Cohomological Field Theories and Generalized Seiberg–Witten Equations

(Joint work with Jürgen Jost)

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Introduction

Gauge theory has led to spectacular advances in mathematics.

Donaldson Theory:

- Anti-self-dual equations: $(F_A)_+ = 0$;
- Gauge group: $\mathrm{SU}(2)$ (or $\mathrm{SO}(3)$).

Seiberg–Witten Theory:

- Seiberg–Witten equations: $(F_A)_+ - \frac{1}{2}\mu(\sigma) = 0, \quad \not{D}_A^+\sigma = 0$;
- Gauge group: $\mathrm{U}(1)$.

There also exist non-abelian generalizations of Seiberg–Witten theory.

The solution space $\mathrm{Sol}(M)$ of these 1st order nonlinear PDEs on a Riemannian manifold M can be interpreted as the zero locus $\mathcal{F}^{-1}(0)$ of a $\mathcal{G}$-equivariant map:

$$\mathcal{F} : \mathcal{E} \to \mathcal{H},$$

where $\mathcal{E} := \Gamma(E)$ (resp., $\mathcal{H} := \Gamma(H)$) is the space of sections of a fiber bundle $E \to M$ (resp., a vector bundle $H \to M$). If H is equipped with a $\mathcal{G}$-invariant bundle metric, one can define

$$S(\Phi) = \int_M |\mathcal{F}(\Phi)|^2 \mathrm{vol}_M, \quad \Phi \in \mathcal{E}. \tag{1}$$

If $M = \mathbb{R}^n$, (1) can be extended to a supersymmetric action functional $\widetilde{S}$ over a (Fréchet) supermanifold $\widetilde{\mathcal{E}}$ with underlying (Fréchet) manifold $i : \mathcal{E} \hookrightarrow \widetilde{\mathcal{E}}$.

$\widetilde{S}$ can be defined for a general Riemannian manifold M at the cost of losing the full super Poincaré symmetries. In such case,

- $\widetilde{\mathcal{E}}$ can be equipped with a compatible $\mathbb{Z}$-grading;
- $\widetilde{S}$ has degree 0 and a remaining degree 1 "scalar" supersymmetry Q.

$(\widetilde{\mathcal{E}}, Q, \widetilde{S})$ is referred to as a **cohomological field theory (CohFT)** by physicists.

After applying a“twisting” procedure, the supersymmetries of the $4D$ $N = 2$ $\mathrm{SU}(2)$ pure super Yang-Mills theory become

- a scalar supersymmetry Q, $Q^2 = 0$;
- a 1-form supersymmetry $K_\mu dx^\mu$, $[Q, K_\mu] = \partial_\mu$;
- an ASD 2-form supersymmetry $H_{\mu\nu} dx^\mu \wedge dx^\nu$.

The twisted pure super Yang-Mills theory is a CohFT (on $\mathbb{R}^4$), known as the Donaldson-Witten theory.

The Seiberg–Witten invariants are closely related to the twisted $4D$ $N = 2$ $\mathrm{U}(1)$ super Yang-Mills theory with matter fields.

It might seem that the cohomology of the differential graded (dg) superalgebra $(C^\infty(\widetilde{\mathcal{E}}), Q)$ is dependent on $\widetilde{S}$ and its supersymmetries. We will show that:

- Both $(C^\infty(\widetilde{\mathcal{E}}), Q)$ and $\widetilde{S}$ can be constructed solely from the data of the 1st order field equation $\mathcal{F} = 0$ and admit a clear mathematical interpretation.[1]
- When applied to the generalized Seiberg-Witten equations on $\mathbb{R}^4$, our construction reproduces the supersymmetric functionals of various CohFTs.

Based on joint work with Jürgen Jost: `arxiv: 2407.04019`.

[1] Our formalism is closely related to the BRST and Mathai–Quillen formalisms of CohFTs.

The general construction

Let $\mathfrak{g}_{dR} = \mathfrak{g} \oplus \mathfrak{g}[-1]$ be a graded Lie superalgebra, whose bracket is induced by the bracket of $\mathfrak{g}$ and the adjoint action of $\mathfrak{g}$ on $\mathfrak{g}[-1]$. $\mathfrak{g}_{dR}$ is a dg Lie superalgebra with the differential

$$0 \to \mathfrak{g}[-1] \xrightarrow{\mathrm{Id}} \mathfrak{g} \to 0.$$

The Weil algebra $W(\mathfrak{g}) = \Lambda(\mathfrak{g}^*) \otimes \mathrm{Sym}(\mathfrak{g}^*)$ is a $\mathfrak{g}_{dR}$-algebra by setting

$$\iota_a \theta^b = \delta_a^b, \quad \iota_a \phi^b = 0,$$
$$\mathrm{Lie}_a \theta^b = -f_{ac}^b \theta^c, \quad \mathrm{Lie}_a \phi^b = -f_{ac}^b \phi^c.$$

Let P be a principal G-bundle. $\Omega(P)$ is also a $\mathfrak{g}_{dR}$-algebra, with ι_a and Lie_a being the usual contractions and Lie derivatives.

The de Rham complex $\Omega(\mathcal{M}) \cong C^\infty(T[1]\mathcal{M})$ of a dg manifold[2] $\mathcal{M}$ with a compatible G-action[3] is a $\mathfrak{g}_{dR}$-algebra. We have the following isomorphism of dg algebras:

$$(CE(\mathfrak{g}_{dR}; C^\infty(T[1]\mathcal{M})), d_{CE}) \cong (W(\mathfrak{g}) \otimes \Omega(\mathcal{M}), d_K), \qquad (2)$$

where,

- $(CE(\mathfrak{g}_{dR}; C^\infty(T[1]\mathcal{M})), d_{CE})$ is the Chevalley–Eilenberg complex of $\mathfrak{g}_{dR}$ with values in $C^\infty(T[1]\mathcal{M})$;
- d_K is the Kalkman differential of the BRST model of the equivariant de Rham cohomology of $\mathcal{M}$.

[2] A dg manifold is a supermanifold $\mathcal{M}$ together with a compatible $\mathbb{Z}$-grading and a degree 1 odd vector field $Q_{\mathcal{M}}$ that squares to 0.

[3] That is, for each $\xi \in \mathfrak{g}$, the fundamental vector field X_ξ over $\mathcal{M}$ induced by ξ commutes with $Q_{\mathcal{M}}$.

Before applying this construction to study CohFTs, let us give a few more definitions. Let $\mathcal{W}$ be a $\mathfrak{g}_{dR}$-algebra. An element $\Theta = \Theta^a \otimes \xi_a \in \mathcal{W} \otimes \mathfrak{g}$ of degree 1 is called a connection of $\mathcal{W}$ if

$$\iota_a \Theta = \xi_a, \quad \mathrm{Lie}_a \Theta = -[\xi_a, \Theta].$$

The curvature of Θ is an element $\Omega = \Omega^a \otimes \xi_a \in \mathcal{W} \otimes \mathfrak{g}$ of degree 2 defined by the formula

$$\Omega = \delta_{\mathcal{W}} \Theta + \frac{1}{2}[\Theta, \Theta].$$

$W(\mathfrak{g})$ admits a canonical connection and curvature, given by the formulas $\theta = \theta^a \otimes \xi_a$ and $\phi = \phi^a \otimes \xi_a$.

The Chern–Weil homomorphism

$$\mathrm{CW}_\Theta : W(\mathfrak{g}) \to \mathcal{W}$$

is defined by sending $\theta^a \mapsto \Theta^a$ and $\phi^a \mapsto \Omega^a$. CW_Θ is a morphism between $\mathfrak{g}_{dR}$-algebras. For $\mathcal{W} = \Omega(P)$ and Θ a connection 1-form on P, CW_Θ gives us the usual Chern–Weil homomorphism.

Let $\mathcal{W}$ and $\mathcal{W}'$ be two $\mathfrak{g}_{dR}$-algebras. Let Θ be a connection of $\mathcal{W}$. The Mathai–Quillen automorphism T_Θ of $\mathcal{W} \otimes \mathcal{W}'$ is defined as

$$T_\Theta = \exp(\Theta^a \otimes \iota_a).$$

For $\mathcal{W} = W(\mathfrak{g})$ and $\mathcal{W}' = \Omega(\mathcal{M})$, T_Θ transforms the Weil differential into the Kalkman differential.

Let $\mathcal{E}_{tot} = \mathcal{E} \times \mathcal{H}$. Consider the following Koszul complex

$$\cdots \xrightarrow{\iota_{\mathcal{F}}} \Gamma(\Lambda^k \mathcal{E}_{tot}^*) \xrightarrow{\iota_{\mathcal{F}}} \cdots \xrightarrow{\iota_{\mathcal{F}}} \Gamma(\mathcal{E}_{tot}^*) \xrightarrow{\iota_{\mathcal{F}}} C^\infty(\mathcal{E}) \to 0,$$

where $\iota_{\mathcal{F}}$ is the contraction by $\mathcal{F}$. This complex is equivalently to an infinite dimensional dg manifold $(\mathcal{E}_{tot}[-1], \iota_{\mathcal{F}})$.

The (minimal) CohFT extension $(\widetilde{\mathcal{E}}, Q, \widetilde{S})$ of $(\mathcal{E}, S)$ is given by

- $\widetilde{\mathcal{E}} = T[1](\mathrm{Lie}(\mathcal{G})[1] \times \mathcal{E}_{tot}[-1])$;
- Q is the Chevalley–Eilenberg differential under the isomorphism

$$C^\infty(\widetilde{\mathcal{E}}) \cong \mathrm{CE}(\mathrm{Lie}(\mathcal{G})_{dR}; C^\infty(T[1](\mathcal{E}_{tot}[-1]))); \qquad (3)$$

Let (Φ, X) denote local coordinates of $\mathcal{E}_{tot}[-1]$. Let (Ψ, B) denote the coordinates of its shifted tangent space. Let (θ, ϕ) denote the coordinates of $T[1](\mathrm{Lie}(\mathcal{G})[1])$. The scalar supersymmetry Q is defined by its action on the fields:

$$
\begin{aligned}
&Q\theta = \phi - \frac{1}{2}[\theta, \theta], \ \ Q\phi = -[\theta, \phi], \\
&Q\Phi = \Psi - \theta\Phi, \ \ Q\Psi = -\theta\Psi + \phi\Phi, \\
&Q\mathsf{X} = \mathsf{B} - \theta\mathsf{X} + \mathcal{F}(\Phi), \ \ Q\mathsf{B} = -\theta\mathsf{B} + \phi\mathsf{X} - \mathrm{Lin}_{\Phi}(\mathcal{F})\Psi,
\end{aligned}
$$

where we use $\mathrm{Lin}_{\Phi}(\mathcal{F})$ to denote the linearization of $\mathcal{F}$ at Φ and $\theta\Phi$ to denote the action of θ on Φ.

- The CohFT action functional $\widetilde{S}$ is defined as

$$\widetilde{S} = Q\left(\int_M \langle \mathsf{X}, \mathsf{B}\rangle \mathrm{vol}_M\right).$$

A direct computation shows that

$$\widetilde{S} = \int_M \left(|\mathsf{B}|^2 + \langle \mathsf{B}, \mathcal{F}\rangle + \langle \mathsf{X}, \mathrm{Lin}_{\Phi}(\mathcal{F})\Psi\rangle - \langle \mathsf{X}, \phi\mathsf{X}\rangle\right) \mathrm{vol}_M.$$

The pullback of $\widetilde{S}$ to $\mathcal{E}_{tot}$ is

$$\widetilde{S}_{Boson} := i^*\widetilde{S} = \int_M \left(|\mathsf{B}|^2 + \langle \mathsf{B}, \mathcal{F}\rangle\right) \mathrm{vol}_M,$$

which gives us a 1st order formulation of (1).

Mathai-Quillen formalism

Let us consider the zero dimensional toy model where M is a point, $\mathcal{E}$ is the frame bundle $\mathrm{Fr}(N)$ over an $2m$-dimensional Riemannian manifold N, $\mathcal{E}_{tot}$ is $\mathrm{Fr}(N) \times \mathbb{R}^{2m}$, and $\mathcal{G} = \mathrm{SO}(2m)$. Note that the $\mathcal{G}$-action on $\mathcal{E}_{tot}$ is free and we have

$$\mathcal{E}_{tot}/\mathcal{G} \cong TN.$$

Therefore, an $\mathcal{G}$-equivariant section $\mathcal{F}$ of $\mathcal{E}_{tot}$ is equivalent to a vector field over N.
The configuration space of the 0-dimensional CohFT is

$$\widetilde{\mathcal{E}} = T[1](\mathfrak{so}(2m)[1] \times \mathrm{Fr}(N) \times \mathbb{R}^{2m}[-1]).$$

Let $\Delta : \mathrm{Fr}(N) \to \mathrm{Fr}(N) \times \mathrm{Fr}(N)$ denote the diagonal embedding. Δ, CW_∇, and T_∇ together induce a homomorphism J between $\mathfrak{so}(2m)_{dR}$-algebras:

$$
\begin{aligned}
J : C^\infty(\widetilde{\mathcal{E}}) \cong W(\mathfrak{so}(2m)) \otimes \Omega(\mathrm{Fr}(N)) \otimes \Omega(\mathbb{R}^{2m}[-1]) \xrightarrow{(CW_\nabla \otimes 1 \otimes 1)} \\
(\Omega(\mathrm{Fr}(N)) \otimes \Omega(\mathrm{Fr}(N))) \otimes \Omega(\mathbb{R}^{2m}[-1]) \xrightarrow{T_\nabla \otimes 1} \\
(\Omega(\mathrm{Fr}(N)) \otimes \Omega(\mathrm{Fr}(N))) \otimes \Omega(\mathbb{R}^{2m}[-1]) \xrightarrow{\Delta^* \otimes 1} \\
\Omega(\mathrm{Fr}(N)) \otimes \Omega(\mathbb{R}^{2m}[-1]) \cong C^\infty(\widetilde{\mathcal{E}}_{MQ}),
\end{aligned}
$$

where

$$\widetilde{\mathcal{E}}_{MQ} := T[1](\mathrm{Fr}(N) \times \mathbb{R}^{2m}[-1]).$$

The cohomological vector field Q_{MQ} of $\widetilde{\mathcal{E}}_{MQ}$ is given by

$$\begin{aligned} &Q_{MQ}\Phi = \Psi, \quad Q_{MQ}\Psi = 0, \\ &Q_{MQ}\mathsf{X} = \mathsf{B} + \mathcal{F} - A_\nabla \mathsf{X}, \quad Q_{MQ}\mathsf{B} = -A_\nabla \mathsf{B} + R_\nabla \mathsf{X} - \nabla\mathcal{F}. \end{aligned}$$

One can check that $Q_{MQ} \circ J = J \circ Q$. The image of the CohFT action functional $\widetilde{S}$ under J is given by

$$\widetilde{S}_{MQ} := J(\widetilde{S}) = |\mathsf{B}|^2 + \langle \mathcal{F}, \mathsf{B}\rangle + \langle \mathsf{X}, \nabla\mathcal{F}\rangle - \langle \mathsf{X}, R_\nabla \mathsf{X}\rangle.$$

The Berezin integral

$$e^{\mathcal{F}}_\nabla(t) := \frac{1}{(2\pi)^{2m}} \int d\mathsf{X} dB \exp(-t\widetilde{S}_{MQ}), \quad t > 0,$$

defines a closed basic $2m$-form on $\mathrm{Fr}(N)$.

Moreover, $\frac{d}{dt}e^{\mathcal{F}}_{\nabla}(t)$ is exact since $\widetilde{S}$ is Q-exact, and

$$\lim_{t \to 0} e^{\mathcal{F}}_{\nabla}(t) = \mathrm{Pf}(\frac{R_{\nabla}}{2\pi}).$$

Thus, $e^{\mathcal{F}}_{\nabla}(t)$ forms a representative of the Euler class of N under the identification $\Omega_{bas}(\mathrm{Fr}(N)) \cong \Omega(N)$.

For a transversal $\mathcal{F}$, a proof of the Poincaré–Hopf theorem can be obtained by letting $t \to \infty$.

Derived Geometric point of view

For a derived geometer, the dg manifold $(\mathcal{E}_{tot}[-1], \iota_{\mathcal{F}})$ provides a concrete model for the derived manifold $\mathrm{Sol}(\mathcal{F})$ of the intersection of $\mathcal{F}$ and the zero section. We may say that:

The study of a CohFT associated to $\mathcal{F}$ is equivalent to the study of the "equivariant de Rham cohomology theory" of $\mathrm{Sol}(\mathcal{F})$.

Applications to the GSW equations

Let $G \subset M_{m\times m}(\mathbb{K})$ be a matrix group containing -1, $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. The Spin^G group is defined as

$$\mathrm{Spin}^G(n) := \mathrm{Spin}(n) \times_{\mathbb{Z}_2} G.$$

There is a short exact sequence

$$\mathrm{Id} \to \mathbb{Z}_2 \to \mathrm{Spin}^G(n) \xrightarrow{\mathrm{Ad}} \mathrm{SO}(n) \times G/\mathbb{Z}_2 \to \mathrm{Id}.$$

Let (M, g) be a compact Riemannian n-manifold. Let L be a $G/\mathbb{Z}_2$-bundle over M. A spin^G structure P on M is a lift of $\mathrm{Fr}(TM \times_M L)$ with the corresponding fiberwise covering map being $\mathrm{Ad} : \mathrm{Spin}^G(n) \to \mathrm{SO}(n) \times G/\mathbb{Z}_2$.

Let S be an irreducible left-module of $\mathrm{Cl}(n) \otimes_{\mathbb{R}} M_{m\times m}(\mathbb{K})$, hence a representation of $\mathrm{Spin}^G(n)$. With a slight abuse of notion, we also use S to denote the corresponding vector bundle over M.
The Levi–Civita connection ∇ on TM and a connection A on L induces a connection ∇_A on S. One can define the twisted Dirac operator on S via the standard formula

$$\not{D}_A \sigma = (e^\mu \otimes 1)\iota_{e_\mu} \nabla_A \sigma,$$

where $\sigma \in \Gamma(S)$ and $\{e_\mu\}_{\mu=1}^n$ is a local orthonormal frame.

For a compact G, we can equip S with a $\mathrm{Spin}^G(n)$-invariant bundle metric $\langle \cdot, \cdot \rangle$ satisfying

$$\langle (e \otimes 1)\sigma_1, (e \otimes 1)\sigma_2 \rangle = \langle \sigma_1, \sigma_2 \rangle$$

for all $\sigma_1, \sigma_2 \in S_x$ and all unit vectors $e \in T_x M$.
Let $\{\epsilon_a\}$ be a orthonormal basis of the Lie algebra $\mathfrak{g}$ of G. We define the following quadratic bundle map

$$\begin{aligned} \mu : S &\to \Lambda^2(M, \mathfrak{g}_E) \\ \sigma &\mapsto \langle e_\mu e_\nu \otimes \epsilon^a \sigma, \sigma \rangle (e^\mu \wedge e^\nu) \otimes \epsilon_a. \end{aligned}$$

The generalized Seiberg–Witten equations on a spin^G manifold M are defined as

$$F_A - \frac{1}{2}\mu(\sigma) = 0, \quad \not{D}_A \sigma = 0.$$

The volume form vol_M induces a chirality operator ω on S. In dimension $n = 4$, $\omega^2 = 1$ and S can be decomposed as $S = S_+ \oplus S_-$. Hence, we can consider the following equations instead:

$$(F_A)_+ - \frac{1}{2}\mu(\sigma) = 0, \quad \not{D}_A^+ \sigma = 0,$$

where $\sigma \in \Gamma(S_+)$.

Example 1

For $G = \mathrm{U}(1)$, we obtain the original Seiberg–Witten equations.

Example 2

For $G = \mathrm{U}(2)$, a spin^G structure is called a spin^u structure. We have

$$\mathrm{U}(2)/\mathbb{Z}_2 \cong \mathrm{U}(1) \times \mathrm{PU}(2) \cong \mathrm{S}^1 \times \mathrm{SO}(3).$$

The relevant GSW equations are called the $\mathrm{SO}(3)$ monopole equations. The ASD $\mathrm{SO}(3)$ connections and the Seiberg–Witten monopoles correspond to the two kinds of fixed points of the S^1-action on $\mathcal{M}(\mathcal{F})$.

To sum up, we have

- $\mathcal{E} = \mathcal{A}(L) \times \Gamma(S^+)$ is the product of the affine space of connections on L and the space of sections of S^+.
- $\mathcal{H} = \Omega^2_+(M, \mathfrak{g}_L) \times \Gamma(S^-)$.
- $\mathcal{F}$ is a $\mathrm{Aut}(L)$-equivariant map sending

$$\Phi = (A, \sigma) \mapsto \mathcal{F}(\Phi) = ((F_A)_+ - \frac{1}{2}\mu(\sigma), \not{D}^+_A \sigma).$$

Using the Weitzenböck formula, one can show that

$$S = \int_M \left(|\nabla_A \sigma|^2 + |(F_A)_+|^2 + \langle \sigma, \mathfrak{R}_M(\sigma) \rangle + \frac{|\mu(\sigma)|^2}{4} \right) \mathrm{vol}_M. \quad (4)$$

The scalar supersymmetry Q of the theory takes the following form:

$$\begin{aligned}
&Q\theta = \phi - \frac{1}{2}[\theta, \theta], \quad Q\phi = -[\theta, \phi], \\
&QA = \psi + d_A\theta, \quad Q\psi = -[\theta, \psi] - d_A\phi, \\
&Q\sigma = \upsilon - \theta\sigma, \quad Q\upsilon = -\theta\upsilon + \phi\sigma, \\
&Q\chi = b - [\theta, \chi] + (F_A)_+ - \frac{1}{2}\mu(\sigma), \\
&Qb = -[\theta, b] + [\phi, \chi] - d_A^+\psi + \mu(\sigma, \upsilon), \\
&Q\xi = h - \theta\xi + \not{D}_A\sigma, \;\; Qh = -\theta h + \phi\xi - \not{D}_A\upsilon - \psi\sigma, \\
&Q\lambda = \eta - [\theta, \lambda], \;\; Q\eta = -[\theta, \eta] + [\phi, \lambda],
\end{aligned}$$

where $\Psi = (\psi, \upsilon)$, $\mathrm{X} = (\chi, \xi)$, and $\mathrm{B} = (b, h)$.

If $M = \mathbb{R}^4$, the theory also has a 1-form supersymmetry $K = e^\mu \wedge K_\mu$:

$$\begin{aligned}
&K\theta = A, \quad K\phi = -\psi, \\
&KA = 2\chi, \quad K\psi = 2(F_A)_- - 2b + \mu(\sigma), \\
&K\sigma = -e^\mu \wedge (e_\mu \xi), \quad K\upsilon = e^\mu \wedge (e_\mu h), \\
&K\chi = 0, \quad Kb = 3d_A\chi - e^\mu \wedge \mu(e_\mu \xi, \sigma), \\
&K\xi = 0, \quad Kh = -e^\mu \wedge \chi_{\mu\nu}(e^\nu \sigma),
\end{aligned}$$

One can check that $[Q, K_\mu] = \partial_\mu$.

Our CohFT construction recovers (partially) the supersymmetric extension $\widetilde{S}$ of the generalized Seiberg-Witten functional.

$$\begin{aligned}\widetilde{S} &= \int_M \mathrm{vol}_M Q\left(\langle b,\chi\rangle + \langle h,\xi\rangle\right)\\ &= \int_M \mathrm{vol}_M \Big(\langle [\phi,\chi] - d_A^+ \upsilon + \mu(\sigma,\psi),\chi\rangle + \langle \phi\xi - \not{D}_A^+\psi - \upsilon\sigma,\xi\rangle\Big)\\ &+ \underbrace{\int_M \mathrm{vol}_M \left(\langle b, b + (F_A)_+ - \frac{1}{2}\mu(\sigma)\rangle + \langle h, h + \not{D}_A^+\sigma\rangle\right)}_{\widetilde{S}_{Boson}}.\end{aligned}$$

If $M = \mathbb{R}^4$, one has

$$K_\mu \left(\int_{\mathbb{R}^4} dx^4 \left(\langle b, \chi \rangle + \langle h, \xi \rangle \right) \right) = \int_{\mathbb{R}^4} dx^4 \langle D_\mu \chi, \chi \rangle = 0,$$

where $D_\mu := \iota_{e_\mu} d_A$. It follows that

$$K_\mu \widetilde{S} = \int_{\mathbb{R}^4} d^4x \, \partial_\mu \left(\langle b, \chi \rangle + \langle h, \xi \rangle \right) = 0.$$

In other words, $\widetilde{S}$ is $(\mathbb{R}^4)_{dR}$-invariant.

Thank you!

Quantization

Step 1 Apply the perturbative Batalin-Vilkovisky quantization for a fixed solution $\Phi \in \mathrm{Sol}(\mathcal{F})$.

Step 2 Globalize the perturbative series over the moduli space $\mathcal{M}(\mathcal{F})$.

Step 3 Integrate the globalized perturbative series over $\mathcal{M}(\mathcal{F})$.

To summarize, the first step defines a map

$$\langle \cdot \rangle_{\mathbf{pert}} : \mathbf{Obs}_{cl}^{\mathcal{F}}(M) \to \mathbb{R} \text{ or } \mathbb{C}$$

for each $\Phi \in \mathrm{Sol}(\mathcal{F})$ and a collection of tangent vectors at $[\Phi] \in \mathcal{M}(\mathcal{F})$.

The second step shows that, for an observable O of degree k, $\langle O \rangle_{\mathbf{pert}}$ defines a differential k-form over $\mathcal{M}(\mathcal{F})$ in good cases. The third step then defines a map

$$\begin{aligned} \langle \cdot \rangle_{\mathbf{nonpert}} : \ \mathbf{Obs}^{\mathcal{F}}_{cl}(M) &\to \mathbb{R} \text{ or } \mathbb{C} \\ O &\mapsto \int_{\mathcal{M}(\mathcal{F})} \langle O \rangle_{\mathbf{pert}}. \end{aligned}$$

We are now ready to describe the final step.

Step 4 Define the "quantum cohomology" of the theory based on the map $\langle \cdot \rangle_{\mathbf{nonpert}}$ and the embeddings of the open disk $B_r(0)$ into M.