\varepsilon(g_i)$ for some $f_1, \dots, f_r \in T_{m+n}$. Find polynomials $f'_1, \dots, f'_r \in K[\xi, \rho]$ such that each $\|f'_i - f_i\| < \|f\|$. Then

$$\left\|\sum_{i=1}^r f_i \iota'_\varepsilon(g_i) - \sum_{i=1}^r f'_i \iota'_\varepsilon(g_i)\right\| < \|f\|,$$

since $\|\iota'_\varepsilon(g_i)\| \leq 1$ for all i. Put $f' := \sum_{i=1}^r f'_i \iota'_\varepsilon(g_i)$. It suffices to prove (3.2.1) for f'.

Since the f'_i are polynomials, $f' = \iota'_\varepsilon(F)$ for some $F \in M$. We wish to apply Lemma 3.2.2. Since $\|S_{m,n}\| = |K|$, we may assume $\|F\|, \|g_1\|, \dots, \|g_r\| = 1$. Hence by Lemma 3.1.2, $\{\widetilde{g}_1, \dots, \widetilde{g}_r\}$ is an $\widetilde{\mathfrak{o}}$-strict generating system of $\widetilde{M}$. Choose $B \in \mathfrak{B}$ such that $F, g_1, \dots, g_r \in (B\langle\xi\rangle[[\rho]])^\ell \cap M$. By iterated application of Lemma 3.2.2 (recall that $\varepsilon \in |K|$, hence $T_{m,n}(\varepsilon)$ and T_{m+n} are isometrically isomorphic) we obtain a sequence $\{a_{ij}\} \subset B\langle\xi\rangle[[\rho]]$ such that $a_{10}, \dots, a_{r0} = 0$ and for every $s \in \mathbb{N}$,

(i) $\left\| \iota'_\varepsilon(F - \sum_{\substack{1\le i\le r\\ 0\le j\le s}} a_{ij} g_i) \right\| \ge \left\| \iota'_\varepsilon(a_{is+1} g_i) \right\|$, and

(ii) $\left\| F - \sum_{\substack{1\le i\le r\\ 0\le j\le s}} a_{ij} g_i \right\| > \left\| F - \sum_{\substack{1\le i\le r\\ 0\le j\le s+1}} a_{ij} g_i \right\|$.

Since $B\langle\xi\rangle[[\rho]]$ is complete in $\|\cdot\|$ and $|B\setminus\{0\}| \subset \mathbb{R}_+\setminus\{0\}$ is discrete, by (ii),

$$F - \sum_{i=1}^r a_i g_i = 0, \quad \text{where} \quad a_i := \sum_{j\ge 0} a_{ij}.$$

Hence $\|f' - \sum_{i=1}^r \iota'_\varepsilon(a_i)\iota'_\varepsilon(g_i)\| = 0 < \|f'\|$. It follows from (i) that $\|f'\| = \|\iota'_\varepsilon(F)\| \ge \max_{i,j} \|\iota'_\varepsilon(a_{ij} g_i)\|$ and hence that

$$\|f'\| = \max_{1\le i\le r} \left\|\iota'_\varepsilon(a_i g_i)\right\|.$$

This concludes the proof in case $\varepsilon \in |K\setminus\{0\}|$.

It remains to treat the case that K is a stable field. Let K' be a finite algebraic extension of K with $\varepsilon \in |K'|$. Let $S'_{m,n} := S_{m,n}(E, K')$ and let $M' := M \cdot S'_{m,n}$. By Lemma 3.1.11, $\{g_1, \ldots, g_r\}$ is a v-strict generating system of M'. Therefore, by the preceding case, $\{\iota_\varepsilon(g_1), \ldots, \iota_\varepsilon(g_r)\}$ is a $\|\cdot\|_{\sup}$-strict generating system of the $T'_{m,n}(\varepsilon)$-module $\iota_\varepsilon(M)\cdot T'_{m,n}(\varepsilon)$.

Let $\{1 = c_1, \ldots, c_s\}$ be a K-Cartesian basis of K', and let $f \in (T_{m,n}(\varepsilon))^\ell \subset (T'_{m,n}(\varepsilon))^\ell$. By the previous case there are $a_1, \ldots, a_r \in T'_{m,n}(\varepsilon)$ such that

$$\left\| f - \sum_{i=1}^r a_i \iota_\varepsilon(g_i) \right\|_{\sup} = \|f\|_{\iota_\varepsilon(M)\cdot T'_{m,n}(\varepsilon)}, \text{ and}$$
$$\|f\|_{\sup} \ge \max_{1\le i\le r} \|a_i \iota_\varepsilon(g_i)\|_{\sup}.$$

For $i = 1, \ldots, r$, write

$$a_i = \sum_{j=1}^s c_j a_{ij} \text{ with } a_{ij} \in T_{m,n}(\varepsilon).$$

Then as in Remark 3.2.1 (iii),

$$\left\| f - \sum_{i=1}^r a_{i1} \iota_\varepsilon(g_i) \right\|_{\sup} = \|f\|_{\iota_\varepsilon(M)\cdot T_{m,n}(\varepsilon)}, \text{ and}$$
$$\|f\|_{\sup} \ge \max_{1\le i\le r} \|a_{i1} \iota_\varepsilon(g_i)\|_{\sup}.$$

Thus $\{\iota_\varepsilon(g_1), \ldots, \iota_\varepsilon(g_r)\}$ is a $\|\cdot\|_{\sup}$-strict generating system of the $T_{m,n}(\varepsilon)$-module $\iota_\varepsilon(M)\cdot T_{m,n}(\varepsilon)$. $\square$

Let M be a submodule of $(S_{m,n})^\ell$. Lemma 3.2.5 uses Theorem 3.2.3 to relate the structure of $\widetilde{M}$ to that of $(\iota_\varepsilon(M)\cdot T_{m,n}(\varepsilon))^\sim$ for ε large enough. Lemma 3.2.6 will be used in Section 4 to prove that $S_{m,n}$ is a UFD.

Definition 3.2.4. — Let M be a submodule of $(S_{m,n})^\ell$, $n \geq 1$, and consider $\widetilde{M} \subset (\widetilde{S}_{m,n})^\ell$. Note that each $f \in (\widetilde{S}_{m,n})^\ell$ can be written uniquely as $f = \sum_{|\nu|\geq \widetilde{\mathrm{o}}(f)} f_\nu(\xi)\rho^\nu$, where each $f_\nu \in (\widetilde{K}[\xi])^\ell$. Define $\Lambda(M)$, the **uniform residue module** of M, to be the $\widetilde{K}[\xi,\rho]$-submodule of $(\widetilde{K}[\xi,\rho])^\ell$ generated by the elements $\sum_{|\nu|=\widetilde{\mathrm{o}}(f)} f_\nu(\xi)\rho^\nu$ for $f \in \widetilde{M}$.

The name uniform residue module is justified by the following lemma.

Lemma 3.2.5. — *Let M be a submodule of $(S_{m,n})^\ell$, $n \geq 1$, and let K' be a complete extension field of K. Suppose $\varepsilon \in |K'|$ with $1 > \varepsilon > \sigma(M)$. Put $N := \iota_\varepsilon(M)\cdot T_{m,n}(\varepsilon,K') \subset (T_{m,n}(\varepsilon,K'))^\ell$. Then $\widetilde{N} = \widetilde{K}'\cdot\Lambda(M)$, where we have identified $\widetilde{T}_{m,n}(\varepsilon,K')$ with $\widetilde{K}'[\xi,\rho]$.*

Proof. — Let $S'_{m,n} := S_{m,n}(E,K')$ and let $M' := S'_{m,n}\cdot M$. Choose a v-strict generating system $\{g_1,\ldots,g_r\}$ of M with $\varepsilon > \max_{1\leq i\leq r}\sigma(g_i)$. By Lemma 3.1.11 (ii), $\{g_1,\ldots,g_r\}$ is a v-strict generating system of M'. Hence by Theorem 3.2.3, $\{\iota_\varepsilon(g_1),\ldots,\iota_\varepsilon(g_r)\}$ is a $\|\cdot\|_{\sup}$-strict generating system of $\iota_\varepsilon(M)\cdot T_{m,n}(\varepsilon,K') = N$. Put $G_i := c^{-\widetilde{\mathrm{o}}(g_i)}\iota_\varepsilon(g_i)$ where $c \in K'$ is chosen with $|c| = \varepsilon$. By Lemma 3.1.4, $\{\widetilde{G}_1,\ldots,\widetilde{G}_r\}$ generates $\widetilde{N}$. □

Lemma 3.2.6. — *Let $I \subset S_{m,n}$ be an ideal. Suppose $\Lambda(I)$ is principal; then I is principal.*

Proof. — For $h \in \widetilde{S}_{m,n}$, let h° denote the leading form in ρ of the power series h. Note that $(hg)^\circ = h^\circ g^\circ$. Choose $h_1,\ldots,h_s \in \widetilde{I}$ such that $\{h_1^\circ,\ldots,h_s^\circ\}$ generates $\Lambda(I)$. Suppose $g \in \widetilde{K}[\xi,\rho]$ generates $\Lambda(I)$. Since each h_i° is a multiple of g, $\deg_\rho g \leq \min_{1\leq i\leq s}(\deg_\rho h_i^\circ) =: d$. Since g is a linear combination of the h_i°, $\widetilde{\mathrm{o}}(g) \geq \min_{1\leq i\leq s}\widetilde{\mathrm{o}}(h_i^\circ) = d$. Hence g is homogeneous in ρ of degree d, and $g = G^\circ$ for some $G \in \widetilde{I}$. By Corollary 2.2.4, it suffices to show that G generates $\widetilde{I}$.

Let $\widetilde{J}$ be the ideal of $\widetilde{S}_{m,n}$ generated by G. Clearly $\widetilde{I} \supset \widetilde{J}$; we will show that $\widetilde{I} = \widetilde{J}$. Suppose there is some $f \in \widetilde{I}\setminus\widetilde{J}$. By Theorem 3.1.3, we may assume that

$$\widetilde{\mathrm{o}}(f-h) \leq \widetilde{\mathrm{o}}(f) \tag{3.2.2}$$

for all $h \in \widetilde{J}$. Since $f^\circ \in \Lambda(I)$, there is some $a \in \widetilde{K}[\xi,\rho]$ such that $f^\circ = ag = (ag)^\circ = a^\circ G^\circ = (aG)^\circ$, contradicting (3.2.2). □

3.3. Contractions from Rational Polydiscs. — In this subsection, we transfer information from $T_{m,n}(\varepsilon)$ back to $S_{m,n}$. The main results are Theorem 3.3.1 and its Corollaries, which show, roughly speaking, how to replace powers of ε with powers of (ρ) for ε near 1. Of course, when K is discretely valued, ε cannot, in general, belong to $|K|$. It is therefore sometimes necessary to extend the ground field as we did in Subsection 3.2.

For $f \in K\langle\xi,\rho\rangle = T_{m+n}$, $n \geq 1$, let $\widetilde{d}(f) := \infty$ if $f = 0$. Otherwise, write $f(\xi) = \Sigma f_\nu(\xi)\rho^\nu$ and let $\widetilde{d}(f)$ be the largest $\ell \in \mathbb{N}$ such that for some ν with $|\nu| = \ell$ we have $\|f\| = \|f_\nu\|$. We call $\widetilde{d}(f)$ the **residue degree** of f. Note that if $\|f\| = 1$, $\widetilde{d}(f)$ is the total degree of $\widetilde{f}$ as a polynomial in ρ.

Let (A, v) be a normed ring and let $f = \Sigma f_\nu \rho^\nu$, $g = \Sigma g_\nu \rho^\nu \in A[[\rho]]$. We say g is a **majorant** of f iff $v(f_\nu) \leq v(g_\nu)$ for all ν.

Let $c \in A$ with $v(c) \leq 1$, and suppose

$$\sum_{|\nu|\leq a} \rho^\nu + \sum_{|\nu|>a} c^{|\nu|-a}\rho^\nu$$

is a majorant of f and

$$\sum_{|\nu|\leq b} \rho^\nu + \sum_{|\nu|>b} c^{|\nu|-b}\rho^\nu$$

is a majorant of g. Put $e := \max\{a, b\}$. Then

(i) $\displaystyle\sum_{|\nu|\leq e} \rho^\nu + \sum_{|\nu|>e} c^{|\nu|-e}\rho^\nu$ is a majorant of $f + g$, and

(ii) $\displaystyle\sum_{|\nu|\leq a+b} \rho^\nu + \sum_{|\nu|>a+b} c^{|\nu|-(a+b)}\rho^\nu$ is a majorant of fg.

Note, for any $f \in S_{m,n}$ with $\|f\| = 1$ and any $c \in K^\circ$, that $f(\xi, c\cdot\rho)$ is majorized by $\sum c^{|\nu|}\rho^\nu$. This fact will be used in the proof of the next theorem, which, for $f \in (S_{m,n})^\ell$ and M a submodule of $(S_{m,n})^\ell$, relates $v_M(f)$ and $\|\iota_\varepsilon(f)\|_{\iota_\varepsilon(M)\cdot T_{m,n}(\varepsilon)}$, when ε is sufficiently large. The proof shows, via the concept of majorization, that if the "slicing" in $(T_{m,n}(\varepsilon))^\ell$ is done carefully, then it pulls back to $(S_{m,n})^\ell$.

Theorem 3.3.1. — *Let M be a submodule of $(S_{m,n})^\ell$, $n \geq 1$, let $\varepsilon \in \sqrt{|K\setminus\{0\}|}$ with $1 > \varepsilon > \sigma(M)$. Then for every $f \in (S_{m,n})^\ell$,*

$$v_M(f) \leq (\|f\|, 2^{-\alpha}),$$

where $\alpha \in \mathbb{N}\cup\{\infty\}$ is the least element such that $\varepsilon^\alpha\|f\| \leq \|\iota_\varepsilon(f)\|_{\iota_\varepsilon(M)\cdot T_{m,n}(\varepsilon)}$. If $\alpha = \infty$, then $v_M(f) = (0, 0)$.

Proof. — Let K' be the completion of the algebraic closure of K, and put $S'_{m,n} := S_{m,n}(E, K')$, $T'_{m,n}(\varepsilon) := T_{m,n}(\varepsilon, K')$ and $M' := S'_{m,n}\cdot M$. By

Lemma 3.1.11, $\sigma(M') \leq \sigma(M)$ and $v_{M'}(f) = v_M(f)$. Certainly,

$$\|\iota_\varepsilon(f)\|_{\iota_\varepsilon(M')\cdot T'_{m,n}(\varepsilon)} \leq \|\iota_\varepsilon(f)\|_{\iota_\varepsilon(M)\cdot T_{m,n}(\varepsilon)}.$$

Therefore, we may assume $K = K'$, so that, in particular, $\varepsilon \in |K|$ and $T_{m,n}(\varepsilon)$ is isometrically isomorphic to T_{m+n}. Choose $c \in K$ with $|c| = \varepsilon$. We may replace ι_ε by ι'_ε as in Remark 3.2.1 (i).

Since $\varepsilon^{\widetilde{\mathrm{o}}(f)}\|f\| \leq \|\iota'_\varepsilon(f)\|$, if $\|\iota'_\varepsilon(f)\| = \|\iota'_\varepsilon(f)\|_{\iota'_\varepsilon(M)\cdot T_{m+n}}$ there is nothing to show. Therefore, we may assume that

$$\|\iota'_\varepsilon(f)\|_{\iota'_\varepsilon(M)\cdot T_{m+n}} < \|\iota'_\varepsilon(f)\|. \tag{3.3.1}$$

We may further assume that $\|f\| = 1$.

Let $\alpha \in \mathbb{N} \cup \{\infty\}$ be the least element such that $\varepsilon^\alpha \leq \|\iota'_\varepsilon(f)\|_{\iota'_\varepsilon(M)\cdot T_{m+n}}$. By (3.3.1), $\alpha > 0$. Fix $\beta \in \mathbb{N}$, $\beta < \alpha$. We must show that

$$v_M(f) < (\|f\|, 2^{-\beta}).$$

Let $\{g_1, \dots, g_r\}$ be a v-strict generating system of M with $\|g_1\| = \cdots = \|g_r\| = 1$ and $\varepsilon > \max_{1\leq i\leq r} \sigma(g_i)$. For $1 \leq i \leq r$, put $G_i := c^{-\widetilde{\mathrm{o}}(g_i)}\iota'_\varepsilon(g_i)$, where $c \in K$ with $|c| = \varepsilon$, and find $B \in \mathfrak{B}$ such that $\iota'_\varepsilon(f), G_1, \dots, G_r \in (B\langle\xi,\rho\rangle)^\ell$. Let $B = B_0 \supset B_1 \supset \dots$ be the natural filtration of B.

***Claim* (A).** — *Let $F \in (B_p\langle\xi,\rho\rangle)^\ell \setminus (B_{p+1}\langle\xi,\rho\rangle)^\ell$ and suppose for some $h \in \iota'_\varepsilon(M)\cdot T_{m+n}$ that $\|F - h\| < \|F\|$. Then there are polynomials $h_i \in B_p[\xi,\rho]$ such that*

(i) $\left\|F - \sum_{i=1}^r h_i G_i\right\| < \|F\|$, *and*

(ii) $\max\{\widetilde{\mathrm{o}}(G_i) + \deg_\rho(h_i) : h_i \neq 0\} = \widetilde{d}(F)$.

Let $\pi_p : B_p \to \widetilde{B}_p \subset \widetilde{K}'$ denote a residue epimorphism (of B-modules), and write $\widetilde{K} = \widetilde{B}_p \oplus V$ for some $\widetilde{B}$-vector space V. Then

$$\widetilde{T}_{m+n} = \widetilde{K}[\xi,\rho] = \widetilde{B}_p[\xi,\rho] \oplus V[\xi,\rho] \tag{3.3.2}$$

as $\widetilde{B}[\xi,\rho]$ modules. Since $\|F - h\| < \|F\|$,

$$\pi_p(F) \in (\iota'_\varepsilon(M) \cdot T_{m+n})^\sim.$$

Since $T_{m,n}(\varepsilon)$ is isometrically isomorphic to T_{m+n}, by Theorem 3.2.3 and Lemma 3.1.4, $\{\widetilde{G}_1, \dots, \widetilde{G}_r\}$ generates $(\iota'_\varepsilon(M) \cdot T_{m+n})^\sim$. Thus there are $\widetilde{h}_i \in \widetilde{K}[\xi,\rho]$ such that

$$\pi_p(F) = \sum_{i=1}^r \widetilde{h}_i \widetilde{G}_i. \tag{3.3.3}$$

By (3.3.2) we may assume that $\widetilde{h}_1, \dots, \widetilde{h}_r \in \widetilde{B}_p[\xi, \rho]$. Furthermore, since each component of each $\widetilde{G}_i$ is either 0 or a sum of monomials of total ρ-degree equal to $\widetilde{\mathrm{o}}(\widetilde{G}_i)$ we may assume that

$$\max\{\widetilde{\mathrm{o}}(G_i) + \widetilde{d}(\widetilde{h}_i) : \widetilde{h}_i \neq 0\} = \widetilde{d}(F).$$

Find $h_1, \dots, h_r \in B_p[\xi, \rho]$ with

$$\max\{\widetilde{\mathrm{o}}(G_i) + \deg_\rho(h_i) : h_i \neq 0\} = \widetilde{d}(F)$$

and $\pi_p(h_i) = \widetilde{h}_i$, $1 \leq i \leq r$. Now by (3.3.3),

$$\pi_p\left(F - \sum_{i=1}^{r} h_i G_i\right) = 0.$$

This proves the claim.

By (3.3.1) and Claim A, there are polynomials $h_{i0} \in B[\xi, \rho]$ such that $\max_{1 \leq i \leq r} \|h_{i0}\| = \|\iota'_\varepsilon(f)\|$, $\left\|\iota'_\varepsilon(f) - \sum_{i=1}^{r} h_{i0} G_i\right\| < \|\iota'_\varepsilon(f)\|$ and $\max\{\widetilde{\mathrm{o}}(G_i) + \deg_\rho(h_{i0})\colon h_{i0} \neq 0\} = \widetilde{d}(\iota'_\varepsilon(f))$. Moreover, since $\sum_\nu c^{|\nu|}\rho^\nu$ majorizes each component of $\iota'_\varepsilon(f)$,

$$\|h_{i0}\| \leq \varepsilon^{\widetilde{d}(\iota'_\varepsilon(f))} \leq \varepsilon^{\widetilde{\mathrm{o}}(G_i)} \cdot \varepsilon^{\deg_\rho(h_{i0})}.$$

In the next claim, we iterate this procedure.

***Claim* (B).** — *There is a finite sequence* $\{h_{ij}\} \subset B[\xi, \rho]$ *such that*

(i) *for each* s, $\left\|\iota'_\varepsilon(f) - \sum_{j=0}^{s} \sum_{i=1}^{r} h_{ij} G_i\right\| < \left\|\iota'_\varepsilon(f) - \sum_{j=0}^{s-1} \sum_{i=1}^{r} h_{ij} G_i\right\|$,

(ii) *for each* s, $\max\limits_{1 \leq i \leq r} \|h_{is}\| = \left\|\iota'_\varepsilon(f) - \sum_{j=0}^{s-1} \sum_{i=1}^{r} h_{ij} G_i\right\|$,

(iii) *for each* i, s, $\sum_{j=0}^{s} h_{ij}$ *is majorized by* $c^{\widetilde{\mathrm{o}}(G_i)} \sum_\nu c^{|\nu|}\rho^\nu$, *and*

(iv) $\left\|\iota'_\varepsilon(f) - \sum_{j \geq 0} \sum_{i=1}^{r} h_{ij} G_i\right\| < \varepsilon^\beta$.

Note that the sum in (iv) *is a finite sum.*

Assume h_{ij}, $1 \leq i \leq r$, $0 \leq j \leq s$, have been chosen so that conditions (i), (ii) and (iii) are satisfied, as they are by $h_{10}, \dots, h_{r0}$. Assume condition (iv) is not satisfied, and find $p \in \mathbb{N}$ so that

$$\iota'_\varepsilon(f) - \sum_{j=0}^{s} \sum_{i=1}^{r} h_{ij} G_i \in (B_p\langle \xi, \rho \rangle)^\ell \setminus (B_{p+1}\langle \xi, \rho \rangle)^\ell. \tag{3.3.4}$$

Since condition (iv) is not satisfied and since $\varepsilon^\beta > \|\iota'_\varepsilon(f)\|_{\iota'_\varepsilon(M)\cdot T_{m+n}}$, we may apply Claim A to $F := \iota'_\varepsilon(f) - \sum_{j=0}^{s}\sum_{i=1}^{r} h_{ij}G_i$. This yields polynomials $h_{is+1} \in B_p[\xi,\rho]$ such that

$$\left\|\iota'_\varepsilon(f) - \sum_{j=0}^{s+1}\sum_{i=0}^{r} h_{ij}G_i\right\| < \left\|\iota'_\varepsilon(f) - \sum_{j=0}^{s}\sum_{i=1}^{r} h_{ij}G_i\right\| \tag{3.3.5}$$

and

$$\max_{1\le i\le r}\{\widetilde{\text{o}}(G_i) + \deg_\rho h_{is+1} : h_{is+1} \neq 0\} = d, \tag{3.3.6}$$

where $d := \widetilde{d}\left(\iota'_\varepsilon(f) - \sum_{j=0}^{s}\sum_{i=1}^{r} h_{ij}G_i\right)$.

By (3.3.5), condition (i) is satisfied for $s+1$. Since $h_{is+1} \in B_p[\xi,\rho]$, by (3.3.4), condition (ii) is also satisfied for $s+1$. To prove (iii) for $s+1$, it suffices to show, for each $1 \le i \le r$, that $\|h_{is+1}\| \le \varepsilon^{\widetilde{\text{o}}(G_i)+\deg_\rho(h_{is+1})}$. If $h_{is+1} = 0$ we are done. Otherwise, by (3.3.6),

$$\deg_\rho(h_{is+1}) \le d - \widetilde{\text{o}}(G_i).$$

By (iii), each component of $\iota'_\varepsilon(f) - \sum_{j=0}^{s}\sum_{i=1}^{r} h_{ij}G_i$ is majorized by $\sum_\nu c^{|\nu|}\rho^\nu$. Therefore, $\left\|\iota'_\varepsilon(f) - \sum_{j=1}^{s}\sum_{i=1}^{r} h_{ij}G_i\right\| \le \varepsilon^d$.

Since (ii) is satisfied for $s+1$, the above yields

$$\begin{aligned}
\|h_{is+1}\| &\le \left\|\iota'_\varepsilon(f) - \sum_{j=0}^{s}\sum_{j=1}^{r} h_{ij}G_i\right\| \\
&\le \varepsilon^d \\
&= \varepsilon^{\widetilde{\text{o}}(G_i)+(d-\widetilde{\text{o}}(G_i))} \\
&\le \varepsilon^{\widetilde{\text{o}}(G_i)+\deg_\rho(h_{is+1})},
\end{aligned}$$

proving that (iii) is satisfied for $s+1$. The claim now follows from the fact that $|B\setminus\{0\}| \subset \mathbb{R}_+\setminus\{0\}$ is discrete.

For $1 \le i \le r$, put

$$h_i := c^{-\widetilde{\text{o}}(G_i)}\sum_{j\ge 0} h_{ij}.$$

Since h_i is a polynomial (recall that the above sum is finite), there is some $h_i^* \in S_{m,n}$ so that $h_i = \iota'_\varepsilon(h_i^*)$. By Claim B (iii), $\max_{1\le i\le r}\|h_i^*\| \le 1$. Write

$$\iota'_\varepsilon(f) - \sum_{j\ge 0}\sum_{i=1}^{r} h_{ij}G_i = \sum_\nu C_\nu(\xi)\rho^\nu.$$

Then

$$f-\sum_{i=1}^{r}h_i^*g_i=\sum_{\nu}c^{-|\nu|}C_\nu(\xi)\rho^\nu.$$

Note that

$$\left\|f-\sum_{i=1}^{r}h_i^*g_i\right\|\leq 1=\|f\|.$$

If $\left\|f-\sum_{i=1}^{r}h_i^*g_i\right\|<1$ we are done. Otherwise, $\left\|f-\sum_{i=1}^{r}h_i^*g_i\right\|=1$, and we want $\widetilde{\mathrm{o}}\left(f-\sum_{i=1}^{r}h_i^*g_i\right)>\beta$. Put $\gamma:=\widetilde{\mathrm{o}}\left(f-\sum_{i=1}^{r}h_i^*g_i\right)$. Then

$$\max_{|\nu|=\gamma}\|c^{-\gamma}C_\nu\|=1;$$

i.e., $\varepsilon^\gamma=\max_{|\nu|=\gamma}\|C_\nu\|\leq\left\|\iota'_\varepsilon(f)-\sum_{j\geq 0}\sum_{i=1}^{r}h_{ij}G_i\right\|<\varepsilon^\beta$. Therefore, $\gamma>\beta$.

Finally, in the case that $\alpha=\infty$, we must show that $v_M(f)=(0,0)$. By Theorem 3.1.3, we may assume that $v(f)=v_M(f)$ and hence $\|f\|=\|f\|_M$. By the above, we have

$$v(f)<(\|f\|,2^{-\beta})$$

for all $\beta\in\mathbb{N}$. Hence $f=0$; i.e., $v_M(f)=(0,0)$. □

Corollary 3.3.2. — *Let M be a submodule of $(S_{m,n})^\ell$, $n\geq 1$, and let $\varepsilon\in\sqrt{|K\setminus\{0\}|}$ with $1>\varepsilon>\sigma(M)$. Then $M=\iota_\varepsilon^{-1}(\iota_\varepsilon(M)\cdot T_{m,n}(\varepsilon))$.*

Proof. — Let $f\in\iota_\varepsilon^{-1}(\iota_\varepsilon(M)\cdot T_{m,n}(\varepsilon))$. Since $\iota_\varepsilon(f)\in\iota_\varepsilon(M)\cdot T_{m,n}(\varepsilon)$, Theorem 3.3.1 with $\alpha=\infty$ yields $v_M(f)=(0,0)$. Hence by Theorem 3.1.3, $f\in M$. □

Corollary 3.3.3. — *Let M be a submodule of $(S_{m,n})^\ell$, $n\geq 1$ and let $f\in(S_{m,n})^\ell$. Then*

$$\|f\|_M=\lim_{\substack{\varepsilon\to 1^-\\ \varepsilon\in\sqrt{|K|}}}\|\iota_\varepsilon(f)\|_{\iota_\varepsilon(M)\cdot T_{m,n}(\varepsilon)}.$$

Indeed, find $h\in M$ so that $v_M(f)=v(f-h)$, and let $F:=f-h$. Then for every $\varepsilon\in\sqrt{|K|}$, if $1>\varepsilon>\sigma(M)$, we have

$$\|f\|_M=\|F\|\geq\|\iota_\varepsilon(f)\|_{\iota_\varepsilon(M)\cdot T_{m,n}(\varepsilon)}\geq\varepsilon^{\widetilde{\mathrm{o}}(F)}\|F\|. \tag{3.3.7}$$

Moreover, when in addition $\varepsilon>\sigma(F)$, equality holds in the rightmost part of (3.3.7).

Proof. — The only assertion that needs proof is

$$\|\iota_\varepsilon(f)\|_{\iota_\varepsilon(M)\cdot T_{m,n}(\varepsilon)} \geq \varepsilon^{\widetilde{o}(F)}\|F\|. \tag{3.3.8}$$

Let $\alpha \in \mathbb{N} \cup \{\infty\}$ be the least element such that

$$\varepsilon^{\alpha}\|F\| < \|\iota_\varepsilon(F)\|_{\iota_\varepsilon(M)\cdot T_{m,n}(\varepsilon)} = \|\iota_\varepsilon(f)\|_{\iota_\varepsilon(M)\cdot T_{m,n}(\varepsilon)}.$$

If (3.3.8) does not hold, $\alpha \geq \widetilde{o}(F) + 1$. So by Theorem 3.3.1,

$$v_M(f) = v(F) = (\|F\|, 2^{-\widetilde{o}(F)}) \leq (\|F\|, 2^{-\widetilde{o}(F)-1}).$$

If $F \neq 0$, this is a contradiction. The additional assertion in the case that $\varepsilon > \sigma(F)$ follows from $\|\iota_\varepsilon(f)\|_{\iota_\varepsilon(M)\cdot T_{m,n}(\varepsilon)} \leq \|\iota_\varepsilon(F)\|_{\sup} = \varepsilon^{\widetilde{o}(F)} \cdot \|F\|$. □

Corollary 3.3.4. — *Let I be an ideal of $S_{m,n}$ and M a submodule of $(S_{m,n}/I)^{\ell}$. Let $\varphi : (S_{m,n})^{\ell} \to (S_{m,n}/I)^{\ell}$ denote the canonical projection and put $N := \varphi^{-1}(M)$. Let $\varepsilon \in \sqrt{|K \setminus \{0\}|}$ with $1 > \varepsilon > \sigma(N)$, and let $f \in (S_{m,n}/I)^{\ell}$. Then $v_M(f) \leq (\|f\|_{I\cdot(S_{m,n})^{\ell}}, 2^{-\alpha})$ where $\alpha \in \mathbb{N} \cup \{\infty\}$ is the least element such that*

$$\varepsilon^{\alpha}\|f\|_{I\cdot(S_{m,n})^{\ell}} \leq \|\iota_\varepsilon(f)\|_{\iota_\varepsilon(M)\cdot(T_{m,n}(\varepsilon)/\iota_\varepsilon(I)\cdot T_{m,n}(\varepsilon))}.$$

In particular, if $\alpha = \infty$ then $v_M(f) = 0$.

Proof. — By Lemma 3.1.4, there is some $F \in (S_{m,n})^{\ell}$ such that $\varphi(F) = f$ and $\|F\| = \|f\|_{I\cdot(S_{m,n})^{\ell}}$. Since

$$\|\iota_\varepsilon(f)\|_{\iota_\varepsilon(M)\cdot(T_{m,n}(\varepsilon)/\iota_\varepsilon(I)\cdot T_{m,n}(\varepsilon))} = \|\iota_\varepsilon(F)\|_{\iota_\varepsilon(N)\cdot T_{m,n}(\varepsilon)}$$

and

$$v_M(f) = v_N(F),$$

the conclusion follows from Theorem 3.3.1. □

3.4. Restrictions to Open Polydiscs. — In previous subsections, we studied properties of the restriction maps $\iota_\varepsilon \colon S_{m,n} \to T_{m,n}(\varepsilon)$ to the closed polydiscs $\operatorname{Max} T_{m,n}(\varepsilon)$. As in [**6**, Section 9.3], the collection $\{\operatorname{Max} T_{m,n}(\varepsilon) \colon \varepsilon \in \sqrt{|K \setminus \{0\}|}\}$ is an admissible open cover of $\cup_\varepsilon \operatorname{Max} T_{m,n}(\varepsilon)$. In fact, as we will see in Subsection 4.1, $\cup_\varepsilon \operatorname{Max} T_{m,n}(\varepsilon) = \operatorname{Max} S_{m,n}$. Properties of the restriction maps ι_ε gave us information about residue norms v_M.

In this subsection, we study properties of restrictions from $\operatorname{Max} S_{m,n}$ to finite unions of disjoint open polydiscs. When the polydiscs have K-rational centers, these restriction maps take the form $\varphi : S_{m,n} \to \oplus_{j=1}^{r} S_{0,m+n}$. Such restrictions are not related in any natural way to admissible covers of $\operatorname{Max} S_{m,n}$. Nonetheless, as we show in Theorems 3.4.3 and 3.4.6, such restrictions are isometries in the residue norms derived from $\|\cdot\|$ and, respectively, I and $\varphi(I)$, provided that the finite collection of open polydiscs is chosen appropriately.

In Subsection 5.5, we prove that for certain reduced quotients $S_{m,n}/I$, the norms $\|\cdot\|_I$ and $\|\cdot\|_{\sup}$ are equivalent. In that subsection we use Theorems 3.4.3 and 3.4.6 to reduce this to the much simpler case of reduced quotients $S_{0,m+n}/I$.

We first treat the case of a restriction to a finite union of disjoint open polydiscs with K-rational centers. The extension to the case of non-K-rational centers is explained in Definition 3.4.4, Lemma 3.4.5 and Theorem 3.4.6.

Definition 3.4.1. — Let $c_1, \ldots, c_r \in (K^\circ)^m$ with $|c_i - c_j| = 1$, $1 \leq i < j \leq r$. For $j = 1, \ldots, r$, consider the ideal I_j of $S_{m,n+m}$ given by

$$I_j := (\xi_1 - c_{j1} - \rho_{n+1}, \ldots, \xi_m - c_{jm} - \rho_{n+m}) \cdot S_{m,n+m}.$$

Put $I := \cap_{j=1}^r I_j$ and define

$$D_{m,n}(c) := S_{m,n+m}/I.$$

Let

$$\omega_c : S_{m,n} \to D_{m,n}(c)$$

be the K-algebra homomorphism induced by the natural inclusion $S_{m,n} \hookrightarrow S_{m,n+m}$.

For $c_1, \ldots, c_r$ as above, consider the open polydiscs

$$\Delta_{m,n}(c_j) := \{(a, b) \in (K')^{m+n} : |a - c_j| < 1 \text{ and } |b| < 1\},$$

where $K' \supset K$ is complete and algebraically closed. Put

$$\Delta_{m,n}(c) := \bigcup_{j=1}^r \Delta_{m,n}(c_j).$$

It is a consequence of the results in Subsection 5.3 that $D_{m,n}(c)$ is the ring of K-quasi-affinoid functions corresponding to the quasi-rational domain $\Delta_{m,n}(c)$, and that ω_c is an inclusion. This justifies regarding ω_c as a restriction to $\Delta_{m,n}(c)$. However, we make no use of the results of Subsection 5.3 here.

It is also a consequence of the results of Subsection 5.3 that $D_{m,n}(c)$ is isomorphic to $\oplus_{j=1}^r S_{0,n+m}$. The next lemma gives a proof of a sharper result.

It is easily checked that the assignments

$$\begin{aligned} \rho_i &\mapsto (\rho_i, \ldots, \rho_i), && 1 \leq i \leq n+m, \\ \xi_i &\mapsto (\rho_{n+i} + c_{1i}, \ldots, \rho_{n+i} + c_{ri}), && 1 \leq i \leq m, \end{aligned}$$

induce a K-algebra homomorphism

$$\chi_c : D_{m,n}(c) \to \bigoplus_{j=1}^r S_{0,n+m}.$$

Lemma 3.4.2. — *χ_c is an isometric isomorphism; in particular,*

$$\|\chi_c(f)\| = \|f\|_I$$

for every $f \in D_{m,n}(c)$.

Proof. — Note, by the Weierstrass Division Theorem, Theorem 2.3.2, that $S_{m,n+m}/I_j = S_{0,n+m}$, $1 \le j \le r$. The fact that χ_c is an isomorphism is now a consequence of [**25**, Theorem 1.4], and the fact that the ideals $I_1, \dots, I_r$ are coprime in pairs.

Since the map $D_{m,n}(c) \to D_{m,n}(c)/I_j \cdot D_{m,n}(c)$ is a contraction, $1 \le j \le r$, it follows that χ_c is a contraction. Thus we may define a $\widetilde{K}$-algebra homomorphism

$$\widetilde{\chi}_c \colon \widetilde{D}_{m,n}(c) \to \bigoplus_{j=1}^{r} \widetilde{S}_{0,n+m},$$

as in the paragraph preceding Lemma 3.1.10. To show that χ_c is an isometry, it suffices to show that $\widetilde{\chi}_c$ is injective.

By Lemma 3.1.4,

$$\widetilde{D}_{m,n}(c) = \widetilde{S}_{m,n+m}/\widetilde{I}.$$

It is not hard to see that

$$\widetilde{I}_j = (\xi_1 - \widetilde{c}_{j1} - \rho_{n+1}, \dots, \xi_m - \widetilde{c}_{jm} - \rho_{n+m}) \cdot \widetilde{S}_{m,n+m},$$

$1 \le j \le r$. (Indeed, there is a linear isometric change of variables under which the image of each ideal I_j is generated by $\xi_1, \dots, \xi_m$.) Because $|c_i - c_j| = 1$, $1 \le i < j \le r$, the ideals $\widetilde{I}_1, \dots, \widetilde{I}_r$ are coprime in pairs. Hence by [**25**, Theorem 1.3],

$$\bigcap_{j=1}^{r} \widetilde{I}_j = \prod_{j=1}^{r} \widetilde{I}_j.$$

We have:

$$\widetilde{I} = \left(\bigcap_{j=1}^{r} I_j\right)^{\sim} \subset \bigcap_{j=1}^{r} \widetilde{I}_j = \prod_{j=1}^{r} \widetilde{I}_j \subset \left(\prod_{j=1}^{r} I_j\right)^{\sim} \subset \left(\bigcap_{j=1}^{r} I_j\right)^{\sim}.$$

Thus $\widetilde{I} = \cap_{j=1}^{r} \widetilde{I}_j$. By [**25**, Theorem 1.4], $\widetilde{\chi}_c$ is an isomorphism. □

From now on, we will also denote by ω_c the map

$$\omega_c : S_{m,n} \to D_{m,n}(c) \xrightarrow{\chi_c} \bigoplus_{j=1}^{r} S_{0,n+m}.$$

Observe that

$$\omega_c(f(\xi,\rho)) = \bigoplus_{j=1}^{r} f(\rho_{n+1}+c_{j1},\ldots,\rho_{n+m}+c_{jm},\rho_1,\ldots,\rho_n).$$

Theorem 3.4.3. — *Let M be a submodule of $(S_{m,n})^\ell$. Suppose there are $c_1,\ldots,c_r \in (K^\circ)^m$ with $|c_i - c_j| = 1$, $1 \le i < j \le r$, such that for every $\mathfrak{p} \in \operatorname{Ass}((\widetilde{S}_{m,n})^\ell/\widetilde{M})$, there is an i, $1 \le i \le r$, with*

$$\mathfrak{m}_i := (\xi - \widetilde{c}_i, \rho) \supset \mathfrak{p},$$

(e.g., suppose K is algebraically closed). Consider the $S_{m,n}$-module homomorphism

$$\varphi : (S_{m,n})^\ell \to \left(\bigoplus_{j=1}^{r} S_{0,n+m}\right)^\ell$$

induced by ω_c. Put $N := \varphi(M)\cdot(\oplus_{j=1}^{r} S_{0,n+m})$. Then:

(i) *If $\{g_1,\ldots,g_s\}$ is a $\|\cdot\|$-strict generating system of M, then $\{\varphi(g_1),\ldots,\varphi(g_s)\}$ is a $\|\cdot\|$-strict generating system of N.*
(ii) *$\|f\|_M = \|\varphi(f)\|_N$ for every $f \in (S_{m,n})^\ell$.*
(iii) *$\varphi^{-1}(N) = M$.*

In particular, under the above assumptions on K, given an ideal I of $S_{m,n}$, there is an isometric embedding $\varphi : S_{m,n}/I \to A$, where A is a finite extension of $S_{0,d}$ and $d = \dim S_{m,n}/I$.

Proof. — (i) This follows from Lemma 3.1.10 (i) once we show that $\widetilde{\omega}_c$ is flat. Applying [**25**, Theorem 7.1], to each of the r maximal ideals of $\oplus_{j=1}^{r}\widetilde{S}_{0,n+m}$, we are reduced to proving that each map

$$(\widetilde{S}_{m,n})_{\mathfrak{m}_j} \to \widetilde{S}_{0,n+m} : f(\xi,\rho) \mapsto f(\rho_{n+1}+\widetilde{c}_{j1},\ldots,\rho_{n+m}+\widetilde{c}_{jm},\rho_1,\ldots,\rho_n)$$

is flat, $1 \le j \le r$. The flatness of these maps is a consequence of the Local Flatness Criterion ([**25**, Theorem 22.3]), because

$$\widetilde{S}_{m,n}/\mathfrak{m}_j^\ell \cong \widetilde{S}_{0,n+m}/(\rho_1,\ldots,\rho_{n+m})^\ell = \widetilde{K}[\rho]/(\rho)^\ell$$

and $\mathfrak{m}_j$ is mapped into $\operatorname{rad}(\widetilde{S}_{0,n+m})$.

(ii) Let $f \in (S_{m,n})^\ell$. By Lemma 3.1.4, we may assume that

$$\|f\| = \|f\|_M = 1,$$

and we must prove that

$$\|\varphi(f)\|_N = 1.$$

In other words, we may assume that $\widetilde{f} \notin \widetilde{M}$ and we must prove that $\widetilde{\varphi}(\widetilde{f}) \notin \widetilde{N}$. By part (i) and Lemma 3.1.4, it suffices to show that

$$\widetilde{\varphi}(\widetilde{f}) \notin \widetilde{\varphi}(\widetilde{M}) \cdot \left(\bigoplus_{j=1}^{r} \widetilde{S}_{0,n+m} \right).$$

Put

$$P := (\widetilde{S}_{m,n})^{\ell} / \widetilde{M},$$

$$A := \widetilde{S}_{m,n}, \quad B := \bigoplus_{j=1}^{r} (\widetilde{S}_{m,n})_{\mathfrak{m}_j}, \quad C := \bigoplus_{j=1}^{r} \widetilde{S}_{0,n+m}.$$

Consider the sequence

$$P \to P \otimes_A B \to (P \otimes_A B) \otimes_B C.$$

We wish to show that the composition is injective. The injectivity of $P \otimes_A B \to (P \otimes_A B) \otimes_B C$ is a consequence of [**25**, Theorem 7.5], because C is a faithfully flat B-algebra (see proof of part (i)). It remains to show that the map

$$P \to \bigoplus_{j=1}^{r} P_{\mathfrak{m}_j} = P \otimes_A B$$

is injective.

Let $x \in P \setminus \{0\}$. We must show for some j, $1 \leq j \leq r$, that

$$\operatorname{Ann}(x) := \{a \in \widetilde{S}_{m,n} \colon ax = 0\} \subset \mathfrak{m}_j.$$

By [**25**, Theorem 6.1], there is some associated prime ideal $\mathfrak{q} \in \operatorname{Ass}(P)$ such that $\operatorname{Ann}(x) \subset \mathfrak{q}$. But we have assumed that $\mathfrak{q} \subset \mathfrak{m}_j$ for some j, $1 \leq j \leq r$. This completes the proof of part (ii).

(iii) This is an immediate consequence of part (ii), above.

The last assertion is now a consequence of Remark 2.3.6 and the observation that $\oplus_{j=1}^{r} S_{0,n+m}$ is a finite $S_{0,n+m}$-algebra. □

In what follows, we treat the case that the centers c may be non-K-rational. Notice that even in the rational case, because K is non-Archimedean, discs do not have uniquely determined centers (indeed, every point of the disc is a center). Hence the rational "centers" actually correspond to points of $\widetilde{K}^m \times \{0\}^n$. In the non-$K$-rational case, they correspond to maximal ideals of $\widetilde{S}_{m,n}$. In other words, for c, $c' \in (K_{\mathrm{alg}}^{\circ})^m$, the rings of K-quasi-affinoid functions on the open unit polydiscs $\Delta_{m,n}(c)$ and $\Delta_{m,n}(c')$ coincide precisely when there is an element γ of the Galois group of K_{alg} over K such that $|c - \gamma(c')| < 1$. This occurs if, and only if, $\mathfrak{m}_{\widetilde{c}} = \mathfrak{m}_{\widetilde{c}'}$, where $\mathfrak{m}_{\widetilde{c}}$ is the maximal ideal of elements of $\widetilde{S}_{m,n}$ vanishing at $(\widetilde{c}, 0)$. (The reader may wish to refer to Subsections 4.1 and 5.3.) This motivates the following definition.

Definition 3.4.4. — Let $c_1, \ldots, c_r \in (K^\circ_{\mathrm{alg}})^m$ satisfy $\mathfrak{m}_{\widetilde{c}_i} \neq \mathfrak{m}_{\widetilde{c}_j}$, $1 \leq i < j \leq r$, and $[K(c) : K] = [\widetilde{K}(\widetilde{c}) : \widetilde{K}]$. For $j = 1, \ldots, r$, write $c_j = (c_{j1}, \ldots, c_{jm})$ and let $f_{j\ell}(\xi_1, \ldots, \xi_\ell)$ be the polynomial monic and of least degree in ξ_ℓ such that $f_{j\ell}(c_{j1}, \ldots, c_{j\ell}) = 0$. We may choose $f_{j\ell} \in K^\circ[\xi_1, \ldots, \xi_\ell]$.

Consider the ideal I_j of $S_{m,n+m}$ given by

$$I_j := (f_{j1}(\xi_1) - \rho_{n+1}, \ldots, f_{jm}(\xi_1, \ldots, \xi_m) - \rho_{n+m}) \cdot S_{m,n+m}.$$

Put $I := \cap_{j=1}^r I_j$ and define

$$D_{m,n}(c) := S_{m,n+m}/I.$$

Let

$$\omega_c : S_{m,n} \to D_{m,n}(c)$$

be the K-algebra homomorphism induced by the natural inclusion $S_{m,n} \hookrightarrow S_{m,n+m}$.

As we remarked above, $D_{m,n}(c)$ is again the ring of K-quasi-affinoid functions on $\Delta_{m,n}(c)$. When c is non-K-rational, the structure of $D_{m,n}(c)$ is only slightly more complicated.

For $i \neq j$, $\mathfrak{m}_{\widetilde{c}_i} \neq \mathfrak{m}_{\widetilde{c}_j}$. It follows from the Nullstellensatz for $\widetilde{K}[T]$ that $\mathfrak{m}_{\widetilde{c}_i} + \mathfrak{m}_{\widetilde{c}_j} = (1)$. Since $\widetilde{I}_i + \widetilde{I}_j + (\rho) \supset \mathfrak{m}_{\widetilde{c}_i} + \mathfrak{m}_{\widetilde{c}_j}$, $\widetilde{I}_i + \widetilde{I}_j$ contains a unit of the form

$$1 + f, \qquad f \in (\rho)\widetilde{S}_{m,n+m}.$$

This implies that the ideals I_j are coprime in pairs. By [**25**, Theorem 1.4], the induced map

$$\chi_c : D_{m,n}(c) \to \bigoplus_{j=1}^r S_{m,n+m}/I_j$$

is a K-algebra isomorphism.

Since $S_{m,n+m}/I_j = D_{m,n}(c)/I_j$, the map χ_c is a contraction. To see that it is an isometry, we show that the induced map

$$\widetilde{\chi}_c : \widetilde{D}_{m,n}(c) \to \bigoplus_{j=1}^r \widetilde{S}_{m,n+m}/\widetilde{I}_j$$

is an isomorphism. This is a consequence of the above-noted fact that the ideals $\widetilde{I}_j$ are coprime in pairs.

Each element $f_{j\ell}(\xi_1, \ldots, \xi_\ell) - \rho_{n+\ell}$ is regular in ξ_ℓ in the sense of Definition 2.3.1. Therefore, by the Weierstrass Division Theorem 2.3.2, each $S_{m,n+m}/I_j$ is a finite, free $S_{0,n+m}$-module.

We have established the following generalization of Lemma 3.4.2.

Lemma 3.4.5. — *With the above notation, χ_c is an isometric isomorphism; in particular,*

$$\|\chi_c(f)\| := \max_{1\leq j\leq r} \|f\|_{I_j} = \|f\|_I$$

for every $f \in D_{m,n}(c)$. Furthermore, there is a finite, torsion-free monomorphism $S_{0,n+m} \to D_{m,n}(c)$.

The generalization of Theorem 3.4.3 is

Theorem 3.4.6. — *Let M be a submodule of $(S_{m,n})^\ell$. Choose $c_1, \ldots, c_r \in (K^\circ_{\mathrm{alg}})^m$ with $\mathfrak{m}_{\widetilde{c}_i} \neq \mathfrak{m}_{\widetilde{c}_j}$, $1 \leq i < j \leq r$, such that for every $\mathfrak{p} \in \mathrm{Ass}\,((\widetilde{S}_{m,n})^\ell/\widetilde{M})$ there is an i, $1 \leq i \leq r$, with*

$$\mathfrak{m}_{\widetilde{c}_i} \supset \mathfrak{p},$$

where $\mathfrak{m}_{\widetilde{c}_i}$ is the maximal ideal of elements of $\widetilde{S}_{m,n}$ that vanish at $(\widetilde{c}_i, 0)$.

Consider the $S_{m,n}$-module homomorphism

$$\varphi : (S_{m,n})^\ell \to \left(\bigoplus_{j=1}^{r} S_{m,n+m}/I_j\right)^\ell$$

induced by $\chi_c \circ \omega_c$. Put $N := \varphi(M) \cdot (\oplus_{j=1}^r S_{m,n+m}/I_j)$. Then:

(i) *If $\{g_1, \ldots, g_s\}$ is a $\|\cdot\|$-strict generating system of M, then $\{\varphi(g_1), \ldots, \varphi(g_s)\}$ is a $\|\cdot\|_I$-strict generating system of N.*

(ii) *$\|f\|_M = \|\varphi(f)\|_N$ for every $f \in (S_{m,n})^\ell$.*

(iii) *$\varphi^{-1}(N) = M$.*

In particular, for any quasi-affinoid algebra $B = S_{m,n}/I$, there is an isometric embedding $\varphi : B \to A$, where A is a finite extension of $S_{0,d}$ and $d = \dim B$.

Proof. — The proof is nearly identical to that of Theorem 3.4.3. Note that each

$$\widetilde{S}_{m,n+m}/\widetilde{I}_j \cong S_{0,n+m}(E, K(c_j))^\sim$$

by the Cohen Structure Theorem [**25**, Theorem 28.3]. □

Remark 3.4.7. — By Corollary 5.1.10, the K-algebra homomorphisms φ of Theorems 3.4.3 and 3.4.6 are isometries in $\|\cdot\|_{\mathrm{sup}}$.

4. The Commutative Algebra of $S_{m,n}$

In this Section, we establish several key algebraic properties of the rings of separated power series. The rings $S_{m,n}$ satisfy a Nullstellensatz (Theorem 4.1.1), they are regular rings of dimension $m+n$ (Corollary 4.2.2), they are excellent when the characteristic of K is zero (Proposition 4.2.3), and sometimes when the characteristic of K is not zero (Example 4.2.4 and Proposition 4.2.5), and they are UFDs (Theorem 4.2.7).

4.1. The Nullstellensatz. — Let A be a K-algebra. We make the following definitions (see [**6**, Definition 3.8.1.2]). Let $\operatorname{Max} A$ denote the collection of all maximal ideals of A, and put

$$\operatorname{Max}_K A := \left\{\mathfrak{m} \in \operatorname{Max} A : A/\mathfrak{m} \text{ is algebraic over } K\right\}.$$

For $\mathfrak{m} \in \operatorname{Max}_K A$ and $f \in A$, denote by $f(\mathfrak{m})$ the image of f under the canonical residue epimorphism $\pi_{\mathfrak{m}} : A \to A/\mathfrak{m}$. Since $A/\mathfrak{m}$ is an algebraic field extension of K and since K is complete in $|\cdot|$, there is a unique extension of $|\cdot|$ to an absolute value on $A/\mathfrak{m}$, which we also denote by $|\cdot|$. Now define the function $\|\cdot\|_{\sup} : A \to \mathbb{R}_+ \cup \{\infty\}$ by

$$\|f\|_{\sup} := \begin{cases} 0 & \text{if } \operatorname{Max}_K A = \varnothing, \\ \sup\limits_{\mathfrak{m} \in \operatorname{Max}_K A} |f(\mathfrak{m})| & \text{if } \operatorname{Max}_K A \neq \varnothing,\ f(\operatorname{Max}_K A) \text{ bounded}, \\ \infty & \text{otherwise.} \end{cases}$$

If $f(\operatorname{Max}_K A)$ is bounded for all $f \in A$, then $\|\cdot\|_{\sup}$ is a K-algebra seminorm on A, called the **supremum seminorm** ([**6**, Lemma 3.8.1.3]). We denote the **nilradical** of an ideal I by $\mathfrak{N}(I) := \{f : f^n \in I \text{ for some } n \in \mathbb{N}\}$.

Theorem 4.1.1. — **(Nullstellensatz)**

(i) *Let I be any proper ideal of $S_{m,n}$, then* $\mathfrak{N}(I) = \bigcap\{\mathfrak{m} \in \operatorname{Max}_K S_{m,n} : \mathfrak{m} \supset I\}$.

(ii) $\operatorname{Max} S_{m,n} = \operatorname{Max}_K S_{m,n}$.

(iii) *Put*

$$U := \{\mathfrak{m} \in \operatorname{Max} K[\xi,\rho] : \max_{1 \le i \le m} |\xi_i(\mathfrak{m})| \le 1,\ \max_{1 \le j \le n} |\rho_j(\mathfrak{m})| < 1\}.$$

Then the map $\mathfrak{m} \mapsto \mathfrak{m} \cdot S_{m,n}$ is a bijective correspondence between U and $\operatorname{Max} S_{m,n}$.

Proof. — Since $S_{m,0} = T_m$, if $n = 0$ we are done by [**6**, Theorem 7.1.2.3, Proposition 7.1.1.1 and Lemma 7.1.1.2]. Assume $n \ge 1$.

(i) Let $I \subset S_{m,n}$ be a proper ideal and let $\varepsilon \in \sqrt{|K \setminus \{0\}|}$ with $\varepsilon > \sigma(I)$. By Corollary 3.3.2, $f^\ell \in I$ if, and only if, $\iota_\varepsilon(f)^\ell \in \iota_\varepsilon(I) \cdot T_{m,n}(\varepsilon)$. Hence $\mathfrak{N}(I) = S_{m,n} \cap \mathfrak{N}(\iota_\varepsilon(I) \cdot T_{m,n}(\varepsilon))$. Therefore (i) follows from the Nullstellensatz for $T_{m,n}(\varepsilon)$ ([**6**, Theorem 7.1.2.3]).

(ii) This is an immediate consequence of (i).

(iii) In case K is algebraically closed this follows immediately from (ii). Otherwise, it follows from (ii) by Faithfully Flat Base Change (Lemma 3.1.11(iii)). Alternatively, (iii) follows immediately from (ii) and the Weierstrass Preparation and Division Theorems as follows.

Let $\mathfrak{m} \in U$. Since $K[\xi, \rho]/\mathfrak{m}$ is algebraic over K, there are polynomials $f_i(\xi_i)$ and $g_j(\rho_j) \in \mathfrak{m}$, $1 \leq i \leq m$, $1 \leq j \leq n$. By [**6**, Proposition 3.8.1.7], we may assume that each f_i is regular in ξ_i and each g_j is regular in ρ_j in the senses of Definition 2.3.1. Applying the Weierstrass Division Theorems (Theorem 2.3.2) yields

$$K[\xi, \rho]/\mathfrak{m} = S_{m,n}/\mathfrak{m} \cdot S_{m,n};$$

hence $\mathfrak{m} \cdot S_{m,n} \in \operatorname{Max} S_{m,n}$.

Conversely, let $\mathfrak{m} \in \operatorname{Max} S_{m,n}$. By (ii), $\mathfrak{m} \in \operatorname{Max}_K S_{m,n}$. Since $S_{m,n}/\mathfrak{m}$ is algebraic over K, we obtain polynomials $f_i(\xi_i)$, $g_j(\rho_j) \in \mathfrak{m}$, $1 \leq i \leq m$, $1 \leq j \leq n$. By the Weierstrass Preparation Theorem (Corollary 2.3.3) we may assume that all $f_i(\xi_i)$ and $g_j(\rho_j)$ are monic polynomials, regular in the senses of Definition 2.3.1. Euclidean Division in $K[\xi, \rho]$ and Weierstrass Division in $S_{m,n}$ yield

$$K[\xi, \rho]/(\mathfrak{m} \cap K[\xi, \rho]) = S_{m,n}/\mathfrak{m}.$$

The fact that $\mathfrak{m} \cap K[\xi, \rho] \in U$ follows from the facts that no f_i nor g_j is a unit. □

Since $\|\cdot\|_{\sup}$ coincides with $\|\cdot\|_\varepsilon$ on $T_{m,n}(\varepsilon)$ ([**6**, Corollary 5.1.4.6]), it follows immediately from Theorem 4.1.1 that $\|\cdot\|_{\sup}$ coincides with $\|\cdot\|$ on $S_{m,n}$. A K-algebra A is called a **Banach function algebra** iff $\|\cdot\|_{\sup}$ is a complete norm on A. Hence when $S_{m,n}$ is complete in $\|\cdot\|$ (cf. Theorem 2.1.3), it is a Banach function algebra. In Subsection 5.5, we show that in many cases, reduced quotients of the $S_{m,n}$ are also Banach function algebras.

Proposition 4.1.2. — *Let $A = S_{m,n}/I$ and $\mathfrak{m} \in \operatorname{Max} A$. Consider the field $K' := A/\mathfrak{m}$, which is complete since it is a finite K-algebra. Then for each representative $f = \sum a_{\mu\nu}\xi^\mu\rho^\nu \in S_{m,n}$ of an element of A:*

(i) *$f(\mathfrak{m}) := f + \mathfrak{m} = \sum a_{\mu\nu}\overline{\xi}^\mu\overline{\rho}^\nu \in K'$, where $\overline{\xi} := \xi + \mathfrak{m}$, $\overline{\rho} := \rho + \mathfrak{m}$.*

(ii) *$|f(\mathfrak{m})| \leq \|f\|_I$. Indeed*

$$|f(\mathfrak{m})| \leq \|f^\ell\|_I^{1/\ell} \text{ for } \ell = 1, 2, \ldots.$$

(iii) *If $f = (f_1 + I) + (f_2 + I)$ where $f_1, f_2 \in S_{m,n}$, $\|f_1\| < 1$, $\|f_2\| \leq 1$ and $f_2 \in (\rho)S^\circ_{m,n}$, then $|f(\mathfrak{m})| < 1$.*

Proof. — (ii) and (iii) follow immediately from (i) and Theorem 4.1.1(iii). (i) is immediate if $K' = K$, since $f(\xi, \rho) - f(\overline{\xi}, \overline{\rho})$ belongs to the maximal ideal

$\{g \in S_{m,n} : g(\overline{\xi}, \overline{\rho}) = 0\}$, which must contain the polynomial generators of $\mathfrak{m}$. Now note that there is a natural inclusion $S_{m,n}(E,K) \hookrightarrow S_{m,n}(E,K')$. $\square$

In the affinoid case, the supremum seminorm behaves well with respect to extension of the ground field. This follows from the Noether Normalization Theorem for affinoid algebras [**6**, Corollary 6.1.2.2], from [**6**, Proposition 6.2.2.4], from [**6**, Lemma 6.2.2.3], and from the fact that $\|f\|_{\sup}$ cannot decrease after extension of the ground field (ground field extensions of affinoid algebras are faithfully flat: see Lemma 3.1.11 (iii)). The supremum seminorms on quotient rings of the $S_{m,n}$ also behave well with respect to ground field extensions, even though, unlike in the affinoid case, the supremum need not be attained.

Proposition 4.1.3. — *Let K' be a complete, valued field extension of K, let $E' \subset (K')^\circ$ be a complete, quasi-Noetherian ring (in characteristic p, let E' be a complete DVR) and put $S_{m,n} := S_{m,n}(E,K)$, $S'_{m,n} := S_{m,n}(E',K')$. Assume $S'_{m,n} \supset S_{m,n}$. Let I be an ideal of $S_{m,n}$ and put $I' := I \cdot S'_{m,n}$. Then for any $f \in S_{m,n}/I$,*

$$\sup\{|f(x)| : x \in \operatorname{Max} S_{m,n}/I\} = \sup\{|f(x)| : x \in \operatorname{Max} S'_{m,n}/I'\}.$$

Indeed, for any $f \in S_{m,n}/I$ and for any $c \in \mathbb{R}$, if $|f(x)| < c$ for all $x \in \operatorname{Max} S_{m,n}/I$ then also $|f(x)| < c$ for all $x \in \operatorname{Max} S'_{m,n}/I'$.

Proof. — Assume $|f(x)| < c$ for all $x \in \operatorname{Max} S_{m,n}/I$ and let $x_0 \in \operatorname{Max} S'_{m,n}/I'$. Let $\varepsilon \in \sqrt{|K \setminus \{0\}|}$ be such that $1 > \varepsilon > \max\{\sigma(I), \sigma(I'), \sigma(x_0)\}$. By the Maximum Modulus Principle [**6**, Proposition 6.2.1.4], we have: $\|\iota_\varepsilon(f)\|_{\sup} < c$, where the supremum is taken over the affinoid variety $\operatorname{Max}(T_{m,n}(\varepsilon)/\iota_\varepsilon(I) \cdot T_{m,n}(\varepsilon))$. By the above observation, it follows that $\|\iota_\varepsilon(f)\|_{\sup} < c$, where this time, the supremum is taken over $\operatorname{Max} T'_{m,n}(\varepsilon)/I' \cdot T'_{m,n}(\varepsilon)$. Thus $|f(x_0)| < c$. $\square$

Remark 4.1.4. — The Maximum Modulus Principle holds for quotients of $T_m = S_{m,0}$ (see [**6**, Proposition 6.2.1.4]), but not, in general, for quotients of $S_{m,n}$, $n > 0$. Nevertheless, for $f \in S_{m,n}/I$,

$$\|f\|_{\sup} \in \sqrt{|K|}.$$

This is a consequence of the quantifier elimination (cf. [**17**, Corollary 7.3.3]), and Proposition 4.1.3. It also follows from the results of this paper (see Corollary 5.1.11).

The following weak form of the Minimum Modulus Principle is an immediate consequence of the Nullstellensatz (Theorem 4.1.1). Let $A = S_{m,n}/I$ and let $f \in A$. If $\inf\{|f(x)| : x \in \operatorname{Max} A\} = 0$ then there is an $x \in \operatorname{Max} A$ such that $f(x) = 0$.

Remark 4.1.5. — Here we give a second proof that $\operatorname{Max} S_{m,n} = \operatorname{Max}_K S_{m,n}$.

We begin by defining an additive valuation w on $S_{m,n}$. Consider $\mathbb{R} \times \mathbb{N}^n$ as an ordered group with coordinatewise addition and lexicographic order. We define a map $w : S_{m,n} \to \mathbb{R} \times \mathbb{N}^n \cup \{\infty\}$ by putting $w(0) := \infty$ and, for $f \in S_{m,n} \setminus \{0\}$, $w(f) := (\alpha, \nu_0)$, where $\alpha \in \mathbb{R}$ and $\nu_0 \in \mathbb{N}^n$ are determined as follows. Write $f = \sum_{\mu,\nu} a_{\mu,\nu} \xi^\mu \rho^\nu = \sum_\nu f_\nu(\xi) \rho^\nu$. Then put $\alpha := \min_{\mu,\nu} \operatorname{ord} a_{\mu\nu}$ (where $\operatorname{ord} : K \to \mathbb{R}$ is the additive valuation corresponding to the absolute value $|\cdot| : K \to \mathbb{R}_+$) and let $\nu_0 \in \mathbb{N}^n$ be the element uniquely determined by the conditions

$$\|f_{\nu_0}\| = \|f\|, \text{ and}$$
$$\|f_\nu\| < \|f\| \text{ for all } \nu < \nu_0 \text{ lexicographically.}$$

We call the multi-index ν_0 the **total residue order** of f, and we call the coefficient $f_{\nu_0}(\xi)$ the **leading coefficient** of f. It is not difficult to show that w is an additive valuation on $S_{m,n}$.

Proposition. — *Each ideal of $S_{m,n}$ is strictly closed in w.*

Proof. — This is proved analogously to Theorem 3.1.3 using the facts that $\|B\langle\xi\rangle[\![\rho]\!] \setminus \{0\}\|$ is discrete and that $\mathbb{N}^n$ with the lexicographic order is well-ordered. We leave the details to the reader. (See also [**17**, Section 2.6].) □

Note that if I is an ideal of $S_{m,n}$ and if $\infty \neq w(f) \geq w(f-h)$ for each $h \in I$, then there is no element h of I with the same total residue order ν_0 as f and such that $\|h_{\nu_0}\| = \|f_{\nu_0}\| > \|f_{\nu_0} - h_{\nu_0}\|$.

Theorem. — $\operatorname{Max} S_{m,n} = \operatorname{Max}_K S_{m,n}$.

Proof. — If there is some $f \in \mathfrak{m}$ which is preregular (in the sense of Definition 2.3.4) in ξ (or ρ) then, after a change of variables among the ξ's (or ρ's), we may assume that f is regular in ξ_m (or in ρ_n). If f is regular in ξ_m (the case that f is regular in ρ_n is similar), then by Weierstrass Division, the map $S_{m-1,n} \to S_{m,n}/\mathfrak{m}$ is finite. Thus $\mathfrak{m}' := \mathfrak{m} \cap S_{m-1,n}$ is maximal, and we are done by induction on the number of variables. We henceforth assume that $\mathfrak{m}$ contains no element which is preregular in any variables.

For each $\nu \in \mathbb{N}^n$, let $\mathfrak{m}_\nu$ be the set in $S_{m,0}$ of leading coefficients of those elements of $\mathfrak{m}$ with total residue order ν. If $\mu_1 \leq \nu_1, \ldots, \mu_n \leq \nu_n$ then $\mathfrak{m}_\mu \subset \mathfrak{m}_\nu$. Let $\widetilde{\mathfrak{m}}_\nu = (\mathfrak{m}_\nu \cap S^\circ_{m,0})/(\mathfrak{m}_\nu \cap S^{\circ\circ}_{m,0})$, if $\mathfrak{m}_\nu \neq \varnothing$ and $\widetilde{\mathfrak{m}}_\nu = (0)$ otherwise. Then $\widetilde{\mathfrak{m}}_\nu$ is an ideal of $\widetilde{S}_{m,0}$. Note that none of the ideals $\widetilde{\mathfrak{m}}_\nu$ can be the unit ideal since then there would be an element of $\mathfrak{m}$ which is preregular in ρ. Since $\mathfrak{m} \neq (0)$, at least one $\tilde{\mathfrak{m}}_\nu \neq (0)$. Moreover, if A is any Noetherian ring and $\{I_\nu\}_{\nu\in\mathbb{N}^n}$ is a family of ideals of A such that $I_\mu \subset I_\nu$ whenever $\mu_1 \leq \nu_1, \ldots, \mu_n \leq \nu_n$, then the family $\{I_\nu\}_{\nu\in\mathbb{N}^n}$ is finite (induct on n).

We can therefore find some $a(\xi) \in S_{m,0}$ with $\|a\| = 1$ such that $\tilde{a} \in \tilde{\mathfrak{m}}_\nu$ for each $\tilde{\mathfrak{m}}_\nu \neq (0)$. Put

$$(4.1.1) \qquad c := a + 1.$$

Since $\|a\| = 1$ and a is not a unit of $S_{m,0}$, it follows that $\|c\| = 1$ and that c is not a unit. Furthermore, $c \notin \mathfrak{m}$ since clearly c is preregular in ξ. Thus there is some $f \in S_{m,n}$ such that $cf - 1 \in \mathfrak{m}$. By the above Proposition, we may assume that for each $h \in \mathfrak{m}$

$$(4.1.2) \qquad w(f) \geq w(f - h).$$

Write $f = \sum f_\nu(\xi)\rho^\nu$, and let f_{ν_0} be the leading coefficient of f. By (4.1.2), there is no $h \in \mathfrak{m}_{\nu_0}$ of total residue order ν_0 with $\|h\| = \|f\|$ and $\|f_{\nu_0} - h_{\nu_0}\| < \|f_{\nu_0}\|$.

Claim. — *$\|f_{\nu_0}\| > 1$ and $\nu_0 \neq 0$.*

If $\|f\| < 1$ then $cf - 1$ is a unit, contradicting the fact that $\mathfrak{m}$ is a proper ideal. Hence $\|f\| \geq 1$. If $\nu_0 = 0$, then since c is not a unit, $cf - 1$ is preregular in ξ, which is a contradiction. Hence $\|f\| \geq 1$ and $\nu_0 \neq 0$. If $\|f\| = 1$ and $\|f_0\| = 1$, then the total residue order of f is 0, a contradiction. If $\|f\| = 1$ and $\|f_0\| < 1$ then $cf - 1 \in \mathfrak{m}$ is a unit, also a contradiction. This proves the claim.

Let $\|f_{\nu_0}\| = |b|$. By the claim, $\frac{1}{b}(cf - 1)$ has total residue order ν_0 and leading coefficient $\frac{c}{b} f_{\nu_0} \in \mathfrak{m}_{\nu_0}$. But by (4.1.1), $cf_{\nu_0} \in \mathfrak{m}_{\nu_0}$ implies $(\frac{1}{b} f_{\nu_0})^\sim \in \tilde{\mathfrak{m}}_{\nu_0}$, contradicting (4.1.2). □

4.2. Completions. — One of the main applications of the Nullstellensatz is to give us information about maximal-adic completions of the $S_{m,n}$. In this subsection, we prove the following facts: $S_{m,n}$ is a regular ring of dimension $m+n$, restriction maps to closed subpolydiscs are flat, $S_{m,n}$ is a UFD, $S_{m,n}$ is excellent in characteristic 0 and sometimes in characteristic $p > 0$, and, when $S_{m,n}$ is a G-ring, radical ideals of $S_{m,n}$ stay radical when they are expanded under restriction maps to closed polydiscs.

Proposition 4.2.1. — *Let $\varepsilon \in \sqrt{|K \setminus \{0\}|}$, $1 > \varepsilon > 0$, let $\mathfrak{M} \in \operatorname{Max} T_{m,n}(\varepsilon)$, put $\mathfrak{m} := K[\xi, \rho] \cap \mathfrak{M}$, and $\mathfrak{N} := \iota_\varepsilon^{-1}(\mathfrak{M}) \in \operatorname{Max} S_{m,n}$. Then ι_ε induces K-algebra isomorphisms*

$$(i) \qquad S_{m,n}/\mathfrak{N}^\ell \cong T_{m,n}(\varepsilon)/\mathfrak{M}^\ell \cong K[\xi, \rho]/\mathfrak{m}^\ell$$

for every $\ell \in \mathbb{N}$.

Let I be an ideal of $S_{m,n}$. Suppose $\mathfrak{M} \in \operatorname{Max} T_{m,n}(\varepsilon)$ with $\mathfrak{M} \supset \iota_\varepsilon(I)$, and put $\mathfrak{N} := \iota_\varepsilon^{-1}(\mathfrak{M})$. Then ι_ε induces K-algebra isomorphisms

$$(ii) \qquad \big(S_{m,n}/I\big)\widehat{_{\mathfrak{N}}} \cong \big(T_{m,n}(\varepsilon)/\iota_\varepsilon(I) \cdot T_{m,n}(\varepsilon)\big)\widehat{_{\mathfrak{M}}},$$

where $\widehat{\ }$ *denotes the maximal-adic completion of a local ring.*

Proof. — (i) is immediate from the Weierstrass Preparation and Division Theorems, and Theorem 4.1.1(ii).

(ii) By part (i), ι_ε induces a K-algebra isomorphism $\widehat{\iota_\varepsilon} : (S_{m,n})\widehat{_{\mathfrak{N}}} \to (T_{m,n}(\varepsilon))\widehat{_{\mathfrak{M}}}$. Part (ii) now follows immediately from [**25**, Theorem 8.11]. □

Corollary 4.2.2. — *For each* $\mathfrak{m} \in \operatorname{Max} S_{m,n}$, $(S_{m,n})_{\mathfrak{m}}$ *is a regular local ring of Krull dimension* $m+n$; *moreover,* $S_{m,n}$ *is a regular ring.*

Proof. — By Hilbert's Nullstellensatz, each $\mathfrak{N} \in \operatorname{Max} K[\xi,\rho]$ can be generated by $m+n$ elements and $\dim K[\xi,\rho]\widehat{_{\mathfrak{N}}} = m+n$; in particular, $K[\xi,\rho]\widehat{_{\mathfrak{N}}}$ is a regular local ring. By Theorem 4.1.1, there is some $\varepsilon \in \sqrt{|K \setminus \{0\}|}$, $1 > \varepsilon > 0$, such that

$$\mathfrak{M} := \iota_\varepsilon(\mathfrak{m}) \cdot T_{m,n}(\varepsilon) \in \operatorname{Max} T_{m,n}(\varepsilon).$$

Now by Proposition 4.2.1,

$$(S_{m,n})\widehat{_{\mathfrak{m}}} \cong (T_{m,n}(\varepsilon))\widehat{_{\mathfrak{M}}} = (K[\xi,\rho])_{K[\xi,\rho]\cap\mathfrak{m}}\widehat{\ },$$

so $\dim(S_{m,n})\widehat{_{\mathfrak{m}}} = m+n$. It follows that $(S_{m,n})_{\mathfrak{m}}$ is a regular local ring of Krull dimension $m+n$. Moreover, by [**25**, Theorem 19.3], $S_{m,n}$ is a regular ring. □

Proposition 4.2.3. — *Assume* $\operatorname{Char} K = 0$. *Then* $S_{m,n}$ *is an excellent ring; in particular, it is a G-ring.*

Proof. — In light of Theorem 4.1.1 and Corollary 4.2.2, this follows directly from [**26**, Theorem 2.7]. □

The next example and proposition show that the situation in characteristic p is more complicated.

Example 4.2.4. — If $\operatorname{Char} K = p \neq 0$, then $S_{m,n} = S_{m,n}(K,E)$ may fail to be a G-ring. Assume, for the moment, that we have found an element $g \in K[\![\rho]\!] \setminus S_{0,1}$ such that $g^p \in S_{0,1}$ (cf. [**28**, Section A1, Example 6]). Put $\mathfrak{m} := (\rho) \cdot S_{0,1}$ and put $R := (S_{0,1})_{\mathfrak{m}}[g]$; if $S_{0,1}$ is a G-ring, so is R (see [**25**, Section 32, p. 260]). Since $R \subset K[\![\rho]\!]$, it is reduced. Put $\mathfrak{M} := \mathfrak{m}R$, and let $\widehat{R}$ denote the $\mathfrak{M}$-adic completion of R. Since $S_{0,1}$ is a UFD, $X^p - g^p$ is irreducible in $(S_{0,1})_{\mathfrak{m}}[X]$; hence

$$R = (S_{0,1})_{\mathfrak{m}}[X]/(X^p - g^p) \text{ and } \widehat{R} = K[X][\![\rho]\!]/(X^p - g^p) \cdot K[X][\![\rho]\!].$$

So $X - g$ is a non-zero nilpotent element of $\widehat{R}$, which is the direct sum of finitely many maximal-adic completions of R ([**25**, Theorem 8.15]). Thus, some maximal-adic completion of R is not reduced. It follows from [**25**, Theorem 32.2 (i)], that R, and hence $S_{0,1}$, cannot be a G-ring. An example of K, E and g can be constructed as follows: let $K := \mathbb{F}_p(t_1, t_2, \dots)((Z))$,

$E := \mathbb{F}_p(t_1^p, t_2^p, \ldots)$ and $g := \sum_{i\geq 0} t_i \rho^i$. In fact, a similar example can be constructed whenever $[E^{\frac{1}{p}} \cap K : E] = \infty$.

Proposition 4.2.5. — *Assume* $\operatorname{Char} K = p$. *Then:*

(i) *if $S_{m,n}$ is a finite extension of $(S_{m,n})^p$, then $S_{m,n}$ is excellent;*
(ii) *if $[K : K^p] < \infty$ and if $E \subset K^\circ$ is a complete DVR which is a finite extension of E^p (e.g., take $E = \mathbb{F}_p \subset K^\circ$), then $S_{m,n}$ is excellent;*
(iii) *if $E \subset K^\circ$ is a DVR and if K' is a complete, perfect, valued field extension of K, then there is a field E' with $E' \subset (K')^\circ$ such that $S_{m,n}(E', K')$ is an excellent and faithfully flat $S_{m,n}(E, K)$-algebra.*

Proof. — (i) By [**38**, Théorème 2.1], it suffices to show that $S_{m,n}$ is universally catenary. But this is an immediate consequence of [**25**, Theorem 31.6 and Corollary 4.2.2].

(ii) Put $S_{m,n} := S_{m,n}(E, K) = S_{m,n}(E^p, K)$. Then, $S_{m,n} = K \otimes_{K^p} S_{m,n}(E^p, K^p)$ is finite over $S_{m,n}(E^p, K^p)$ and by the Weierstrass Division Theorem 2.3.2, $S_{m,n}$ is finite over $(S_{m,n})^p$. Now apply part (i).

(iii) Lift $\tilde{K}'$ to $(K')^\circ$ by extending the lifting of $\tilde{E}$ given by E (see Remark 2.1.4 (iv)). By part (ii), $S_{m,n}(E', K')$ is excellent, and by Lemma 3.1.11 (i), it is faithfully flat over $S_{m,n}(E, K)$. □

A useful property of reduced G-rings is that they are analytically unramified in the sense of [**28**]. The next proposition shows that reduced quotients of $S_{m,n}$ are analytically unramified in a different sense, when $S_{m,n}$ is a G-ring. Example 4.2.4 shows what goes wrong if $S_{m,n}$ is a not a G-ring.

Proposition 4.2.6. — *Let I be an ideal of $S_{m,n}$, $n \geq 1$, and let $\varepsilon \in \sqrt{|K \setminus \{0\}|}$, $1 > \varepsilon > 0$. If $\varepsilon > \sigma(I)$ and $T_{m,n}(\varepsilon)/\iota_\varepsilon(I) \cdot T_{m,n}(\varepsilon)$ is reduced then $S_{m,n}/I$ is reduced. Suppose $S_{m,n}$ is a G-ring (e.g., use Proposition 4.2.3 or Proposition 4.2.5 (ii)). If $S_{m,n}/I$ is reduced then $T_{m,n}(\varepsilon)/\iota_\varepsilon(I) \cdot T_{m,n}(\varepsilon)$ is reduced.*

Proof. — Suppose $T_{m,n}(\varepsilon)/\iota_\varepsilon(I) \cdot T_{m,n}(\varepsilon)$ is reduced and suppose $f^r \in I$ for some $f \in S_{m,n}$; then $\iota_\varepsilon(f) \in \iota_\varepsilon(I) \cdot T_{m,n}(\varepsilon)$. Hence by Corollary 3.3.2, $f \in I$. Therefore, $S_{m,n}/I$ is reduced.

Suppose $S_{m,n}/I$ is reduced and that $S_{m,n}$ is a G-ring; we must prove that $T_{m,n}(\varepsilon)/\iota_\varepsilon(I) \cdot T_{m,n}(\varepsilon)$ is reduced. For this, it suffices to prove that $\big(T_{m,n}(\varepsilon)/\iota_\varepsilon(I) \cdot T_{m,n}(\varepsilon)\big)_{\mathfrak{m}}$ is reduced for every $\mathfrak{m} \in \operatorname{Max}\big(T_{m,n}(\varepsilon)/\iota_\varepsilon(I) \cdot T_{m,n}(\varepsilon)\big)$. Indeed, let A be a ring such that $A_{\mathfrak{m}}$ is reduced for every $\mathfrak{m} \in \operatorname{Max} A$, and suppose $f^r = 0$. Then $f \in \operatorname{Ker}(A \to A_{\mathfrak{m}})$ for every $\mathfrak{m} \in \operatorname{Max} A$. Consider the ideal $\mathfrak{a} := \{a \in A : af = 0\}$. If $\mathfrak{a} = (1)$, then $f = 0$, and we are done; otherwise, $\mathfrak{a} \subset \mathfrak{m}$ for some $\mathfrak{m} \in \operatorname{Max} A$. Hence $f \notin \operatorname{Ker}(A \to A_{\mathfrak{m}})$, a contradiction. Furthermore, by the Krull Intersection Theorem ([**25**, Theorem 8.10]), $\operatorname{Ker}(A \to \widehat{A}) = (0)$ for any Noetherian local ring A. Hence it suffices to prove that $\big(T_{m,n}(\varepsilon)/\iota_\varepsilon(I)\cdot T_{m,n}(\varepsilon)\big)\widehat{\ }_{\mathfrak{m}}$ is reduced for every $\mathfrak{m} \in \operatorname{Max}(T_{m,n}(\varepsilon)/\iota_\varepsilon(I)\cdot T_{m,n}(\varepsilon))$.

Let $\mathfrak{m} \in \operatorname{Max}(T_{m,n}(\varepsilon)/\iota_\varepsilon(I)\cdot T_{m,n}(\varepsilon))$, and put $\mathfrak{N} := S_{m,n}\cap\mathfrak{m} \in \operatorname{Max} S_{m,n}/I$. Since $S_{m,n}/I$ is reduced, so is $(S_{m,n}/I)_{\mathfrak{N}}$. Indeed, let A be a reduced ring and let $\mathfrak{m} \in \operatorname{Max} A$. If $f^r \in \operatorname{Ker}(A \to A_{\mathfrak{m}})$ then for some $a \in A\setminus\mathfrak{m}$, $af^r = 0$; whence $(af)^r = 0$. But A is reduced, so $af = 0$; i.e., $f \in \operatorname{Ker}(A \to A_{\mathfrak{m}})$. Now any quotient or localization of a G-ring is again a G-ring, so $(S_{m,n}/I)_{\mathfrak{N}}$ is a reduced G-ring. Thus

$$(S_{m,n}/I)_{\mathfrak{N}} \to (S_{m,n}/I)\widehat{\ }_{\mathfrak{N}}$$

is regular; in particular, it is faithfully flat. By [**25**, Theorem 32.2], $(S_{m,n}/I)\widehat{\ }_{\mathfrak{N}}$ is reduced. Then $(T_{m,n}(\varepsilon)/\iota_\varepsilon(I)\cdot T_{m,n}(\varepsilon))\widehat{\ }_{\mathfrak{m}}$ is reduced by Proposition 4.2.1. Since this holds for every $\mathfrak{m} \in \operatorname{Max}(T_{m,n}(\varepsilon)/\iota_\varepsilon(I)\cdot T_{m,n}(\varepsilon))$, we have proved that $T_{m,n}(\varepsilon)/\iota_\varepsilon(I)\cdot T_{m,n}(\varepsilon)$ is reduced. $\square$

Theorem 4.2.7. — *$S_{m,n}$ is a UFD.*

Proof. — A Noetherian integral domain is a UFD if, and only if, every height 1 prime is principal ([**25**, Theorem 20.1]). Let P be a height 1 prime ideal of $S_{m,n}$; we must prove that P is principal. By Lemma 3.2.6, it suffices to prove that the uniform residue ideal $\Lambda(P)$ is principal. Let K' be a finite algebraic extension of K such that $\widetilde{K}' = \widetilde{K}$, let $S'_{m,n} := S_{m,n}(E, K')$ and let $P' := P\cdot S'_{m,n}$. By Lemma 3.1.11, $\widetilde{P}' = \widetilde{P}\cdot\widetilde{S}'_{m,n} = \widetilde{P}$; hence $\Lambda(P') = \Lambda(P)$. It suffices to prove that $\Lambda(P')$ is principal.

Fix a finite algebraic extension K' of K such that for some $\varepsilon \in |K'|$, $1 > \varepsilon > \sigma(P)$, and $\widetilde{K}' = \widetilde{K}$.

Claim. — *For every $\mathfrak{n}' \in \operatorname{Max} S'_{m,n}$, $P'\cdot(S'_{m,n})\widehat{\ }_{\mathfrak{n}'}$ is a principal ideal.*

Let $\mathfrak{n}' \in \operatorname{Max} S'_{m,n}$ and put $\mathfrak{n} := \mathfrak{n}'\cap S_{m,n}$. Since $S'_{m,n}$ is finite over $S_{m,n}$, $\mathfrak{n} \in \operatorname{Max} S_{m,n}$. By Corollary 4.2.2, $S_{m,n}$ is a regular ring. Hence by [**25**, Theorem 20.3], $(S_{m,n})_{\mathfrak{n}}$ is a UFD. If $\mathfrak{n} \supset P$ then $\operatorname{ht} P\cdot(S_{m,n})_{\mathfrak{n}} = 1$, and if $\mathfrak{n} \not\supset P$ then $P\cdot(S_{m,n})_{\mathfrak{n}} = (1)$. Thus, the ideals $P\cdot(S_{m,n})_{\mathfrak{n}}$, $P'\cdot(S'_{m,n})_{\mathfrak{n}'}$ and $P'\cdot(S'_{m,n})\widehat{\ }_{\mathfrak{n}'}$ are all principal. This proves the claim.

Let $T'_{m,n}(\varepsilon) := T_{m,n}(\varepsilon, K')$. By the Claim and by Proposition 4.2.1, $\iota_\varepsilon(P')\cdot(T'_{m,n}(\varepsilon))\widehat{\ }_{\mathfrak{m}}$ is a principal ideal of $(T'_{m,n}(\varepsilon))\widehat{\ }_{\mathfrak{m}}$ for every $\mathfrak{m} \in \operatorname{Max} T'_{m,n}(\varepsilon)$. By [**25**, Exercise 8.3], $\iota_\varepsilon(P')\cdot(T'_{m,n}(\varepsilon))_{\mathfrak{m}}$ is a principal ideal, hence a free $(T'_{m,n}(\varepsilon))_{\mathfrak{m}}$-module for every $\mathfrak{m} \in \operatorname{Max} T'_{m,n}(\varepsilon)$. By [**25**, Theorem 7.12], $\iota_\varepsilon(P)\cdot T'_{m,n}(\varepsilon)$ is a projective ideal. But $T'_{m,n}(\varepsilon)$ is isomorphic to $T_{m+n}(K')$, which by

[**6**, Theorem 5.2.6.1], is a UFD. Hence by [**25**, Theorem 20.7], $\iota_\varepsilon(P)\cdot T'_{m,n}(\varepsilon)$ is principal. By Lemma 3.2.5, this implies that $\Lambda(P')$ is principal, as desired. $\square$

In the next lemma we collect together some facts on flatness.

Lemma 4.2.8. — *Let $\varepsilon \in \sqrt{|K\setminus\{0\}|}$ with $1 > \varepsilon > 0$. Let K' be a complete, valued field extension of K, let $E' \subset (K')^\circ$ be a complete, quasi-Noetherian ring, and put $S_{m,n} := S_{m,n}(E,K)$, $S'_{m,n} := S_{m,n}(E',K')$. Assume $S'_{m,n} \supset S_{m,n}$; e.g., take $E' \supset E$.*

(i) *The inclusion $\iota_\varepsilon : S_{m,n} \to T_{m,n}(\varepsilon)$ is flat*

The following inclusions are faithfully flat:

(ii) $S_{m,n}(E,K)^\circ \to S_{m,n}(E',K')^\circ$

(iii) $S_{m,n}(E,K) \to S_{m,n}(E',K')$

(iv) $S_{m,n}(E,K)^\sim \to S_{m,n}(E',K')^\sim$

(v) $T_{m,n}(\varepsilon) \to T'_{m,n}(\varepsilon)$

Proof. — (i) Consider the map $\iota_\varepsilon : S_{m,n} \to T_{m,n}(\varepsilon)$. Let $\mathfrak{M}$ be a maximal ideal of $T_{m,n}(\varepsilon)$, put $\mathfrak{m} := \iota_\varepsilon^{-1}(\mathfrak{M})$, $A := (S_{m,n})_{\mathfrak{m}}$ and $B := (T_{m,n}(\varepsilon))_{\mathfrak{M}}$. By [**25**, Theorem 7.1], it suffices to show that the induced map $\iota_\varepsilon : A \to B$ is flat. Let $\widehat{A}$, $\widehat{B}$ be the maximal-adic completions, respectively, of the local rings A, B. By Proposition 4.2.1 (ii), $\widehat{A} \cong \widehat{B}$, and by [**25**, Theorem 8.14], $A \to \widehat{A} \cong \widehat{B}$ and $B \to \widehat{B}$ are faithfully flat. Part (i) now follows by descent.

(ii), (iii) and (iv) are Lemma 3.1.11 (iv), (iii) and (i), respectively.

(v) For some $s \in \mathbb{N}$, $\varepsilon^s \in |K|$. Let $c \in K$ with $|c| = \varepsilon^s$, and let I be the ideal of T_{m+2n} generated by $\rho_i^s - \rho_{i+n}c$, $1 \le i \le n$. By [**6**, Theorem 6.1.5.4],

$$T_{m,n}(\varepsilon) = T_{m+2n}/I \text{ and } T'_{m,n}(\varepsilon) = T_{m+2n}/I\cdot T'_{m+2n}.$$

It therefore suffices to show that the inclusion $T_m \to T'_m$ is faithfully flat. But this is Lemma 3.1.11 (iii) with $n = 0$. $\square$

Note that the inclusion $S^\circ_{m,n} \hookrightarrow T_{m,n}(\varepsilon)^\circ$ is not flat. Indeed, find $c \in K$ and $\ell \in \mathbb{N}$ such that $|c| = \varepsilon^\ell$. Let

$$\begin{aligned} M &:= \{(f,g) \in (S^\circ_{m,n})^2 : cf + \rho^\ell g = 0\}, \text{ and} \\ N &:= \{(f,g) \in (T_{m,n}(\varepsilon)^\circ)^2 : cf + \rho^\ell g = 0\}. \end{aligned}$$

If $S^\circ_{m,n} \hookrightarrow T_{m,n}(\varepsilon)^\circ$ were flat, then $N = \iota_\varepsilon(M)\cdot T_{m,n}(\varepsilon)^\circ$. But $(\frac{\rho^\ell}{c}, -1) \in N \setminus \iota_\varepsilon(M)\cdot T_{m,n}(\varepsilon)^\circ$.

5. The Supremum Semi-Norm and Open Domains

In this section, we investigate algebraic and topological relations between residue norms and the supremum seminorm on a quasi-affinoid algebra (i.e., a quotient ring $S_{m,n}/I$). The key topological concepts are power-boundedness and quasi-nilpotence (see Definition 5.1.7). The first main result is Theorem 5.1.5, which asserts that each $h \in S_{m,n}/I$ with $\|h\|_{\sup} \leq 1$ is integral over the subring of all $a \in S_{m,n}/I$ with $\|a\|_I \leq 1$. Moreover, if $|h(x)| < 1$ for all $x \in \operatorname{Max} S_{m,n}/I$, then h is integral over the set of all $a \in S_{m,n}/I$ with $v_I(a) \leq (1,1)$. It then follows (Corollary 5.1.8) for $f \in S_{m,n}/I$ that f is power bounded if, and only if, $\|f\|_{\sup} \leq 1$, and that f is quasi-nilpotent if, and only if, $|f(x)| < 1$ for all $x \in \operatorname{Max} S_{m,n}/I$. These are the quasi-affinoid analogues of well-known properties of affinoid algebras. In Subsection 5.2 we use the results of Subsection 5.1 to show that K-algebra homomorphisms are continuous (Theorem 5.2.3). Hence all residue norms on a quasi-affinoid algebra are equivalent (Corollary 5.2.4); i.e., the topology of a quasi-affinoid algebra is independent of presentation. We also prove an Extension Lemma (Theorem 5.2.6) for quasi-affinoid maps. The results of Subsection 5.1 also lead, as in the affinoid case, to a satisfactory theory of open quasi-affinoid subdomains. In particular, in Subsection 5.3 we define quasi-rational subdomains (Definition 5.3.3), and show, using the Extension Lemma (Theorem 5.2.6), that they are quasi-affinoid subdomains. Subsection 5.4 contains the definition and elementary properties of the "tensor product" in the quasi-affinoid category. In Subsection 5.5 we show when $\operatorname{Char} K = 0$ and in many cases when $\operatorname{Char} K = p$, that if $S_{m,n}/I$ is reduced then the residue norm $\|\cdot\|_I$ and the supremum norm $\|\cdot\|_{\sup}$ are equivalent. If in addition E is such that $S_{m,n}$ is complete then $S_{m,n}/I$ is a Banach function algebra.

5.1. Relations with the Supremum Seminorm. — The first step towards proving Theorem 5.1.5 is an analogue of that theorem for $T_{m,n}(\varepsilon)/\iota_\varepsilon(I) \cdot T_{m,n}(\varepsilon)$ uniformly in ε, where ε is a sufficiently large element of $\sqrt{|K \setminus \{0\}|}$.

Let A be a Noetherian ring and let $I \subset A$ be an ideal. For $r = 0, 1, \ldots,$ let I_r denote the intersection of all minimal prime divisors of I of height r (if there are none, put $I_r := (1)$.) Clearly, $\mathfrak{N}(I) = \cap_{r \geq 0} I_r$, where $\mathfrak{N}(I)$ denotes the nilradical of I, and each I_r is a radical ideal. The ideals I_r are the **equidimensional components** of the ideal I.

In Lemma 5.1.1 we show that the ideals $\iota_\varepsilon(I_r) \cdot T_{m,n}(\varepsilon)$ generate the equidimensional components of the ideal $\iota_\varepsilon(I) \cdot T_{m,n}(\varepsilon)$, in the case that $S_{m,n}$ is a G-ring. This is important in applying [**6**, Proposition 6.2.2.2], in a uniform way.

Lemma 5.1.1. — *Let I be an ideal of $S_{m,n}$, $n \geq 1$, and let $\varepsilon \in \sqrt{|K \setminus \{0\}|}$, $1 > \varepsilon > 0$. Put $J := \iota_\varepsilon(I) \cdot T_{m,n}(\varepsilon)$. Then $J_r = \mathfrak{N}(\iota_\varepsilon(I_r) \cdot T_{m,n}(\varepsilon))$, $r \geq 0$. Thus, if $S_{m,n}$ is a G-ring, then $J_r = \iota_\varepsilon(I_r) \cdot T_{m,n}$, $r \geq 0$.*

Proof. — Since J_r is a radical ideal, by the Nullstellensatz (Theorem 4.1.1), it suffices to show, for each $\mathfrak{m} \in \operatorname{Max} T_{m,n}(\varepsilon)$, that $\mathfrak{m} \supset J_r$ if, and only if, $\iota_\varepsilon^{-1}(\mathfrak{m}) \supset I_r$.

Let A be any Noetherian ring, let $I \subset A$ be an ideal, and let $\mathfrak{m} \in \operatorname{Max} A$. By [**25**, Theorem 6.2], $\mathfrak{m} \supset P \supset I$ is a prime divisor of I if, and only if, $P \cdot A_\mathfrak{m}$ is a prime divisor of $I \cdot A_\mathfrak{m}$. Thus, $\mathfrak{m} \supset I_r$ if, and only if, $I \cdot A_\mathfrak{m}$ has a minimal prime divisor of height r.

Claim. — *Let $I \subset A$ be an ideal, and let $\mathfrak{m} \in \operatorname{Max} A$. Then $\mathfrak{m} \supset I_r$ if, and only if, $I \cdot (A_\mathfrak{m})\widehat{\ }$ has a minimal prime divisor of height r.*

By the foregoing, we may assume that A is a local ring with maximal ideal $\mathfrak{m}$, and we must show that I has a minimal prime divisor of height r if, and only if, $I \cdot \widehat{A}$ has one. (As usual, $\widehat{A}$ denotes the maximal-adic completion of A.)

Let $\mathfrak{p} \in \operatorname{Spec} A$ and let $\mathfrak{P} \in \operatorname{Spec} \widehat{A}$ be a minimal prime divisor of $\mathfrak{p} \cdot \widehat{A}$; we will show that $\operatorname{ht}\mathfrak{P} = \operatorname{ht}\mathfrak{p}$. Since $\widehat{A}$ is flat over A ([**25**, Theorem 8.8]), this follows from [**25**, Theorem 15.1 (ii)], if we can show that $\mathfrak{p} = \mathfrak{P} \cap A$. By the Going-Down Theorem ([**25**, Theorem 9.5]), there is some $\mathfrak{Q} \in \operatorname{Spec} \widehat{A}$ such that $\mathfrak{Q} \subset \mathfrak{P}$ and $\mathfrak{Q} \cap A = \mathfrak{p}$; hence $\mathfrak{P} \supset \mathfrak{Q} \supset \mathfrak{p} \cdot \widehat{A}$. Since $\mathfrak{P}$ is a minimal prime divisor of $\mathfrak{p} \cdot \widehat{A}$, $\mathfrak{Q} = \mathfrak{P}$. Therefore, $\mathfrak{p} = \mathfrak{P} \cap A$, as desired.

Suppose $\mathfrak{p} \in \operatorname{Spec} A$ is a minimal prime divisor of I of height r, and let $\mathfrak{P} \in \operatorname{Spec} \widehat{A}$ be a minimal prime divisor of $\mathfrak{p} \cdot \widehat{A}$. Then $\operatorname{ht}\mathfrak{P} = \operatorname{ht}\mathfrak{p} = r$. We will show that $\mathfrak{P}$ is a minimal prime divisor of $I \cdot \widehat{A}$. If $\mathfrak{P} \supset \mathfrak{Q} \supset I \cdot \widehat{A}$ for some $\mathfrak{Q} \in \operatorname{Spec} \widehat{A}$, then

$$\mathfrak{p} = \mathfrak{P} \cap A \supset \mathfrak{Q} \cap A \supset I.$$

Since $\mathfrak{p}$ is a minimal prime divisor of I, $\mathfrak{p} = \mathfrak{Q} \cap A$; i.e., $\mathfrak{Q} \supset \mathfrak{p} \cdot \widehat{A}$. Since $\mathfrak{P}$ is a minimal prime divisor of $\mathfrak{p} \cdot \widehat{A}$, $\mathfrak{Q} = \mathfrak{P}$. Thus $\mathfrak{P}$ is a minimal prime divisor of $I \cdot \widehat{A}$.

Suppose $\mathfrak{P} \in \operatorname{Spec} \widehat{A}$ is a minimal prime divisor of $I \cdot \widehat{A}$ of height r, and put $\mathfrak{p} := \mathfrak{P} \cap A$. Then $\mathfrak{P}$ is a minimal prime divisor of $\mathfrak{p} \cdot \widehat{A}$, so $\operatorname{ht}\mathfrak{p} = r$. We will show that $\mathfrak{p}$ is a minimal prime divisor of I. If $\mathfrak{p} \supset \mathfrak{q} \supset I$ for some $\mathfrak{q} \in \operatorname{Spec} A$, then by the Going-Down Theorem ([**25**, Theorem 9.5]), there is some $\mathfrak{Q} \in \operatorname{Spec} \widehat{A}$ with $\mathfrak{P} \supset \mathfrak{Q}$ and $\mathfrak{q} = \mathfrak{Q} \cap A$. Since $\mathfrak{P}$ is a minimal prime divisor of $I \cdot \widehat{A}$, $\mathfrak{Q} = \mathfrak{P}$, so $\mathfrak{q} = \mathfrak{p}$. Therefore, $\mathfrak{p}$ is a minimal prime divisor of I, proving the claim.

Let $\mathfrak{m} \in \operatorname{Max} T_{m,n}(\varepsilon)$ and put $\mathfrak{n} := \iota_\varepsilon^{-1}(\mathfrak{m})$. By the Claim, and by Proposition 4.2.1,

$$\begin{aligned} \mathfrak{m} \supset J_r \quad &\Leftrightarrow \quad J \cdot (T_{m,n}(\varepsilon))\widehat{_\mathfrak{m}} \text{ has a minimal prime divisor of height } r \\ &\Leftrightarrow \quad I \cdot (S_{m,n})\widehat{_\mathfrak{n}} \text{ has a minimal prime divisor of height } r \\ &\Leftrightarrow \quad \mathfrak{n} \supset I_r, \end{aligned}$$

as desired. The last assertion of the lemma follows from Proposition 4.2.6. □

Let $\Lambda(I_r)$ be the uniform residue ideal of an equidimensional component I_r. The next proposition allows us to lift a Noether normalization map $\widetilde{T}_d \to \widetilde{T}_m/\Lambda(I_r)$ to affinoid algebras corresponding to the restriction of $S_{m,n}/I$ to closed polydiscs $\operatorname{Max} T_{m,n}(\varepsilon)$, uniformly in ε for ε large enough.

Proposition 5.1.2. — (cf. [**4**, Satz 3.1].) *Let $\varphi : T_d \to T_m$ be a K-algebra homomorphism, let I be an ideal of T_m, and let $\psi : T_d \to T_m/I$ be the composition of φ with the canonical projection $T_m \to T_m/I$. Now by* [**6**, Section 6.3], *φ induces a $\widetilde{K}$-algebra homomorphism $\widetilde{\varphi} : \widetilde{T}_d \to \widetilde{T}_m$. Let $\widetilde{\tau} : \widetilde{T}_d \to \widetilde{T}_m/\widetilde{I}$ be the composition of $\widetilde{\varphi}$ with the canonical projection $\widetilde{T}_m \to \widetilde{T}_m/\widetilde{I}$. Suppose that $\widetilde{\tau}$ is a finite monomorphism and that the $\widetilde{T}_d$-module $\widetilde{T}_m/\widetilde{I}$ can be generated by r elements. Then ψ is a finite monomorphism and the T_d-module T_m/I can be generated by r elements.*

Proof. — Put $J := \operatorname{Ker} \psi \subset T_d$; we will show that $J = (0)$. Let $f \in J$, $\|f\| \leq 1$. Since $f \in J$, $\varphi(f) \in I$; hence $\widetilde{\varphi}(\widetilde{f}) = \varphi(f)^\sim \in \widetilde{I}$. This implies $\widetilde{J} \subset \operatorname{Ker} \widetilde{\tau} = (0)$. Thus by Lemma 3.1.4, $J = (0)$; i.e., ψ is a monomorphism.

Find $G_1, \dots, G_r, g_1, \dots, g_s \in T_m^\circ$, with $g_1, \dots, g_s \in I$, such that the images of $\widetilde{G}_1, \dots, \widetilde{G}_r$ in $\widetilde{T}_m/\widetilde{I}$ generate the $\widetilde{T}_d$-module $\widetilde{T}_m/\widetilde{I}$, and $\{\widetilde{g}_1, \dots, \widetilde{g}_s\}$ generates the ideal $\widetilde{I}$. We will show that the images of $G_1, \dots, G_r$ in T_m/I generate the T_d-module T_m/I. Indeed, let $f \in T_m$; we will find $H_1, \dots, H_r \in T_d$ and $h_1, \dots, h_s \in T_m$ such that

$$f - \sum_{j=1}^{r} \varphi(H_j) G_j = \sum_{j=1}^{s} h_j g_j.$$

We may take $\|f\| \leq 1$. Let $B \in \mathfrak{B}$ with

$$f, \varphi(\xi_1), \dots, \varphi(\xi_d), G_1, \dots, G_r, g_1, \dots, g_s \in B\langle\xi\rangle \subset T_m.$$

Let $B = B_0 \supset B_1 \supset \cdots$ be the natural filtration of B.

Claim. — *Let $F \in B_p\langle\xi\rangle \setminus B_{p+1}\langle\xi\rangle \subset T_m$. There are $H_1, \ldots, H_r \in B_p\langle\xi\rangle \subset T_d$ and $h_1, \ldots, h_s \in B_p\langle\xi\rangle \subset T_m$ such that*

$$F - \sum_{j=1}^{r} \varphi(H_j)G_j - \sum_{j=1}^{s} h_j g_j \in B_{p+1}\langle\xi\rangle \subset T_m.$$

Let $\pi_p : B_p \to \widetilde{B}_p \subset \widetilde{K}$ denote a residue epimorphism, and write $\widetilde{K} = \widetilde{B}_p \oplus V$ for some $\widetilde{B}$-vector space V. Then

$$\begin{aligned} \widetilde{T}_m &= \widetilde{K}[\xi_1, \ldots, \xi_m] = \widetilde{B}_p[\xi] \oplus V[\xi] \text{ and} \\ \widetilde{T}_d &= \widetilde{K}[\xi_1, \ldots, \xi_d] = \widetilde{B}_p[\xi] \oplus V[\xi] \end{aligned} \tag{5.1.1}$$

as $\widetilde{B}[\xi]$-modules. Furthermore, since $\widetilde{\varphi}(\xi_1), \ldots, \widetilde{\varphi}(\xi_d) \in \widetilde{B}[\xi]$,

$$\begin{aligned} &\widetilde{\varphi}(\widetilde{B}_p[\xi]) \subset \widetilde{B}_p[\xi] \text{ and} \\ &\widetilde{\varphi}(V[\xi]) \subset V[\xi]. \end{aligned} \tag{5.1.2}$$

Since the images of $\widetilde{G}_1, \ldots, \widetilde{G}_r$ in $\widetilde{T}_m/\widetilde{I}$ generate the $\widetilde{T}_d$-module $\widetilde{T}_m/\widetilde{I}$, and since $\{\widetilde{g}_1, \ldots, \widetilde{g}_s\}$ generates the ideal $\widetilde{I}$ in $\widetilde{T}_m$, there are $\widetilde{H}_1, \ldots, \widetilde{H}_r \in \widetilde{T}_d$ and $\widetilde{h}_1, \ldots, \widetilde{h}_s \in \widetilde{T}_m$ such that

$$\pi_p(F) - \sum_{j=1}^{r} \widetilde{\varphi}(\widetilde{H}_j)\widetilde{G}_j - \sum_{j=1}^{s} \widetilde{h}_j \widetilde{g}_j = 0. \tag{5.1.3}$$

By (5.1.1) and (5.1.2), we may assume

$$\begin{aligned} &\widetilde{H}_1, \ldots, \widetilde{H}_r \in \widetilde{B}_p[\xi] \subset \widetilde{T}_d \text{ and} \\ &\widetilde{h}_1, \ldots, \widetilde{h}_s \in \widetilde{B}_p[\xi] \subset \widetilde{T}_m. \end{aligned}$$

Find $H_1, \ldots, H_r \in B_p\langle\xi\rangle \subset T_d$ and $h_1, \ldots, h_s \in B_p\langle\xi\rangle \subset T_m$ so that

$$\begin{aligned} &\pi_p(H_1) = \widetilde{H}_1, \ldots, \pi_p(H_r) = \widetilde{H}_r \text{ and} \\ &\pi_p(h_1) = \widetilde{h}_1, \ldots, \pi_p(h_s) = \widetilde{h}_s. \end{aligned}$$

By (5.1.3),

$$F - \sum_{j=1}^{r} \varphi(H_j)G_j - \sum_{j=1}^{s} h_j g_j \in B_{p+1}\langle\xi\rangle \subset T_m.$$

This proves the claim.

Now, $|B \setminus \{0\}| \subset \mathbb{R}_+ \setminus \{0\}$ is discrete, and $B\langle\xi\rangle$ is complete. Thus since φ is continuous ([**6**, Theorem 6.1.3.1]), iterated application of the Claim yields the desired result. □

The following lemma is a key step towards proving Theorem 5.1.5.

Lemma 5.1.3. — *Assume $S_{m,n}$ is a G-ring (e.g., use Proposition 4.2.3 or Proposition 4.2.5 (ii)), and let I be an ideal of $S_{m,n}$. Then there is an $e \in \mathbb{N}$ such that for every $\varepsilon \in |K|$ with $1 > |\varepsilon| > \sigma(I)$ and for every $f \in S_{m,n}/I$, $\iota'_\varepsilon(f) \in T_{m+n}/\iota'_\varepsilon(I) \cdot T_{m+n}$ satisfies an equation of the form*

$$t^e + a_1 t^{e-1} + \cdots + a_e = 0$$

where the $a_i \in T_{m+n}/\iota'_\varepsilon(I) \cdot T_{m+n}$ satisfy $\max_{1 \le i \le e} \|a_i\|^{1/i}_{\iota'_\varepsilon(I) \cdot T_{m+n}} = \|\iota'_\varepsilon(f)\|_{\sup}$.

Proof. — Let $\Lambda(I)$ be the uniform residue ideal of I as in Definition 3.2.4. By Noether Normalization, there is a $\widetilde{K}$-algebra homomorphism $\widetilde{\varphi} : \widetilde{T}_d \to \widetilde{T}_{m+n}$ such that $\widetilde{\tau}\colon \widetilde{T}_d \to \widetilde{T}_{m+n}/\Lambda(I)$ is a finite monomorphism where $\widetilde{\tau}$ is the composition of $\widetilde{\varphi}$ with the canonical projection $\widetilde{T}_{m+n} \to \widetilde{T}_{m+n}/\Lambda(I)$. Let $I_0, I_1, \ldots,$ be defined as in Lemma 5.1.1. Since $I \subset I_r$ for $r \ge 0$, $\Lambda(I) \subset \Lambda(I_r) \subset \widetilde{T}_{m+n}$, $r \ge 0$. Thus by Noether Normalization, for $r \ge 0$, there is a $\widetilde{K}$-algebra homomorphism $\widetilde{\varphi}_r : \widetilde{T}_{d_r} \to \widetilde{T}_d$ such that $\widetilde{\tau}_r : \widetilde{T}_{d_r} \to \widetilde{T}_{m+n}/\Lambda(I_r)$ is a finite monomorphism, where $\widetilde{\tau}_r$ is the composition of $\widetilde{\varphi} \circ \widetilde{\varphi}_r$ with the canonical projection $\widetilde{T}_{m+n} \to \widetilde{T}_{m+n}/\Lambda(I_r)$. Suppose the $\widetilde{T}_{d_r}$-module $\widetilde{T}_{m+n}/\Lambda(I_r)$ is generated by e_r elements, $r \ge 0$, and find $\alpha \in \mathbb{N}$ such that $\mathfrak{N}(I)^\alpha \subset I$ (where $\mathfrak{N}$ denotes the nilradical). Put

$$e := \alpha \sum_{r=0}^{m+n} e_r.$$

We will show that e is the exponent sought in the lemma. Fix $\varepsilon \in |K|$, $1 > \varepsilon > \sigma(I)$. By [**6**, Proposition 6.1.1.4], there are K-algebra homomorphisms $\varphi : T_d \to T_{m+n}$ and $\varphi_r : T_{d_r} \to T_d$, $0 \le r \le m+n$, that correspond modulo $K^{\circ\circ}$, respectively, to $\widetilde{\varphi} : \widetilde{T}_d \to \widetilde{T}_{m+n}$ and $\widetilde{\varphi}_r : \widetilde{T}_{d_r} \to \widetilde{T}_d$. Put $J := \iota'_\varepsilon(I) \cdot T_{m+n}$. Let $\psi : T_d \to T_{m+n}/J$ and $\psi_r : T_{d_r} \to T_{m+n}/\iota'_\varepsilon(I_r) \cdot T_{m+n}$, $0 \le r \le m+n$, be defined, respectively, by composing φ with the canonical projection $T_{m+n} \to T_{m+n}/J$ and by composing $\varphi \circ \varphi_r$ with the canonical projection $T_{m+n} \to T_{m+n}/\iota'_\varepsilon(I_r) \cdot T_{m+n}$. Since $\widetilde{\tau}$, $\widetilde{\tau}_0, \ldots, \widetilde{\tau}_{m+n}$ are finite monomorphisms, by Proposition 5.1.2, each of ψ, $\psi_0, \ldots, \psi_{m+n}$ is a finite monomorphism, moreover the T_{d_r}-module $T_{m+n}/\iota'_\varepsilon(I_r) \cdot T_{m+n}$ is generated by e_r elements, $0 \le r \le m+n$. By Lemma 5.1.1, $J_r = \iota'_\varepsilon(I_r) \cdot T_{m+n}$. Since each J_r is a radical ideal and since $\operatorname{ht} \mathfrak{p} = r$ for every prime divisor $\mathfrak{p}$ of J_r, each ψ_r is a finite torsion-free monomorphism.

Fix $f \in S_{m,n}/I$ with $\|f\|_{\sup} \le 1$, and put $F := \iota'_\varepsilon(f)$. For $0 \le r \le m+n$, let $Q_r \in T_{d_r}[t]$ be the monic polynomial of least degree such that $Q_r(F)$ vanishes in T_{m+n}/J_r. Write

$$Q_r = t^{\ell_r} + a_{r1} t^{\ell_r - 1} + \cdots + a_{r\ell_r}.$$

Since ψ_r is a finite, torsion-free monomorphism, by [**6**, Proposition 6.2.2.2],

$$\max_{1\leq i\leq \ell_r} \|a_{ri}\|^{1/i} = \|F\|_{\sup}.$$

Furthermore, by the Cayley-Hamilton Theorem [**25**, Theorem 2.1], $\ell_r = \deg Q_r \leq e_r$.

We may regard each Q_r as an element of $T_d[t]$ via the K-algebra homomorphism φ_r. Put

$$Q := \left(\prod_{r=0}^{m+n} Q_r\right)^{\alpha} = t^{\ell} + a_1 t^{\ell-1} + \cdots + a_{\ell}.$$

By [**6**, Corollary 3.2.1.6], $\max_{1\leq i\leq \ell} \|a_i\|^{1/i} = \|F\|_{\sup}$, $\ell \leq e$, and by Proposition 4.2.6, $Q(F)$ vanishes in T_{m+n}/J. It follows that $\iota'_{\varepsilon}(f)$ satisfies the equation

$$t^e + a_1 t^{e-1} + \cdots + a_{\ell} t^{e-\ell} = 0,$$

as desired. □

In Lemma 5.1.3, we assumed that $\varepsilon \in |K|$ and that $S_{m,n}$ is a G-ring in order to make some computations. Under these assumptions we obtained monic polynomials of degree e over $T_{m,n}(\varepsilon)$ satisfied by $h \in S_{m,n}/I$. The coefficients of these polynomials, in addition, satisfy certain estimates depending on $\|h\|_{\sup}$. In Lemma 5.1.4 we show that the computations of Lemma 5.1.3 are not affected by ground field extensions; i.e., they remain valid for $\varepsilon \in \sqrt{|K \setminus \{0\}|}$ and whether or not $S_{m,n}$ is a G-ring. This allows us to transfer the data back to $S_{m,n}$ by examining the module M of relations among $h^e, h^{e-1}, \ldots, 1$.

Lemma 5.1.4. — *Let I be an ideal of $S_{m,n}$ and let M be a submodule of $(S_{m,n}/I)^{\ell}$. Let K' be a complete, valued extension field of K, let $E' \subset (K')^{\circ}$ be a complete, quasi-Noetherian ring with $E' \supset E$ (recall, if* $\operatorname{Char} K = p > 0$, *we assume E' is also a DVR), and put*

$$\begin{aligned} S'_{m,n} &:= S_{m,n}(E', K') \supset S_{m,n}, \\ I' &:= I \cdot S'_{m,n}, \text{ and} \\ M' &:= M \cdot (S'_{m,n}/I') \subset (S'_{m,n}/I')^{\ell}. \end{aligned}$$

By φ denote the canonical projections

$$\begin{aligned} &(S'_{m,n})^{\ell} \to (S'_{m,n}/I')^{\ell}, \text{ and} \\ &(S_{m,n})^{\ell} \to (S_{m,n}/I)^{\ell}. \end{aligned}$$

Put

$$N := \varphi^{-1}(M), \text{ and}$$
$$N' := \varphi^{-1}(M') = N \cdot S'_{m,n},$$

and let $\varepsilon \in |K'|$ *with* $1 > \varepsilon > \sigma(N')$. *Put*

$$T'_{m+n} := K'\langle \xi, \rho \rangle.$$

By π *denote projection of an* ℓ*-tuple on the first coordinate. Suppose there is some*

$$f \in \iota'_\varepsilon(M') \cdot (T'_{m+n}/\iota'_\varepsilon(I') \cdot T'_{m+n})$$

with $\|f\|_{\iota'_\varepsilon(I')\cdot T'_{m+n}} \leq 1$ *and* $\pi(f) = 1$. *Then there is some* $F \in M$ *with* $\|F\|_I \leq 1$ *and* $\pi(F) = 1$.

Proof. — It suffices to show that $\pi(\widetilde{N})$ is the unit ideal; indeed, by Lemma 3.1.11, it suffices to show that $\pi(\widetilde{N}')$ is the unit ideal. Let $\Lambda(N')$ be the uniform residue module of N' as in Definition 3.2.4. It suffices to show that $\pi(\Lambda(N'))$ is the unit ideal. Denote also by φ the canonical projection

$$(T'_{m+n})^\ell \to (T'_{m+n}/\iota'_\varepsilon(I') \cdot T'_{m+n})^\ell.$$

By Lemma 3.1.4 with $n = 0$, there is some

$$F \in \varphi^{-1}(\iota'_\varepsilon(M') \cdot (T'_{m+n}/\iota'_\varepsilon(I') \cdot T'_{m+n}))$$

with $\|F\| = \|f\|_{\iota'_\varepsilon(I')\cdot T'_{m+n}} \leq 1$ and $\pi(F) = 1 + h$ for some $h \in \iota'_\varepsilon(I') \cdot T'_{m+n}$. Since $(h, 0, \ldots, 0) \in \operatorname{Ker} \varphi$, we may assume that $\pi(F) = 1$. Since

$$\varphi^{-1}(\iota'_\varepsilon(M') \cdot (T'_{m+n}/\iota'_\varepsilon(I') \cdot T'_{m+n})) = \iota'_\varepsilon(N') \cdot T'_{m+n},$$

by Lemma 3.2.5, $\widetilde{F} \in \Lambda(N')$. □

Theorem 5.1.5. — *Let* I *be an ideal of* $S_{m,n}$. *There is an* $e \in \mathbb{N}$ *such that each* $h \in S_{m,n}/I$ *with* $\|h\|_{\sup} \leq 1$ *satisfies a polynomial equation of the form*

$$t^e + a_1 t^{e-1} + \cdots + a_e = 0,$$

where $a_1, \ldots, a_e \in S_{m,n}/I$ *and each* $\|a_i\|_I \leq 1$. *Moreover, if* $|h(x)| < 1$ *for all* $x \in \operatorname{Max} S_{m,n}/I$ *then each* $v_I(a_i) < (1, 1)$.

Proof. — Write $S_{m,n} := S_{m,n}(E, K)$. Let K' be the completion of the algebraic closure of K. If $\operatorname{Char} K = 0$, let $E' := E$ and if $\operatorname{Char} K = p > 0$, we use Remark 2.1.4 to find $E' \supset E$ as in Proposition 4.2.5 (iii). Hence $S'_{m,n} := S_{m,n}(E', K')$ is a G-ring by Proposition 4.2.3 or Proposition 4.2.5 (iii). Let $I' := I \cdot S'_{m,n}$. By Proposition 4.1.3, $\|h\|_{\sup} \leq 1$, where the supremum is computed in $\operatorname{Max} S'_{m,n}/I'$.

Applying Lemma 5.1.3 to $S'_{m,n}/I'$ yields an integer e. Put

$$M := \left\{ (a_0, \dots, a_e) \in (S_{m,n}/I)^{e+1} : \sum_{i=0}^{e} a_i h^{e-i} = 0 \right\},$$
$$M' := \left\{ (a_0, \dots, a_e) \in (S'_{m,n}/I')^{e+1} : \sum_{i=0}^{e} a_i h^{e-i} = 0 \right\},$$
$$M_0 := \{(a_0, \dots, a_e) \in M : a_0 = 0\}, \text{ and}$$
$$M'_0 := \{(a_0, \dots, a_e) \in M' : a_0 = 0\}.$$

Choose $\varepsilon \in |K'|$ with $1 > \varepsilon > 0$ and ε suitably large, as in Lemma 5.1.3, and put

$$L' := \left\{ (b_0, \dots, b_e) \in (T'_{m+n}/\iota'_\varepsilon(I') \cdot T'_{m+n})^{e+1} : \sum_{i=0}^{e} b_i \iota'_\varepsilon(h^{e-i}) = 0 \right\},$$
$$L'_0 := \{(b_0, \dots, b_e) \in L' : b_0 = 0\}.$$

Since T'_{m+n} is isometrically isomorphic to $T_{m,n}(\varepsilon, K')$, by Lemma 4.2.8 (i) and (ii), we have:

$$M' = M \cdot (S'_{m,n}/I'),$$
$$M'_0 = M_0 \cdot (S'_{m,n}/I'),$$
$$L' = \iota'_\varepsilon(M') \cdot (T'_{m+n}/\iota'_\varepsilon(I') \cdot T'_{m+n}), \text{ and}$$
$$L'_0 = \iota'_\varepsilon(M'_0) \cdot (T'_{m+n}/\iota'_\varepsilon(I') \cdot T'_{m+n}).$$

Lemma 5.1.3 yields

$$b_1, \dots, b_e \in T'_{m+n}/\iota'_\varepsilon(I') \cdot T'_{m+n}$$

such that

$$\max_{1 \le i \le e} (\|b_i\|_{\iota'_\varepsilon(I') \cdot T'_{m+n}})^{\frac{1}{i}} = \|\iota'_\varepsilon(h)\|_{\sup} \le 1, \text{ and}$$
$$(1, b_1, \dots, b_e) \in L'.$$

Lemma 5.1.4 implies that there are $a_1, \dots, a_e \in S_{m,n}/I$ such that

$$\|a_i\|_I \le 1, \quad 1 \le i \le e, \text{ and}$$
$$(1, a_1, \dots, a_e) \in M.$$

This proves the first assertion.

Suppose now that $|h(x)| < 1$ for all $x \in \operatorname{Max} S_{m,n}/I$; then the same inequality holds for $x \in \operatorname{Max} S'_{m,n}/I'$ by Proposition 4.1.3. Hence $\|\iota'_\varepsilon(h)\|_{\sup} < 1$. Since

$$\iota'_\varepsilon((1, a_1, \dots, a_e)) - (1, b_1, \dots, b_e) \in L'_0,$$

we get

$$\|\iota'_\varepsilon((0,a_1,\ldots,a_e))\|_{L'_0} \leq \|(0,b_1,\ldots,b_e)\|_{\iota'_\varepsilon(I')\cdot T'_{m+n}} < 1.$$

By Corollary 3.3.4, this yields

$$v_{M'_0}((0,a_1,\ldots,a_e)) < (1,1).$$

Hence by Lemma 3.1.11(ii),

$$v_{M_0}((0,a_1,\ldots,a_e)) < (1,1),$$

as desired. □

Remark 5.1.6. — Let I be an ideal of $S_{m,n}$ and define the seminorm $v_{\sup}$: $S_{m,n}/I \to \mathbb{R}_+ \times \mathbb{R}_+$ by

$$v_{\sup}(h) := (\|h\|_{\sup}, 2^{-\alpha}),$$

where $\alpha := \inf\{\beta \in \mathbb{R}_+ : \exists\varepsilon_0 \in \sqrt{|K\setminus\{0\}|}\ \forall\varepsilon \in \sqrt{|K\setminus\{0\}|}$ with $1 > \varepsilon > \varepsilon_0, \varepsilon^\beta\|h\|_{\sup} \leq \|\iota_\varepsilon(h)\|_{\sup}\}$. In fact $\alpha \in \sqrt{|K\setminus\{0\}|}$. Indeed if $\|h\|_{\sup} \neq 0$, the function

$$\varepsilon \mapsto \|\iota_\varepsilon(h)\|_{\sup}/\|h\|_{\sup}$$

is a definable function of ε, in the sense of [**17**] and [**23**]. By the analytic elimination theorem of [**23**, Corollary 4.3] it follows immediately that $\alpha \in \sqrt{|K\setminus\{0\}|}$ and that $\varepsilon^\alpha\|h\|_{\sup} = \|\iota_\varepsilon(h)\|_{\sup}$ for $\varepsilon < 1$ but sufficiently large.

There is an $e \in \mathbb{N}$ such that each $h \in S_{m,n}/I$ satisfies a polynomial equation of the form

$$t^e + a_1 t^{e-1} + \cdots + a_e = 0$$

where $a_1,\ldots,a_e \in S_{m,n}/I$ and $\max_{1\leq i\leq e} v_I(a_i)^{1/i} \leq v_{\sup}(h)$.

Definition 5.1.7. — Let I be an ideal of $S_{m,n}$. An element $f \in S_{m,n}/I$ is called **power-bounded** iff the set $\{\|f^\ell\|_I : \ell \in \mathbb{N}\} \subset \mathbb{R}$ is bounded. By $\mathfrak{b}(S_{m,n}/I)$ denote the set of all power-bounded elements; it is a subring of $S_{m,n}/I$. An element $f \in S_{m,n}/I$ is called **topologically nilpotent** iff $\{\|f^\ell\|_I : \ell \in \mathbb{N}\}$ is a zero sequence. By $\mathfrak{t}(S_{m,n}/I)$ denote the set of topologically nilpotent elements; it is an ideal of $\mathfrak{b}(S_{m,n}/I)$. An element $f \in S_{m,n}/I$ is called **quasi-nilpotent** iff for some $\ell \in \mathbb{N}$, $f^\ell \in \mathfrak{t} + (\rho)\mathfrak{b}$. By $\mathfrak{q}(S_{m,n}/I)$ denote the set of quasi-nilpotent elements; it is an ideal of $\mathfrak{b}(S_{m,n}/I)$.

Note that, even in the case $n = 0$, i.e., the affinoid case, the set $\{\|f^\ell\|_I : \ell \in \mathbb{N}\}$ appearing in Definition 5.1.7, while bounded, may not be bounded by 1. The element $\rho \in S_{0,1}$ is quasi-nilpotent, but not topologically nilpotent.

Corollary 5.1.8. — *Let I be an ideal of $S_{m,n}$ and let $f \in S_{m,n}/I$. Then f is power-bounded if, and only if, $\|f\|_{\sup} \leq 1$, f is topologically nilpotent if, and only if, $\|f\|_{\sup} < 1$, and f is quasi-nilpotent if, and only if, $|f(x)| < 1$ for all $x \in \operatorname{Max} S_{m,n}/I$. Hence, in the notation of Theorem 5.1.5, each $a_i f^{e-i} \in \mathfrak{q}(S_{m,n}/I)$.*

Proof. — The 'only if' statements are immediate consequences of Proposition 4.1.2.

Suppose $\|f\|_{\sup} \leq 1$. By Theorem 5.1.5

$$f^e = a_1 f^{e-1} + \cdots + a_e \tag{5.1.4}$$

for some $a_1, \ldots, a_e \in S_{m,n}/I$ with each $\|a_i\|_I \leq 1$. Then for every $\ell \in \mathbb{N}$ there are $b_1, \ldots, b_e \in S_{m,n}/I$ with each $\|b_i\|_I \leq 1$ such that

$$f^\ell = b_1 f^{e-1} + \cdots + b_e.$$

Thus $\{\|f^\ell\|_I : \ell \in \mathbb{N}\}$ is bounded by $\max\{\|f^i\|_I : 0 \leq i \leq e-1\}$, and f is power-bounded.

Suppose in addition that $|f(x)| < 1$ for all $x \in \operatorname{Max} S_{m,n}/I$. Then by Theorem 5.1.5, in (5.1.4) we may take each $v_I(a_i) \leq (1,1)$. By Theorem 3.1.3 each $a_i \in \mathfrak{t}(S_{m,n}/I) + (\rho)\mathfrak{b}(S_{m,n}/I)$. To conclude the proof note that since each $\|f^i\|_{\sup} \leq 1$, each $f^i \in \mathfrak{b}(S_{m,n}/I)$. Hence each $a_i f^{e-i} \in \mathfrak{q}(S_{m,n}/I)$. □

Remark 5.1.9. — The result of Corollary 5.1.8 is much easier to prove if one makes the strong additional assumption that $\|f\|_I \leq 1$. In particular:

Lemma. — *Let I be an ideal of $S_{m,n}$. There is an $\ell \in \mathbb{N}$ such that for all $f \in S_{m,n}$ with $\|f\| \leq 1$ and $|f(x)| < 1$ for all $x \in \operatorname{Max} S_{m,n}/I$, we have:*

(i) *for all $\varepsilon \in |K|$ with $1 > \varepsilon > \sigma(I)$, $\|\iota'_\varepsilon(f^\ell)\|_{\iota'_\varepsilon(I)\cdot T_{m+n}} < 1$, and*

(ii) *$v_I(f^\ell) < (1,1)$.*

Proof. — (i) Let $\Lambda(I) \subset \widetilde{T}_{m+n}$ be the uniform residue ideal of I. Let $\mathfrak{N} := \mathfrak{N}(\Lambda(I)) \subset \widetilde{T}_{m+n}$ be the nilradical of $\Lambda(I)$. Then there is some $\ell \in \mathbb{N}$ such that $\mathfrak{N}^\ell \subset \Lambda(I)$. By $\sim\colon T^\circ_{m+n} \to \widetilde{T}_{m+n}$ denote the canonical residue epimorphism. It suffices to show that $\iota'_\varepsilon(f)^\sim \in \mathfrak{N}$. Fix $\varepsilon \in |K|$ with $1 > \varepsilon > \sigma(I)$, and by F denote the image of $\iota'_\varepsilon(f)$ in $T_{m+n}/\iota'_\varepsilon(I) \cdot T_{m+n}$; then $\|F\|_{\sup} < 1$. By [**6**, Proposition 6.2.3.2], F is topologically nilpotent; i.e., $\lim_{r\to\infty} \|\iota'_\varepsilon(f)^r\|_{\iota'_\varepsilon(I)\cdot T_{m+n}} = 0$. Hence $\iota'_\varepsilon(f)^\sim \in \mathfrak{N}$.

(ii) By Proposition 4.1.3 and Lemma 3.1.11(ii) we may assume that $|K|$ is not discrete. Let ℓ be as in part (i) and put $F := f^\ell$. If $\widetilde{o}(F) > 0$ or $\|F\| < 1$, we are done. Therefore, assume that $\|F\| = 1$ and $\widetilde{o}(F) = 0$. Let $\{g_1, \ldots, g_r\} \subset I$ be a v-strict generating system with $\|g_1\| = \cdots = \|g_r\| = 1$, and let $\varepsilon \in |K|$ satisfy $1 > \varepsilon > \max_{1\leq i\leq r} \sigma(g_i)$. Since $\widetilde{o}(F) = 0$, it follows that $\|\iota'_\varepsilon(F)\| = 1$ and $\widetilde{d}(\iota'_\varepsilon(F)) = 0$. By the choice of ℓ, $\|\iota'_\varepsilon(F)\|_{\iota'_\varepsilon(I)\cdot T_{m+n}} < 1$. So by Claim A of the proof of Theorem 3.3.1, there are polynomials $h_1, \dots, h_r \in K^\circ[\xi]$ such that $\|\iota'_\varepsilon(F) - \sum_{i=1}^r h_i \varepsilon^{-\widetilde{o}(g_i)} \iota'_\varepsilon(g_i)\| < 1$, and such that $h_i = 0$ for all i with $\widetilde{o}(g_i) > 0$. This implies that $v(F - \sum_{i=1}^r h_i g_i) < (1,1)$; i.e., $v_I(F) < (1,1)$. □

Corollary 5.1.10. — *Let I be an ideal of $S_{m,n}$ and let $f \in S_{m,n}/I$. Then*

$$\|f\|_{\sup} = \inf_{\ell \in \mathbb{N}} \|f^\ell\|_I^{1/\ell} = \lim_{\ell \to \infty} \|f^\ell\|_I^{1/\ell}.$$

In particular if $\varphi : S_{m,n}/I \to S_{m',n'}/I'$ is a K-algebra homomorphism which is an isometry with respect to $\|\cdot\|_I$ and $\|\cdot\|_{I'}$, then φ is an isometry with respect to $\|\cdot\|_{\sup}$.

Proof. — The last equality is given in [**6**, Section 1.3.2]. We prove the first equality. Let $\mathfrak{m} \in \operatorname{Max}_K S_{m,n}/I$. By Proposition 4.1.2

$$|f(\mathfrak{m})| \le \|f^\ell\|_I^{1/\ell},$$

for $\ell \in \mathbb{N}$. Hence $\|f\|_{\sup} \le \inf_{\ell \in \mathbb{N}} \|f^\ell\|_I^{1/\ell}$. Suppose that $\|f\|_{\sup} < \inf_{\ell \in \mathbb{N}} \|f^\ell\|_I^{1/\ell}$. Then for some $N \in \mathbb{N}$, $\alpha \in K$ and all $\ell \in \mathbb{N}$

$$\|f^N\|_{\sup} < |\alpha| < \|f^{N\ell}\|_I^{1/N\ell},$$

since $\sqrt{|K|}$ is dense in $\mathbb{R}_+$. Put $F := \frac{1}{\alpha} f^N$. Then for all $\ell \in \mathbb{N}$

$$\|F\|_{\sup} < 1 < \|F^\ell\|_I^{1/\ell}.$$

This contradicts Corollary 5.1.8 since F is not topologically nilpotent though $\|F\|_{\sup} < 1$. □

Corollary 5.1.11. — *Let $f \in S_{m,n}/I$. Then $\|f\|_{\sup} \in \sqrt{|K|}$.*

Proof. — If $m = 0$, the result follows from Noether normalization for quotients of $S_{0,n}$ (Remark 2.3.6) and [**6**, Proposition 3.8.1.7]. We reduce to this case.

By Theorem 3.4.6, there are $m', n' \in \mathbb{N}$, an ideal J of $S_{m',n'}$ and a K-algebra homomorphism

$$\varphi : S_{m,n}/I \to S_{m',n'}/J$$

such that (i) φ is an isometry with respect to $\|\cdot\|_I$ and $\|\cdot\|_J$, and (ii) $S_{m',n'}/J$ is a finite $S_{0,d}$-algebra for some $d \in \mathbb{N}$. By (i) and Corollary 5.1.10, φ is an isometry in $\|\cdot\|_{\sup}$. Now (ii) permits us to reduce to the case above. □

5.2. Continuity and Extension of Homomorphisms. — In this subsection we prove that K-algebra homomorphisms between quasi-affinoid algebras are continuous, i.e., bounded (Theorem 5.2.3). It follows that all residue norms on a quasi-affinoid algebra are equivalent (Corollary 5.2.4). We also prove an Extension Lemma (Theorem 5.2.6) for quasi-affinoid maps.

Depending on the choice of E, $S_{m,n}$ may not be complete in $\|\cdot\|$ (see Theorem 2.1.3). Hence the results of this subsection do not follow from [**6**, Theorem 3.7.5.1]. Nevertheless $S^{\circ}_{m,n}$ is the direct limit of rings $B\langle\xi\rangle[\![\rho]\!]$ that are complete both in $\|\cdot\|$ and (ρ)-adically. Furthermore (Corollary 2.2.6 and Theorem 2.3.2) the operations of factoring $S_{m,n}$ by an ideal and Weierstrass Division respect the decomposition of $S_{m,n}$ as the direct limit of the $B\langle\xi\rangle[\![\rho]\!]$.

We first establish the continuity of K-algebra homomorphisms from quasi-affinoid algebras to affinoid algebras.

Lemma 5.2.1. — *Let $\varphi : S_{m,n}/I \to S_{m',0}/J =: A$ be a K-algebra homomorphism. Then φ is continuous with respect to $\|\cdot\|_I$ and $\|\cdot\|_J$, and is uniquely determined by its values on $\xi_i + I$ and $\rho_j + I$, $i = 1,\dots,m$; $j = 1,\dots,n$.*

Proof. — *Continuity.* It is sufficient to consider the case $I = (0)$. Since φ is a K-algebra homomorphism it follows from [**6**, Propositions 6.2.3.1 and 6.2.3.2] that the $\varphi(\xi_i)$ are power-bounded and the $\varphi(\rho_j)$ are topologically nilpotent (i.e., the set $\|\varphi(\xi_i)^k\|_J$ is bounded and for each j, $\|\varphi(\rho_j)^k\|_J \to 0$ as $k \to \infty$). Therefore we may put

$$M := \max\{\|\varphi(\xi^\mu\rho^\nu)\|_J : \mu \in \mathbb{N}^m, \nu \in \mathbb{N}^n\}.$$

***Claim* (A).** — *Let $M' \in \mathbb{R}$, $B \in \mathfrak{B}$. If*

$$\|\varphi(f)\|_J \le M'\|f\|$$

for all $f \in B\langle\xi\rangle[\![\rho]\!]$, then in fact

$$\|\varphi(f)\|_J \le M\|f\|$$

for all $f \in B\langle\xi\rangle[\![\rho]\!]$.

Choose $\alpha \in \mathbb{N}$ so that for $|\nu| = \alpha$ we have $\|\varphi(\rho^\nu)\|_J < M/M'$. Let $f \in B\langle\xi\rangle[\![\rho]\!]$ and write

$$f = p(\xi,\rho) + f_0(\xi,\rho) + \sum_{|\nu_i|=\alpha} \rho^{\nu_i} f_i(\xi,\rho)$$

where the p, f_0, $f_i \in B\langle\xi\rangle[\![\rho]\!]$ satisfy

- p is a polynomial and $\|p\| \le \|f\|$,
- $\|f_0\| \le \left(\frac{M}{M'}\right)\|f\|$, and
- $\|f_i\| \le \|f\|$ for all i.

(In other words choose a polynomial p such that $f - p \in (B_i + (\rho)^\alpha)B\langle\xi\rangle[\![\rho]\!]$ for some i with $|B_i| \subset [0, M/M']$.) Then

$$\varphi(f) = p(\varphi(\xi), \varphi(\rho)) + \varphi(f_0) + \sum_{|\nu_i|=\alpha} \varphi(\rho)^{\nu_i}\varphi(f_i)$$

and

$$\begin{aligned} \|\varphi(f)\|_J &\leq \max\left\{M\|p\|, M'\|f_0\|, \frac{M}{M'}M'\|f_i\|\right\} \\ &\leq M\|f\|. \end{aligned}$$

Claim A is proved.

By Proposition 2.1.5 there is a complete, discretely valued subfield $F \subset K$ such that

$$S_{m,n} = \varinjlim_{F^\circ \subset B \in \mathfrak{B}} F\widehat{\otimes}_{F^\circ} B\langle\xi\rangle[\![\rho]\!].$$

Once we prove that each map

$$\varphi|_{F\widehat{\otimes}_{F^\circ}B\langle\xi\rangle[\![\rho]\!]} : F\widehat{\otimes}_{F^\circ}B\langle\xi\rangle[\![\rho]\!] \to A$$

of F-Banach Algebras is bounded, it will follow from Claim A that $\varphi : S_{m,n} \to A$ is also bounded. It remains to prove

***Claim* (B).** — *The restriction* $\varphi|_{F\widehat{\otimes}_{F^\circ}B\langle\xi\rangle[\![\rho]\!]} : F\widehat{\otimes}_{F^\circ}B\langle\xi\rangle[\![\rho]\!] \to A$ *is bounded.*

Since it is affinoid, A is certainly also an F-Banach Algebra. By the Closed Graph Theorem ([**6**, Section 2.8.1] or [**7**]) it is thus sufficient to prove that if the $v_n \in F\widehat{\otimes}_{F^\circ}B\langle\xi\rangle[\![\rho]\!]$ satisfy $\lim v_n = 0$ and $\lim \varphi(v_n) = w \in A$, then $w = 0$. We follow the proof of [**6**, Proposition 3.7.5.1]. Let $\mathfrak{b} = \mathfrak{m}^N$ for some maximal ideal $\mathfrak{m} \in \operatorname{Max} A$ and $N \in \mathbb{N}$. Let $\mathfrak{a} = \varphi^{-1}(\mathfrak{b}) \subset S_{m,n}$. Consider the commutative diagram

$$\begin{tikzcd} S_{m,n} \arrow[r, "\varphi"] \arrow[d, "\pi"'] \arrow[dr, "\psi"] & A \arrow[d, "\beta"] \\ S_{m,n}/\mathfrak{a} \arrow[r, "\overline{\varphi}"] & A/\mathfrak{b} \end{tikzcd}$$

where π and β are the canonical projections, $\overline{\varphi}$ is the induced map and ψ is $\overline{\varphi} \circ \pi$. Note that π and β are contractions, and that $\overline{\varphi}$ is continuous since by Proposition 4.2.1, $S_{m,n}/\mathfrak{a}$ and $A/\mathfrak{b}$ are finite dimensional K-algebras. Hence ψ is continuous and $\beta(w) = 0$. Since this is true for all $\mathfrak{m} \in \operatorname{Max} A$ and all $N \in \mathbb{N}$, by the Krull Intersection Theorem, $w = 0$. (Suppose $w \in \mathfrak{m}^N$ for all $\mathfrak{m} \in \operatorname{Max} A$, and let J be the ideal of all $x \in A$ such that $xw = 0$. Fix $\mathfrak{m} \in \operatorname{Max} A$. By the Krull Intersection Theorem [**25**, Theorem 8.10(i)], the image of w in the localization $A_{\mathfrak{m}}$ is zero. Thus, $J \not\subset \mathfrak{m}$. Since this holds for all $\mathfrak{m} \in \operatorname{Max} A$, $J = (1)$; i.e., $w = 0$.) This proves Claim B and hence φ is continuous.

Uniqueness: This follows directly from Claim A: suppose φ and ψ agree on the $\xi_i + I$ and $\rho_j + I$. Put $\Phi := \varphi - \psi$. Now apply Claim A, with $M = 0$, to Φ. $\square$

Next we show that there are continuous K-algebra homomorphisms $S_{m,n} \to S_{m',n'}/I'$ sending the ξ_i (respectively ρ_j) to any specified power-bounded (respectively quasi-nilpotent) elements of $S_{m',n'}/I'$.

Lemma 5.2.2. — *Let $f_i \in S_{m',n'}/I'$, $i = 1, \ldots, m$, be power-bounded and let $g_j \in S_{m',n'}/I'$, $j = 1, \ldots, n$, be quasi-nilpotent. There is a K-algebra homomorphism,*

$$\varphi : S_{m,n} \to S_{m',n'}/I',$$

continuous in $\|\cdot\|$ and $\|\cdot\|_{I'}$, such that $\varphi(\xi_i) = f_i$ and $\varphi(\rho_j) = g_j$ for $i = 1, \ldots, m$; $j = 1, \ldots, n$.

Proof. — Since the f_i are power-bounded, by Theorem 5.1.5, there are $a_{ij} \in S_{m',n'}/I'$, $1 \le i \le m$, $1 \le j \le e$, with each $\|a_{ij}\|_{I'} \le 1$ such that

$$f_i^e + a_{i1} f_i^{e-1} + \cdots + a_{ie} = 0, \qquad 1 \le i \le m.$$

Similarly, there are $b_{ij} \in S_{m',n'}/I'$, $1 \le i \le n$, $1 \le j \le e$, with each $v_{I'}(b_{ij}) < (1,1)$ such that

$$g_i^e + b_{i1} g_i^{e-1} + \cdots + b_{ie} = 0, \qquad 1 \le i \le n.$$

By Theorem 3.1.3, there are A_{ij}, $B_{ij} \in S_{m',n'}$ such that $v(A_{ij}) = v_{I'}(a_{ij})$, $v(B_{ij}) = v_{I'}(b_{ij})$, $a_{ij} = A_{ij} + I$, and $b_{ij} = B_{ij} + I$. Put

$$P_i(\xi_{m'+i}) := \xi_{m'+i}^e + A_{i1}\xi_{m'+i}^{e-1} + \cdots + A_{ie}, \qquad i = 1, \ldots, m,$$
$$Q_i(\rho_{n'+i}) := \rho_{n'+i}^e + B_{i1}\rho_{n'+i}^{e-1} + \cdots + B_{ie}, \qquad i = 1, \ldots, n.$$

Note that each P_i is regular in $\xi_{m'+i}$ of degree e and each Q_i is regular in $\rho_{n'+i}$ of degree e. Let $\psi_0 : S_{m,n} \hookrightarrow S_{m'+m,n'+n}$ be the inclusion defined by $\xi_i \mapsto \xi_{m'+i}$, $\rho_j \mapsto \rho_{n'+j}$, $i = 1, \ldots, m; j = 1, \ldots, n$. By Weierstrass Division (Theorem 2.3.2) there is a unique K-algebra homomorphism

$$\psi_1 \colon S_{m'+m,n'+n} \to S_{m',n'}[\xi_{m'+1}, \ldots, \xi_{m'+m}, \rho_{n'+1}, \ldots, \rho_{n'+n}]/(P, Q)$$

with $\operatorname{Ker}\psi_1 = (P, Q) \cdot S_{m'+n,n'+n}$. Furthermore, by Weierstrass Division, ψ is continuous and the range of ψ_1 is a Cartesian $S_{m',n'}$-module (see [**6**, Definition 5.2.7.3]). Let $\psi_2 : S_{m',n'}[\xi_{m'+1}, \ldots, \xi_{m'+m}, \rho_{n'+1}, \ldots, \rho_{n'+n}]/(P, Q) \to$

$S_{m',n'}/I'$ be the unique K-algebra homomorphism that sends $S_{m',n'} \ni f \mapsto f + I'$, $\xi_{m'+i} \mapsto f_i$ and $\rho_{n'+j} \mapsto g_j$, $i = 1, \ldots, m$, $j = 1, \ldots, n$.

Since ψ_0 is an isometry in $\|\cdot\|$, ψ_1 is a contraction and

$$S_{m',n'}[\xi_{m'+1}, \ldots, \xi_{m'+m}, \rho_{n'+1}, \ldots, \rho_{n'+n}]/(P, Q)$$

is a Cartesian $S_{m',n'}$–module, ψ_2 is continuous. Take $\varphi := \psi_2 \circ \psi_1 \circ \psi_0$. □

Theorem 5.2.3. — *Let $\varphi : S_{m,n}/I \to S_{m',n'}/I'$ be a K-algebra homomorphism. Then φ is continuous with respect to $\|\cdot\|_I$ and $\|\cdot\|_{I'}$, and is uniquely determined by the values $\varphi(\xi_i + I)$, $\varphi(\rho_j + I)$, $i = 1, \ldots, m; j = 1, \ldots, n$.*

Proof. — It is sufficient to take $I = (0)$. Let $\varphi' : S_{m,n} \to S_{m',n'}/I'$ be the continuous K-algebra homomorphism provided by Lemma 5.2.2 with $\varphi'(\xi_i) = \varphi(\xi_i)$ and $\varphi'(\rho_j) = \varphi(\rho_j)$, $i = 1, \ldots, m$; $j = 1, \ldots, n$. By Corollary 3.3.2, there is an $\varepsilon \in \sqrt{|K \setminus \{0\}|}$ such that

$$S_{m',n'}/I' \xrightarrow{\iota_\varepsilon} T_{m',n'}(\varepsilon)/\iota_\varepsilon(I') \cdot T_{m',n'}(\varepsilon)$$

is an inclusion. By Lemma 5.2.1, $\iota_\varepsilon \circ \varphi = \iota_\varepsilon \circ \varphi'$. Since ι_ε is an inclusion $\varphi = \varphi'$, and thus φ is continuous. □

In general a quasi-affinoid algebra has many representations as a quotient of an $S_{m,n}$. The residue norms corresponding to different representations may be different. However all these norms are equivalent, i.e., they induce the same topology.

Corollary 5.2.4. — *If $S_{m,n}/I \simeq S_{m',n'}/I'$ as K-algebras then the two norms $\|\cdot\|_I$ and $\|\cdot\|_{I'}$ are equivalent; i.e., they induce the same topology.*

Remark 5.2.5. — Let $c \in K^{\circ\circ}$. The $(c)+(\rho)$–adic topology on $S^\circ_{m,n}$ induces a topology on $S_{m,n}$ and on any quotient. A K–algebra homomorphism

$$\varphi : S_{m,n} \to S_{m',n'}/I'$$

is also continuous with respect to such topologies. In other words, if $f = \sum a_{\mu\nu}\xi^\mu\rho^\nu \in S^\circ_{m,n}$, then by the above arguments, $\sum a_{\mu\nu}\varphi(\xi)^\mu\varphi(\rho)^\nu$ converges to $\varphi(f)$.

Theorem 5.2.6. — **(Extension Lemma,** cf. Remark 5.2.8.**)**
Let $\varphi : S_{m,n}/I \to S_{m',n'}/I'$ be a K-algebra homomorphism, let $f_1, \ldots, f_M \in S_{m',n'}/I'$ be power-bounded and let $g_1, \ldots, g_N \in S_{m',n'}/I'$ be quasi-nilpotent. Then there is a unique K-algebra homomorphism

$$\psi : S_{m+M,n+N}/I \cdot S_{m+M,n+N} \to S_{m',n'}/I'$$

such that $\psi(\xi_{m+i}) = f_i$, $1 \le i \le M$, $\psi(\rho_j) = g_j$, $1 \le j \le N$, and the following diagram commutes:

$$\begin{tikzcd} S_{m,n}/I \arrow[r, "\varphi"] \arrow[d] & S_{m',n'}/I' \\ S_{m+M,n+N}/I \cdot S_{m+M,n+N} \arrow[ur, dotted, "\psi"'] & \end{tikzcd}$$

Proof. — By Lemma 5.2.2 there is a K–algebra homomorphism

$$\psi' : S_{m+M,n+N} \to S_{m',n'}/I'$$

such that

$$\begin{aligned} \psi'(\xi_i) &= \varphi(\xi_i + I), && i = 1, \ldots, m, \\ \psi'(\rho_j) &= \varphi(\rho_j + I), && j = 1, \ldots, n, \\ \psi'(\xi_{m+i}) &= f_i, && i = 1, \ldots, M, \\ \psi'(\rho_{m+j}) &= g_j, && j = 1, \ldots, N. \end{aligned}$$

By Theorem 5.2.3,

$$\psi'|_{S_{m,n}} = \varphi \circ \pi,$$

where

$$\pi : S_{m,n} \to S_{m,n}/I$$

is the canonical projection. Hence $I \subset \operatorname{Ker} \psi'$ and ψ' gives rise to a K–algebra homomorphism

$$\psi : S_{m+M,n+N}/I \cdot S_{m+M,n+N} \to S_{m',n'}/I'.$$

That $\psi|_{S_{m,n}/I} = \varphi$ and that ψ is unique follow immediately from Theorem 5.2.3. □

For notational convenience we make the following definition:

Definition 5.2.7. — Fix the pair (E, K) and let A be a quasi-affinoid algebra, say $A = S_{m',n'}(E,K)/I$. We define

$$A\langle \xi_1, \ldots, \xi_m \rangle [\![\rho_1, \ldots, \rho_n]\!]_s := S_{m'+m,n'+n}/I \cdot S_{m'+m,n'+n}$$

where we regard

$$S_{m',n'} = K\langle \eta_1, \ldots, \eta_{m'} \rangle [\![\tau_1, \ldots, \tau_{n'}]\!]_s$$

and

$$S_{m'+m,n'+n} = K\langle \eta_1, \ldots, \eta_{m'}, \xi_1, \ldots, \xi_m \rangle [\![\tau_1, \ldots, \tau_{n'}, \rho_1, \ldots, \rho_n]\!]_s.$$

By the Extension Lemma, Theorem 5.2.6, $A\langle\xi_1,\ldots,\xi_m\rangle[\![\rho_1,\ldots,\rho_n]\!]_s$ is independent of the presentation of A.

We will show that that

$$A\langle\xi\rangle[\![\rho]\!]_s \subset A[\![\xi,\rho]\!]$$

via the K-algebra homomorphism

$$\varphi : S_{m'+m,n'+n} \to A[\![\xi,\rho]\!] : \sum f_{\mu\nu}\xi^\mu\rho^\nu \mapsto \sum(f_{\mu\nu}+I)\xi^\mu\rho^\nu.$$

Indeed, it suffices to verify

$$\operatorname{Ker}\varphi \subset I\cdot S_{m'+m,n'+n}.$$

Let $f = \sum f_{\mu\nu}\xi^\mu\rho^\nu \in \operatorname{Ker}\varphi$; without loss of generality $\|f\| = 1$. Hence $f \in B\langle\eta,\xi\rangle[\![\tau,\rho]\!]$ for some $B \in \mathfrak{B}$. By Lemma 3.1.6, there are $s \in \mathbb{N}$, $B \subset B' \in \mathfrak{B}$ and $h_{\mu\nu} \in B'\langle\eta,\xi\rangle[\![\tau,\rho]\!]$ such that

$$f = \sum_{|\mu|+|\nu|\le s} f_{\mu\nu}h_{\mu\nu}.$$

Since each $f_{\mu\nu} \in I$, it follows that $f \in I\cdot S_{m'+m,n'+n}$, as desired.

Let $\psi : S_{m',n'} \to A[\![\xi,\rho]\!]$ be the composition of φ with the obvious inclusion $S_{m',n'} \hookrightarrow S_{m'+m,n'+n}$. Since $\operatorname{Ker}\psi = I$, it follows that

$$A \to A\langle\xi\rangle[\![\rho]\!]_s$$

is injective.

Remark 5.2.8. — Here we rephrase the Extension Lemma (Theorem 5.2.6) in terms of the notation introduced in Definition 5.2.7.

Let $\varphi : A \to B$ be a K-algebra homomorphism of quasi-affinoid algebras A and B. Suppose $f_1,\ldots,f_m \in B$ are power-bounded and $g_1,\ldots,g_n \in B$ are quasi-nilpotent. Then there is a unique K-algebra homomorphism $\psi : A\langle\xi\rangle[\![\rho]\!]_s \to B$ such that $\psi(\xi_i) = f_i$ and $\psi(\rho_j) = g_j$, $1 \le i \le m$, $1 \le j \le n$, and the following diagram commutes:

$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[d] & B \\ A\langle\xi\rangle[\![\rho]\!]_s \arrow[ur, dotted, "\psi"'] & \end{tikzcd}$$

In particular, it follows that there are $m,n \in \mathbb{N}$ and a surjection of A-algebras

$$A\langle\xi_1,\ldots,\xi_m\rangle[\![\rho_1,\ldots,\rho_n]\!]_s \to B,$$

and hence for some ideal I,

$$B \simeq A\langle \xi_1, \ldots, \xi_m \rangle [\![\rho_1, \ldots, \rho_n]\!]_s / I.$$

5.3. Quasi-Rational Domains. — By analogy with [**6**, Section 6.1.4], we define generalized rings of fractions in the quasi-affinoid setting. This leads, in Definition 5.3.3, to the construction of quasi-rational domains and, by iterating, R-domains. Example 5.3.7 shows that R-domains are more general than quasi-rational domains, in contrast to the affinoid case. Nevertheless the Extension Lemma (Theorem 5.2.6) shows that generalized rings of fractions are well-defined and that the association of a generalized ring of fractions with a quasi-rational domain provides it with a canonical ring of quasi-affinoid functions. Thus quasi-rational subdomains (and by iteration, R-subdomains) are examples of quasi-affinoid subdomains (the formal generalization to the quasi-affinoid category of the notion of affinoid subdomains). This provides a foundation for a theory of quasi-affinoid varieties (see [**22**]). We end this subsection proving in Proposition 5.3.8 that a quasi-affinoid algebra is affinoid if, and only if, it satisfies the Maximum Modulus Principle.

Definition 5.3.1. — Let A be a quasi-affinoid algebra, say $A = S_{m,n}/I$, and let $f_1, \ldots, f_M$; $g_1, \ldots, g_N$; $h \in A$. Define the **generalized ring of fractions** $A\langle f/h \rangle [\![g/h]\!]_s$ to be the quotient ring

$$A \left\langle \frac{f}{h} \right\rangle \left[\!\!\left[\frac{g}{h} \right]\!\!\right]_s := S_{m+M,n+N}/J,$$

where J is the ideal of $S_{m+M,n+N}$ generated by the elements of I and the elements

$$H\xi_{m+i} - F_i, \quad H\rho_{n+j} - G_j, \quad 1 \leq i \leq M, \quad 1 \leq j \leq N,$$

where the F_i, G_j, $H \in S_{m,n}$ satisfy $f_i = F_i + I$, $g_j = G_j + I$, $h = H + I$, $1 \leq i \leq M$, $1 \leq j \leq N$. By Theorem 5.2.6 any isomorphism $S_{m,n}/I \to S_{m',n'}/I'$ extends to an isomorphism $S_{m+M,n+N}/I \cdot S_{m+M,n+N} \to S_{m'+M,n'+N}/I' \cdot S_{m'+M,n'+N}$ sending ξ_{m+i} to $\xi_{m'+i}$ and ρ_{n+j} to $\rho_{n'+j}$. It follows that $A\langle \frac{f}{h} \rangle [\![\frac{g}{h}]\!]_s$ is well-defined.

Let f, g, h be as in Definition 5.3.1. In general, $\operatorname{Max} A\langle \frac{f}{h} \rangle [\![\frac{g}{h}]\!]_s$ is neither open in $\operatorname{Max} A$ nor does it satisfy the Universal Property of [**6**, Section 7.2.2] (see Definition 5.3.4 below). With the additional restriction that f, g, h generate the unit ideal of A (see Definition 5.3.3, below) the following Universal Property is satisfied.

Proposition 5.3.2. — *Let A be a quasi-affinoid algebra, let $f_1, \ldots, f_M$; $g_1, \ldots, g_N$; $h \in A$, and put*

$$A' := A\left\langle \frac{f}{h} \right\rangle \left[\!\left[\frac{g}{h} \right]\!\right]_s .$$

Suppose $\psi : A \to B$ is a K-algebra homomorphism into a K-quasi-affinoid algebra B such that

(i) *$\psi(h)$ is a unit,*
(ii) *$\psi(f_i)/\psi(h)$ is power-bounded, $1 \le i \le M$, and*
(iii) *$\psi(g_j)/\psi(h)$ is quasi-nilpotent, $1 \le j \le N$.*

Then there is a unique K-algebra homomorphism $\psi' : A' \to B$ such that

$$\begin{tikzcd} & A' \arrow[dr, dotted, "\psi'"] & \\ A \arrow[ur] \arrow[rr, "\psi"'] & & B \end{tikzcd}$$

commutes. In particular, if $\{f, g, h\}$ generates the unit ideal of A and if $\operatorname{Max} B \subset \operatorname{Max} A'$ *(as subsets of* $\operatorname{Max} A$*) then by Corollary 5.1.8 and the Nullstellensatz, Theorem 4.1.1, conditions (i), (ii) and (iii) are all satisfied.*

Proof. — Immediate from Theorem 5.2.6. □

Definition 5.3.3. — Let A be a quasi-affinoid algebra and put $X := \operatorname{Max} A$. A **quasi-rational subdomain** of X is a subset $U \subset X$ of the form

$$U = \operatorname{Max}\left(A\left\langle \frac{f}{h} \right\rangle \left[\!\left[\frac{g}{h} \right]\!\right]_s \right)$$

where $f_1, \ldots, f_M$; $g_1, \ldots, g_N$; $h \in A$ generate the unit ideal. The class of **R-subdomains** of X is defined inductively as follows. Any quasi-rational subdomain of X is an R-subdomain of X. If $U \subset X$ is an R-subdomain of X and if $V \subset U$ is a quasi-rational subdomain of U, then $V \subset X$ is an R-subdomain of X.

Suppose $U = \operatorname{Max}(A\langle \frac{f}{h} \rangle [\![\frac{g}{h}]\!]_s)$ is a quasi-rational subdomain of $X = \operatorname{Max} A$. Then

$$U = \{x \in X : |f_i(x)| \le |h(x)|, |g_j(x)| < |h(x)|, 1 \le i \le M, 1 \le j \le N\}.$$

To see this, write $A = S_{m,n}/I$ and $A\langle \frac{f}{h} \rangle [\![\frac{g}{h}]\!]_s = S_{m+M,n+N}/J$, where J is generated by the elements of I together with the elements of the form

$$H\xi_{m+i} - F_i, \quad H\rho_{n+j} - G_j, \quad 1 \le i \le M, \quad 1 \le j \le N,$$

where the F_i, G_j, $H \in S_{m,n}$ satisfy $f_i = F_i + I$, $g_j = G_j$, $h = H + I$, $1 \le i \le M$, $1 \le j \le N$. The elements of U correspond naturally to the maximal ideals of $S_{m+M,n+N}$ that contain J. Let x be such a maximal ideal. By the Nullstellensatz (Theorem 4.1.1),

$$|\xi_{m+i}(x)| \le 1 \text{ and } |\rho_{n+j}(x)| < 1.$$

The description of U above then follows immediately from $h(x)\xi_{m+i}(x) - f_i(x) = 0$ and $h(x)\rho_{n+j}(x) - g_j(x) = 0$ and from the fact that $h(x) \neq 0$. The fact that $h(x) \neq 0$ for all $x \in U$ also guarantees that U is an open and closed subset of X when X is endowed with the canonical (metric) topology (see [**6**, Section 7.2.1]).

As in the affinoid case, one easily proves (cf. [**6**, Proposition 7.2.3.7]) that the intersection of quasi-rational domains is a quasi-rational domain. In contrast to the affinoid case, the complement of a quasi-rational domain is a finite union of quasi-rational domains. To see this, consider the quasi-rational domain

$$U = \operatorname{Max}\left(A\left\langle\frac{f}{h}\right\rangle\left[\!\left[\frac{g}{h}\right]\!\right]_s\right),$$

where the f, g, h generate the unit ideal of A. Note that h is a unit of $A\langle\frac{f}{h}\rangle[\![\frac{g}{h}]\!]_s$. Choose $\frac{1}{c} \in K$ with

$$\left|\frac{1}{c}\right| \ge \left\|\frac{1}{h}\right\|_{\sup}; \text{ i.e.,}$$
$$|c| \le |h(x)|, \text{ for all } x \in U.$$

Then

$$U = \{x \in \operatorname{Max} A\colon |f_i(x)| \le |h(x)|, |g_j(x)| < |h(x)|, \\ |c| \le |h(x)|, 1 \le i \le M, 1 \le j \le N\}.$$

Hence

$$\begin{aligned}\operatorname{Max} A \setminus U \quad = \quad & \{x \in \operatorname{Max} A\colon |h(x)| < |c|\} \cup \\ & \bigcup_i \{x \in \operatorname{Max} A\colon |h(x)| < |f_i(x)|, |c| < |f_i(x)|\} \cup \\ & \bigcup_j \{x \in \operatorname{Max} A\colon |h(x)| \le |g_j(x)|, |c| \le |g_j(x)|\}.\end{aligned}$$

By induction, a finite intersection of R-domains is an R-domain and the complement of an R-domain is a finite union of R-domains.

Definition 5.3.4. — Let A and B be K-quasi-affinoid algebras. A **K-quasi-affinoid map**

$$(\Phi, \varphi) : (\operatorname{Max} B, B) \to (\operatorname{Max} A, A)$$

is a map $\Phi : \operatorname{Max} B \to \operatorname{Max} A$ induced by a K-algebra homomorphism $\varphi : A \to B$ via the Nullstellensatz, Theorem 4.1.1. Let U be a subset of $\operatorname{Max} A$. Following [**6**, Section 7.2.2], and suppressing mention of φ, we say that a quasi-affinoid map $\Phi : \operatorname{Max} A' \to \operatorname{Max} A$ **represents all quasi-affinoid maps into** U if $\Phi(\operatorname{Max} A') \subset U$ and if, for any quasi-affinoid map $\Psi : \operatorname{Max} B \to \operatorname{Max} A$ with $\Psi(\operatorname{Max} B) \subset U$, there exists a unique quasi-affinoid map $\Psi' : \operatorname{Max} B \to \operatorname{Max} A'$ such that $\Psi = \Phi \circ \Psi'$; i.e., such that

$$\begin{tikzcd} & \operatorname{Max} A' \arrow[dr, "\Phi"] & \\ \operatorname{Max} B \arrow[ur, dotted, "\Psi'"] \arrow[rr, "\Psi"'] & & \operatorname{Max} A \end{tikzcd}$$

commutes. A subset $U \subset \operatorname{Max} A$ is called a **quasi-affinoid subdomain of** $\operatorname{Max} A$ if there exists a quasi-affinoid map $\varphi : \operatorname{Max} A' \to \operatorname{Max} A$ representing all quasi-affinoid maps into U.

As in [**6**, Section 7.2.2], the above universal property has useful formal consequences which are proved in Proposition 5.3.6. In addition it allows us to associate to every quasi-affinoid subdomain U of $\operatorname{Max} A$ a canonical A-algebra of quasi-affinoid functions $\mathcal{O}(U)$. Indeed if $\Phi : \operatorname{Max} A' \to \operatorname{Max} A$ represents all quasi-affinoid maps into U, then $\mathcal{O}(U) := A'$. Reversing the arrows in Proposition 5.3.2 yields many examples of quasi-affinoid subdomains.

Theorem 5.3.5. — *Let A be a quasi-affinoid algebra and let $U \subset \operatorname{Max} A$ be a quasi-rational subdomain, $U = \operatorname{Max} A\langle \frac{f}{h}\rangle [\![\frac{g}{h}]\!]_s$, where the f, g, h generate the unit ideal of A. The inclusion*

$$\operatorname{Max}\left(A\left\langle \frac{f}{h}\right\rangle \left[\!\left[\frac{g}{h}\right]\!\right]_s\right) \to \operatorname{Max} A$$

represents all quasi-affinoid maps into U. Thus every R-subdomain is a quasi-affinoid subdomain.

To every R-subdomain U of $\operatorname{Max} A$, we have thus associated the canonical A-algebra of quasi-affinoid functions $\mathcal{O}(U)$ such that $\operatorname{Max} \mathcal{O}(U) \to \operatorname{Max} A$ represents all quasi-affinoid maps into U. In particular, if $U \subset \operatorname{Max} A$ is the quasi-rational subdomain defined by

$$U = \{x \in \operatorname{Max} A : |f_i(x)| \leq |h(x)|, |g_j(x)| < |h(x)|, 1 \leq i \leq M, 1 \leq j \leq N\},$$

where $\{f, g, h\}$ generates the unit ideal of A, then $\mathcal{O}(U) = A\langle \frac{f}{h}\rangle [\![\frac{g}{h}]\!]_s$ is independent of the above presentation. In other words, if $U \subset \operatorname{Max} A$ is a quasi-rational subdomain, $f', g', h' \in \mathcal{O}(U)$ have no common zero and

$$|f'(x)| \leq |h'(x)|, \qquad |g'(x)| < |h'(x)|$$

for all $x \in U$, then

$$\mathcal{O}(U) = \mathcal{O}(U) \left\langle \frac{f'}{h'} \right\rangle \left[\left[\frac{g'}{h'} \right]\right]_s .$$

By induction, the same holds for R-subdomains of $\operatorname{Max} A$. This fact is a key step in developing a natural theory of quasi-affinoid varieties, as will be seen in [**22**]. A special case of this result was proved in [**18**, Theorem 3.6]. The proof of the main result of [**18**] can be simplified considerably using Theorem 5.3.5.

Proposition 5.3.6. — (cf. [**6**, Proposition 7.2.2.1].) *Let A be a quasi-affinoid algebra, let $U \subset \operatorname{Max} A$ and suppose $(\Phi, \varphi) : (\operatorname{Max} A', A') \to (\operatorname{Max} A, A)$ is a quasi-affinoid map representing all quasi-affinoid maps into U. Then*

(i) *Φ is injective and satisfies $\Phi(\operatorname{Max} A') = U$;*
(ii) *for $x \in \operatorname{Max} A'$ and $n \in \mathbb{N}$, the map $\varphi : A \to A'$ induces an isomorphism $A/\Phi(x)^n \to A'/x^n$;*
(iii) *for $x \in \operatorname{Max} A'$, $x = \varphi(\Phi(x)) \cdot A'$.*

Proof. — Let $y \in U$ and consider the commutative diagram

$$\begin{tikzcd}
A \arrow[r, "\varphi"] \arrow[d, "\pi"'] & A' \arrow[d, "\pi'"] \arrow[dl, dotted, "\sigma"'] \\
A/y^n \arrow[r, "\psi"] & A'/\varphi(y^n) \cdot A'
\end{tikzcd}$$

where π and π' denote the canonical projections and ψ is induced by φ. Since Φ represents all affinoid maps into U, there exists a unique homomorphism $\sigma : A' \to A/y^n$ making the upper triangle commute.

Thus both maps π' and $\psi \circ \sigma$ make

$$\begin{tikzcd}
& A'/\varphi(y^n) \cdot A' & \\
A \arrow[ur, "\psi \circ \pi"] \arrow[rr, "\varphi"'] & & A' \arrow[ul, dotted]
\end{tikzcd}$$

commute. Due to the universal property of φ, they must be equal; i.e., the lower triangle in the above diagram commutes.

Since π' is surjective, so is ψ. Furthermore, σ is surjective because π is. Since the upper triangle commutes, $\operatorname{Ker}\pi' = \varphi(y^n \cdot A') \subset \operatorname{Ker}\sigma$. Hence ψ must be bijective. Taking $n = 1$, we see that $\varphi(y)\cdot A'$ must be a maximal ideal of A'. Thus $\Phi^{-1}(y)$ consists precisely of one element, $\varphi(y)\cdot A'$. This proves (i) and (iii). Moreover (ii) must hold because $x^n = y^n \cdot A'$ where $y = \Phi(x)$, and because ψ is bijective. □

Example 5.3.7. — In the affinoid case, a rational subdomain V of a rational subdomain U of an affinoid variety X is itself a rational subdomain of X (see [**6**, Section 7.2.4]). This transitivity property is not in general true in the quasi-affinoid case.

First note that the quasi-rational subdomains U of the affinoid variety $\operatorname{Max} S_{m,0}$ are all of the form

$$U = \operatorname{Max} S_{m,0}\left\langle \frac{f}{h}\right\rangle \left[\!\left[\frac{g}{h}\right]\!\right]_s$$

where $f_1, \ldots, f_M$, $g_1, \ldots, g_N$, h are polynomials. That is because h is a unit of $S_{m,0}\langle\frac{f}{h}\rangle[\![\frac{g}{h}]\!]_s$ (recall that the ideal generated by f, g and h contains 1).

Let $K = \mathbb{C}_p$, the completion of the algebraic closure of the p-adic field $\mathbb{Q}_p$. Note that $\widetilde{K}$ and K°/aK° are countably infinite for every $a \in K^{\circ\circ}\setminus\{0\}$. Let $E \subset K^\circ$ be a DVR such that $\widetilde{E} = \widetilde{K}$, and put $S_{m,n} := S_{m,n}(E, K)$.

We will show that every quasi-rational subdomain of $\operatorname{Max} S_{2,0}$ has a property (see lemma below) that is not possessed by the set

$$U = \{(\xi, \rho) \in \operatorname{Max} S_{1,1} : |\xi - f(\rho)| < \varepsilon\},$$

for a suitable choice of $f \in S_{0,1}$ and $\varepsilon \in |K|\setminus\{0\}$. The failure of the transitivity property for quasi-rational subdomains follows, since U is a quasi-rational subdomain of $\operatorname{Max} S_{1,1}$, which is a quasi-rational subdomain of $\operatorname{Max} S_{2,0}$. By

$$\pi : \operatorname{Max} S_{2,0} \to \operatorname{Max} S_{1,0}$$

denote the map induced by the obvious inclusion $S_{1,0} \to S_{2,0}$.

Lemma. — *Let $U \subset \operatorname{Max} S_{2,0}$ be a quasi-rational subdomain such that $\pi(U)$ contains an annulus of the form*

$$\{x \in \operatorname{Max} S_{1,0} \colon \delta < |x| < 1\}, \qquad 0 < \delta < 1. \tag{5.3.1}$$

Then there is a polynomial $P \in K[\xi_1, \xi_2]\setminus\{0\}$ such that $\pi(U \cap \{x \in \operatorname{Max} S_{2,0} \colon P(x) = 0\})$ contains a set of form (5.3.1).

Proof. — The set U is definable in the language of valued fields with constants in K. The statement that $\pi(U)$ contains a set of form (5.3.1) is true over any (algebraically closed) valued field extending K because the theory of algebraically closed valued fields is model complete [**40**].

In particular, it is true over the algebraic closure F of the field $K(\xi_1)$, where the valuation $|\cdot|$ on F extends that on $K \subset F$ and

$$1 - \frac{1}{n} < |\xi_1| < 1$$

for all $n \in \mathbb{N}$. Hence there is a $b \in F$ such that $(\xi_1, b) \in U$. Let $P(\xi_1, \xi_2) \in K[\xi_1, \xi_2] \subset F[\xi_2]$ be any nonzero polynomial that vanishes at b.

If

$$\pi(U \cap \{x \in \operatorname{Max} S_{2,0} : P(x) = 0\})$$

does not contain a set of form (5.3.1), then by the Quantifier Elimination Theorem for the theory of algebraically closed valued fields [**40**],

$$\pi(U \cap \{x \in \operatorname{Max} S_{2,0} : P(x) = 0\}) \subset \{x \in \operatorname{Max} S_{1,0} : |x| < \delta\}$$

for some $\delta \in |K|$, $\delta < 1$. But this is not true over F, contradicting the fact that, by model completeness, K is an elementary submodel of F. $\square$

The following construction completes the example. For every $\varepsilon \in |K \setminus \{0\}|$, $\varepsilon < 1$, there is an $f \in S_{0,1}$ such that for every $P \in K[\xi_1, \xi_2] \setminus \{0\}$,

$$\pi\left(\{(\xi, \rho) \in \operatorname{Max} S_{1,1} : P(\xi, \rho) = 0 \text{ and } |\xi - f(\rho)| < \varepsilon\}\right)$$

contains no set of form (5.3.1).

Let P_i be an enumeration of polynomials in $K^\circ[\xi_1, \xi_2]$ such that for every $P \in K^\circ[\xi_1, \xi_2]$ there are infinitely many $i \in \mathbb{N}$ with $\|P - P_i\| < \varepsilon$. We inductively define sequences $\{n_i\} \subset \mathbb{N}$, $\{\rho_i\} \subset K^{\circ\circ}$ and $\{a_i\} \subset E$ such that $n_i \to \infty$ and $|\rho_i| \to 1$. Suppose $a_0, \ldots, a_{\ell-1}$; $n_0, \ldots, n_{\ell-1}$; $\rho_0, \ldots, \rho_{\ell-1}$ have been chosen and put

$$f_\ell := \sum_{i=0}^{\ell-1} a_i \rho^{n_i}.$$

Choose $n_\ell > n_{\ell-1}$ such that $|\rho_i^{n_\ell}| < \varepsilon$ for all $i < \ell$. Choose $\rho_\ell \in K^{\circ\circ}$ such that $|\rho_\ell^{n_\ell}| > \varepsilon$. Suppose $b_1, \ldots, b_r$ are all the roots of $P_\ell(\xi_2, \rho_\ell) = 0$. Choose $a_\ell \in E$ such that

$$\left|\sum_{i=0}^{\ell} a_i \rho_\ell^{n_i} - b_j\right| > \varepsilon$$

for $j = 1, \ldots, r$.

Put

$$f := \sum_{i \geq 0} a_i \rho^{n_i},$$

and let $P \in K^\circ[\xi_1, \xi_2] \setminus \{0\}$. There are infinitely many $i \in \mathbb{N}$ such that $\|P - P_i\| < \varepsilon$, and

$$\rho_i \notin \pi\left(\{(\xi,\rho) \in \operatorname{Max} S_{1,1} \colon P(\xi,\rho) = 0 \text{ and } |\xi - f(\rho)| < \varepsilon\}\right)$$

for each such i.

We include the next propositions for completeness. Proposition 5.3.8 gives conditions under which a quasi-affinoid algebra is actually affinoid (i.e., is a quotient of an $S_{m,0}$). Proposition 5.3.9 gives conditions under which a quasi-affinoid algebra is a quotient of an $S_{0,n}$.

Proposition 5.3.8. — *Let $A = S_{m,n}/I$ be a quasi-affinoid algebra. The following are equivalent:*

(i) *A is an affinoid algebra,*
(ii) *A satisfies the Maximum Modulus Principle*
(iii) *$\|\rho_i\|_{\sup}$ is attained for all $1 \le i \le n$,*
(iv) *$\|\rho_i\|_{\sup} < 1$ for all $1 \le i \le n$.*

Proof. — (i)⇒(ii), (ii)⇒(iii) and (iii)⇒(iv) are immediate from [**6**, Proposition 6.2.1.4], and the Nullstellensatz, Theorem 4.1.1. To see that (iv)⇒(i) observe that if

$$\|\rho_i\|_{\sup} \le \varepsilon < 1 \text{ for all } 1 \le i \le n$$

and $\varepsilon \in \sqrt{|K \setminus \{0\}|}$, say $\varepsilon^r = |c|$, $c \in K^{\circ\circ}$, then by Theorem 5.3.5

$$A\left\langle \frac{\rho_1^r}{c}, \dots, \frac{\rho_n^r}{c} \right\rangle = A$$

and

$$\begin{aligned} A\left\langle \frac{\rho_1^r}{c}, \dots, \frac{\rho_n^r}{c} \right\rangle &= (S_{m,n}/I)\left\langle \frac{\rho_1^r}{c}, \dots, \frac{\rho_n^r}{c} \right\rangle \\ &= S_{m,n}\left\langle \frac{\rho_1^r}{c}, \dots, \frac{\rho_n^n}{c} \right\rangle \Big/ I \cdot S_{m,n}\left\langle \frac{\rho_1^r}{c}, \dots, \frac{\rho_n^r}{c} \right\rangle. \end{aligned}$$

By the Weierstrass Division Theorem, Theorem 2.3.2, $S_{m,n}\langle \frac{\rho_1^r}{c}, \dots, \frac{\rho_n^r}{c}\rangle$ is affinoid. □

Proposition 5.3.9. — *Assume that K is algebraically closed and let $A = S_{m,n}/I$ be a quasi-affinoid algebra. The following are equivalent:*

(i) *$A \simeq S_{0,\ell}/J$ for some ℓ, J.*
(ii) *For every $f \in A$, each set*

$$\{x \in \operatorname{Max} A \colon |f(x)| = \|f\|_{\sup}\},$$
$$\{x \in \operatorname{Max} A \colon |f(x)| < \|f\|_{\sup}\},$$

is Zariski-closed; hence is a union of Zariski-connected components of $\operatorname{Max} A$.

(iii) *Let* $\pi : S_{m,n} \to S_{m,n}/I = A$ *be the canonical projection and let* N *be the number of Zariski-connected components of* $\operatorname{Max} A$. *Then there are* $c_{ij} \in K^\circ$, $1 \leq i \leq m$, $1 \leq j \leq N$, *such that each*

$$\prod_{j=1}^{N} (\pi(\xi_i) - c_{ij})$$

is quasi-nilpotent. (In other words, as a subset of $\operatorname{Max} S_{m,n}$, $\operatorname{Max} A$ *is contained in a finite union of open unit polydiscs, namely, those with centers* $(c_{ij}, \dots, c_{mj}) \times 0$.*)*

Proof. — (i)$\Rightarrow$(ii). Let $\mathfrak{p}$ be a minimal prime ideal of A. By Remark 2.3.6, there is a finite, torsion-free monomorphism

$$\varphi : S_{0,d} \to A/\mathfrak{p}.$$

Let $f \in A$ and let $q(f)$ be the integral equation of minimal degree for f over $S_{0,d}$, where

$$q = X^s + b_1 X^{s-1} + \cdots + b_s \in S_{0,d}[X],$$

as in [**6**, Proposition 3.8.1.7]. Following the argument of [**6**, Proposition 3.8.1.7], for every $y \in \operatorname{Max} S_{0,d}$,

$$\|\overline{f}_y\|_{\sup} = \max_{\substack{\varphi(y) \subset x \\ x \in \operatorname{Max} A}} |f(x)| = \max_{1 \leq i \leq s} |b_i(y)|^{\frac{1}{i}},$$

and

$$\|f\|_{\sup} = \max_{1 \leq i \leq s} \|b_i\|_{\sup}^{\frac{1}{i}},$$

where $\overline{f}_y$ is the residue class of f in the quotient of $A/\varphi(y) \cdot A$ by its nilradical. Since each $b_i \in S_{0,d}$, either

$$|b_i(y)| < \|b_i\|_{\sup} = \|b_i\| \tag{5.3.2}$$

for all $y \in \operatorname{Max} S_{0,d}$, or

$$|b_i(y)| = \|b_i\|_{\sup} = \|b_i\| \tag{5.3.3}$$

for all $y \in \operatorname{Max} S_{0,d}$. If (5.3.2) holds for every i such that

$$\|b_i\|^{\frac{1}{i}} = \|f\|_{\sup},$$

then $|f(x)| < \|f\|_{\sup}$ for all $x \in \operatorname{Max} A/\mathfrak{p}$. Otherwise, there is some i_0 such that

$$\|b_{i_0}\|^{\frac{1}{i_0}} = \|f\|_{\sup} \text{ and } |b_{i_0}(y)| = \|b_{i_0}\|$$

for all $y \in \operatorname{Max} S_{0,d}$. In this case, $|f(x)| = \|f\|_{\sup}$ for all $x \in \operatorname{Max} A/\mathfrak{p}$. This shows that each set

$$\{x \in \operatorname{Max} A/\mathfrak{p} \colon |f(x)| = \|f\|_{\sup}\},$$
$$\{x \in \operatorname{Max} A/\mathfrak{p} \colon |f(x)| < \|f\|_{\sup}\},$$

is Zariski-closed. Taking the union over the finitely many minimal prime ideals of A, (ii) follows.

(ii)⇒(iii). Let $X_1, \dots, X_N$ be the Zariski-connected components of $\operatorname{Max} A$, choose $x_j \in X_j$, $1 \le j \le N$, and put

$$c_{ij} := \xi_i(x_j).$$

Part (iii) follows by applying part (ii) to each $\xi_i - c_{ij}$.

(iii)⇒(i). Put

$$g_i := \prod_{j=1}^{N} (\xi_i - c_{ij});$$

then by the Extension Lemma, Theorem 5.2.6, there is a K-algebra homomorphism $\psi : S_{0,m+n} \to A$ such that

$$\psi(\rho_i) = \pi(\rho_i), \qquad 1 \le i \le n, \text{ and}$$
$$\psi(\rho_{n+i}) = \pi(g_i), \qquad 1 \le i \le m.$$

It follows from the Weierstrass Division Theorem, Theorem 2.3.2, that ψ is finite. Thus, after a homothety, part (i) follows. □

5.4. Tensor Products. — In this subsection we prove that tensor products exist in the category of quasi-affinoid algebras with K-algebra homomorphisms. These results will be needed in [**22**] when we discuss fiber products of quasi-affinoid varieties.

Lemma 5.4.1. — (i) *If A is a quasi-affinoid algebra and $\varphi : A \to B$ is a finite K-algebra homomorphism, then B is quasi-affinoid.*

(ii) *If A and B are quasi-affinoid algebras then so is the ring-theoretic direct sum $A \oplus B$.*

Proof. — (i) We may take $A = S_{m,n}$. Let $b_1, \dots, b_\ell \in B$ be such that $B = \sum_{i=1}^{\ell} \varphi(S_{m,n}) b_i$. For each i, let $A_{ij} \in S_{m,n}$ be such that $b_i^{n_i} + \varphi(A_{i1}) b_i^{n_i - 1} + \cdots + \varphi(A_{in_i}) = 0$. Replacing b_i by cb_i for a suitable nonzero $c \in K^\circ$ we may assume that $\|A_{ij}\| \le 1$. Let $P_i \in S_{m+\ell,n}$ be defined by

$$P_i(\eta_i) = \eta_i^{n_i} + A_{i1}\eta_i^{n_i - 1} + \cdots + A_{in_i},$$

where $S_{m+\ell,n} = K\langle \xi, \eta \rangle [\![\rho]\!]_s$. Then P_i is regular in η_i. Let

$$\pi : S_{m+\ell,n} \to S_{m+\ell,n}/(P_1, \dots, P_\ell)$$

be the canonical projection, and consider the K-algebra homomorphism

$$\psi : S_{m,n}[\eta_1, \ldots, \eta_\ell] \to B : \Sigma f_\mu \eta^\mu \mapsto \Sigma \varphi(f_\mu) b^\mu.$$

By the Weierstrass Division Theorem (Theorem 2.3.2),

$$S_{m+\ell,n}/(P_1, \ldots, P_\ell) \simeq S_{m,n}[\eta_1, \ldots, \eta_\ell]/(P_1, \ldots, P_\ell).$$

The K-algebra homomorphism

$$S_{m+\ell,n} \to S_{m+\ell,n}/(P_1, \ldots, P_\ell) \to S_{m,n}[\eta_1, \ldots, \eta_\ell]/(P_1, \ldots, P_\ell) \to B$$

is surjective, as required.
(ii) It is sufficient to consider $A = B = S_{m,n}$. The diagonal map $S_{m,n} \to S_{m,n} \oplus S_{m,n}$ is a finite K-algebra homomorphism, so the result follows from part (i). □

Definition 5.4.2. — Let A, B_1, B_2 be quasi-affinoid algebras and let B_1, B_2 be A-algebras via homomorphisms $\varphi_i : A \to B_i$, $i = 1, 2$. By Remark 5.2.8, we can write

$$B_1 = A\langle \xi_1, \ldots, \xi_{m_1} \rangle [\![\rho_1, \ldots, \rho_{n_1}]\!]_s / I_1 \text{ and}$$
$$B_2 = A\langle \xi_{m_1+1}, \ldots, \xi_{m_1+m_2} \rangle [\![\rho_{n_1+1}, \ldots, \rho_{n_1+n_2}]\!]_s / I_2.$$

We define the **separated tensor product of** B_1 **and** B_2 **over** A by

$$B_1 \otimes^s_A B_2 := A\langle \xi_1, \ldots, \xi_{m_1+m_2} \rangle [\![\rho_1, \ldots, \rho_{n_1+n_2}]\!]_s / (I_1 + I_2).$$

By the Extension Lemma (Theorem 5.2.6), $B_1 \otimes^s_A B_2$ is independent of the presentations of B_1 and B_2. The inclusions

$$A\langle \xi_1, \ldots, \xi_{m_1} \rangle [\![\rho_1, \ldots, \rho_{n_1}]\!]_s \to A\langle \xi_1, \ldots, \xi_{m_1+m_2} \rangle [\![\rho_1, \ldots, \rho_{n_1+n_2}]\!]_s,$$
$$\xi_i \mapsto \xi_i, \quad \rho_j \mapsto \rho_j, \quad i = 1, \ldots, m_1, \quad j = 1, \ldots, n_1;$$
$$A\langle \xi_{m_1+1}, \ldots, \xi_{m_1+m_2} \rangle [\![\rho_{n_1+1}, \ldots, \rho_{n_1+n_2}]\!]_s \to$$
$$A\langle \xi_1, \ldots, \xi_{m_1+m_2} \rangle [\![\rho_1, \ldots, \rho_{n_1+n_2}]\!]_s,$$
$$\xi_{m_1+i} \mapsto \xi_{m_1+i}, \quad \rho_{n_1+j} \mapsto \rho_{n_1+j} \quad i = 1, \ldots, m_2, \quad j = 1, \ldots, n_2,$$

define canonical homomorphisms

$$\sigma_i : B_i \to B_1 \otimes^s_A B_2.$$

The next proposition shows that $B_1 \otimes^s_A B_2$ satisfies the universal property in the category of quasi-affinoid algebras that justifies calling it a tensor product.

Proposition 5.4.3. — *Let $\varphi_i : A \to B_i$, $i = 1, 2$, be K-algebra homomorphisms of quasi-affinoid algebras and let $\psi_i : B_i \to D$ be A-algebra homomorphisms of quasi-affinoid algebras. Then there is a unique A-algebra homomorphism $\psi : B_1 \otimes^s_A B_2 \to D$ such that*

$$\begin{tikzcd}
 & B_1 \arrow[dl, "\sigma_1"'] \arrow[dr, "\psi_1"] & \\
B_1 \otimes^s_A B_2 \arrow[rr, dotted, "\psi"] & & D \\
 & B_2 \arrow[ul, "\sigma_2"] \arrow[ur, "\psi_2"'] &
\end{tikzcd}$$

commutes, where the $\sigma_i : B_i \to B_1 \otimes^s_A B_2$ *are the homomorphisms given in Definition 5.4.2.*

Proof. — By the Extension Lemma (Theorem 5.2.6 or Remark 5.2.8) there is a unique $\psi' : A\langle\xi_1, \ldots, \xi_{m_1+m_2}\rangle[\![\rho_1, \ldots, \rho_{n_1+n_2}]\!] \to D$ that extends $\psi_1 \circ \varphi_1 = \psi_2 \circ \varphi_2$ such that

$$\begin{aligned}
\psi'(\xi_i) &= \psi_1(\xi_i), & i &= 1, \ldots, m_1, \\
\psi'(\rho_j) &= \psi_1(\rho_j), & j &= 1, \ldots, n_1, \\
\psi'(\xi_{m_1+i}) &= \psi_2(\xi_{m_1+i}), & i &= 1, \ldots, m_2, \\
\psi'(\rho_{n_1+j}) &= \psi_2(\rho_{n_1+j}), & j &= 1, \ldots, n_2.
\end{aligned}$$

Since $(I_1 + I_2) \subset \mathrm{Ker}\,(\psi')$, the result follows. $\square$

Remark 5.4.4. — (i) If A, B_1, B_2 are affinoid then it follows from the above Proposition and the universal property of the complete tensor product ([**6**, Proposition 3.1.1.2]) that $B_1 \otimes^s_A B_2 = B_1 \widehat{\otimes}_A B_2$.
(ii) In general, $B_1 \otimes^s_A B_2 \neq B_1 \widehat{\otimes}_A B_2$. In the case that the $S_{m,n}(E, K)$ are complete, we have $B_1 \otimes^s_A B_2 \supset B_1 \widehat{\otimes}_A B_2$. This follows from the universal property of $\widehat{\otimes}_A$. In all cases we have $S_{0,1} \otimes^s_K S_{0,1} \not\subset S_{0,1} \widehat{\otimes}_K S_{0,1}$ since $\sum_i (\rho_1\rho_2)^i \in (S_{0,1} \otimes^s_K S_{0,1}) \setminus (S_{0,1} \widehat{\otimes}_K S_{0,1})$.

The following important examples of separated tensor products are computed directly from Definition 5.4.2.

Corollary 5.4.5. —

$$S_{m_1,n_1} \otimes^s_K S_{m_2,n_2} = S_{m_1+m_2,n_1+n_2},$$

and if A is a quasi-affinoid algebra,

$$A \otimes^s_K S_{m,n} = A\langle\xi\rangle[\![\rho]\!]_s.$$

The following two propositions are easy consequences of the definition and the universal property of the separated tensor product (cf. [**6**, Propositions 6.1.1.10 and 6.1.1.11]).

Proposition 5.4.6. — *Let A', A, B_1, B_2 be quasi-affinoid algebras and assume that the B_i are both A and A'-algebras via homomorphisms $A' \to A$ and $A \to B_1$, $A \to B_2$. Then the canonical homomorphism*

$$B_1 \otimes^s_{A'} B_2 \to B_1 \otimes^s_A B_2$$

is surjective.

Proposition 5.4.7. — *Let A, B_1, B_2 be quasi-affinoid algebras and assume that B_1, B_2 are A-algebras via homomorphisms $A \to B_i$, $i = 1, 2$. Let $\mathfrak{b}_i \subset B_i$, $i = 1, 2$ be ideals and denote by $(\mathfrak{b}_1, \mathfrak{b}_2)$ the ideal in $B_1 \otimes^s_A B_2$ generated by the images of $\mathfrak{b}_1$ and $\mathfrak{b}_2$. Then the canonical map $\pi : B_1 \otimes^s_A B_2 \to B_1/\mathfrak{b}_1 \otimes^s_A B_2/\mathfrak{b}_2$ is surjective and satisfies* $\operatorname{Ker} \pi = (\mathfrak{b}_1, \mathfrak{b}_2)$. *Hence $(B_1 \otimes^s_A B_2)/(\mathfrak{b}_1, \mathfrak{b}_2) \simeq B_1/\mathfrak{b}_1 \otimes^s_A B_2/\mathfrak{b}_2$.*

It follows from Lemma 5.4.1 and Proposition 5.4.7 that base change preserves finite (respectively surjective) morphisms.

Proposition 5.4.8. — *Let A and B be quasi-affinoid algebras. Let $\varphi : A \to B$ be a K-algebra homomorphism and let C be a quasi-affinoid A-algebra. If φ is finite (respectively surjective) then the induced map $C \to B \otimes^s_A C$ is finite (respectively surjective).*

Proof. — Suppose B is a finite A-module via φ. It follows from the right-exactness of the ordinary tensor product that $B \otimes_A C$ is a finite C-module. By Lemma 5.4.1 $B \otimes_A C$ is a quasi-affinoid algebra. It therefore follows from the universal property for tensor products that $B \otimes^s_A C = B \otimes_A C$. In particular, $C \to B \otimes^s_A C$ is finite.

If φ is surjective, then we may write $B = A/I$, where $I := \operatorname{Ker} \varphi$. Then by Proposition 5.4.7,

$$B \otimes^s_A C = A/I \otimes^s_A C/(0) \cong (A \otimes^s_A C)/(I, (0)),$$

which is a quotient of C. Therefore $C \to B \otimes^s_A C$ is surjective. □

A small extension of Definition 5.4.2 yields a ground field extension functor for quasi-affinoid algebras.

Definition 5.4.9. — Let (E, K), (E', K') be such that $S_{m,n}(E, K) \subset S_{m,n}(E', K')$ and let $A := S_{m,n}(E, K)/I$. We say that the K'-affinoid algebra

$$A' = S_{0,0}(E', K') \otimes^s_{S_{0,0}(E,K)} A := S_{m,n}(E', K')/I \cdot S_{m,n}(E', K')$$

results from A by **ground field extension** from (E, K) to (E', K').

Proposition 5.4.10. — *The canonical homomorphism*

$$A \to S_{0,0}(E', K') \otimes^s_{S_{0,0}(E,K)} A$$

is a faithfully flat norm-preserving monomorphism both in $\|\cdot\|_I$ *and* $\|\cdot\|_{I\cdot S_{m,n}(E',K')}$ *and in* $\|\cdot\|_{\sup}$.

Proof. — Immediate from Lemma 3.1.11 and Proposition 4.1.3. □

5.5. Banach Function Algebras. — Each representation of a quasi-affinoid algebra A as a quotient $S_{m,n}/I$ yields the K-algebra norm $\|\cdot\|_I$, which by Lemma 3.1.4, is complete if $S_{m,n}$ is. We saw (Corollary 5.2.4) that even though A may not be complete, all these norms are equivalent. By the Nullstellensatz, Theorem 4.1.1, if A is reduced then $\|\cdot\|_{\sup}$ is a norm on A. In this subsection we shall show when $\operatorname{Char} K = 0$ (Theorem 5.5.3) and often when $\operatorname{Char} K = p \neq 0$ (Theorem 5.5.4) that if A is reduced, $\|\cdot\|_{\sup}$ is equivalent to the residue norms $\|\cdot\|_I$. It follows that if in addition E and K are such that A is complete in $\|\cdot\|_I$ then A is complete in $\|\cdot\|_{\sup}$, i.e., it is a Banach function algebra.

The obstruction to following the argument of [**6**, Theorem 6.2.4.1], is, as usual, the lack of a suitable Noether Normalization for quasi-affinoid algebras. Theorems 3.4.3 and 3.4.6 allow us to reduce the problem to considering quotient rings of $S_{0,n+m}$, for which a Noether Normalization is available. The fact that the quotients of $S_{0,n+m}$ so obtained are reduced is guaranteed when the $S_{m,n}$ are excellent.

Lemma 5.5.1. — *Suppose K and E are such that the $S_{m,n}$ are complete and the fields of fractions of the $S_{0,n}(E,K)$ are weakly stable. Let A be a reduced quasi-affinoid algebra. If there is a finite K-algebra homomorphism $S_{0,n}/I \to A$ then A is a Banach function algebra.*

Proof. — As in the proof of [**6**, Theorem 6.2.4.1], we use Noether Normalization for quotients of $S_{0,n}$ (Remark 2.3.6) to reduce to the case that $I = (0)$ and $S_{0,n} \to A$ is a finite, torsion-free monomorphism.

Note that $S_{0,n}$ is integrally closed (for example, apply Theorem 4.2.7 or use Noether Normalization as in [**6**, Theorem 5.2.6.1]). Since, in addition, we have assumed that $Q(S_{0,n})$ is a weakly stable field ([**6**, Definition 3.5.2.1]), we may apply [**6**, Theorem 3.8.3.7]. □

Proposition 5.5.2. — *Under any of the conditions*

(i) $\operatorname{Char} K = 0$

(ii) $\operatorname{Char} K = p \neq 0$ *and* $S_{m,n}(E,K) \simeq \oplus_{i=1}^N (S_{m,n}(E,K))^p$ *as normed* K^p*-algebras,*

(iii) $\operatorname{Char} K = p \neq 0$, $[K : K^p] < \infty$ *and* $[\widetilde{E} : \widetilde{E}^p] < \infty$

the fields of fractions of the rings $S_{m,n}(E,K)$ are weakly stable.

Proof. — When $\operatorname{Char} K = 0$, this is [**6**, Proposition 3.5.1.4]. Note that condition (iii) implies condition (ii) because K is complete (use [**6**, Proposition 2.3.3.4]). Thus it remains only to verify case (ii), which follows from [**6**, Lemma 3.5.3.2]. □

Note that under any of the conditions of Proposition 5.5.2, the rings $S_{m,n}(E,K)$ are excellent (see Propositions 4.2.3 and 4.2.5).

In characteristic zero, we show in Theorem 5.5.3 that the supremum norm of a reduced quasi-affinoid algebra A is equivalent to the residue norm arising from any presentation of A as a quotient of a ring of separated power series. In some cases this is an extension of Corollary 5.2.4, which establishes the equivalence of all the residue norms (whether or not A is reduced and of characteristic zero). In characteristic p, our results are less complete (see Theorem 5.5.4). The proofs of Theorems 5.5.3 and 5.5.4 rely on restriction to finite disjoint unions of open polydiscs, for which one has a Noether Normalization. In the proof of Theorem 5.5.3, we reduce to the case of polydiscs with rational centers. The proof of Theorem 5.5.4 does not depend on the characteristic.

Theorem 5.5.3. — *Suppose that* $\operatorname{Char} K = 0$ *and that* $A = S_{m,n}(E,K)/I$ *is a reduced quasi-affinoid algebra. Then* $\|\cdot\|_I$ *and* $\|\cdot\|_{\sup}$ *on* A *are equivalent. If in addition* A *is complete in* $\|\cdot\|_I$*, then* A *is a Banach function algebra.*

Proof. — Let $E' \supset E$ be as in Theorem 2.1.3 (ii) so that the $S_{m,n}(E',K)$ are complete. By Propositions 4.2.3 and 4.2.6, $A' = S_{m,n}(E',K)/I \cdot S_{m,n}(E',K)$ is reduced, since $T_{m,n}(\varepsilon) = T_{m,n}(\varepsilon, K)$ does not depend on E or E'. By Proposition 4.1.3 and Lemma 3.1.11 the map

$$S_{m,n}(E,K)/I \to S_{m,n}(E',K)/I \cdot S_{m,n}(E',K)$$

is an inclusion which is an isometry in both the supremum and residue norms. Hence it is sufficient to prove the equivalence of $\|\cdot\|_I$ and $\|\cdot\|_{\sup}$ when E is such that the $S_{m,n}(E,K)$ are complete.

Let K' be a finite extension of K such that there are $c_1, \ldots, c_r \in ((K')^\circ)^m$ with $|c_i - c_j| = 1$, $1 \le i < j \le r$, such that for every

$$\mathfrak{p} \in \operatorname{Ass}\,(S_{m,n}(E, \widehat{K}_{\text{alg}})^\sim / \widetilde{I} \cdot S_{m,n}(E, \widehat{K}_{\text{alg}})^\sim)$$

there is an i, $1 \le i \le r$, with

$$\mathfrak{m}_{\widetilde{c}_i} = (\xi - \widetilde{c}_i, \rho) \supset \mathfrak{p},$$

where $\widehat{K}_{\text{alg}}$ is the completion of the algebraic closure of K.

Let $S'_{m,n} := S_{m,n}(E,K')$ and $I' := I \cdot S'_{m,n}$. Observe that $S'_{m,n}/I'$ is reduced. (Indeed, we may write $K' = K(\alpha)$, so every $f \in S'_{m,n}$ may be written in the form

$$f = \sum_{j=0}^{d-1} f_j \alpha^j,$$

for $f_j \in S_{m,n}$. Let $\sigma_0, \ldots, \sigma_{d-1}$ be the distinct embeddings of K' over K in an algebraic closure of K and let $\alpha_i := \sigma_i(\alpha)$, $0 \leq i \leq d-1$. Then $\det(\alpha_i^j) = \Pi_{i \neq j}(\alpha_i - \alpha_j) \neq 0$. It follows that the f_j are linear combinations of the $\sigma_i(f)$. Hence, if $f \in \sqrt{I'}$, so is each f_j. But the map $S_{m,n} \to S'_{m,n}$ is faithfully flat (Lemma 4.2.8(iii)), so each $f_j \in \sqrt{I} = I$. It follows that $f \in I'$.) Now, by Proposition 4.1.3 and Lemma 3.1.11(ii), the map $S_{m,n}/I \to S'_{m,n}/I'$ is an inclusion and an isometry in both the supremum norm and the residue norm. Since $S_{m,n}/I$ is complete in $\|\cdot\|_I$, it therefore suffices to prove the theorem for $S'_{m,n}/I'$. Note that all the $S_{m',n'}(E, K')$ are complete

By Theorem 3.4.3(ii), the map

$$\psi : S'_{m,n}/I' \to \left(\oplus_{j=1}^r S'_{0,n+m}\right) / \omega_c(I') \cdot \left(\oplus_{j=1}^r S'_{0,n+m}\right)$$

is an isometry in the residue norms. By Proposition 4.2.3 and [**25**, Theorem 32.2],

$$\left(\oplus_{j=1}^r S'_{0,n+m}\right) / \omega_c(I') \cdot \left(\oplus_{j=1}^r S'_{0,n+m}\right)$$

is reduced. Since ψ is a contraction with respect to $\|\cdot\|_{\sup}$, it suffices to prove the theorem for this ring. That is Lemma 5.5.1. □

Theorem 5.5.4. — *Suppose that the rings $S_{m,n}(E, K)$ are excellent (see Proposition 4.2.3 or 4.2.5) and that at least one of the following two conditions is satisfied:*

(i) *K is perfect*
(ii) *There is an E', $E \subset E'$, such that the fields of fractions of the $S_{0,n}(E', K)$ are weakly stable, and the $S_{0,n}(E', K)$ are complete.*

Let $A = S_{m,n}(E, K)/I$ be reduced. Then on A the norms $\|\cdot\|_I$ and $\|\cdot\|_{\sup}$ are equivalent. If in addition A is complete in $\|\cdot\|_I$ then A is a Banach function algebra.

Proof. — We may assume (see Remark 2.1.4(i)) that E is a field. We now show that (i) implies (ii). In the case that K is perfect there is an $E' \supset E$ such that $S_{m,n}(E', K)$ is complete (see Theorem 2.1.3(ii)). Since K is perfect, we may extend further so that E' is perfect. Then, by Proposition 5.5.2 the fields of fractions of the $S_{0,n}(E', K)$ are also weakly stable.

Choose $c_1, \ldots, c_r \in (K_{\text{alg}}^\circ)^m$ with $\mathfrak{m}_{\widetilde{c}_i} \neq \mathfrak{m}_{\widetilde{c}_j}$, $1 \leq i < j \leq r$, such that for every $\mathfrak{p} \in \operatorname{Ass}(\widetilde{S}_{m,n}/\widetilde{I})$ there is some i, $1 \leq i \leq r$, with

$$\mathfrak{m}_{\widetilde{c}_i} \supset \mathfrak{p}.$$

(The $\widetilde{m}_{\widetilde{c}_i}$ are the maximal ideals of $\widetilde{S}_{m,n}$ corresponding to $\widetilde{c}_i$ as in Definition 3.4.4.)

By Theorem 3.4.6(ii), the map

$$\psi : S_{m,n}/I \to D_{m,n}(c)/\omega_c(I)$$

is an isometry in the residue norms $\|\cdot\|_I$ and $\|\cdot\|_{\omega_c(I)}$. Since $S_{m,n}(E,K)$ is excellent, by [**25**, Theorem 32.2], $D_{m,n}(c)/\omega_c(I)$ is reduced. Since ψ is a contraction with respect to $\|\cdot\|_{\sup}$, it suffices to prove the theorem for that ring. Recall that $D_{m,n}(c) = S_{m,n+m}(E,K)/J$ for some ideal J. Let

$$D'_{m,n}(c) := S_{m,n+m}(E',K)/J \cdot S_{m,n+m}(E',K).$$

Then $D'_{m,n}(c)/\omega_c(I) \cdot D'_{m,n}(c)$ is reduced since the maximal-adic completions of all its local rings coincide with those of the reduced, excellent ring $D_{m,n}(c)/\omega_c(I)$. By Proposition 4.1.3 and Lemma 3.1.11, the map

$$D_{m,n}(c)/\omega_c(I) \to D'_{m,n}(c)/\omega_c(I).D'_{m,n}(c)$$

is an inclusion which is an isometry in both the supremum and residue norms. Hence it suffices to prove the equivalence of the residue norm and the supremum norm on $D'_{m,n}(c)/\omega_c(I) \cdot D'_{m,n}(c)$. By Lemma 3.4.5, this ring is a finite extension of a quotient of a ring $S_{0,d}(E',K)$. Now apply Lemma 5.5.1. $\square$

6. A Finiteness Theorem

In Subsection 6.1 we prove a finiteness theorem, which is a weak analogue of Zariski's Main Theorem, for quasi-finite maps, and in Subsection 6.2 we apply this finiteness theorem to show that every quasi-affinoid subdomain is a finite union of R-subdomains.

6.1. A Finiteness Theorem. — In applications ([**2**], [**16**], [**17**], [**18**], [**19**], [**20**], [**21**] and [**23**]), certain weaker forms of Noether Normalization have proved useful. We collect two examples here. Recall that we showed in Subsection 5.3 that we associate canonically with each R-domain $U \subset \operatorname{Max} A$, the A-algebra of quasi-affinoid functions $\mathcal{O}(U)$.

We call a quasi-affinoid map $\pi : \operatorname{Max} B \to \operatorname{Max} A$ **finite** if, and only if, B is a finite A-module via the induced map $\pi^* : A \to B$.

Proposition 6.1.1. — *Let $\pi : \operatorname{Max} B \to \operatorname{Max} A$ be a quasi-affinoid map. Suppose $U_1, \ldots, U_n$ is a cover of* $\operatorname{Max} B$ *by R-subdomains. If each $\pi|_{U_i} : U_i \to$* $\operatorname{Max} A$ *is finite then π is finite.*

Proof. — By Proposition 5.3.6(ii) and the Krull Intersection Theorem ([**25**, Theorem 8.10]), the natural map

$$B \to \prod_{i=0}^{n} \mathcal{O}(U_i)$$

is injective. Each $\mathcal{O}(U_i)$ is a finite A-module; hence B, being a submodule of the finite A-module $\Pi\mathcal{O}(U_i)$, is a finite A-module as well. □

Let $\pi : \operatorname{Max} B \to \operatorname{Max} A$ be a quasi-affinoid map. If $U \subset \operatorname{Max} A$ is an R-domain defined by inequalities among $f_1, \ldots, f_\ell$ then $\pi^{-1}(U) \subset \operatorname{Max} B$ is an R-domain defined by the corresponding inequalities among $f_1 \circ \pi, \ldots, f_\ell \circ \pi$.

The affinoid analog of the following is false; see Example 6.1.3.

Theorem 6.1.2. — **(Finiteness Theorem)** *Let $\pi : \operatorname{Max} B \to \operatorname{Max} A$ be a quasi-affinoid map which is finite-to-one. There exists a finite cover of* $\operatorname{Max} A$ *by R-domains U_i such that each map*

$$\pi|_{\pi^{-1}(U_i)} : \pi^{-1}(U_i) \to U_i$$

is finite. (Note: We do not assume that π is surjective.)

Proof. — Let $\varphi : A \to B$ be the K-algebra homomorphism corresponding to π. Since B is quasi-affinoid, there is a K-algebra epimorphism

$$S_{m,n} \to B.$$

The images of $\xi_1, \dots, \xi_m$ (respectively, $\rho_1, \dots, \rho_n$) in B are power-bounded (respectively, quasi-nilpotent). By Remark 5.2.8, this induces a unique K-algebra homomorphism ψ such that the following diagram commutes

$$\begin{tikzcd}
 & A\langle \xi_1, \dots, \xi_m \rangle [\![\rho_1, \dots, \rho_n]\!]_s \arrow[dr, "\Psi"] & \\
A \arrow[ur] \arrow[rr, "\varphi"'] & & B
\end{tikzcd}$$

Since $S_{m,n} \to B$ is surjective, so is ψ.

Let

$$I := \operatorname{Ker} \psi;$$

then

$$B \cong A\langle \xi_1, \dots, \xi_m \rangle [\![\rho_1, \dots, \rho_n]\!]_s / I,$$

and we may therefore assume that the original map φ is of the form

$$A \xrightarrow{\varphi} A\langle \xi \rangle [\![\rho]\!]_s / I.$$

The proof proceeds by induction on (m, n), ordered lexicographically. Assume $m + n > 0$. (If $m + n = 0$, then $B = K$ and the K-algebra homomorphism φ, being surjective, is finite.)

Let $f_1, \dots, f_\ell$ generate I, and write

$$f_i = \sum a_{i\mu\nu} \xi^\mu \rho^\nu, \quad 1 \le i \le \ell,$$

where each $a_{i\mu\nu} \in A$. Since π is finite-to-one, $\{a_{i\mu\nu}\}$ generates the unit ideal of A.

Writing A as a quotient of a ring of separated power series and applying Lemma 3.1.6 to pre-images of the f_i, we obtain a finite index set $J \subset \mathbb{N}^m \times \mathbb{N}^n$ such that for each $x \in \operatorname{Max} A$ there is an i_0, $1 \le i_0 \le \ell$, and an index $(\mu_0, \nu_0) \in J$ such that

$$\begin{aligned}
&|a_{i_0\mu_0\nu_0}(x)| \ge |a_{i\mu\nu}(x)| && \text{for all } i, \mu, \nu \\
(6.1.1) \qquad &|a_{i_0\mu_0\nu_0}(x)| > |a_{i_0\mu\nu}(x)| && \text{for all } \nu < \nu_0 \text{ and all } \mu \\
&|a_{i_0\mu_0\nu_0}(x)| > |a_{i_0\mu\nu_0}(x)| && \text{for all } \mu > \mu_0.
\end{aligned}$$

(Note, in particular, that (6.1.1) guarantees that $\{a_{i\mu\nu} : 1 \le i \le \ell, (\mu, \nu) \in J\}$ generates the unit ideal of A.)

Fix i_0, $1 \leq i_0 \leq \ell$, and $(\mu_0, \nu_0) \in J$. Let $U_{i_0\mu_0\nu_0}$ be the set of points $x \in \operatorname{Max} A$ such that

$$\begin{aligned} |a_{i_0\mu_0\nu_0}(x)| &\geq |a_{i\mu\nu}(x)| && \text{for all } 1 \leq i \leq \ell \text{ and } (\mu,\nu) \in J \\ |a_{i_0\mu_0\nu_0}(x)| &> |a_{i_0\mu\nu}(x)| && \text{for all } (\mu,\nu) \in J \text{ with } \nu < \nu_0 \\ |a_{i_0\mu_0\nu_0}(x)| &> |a_{i_0\mu\nu_0}(x)| && \text{for all } (\mu,\nu) \in J \text{ with } \mu > \mu_0. \end{aligned}$$

As in Subsection 5.3, $U_{i_0\mu_0\nu_0}$ is a quasi-rational subdomain of $\operatorname{Max} A$, which is in fact equal to the set of points $x \in \operatorname{Max} A$ where (6.1.1) holds. Furthermore, the $U_{i_0\mu_0\nu_0}$ cover $\operatorname{Max} A$.

We may now replace A by $\mathcal{O}(U_{i_0\mu_0\nu_0})$ and B by

$$\mathcal{O}(U_{i_0\mu_0\nu_0})\langle\xi\rangle[\![\rho]\!]_s / I \cdot \mathcal{O}(U_{i_0\mu_0\nu_0})\langle\xi\rangle[\![\rho]\!]_s.$$

Replacing f_{i_0} by $a^{-1}_{i_0\mu_0\nu_0} f_{i_0}$, we may assume that $a_{i_0\mu_0\nu_0} = 1$. Put

$$f_{i_0\nu_0} := \sum_{\mu} a_{i_0\mu\nu_0}\xi^{\mu};$$

then $f_{i_0\nu_0}$ is preregular in ξ (cf. Definition 2.3.7).

The two quasi-rational subdomains

$$V := \{y \in \operatorname{Max} B : |f_{i_0\nu_0}(y)| = 1\} \text{ and } W := \{y \in \operatorname{Max} B : |f_{i_0\nu_0}(y)| < 1\}$$

cover $\operatorname{Max} B$, and each restriction $\pi|_V$ and $\pi|_W$ is finite-to-one. By Proposition 5.3.6(ii) and the Krull Intersection Theorem ([**25**, Theorem 8.10]), the natural map

$$B \to \mathcal{O}(V) \oplus \mathcal{O}(W)$$

is injective. Hence it suffices to treat the maps $A \to \mathcal{O}(V)$ and $A \to \mathcal{O}(W)$.

***Case* (A).** — $A \to \mathcal{O}(V)$.

Observe that

$$\mathcal{O}(V) = A\langle\xi_1, \dots, \xi_{m+1}\rangle[\![\rho_1, \dots, \rho_n]\!]_s / J,$$

where J is the ideal generated by I and the element

$$F := \xi_{m+1} f_{i_0\nu_0} - 1.$$

Put

$$G := \rho^{\nu_0} + \sum_{\substack{\nu \neq \nu_0 \\ \mu}} a_{i_0\mu\nu}\xi_{m+1}\xi^{\mu}\rho^{\nu} \equiv \xi_{m+1} f_{i_0} \mod J;$$

in particular, $G \in J$. By (6.1.1), after a change of variables among the ρ's, we can assume that G is regular in ρ_n (in the sense of Definition 2.3.7). Similarly, after a change of variables among the ξ's, we can assume that F is regular in

ξ_{m+1}. Applying Theorem 2.3.8, first to divide by G, then by F, shows that $\mathcal{O}(V)$ is a finite extension of an A-algebra of the form

$$B' := A\langle\xi_1, \dots, \xi_m\rangle[\![\rho_1, \dots, \rho_{n-1}]\!]_s/I'.$$

Since $\mathcal{O}(V)$ is a finite extension of the A-algebra B', the map

$$\operatorname{Max} B' \to \operatorname{Max} A$$

is finite-to-one. Furthermore, $(m, n-1) < (m, n)$. We are done by induction.

***Case* (B)**. — $A \to \mathcal{O}(W)$.

Observe that

$$\mathcal{O}(W) = A\langle\xi_1, \dots, \xi_m\rangle[\![\rho_1, \dots, \rho_{n+1}]\!]_s/J,$$

where J is generated by I and the element

$$F := f_{i_0\nu_0} - \rho_{n+1}.$$

By (6.1.1), after a change of variables among the ξ's, F is regular in ξ_m (in the sense of Definition 2.3.7). By Theorem 2.3.8, $\mathcal{O}(W)$ is a finite extension of an A-algebra of the form

$$B' := A\langle\xi_1, \dots, \xi_{m-1}\rangle[\![\rho_1, \dots, \rho_{n+1}]\!]_s/I'.$$

Since $\mathcal{O}(W)$ is a finite extension of the A-algebra B', the map

$$\operatorname{Max} B' \to \operatorname{Max} A$$

is finite-to-one. Furthermore, $(m-1, n+1) < (m, n)$, completing Case B.

To complete the proof, we pass to a common refinement of the covers by R-domains obtained in the above two cases, observing that the intersection of R-domains is an R-domain, and that if $\pi : \operatorname{Max} B \to \operatorname{Max} A$ is finite, so is $\pi|_{\pi^{-1}(U)} : \pi^{-1}(U) \to U$ for any R-subdomain U of $\operatorname{Max} A$. □

Example 6.1.3. — The affinoid map induced by

$$\varphi : K\langle\xi\rangle \to K\langle\xi, \eta\rangle/(\xi\eta^2 + \eta + 1)$$

is finite-to-one. But if $\operatorname{Char} \widetilde{K} \neq 2$, φ is not finite. Indeed, if it were, the polynomial $\xi\eta^2 + \eta + 1$, being prime, would have to divide a monic polynomial in $K\langle\xi\rangle[\eta]$. Since ξ is not a unit, φ cannot be finite.

Now, suppose there is a finite cover of $\operatorname{Max} K\langle\xi\rangle$ by affinoid rational subdomains U such that each induced map

$$\mathcal{O}(U) \to \mathcal{O}(U)\widehat{\otimes}_{K\langle\xi\rangle} K\langle\xi, \eta\rangle/(\xi\eta^2 + \eta + 1)$$

is finite. Then the affinoid map induced by φ is proper by [**6**, Proposition 9.6.2.5], and [**6**, Proposition 9.6.2.3]. It then follows from [**6**, Corollary 9.6.3.6], that φ is finite, a contradiction. This shows that the analogue of

Theorem 6.1.2 does not hold in the affinoid case. Indeed the covering obtained is not in general admissible in the sense of [**22**].

6.2. An Application to Quasi-Affinoid Domains. — In this subsection we apply Theorem 6.1.2 to prove that every quasi-affinoid subdomain is a finite union of R-subdomains. As a corollary we deduce that every quasi-affinoid subdomain is open.

Lemma 6.2.1. — *Let A and B be commutative rings and let $\varphi : A \to B$ be a finite homomorphism.*

(i) *Suppose that for every maximal ideal $\mathfrak{M}$ of B, the induced map*

$$A_{\mathfrak{m}} \to B \otimes_A A_{\mathfrak{m}}$$

is surjective, where $\mathfrak{m} := A \cap \mathfrak{M}$. Then φ is surjective and $\operatorname{Spec} B$ *is a closed subset of* $\operatorname{Spec} A$.

(ii) *Suppose that for every maximal ideal $\mathfrak{M}$ of B, the induced map*

$$A_{\mathfrak{m}} \to B \otimes_A A_{\mathfrak{m}}$$

is bijective, where $\mathfrak{m} := A \cap \mathfrak{M}$. Then $\operatorname{Spec} B$ *is an open subset of* $\operatorname{Spec} A$.

Proof. — (i) For every $\mathfrak{m} \in \operatorname{Max} A$ the map

$$A_{\mathfrak{m}} \to B \otimes_A A_{\mathfrak{m}}$$

is surjective. This is true by assumption when $\mathfrak{m} = A \cap \mathfrak{M}$ for some $\mathfrak{M} \in \operatorname{Max} B$. It only remains to treat the other elements of $\operatorname{Max} A$. Let $\mathfrak{m} \in \operatorname{Max} A$ be such an ideal. By [**25**, Theorem 9.3], there is an $a \in \operatorname{Ker} \varphi$ such that $a \notin \mathfrak{m}$. Since a annihilates the A-module B and the image of a in $A_{\mathfrak{m}}$ is nonzero, it follows that $B \otimes_A A_{\mathfrak{m}} = (0)$. Thus the map $A_{\mathfrak{m}} \to B \otimes_A A_{\mathfrak{m}}$ is surjective.

Now let $b \in B$ and consider the ideal

$$I := \{a \in A : ab \in \varphi(A)\}.$$

We will show that I is the unit ideal. Suppose not. Then there is an $\mathfrak{m} \in \operatorname{Max} A$ such that $I \subset \mathfrak{m}$. But $A_{\mathfrak{m}} \to B \otimes_A A_{\mathfrak{m}}$ is surjective so $IA_{\mathfrak{m}}$ is the unit ideal, a contradiction. This proves that φ is surjective. By [**25**, Theorem 9.3], $\operatorname{Spec} B \cap \operatorname{Spec} A = V(\operatorname{Ker} \varphi)$. Hence $\operatorname{Spec} B$ is a closed subset of $\operatorname{Spec} A$.

(ii) Since we are only concerned with prime ideals, it is no loss of generality to assume that A and B are both reduced, i.e. have no nonzero nilpotent elements. It suffices to show that B is a direct summand of A.

By part (i), φ is surjective, so $B = A/I$ where $I := \operatorname{Ker} \varphi$. Since B is reduced, I is the intersection of some prime ideals of A. Let J be the intersection of the unit ideal with all the minimal prime ideals of A that do not contain I. We will show that

$$A = A/I \oplus A/J.$$

This is obvious if $J = (1)$. So assume $J \neq (1)$. By [**25**, Theorem 1.4], it suffices to show that $I + J$ is the unit ideal of A. Suppose not. Then there is an $\mathfrak{m} \in \operatorname{Max} A$ such that $\mathfrak{m} \supset I + J$; in particular $\mathfrak{m} \supset J$. Since J is an intersection of minimal prime ideals of A, at least one such prime must be contained in $\mathfrak{m}$. In other words, there is a minimal prime ideal $\mathfrak{p}$ of A contained in $\mathfrak{m}$ that does not contain I. We show that $\mathfrak{p}A_{\mathfrak{m}} \not\supset IA_{\mathfrak{m}}$. Let $a \in I \setminus \mathfrak{p}$; if $\mathfrak{p}A_{\mathfrak{m}} \supset IA_{\mathfrak{m}}$, then $a = \sum_{i=1}^{r} \frac{g_i}{s}$ for some $s \in A \setminus \mathfrak{m}$ and $g_i \in \mathfrak{p}$. Thus $sa \in \mathfrak{p}$ and $s, a, \notin \mathfrak{p}$, a contradiction. So, $\mathfrak{p}A_{\mathfrak{m}}$ is a minimal prime ideal of $A_{\mathfrak{m}}$ that does not contain $IA_{\mathfrak{m}}$. But by assumption $A_{\mathfrak{m}} = A_{\mathfrak{m}}/IA_{\mathfrak{m}}$; i.e. $IA_{\mathfrak{m}} = (0)$. In particular, since $A_{\mathfrak{m}}$ is reduced, $I \cdot A_{\mathfrak{m}}$ is the intersection of all the minimal prime ideals of $A_{\mathfrak{m}}$, a contradiction. Thus $I + J$ is the unit ideal of A. $\square$

Recall that in Subsection 5.3 we showed that every R-subdomain is a quasi-affinoid subdomain.

Theorem 6.2.2. — *Let A be a quasi-affinoid algebra and let $U \subset \operatorname{Max} A$ be a quasi-affinoid subdomain. Then U is a finite union of R-subdomains of $\operatorname{Max} A$.*

Proof. — Let $B := \mathcal{O}(U)$, and let $\pi : \operatorname{Max} B \to \operatorname{Max} A$ be the canonical inclusion. By Theorem 6.1.2 there is a finite cover of $\operatorname{Max} A$ by R-subdomains U_i such that each map

$$\pi|_{\pi^{-1}(U_i)} : \pi^{-1}(U_i) \to U_i$$

is finite. Thus, without loss of generality, we assume that $\pi : \operatorname{Max} B \to \operatorname{Max} A$ is finite.

We will apply Lemma 6.2.1 to show that U is a Zariski-open and -closed subset of $\operatorname{Max} A$. Let $\mathfrak{M} \in \operatorname{Max} B$, and put $\mathfrak{m} := A \cap \mathfrak{M}$. We wish to show that $A_{\mathfrak{m}} \to B \otimes_A A_{\mathfrak{m}}$ is bijective. Since $B \otimes_A A_{\mathfrak{m}}$ is a finite $A_{\mathfrak{m}}$-module, this follows from Nakayama's Lemma [**25**, Theorem 2.3], once we know that $B \otimes_A (A_{\mathfrak{m}}/\mathfrak{m}A_{\mathfrak{m}}) \cong A_{\mathfrak{m}}/\mathfrak{m}A_{\mathfrak{m}}$. Indeed,

$$B \otimes_A (A_{\mathfrak{m}}/\mathfrak{m}A_{\mathfrak{m}}) = B \otimes_A A/\mathfrak{m} = B/\mathfrak{m}B = B/\mathfrak{M} = A/\mathfrak{m} = A_{\mathfrak{m}}/\mathfrak{m}A_{\mathfrak{m}},$$

by Proposition 5.3.6 (ii) and (iii).

By Lemma 6.2.1, U is a Zariski-open and -closed subset of $\operatorname{Max} A$, thus there is some $f \in A$ such that $f|_U \equiv 0$ and $f|_{\operatorname{Max} A \setminus U} \equiv 1$. So $U = \{x \in \operatorname{Max} A : |f(x)| \leq \frac{1}{2}\}$ is an R-subdomain of $\operatorname{Max} A$. $\square$

Note that the covering of U given by Theorem 6.2.2 is not necessarily a quasi-affinoid covering in the sense of [**22**]; nonetheless Theorem 6.2.2 does show that quasi-affinoid subdomains are well-behaved. In particular the following openness theorem (cf. [**6**, Theorem 7.2.5.3]) is an immediate consequence.

Corollary 6.2.3. — **(Openness Theorem)** *Let A be a quasi-affinoid algebra. All quasi-affinoid subdomains of A are open in the canonical topology on* $\operatorname{Max} A$ *derived from the absolute value* $|\cdot| : K \to \mathbb{R}_+$.

Proof. — As we remarked in Subsection 5.3 all R-subdomains of $\operatorname{Max} A$ are open. $\square$

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