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DISKS, DUALITY AND Θ-CATEGORIES

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The goal of this note is to define a simple concept of infinite dimensional (weak) categories that makes possible the development of a theory [BD,B,HMP,BFSV,T]. Our method is to extend simplicial homotopy theory to an appropriate concept of cellular sets. Our concept of higher categories is imitating the concept of restricted Kan complexes of Boardman and Vogt [BV]. A *cellular set* is a contravariant functor $X : \Theta \to Sets$ on a category Θ of cells. For defining the objects of Θ we use Batanin's idea of representing higher cells by finite trees [B]. To define the arrows we introduce a dual category $\mathcal{D}$ of finite disks. A *disk* is a combinatorial structure that generalises to higher dimension the idea of an interval (a linearly ordered set with distinct first and last elements). The euclidian n-ball $B^n \subset \mathrm{R}^n$ is an example of n-disk. Every finite disk D has a dual $D^{\check{}} = hom(D, B^{\infty})$ which is a polyhedral ball. This defines the geometric realisation of the cells from which we obtain a geometric realisation functor from the category θSet of cellular sets to CW-complexes. We define the concepts of horns and of *inner* horns. A θ*-category* is then defined to be a cellular set for which *inner* horns can be filled. The category of finite disks has an increasing filtration $Disk^1 \subset Disk^2 \subset \cdots$ defined by disk dimension and there is a corresponding *level* filtration $\theta^1 Sets \subset \theta^2 Sets \subset \cdots$ on the category of cellular sets. A θ-category of level $\leq n$ is called a θ^n-category: it is an infinite dimensional (weak) category for which all cells of dimension $> n$ are invertibles. A θ^1-category is a restricted Kan complex. A concept of (weak) n-categories can be obtained by further truncating θ^n-categories to cells of dimension $\cancel{\leq n+1}$. In a second note we shall prove that θCat, the category of θ-categories, is cartesian closed. We shall define a concept of h-isomorphism for arrows in θ-categories and prove that a θ-category is a (cellular) Kan complex iff all its cells are h-invertible. We shall define the *fundamental* n*-category* $\tau_n X$ of a θ-category X and homotopy groups $\pi_n(X, f)$ for cells $f \in X$ and $n > \dim f$. We shall prove a Whitehead theorem.

In what follows we shall define most concepts mentioned aboved.

§ 1.1 Simplicial sets and intervals

We begin by recalling the connection between simplicial sets and the theory of intervals. It can be phrased in topos theoretic terms as in [MM] but we shall limit the discussion to its combinatorial aspects. The category of finite non-empty ordinals and order preserving maps is denoted Δ. It is standart to denote the ordinal $n+1 = \{0, \ldots, n\}$ by $[n]$. Recall that a simplicial set is a *contravariant* functor $X : \Delta \to Sets$. The opposite category Δ^o has a direct combinatorial description with the category of finite strict intervals. Let us recall this duality. An *interval* is a linearly ordered set having a first and a last element respectively denoted 0 and 1, or $\perp$ and $\top$. If $0 \neq 1$ the interval is said to be *strict*. A *morphism* is a map $f : I \to J$ preserving the order ($\leq$) and such that $f(0) = 0$ and $f(1) = 1$. We shall denote by $\mathcal{I}$ the category of finite strict intervals. Let us describe the duality

$$(-)^* : \Delta^o \simeq \mathcal{I} \qquad (-)^{\check{}} : \mathcal{I}^o \simeq \Delta.$$

[1]?

By definition $[n]^* = \Delta([n],[1]) = [n+1]$ where for $f, g : [n] \to [1]$ we put $f \leq g$ iff $f(i) \leq g(i)$ for every $i \in [n]$. Similarly, $I^{\vee} = \mathcal{I}(I,[2])$. We shall view the interval $[n]^* = [n+1]$ as the set of *Dedekind cuts* on $[n]$. For example, $[3]$ has 5 cuts, hence $[3]^* = [4]$:

$$|\,0123, \quad 0\,|\,123, \quad 01\,|\,23, \quad 012\,|\,3 \quad 0123\,|\,.$$

Similarly, the dual $I^{\vee}$ is the set of *inner* cuts of I. For example, $[4]$ has four inner cuts:

$$0*1234, \quad 01*234, \quad 012*34 \quad 0123*4.$$

The duality is illustrated by the picture:

$$|*|*|*|*|\,.$$

The duals f^* or $f^{\vee}$ of a map $f : [n] \to [m]$ is obtained by taking inverse images of cuts. Notice that if $I = [n]^*$ then the set $hom(I, B^1)$ is the *geometric simplex* of dimension n where B^1 denotes the real interval $[0,1]$. It follows from the duality that a simplicial set could be defined as a *covariant* functor $X : \mathcal{I} \to Sets$.

§ 1.2 Disks

We now describe a combinatorial concept of *n-disks*, an interval being a 1-disk. The euclidian ball $B^n = \{x \in \mathbf{R}^n : \| x \| \leq 1\}$ is the main geometric example of n-disk. Observe that the projection $p_n : B^n \to [-1,1]$ is surjective and that its fibers $p^{-1}(x)$ are $n-1$-disks except for $x = \pm 1$ where they degenerate to a point. There is a complementary view with the projection $q : B^n \to B^{n-1}$. The fiber $q^{-1}(x)$ is a 1-disk except if $x \in \partial B^{n-1}$ where it degenerates to a point. We shall say that q is a *bundle of intervals.* If we order the coordinates we obtain a sequence of bundles of intervals:

$$1 \leftarrow B^1 \leftarrow B^2 \leftarrow \cdots B^{n-1} \leftarrow B^n$$

Each projection $p : B^{n+1} \to B^n$ has two canonical sections $d_0, d_1 : B^n \to B^{n+1}$ that correspond to the bottom and top hemispheres. We have $d_0(x) = d_1(x)$ iff $x \in \partial B^n$. Observe also that $\partial B^{n+1} = d_0(B^n) \cup d_1(B^n)$.

We now define the combinatorial concept of bundles of intervals in the category *Sets*. Shortly, a *bundle of intervals over $B \in Sets$* is an interval object in the category $Sets/B$. More explicitly, it is a map $p : X \to B$ whose fibers have an interval structure. It has then a top and a bottom section $d_0, d_1 : B \to X$ and we have $x \in [d_0(px), d_1(px)]$ for every $x \in X$. We shall say that the equaliser of d_0 and d_1 is the *singular set* of the bundle.

Definition 1: A *disk D* of dimension $\leq N$ is a sequence of length N of bundles of intervals:

$$1 = D_0 \leftarrow D_1 \leftarrow D_2 \leftarrow \cdots D_{N-1} \leftarrow D_N$$

such that the singular set of $p : D_{n+1} \to D_n$ is equal to $d_0(D_{n-1}) \cup d_1(D_{n-1})$ for every $0 \leq n < N$. If $n = 0$ this condition means that D_1 is a strict interval.

Disks of dimension $\leq \infty$ are defined by using infinite sequences. It follows from the definition that each map $d_0, d_1 : D_{i-1} \to D_i$ equalizes the pair $d_0, d_1 : D_i \to D_{i+1}$, hence $d_0 d_0 = d_1 d_0$ and $d_0 d_1 = d_1 d_1$. We define the *boundary* ∂D_n to be $d_0(D_{n-1}) \cup d_1(D_{n-1})$ and the *interior* $\imath(D_n)$ to be $D_n \backslash \partial D_n$. By convention $\partial D_0 = \emptyset$.

lemma 1. *We have $p(\imath(D_n)) \subseteq \imath(D_{n-1})$ for every n.*

Proof: If $p(x) \in \partial D_{n-1}$ then $d_0(px) = d_1(px)$ by the condition in the definition. But $x \in [d_0(px), d_1(px)]$ and it follows that $x = d_0(px) \in \partial D_n$. QED

By restricting the projections to the interiors we obtain a sequence of maps

$$1 = \imath(D_0) \leftarrow \imath(D_1) \leftarrow \imath(D_2) \leftarrow \imath(D_3) \leftarrow \cdots$$

This is a (generalised) *planar tree* since the fibers of each projection $\imath(D_i) \to \imath(D_{i-1})$ are linearly ordered. We shall say that it is the *interior* of D and denote it by $\imath(D)$. We define the *level* of D to be the height of $\imath(D)$. We define the *volume* $\mid D \mid$ of D to be the cardinality of $\imath(D)$ *minus* 1 (we do not include $D_0 = 1$ in counting the volume). If $\mid D \mid < \infty$ we shall say that D is *finite* .

Proposition 1. *A disk is determined by its interior (up to unique isomorphism). Any tree is the interior of a disk.*

Proof: It follows from definition 1 that the square

$$\begin{tikzcd}
\partial D_{n-1} \arrow[r, "d_1"] \arrow[d, "d_0"'] & d_1(D_{n-1}) \arrow[d] \\
d_0(D_{n-1}) \arrow[r] & \partial D_n
\end{tikzcd}$$

is cartesian. It is thus a pushout square and we have $\partial D_n \simeq D_{n-1} \sqcup_{\partial D_{n-1}} D_{n-1}$. The result then follows from the formula $D_n = \imath(D_n) \sqcup \partial D_n$ by induction on n.

A *morphism* $S \to T$ of planar trees is a sequence of maps $f_n : S_n \to T_n$ commuting with the projections

$$\begin{tikzcd}
1 & S_1 \arrow[l] \arrow[d, "f_1"'] & S_2 \arrow[l] \arrow[d, "f_2"'] & S_3 \arrow[l] \arrow[d, "f_3"'] & \cdots \arrow[l] \\
 & & & & \cdots \\
1 & T_1 \arrow[l] & T_2 \arrow[l] & T_3 \arrow[l] & \cdots \arrow[l]
\end{tikzcd}$$

and respecting the order of the fibers. A *morphism* $D \to D'$ of disks is a morphism of planar trees that respects the endpoints of each fiber. For any planar tree S we shall denote by $\overline{S}$ the unique disk such that $S = \imath(\overline{S})$. This is a functorial construction.

Proposition 2. *The functor $S \mapsto \overline{S}$ is left adjoint to the forgetful functor from disks to planar trees.*

Remark: The construction $S \mapsto \overline{S}$ is a monad on the category of planar trees. The category of algebras for this monad is the category of disks. It follows from proposition 1 that every disk is free on its interior. Hence the category of disks is also a Kleisly category.

A map $f : D \to D'$ between disks is *open* if $f(\imath(D)) \subseteq \imath(D')$. The category of planar trees is equivalent to the category of disks with open maps.

The *suspension* $\sigma(T)$ of a planar tree is defined by shifting : $\sigma(T)_i = T_{i-1}$ for $i > 0$ where $T_0 = 1$. There is also the operation of *concatenation* for trees: $(S \star T)_i = S_i \sqcup T_i$ for $i > 0$ where $\sqcup$ is disjoint union for $i > 1$ and the ordinal sum for $i = 1$. The *suspension* $S(D)$ of a disk is the disk

$$1 \leftarrow [2] \leftarrow D_1 \sqcup [1] \leftarrow D_2 \sqcup [1] \leftarrow \cdots$$

where the map $D_1 \sqcup [1] \to [2]$ sends D_1 to the mid point of $[2]$ and is equal to $d_1 : [1] \to [2]$ on $[1]$. All the other maps are the obvious ones. The suspension of disks is defined so that $S(\overline{T}) = \overline{\sigma(T)}$. These operations are functorial on disks and trees. There is also an operation of *composition* $\circ$ for disks so that $\overline{S} \circ \overline{T} = \overline{(S \star T)}$ but it has a limited functoriality (it is of course functorial for open maps).

§ 1.3 Duality and Cellular sets

For the next result we need the infinite dimensional euclidian ball B^∞:

$$1 \leftarrow B^1 \leftarrow B^2 \leftarrow \cdots .$$

Proposition 3. *Let D be a finite disk. Then the set $hom(D, B^\infty)$ has the structure of an euclidian ball of dimension $\mid D \mid$.*

Proof : We only sketch the construction. Let us denote respectively by $hom_d(D, D')$ and $hom_t(T'T')$ the maps in the categories of disks and trees. If $T = \imath(D)$ then we have $hom_d(D, B^\infty) = hom_t(T, B^\infty)$ by adjointness. We shall describe the polyhedron $R(T) = hom_t(T, B^\infty)$ by structural induction on trees. If $T = 0$ then $R(T) = 1$. If $T = \sigma(A)$ then $R(T)$ is obtained as a pushout square

$$\begin{tikzcd}
\{-1,1\} \times R(A) \arrow[r] \arrow[d, "p_1"'] & {[-1,1] \times R(A)} \arrow[d] \\
\{-1,1\} \arrow[r] & R(T)
\end{tikzcd}$$

and this shows that $R(T)$ is the *suspension* $SR(A)$ of $R(A)$. Observe now that there is a structure map $q : SR(A) \to [-1, 1]$. If $T = \sigma A_1 \star \cdots \star \sigma A_n$ then there is a cartesian square

$$\begin{tikzcd}
R(T) \arrow[r] \arrow[d] & \prod_{i=1}^{n} SR(A_i) \arrow[d, "q^{\times n}"] \\
\Delta^n \arrow[r] & {[-1,1]^n}
\end{tikzcd}$$

where $(x_1, \ldots, x_n) \in \Delta^n$ iff $x_1 \leq \cdots \leq x_n$.

We shall denote by *Disks* (resp. *Trees*) the category of finite disks (resp. of finite trees). We define the category Θ of *Batanin cells* to be the opposite of *Disks*. We shall write $C = D^o$ and $D = C^o$ for a disk and its opposite cell. For any $C \in \Theta$ let us put $R(C) = hom(C^o, B^\infty)$. This define the *geometric realisation* functor

$$R : \Theta \to CW\text{-complexes.}$$

The *dimension* of $C = D^o$ is defined to be $\mid D \mid$. If t is a finite tree we shall denote the corresponding disk by $\bar{t}$ and the dual cell by $[t] = (\bar{t})^o$. The dimension of $[t]$ is thus the cardinality of t (not counting the root). We shall denote by 0 the null tree (with only the root) . Recall the set of planar (rooted) trees can be constructed inductively from 0 by using the operation of (non-empty) concatenation $*$ and the operation of succession σ. Let us put $1 = \sigma(0)$ and $n + 1 = n \star 1$. The disk $\bar{n}$ is the interval $\{0, 1, 2, \ldots, n, n+1\}$ and its dual is $[n]$. The cells $[n]$ for $n \in \mathrm{N}$ are the classical simplicial cells. The tree $\sigma^n(0)$ is a chain of n nodes. The corresponding cell has dimension n and we shall denote it δ_n. From the inclusion $\sigma^{n-1}(0) \subset \sigma^n(0)$ we obtain an arrow $\delta_n \to \delta_{n-1}$.

We define a *cellular set* to be a functor $X : Disks \to Sets$, that is, a functor $X : \Theta^o \to Sets$. We shall denote by $\theta Sets$ the category of cellular sets. We shall use the Yoneda embedding $\Theta \subset \theta Sets$ to identify Θ with a full subcategory of $\theta Sets$. By Kan extension we obtain a geometric realisation functor $R : \theta Sets \to CW\text{-complexes}$.

Theorem 1. *In the category $\theta Sets$ the sequence $\delta = (\delta_n)$*

$$1 = \delta_0 \leftarrow \delta_1 \leftarrow \delta_2 \leftarrow \cdots$$

has the structure of a disk. It is the universal disk in a topos theoretic sense.

For the purpose of the discussion we shall say that a functor between two Grothendieck toposes is a *realisation functor* if it preserves colimits and finite limits (that is, it is the inverse image part of a geometric morphism). The universal property in theorem 1 means that for any topos $\mathcal{E}$ and for any disk $D \in \mathcal{E}$ there is a unique realisation functor $r : \theta Sets \to \mathcal{E}$ such that $r(\delta) = D$. For example the topos $SSets$ of simplicial sets classifies strict intervals [MM]. There is a realisation functor $SSets \to \theta Sets$ that transforms the classical simplicial interval $\Delta[1]$ into δ_1. The category $\theta Sets$ is equipped with many operations that can be defined from its universal property. As an example, the truncated sequence $\delta_1 \leftarrow \delta_2 \leftarrow \cdots$ is almost a disk in the category $\theta Sets/\delta_1$. It is not only because the bundle of intervals $\delta_2 \to \delta_1$ degenerates on $\partial\delta_1 \neq \emptyset$. This is easy to cure with the subtopos $\theta Sets/\imath(\delta_1)$. The objects of this category are the maps $X \to \delta_1$ that are isomorphism (if pulled back) over $\partial\delta_1 \subset \delta_1$. The operation of *suspension* $S : \theta Sets \to \theta Sets/\imath(\delta_1)$ is the realisation functor that send δ to this truncated sequence. We obtain a suspension functor $S : \theta Sets \to \theta Sets$ by further composing with the forgeful functor $\theta Sets/\imath(\delta_1) \to \theta Sets$. We then have $S^n(1) = \delta_n$ for every $n \geq 0$ and this indicate that the topos $\theta Sets$ is essentially generated by an abstract operation of suspension. It might be interesting to investigate the properties of such an operation. There are many realisation functors from $\theta Sets$ to $SSets$. Let us define the suspension of a simplicial set $X \in SSets$ by putting $S(X) = \Delta[1] \times X/\sim$ where the equivalence relation $\sim$ is collapsing $\{0\} \times X$ to $\{0\}$ and $\{1\} \times X$ to $\{1\}$. We obtain an infinite disk $S^0(pt) \leftarrow S^1(pt) \leftarrow \cdots$ in $SSets$. We could also choose the suspension defined by the formula $S(X) = X \star 1/X$ where $\star$ is the simplical join.

To identify the horns in $\theta Sets$ we need to understand the (codimension 1) faces of a cell $C \in \Theta$. A (codimension 1) face of $C = [t]$ is a quotient of $\overline{t}$ that identifies exactly two points. The tree t is a disjoint union of intervals $J(x) = p^{-1}(x)$. Let us say that a pair $\{a, b\}$ of vertices of t is *contractible* if a and b are adjacents to each other in the same interval of t. In this case we obtain a quotient tree $t/\{a, b\}$ by first identifying a with b and then *schuffling* the intervals $J(a)$ and $J(b)$. Let us called such a quotient an *inner contraction.*

Let us say that a vertex $a \in t$ is *bounding* if it is a leaf and an extremal point of its interval. In this case we shall say that a vertex $a' \in \partial\overline{t}$ adjacent to a in $\overline{t}$ is a *compagnon* of a. We can then form a quotient disk $\overline{t}/\{a, a'\}$. Its interior is the tree $s = t\backslash\{a\}$ obtained by deleting a since a is now identified to a boundary point. Let us call such a quotient *boundary contraction.* Notice that the map $\overline{t} \to \overline{s}$ depends on the choice of a'.

Proposition 4. *Let t be a finite planar tree. Then the faces of codimension 1 of $[t]$ are of the following two kinds:*

(i) the faces $[s] \subset [t]$ where $\overline{t} \to \overline{s}$ is an inner contraction $t/\{a, b\}$;

(ii) the faces $[s] \subset [t]$ where $\overline{t} \to \overline{s}$ is a boundary contraction $t\backslash\{a, a'\}$.

We call a face of type (i) *inner* and of type (ii) *outer.* A codimension 1 face $[s] \subset [t]$ is inner iff the dual map $\overline{t} \to \overline{s}$ is open. The *boundary* $\partial[t]$ of the cell $[t]$ is the union of its faces of codimension 1. If $[s]$ is a face of codimension 1 the *horn* $\Lambda^s[t]$ is the union of the faces of codimension 1 other than $[s]$. If X is a cellular set a map $\xi : \Lambda^s[t] \to X$ is a *horn in X*. A *filler* for ξ is an extension $\xi' : [t] \to X$.

We say that a horn $\Lambda^s[t] \subset [t]$ is *inner* if $[s]$ is an inner face of $[t]$.

Definition 2: A *θ-category* (resp. a *cellular Kan complex*) is a cellular set X in which every inner horn (resp. every horn) $\xi : \Lambda^s[t] \to X$ has a filler.

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