# Topological Wick Rotation and Holographic Dualities

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(1) a summary of joint works with Hao Zheng 1705.01087,1905.04924,1912.01760, 2011.02859 ,
and joint works with Tian Lan, Xiao-Gang Wen, Zhi-Hao Zhang, Hao Zheng:2003.08898, 2005.14178, 2108.08835.
(2) an expository work joint with Hao Zheng, Topological Wick Rotation and Holographic Dualities, in preparation.

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2. The AdS/CFT dictionary (for correlation functions) is only defined when the boundary condition of 1-dimensional higher theory (AdS-side) is specified:

$$\left.\frac{\delta^n S_{\mathrm{SUGRA}}^{\mathrm{on-shell}}}{\delta f_{(0)}(x_1)\cdots f_{(0)}(x_n)}\right|_{f_{(0)}=0} = (-1)^{n+1}\langle \mathcal{O}(x_1)\cdots\mathcal{O}(x_2)\rangle_{\mathrm{CFT}},$$

where the boundary condition $\lim_{z\to 0} z^{-d-\Delta}F(x,z) = f_0(z)$ of the bulk field $F(x,z)$ becomes the source term $f_{(0)}(x)$ on the boundary.

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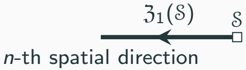
The holographic dualities we discuss today share the same two properties as the AdS/CFT dictionary.

Today we discuss a new type of holographic dualities based on the idea of Topological Wick Rotation K.-Zheng:1705.01087, 1905.04924, 1912.01760, 2011.02859: ($n$D is the spacetime dimension.)





| | |
|---|---|
| an $n$+1D topological order with a gapped boundary | an $n$D quantum liquid (SPT/SET/SSB/gapless) with an internal symmetry of finite type |
| $\mathcal{S}$ is the category of topological defects on the boundary | $\mathcal{S}$ is the category of topological defects { the superselection sectors of states } |
| $\mathfrak{Z}_1(\mathcal{S})$ is the category of topological defects in the bulk K.-Wen-Zheng:1502.01690,1702.00673 | $\mathfrak{Z}_1(\mathcal{S})$ is the category of topological sectors of operators { spaces of non-local operators invariant under LOA} |
| $\mathfrak{Z}_1(\mathcal{S})$ naturally acts on $\mathcal{S}$ | $\mathfrak{Z}_1(\mathcal{S})$-action on $\mathcal{S} \rightsquigarrow$ an enriched category ${}^{\mathfrak{Z}_1(\mathcal{S})}\mathcal{S}$ |



| | |
|---|---|
| a 3D topological order with a gapped boundary | a 2D rational CFT (up to VOA's) |
| 3D toric code model with two gapped boundaries | two gapped phases in 2D Ising chain |
| 3D finite gauge theories with gapped boundaries | 2D gapped quantum liquids (SPT/SSB orders) |
| $n$+1D gauge theories with gapped boundaries | $n$D gapped quantum liquids (SPT/SET/SSB) |
| gauge symmetries | global symmetries |
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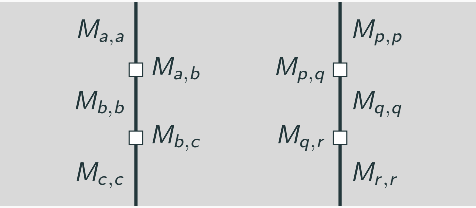
| further generalizations | |

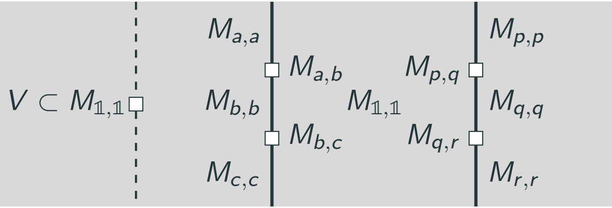
# 3D-2D duality through the theory of gapless edges of 2+1D topological orders

Consider a 2+1D topological order $(\mathcal{C}, c)$, where $\mathcal{C}$ is a modular tensor category (MTC) and $c$ is the chiral central charge. Consider a gapped or gapless edge of $(\mathcal{C}, c)$ and its 2D worldsheet.

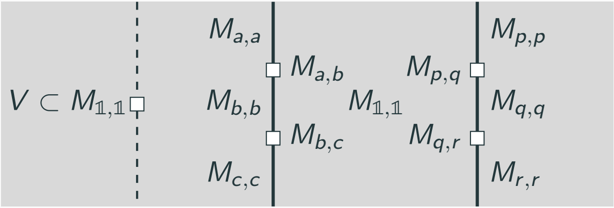



The macroscopic observables on the 1+1D worldsheet of the edge form an enriched fusion category ${}^{\mathcal{B}}\mathcal{S}$ K.-Zheng:1705.01087, 1905.04924, 2011.02859, K.-Wen-Zheng:2108.08835, where

1. $a, b, c \in \mathcal{S}$ are the labels of topological defect lines (TDL);
2. $M_{a,b}$ is the spaces of fields living on the 0D defect junction; in particular, it means that the space of fields living on the TDL label by '$a$' is just $M_{a,a}$;
3. OPE $M_{b,c} \otimes_{\mathbb{C}} M_{a,b} \xrightarrow{\circ} M_{a,c}$ of defect fields defines a kind of 'composition map' such that all $\{M_{a,b}\}_{a,b\in\mathcal{S}}$ together form a structure similar to that of a category ($\{\hom(a,b)\}_{a,b\in\mathcal{C}}$).
4. We will show next that $M_{a,b}, \circ \in \mathcal{B}$ and we obtain an enriched fusion category ${}^{\mathcal{B}}\mathcal{S}$.




Let $\mathbb{1} \in \mathcal{S}$ be the label of the trivial TDL. Then fields in $M_{\mathbb{1},\mathbb{1}}$ can live in the 2-cell.

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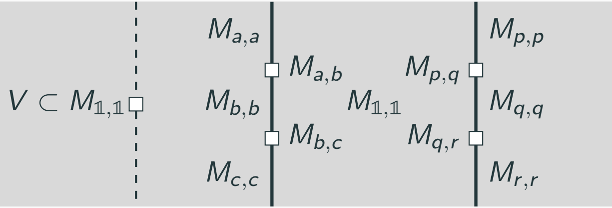


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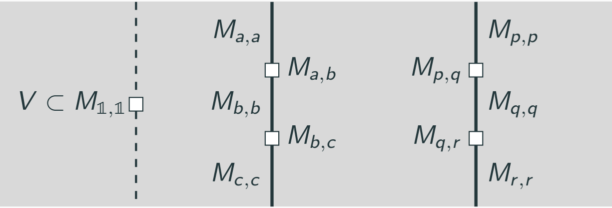
Moore-Seiberg:1989, Huang:math/0502533

1. $V$ is called "chiral symmetry" if the edge is chiral and is defined mathematically by a vertex operator algebra (VOA);
2. $V$ is called "non-chiral symmetry" if the edge is non-chiral and is defined mathematically by a so-called full field algebra (a non-chiral analogue of VOA). Huang-K.:math/0511328




Since $V$ is transparent on the entire 1+1D worldsheet, $M_{a,b}$ is clearly a $V$-module, i.e. $M_{a,b} \in \mathrm{Mod}_V$, where $\mathrm{Mod}_V$ denotes the category of $V$-modules (with a 2-dimensional $V$-action). $M_{b,c} \otimes_{\mathbb{C}} M_{a,b} \xrightarrow{\circ} M_{a,c}$ is a morphism in $\mathrm{Mod}_V$. As a consequence, we obtain an $\mathrm{Mod}_V$-enriched fusion category, where $\mathrm{hom}(a,b) = M_{a,b}$ and $\otimes$ is the horizontal fusion of TDLs.

It is convenient to introduce an abstract MTC $\mathcal{B}$ and a braided equivalence $\phi : \mathrm{Mod}_V \xrightarrow{\simeq} \mathcal{B}$. Then $(\phi, {}^{\mathcal{B}}\mathcal{S})$ can encode different enrichments ${}^{\mathrm{Mod}_V}_{\phi}\mathcal{S}$.

It turns out that all correlation functions and the OPE among defect fields can be recovered from the triple $(V, \phi, {}^{\mathcal{B}}\mathcal{S})$. Huang:arXiv:math/0502533, Fuchs-Runkel-Schweigert:2002-2006, Davydov-K.-Runkel:1307.5956, K.-Zheng:1705.01087

**Theorem (K.-Zheng:1705.01087, 1905.04924, 1912.01760)**

*The gapped/gapless boundaries of a 2+1D topological order $(\mathcal{C}, c)$ can be completely characterized or classified by the triples $(V, \phi, {}^{\mathcal{B}}\mathcal{S})$, where*

1. *$V$ is the chiral/non-chiral symmetry:*
   - *for a chiral gapless boundary, $V$ is a rational VOA* Huang:2008 *of central charge $c$;*
   - *for a non-chiral gapless boundary, $V$ is a rational full field algebra $(c_L - c_R = c)$* K.-Huang:2006*;*
   - *When $V = \mathbb{C}$, it describes a gapped boundary.*
2. *$\phi : \mathrm{Mod}_V \xrightarrow{\simeq} \mathcal{B}$ is a braided equivalence.*
3. *$\mathcal{S}$ is a fusion category equipped with a braided equivalence $\mathcal{C} \boxtimes \overline{\mathcal{B}} \simeq \mathfrak{Z}_1(\mathcal{S})$, which determines ${}^{\mathcal{B}}\mathcal{S}$ via the so-called canonical construction.* Morrison-Penneys:1701.00567

**Theorem (K.-Zheng:1704.01447, K.-Yuan-Zhang-Zheng:2104.03121)**

*The bulk is the center of a boundary, i.e. $\mathcal{C} \simeq \mathfrak{Z}_1({}^{\mathcal{B}}\mathcal{S})$.*

## Theorem (K.-Zheng:1705.01087, 1905.04924, 1912.01760)

*All gapped/gapless edges of 2+1D topological order* $(\mathcal{C}, c)$ *can be obtained by performing the following* *topological Wick rotation* *trick then adding the data of chiral/non-chiral symmetry by hands* $(V, \mathrm{Mod}_V \xrightarrow{\phi} \mathcal{B}, {}^{\mathcal{B}}\mathcal{S})$.


*When* $(\mathcal{C}, c)$ *is trivial, i.e.* $(\mathcal{C}, c) = (\mathrm{Vec}, 0)$, *we obtain a holographic duality.*

**Examples of non-chiral gapless edges**:

- Three modular tensor categories (MTC):
  1. $\mathsf{Is} := \mathrm{Mod}_{V_{\mathsf{Is}}}$ where $V_{\mathsf{Is}}$ is the Ising VOA with the central charge $c = \frac{1}{2}$. It has three simple objects $\mathbb{1}, \psi, \sigma$ (i.e. $\mathbb{1} = V_{\mathsf{Is}}$) and the following fusion rules:

$$\psi \otimes \psi = \mathbb{1}, \quad \psi \otimes \sigma = \sigma, \quad \sigma \otimes \sigma = \mathbb{1} \oplus \psi.$$

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  2. $\mathfrak{Z}_1(\mathsf{Is}) \simeq \mathsf{Is} \boxtimes \overline{\mathsf{Is}}$. It has 9 simple objects: $\mathbb{1} \boxtimes \mathbb{1}, \mathbb{1} \boxtimes \psi, \mathbb{1} \boxtimes \sigma, \psi \boxtimes \mathbb{1}, \cdots$.

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  3. $\mathsf{TC}$ = the MTC of toric code. It has four simple objects $1, e, m, f$ and the following fusion rules:

$$e \otimes e = m \otimes m = f \otimes f = 1, \quad m \otimes e = f.$$

  4. $\mathrm{Vec}$ = the category of finite dimensional vector spaces, i.e. the trivial MTC.

- Three non-chiral symmetries (i.e. full field algebras Huang-K.:math/0511328):
  1. $P = V_{\mathrm{Is}} \otimes_{\mathbb{C}} \overline{V_{\mathrm{Is}}} = \mathbb{1} \boxtimes \mathbb{1} \in \mathrm{Is} \boxtimes \overline{\mathrm{Is}} = \mathfrak{Z}_1(\mathrm{Is})$
  2. $Q = \mathbb{1} \boxtimes \mathbb{1} \oplus \psi \boxtimes \psi \in \mathfrak{Z}_1(\mathrm{Is})$
  3. $R = \mathbb{1} \boxtimes \mathbb{1} \oplus \psi \boxtimes \psi \oplus \sigma \boxtimes \sigma \in \mathfrak{Z}_1(\mathrm{Is})$

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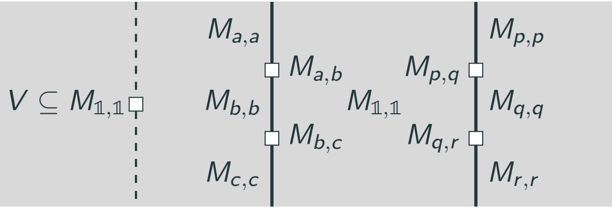
- $P, Q, R$ are condensable algebras in $\mathfrak{Z}_1(\mathsf{Is})$ and $R$ is a Lagrangian algebra.
    1. $\phi_P : \mathrm{Mod}_P \xrightarrow{\simeq} \mathfrak{Z}_1(\mathsf{Is})$: condensing $P$ gives $\mathfrak{Z}_1(\mathsf{Is})$;
    2. $\phi_Q : \mathrm{Mod}_Q \xrightarrow{\simeq} \mathrm{TC}$: condensing $Q$ in $\mathfrak{Z}_1(\mathsf{Is})$ gives toric code TC Bais-Slingerland:0808.0627, Chen-Jian-K.-You-Zheng:1903.12334;
    3. $\phi_R : \mathrm{Mod}_R \xrightarrow{\simeq} \mathrm{Vec}$: condensing $R$ in $\mathfrak{Z}_1(\mathsf{Is})$ gives the trivial phase.

$P = V_{\mathrm{Is}} \otimes_{\mathbb{C}} \overline{V_{\mathrm{Is}}} = \mathbb{1} \boxtimes \mathbb{1}$
$Q = \mathbb{1} \boxtimes \mathbb{1} \oplus \psi \boxtimes \psi$
$R = \mathbb{1} \boxtimes \mathbb{1} \oplus \psi \boxtimes \psi \oplus \sigma \boxtimes \sigma$
$\phi_P : \mathrm{Mod}_P \xrightarrow{\simeq} \mathfrak{Z}_1(\mathrm{Is})$
$\phi_Q : \mathrm{Mod}_Q \xrightarrow{\simeq} \mathrm{TC}$
$\phi_R : \mathrm{Mod}_R \xrightarrow{\simeq} \mathrm{Vec}$




- Four anomaly-free 1+1D gapless quantum liquids defined by triples: i.e. its 2+1D bulk topological order is trivial: $(\mathcal{C}, c) = (\mathrm{Vec}, 0)$ and $\mathfrak{Z}_1({}^{\mathcal{B}}\mathcal{S}) \simeq \mathrm{Vec}$.
  1. $(P, \phi_P, {}^{\mathfrak{Z}_1(\mathrm{Is})}\mathrm{Is})$: in this case $V = P \subsetneq R = M_{\mathbb{1},\mathbb{1}}$;
  2. $(Q, \phi_Q, {}^{\mathrm{TC}}\mathrm{Rep}(\mathbb{Z}_2))$: in this case $V = Q \subsetneq R = M_{\mathbb{1},\mathbb{1}}$;
  3. $(Q, \phi_Q, {}^{\mathrm{TC}}\mathrm{Vec}_{\mathbb{Z}_2})$: in this case $V = Q \subsetneq R = M_{\mathbb{1},\mathbb{1}}$;
  4. $(R, \phi_R, {}^{\mathrm{Vec}}\mathrm{Vec})$: in this case $V = R = M_{\mathbb{1},\mathbb{1}}$.

  In all 4 cases, the space of non-chiral fields living on each 2-cells (i.e. $M_{\mathbb{1},\mathbb{1}}$) is given by the same modular-invariant closed CFT $R$.

- Gappable gapless edges of 2+1D toric code: $\mathrm{TC} = \mathfrak{Z}_1({}^{\mathcal{B}}\mathcal{S})$,

  Chen-Jian-K.-You-Zheng:arXiv:1903.12334, K.-Zheng:1912.01760

  1. $(P, \phi_P, {}^{\mathfrak{Z}_1(\mathrm{Is})}\mathcal{S})$, where $\mathcal{S} = (\mathfrak{Z}_1(\mathrm{Is}))_Q$ is the fusion category of the right $Q$-modules in $\mathfrak{Z}_1(\mathrm{Is})$. $\mathcal{S}$ has 6 simple objects $\mathbb{1}, e, m, f, \chi_\pm$, where $\mathbb{1}, e, m, f$ can be identified with 4 anyons in the bulk and $\chi_\pm$ can be identified with two twist defects in the bulk.
  2. $(Q, \phi_Q, {}^{\mathrm{TC}}\mathrm{TC})$ = the canoncial gapless edge;
  3. $(R, \phi_R, {}^{\mathrm{Vec}}\mathrm{Rep}(\mathbb{Z}_2) = \mathrm{Rep}(\mathbb{Z}_2))$, where $\phi : \mathrm{Mod}_R \xrightarrow{\simeq} \mathrm{Vec}$. Moreover, we have

$$(R, \phi_R, \mathrm{Rep}(\mathbb{Z}_2)) = \underbrace{(\mathbb{C}, \mathrm{id}, \mathrm{Rep}(\mathbb{Z}_2))}_{\text{the smooth gapped edge}} \boxtimes \underbrace{(R, \phi_R, {}^{\mathrm{Vec}}\mathrm{Vec})}_{\text{an anomaly-free 2D gapless liquid}}.$$

  4. $(R, \phi_R, {}^{\mathrm{Vec}}\mathrm{Vec}_{\mathbb{Z}_2})$:

$$(R, \phi_R, \mathrm{Vec}_{\mathbb{Z}_2}) = \underbrace{(\mathbb{C}, \mathrm{id}, \mathrm{Vec}_{\mathbb{Z}_2})}_{\text{the rough gapped edge}} \boxtimes \underbrace{(R, \phi_R, {}^{\mathrm{Vec}}\mathrm{Vec})}_{\text{an anomaly-free 2D gapless liquid}}$$

  5. (smooth/rough gapped edge) $\boxtimes$ (any anomaly-free 2D gapless liquid).

We can see that the data in the triple play somewhat different roles.

- Since ${}^{\mathcal{B}}\mathcal{S}$ encodes all the topological or categorical data, we called ${}^{\mathcal{B}}\mathcal{S}$ the topological skeleton of the edge.
- We call $(V, \phi)$ the local quantum symmetry of the edge (e.g. VOA or full field algebra or the conformal net of local operator algebras (LOA)).

Note that the topological skeleton ${}^{\mathcal{B}}\mathcal{S}$ is an abstract enriched fusion category. It acquires a precise physical meaning only after we identified $\mathcal{B}$ with $\mathrm{Mod}_V$ via the braided equivalence $\phi : \mathrm{Mod}_V \xrightarrow{\simeq} \mathcal{B}$.

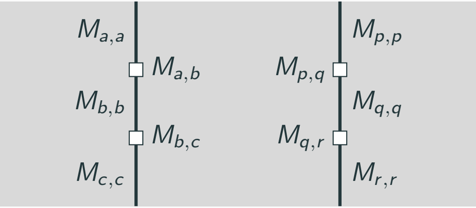
This suggests that a gapped/gapless quantum liquid $\mathcal{X}$ can be described by two types of data, i.e. $\mathcal{X} = (\mathcal{X}_{\mathrm{lqs}}, \mathcal{X}_{\mathrm{sk}})$ (e.g. $\mathcal{X}_{\mathrm{lqs}} = (V, \phi)$ and $\mathcal{X}_{\mathrm{sk}} = {}^{\mathcal{B}}\mathcal{S}$). One justification of this statement is the fact that the same topological skeleton $\mathcal{X}_{\mathrm{sk}} = {}^{\mathcal{B}}\mathcal{S}$ can be realized by gapped phases. Thorngren-Wang:1912.02817, K.-Lan-Wen-Zhang-Zheng:2005.14178, K.-Wen-Zheng:2108.08835, Inamura:2110.12882, Lootens-Delcamp-Ortiz-Verstraete:2112.09091, Xu-Zhang:2205.09656,

- 3D Chern-Simons-Witten Theories $\leftrightarrow$ 2D WZW CFT's; Witten:1989
- $\mathrm{Mod}_V$ = MTC = 3D Reshetikhin-Turaev TQFT; Moore-Seiberg:1989, Reshetikhin-Turaev:1991
- Construct correlators in 2D CFT's from 3D TQFTs; Fuchs-Runkel-Schweigert:2002-2006
- Lagrangian algebras in 3D anyon condensation $\leftrightarrow$ modular-invariant partition functions in 2D CFT K.-Runkel:0807.3356, Müger:0909.2537, Davydov-Müger-Nikshych-Ostrik:1009.2117, K.:1307.8244
- Dualities explicitly constructed from classical statistical models; Aasen-Mong-Fendley:1601.07185, Aasen-Fendley-Mong:2008.08598.
- Strange correlators mapping 3D TQFTs to 2D RCFTs: Bal-Williamson-Vanhove-Bultinck-Haegeman-Verstraete:1801.05959,
- String-net construction of 2D correlators: Schweigert-Yang:1911.10147, Traube:2009.11895, Fuchs-Schweigert-Yang:2112.12708

The story told in this talk is unique in that topological Wick rotation and enriched categories are used in no other places.

# 3D-2D holographic dualities when 2D phases are gapped

Ising chain: $\mathcal{H}_{tot} = \otimes_{i\in\mathbb{Z}}\mathbb{C}_i^2$, $H = -\sum_i BX_i - \sum_i JZ_iZ_{i+1}$, where $B$ and $J$ are the coupling constants, $X_i$ and $Z_i$ are Pauli matrices. $U = \otimes_i X_i$ is a global onsite $\mathbb{Z}_2$-symmetry.




The macroscopic observables on the 2D spacetime form an enriched fusion category ${}^{\mathcal{B}}\mathcal{S}$

- Symmetric phase ($J = 0$): two superselection sectors of states (or TDL's) form a fusion category $\mathrm{Rep}(\mathbb{Z}_2)$ (i.e the category of finite dim. $\mathbb{Z}_2$-representations). $X|\pm\rangle = \pm|\pm\rangle$,

$$\{\mathbb{1} \ni |\cdots +++\cdots\rangle, \quad e \ni |\cdots ++-++\cdots\rangle\} = \mathrm{Rep}(\mathbb{Z}_2).$$

In this gapped case, what replace the chiral/non-chiral symmetry $V$ is the topological net of $U$-symmetric local operators, and $\mathrm{Mod}_V$ becomes the category of the sectors of this net, and $\mathrm{Mod}_V = \mathcal{B} = \mathfrak{Z}_1(\mathrm{Rep}(\mathbb{Z}_2)) = \mathsf{TC}$. K.-Zheng:2201.05726 Roughly speaking, they are the sectors of $U$-symmetric non-local operators (i.e. the spaces of unconfined non-local operators invariant under the action of symmetric local operators). Ji-Wen:1912.13492, K.-Wen-Zheng:2108.08835

1. $\mathbb{1}$ consists of $U$-symmetric local operators such as $1, Z_i Z_{i+1}$;
2. $m$ consists of $m_i = \otimes_{k\leq i} X_k$ and $Um_i$;
3. $e$ consists of $Z_i \approx Z_{i,\infty} = \otimes_{k\geq i}(Z_k Z_{k+1})$;
4. $f$ consists of $m_i Z_j$.

Then $M_{a,b}$ is precisely the space of operators that map the sector $a$ of states (or TDL $a$) to the sector $b$ of states (or TDL $b$).

$$M_{\mathbb{1},\mathbb{1}} = \mathbb{1} \oplus m \quad \Leftarrow \quad m_j|\cdots+++\cdots\rangle = |\cdots+++\cdots\rangle.$$
$$M_{\mathbb{1},e} = M_{e,\mathbb{1}} = e \oplus f, \qquad M_{e,e} = \mathbb{1} \oplus m.$$

$\{M_{a,b}\}_{a,b=\mathbb{1},e}$ form an enriched fusion category ${}^{\mathcal{B}}\mathcal{S} = {}^{\mathsf{TC}}\mathrm{Rep}(\mathbb{Z}_2)$, i.e. $\hom_{{}^{\mathcal{B}}\mathcal{S}}(a,b) = M_{a,b}$.

**Theorem** (K.-Wen-Zheng:2108.08835)

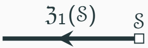
1. *the symmetric phase of Ising chain* ($J=0$): ${}^{\mathcal{B}}\mathcal{S} = {}^{\mathsf{TC}}\mathrm{Rep}(\mathbb{Z}_2)$;
2. *the symmetry-breaking phase of Ising chain* ($B=0$): ${}^{\mathcal{B}}\mathcal{S} = {}^{\mathsf{TC}}\mathrm{Vec}_{\mathbb{Z}_2}$.

*In both case,* $\mathcal{S}$ *is the set (or better the category) of the labels of TDL's.*

1. $\mathsf{TC} \simeq \mathfrak{Z}_1(\mathrm{Rep}(\mathbb{Z}_2)) \simeq \mathfrak{Z}_1(\mathrm{Vec}_{\mathbb{Z}_2})$ is the category of particles in the bulk of the 2+1D toric code model.
2. $\mathrm{Rep}(\mathbb{Z}_2)$ the category of particles on the smooth boundary of the toric code.
3. $\mathrm{Vec}_{\mathbb{Z}_2}$ is the category of particles on the rough boundary. of the toric code.

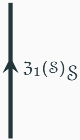
$$\boxed{\text{2+1D toric code model + gapped boundaries}} \xrightarrow{\text{TWR}} \boxed{\text{1+1D Ising model}}.$$

$$(\mathsf{TC}, \mathrm{Rep}(\mathbb{Z}_2)), \quad (\mathsf{TC}, \mathrm{Vec}_{\mathbb{Z}_2}) \quad \mapsto \quad {}^{\mathsf{TC}}\mathrm{Rep}(\mathbb{Z}_2), \quad {}^{\mathsf{TC}}\mathrm{Vec}_{\mathbb{Z}_2}.$$




*n*-th spatial direction

the time direction




a 2+1D finite gauge theories + a gapped boundary

$G$=gauge symmetries

$\mathfrak{Z}_1(\mathcal{S}) = \mathfrak{Z}_1(\mathrm{Rep}(G))$

$\mathcal{S} = \mathfrak{Z}_1(\mathrm{Rep}(G))_{A_{(H,\omega)}}$

$A_{(H,\omega)}$ = Lagrangian algebra associated to the bdy

$(G, H, \omega)$

a 1+1D gapped phase (SPT/SSB orders)

$G$=global symmetries

$^{\mathfrak{Z}_1(\mathrm{Rep}(G))}(\mathfrak{Z}_1(\mathrm{Rep}(G))_{A_{(H,\omega)}})$

classification of 2D SPT/SET/SSB orders

Chen-Gu-Wen:1008.3745, Schuch-Pérez-García-Cirac:1010.3732

Freed-Teleman:1806.00008

K.-Zheng:2011.02859, K.-Wen-Zheng:2108.08835

We have seen that ${}^{\mathfrak{Z}_1(\mathcal{S})}\mathcal{S}$ is realized as the "topological skeleton" gapless 1+1D quantum liquids (rational CFT's).

**Question**: Can the same ${}^{\mathfrak{Z}_1(\mathcal{S})}\mathcal{S}$ (or $\mathcal{S}$) defines the "topological skeleton" of a gapped 1+1D phase?

We have seen that ${}^{\mathfrak{Z}_1(\mathcal{S})}\mathcal{S}$ is realized as the "topological skeleton" gapless 1+1D quantum liquids (rational CFT's).

**Question**: Can the same ${}^{\mathfrak{Z}_1(\mathcal{S})}\mathcal{S}$ (or $\mathcal{S}$) defines the "topological skeleton" of a gapped 1+1D phase?

The answer is Yes! $\mathcal{S}$ (or ${}^{\mathfrak{Z}_1(\mathcal{S})}\mathcal{S}$) can be realized as the "topological skeleton" of a 1+1D gapped quantum liquid (equipped with the algebraic higher symmetry $\mathcal{S}$).

Thorngren-Wang:1912.02817, K.-Lan-Wen-Zhang-Zheng:2005.14178, Inamura:2110.12882, Lootens-Delcamp-Ortiz-Verstraete:2112.09091, These dualities in this section were explicitly/implicitly discovered by different groups of people in different contexts. An incomplete list:

- **Generalized Kramers-Wannier dualities**: Freed-Teleman:1806.00008, Lootens-Delcamp-Ortiz-Verstraete:2112.09091.
- **Categorical and fusion Symmetries**: Thorngren-Wang:1912.02817, Ji-Wen:1912.13492, K.-Lan-Wen-Zhang-Zheng:2003.08898,2005.14178, Huang-Lin-Seifnashri:2110.02958, Albert-Aasen-Xu-Ji-Alicea-Preskill:2111.12096, Chatterjee-Wen:2203.03596,2205.06244, Liu-Ji:2208.09101,
- **Topological Wick Rotation**: K.-Zheng:1705.01087, 1905.04924, 1912.01760, K.-Zheng:2011.02859, K-Wen-Zheng:2108.08835, Xu-Zhang:2205.09656, Lu-Yang:2208.01572, K.-Zhang-Zheng:in preparation
- **Classical Statistical Models**: Aasen-Mong-Fendley:1601.07185, Aasen-Fendley-Mong:2008.08598.
- Gaiotto-Kulp:2008.05960, Bhardwaj-Lee-Tachikawa:2009.10099, Inamura:2110.12882, Apruzzi-Bonetti-Etxebarria-Hosseini-Schafer-Nameki:2112.02092, Apruzzi:2203.10063, Moradi-Moosavian-Tiwari:2207.10712, ...

# $n+1$D-$n$D holographic dualities

## Theorem

*There is a holographic duality* K.-Lan-Wen-Zhang-Zheng:2003.08898,2005.14178:

*(an n+1D top. orders + a gapped bdy)* $\mapsto$ *(an nD gapped quantum liquid)*

*which is 'more completely' defined by topological Wick rotation:* K.-Zheng:2011.02859:


*n+1D top. order with a gapped bdy*

$(\mathfrak{Z}_1(\mathcal{S}), \mathcal{S})$

*nD gapped quantum liquids*

${}^{\mathfrak{Z}_1(\mathcal{S})}\mathcal{S}$

1. Consider a 3D SPT order with an onsite symmetry defined by a finite group $G$. In this case, the category of $G$-charges is $\mathrm{Rep}(G)$. It seems that there is no further particle-like "macroscopic observables" in such a SPT order. If a 3D SPT is just described by $\mathrm{Rep}(G)$, it contradicts to the well-known $H^3(G, U(1))$ classification of 3D SPT orders.

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2. Idea of gauging: there are two ways to gauge the symmetry:
    2.1 by invertible domain walls or twist defects $\rightsquigarrow$ $G$-crossed braided fusion categories Barkeshli-Bonderson-Cheng-Wang:1410.4540;
    2.2 by additional topological anyons (gauge fluxes) $\rightsquigarrow$ a modular tensor category minimally extending $\mathrm{Rep}(G)$ (i.e. a minimal modular extension of $\mathrm{Rep}(G)$) Levin-Gu:1202.3120, Lan-K.-Wen:1507.04673,1602.05936,1602.05946

   The gauging idea is successful in that it recovers the $H^3(G, U(1))$ classification of 2+1D SPT orders.

   However, it does not give an intrinsic description of the SPT orders because the SPT order should have a well-defined mathematical description before we gauge it.

3 Idea of boundary-bulk relation: K.-Lan-Wen-Zhang-Zheng:2003.08898,2005.14178 Since $G$-charges are symmetric and can not detect each other via double braidings, $\mathrm{Rep}(G)$ is in some sense 'anomalous'. According to boundary-bulk relation, this 'anomaly' can be cancalled by "its 3+1D bulk". However, it does not solve the problem because "its 3+1D bulk" is uniquely fixed. Where does extra data come from in order to recover the $H^3(G, U(1))$-classification?

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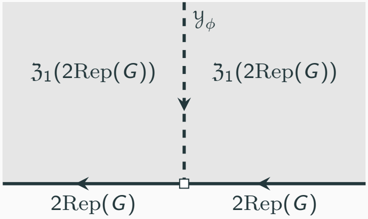
4 It turns out that there are multiple ways to identify the 3+1D bulk with itself via braided autoequivalences $\phi$ (or invertible domain walls $\mathcal{Y}_\phi$) preserving the symmetry charges $2\mathrm{Rep}(G)$ (including descendant $G$-charges Douglas-Reutter:1812.11933) in $\mathfrak{Z}_1(2\mathrm{Rep}(G))$.




$$\{\phi\} \simeq H^3(G, U(1))$$

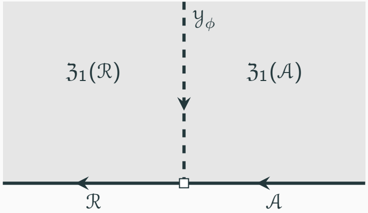
K.-Lan-Wen-Zhang-Zheng:2003.08898, K.-Zheng:2107.03858

5 Above arguments work SPT/SET orders in all dimensions.

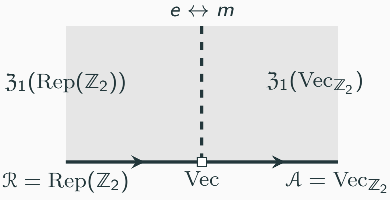
**Theorem (K.-Lan-Wen-Zhang-Zheng:2003.08898, K.-Zheng:2011.02859)**

*An $n+1$D SPT/SET/SSB order with the higher symmetry $\mathcal{R} = n\mathrm{Rep}(G), n\mathrm{Rep}(G,z)$ can be completely characterized by:*

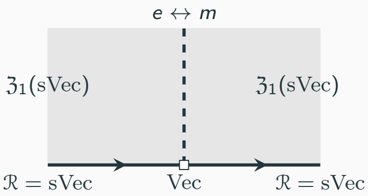
- *a fusion n-category $\mathcal{A}$ equipped with a braided embedding $\mathcal{R} \hookrightarrow \mathfrak{Z}_1(\mathcal{A})$*
- *a braided equivalence $\phi : \mathfrak{Z}_1(\mathcal{R}) \xrightarrow{\simeq} \mathfrak{Z}_1(\mathcal{A})$*



For example, for 1+1D gapped quantum liquids with an internal $\mathbb{Z}_2$-symmetry, i.e. $\mathcal{R} = \mathrm{Rep}(\mathbb{Z}_2)$ or $\mathcal{R} = \mathrm{Rep}(\mathbb{Z}, z) = \mathrm{sVec}$.




bosonic $\mathbb{Z}_2$-symmetry



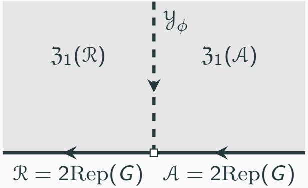

fermionic $\mathbb{Z}_2$-symmetry

Note that $\mathrm{Rep}(\mathbb{Z}_2) = \mathrm{Vec}_{\mathbb{Z}_2} = \mathrm{sVec}$ as fusion categories. However, $\mathrm{Rep}(\mathbb{Z}_2) = \{\mathbb{1}, e\}$, $\mathrm{Vec}_{\mathbb{Z}_2} = \{\mathbb{1}, m\}$, $\mathrm{sVec} = \{\mathbb{1}, f\}$ as subcategories of $\mathsf{TC} = \{\mathbb{1}, e, m, f\}$ are different.

1. In the bosonic case, the $(e \leftrightarrow m)$-duality breaks the $\mathbb{Z}_2$-symmetry. Hence, one phase in Ising chain is the trivial SPT order and the other is a symmetry-breaking order.
2. In the fermionic case, the $(e \leftrightarrow m)$-duality preserves the fermion parity (i.e. $f \mapsto f$). Hence, one phase in Kitaev chain is the trivial fermionic SPT order and the other is a non-trivial fermionic SPT order.

6 Although such a characterization of SPT orders are successful in recovering the usual classification of SPT orders, it is still not an intrinsic description of an $n$D gapped quantum liquid because, in a lattice model realization, the $n$+1D bulk is completely empty.

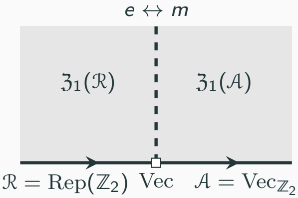
7 the last idea needed to complete the story is topological Wick rotation. This leads to the enriched-higher-category description of the topological skeleton of an $n$D gapped quantum liquid.




$\xrightarrow{\text{Top. Wick Rotation}}$

enriched fusion 2-categories

$${}^{\mathfrak{Z}_1(2\mathrm{Rep}(G))}_{\phi}2\mathrm{Rep}(G)$$



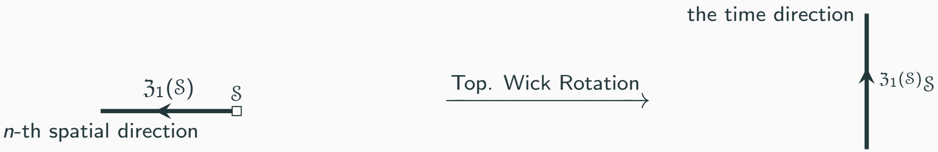

$\xrightarrow{\text{Top. Wick Rotation}}$

enriched fusion 1-categories

$${}^{\mathfrak{Z}_1(\mathrm{Rep}(G))}\mathrm{Rep}(G),$$

$${}^{\mathfrak{Z}_1(\mathrm{Rep}(G))}_{e\leftrightarrow m}\mathrm{Rep}(G) = {}^{\mathfrak{Z}_1(\mathrm{Rep}(G))}\mathrm{Vec}_{\mathbb{Z}_2}$$

6 Generalizing $\mathcal{R}$ to any fusion $(n-1)$-category , we obtain a characterization of an $n$D gapped quantum liquid by (an $n+1$D topological order + a gapped boundary).




$n+1$D finite gauge theory with a gapped bdy

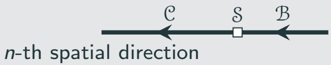
$n$D gapped quantum liquids

with a finite onsite symmetry $G$

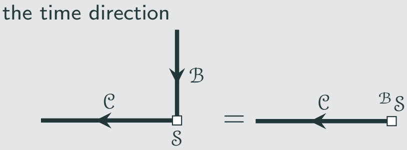
For $\mathcal{S} = n\mathrm{Rep}(G)$, explicit demonstrations via lattice models were known in some cases Freed-Teleman:1806.00008, Ji-Wen:1912.13492, Chatterjee-Wen:2203.03596, Liu-Ji:2208.09101, ... and more is coming for both gapped/gapless quantum liquids K.-Zhang-Zheng:in preparation, Chen-Zhang-Ji-Wang-Shen-Zeng-Hung:in preparation

## Generalized holographic duality with anomalies: K.-Zheng:1905.04924

For an $n$+1D topological order $\mathcal{C}$,




$\mathcal{S}$ is an "anomalous gapped boundaries" of $\mathcal{C}$




$n$D potentially gapless boundaries of $\mathcal{C}$

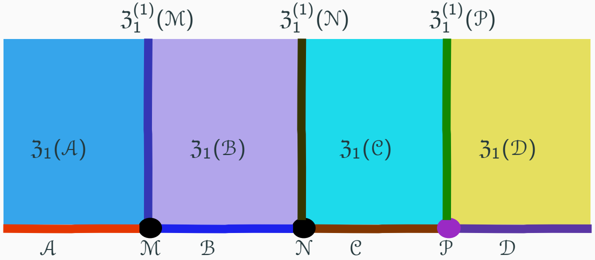
Moreover, the boundary-bulk relation holds, i.e. $\mathcal{C} \simeq \mathfrak{Z}_1({}^{\mathcal{B}}\mathcal{S})$. K.-Zheng:in preparation

# Advantages and consequences of topological Wick rotation

Why we emphasize the “rotation” instead of just a holographic dictionary? It is because ‘rotation’ suggests that all geometric intuitions are preserved like a real rotation.

1. The holographic dictionary automatically extends to higher codimensional domain walls on both sides.
2. The holographic map is compatible with the fusions of domain walls

In other words, this holographic duality is actually functorial.



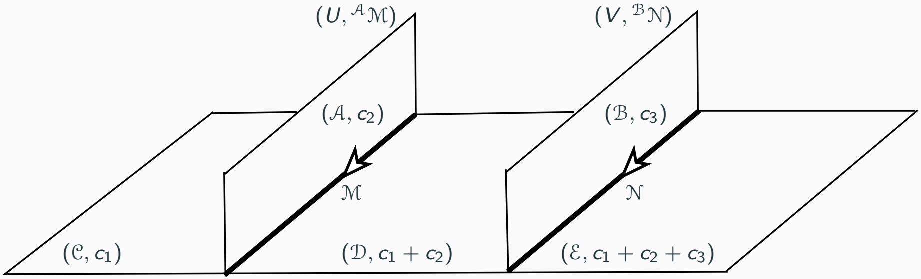

More precisely, topological Wick rotation gives an equivalence between the category of $n$+1D non-chiral topological orders (+ a gapped boundary) and the category of $n$D quantum liquids defined by

$$(\mathfrak{Z}_1(\mathcal{A}), \mathcal{A}) \mapsto {}^{\mathfrak{Z}_1(\mathcal{A})}\mathcal{A}, \qquad (\mathfrak{Z}_1^{(1)}(\mathcal{M}), \mathcal{M}) \mapsto {}^{\mathfrak{Z}_1^{(1)}(\mathcal{M})}\mathcal{M}$$

$${}^{\mathfrak{Z}_1^{(1)}(\mathcal{M}) \boxtimes_{\mathfrak{Z}_1(\mathcal{B})} \mathfrak{Z}_1^{(1)}(\mathcal{N})}(\mathcal{M} \boxtimes_{\mathcal{B}} \mathcal{N}) = ({}^{\mathfrak{Z}_1^{(1)}(\mathcal{M})}\mathcal{M}) \boxtimes_{{}^{\mathfrak{Z}_1(\mathcal{B})}\mathcal{B}} ({}^{\mathfrak{Z}_1^{(1)}(\mathcal{N})}\mathcal{N})$$

Our holographic dictionary automatically includes a dictionary map of dualities because many dualities can be realized by invertible domain walls. (e.g. e-m duality $\mapsto$ KW-duality Freed-Teleman:1806.00008, Ji-Wen:1912.13492)

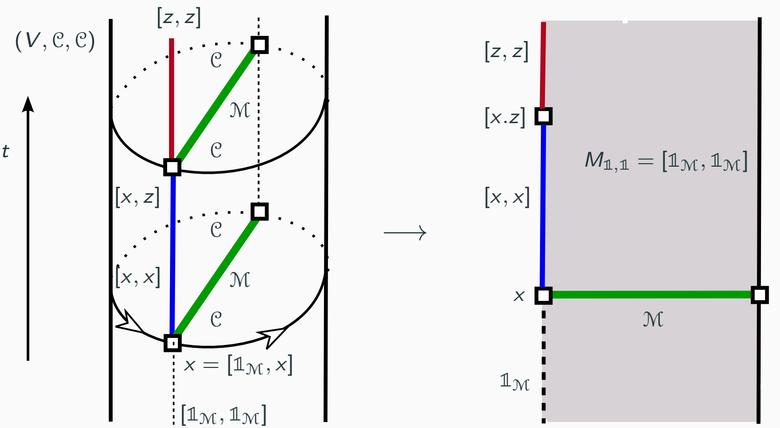
TWR automatically gives a description/classification of gapless domain walls between two bulk phases and a way to compute the fusion of two gapless domain walls.




$$(U, {}^{\mathcal{A}}\mathcal{M}) \boxtimes_{(\mathcal{D}, c_1+c_2)} (V, {}^{\mathcal{B}}\mathcal{N}) = (U \otimes_{\mathbb{C}} V, {}^{\mathcal{A}\boxtimes\mathcal{B}}\mathcal{M} \boxtimes_{\mathcal{D}} \mathcal{N}),$$

where $U, V$ are local quantum symmetries (e.g. topological/conformal nets, VOAs or full field algebras)

As an example, the fusion of a gapless edge of 2+1D topological orders with an gapless edge reproduces all modular invariant 1+1D CFT's as $M_{\mathbb{1},\mathbb{1}}$ after the fusion.

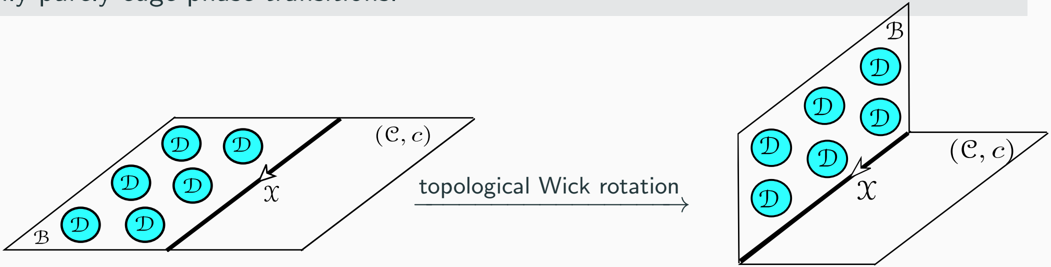



$$(V, {}^{\mathcal{C}}\mathcal{C}) \boxtimes_{(\mathcal{C},c)} (\mathbb{C}, {}^{\mathrm{Vec}}\mathcal{M}) \boxtimes_{(\mathcal{C},c)} (\overline{V}, {}^{\overline{\mathcal{C}}}\mathcal{C}^{\mathrm{rev}}) = (V \otimes_\mathbb{C} \overline{V}, {}^{\mathcal{C}\boxtimes\overline{\mathcal{C}}}\mathcal{M}).$$

(Skip unless people ask) Topological Wick Rotation also suggests that it maps phase diagrams to phase diagrams. Chen-Jian-K.-You-Zheng:arXiv:1903.12334, K.-Zheng:1912.01760, Chatterjee-Wen:2205.06244, Moradi-Moosavian-Tiwari:2207.10712,

**Conjecture**

The phase diagram of all $n$D gapped liquids with a fixed and unbroken internal symmetry (e.g. an onsite symmetry, categorical symmetry) can be identified with the phase diagram of all gapped boundaries of a fixed bulk $n$+1D topological order with only purely edge phase transitions.



# Conclusions and outlooks

- We have seen a holographic duality based on topological Wick rotation. This picture is perhaps a crucial ingredient of a unified mathematical theory of both gapped and gapless phases.

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- We hope that this talk can convince readers that topological Wick rotation can provide some new insights other than the usual holographic dictionary. More applications of topological Wick rotation are yet to come. K.-Zheng:in preparation
- One of our main goals behind all these studies is the yet-unknown mathematical theory of higher dimensional conformal field theories. We hope that we can tell you more about it in the future.

**Relation to AdS/CFT dualities**: We intend to propose that we should regard the AdS/CFT dualities and the topological holographic dualities discussed in this talk as parts of a unified framework. However, our discussion of their analogy so far is limited to two features: (Gauge symmetries $\leftrightarrow$ Global symmetries) and (1-dim higher side:bulk+a boundary condition).

1. Topological defects are missing from the AdS/CFT dictionary??

$$\boxed{\text{topological defects on AdS-side}} \xrightarrow{TWR??} \boxed{\text{topological skeleton on CFT-side}}$$

2. Local quantum symmetries or LOA (needed to recover correlation functions) are completely missing from the topological holographic dictionary so far.

   2.1 AdS/CFT dictionary (in a topological twist sector): LOA's on the AdS-side and the CFT-side are Koszul dual Costello-Li:1606.00365, Costello:1705.02500, Costello-Paquette:2001.02177

   2.2 Topological holographic dictionary on LOA:

$$\boxed{\text{Hopf-type LOA in 3D QD or LW model}} \xrightarrow{\text{`Koszul dual'}} \boxed{E_2\text{-type LOA in 2D CFT}}$$

$$\boxed{\text{topological 2-net from Levin-Wen model}} \longleftrightarrow \boxed{\text{conformal 2-net from 2D CFT}}$$

K.-Zheng:2201.05726

3 AdS/CFT dictionary: $n+1$-spatial dimension in the AdS-side is the dimension of RG flow in the CFT-side.

Topological holographic dictionary: "Recursive application of the RG steps (i.e. cross-graining in the bulk theory) would result in a collection of tetrahedra that discretize a Euclidean AdS space." Chen-Zhang-Ji-Wang-Shen-Zeng-Hung, "Exact Holographic Tensor Networks – Consturcting $\mathrm{CFT}_D$ from $\mathrm{TQFT}_{D+1}$", in preparation

4 Any more edvidence?

Thank you !