# Spectra and stable homotopy theory

Lectures delivered by Michael Hopkins
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## Contents

# Introduction

Michael Hopkins taught a course (Math 256y) on spectra and stable homotopy theory at Harvard in Fall 2012. These are my "live-TEXed" notes from the course.

Conventions are as follows: Each lecture gets its own "chapter," and appears in the table of contents with the date.

Of course, these notes are not a faithful representation of the course, either in the mathematics itself or in the quotes, jokes, and philosophical musings; in particular, the errors are my fault. By the same token, any virtues in the notes are to be credited to the lecturer and not the scribe. Thanks to Emily Riehl and Arnav Tripathy for pointing out several mistakes.

Please email corrections to `amathew@college.harvard.edu`.

# Lecture 1
# 9/5

## §1 Administrative announcements

This is a class taught by Michael Hopkins, whose office in 508. Office hours are Wednesdays 3-4. After this class, we'll be in room 507. That makes me happy, because I'm taller than everyone in that class. There will be no class meeting on Monday, September 10, 2012. In local coordinates, Monday.

I've had a sense that people are interested in learning about more classical and computational topics. As in the course outline, I want to roughly focus this on a story about computations of the homotopy groups of spheres. I'll outline that story a little today. We're going to use this to segue into other classical topics in the subject.

I put down that 231br was a prerequisite for this class. We're going to use tools like the Steenrod algebra, Serre classes, etc. I don't want to lose anyone, since they weren't covered in the last semester of 231br.

## §2 Introduction

There are three classical theorems in homotopy theory.

1. The Hopf invariant one problem. What's that? Start with a map $f : S^{2n-1} \to S^n$; the **Hopf invariant** is defined by forming the mapping cone $X = S^n \cup_f e^{2n}$, and the cohomology $\widetilde{H}^*(X) = \mathbb{Z}$ when $* = n, 2n$ and zero otherwise. These are definite spheres, so we choose definite generators

$$x \in H^n(X), \quad y \in H^{2n}(X),$$

then we can look at the cup product structure. We have that $x^2 = H(f)y$ where $H(f) \in \mathbb{Z}$. That number $H(f)$ depends only on the *homotopy class* of $f$, and it's called the **Hopf invariant** of $f$.

The Hopf invariant was first introduced by Hopf, although he didn't call it the Hopf invariant, I think. He used it to show that the map $S^3 \to S^2$ that he constructed was not homotopic to the constant map. In other words, he used it to show that the homotopy groups of spheres were not the homology groups.

The problem of understanding for which $n$ does there exist a map $f : S^{2n-1} \to S^n$ with Hopf invariant one was a big one. It got kicked around a lot, and was regarded as one of the most important problems in algebraic topology until Adams solved it in the late 1950s.

**1.1 Theorem** (Adams)**.** *There exists a map $f : S^{2n-1} \to S^n$ with Hopf invariant one only when $n = 2, 4, 8$.*

This was related to many other questions. It was related to the existence of division algebra structures on euclidean space $\mathbb{R}^m$, and other things. Adams's papers explain the implications of his theorem. It's one of those papers whose title is the main theorem, it's called "On the non-existence of elements of Hopf invariant one."

Adams's original proof used the Adams spectral sequence and was quite illuminating. Later on Atiyah and Adams gave a much simpler proof, although I think a lot of people who work with this stuff think that the simpler proof doesn't really give you the reason.

2. The **vector field problem**. What is the maximum number of linearly independent vector fields on a sphere $S^{n-1}$? In a first course on algebraic topology, you prove that an even sphere has no nonvanishing vector fields (the hairy ball theorem). It's an application of the notion of a degree of a map. Odd spheres are harder. $S^1$ has one, $S^3$ has three, $S^7$ has seven, but the remaining odd spheres are more mysterious.

   This problem was also solved by Adams.

   **1.2 Theorem** (Adams)**.** *$S^{n-1}$ has $\rho(n)-1$ linearly independent vector fields, but not $\rho(n)$. Here $\rho(n)$ is the **Radon-Hurwitz number**: if $n=(2a+1)2^b$ and $b=c+4d, d\in[0,3]$, then*
   $$\rho(n)=2^c+8d.$$

   Later we'll try to understand $\rho(n)$ better. Today it will play no role.

   We'll spend some time discussing how the vector field problem is solved. Adams solved it by studying $K$-theory, and it was one of the first applications. That'll be a unit in the class.

3. The third of these classical problems in algebraic topology was the **Kervaire invariant problem.** I'm going to say more about this from the point of view of homotopy theory in a little bit, but this is a problem that originates in differential topology. In which dimensions $n$ does there exist a smooth stably framed manifold with Kervaire invariant one?

   **1.3 Theorem** (Hill, Hopkins, Ravenel)**.** *Only when $n=2,6,14,30,62$, and possibly 126, can there exist such a manifold.*

   I won't say much about our solution to the problem, but I'd like to put it up as one of the classic three problems.

These were really the three long-lasting and hallmark questions in homotopy theory, and there were several reasons for wanting to know the answer. In its original incarnation, it had to do with detecting maps of spheres, although it connected to questions about multiplications on $\mathbb{R}^n$ and other structures. The vector fields of spheres problem is "almost" a recreational problem. It might help to know that $S^5$ has 2 and not 3 linearly independent vector fields, but it's hard to know what to do with it. The Kervaire invariant problem occurred as a thorn in surgery theory and an issue people couldn't really get around, but people got good at avoiding it. It still plays a certain role, though.

I bring them up because they all get united in some basic questions about the homotopy groups of spheres, and the goal of the course is to flesh out this story as much as I can. How do these come together?

## §3 The EHP sequence

There's something called the **EHP sequence,** first invented by Whitehead but really developed by James. For this, we localize everything at 2. James showed:

**1.4 Theorem** (James)**.** *There is an exact sequence*

$$\pi_k(S^n) \to \pi_{k+1}(S^{n+1}) \to \pi_{k+1}(S^{2n+1}) \to \pi_{k-1}(S^n) \to \pi_k(S^{n+1}) \to \pi_k(S^{2n+1}) \to \dots.$$

We'd like to think of this as an exact couple. If you know about exact couples and spectral sequences,

$$\begin{tikzcd} \pi_k S^n \arrow[rr, "E"] & & \pi_{k+1} S^{n+1} \arrow[dl, "H"] \\ & \pi_{k+1} S^{2n+1} \arrow[ul, "P"] & \end{tikzcd}$$

Here are the maps:

- $E$ is the first letter for the German word for suspension. It suspends a map $f : S^k \to S^n$ to $\Sigma f : S^{k+1} \to S^{n+1}$.
- $H$ is a bit of a surprise. See below.

Let's note that when $k + 1 < 2n$, the groups $\pi_{k+1} S^{2n+1}$ are zero, and we get that $\pi_k S^n \to \pi_{k+1} S^{n+1}$ is an isomorphism. Note that this is precisely the Freudenthal suspension theorem. If you're going around calculating the homotopy groups of spheres, you're going to get the same thing a lot.

**1.5 Definition.** The **stable range** is when $k+1 \leq 2n$ and $\pi_k(S^n)$ is the same as the colimit $\varinjlim \pi_{k+j}(S^{n+j})$. These colimits are called the **stable homotopy groups of spheres.**

So once you are in the stable range, the homotopy groups stabilize. That's one little lesson on which we'll expand on a great deal later in the course.

What happens when $\pi_{k+1} S^{2n+1}$ first appears and is nonzero? Let's take $k+1 = 2n+1$, so that $\pi_{2n+1}(S^{2n+1}) = \mathbb{Z}$. The exact sequence runs

$$\pi_{2n}(S^n) \xrightarrow{E} \pi_{2n+1}(S^{n+1}) \xrightarrow{H} \pi_{2n+1}(S^{2n+1}) \simeq \mathbb{Z}.$$

The surprising thing is that this is precisely the Hopf invariant.

**1.6 Theorem.** *The map* $\pi_{2n+1}(S^{n+1}) \to \pi_{2n+1}(S^{2n+1}) \simeq \mathbb{Z}$ *is precisely the Hopf invariant.*

In particular, the exact sequence of James generalizes the Hopf invariant! It's not at all obvious how the Hopf invariant would lead to an exact sequence involving stabilization. This is a wonderful surprise, and it starts to tell you how you might generalize the Hopf invariant to maps of spheres in different dimensions. It also tells you about the role that the Hopf invariant might play. Hopf invariant one is equivalent to:

1. $H : \pi_{2n+1}(S^{n+1}) \to \pi_{2n+1}(S^{2n+1})$ is a surjection.
2. It tells you when the map $P$ out of $\pi_{2n+1}(S^{2n+1})$ is zero.

The idea of using the EHP sequence is to somehow start the homotopy groups of $S^1$ plus a little extra and then inductively calculate the higher homotopy groups of spheres. I haven't explained how you might get organized to think this way—we'll talk about it in the next couple of lectures. But you want to imagine that it is a way to make conjectures (about this long exact sequence) that will help calculate the homotopy groups of spheres. The first real calculations of the homotopy groups of spheres were done by Toda, using this sequence. Understanding what carrying out what this calculational program requires will occupy us for parts of this course.

Let's now describe the other two maps in the $EHP$ sequence in this special dimensional case.

- $H$ is secretly the Hopf invariant (or a generalization thereof).
- The map $P : \pi_{2n+1}(S^{2n+1}) \to \pi_{2n-1}(S^n)$ has the property that 1 goes to the **Whitehead product** of $[\iota, \iota]$ (here $\iota \in \pi_n(S^n)$ is the identity). That's some magic element in $\pi_{2n-1}(S^n)$.

  Let's review what this is. Given a map $f : S^{a+1} \to S^n$ and $g : S^{b+1} \to S^n$, then we can combine them to get a map $S^{a+1} \vee S^{b+1} \to S^n$. There is a nice map $S^{a+b+1} \to S^{a+1} \vee S^{b+1}$ by looking at the cell decomposition of $S^{a+1} \times S^{b+1}$, which starts with $S^{a+1} \vee S^{b+1} \cup_p e^{a+b+2}$ where $p$ is an attaching map

  $$p : S^{a+b+1} \to S^{a+1} \vee S^{b+1};$$

  it's the attaching map for the next cell in the product. One can give a formula for it.

  So, given maps $f : S^{a+1} \to S^n$ and $g : S^{b+1} \to S^n$, we can form a new map

  $$S^{a+b+1} \xrightarrow{p} S^{a+1} \vee S^{b+1} \xrightarrow{f \vee g} S^n,$$

  which is the **Whitehead product** $[f, g]$ (which makes the homotopy groups of spheres into a graded Lie algebra). That's what this means.

Let's recapitulate where we got. We are imagining that we can use this $EHP$ sequence to calculate all the homotopy groups of spheres. What problems do we have when carrying this out? One is to understand what the image of the Hopf map in the EHP sequence; that's the Hopf invariant one problem. The next problem is, when the Hopf invariant one doesn't exist, to understand the image of $P : \pi_{2n+1}(S^{2n+1}) \to \pi_{2n-1}(S^n)$. There are two questions that arise.

1. For which $k$ is $[\iota, \iota] \in \pi_{2n-1}(S^n)$ in the image of $E^k$? How divisible is it by $E$? This is a natural problem if you try to turn this into an exact couple.

2. Is $[\iota, \iota]$ divisible by two? That is, does $[\iota, \iota]$ generate a summand of this group $\pi_{2n-1}(S^n)$?

These are two fundamental issues to deal with while carrying out this exact sequence.

Here are the theorems.

**1.7 Theorem** (Toda(?))**.** *$[\iota, \iota]$ is in the image of $E^k$ if and only if the $n-1$ (this might be $n$ instead) sphere has $k$ linearly independent vector fields.*

In particular, the vector field problem has an important role in understanding the $EHP$ sequence. This is a really important part of this course. We're going to see that it is the true meaning of the vector fields on spheres problem, and it explains why the solution works the way it does.

**1.8 Theorem** (To be stated later fully)**.** *The Whitehead square $[\iota, \iota]$ is divisible by 2 in $\pi_{2n-1}(S^n)$ under the following conditions:*

1. *When $n$ is even, this is a version of the Hopf invariant one problem (to be explained another time; it's because the Hopf invariant of $[\iota, \iota]$ is divisible by 2).*

2. *When $n$ is even, this is equivalent to the Kervaire invariant problem.*

Anyway, the organizational principle is that all these hallmark problems in algebraic topology are all aspects of the $EHP$ sequence. Another theme that I hope to cover in the course is the following. In the *meta*stable range, the $EHP$ sequence also gives information on the homotopy groups: it turns the metastable homotopy of the sphere into the stable homotopy of $\mathbb{RP}^\infty$. All these problems have a manifestation in terms of the stable homotopy groups of $\mathbb{RP}^\infty$. Moreover, this picture leads to an understanding of the image of $J$ in the $EHP$ sequence, which might appear later in this course.

# Lecture 2
# 9/7

(Reminder: no class on Monday.)

So, last time I gave this overview of what I want to do in the course, relating these computational questions about the homotopy groups of spheres to classical problems in algebraic topology. One of our goals is to really flesh out that relationship. Today, and on Wednesday, I'd like to spell out how James constructed this EHP sequence. We'll play around with various aspects of it for a while.

## §1 Suspension and loops

Let's recall what the EHP sequence is. It is a long exact sequence (2-locally)

$$\pi_k S^n \to \pi_{k+1} S^{n+1} \to \pi_{k+1} S^{2n+1} \to \dots,$$

which we'd like to derive from a *fibration sequence.* In other words, we'd like to construct a fibration $F \to X \to B$ whose long exact sequence in homotopy groups was exactly this. In order to do this, we'd need to get at least one of the maps here via spaces. This looks hard, because any map $S^{n+1} \to S^{2n+1}$ is nullhomotopic!

However, observe that

$$\pi_{k+1} S^{n+1} = \pi_k \Omega S^{n+1},$$

where $\Omega$ means "loop space:" the space of maps $\omega : [0,1] \to X$ with $\omega(0) = \omega(1) = *$. I assume you know this, but recall that there is another construction called $\Sigma$ ("suspension") of a pointed space. These functors (on the category of pointed spaces) are adjoint: we have

$$\mathrm{Hom}(A, \Omega X) = \mathrm{Hom}(\Sigma A, X),$$

which is true at the point-set level as well as at the homotopy level. So, observe that there is an isomorphism,

$$\mathrm{Hom}(\Sigma A, \Sigma A) \to \mathrm{Hom}(A, \Omega \Sigma A)$$

and the identity $\Sigma A \to \Sigma A$ corresponds to an adjunction map $\iota : A \to \Omega \Sigma A$. Here $\iota$ is a *canonical map*, and we even know what it is: $a \in A$ goes to the path $\gamma_a$ which at time $t$ is the point $(a, t) \in \Sigma A$. So we can get formulas, but maybe it's not so useful for now.

Let's take $A = S^n$, now. Then $\Sigma A = S^{n+1}$ and we have a natural map

$$S^n \to \Omega S^{n+1},$$

arising from the above induction. That induces a map

$$\pi_k(S^n) \to \pi_k(\Omega S^{n+1}) \simeq \pi_{k+1}(S^{n+1}).$$

I leave it to you to check that this map is precisely the suspension map $E$. In particular, one of the maps ($E$) in the EHP sequence comes from a map of spaces.

## §2 Homotopy fibers

Now we might guess that we can get the EHP sequence from taking the homotopy fiber of the map $S^n \to \Omega S^{n+1}$.

**2.1 Definition.** Given $X \to B$, we define the **homotopy fiber** to be the pull-back $X \times_B PB$: in other words, it is the space of pairs $(\gamma, x)$ such that $\gamma$ is a path in $B$ (starting from the basepoint) and $x \in X$, and such that $\gamma$ ends at the image of $x$.

In particular, if $F$ is the homotopy fiber of $X \to B$, we get a long exact sequence from the "fiber sequence" $F \to X \to B$. So anytime we run into a map in homotopy theory, we can get a long exact sequence. We might thus hope to study the homotopy fiber $F$ of $S^n \to \Omega S^{n+1}$, and we'd then get a long exact sequence

$$\pi_k F \to \pi_k(S^n) \to \pi_{k+1}(S^{n+1}) \to \pi_{k-1}(F) \to \dots.$$

That's not too bad, but remember that we want to show (for the EHP sequence) that $\pi_{k-1}F \simeq \pi_{k+1}S^{2n+1}$. This tells us that we expect that

$$F \simeq \Omega^2 S^{2n+1}.$$

That would give us the right homotopy groups. We could calculate the rather complicated homology of $\Omega^2 S^{2n+1}$ and the homology of $F$ (via the Serre spectral sequence), but it's tough to write down a map, and it's not how James did it. We will later see that this is *true*, but not a viable approach for understanding the EHP sequence.

## §3 Shifting the sequence

I also remind you that when you cook up a fibration sequence

$$F \to X \to B,$$

then we can extend it by forming the homotopy fiber of $F \to X$, and so forth: we extend the fiber sequence

$$\cdots \to \Omega X \to \Omega B \to F \to X \to B,$$

and the whole thing keeps continuing. We could get a long exact sequence out of any triple here, so we could get a long exact sequence in lots of different ways.

We're staring at the sequence

$$\Omega^2 S^{2n+1} \to S^n \to \Omega S^{n+1},$$

or at least, we hope to get that. But maybe the fiber sequence is a shift of something else. Maybe where we are is in a different place in the sequence, and instead maybe we have a map

$$\Omega S^{n+1} \to \Omega S^{2n+1}$$

whose fiber is $S^n$. That would give us the homotopy fiber sequence we desire.

That's what James did. James produced a very interesting map

$$\Omega S^{n+1} \to \Omega S^{2n+1},$$

and showed that the homotopy fiber of that had the mod 2 homology of $S^n$. From there, it was fairly easy to get the EHP sequence. I'll remind you of some of the apparatus that goes into that.

## §4 The James construction

Our first task that we have to solve though, is to produce the James map $\Omega S^{n+1} \to \Omega S^{2n+1}$ and then to say something about the homotopy fiber. This is the sort of story of the rest of this lecture, and Wednesday's lecture. It's to understand something about loop spaces and the rest of that. I was a student of Ioan James (I had two thesis advisors, Ioan James and Mark Mahowald). James came up with a way of describing the loop space of a suspension, and he told me about it.

Here's a bit of the history. Marston Morse, using Morse theory, understood the homology of $\Omega S^{n+1}$, $H_*(\Omega S^{n+1})$. Nowadays you can do it with the Serre spectral sequence. But you can also do it by using Morse theory to give a cell decomposition of $\Omega S^{n+1}$ with a cell decomposition in each multiple of $n$. That is,

$$\Omega S^{n+1} \simeq S^n \cup e^{2n} \cup e^{3n} \cup \dots .$$

E. Pitcher showed that the first attaching map $S^{2n-1} \to S^n$ is precisely the Whitehead product. Thus, what is the mapping cone of the Whitehead product look like?

**Question.** What does the mapping cone of the Whitehead product $[\iota, \iota] \in \pi_{2n-1}(S^n)$ look like?

We might as well calculate its cohomology *ring* structure, and thus the Hopf invariant. But we might also understand a little better what the geometry of this space $\Omega S^{n+1}$ is. Our goal, for reasons I haven't really motivated, is to produce an interesting map out of $\Omega S^{n+1}$. The better we can understand these attaching maps, the better our chance of making such a map. We can start by understanding the mapping cone of the Whitehead square.

If you look at what the space means, it gives you a good idea. What is the definition of the Whitehead product again? We're supposed to take the map

$$S^{2n-1} \to S^n \vee S^n \xrightarrow{\nabla} S^n,$$

where $S^{2n-1} \to S^n \vee S^n$ has the property that its mapping cone is $S^n \times S^n$ (that's the description of the Whitehead product in the previous lecture). In particular, the mapping cone of the Whitehead product is what you get by taking $S^n \times S^n$ (a torus) and folding two axes together (that is, crushing $S^n \vee S^n$ to $S^n$).

Anyway, we have

$$H^n(S^n \times S^n) = \mathbb{Z}a \oplus \mathbb{Z}b,$$

and $H^{2n}(S^n \times S^n) = ab$. Observe that $x \in H^n(S^n; \mathbb{Z})$ pulls back to $a + b$ in $S^n \times S^n$ and consequently $x^2$ goes to $(a + b)^2$. Now,

$$(a + b)^2 = a^2 + b^2 + ab + ba = ab + ba = (1 + (-1)^n)ab.$$

In particular, the Hopf invariant is,

$$H([\iota, \iota]) = \begin{cases} 0 & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases} .$$

That's something I claimed in the previous lecture, in relation to the question of dividing the Whitehead square by two.

The geometric description was a little cumbersome, but the algebra is easier. $S^n \cup_{[\iota,\iota]} e^{2n}$ is the quotient of all pairs $\alpha, \beta$ modulo the relation $(a, *) = a, (*, b) = b$. That's supposed to remind you of forming *formal products* of an element in the sphere with another element in the sphere, with the basepoint the identity. That inspired James to look at the rest of the whole free associative monoid on the sphere. In other words, Pitcher's result is the beginning of writing down the whole free associative monoid on the sphere.

Motivated by this, James produced for a pointed space $(X, *)$ a space $JX$ which is the free associative monoid on $X$ with $*$ the identity.

**2.2 Definition.** The **James construction** $JX$ is the free associative topological monoid on $X$ with $*$ the unit.

## §5 Relation with the loopspace on a suspension

The James construction turned out to be an extremely important construction. What James did was to relate this to the loopspace of a suspension of $X$. If we could make $\Omega\Sigma X$ into a monoid, then the adjunction map

$$X \to \Omega\Sigma X$$

would extend uniquely to a map of monoids

$$JX \to \Omega\Sigma X.$$

We'd like to say that $\Omega\Sigma X$ is a topological monoid under catenation of loops, and then the map (of topological monoids!) $JX \to \Omega\Sigma X$ is "formal." That's good, and we'll calculate the homology of both sides and show that it's a homology isomorphism.

Let's stop and think about this statement. The space $\Omega\Sigma X$ is a complicated space: it doesn't come to you with a cell decomposition. Once you have the Serre spectral sequence, you can say something about the homology, but it doesn't have much geometric context. However, $JX$ can be built by an explicit construction: a quotient of a disjoint union of products of $X$ modulo some explicit relations. You can use combinatorial methods to understand $JX$, and to produce maps from it. It gives you a lot of insight into what $\Omega\Sigma X$ looks like.

Back in the 50s when James did this, $JX$ looks like a free monoid, and you can manipulate it using the methods of combinatorial group theory (collecting words and so forth); many of the original means of analyzing this space used them. Once the basic theorems were proved, topologists found sneakier and quicker ways of proving them. I have a different motivation in mind, so I'll give a quicker proof than the more classical combinatorial ones, although there's still some value in going back and looking at them. There was an era when learning about $\Omega\Sigma X$ was learning about commutators, but now we handle it with different techniques.

My goal is to tell you what's wrong with the above sketched argument, and then to calculate the homology of $\Omega\Sigma X$.

## §6 Moore loops

The argument in the previous section is wrong. In producing $JX \to \Omega\Sigma X$, we have a problem: loop concatenation on $\Omega\Sigma X$ is *homotopy associative*, not literally associative on the nose. So $\Omega\Sigma X$ is not a topological monoid.

Given paths $f, g, h \in \Omega Y$ for a space $Y$ (say $\Sigma X$), and if we write $*$ for loop concatenation, then the path $f * g \in \Omega Y$ is defined as

$$(f * g)(t) = \begin{cases} f(2t) & 0 \leq t \leq 1/2 \\ g(2t-1) & 1/2 \leq t \leq 1 \end{cases}.$$

So in particular, $(f * g) * h$ and $f * (g * h)$ are basically the same path but traversed at different speeds. So the loop multiplication is not actually associative.

That looks like a problem, but fortunately it's only a minor problem. There are lots of ways of remedying this. Later on in the course we are going to study $n$-fold loop spaces, and in order to understand the relations this little trick I'm going to now introduce won't be enough. The idea is that a loop space is not just homotopy associative, but infinitely homotopy associative, and there are general strictification results.

But here there's a trick, due to Moore. We will define a space $\Omega_m X \subset \mathrm{Map}([0, \infty], X) \times (0, \infty)$.

**2.3 Definition.** $\Omega_m X$ (the space of **Moore loops**) consists of all pairs $(f, t)$ such that $f(0) = *$ and $f(s) = *$ for $s \geq t$.

This is just a fancy way of avoiding the rescaling involved in the previous construction. If you don't have to rescale, then the rescaling involved previously becomes unnecessary, and we can get associativity on the nose.

I leave it to you to produce a map $\Omega X \subset \Omega_m X$ which is a homotopy equivalence. I also leave the following to you:

**2.4 Proposition.** *$\Omega_m X$ is an associative topological monoid.*

(Given $(f_1, t_1)$ and $(f_2, t_2)$, you form their product by first running $f_1$ from 0 to $t_1$ and $f_2$ from $t_1$ to $t_1 + t_2$.) This is a great trick, and I think a really good technical trick that saved a lot of technical headaches. Every loop space is homotopy equivalent to a monoid.

In particular, we get a map

$$X \to \Omega X \to \Omega_m X,$$

and consequently a map *of monoids*

$$JX \to \Omega_m \Sigma X.$$

That's an important map.

Our goal is:

**2.5 Theorem.** *The map $JX \to \Omega_m \Sigma X$ is a homology isomorphism.*

Let's think a little about the homology of $JX$. Fix a field $k$, and write $H_*$ for homology with coefficients in $k$. The tensor product will mean over that field. The James construction is coming to us *filtered*: it's got the basepoint $*$, and then $X$, and then $J_2X$ (all words of length $\leq 2$), and then $J_3X$, and so forth. In general, $J_nX$ is a quotient of $X^n$, or more accurately a quotient of

$$* \sqcup X \sqcup X^2 \sqcup \cdots \sqcup X^n / \sim,$$

modulo an appropriate equivalence relation $\sim$. Anyway, it corresponds to words of length $\leq n$, and there is a filtration

$$* \subset X \subset J_2X \subset J_3X \subset \cdots \subset J_nX \subset \cdots \subset JX.$$

In general, we can write $J_nX$ as a quotient of $J_{n-1}X \times X$.

Anyway, you should try to work the following out:

**2.6 Theorem.** *$H_*(JX)$ is the tensor algebra $T(\widetilde{H}_*(X))$.*

To see this, you should use the Künneth theorem, since we have field coefficients.

# Lecture 3
# 9/12

## §1 Recap of the James construction

So, last time I described the James construction $JX$ (for $(X, *)$ a pointed space), which was the *free associative monoid on* $X$ with the basepoint as the unit. We could describe $JX = \bigsqcup_{n \geq 0} X^n / \sim$, modulo a suitable equivalence relation: $(x_1, \dots, x_n) = (x_1, \dots, \hat{x}_i, \dots x_n)$ with the $i$th coordinate removed if $x_i = *$. Now $JX$ is filtered, and let $J_nX$ denote $\bigsqcup_{i \leq n} X^n / \sim$, so words of length $\leq n$.

We had a filtration

$$J_0X \subset J_1X \subset J_2X \subset \dots,$$

where $J_nX / J_{n-1}X \simeq X^{\wedge n}$, as is easily seen. From this, it is fairly easy to calculate the homology of $JX$.

*Let's assume field coefficients throughout, so that* $H_*(X \times Y) \simeq H_*(X) \otimes H_*(Y)$. So let's see. The one thing is that $H_*(JX)$ is an *algebra,* as $JX$ is a monoid. It contains $\widetilde{H}_*(X)$; that is, there is a map

$$\widetilde{H}_*(X) \to H_*(JX),$$

via the map $X \to JX$. Let $T(V) = \bigoplus V^{\otimes i}$ denote the tensor algebra on a vector space $V$. This extends to a map

$$T(\widetilde{H}_*(X)) \to H_*(JX),$$

because $H_*(JX)$ is an algebra.

**3.1 Theorem.** *$T(\widetilde{H}_*(X)) \to H_*(JX)$ is an isomorphism.*

*Proof.* Let $V = \widetilde{H}_*(JX)$. We can filter $T(V)$ via subspaces $T_nV = \bigoplus_{i \leq n} V^{\otimes i}$. We get maps

$$T_nV \to H_*(J_nX),$$

which are compatible as $n$ varies. In fact, we have a diagram

$$\begin{tikzcd} T_{n-1}(V) \arrow[r] \arrow[d] & H_*(J_{n-1}V) \arrow[d] \\ T_n(V) \arrow[r] \arrow[d] & H_*(J_nX) \arrow[d] \\ V^{\otimes n} \arrow[r] & H_*(J_nX, J_{n-1}X) \simeq V^{\otimes n} \end{tikzcd},$$

and assuming a mild topology condition to get $H_*(J_nX, J_{n-1}X) \simeq \widetilde{H}_*(J_nX/J_{n-1}X) \simeq \widetilde{H}_*(X^{\wedge n})$.

Let's look at this diagram. We want to prove that $T_n(V) \to H_*(J_nX)$ is an isomorphism. We can prove this by induction on $n$; when $n = 1$ it is obvious. If we assume that $T_{n-1}(V) \to H_*(J_{n-1}X)$ is an isomorphism, and we know that the bottom map is an isomorphism, we can use the long exact sequence of a pair. Now use the long exact sequence in homology to work up on $n$ (observe that the map $H_*(J_nX) \to V^{\otimes n}$ is a surjection just by the diagram, so it's really a short exact sequence). ▲

**Remark.** Alternatively, one could run a spectral sequence argument on the filtered space $JX$, although it doesn't seem to make things easier. Note that we'll review spectral sequences in a couple of days, so till then we'll do a few arguments the long way.

## §2 The homology on $\Omega\Sigma X$

We have a canonical map

$$X \mapsto \Omega\Sigma X,$$

which gives a map

$$\widetilde{H}_*(X) \to H_*(\Omega\Sigma X),$$

and the target is an algebra, so that if $V = \widetilde{H}_*(X)$ as before, we get a map

$$T(V) \to H_*(\Omega\Sigma X).$$

**3.2 Theorem.** *The map $T(V) \to H_*(\Omega\Sigma X)$ above is an isomorphism.*

*Proof.* This would be relatively easy to do with the Serre spectral sequence for the path loop fibration. But I want to use this for a couple of different purposes. Let's study $\Omega\Sigma X$ by studying the fibration

$$\Omega\Sigma X \to P\Sigma X \to \Sigma X,$$

where $P\Sigma X$ is the space of (based) paths in $\Sigma X$. We could work out the homology of $\Omega\Sigma X$ from the homology of $\Sigma X$ and that of the (contractible) $P\Sigma X$.

But I'd like to extract a picture from this. Imagine the suspension of $X$: it splits into a *positive cone* $C_+X \subset \Sigma X$ and a *negative cone* $C_-X \subset \Sigma X$, which are (contractible) open sets whose intersection is homotopy equivalent to $X$: for instance, $C_+X \cap C_-X =$

$(-\epsilon, \epsilon) \times X$. Now one of the first things you do when you learn the Mayer-Vietoris sequence is to use this to calculate the homology of $\Sigma X$.

Consider the evaluation map (evaluation at 1)

$$p : P\Sigma X \to \Sigma X,$$

whose fiber is $\Omega\Sigma X$. Consider the inverse images $p^{-1}(C_+X)$ and $p^{-1}(C_-X)$. We write

$$P_\pm \Sigma X = p^{-1}(C_\pm X).$$

That leads to a diagram which we are going to study. We will study $H_*(\Omega\Sigma X)$ using the diagram

$$\begin{tikzcd} P_+ \cap P_- \arrow[r] \arrow[d] & P_+\Sigma X \arrow[d] \\ P_-\Sigma X \arrow[r] & P\Sigma X \end{tikzcd}.$$

Now we'll study the Mayer-Vietoris sequence for this diagram. Let's look at these spaces and try to understand their homotopy type.

Consider $P_+\Sigma X$: this consists of the space of paths $\gamma$ such that $\gamma(0) = *$ and $\gamma(1) \in C_+X$. The path starts at the basepoint, can go anywhere, but has to land in the positive cone. Note that $C_+X$ is contractible. In other words, we have a pullback

$$\begin{tikzcd} P_+\Sigma X \arrow[r] \arrow[d] & P\Sigma X \arrow[d] \\ C_+X \arrow[r] & \Sigma X \end{tikzcd},$$

and note that it is also a homotopy pull-back. Since $C_+X$ is contractible, we find that

$$P_+\Sigma X \simeq \Omega\Sigma X.$$

We could very easily provide a homotopy equivalence by taking a loop that extends in the positive cone and extending it on any homotopy that contracts the positive cone onto the base point. Similarly,

$$P_-\Sigma X \simeq \Omega\Sigma X.$$

The last thing to understand is $P_+\Sigma X \cap P_-\Sigma X$. To get this, we have a homotopy pullback

$$\begin{tikzcd} P_+ \cap P_- \arrow[r] \arrow[d] & P\Sigma X \arrow[d] \\ X \arrow[r] & \Sigma X \end{tikzcd}.$$

However, $X \to \Sigma X$ is nullhomotopic (it's the inclusion of the equator), so the homotopy pull-back is the same as one would have gotten by taking the constant map $X \to \Sigma X$. In particular, we get for the pull-back a homotopy equivalence

$$P_+ \cap P_- \simeq X \times \Omega\Sigma X.$$

Maybe you're not used to these ideas, so let me tell you what you'd have to do to actually prove this. Choose a homotopy $X \times [0,1] \xrightarrow{H} X$ from the inclusion to the constant map. We have a diagram

$$\begin{tikzcd} E \arrow[r] \arrow[d] & P\Sigma X \arrow[d] \\ X \times [0,1] \arrow[r, "H"] & \Sigma X \end{tikzcd},$$

where $E$ is the pull-back. We could include $X \times \{t\}$ in $X \times [0,1]$ for any $t$, a homotopy equivalence. In particular, the map $E_t \to E$ is a homotopy equivalence,

$$\begin{tikzcd} E_t \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & P\Sigma X \arrow[d] \\ X \times \{t\} \arrow[r] & X \times [0,1] \arrow[r, "H"] & \Sigma X \end{tikzcd},$$

and now you compare $E_t$ for $t = 0, 1$: these maps are respectively the inclusion $X \to \Sigma X$ and the constant map. In particular, the two $E_0, E_1$ are just $X \times \Omega\Sigma X$ and $P_+ \cap P_-$. I'm not using anything very deep, but it'd probably be a good idea to really try to understand this argument. I'm not using anything deep, but I'm using a pretty good fluency in the notion of a fibration.

OK, so now I want to put up something here. We've been studying the diagram

$$\begin{tikzcd} P_+ \cap P_- \arrow[r] \arrow[d] & P_+ \arrow[d] \\ P_- \arrow[r] & P\Sigma X \end{tikzcd},$$

and we can now write down what these spaces are homotopy equivalent to:

$$\begin{tikzcd} X \times \Omega\Sigma X \arrow[r] \arrow[d] & \Omega\Sigma X \arrow[d] \\ \Omega\Sigma X \arrow[r] & * \end{tikzcd}.$$

This is a homotopy pushout square.

Here's a subtlety: there's something to think through. I'm going to advertise this as an exercise. But this is an important little point, about principle bundles over suspensions. I went through this quickly, but there was a choice to be made, and I went through it sufficiently quickly that you probably didn't notice the choice. The nullhomotopies of $X \to \Sigma X$ involved running through either $C_+X$ or $C_-X$; let's choose $C_+X$. In other words, let's use the contractibility of $C_+X$ rather than that of $C_-X$.

**Exercise:** With the choice of $C_+$ contractibility, the horizontal map in the above diagram $(P_+ \cap P_-) \to P_+$

$$X \times \Omega\Sigma X \to \Omega\Sigma X$$

is the projection map, while the vertical map $P_+ \cap P_- \to P_-$

$$X \times \Omega\Sigma X \to \Omega\Sigma X$$

is the twisting map (where $X$ goes into $\Omega\Sigma X$ and then you multiply: $X \times \Omega\Sigma X \to \Omega\Sigma X \times \Omega\Sigma X \to \Omega\Sigma X$). This is something important to understand.

Finally, after all this, we'd like to study this diagram using the Mayer-Vietoris sequence in *reduced* homology. In particular, we get that the map

$$\widetilde{H}_*(X \times \Omega\Sigma X) \to \widetilde{H}_*(\Omega\Sigma X) \oplus \widetilde{H}_*(\Omega\Sigma X),$$

is an *isomorphism*, where one of the two maps $\widetilde{H}_*(X \times \Omega\Sigma X) \to \widetilde{H}_*(\Omega\Sigma X)$ come from projection and the other comes from the action map.

The projection map $\widetilde{H}_*(X \times \Omega\Sigma X) \to \widetilde{H}_*(\Omega\Sigma X)$ has a section (coming from the section $\Omega\Sigma X \to X \times \Omega\Sigma X$ from the basepoint). Consider a diagram

$$\begin{tikzcd}
0 \arrow[r] & \widetilde{H}_*(\Omega\Sigma X) \arrow[r] \arrow[d] & \widetilde{H}_*(\Omega\Sigma X) \arrow[d] \\
& \widetilde{H}_*(X \times \Omega\Sigma X) \arrow[r] \arrow[d] & \widetilde{H}_*(\Omega\Sigma X) \oplus \widetilde{H}_*(\Omega\Sigma X) \arrow[d] \\
0 \arrow[r] & C \arrow[r] & \widetilde{H}_*(\Omega\Sigma X)
\end{tikzcd}.$$

Note that $\widetilde{H}_*(X \times \Omega\Sigma X) \simeq \widetilde{H}_*(X) \oplus \widetilde{H}_*(\Omega\Sigma X) \oplus \left(\widetilde{H}_*(X) \otimes \widetilde{H}_*(\Omega\Sigma X)\right)$. In particular, the map $\widetilde{H}_*(\Omega\Sigma X) \to \widetilde{H}_*(X \times \Omega\Sigma X)$ is the inclusion of a factor, and we find that

$$C = \widetilde{H}_*(X) \oplus \left(\widetilde{H}_*(X) \otimes \widetilde{H}_*(\Omega\Sigma X)\right).$$

We find that that's isomorphic to $\widetilde{H}_*(\Omega\Sigma X)$. Moreover, the map

$$C \to \widetilde{H}_*(\Omega\Sigma X)$$

comes from both the inclusion and the twisting map.

This may not seem like much, but we find that $H_*(\Omega\Sigma X)$ is an algebra which has this recursive property that we've just seen. For instance, we could just do this: substitute the identity into itself! We could abstract upon this.

**3.3 Lemma.** *Let $A$ be an augmented algebra (with augmentation ideal $\overline{A}$) and consider a map $V \to \overline{A}$ with the property that*

$$V \oplus (V \otimes \overline{A}) \simeq \overline{A}.$$

*This implies that $A \simeq T(V)$.*

This lemma implies the result. We could prove this lemma purely algebraically, or substitute the identity into itself. We get that

$$V \oplus (V \otimes V) \oplus (V \otimes V \otimes \overline{A}) \simeq \overline{A},$$

and in the limit, we get the desired form of $A$.

The conclusion is that $H_*(\Omega\Sigma X)$ is the tensor algebra on $\widetilde{H}_*(X)$, as in the theorem. ▲

**3.4 Corollary.** *The map $JX \to \Omega\Sigma X$ is an isomorphism in homology.*

By the Hurewicz theorem, we find that the map $JX \to \Omega\Sigma X$ is a weak equivalence.

## §3 To be fixed later

I would like to finish today by deducing a different, topological statement, out of this. We calculated this without using very much. We got this identity and this recursive relation, but we never had to think about what anything really looked like. This is usually a clue that there is a more geometric statement that implies this. This is probably a general rule in math. There's a more fundamental geometric statement.

This geometric statement was originally proved by using this construction $JX$ and the equivalence $JX \simeq \Omega\Sigma X$. I'm going to deduce it from a diagram, a trick due to Ganea. Let's consider again the diagram

$$\begin{tikzcd} X \times \Omega\Sigma X \arrow[r] \arrow[d] & \Omega\Sigma X \arrow[d] \\ \Omega\Sigma X \arrow[r] & * \end{tikzcd} .$$

Consider the mapping cones of both maps. Since the map is a homotopy pushout, the mapping cones are homotopy equivalent.

The mapping cone $\mathrm{Cone}(X \times \Omega\Sigma X \overset{\pi_2}{\to} \Omega\Sigma X)$ is $\Sigma\Omega\Sigma X$. We'll deduce from this. Let $X_+ = X \sqcup *$ where $*$ is a new basepoint, so that $(X \times Y)_+ = X_+ \wedge Y$. The smash product commutes with taking mapping cones. The mapping cone of $X_+ \to S^0$ is $\Sigma X$. Now smash with $Y_+$ to get a new cofiber sequence,

$$X_+ \wedge Y_+ \to Y_+ \to \Sigma X \wedge Y_+.$$

So in particular, we have a cofiber sequence

$$(X \times Y)_+ \to Y_+ \to \Sigma X \wedge Y_+.$$

That implies that the cone of $X \times \Omega\Sigma X \to \Omega\Sigma X$ is homotopy equivalent to $\Sigma X \wedge (\Omega\Sigma X)_+$. In particular,

$$\Omega\Sigma X \simeq S^1 \wedge X \wedge (\Omega\Sigma X)_+.$$

I'm out of time, so let me just say: technical basepoint issues. That tells me that $X \wedge S^1$. Anyway, that's $X \wedge \Sigma(\Omega\Sigma X_+)$, and now we're in a position of substituting this identity.

**3.5 Corollary** (James splitting)**.** $\Sigma\Omega\Sigma X \simeq \Sigma \bigvee_{j=1}^{\infty} X^{\wedge j}$.

We're going to play a lot with this in the next class.

# Lecture 4
# 9/14

I keep digging and found some resources which I'll put up on the website on the weekend. I'll attend to it this weekend and hopefully put up a problem set.

We were trying to set up the EHP sequence and just consider that as a device for calculating the homotopy groups of spheres, but we'll primarily consider the questions that arise in doing so. Let me remind you what we did in the last class.

## §1 Recap

I looked at the path loop fibration

$$\Omega\Sigma X \to P\Sigma X \to \Sigma X = C_+X \cup_X C_-X,$$

where $C_+X, C_-X$ were contractible. We saw that the pull-back $P_+\Sigma X = P\Sigma X \times_{\Sigma X} C_+X$ was $\Omega\Sigma X$. We saw that there was a commutative homotopy pushout square

$$\begin{tikzcd} X \times \Omega\Sigma X \arrow[r] \arrow[d] & \Omega\Sigma X \arrow[d] \\ \Omega\Sigma X \arrow[r] & * \end{tikzcd}$$

and there's something like this for general principal bundles. The two maps $X \times \Omega\Sigma X \to \Omega\Sigma X$ came from projection and a map we called $\tau$. We used this diagram in a couple of ways:

1. First, we used it to calculate $H_*(\Omega\Sigma X)$.
2. Next, we used it to analyze the homotopy type. Consider the mapping cones of both rows, we get a map

$$\Sigma X_+ \wedge \Omega\Sigma X \to \Sigma\Omega\Sigma X$$

of mapping cones, which is an equivalence, since the diagram was homotopy cocartesian. Since $\Sigma X_+ \wedge \Omega\Sigma X = X_+ \wedge \Sigma\Omega\Sigma X$, we iterate this over and over and get the **James splitting**

$$\Sigma\Omega\Sigma X \simeq \Sigma \bigvee_{n\geq 1} X^{\wedge n}.$$

There are some good things about the way I did this and some bad things. In the problem set, I'm going to ask you a lot of questions about this—I'll give you a flavor of them in a minute. However, I haven't been very explicit about what these maps are. There's some work to unpack everything.

## §2 James-Hopf maps

We understand what $\Sigma\Omega\Sigma X \to \Sigma \bigvee_{n\geq 1} X^{\wedge n}$ does in homology.

**4.1 Definition.** The **James-Hopf maps** (which generalize the Hopf invariant) come from taking this splitting

$$\Sigma\Omega\Sigma X \to \Sigma \bigvee_{n\geq 1} X^{\wedge n} \xrightarrow{p} \Sigma X^{\wedge n}$$

and then taking the adjoint map

$$\Omega\Sigma X \to \Omega\Sigma X^{\wedge n},$$

which is called the James-Hopf map.

It's related to the Hopf invariant when $n = 2$ and $X$ a sphere. On Monday I'll start to explain how that works, but you might puzzle it out yourself.

**4.2 Example.** Here is an example of a little exercise. I plan to put this on the imaginary problem set with a little coaching, but it's worth thinking about even at this point. Let's try to understand something about this map. Consider the map

$$\Omega\Sigma X \to \Omega\Sigma X \wedge X.$$

Recall that $[A, \Omega\Sigma X] = [\Sigma A, \Sigma X]$. Moreover, $[A, \Omega\Sigma X \wedge X] = [\Sigma A, \Sigma X \wedge X]$. Somehow, given a map

$$\Sigma A \to \Sigma X$$

we get a map

$$\Sigma A \to \Sigma X \wedge X.$$

There's some construction which produces this map, which we haven't thought of yet. An exercise would be to *describe* this construction, which is not a very easy thing to do given what I've told you so far.

**4.3 Example.** Here's another exercise. Given $f : \Sigma A \to \Sigma X$, define $H(f) : \Sigma A \to \Sigma X \wedge X$ by the construction of the James-Hopf. Show that $H(f) = 0$ iff the following diagram commutes:

Recall that $\Sigma A$ is a cogroup object in the homotopy category, via the "throttle" map $\psi_A : \Sigma A \to \Sigma A \vee \Sigma A$.

$$\begin{tikzcd}
\Sigma A \arrow[r, "f"] \arrow[d, "\psi_A"] & \Sigma X \arrow[d, "\psi_A"] \\
\Sigma A \vee \Sigma A \arrow[r, "f \vee f"] & \Sigma X \vee \Sigma X
\end{tikzcd}.$$

I'll either give a more guided exercise or come back and tell you about it. But note that something nontrivial has already happened with this James splitting: we get new maps out of old maps that we wouldn't have been able to think about before. This is part of the way of life that we're going to go over the next week.

You can think of the commutativity of the above diagram as the first obstruction to $f : \Sigma A \to \Sigma X$ being a suspension.

## §3 The induced map in homology

The rest of the talk today will be on the induced map in homology

$$H_*(\Omega\Sigma X) \to H_*(\Omega\Sigma X^{\wedge n}).$$

You might think this isn't a very hard problem. We saw that

$$H_*(\Omega\Sigma X) = T(\widetilde{H}_*(X))$$

and

$$H_*(\Omega\Sigma X^{\wedge n}) = T(\widetilde{H}_*(X)^{\otimes n}).$$

You might think that such a map is determined by where the algebra generators go, i.e. by what happens to $\widetilde{H}_*(X)$. Because, after all, these are the algebra generators for $H_*(\Omega\Sigma X)$. However, this might fail: you might not think that the map

$$H_*(\Omega\Sigma X) \to H_*(\Omega\Sigma X^{\wedge n})$$

is an algebra map. In fact, it is not an algebra map. The map

$$\Omega\Sigma X \to \Omega\Sigma X^{\wedge n}$$

is **not a loop map**, so the map on homologies is not necessarily an algebra. The James-Hopf maps are maps between loop spaces which are *not* loop maps.

But that's OK. There's still something which we can do. We could look at the map in *cohomology*

$$H^*(\Omega\Sigma X^{\wedge n}) \to H^*(\Omega\Sigma X),$$

which is a *ring homomorphism:* that would be good, as we could leverage information in low dimensions to information in high dimensions. As such, we need to understand the ring structure in $H^*(\Omega\Sigma Y)$ for a space $Y$. We'd like to study the multiplicative structure via homology.

## §4 Coalgebras

If I have a space $A$, and I look at the diagonal map

$$A \to A \times A,$$

the Künneth formula and pull-back gives the ring structure $H^*(A) \otimes H^*(A) \to H^*(A)$. If I want to study the map on homology, I look at the induced map

$$H_*(A) \to H_*(A) \otimes H_*(A).$$

This makes homology into a **coalgebra.**

**4.4 Definition.** A **coalgebra** is a vector space $V$ with a comultiplication $V \to V \otimes V$ and a counit $V \xrightarrow{\epsilon} k$ which satisfy coassociativity and comultiplicativity.

The commutative diagrams you have to write down for a coalgebra are the opposite of the diagrams you have to write down for an algebra.

Note as such that the map

$$H_*(\Omega\Sigma X) \to H_*(\Omega\Sigma X^{\wedge n})$$

is a map of coalgebras. Any map of spaces gives me a map of coalgebras, and from that, we can usually work out the effect of the map in homology. I want to do this *explicitly* in the case $X = S^m$.

So I need to be able to understand the coalgebra structure here. What is the coproduct map

$$H_*(\Omega\Sigma X) \to H_*(\Omega\Sigma X)^{\otimes 2}?$$

The first fact is that the diagonal map $\Omega\Sigma X \xrightarrow{\Delta} \Omega\Sigma X$ *is a loop map.* Therefore, the coproduct map is a ring homomorphism. And hence, you might think it's determined by what it does on $\widetilde{H}_*(X)$, the generators. In fact, it is determined by what it does on the generators. How can we figure out what it does on the algebra generators? That's kind of easy too, because this diagram commutes:

$$\begin{tikzcd} X \arrow[r] \arrow[d] & X \times X \arrow[d] \\ \Omega\Sigma X \arrow[r] & \Omega\Sigma X \times \Omega\Sigma X \end{tikzcd},$$

and in particular the effect on the generators is given by the coproduct on $H_*(X)$ itself. If we knew the cohomology *ring* of $X$, we could determine the coproduct on $H_*(X)$, and then we could determine everything for $\Omega\Sigma X$.

**4.5 Example.** Let $X = S^n$. I'll break it into even and odd in a moment. Then

$$\widetilde{H}_*(X) \simeq k$$

in degree $* = n$ and zero otherwise (where we use coefficients in the field $k$). Let's call $x_n \in \widetilde{H}_*(S^n)$ the generator. The coproduct in $x_n$ has to be

$$x_n \mapsto x_n \otimes 1 + 1 \otimes x_n,$$

because that's the only possibility. In fact, $x_n$ maps to something in the group $\widetilde{H}_n(S^n \times S^n)$, which is free on $x_n \otimes 1, 1 \otimes x_n$, and after projecting on each factor we have to get $x_n$ again. Something more general happens in a space which is a suspension: the coproduct is always of this form.

We have

$$H_*(\Omega\Sigma S^n) = T(\widetilde{H}_*(S^n)) = k[x_n].$$

It's not actually commutative if I think about graded commutativity. If I think of $n$ odd, then the ring is not graded commutative.

Now the coproduct is

$$x_n \mapsto x_n \otimes 1 + 1 \otimes x_n,$$

and consequently

$$x_n^k \mapsto (x_n \otimes 1 + 1 \otimes x_n)^k.$$

I'm supposed to expand this out. You might think that that's the sum $\sum_{i+j=n} \binom{n}{j} x_n^i \otimes x_n^j$, but that's not necessarily true. For instance,

$$(x_n \otimes 1 + 1 \otimes x_n)^2 = x_n^2 \otimes 1 + 1 \otimes x_n^2 + (1 \otimes x_n)(x_n \otimes 1) + (x_n \otimes 1)(1 \otimes x_n),$$

and there is a sign trick that happens: we get

$$(x_n \otimes 1 + 1 \otimes x_n)^2 = x_n^2 \otimes 1 + (1 + (-1)^n)(x_n \otimes x_n) + 1 \otimes x_n^2.$$

If you want to avoid the sign issue, you can either work with $n$ even or take $k = \mathbb{F}_2$.

For the heck of it, let's suppose $n$ is even. Then $H_*(\Omega S^{n+1})$ is a polynomial algebra $k[x_n]$ with

$$x_n^k \mapsto \sum_{i+j=k} \binom{k}{i} x_n^i \otimes x_n^j = (x_n \otimes 1 + 1 \otimes x_n)^k.$$

When $n$ is even, $2n$ is also even, and we want to focus on the first James-Hopf map, that is the map

$$\Omega S^{n+1} \to \Omega S^{2n+1}.$$

We have $H_*(\Omega S^{n+1}) \simeq k[x_n]$ and $H_*(\Omega S^{2n+1}) = k[y_{2n}]$. Observe that

$$x_n \mapsto 0, \quad x_n^2 \mapsto y_{2n}$$

because the following diagram commutes:

$$\begin{tikzcd} S^n \times S^n \arrow[r] \arrow[d] & S^n \wedge S^n \arrow[d] \\ \Omega S^{2n+1} \arrow[r] & \Omega\Sigma S^n \wedge S^n. \end{tikzcd}$$

In general, for $X$ a space, we have a commutative square:

$$\begin{tikzcd} X^k \arrow[r] \arrow[d] & X^{\wedge k} \arrow[d] \\ \Omega\Sigma X \arrow[r] & \Omega\Sigma X^{\wedge k} \end{tikzcd},$$

which is easy to deduce. Taking higher iterated maps is much more complicated.

For example, we find

$$x_n^3 \mapsto 0 \in H_{3n}(\Omega S^{2n+1}) = 0.$$

However,

$$x_n^4 \mapsto \lambda y_{2n}^2.$$

In order to figure out what $\lambda$ is, we figure out the comultiplication. Under the comultiplication,

$$x_n^4 \mapsto x_n^4 \otimes 1 + 4x_n^3 \otimes x_n + 6x_n^2 \otimes x_n^2 + 4x_n \otimes x_n^3 + 1 \otimes x_n^4.$$

This map goes to

$$\lambda y_{2n}^2 \otimes 1 + 6y_{2n} \otimes y_{2n} + \lambda \otimes y_{2n}^2.$$

However, the coproduct $\Delta(\lambda y_{2n}^2)$ can be expanded to

$$\Delta(\lambda y_{2n}^2) = \lambda y_{2n}^2 \otimes 1 + 2\lambda y_{2n} \otimes y_{2n} + \lambda(1 \otimes y_{2n}^2),$$

and from this we get

$$\boxed{\lambda = 3.}$$

**4.6 Example.** Let's now try an arbitrary element $x_n^{2k}$, which has to go to $\lambda y_{2n}^k$, and our goal is to determine $\lambda$. The strategy is to go all the way to the $k$-fold comultiplication. That's going to go to (this is a little confusing for the coalgebra)

$$\Delta^k(x_n^k) = (x_n \otimes 1 \cdots \otimes 1 + 1 \otimes x_n \otimes 1 \otimes \cdots \otimes 1 + \dots)^k.$$

That turns out to be the better thing to look at. I want to wrap this up: there are all kinds of monomials in here, but the interesting ones to compare are where $x_n$ occurs squared and $y_{2n}$ does. So let's compare the coefficient of

$$x_n^2 \otimes x_n^2 \cdots \otimes x_n^2, \quad y_{2n} \otimes \cdots \otimes y_{2n}.$$

The first thing is $\binom{2k}{2\ 2 \dots 2}$ (where there are $k$ two's), so

$$\frac{(2k)!}{2^k}.$$

The coefficient on the other side is

$$\lambda \binom{k}{1\ 1 \dots\ 1} = \lambda k!,$$

so that

$$\lambda = \frac{(2k)!}{2^k k!} = 1 \times 3 \times \cdots \times (2k-1).$$

The important point for us is that $\lambda$ is *odd.* Next class, we will exploit that and finally establish the fibration sequence defining the EHP sequence.

# Lecture 5 9/17

I struggled today: there are two thorny topics to get through. I thought I had a nice way of avoiding the Serre ss in one, but I don't. In any event, I'm going to need to use spectral sequences throughout the course. So I'll explain what the issue is, and give you a quick overview of the Serre spectral sequence. We're going to need that technique in the future.

## §1 Recap

OK, so let me remind you where we are, and the next thing we're proving. I produced this map

$$\Omega\Sigma X \to \Omega\Sigma X \wedge X,$$

and a lot of the fun's going to begin when we start to analyze this map, called the **James-Hopf map.** We were particularly interested in it when $X$ was a sphere $S^n$. In the last lecture, we calculated the homology of both sides, and showed that this was a decent map. I emphasized the case when $n$ was even.

1. $H_*(\Omega\Sigma S^n) \simeq k[x_n]$, $H_*(\Omega\Sigma S^{2n}) = k[y_{2n}]$.
2. The map
$$k[x_n] \to k[y_{2n}]$$
is not a ring homomorphism, but we saw that $x_n^j \mapsto 0$ if $j$ is odd and
$$x_n^{2j} \mapsto (\text{odd}) y_{2n}^j.$$

## §2 Goals

Let $F$ be the homotopy fiber of $\Omega\Sigma S^n \to \Omega\Sigma S^{2n}$. The adjunction map $S^n \to \Omega\Sigma S^n$ factors through $F$. We want to show that the map

$$H_*(S^n) \to H_*(F)$$

is an isomorphism, *whenever we are localized at 2*. By the mod 2 Hurewicz theorem, we find that there is an equivalence

$$\pi_*(S^n)_{(2)} \simeq \pi_*(F)_{(2)}.$$

Even if I could trick this out and prove the result without the Serre ss, most people (including me) prove the mod $\mathcal{C}$ Hurewicz theorem using the Serre ss. Apparently tom Dieck does these theorems without the Serre ss.

If you don't mind, maybe you'll humor me and I'll take you through a little of the analysis of this fibration and why you really need spectral sequences.

Let's consider something more general. Suppose we have a fibration $X \to B$ and a space $F \to X$. Consider two situations:

1. This is a fiber sequence $F \to X \to B$ and $B$ is connected. We're also going to suppose that we have a map $H_*(X) \to H_*(F)$ which splits the map $H_*(F) \to H_*(X)$. The conclusion is that the composite

   $$H_*(X) \xrightarrow{\Delta} H_*(X) \otimes H_*(X) \to H_*(B) \otimes H_*(F)$$

   is an isomorphism.

   This is a typical result you might prove with the Serre ss. We're not quite in this situation with the James-Hopf maps, but we're almost are, since we don't yet know that $S^n$ is the homotopy fiber of the James-Hopf map $\Omega\Sigma S^n \to \Omega\Sigma S^{2n}$.

   Let's try to prove this.

   (a) Suppose first $X \simeq F \times B$. This should be easy, and it is: it's a matter of pure algebra. It's easiest to state and prove this in the language of cohomology rather than cohomology. So let's work in cohomology, since products are easier to think about than coproducts. I have a vector space map

   $$p : H^*(F) \to H^*(F \times B) \simeq H^*(F) \otimes H^*(B)$$

   by the Künneth formula, and I have a ring homomorphism

   $$H^*(B) \to H^*(F \times B), \quad a \mapsto 1 \otimes a.$$

   We know that $H^0(B) = k$ and that the first map is a splitting, so that the composite

   $$H^*(F) \xrightarrow{p} H^*(F) \otimes H^*(B) \to H^*(F \times *)$$

   is the identity. So in particular,

   $$p(x) = x \otimes 1 + \sum x_i' \otimes x_i'', \quad \dim x_i'' > 0.$$

I want to conclude that the map

$$H^*(F) \otimes H^*(B) \to H^*(F) \otimes H^*(B), \quad x \otimes b \mapsto x \otimes b + \sum x_i' \otimes x_i'' b$$

is an isomorphism. But that's pretty easy: it's almost the identity map. If we filter $H^*(B)$ by degree, then this map is an isomorphism (in fact, the identity) on the associated graded. I just want to emphasize that I'm doing something here where there is a filtration on the cohomology of $B$. So that's fairly straightforward.

(b) ??

2. Here's the other situation. The situation we're actually in is:

**5.1 Theorem.** *We have a map $f : X \to B$ with homotopy fiber $F_f$, and with $B$ simply connected. We also have a map $F \to X$ whose composite with $f$ is null, which gives us a map*

$$F \to F_f$$

*We have a map of vector spaces*

$$H_*(X) \to H_*(F)$$

*splitting the map $H_*(F) \to H_*(X)$. Also, we have that the map $H_*(X) \to H_*(X) \otimes H_*(X) \to H_*(F) \otimes H_*(B)$ is an isomorphism. The conclusion is that*

$$H_*(F) \to H_*(F_f)$$

*is an isomorphism.*

It seems we have to use the Serre ss here. This is the thing I want, and that's the situation we're in with the James-Hopf maps. Once we've done this, we will have established the EHP sequence, a 2-local fiber sequence

$$S^n \to \Omega S^{n+1} \to \Omega S^{2n+1}.$$

*Proof.* The first place in an algebraic topology course where you encounter a spectral sequence but are not told about it is in the cellular chain complex. Let $B$ be a CW complex. Then you introduce the cellular chains on $B$. We have

$$C_n^{\text{cell}}(B) = H_n(B^{(n)}, B^{(n-1)}),$$

and we have a differential

$$H_n(B^{(n)}, B^{(n-1)}) \to H_{n-1}(B^{(n-1)}) \to H_{n-1}(B^{(n-1)}, B^{(n-2)}),$$

and you prove (almost axiomatically) that the homology of $B$ is the homology of this cellular complex. All you need is excision, the calculation of the homology of spheres, and the long exact sequence of a pair. In fact, this chain complex, it's not always pointed out, is a bit of a funny one. There isn't a natural way to relate this to the singular chain complex. If you know about the AHSS, you'll know that there can't be a functorial relationship between cellular and singular chains. It's a bit strange: you use this in the first semester to calculate, but it's relating to homology in a special case. It's a special case of a spectral sequence.

I want to try the same thing. Consider a Serre fibration $X \to B$, and suppose that $B$ is a CW complex. I'm reminding you of what the quickest way to explain the Serre ss. I also know that, from lecturing on it, it's not the quickest way of constructing it actually. This is just a quick way to think about it. So for instance, the Serre ss doesn't really require $B$ to be a CW complex, and it's easier to construct in terms of singular than cellular homology.

What I want to do is to look at the $n$-skeleton $B^{(n)}$ and form the pull-back,

$$\begin{tikzcd} X^{[n]} \arrow[r] \arrow[d] & X \arrow[d] \\ B^{(n)} \arrow[r] & B \end{tikzcd},$$

where $X^{[n]}$ is not to be confused with the $n$-skeleton of $X$. Let's try to use this method for calculating the homology of $B$ and use that to calculate the homology of $X$. We need to understand

$$H_*(X^{[n]}, X^{[n-1]})$$

to start with, which isn't so bad: it's a pull-back of the pair $(B^{(n)}, B^{(n-1)})$. Sitting over it I have $(X^{[n]}, X^{[n-1]})$. By excision or relative homeomorphism, the homology of $(B^{(n)}, B^{(n-1)})$ is the same as the disjoint union of a bunch of pairs $(D^n, S^{n-1})$, one for each $n$-cell of $B$. There's a map of pairs

$$\bigsqcup (D^n, S^{n-1}) \to (B^{(n)}, B^{(n-1)})$$

which is an isomorphism of homology. Pulling back the fibration over $X$, we get something homotopy equivalent to a direct sum of copies of $H_*(F \times D^n, F \times S^{n-1})$.

I can write this more functorially. The homology of $H_*(X^{[n]}, X^{[n-1]})$ is the set of cellular $n$-chains in $B$, tensored with the homology of $F$. I can write that more canonically as

$$H_*(X^{[n]}, X^{[n-1]}) \simeq C_n^{\text{cell}}(B) \otimes H_*(F),$$

and the reason that this ends up being a bit of a hand-wavy approach is this fact which requires a tough proof: when $B$ is simply connected, the connecting homomorphism

$$H_*(X^{[n]}, X^{[n-1]}) \to H_*(X^{[n-1]}) \to H_*(X^{[n-1]}, X^{[n-2]})$$

is the cellular chain map of $C_n^{\text{cell}}(B)$ tensored with the identity. There's even a statement when $B$ is not simply connected. Then we get this long exact sequence for the triple

$$X^{[n-2]} \subset X^{[n-1]} \subset X^{[n]}$$

and from there we can calculate $H_*(X^{[n]}, X^{[n-2]})$. Inductively, we could try to calculate $H_*(X^{[n]}, X^{[n-k]})$ and in the end, what you're doing is working with the spectral sequence. That's one way of thinking about what a spectral sequence is.

Now I'm going to speed up and tell you how this works out. This will probably be very hard to follow if you've never seen spectral sequences before — I'm just going to get through it. So what's a spectral sequence? A spectral sequence is a sequence of groups

$$(E_r, d_r)$$

where $E_r$ is an abelian group (usually bigraded) and $d_r : E_r \to E_r$ is a differential. There are isomorphisms

$$E_{r+1} \simeq H(E_r, d_r).$$

In the case of the Serre spectral sequence, the theorem about the Serre spectral sequence is that there is a spectral sequence (which organizes all these long exact sequences):

**5.2 Theorem** (Serre)**.** *Suppose $F \to X \to B$ is a Serre fibration with $B$ simply connected. Then there is a spectral sequence*

$$E_2^{p,q} \simeq H_p(B, H_q(F)) \implies H_{p+q}(X).$$

*In fact, $E_1^{p,q} \simeq C_*^{\mathrm{cell}}(B) \otimes H_*(F)$ and $d_1 = d^{\mathrm{cell}}$ when $B$ is a CW complex.*

Usually when you draw one of these spectral sequences, you just draw a bigraded group where in position $p, q$ you have $H_p(B; H_q(F))$. The $E_r$ term measures the homology $H_*(X^{[n]}, X^{[n-r]})$ (for all $n$). The sense in which I say "measures" is a more subtle aspect of a spectral sequence. There are a lot of ways in which people use spectral sequences, sometimes the $E_2$ page tells you the entire answer, or sometimes the evolution of the spectral sequence is of interest. In the latter case, you need to understand what the notion "measures" means. We'll come back to these aspects of spectral sequences later in the course.

A spectral sequence at this level of generality is only a marginally useful notion: usually there's a mechanism that produces the differentials $d_r$. You don't just get a sequence of random new terms every time. In the Serre spectral sequence, there are two gradings, and the differentials respect the grading. We have:

$$d_r : E_r^{p,q} \to E_r^{p-r,q+r-1}.$$

These groups are being related to $H_{p+q}(X)$, and the total degree of the differential is always $-1$, just like in the chain complex. One thing that even those of us who are making mistakes made sure of was that $d_r$ had degree $-1$.

What does this notion of "convergence" mean? If I'm in a given box (bigraded piece), after a while the groups stabilize because we are in a first quadrant spectral sequence. As you move through the spectral sequence, you're replacing each term by the kernel of $d$ mod the image of $d$, and each box gets replaced by a subquotient. Eventually they reach a stationary value, because the kernel of the zero map is the whole thing, and the image of the zero map is zero. In other words,

$$E_r^{p,q}, r \gg 0$$

is fixed, and we write that as $E_\infty^{p,q}$. The "convergence" now means that $H_n(X)$ has a *filtration* with associated graded $\bigoplus_{p+q=n} E_\infty^{p,q}$.

At any rate, let's look at how this works in the situation that we were in, in the last minute. I have a map $X \to B$ with homotopy fiber $F_f$ and a space I called $F$. We had a map $H_*(X) \to H_*(F)$ which was a section, and which gave us an isomorphism

$$H_*(X) \simeq H_*(F) \otimes H_*(B).$$

We wanted to conclude that $H_*(F) \to H_*(F_f)$ was an isomorphism. I think I'm not going to be able to spell out all the intracies of this in just five minutes. If you've played with the Serre ss, you'll understand this argument. If not, it's a good argument to get yourself acquainted with some of the formalities.

Notice that the map

$$H_*(X) \to H_*(B)$$

has to be surjective, because $H_*(X) \to H_*(F) \otimes H_*(B)$ was an isomorphism. This means that all the differentials out of $H_*(B)$ in the bottom row must be zero. Therefore, for instance, $H_1 X$ is formed from $H_1 B$ and $H_1 F_f$. Or in other words,

$$H_*(B) \otimes H_*(F_f) \simeq H_*(X) \quad \text{in degrees} \leq 1.$$

This is enough to imply that $H_1(F) \to H_1(F_f)$ is an isomorphism. That's enough to tell us that there are no differentials which can come out of the second row. And you can run the argument over and over with 1 replaced by 2, and keep going. I know that was very quick. If you're familiar with the Serre ss, you've probably seen that. But I wanted to contrast this with the other situation we were in. This was proved by induction on the homology degree of $F$, which is hard to access without the Serre ss. ▲

# Lecture 6
# 9/19

## §1 The EHPss

We still have a lot of analysis to do just to set up the EHP sequence, but I want to take a minute in today's class and explain how one uses it to inductively calculate the homotopy groups of spheres. The idea was to use this as a question-generating device and then we'd scurry off and do other things. We need to organize the EHP sequence.

The EHP sequence is a (2-localized) collection of long exact sequences

$$\dots \pi_k S^n \to \pi_{k+1} S^{n+1} \to \pi_{k+1} S^{2n+1} \to \dots;$$

there's a whole bunch of these, as $n$ varies over the spheres. These together form an *exact couple* and hence a spectral sequence. I want to get into what this spectral sequence looks like. There's a sort of recursive aspect to the spectral sequence that enables you to make a lot of caclulations. I'll do that today; it's rather nice.

Let's write this as an exact couple.

$$\begin{tikzcd} \pi_* S^n \arrow[rr] \arrow[dr] & & \pi_{*+1} S^{n+1} \arrow[dl] , \\ & \pi_* S^{2n+1} & \end{tikzcd}$$

where the "$E^1$" term is $\pi_* S^{2n+1}$ (the homotopy groups of odd spheres) and the abutment is the colimit of the maps on the "$D$ terms," $\varinjlim \pi_{n+k} S^n$, the stable homotopy groups of spheres. You don't really use this as a method of starting with the homotopy groups of odd spheres and calculating the stable homotopy groups of spheres. Instead it's the process of the spectral sequence which enables you to make calculations and which reveals a very beautiful story which we're going to get to.

Before we go forward, we have to make a bunch of decisions of how to draw these things on the page.

## §2 The spectral sequence for a double complex

Let's first talk about an easier type of spectral sequence, and that's the spectral sequence of a *double chain complex.* I'm going to suppose I have a chain complex which *starts*

$$\begin{tikzcd}
{} \arrow[d] & {} \arrow[d] & {} \arrow[d] \\
C_{20} \arrow[d] & C_{21} \arrow[l] \arrow[d] & C_{22} \arrow[l] \arrow[d] \\
C_{10} & C_{11} \arrow[l] & C_{12} \arrow[l] \\
C_{00} & C_{01} \arrow[l] & C_{02} \arrow[l]
\end{tikzcd}$$

which means that the maps are differentials and the maps square to zero. I'm not giving these names because that would be a drag. Out of this you make a *total chain complex.* It's like

$$C_{00} \leftarrow C_{10} \oplus C_{10} \leftarrow C_{20} \oplus C_{11} \oplus C_{02} \dots .$$

In general,

$$\mathrm{Tot}(C)_n = \bigoplus_{i+j=n} C_{ij}$$

and the differentials come from the vertical and horizontal differentials, *but you have to alternate the sign.*

Alright, now, out of a double chain complex there is a spectral sequence. You make a spectral sequence out of a double chain complex by *filtering it.* So we would take our double chain complex $C_{ij}$. We'd just take the first $n$ columns and cut it off there; there's a sub double complex. Given $C_{\bullet\bullet}$, we have

$$F_n C_{\bullet\bullet} \subset C_{\bullet\bullet}$$

where $F_n$ consists of the first $n$ columns. Then we have a filtration on $C_{\bullet\bullet}$ and $F_n/F_{n-1}$ is just the $n$th column, as a chain complex. That gives a long exact sequence of homology groups. For instance, we have an exact couple

$$\begin{tikzcd}
H_*(F_*) \arrow[rr] \arrow[dr] & & H_*(F_{*+1}) \arrow[dl] \\
& H_*(C_{\mathrm{col}}) &
\end{tikzcd},$$

and thus we get a spectral sequence starting (in $E_1$) with the homologies of the columns, and converging to $H_*(\mathrm{Tot}(C_{\bullet\bullet})$.

The spectral sequence of a double chain complex is a really good example to think about. The Serre ss can be constructed as an ss of a double chain complex. I want to talk about this because even if you don't know about spectral sequence, the process of the spectral sequence gives an algorithm for computing the homology of the total chain complex, and it's possible to understand that algorithm.

OK, so let's imagine that we know the $E_1$ page of this spectral sequence. That is, we know the homologies of each of the columns. Let's suppose we have an element in the homology of one of these columns, so an element in the $E_1$ page of the ss. Say $(n, k)$ dimensions. We can represent $x$ by an element $\overline{x}$ in $C_n$ which is a *cycle*, with respect to the *vertical* differential.

Over here, let's imagine the double chain complex itself.

What happens to $\overline{x}$ under the total differential? There's a vertical and a horizontal component. The horizontal component might be nonzero, though the vertical element is zero. That gives an element $d_1\overline{x}$ in the chain complex. Back here, with our friends running the spectral sequence, the element $d_1\overline{x}$ is a cycle in $C_{n-1,k}$, because these diagrams are supposed to commute. So $d_1\overline{x}$ represents a homology class, which I'll call $d_1 x$, in $H_k(C_{n-1,*})$. If that's not zero, that's a differential and we have to replace the group in the spectral sequence by the homology of this differential and we don't learn anything more about spectral sequences.

To go further, suppose that $d_1 x = 0 \in H_*(C_{n-1,*})$. That doesn't mean that $d_1\overline{x} = 0$, just that $d_1\overline{x}$ is a boundary. We can find a class $c_1$ whose vertical differential is $d_1\overline{x}$. So we can choose a $c_1 \in C_{n-1,k+1}$ such that the vertical differential of $c_1$ is the same as $d_1\overline{x}$. Now let's look at the total differential of $c_1$: it has vertical and horizontal components. The horizontal component $d^{\mathrm{hor}}(c_1)$ is a cycle, and it represents another homology class in $H_{k+1}(C_{n-2,*})$ which I'm going to call $d_2 x$.

So this is the basic process. There were some choices involved. I could have added to this any element which went to zero there, and that's exactly what happens in a spectral sequence. I'm going to stop with this story, but the idea is that one can continue this process over and over. So this is the *algorithm* that the spectral sequence is doing. You can learn about the great applications of the Serre ss, but those are mostly where the ss collapses. But if you want to deal with a ss where not much collapses, like the EHPss...

Note also that we could use this strategy to get a spectral sequence for computing the homology of $F_n C_{\bullet\bullet}$.

OK. So that's a bit of advice about learning spectral sequences. It's good if you get lost, a little bit—I recommend really thinking through this algorithm and understanding at which point are things well-defined. All those little questions are informative to think through.

## §3 Back to the EHPss

OK. So, the EHP sequence. We got one problem straight off the bat. And that is, I want to think of $\pi_* S^n$ as the *homology* of a chain complex. There's actually—well, we'll come back to that later—a way, but let's just imagine that we can do that. So

$\pi_* S^n = H_* F_n$ and these $F_n$ chain complexes sit inside each other

$$F_n \subset F_{n+1} \subset \dots$$

and $H_*(\varinjlim F_n) = \pi_*^s S^0$.

One problem is the index. $\pi_k S^n$ isn't the $k$th homology of some chain complex because then the map would have to raise degrees. Let's imagine

$$\pi_{n+k}(S^n) = H_k(F_n)$$

for a chain complex $F_n$. Then, if I were drawing it like this, we would get a spectral sequence where the $E_1$ term would be $H_*(F_n/F_{n-1})$ which would be the homotopy groups of an odd sphere. The $E_1$ term would be $\pi_{n+k}(S^{2n-1})$.

Let's make a chart and try to imagine what this spectral sequence looks like. In a given column, I'm supposed to write down the homotopy groups of odd spheres. In the first vertical column, we'd have (from bottom to top) $\pi_1 S^1$, $\pi_2 S^1, \dots,$, and then we'd get the homotopy groups of $S^3$, starting with $\pi_2 S^3$. The groups $\pi_{2k+1}(S^{2k+1})$ are on the diagonal. **Wait, is this right? There was some debate in class.**

Anyway, the first differential goes

$$\pi_{n+k}(S^{2n-1}) \to \pi_{n+k-2}(S^{2n-3}).$$

I have something messed up in the way I indexed this. Topologists rewrite the indices a little differently anyway. What's supposed to happen is that the $d_1$ is supposed to go horizontally. However you do this, there's a problem. It's not a bad problem, but if you're analyzing a spectral sequence like this and it's not going to collapse and there's really something going on, you want to be able to assess visually what the situation is. Given an element, I want to know what the possible differentials are on an element, and what the possible groups contributing to a given group. The problem with this indexing is that everything happens on a diagonal.

Topologists, starting with Adams, reindex these things. It's just a linear change of coordinates in the plane. We use *Adams indexing*. In the Adams indexing, you arrange things so that all the groups in a given *column* contribute to the same group that you're computing in the total complex. My point, which I'm going to be rushing through in the next few minutes, is to show you the recursive nature of the sequence. The differentials go over from one column to the next. This requires a "shearing" of the coordinate system. I'm going to rewrite the EHP ss in a way that works with Adams indexing, and we'll talk a little about what the spectral sequence is telling you.

What the spectral sequence tells you is that you can calculate the homotopy groups of a *given* sphere by *truncating* the spectral sequence at a given column. If you truncate the homotopy groups at the first $m$ columns, then—as in the $F_m$'s—you can compute the homotopy of $S^m$. You can already see that there's an opportunity for feeding the information back into itself.

What I want to do in the last ten minutes is to start going through a chart which you can find in Ravenel's green book. **Holds up a copy.** This is proof that the book used to be green, but if you buy it now it's going to be red. It's a red shift. In the free version that you download, it's on page 27. It's in the first chapter. Ravenel Adams indexes this.

**(to be added)**

One can use this to show:

$$\pi_*(S^{2n+1}) \otimes \mathbb{Q} = \begin{cases} 0 & \text{if } * > 2n+1 \\ \mathbb{Q} & \text{if } * = 2n+1 \end{cases}.$$

The whole spectral sequence can inductively shown to be basically zero after tensoring with $\mathbb{Q}$. On the other hand, when $* = 2n$, we get something a little more complicated. More generally,

$$\pi_*(S^{2n}) \otimes \mathbb{Q} = \begin{cases} \mathbb{Q} & \text{if } * = 2n, 4n-1 \\ 0 & \text{otherwise} \end{cases}.$$

We're going to pick this up much more carefully next time.

# Lecture 7
# 9/21

## §1 A fix

So, I want to continue going through the EHP sequence and how it's used to calculate. I want to explain this chart that's in Ravenel's book. First let me clear up something I got confused about in the last lecture. So I was imagining we were looking at a double chain complex $C_{\bullet\bullet}$, which I drew as if on a piece of graph theory. I filtered that, calling $F_pC$ the $p$th piece. We got an exact couple,

$$\begin{tikzcd} H_*(F_{p-1}(C)) \arrow[rr] & & H_*(F_pC) \arrow[dl] \\ & H_*(F_p/F_{p-1}) \arrow[ul] & \end{tikzcd},$$

and the thing $H_*(F_p/F_{p-1})$ is the $E_1$ term of the spectral sequence. I had said that $F_p/F_{p-1}$ is exactly the $p$th column $C_{p*}$, and that's true, but that column's not put in the degree you're seeing it. For instance, $C_{p0}$ has total degree zero. The $p$th homology of $F_p/F_{p-1}$ is contributed to by the group $C_{p0}$. So

$$H_*(F_p/F_{p-1}) \simeq H_*(C_{p*})$$

is true, but with a shift of degrees.

In fact,

$$H_k(F_p/F_{p-1}) \simeq H_{k+p}(C_{p,*}).$$

So that was my mistake—I forgot that there was a change in degrees, and if you make that you'll get the standard conventions in a spectral sequence.

## §2 The EHP sequence

I'm now going to follow Ravenel's indexing. So recall: we're taking the sequence of homotopy groups of spheres. We consider the exact couple

$$\begin{tikzcd} \pi_{p+k}(S^p) \arrow[rr] & & \pi_{p+1+k}(S^{p+1}) \arrow[dl] \\ & \pi_{p+1+k}(S^{2p+1}) \arrow[ul] & \end{tikzcd}$$

given by the EHP sequence (2-locally). The $E_1$ term of this spectral sequence is the homotopy groups of odd spheres. The $E_\infty$ page corresponds to the stable homotopy groups of spheres.

If you truncate this spectral sequence, you get a spectral sequence converging to the homotopy groups of any sphere.

Let's put this together in a chart, and I'm going to index it in the way homotopy theory people index it: I'm going to follow the Adams indexing convention rather than the Serre one.

We place the homotopy groups of the odd spheres in the rows. Let's start with the knowledge that $\pi_n(S^n) = \mathbb{Z}$.

| Abutment: $\pi_k^s(S^0)$ | | $\mathbb{Z}$ | | | | | |
|---|---|---|---|---|---|---|---|
| | $n$ | | | | | | |
| $S^1$ | 1 | $\mathbb{Z}\lambda_0$ | | | | | |
| $S^3$ | 2 | | $\mathbb{Z}$ | | | | |
| $S^5$ | 3 | | | $\mathbb{Z}$ | | | |
| $S^7$ | 4 | | | | $\mathbb{Z}$ | | |
| $S^9$ | 5 | | | | | $\mathbb{Z}$ | |
| $S^{11}$ | 6 | | | | | | $\mathbb{Z}$ |

Below the diagonal, everything is zero, and we don't know what the stuff above the diagonal is at this point. We will calculate it recursively. The truncated spectral sequence would go for instance, from $\pi_*S^1 \oplus \pi_*S^3 \oplus \pi_*S^5 \oplus \pi_*S^9$ and it would converge to $\pi_*S^4$. This sets up an amazing recursive relationship between the homotopy groups of spheres and lets you calculate very far.

It's much easier if I write down generators of groups than groups. So instead of writing down $\mathbb{Z}$'s, let's call the generators $\lambda_i$. So let's change this to:

| Abutment: $\pi_k^s(S^0)$ | | $\mathbb{Z}$ | | | | | |
|---|---|---|---|---|---|---|---|
| | $n$ | | | | | | |
| $S^1$ | 1 | $\lambda_0$ | 0 | 0 | 0 | 0 | 0 |
| $S^3$ | 2 | | $\lambda_1$ | | | | |
| $S^5$ | 3 | | | $\lambda_2$ | | | |
| $S^7$ | 4 | | | | $\lambda_3$ | | |
| $S^9$ | 5 | | | | | $\lambda_4$ | |
| $S^{11}$ | 6 | | | | | | $\lambda_5$ |

In the last class, we worked out the diagonal differentials $d_1$. I'll just remind you what that was. These are maps

$$\pi_{2n+1}(S^{2n+1}) \to \pi_{2n-1}(S^{2n-1})$$

which sends the generator to the Hopf invariant of the Whitehead product, i.e. 2 if $n$ is even and zero if $n$ is odd. So every other $d_1$ is multiplication by 2 on the $\lambda_i$. That is,

$$d_1(\lambda_2) = 2\lambda_1$$

and so forth. After this differential hits, we know that the *first* stable homotopy groups of spheres is $\mathbb{Z}/2$, and we even know something about $\pi_3(S^2)$. We learn the following:

1. $\pi_1^{st}(S^0) = \mathbb{Z}/2$.
2. $\pi_3(S^2) = \mathbb{Z}$, $\pi_4(S^3) = \mathbb{Z}/2$, and then we're in the stable range. (This we learn from the truncation of the spectral sequence.) This is a bit confusing. If we want to read of $\pi_*(S^3)$ from this spectral sequence, we take the $E_\infty$ page of the truncated spectral sequence, and look at the various columns.

So we've learned something. We know the next stable group. And that means we can continue writing down the spectral sequence. Let's try to start filling in holes here. We're using the spectral sequence to go back and calculate the $E_1$ term.

| Abutment: $\pi_k^s(S^0)$ | | $\mathbb{Z}$ | $\mathbb{Z}/2$ | | | | | |
|---|---|---|---|---|---|---|---|---|
| | $n$ | | | | | | | |
| $S^1$ | 1 | $\lambda_0$ | 0 | 0 | 0 | 0 | 0 | |
| $S^3$ | 2 | | $\lambda_1$ | $\mathbb{Z}/2$ | | | | |
| $S^5$ | 3 | | | $\lambda_2$ | $\mathbb{Z}/2$ | | | |
| $S^7$ | 4 | | | | $\lambda_3$ | $\mathbb{Z}/2$ | | |
| $S^9$ | 5 | | | | | $\lambda_4$ | $\mathbb{Z}/2$ | |
| $S^{11}$ | 6 | | | | | | $\lambda_5$ | $\mathbb{Z}/2$ |

Now we get a bunch of maps between $\mathbb{Z}/2$'s which are differentials $d_1$ in the EHPss. The standard name for the generator of $\pi_1^s(S^0) = \mathbb{Z}/2$ is $\eta$. What I'd like to say that the maps $\mathbb{Z}/2 \to \mathbb{Z}/2$ are zero. We run into a funny question.

**Question.** Let's say I have a map $\alpha : S^{n+k} \to S^n$, so $\alpha \in \pi_{n+k}(S^n)$. The set of maps between spheres of the same dimension is the integers. Given an integer $d$, we can do two things with $\alpha$:

1. Compose with the degree $d$ map $S^{n+k} \to S^{n+k}$.
2. Compose with the degree $d$ map $S^n \to S^n$.

You might think that those are the same. But they're not. The first map, by *definition of addition*, is $d\alpha$. The second map is not necessarily $d\alpha$. Let me just give you a simple example where this is not the case.

**7.1 Example.** Consider the Hopf map $H : S^3 \to S^2$. Compose with the degree $d$ map $S^2 \to S^2$. How do we make the diagram commute?

$$\begin{tikzcd} S^3 \arrow[r, "?"] \arrow[d, "H"] & S^3 \arrow[d, "H"] \\ S^2 \arrow[r, "d"] & S^2 \end{tikzcd}.$$

We might think that $S^3 \to S^3$ should be $d$. But that's not the case. Look at the mapping cones; we get an endomorphism of the mapping cone $\mathbb{CP}^2$. It sends the generator in degree two to $d$ times it, so it multiples the top generator by $d^2$. So the ? map should be multiplication by $d^2$!

So that raises a question:

**Question.** What does the degree $d$ map $S^n \to S^n$ (on the *wrong side*) do in the homotopy groups of spheres?

We need to take this information to leverage that into the differentials.

Anyway, the differentials

$$d_1 : \mathbb{Z}/2 \to \mathbb{Z}/2$$

in the portion of the spectral sequence thus drawn are all zero. I want to name these generators rather than labeling groups. There's something good to do here, but let's think about what the classes are. What do I know about it from this spectral sequence. Let's call it $g$.

| Abutment: $\pi_k^s(S^0)$ | | $\mathbb{Z}$ | $\mathbb{Z}/2$ | | | | | |
|---|---|---|---|---|---|---|---|---|
| | $n$ | | | | | | | |
| $S^1$ | 1 | $\lambda_0$ | 0 | 0 | 0 | 0 | 0 | |
| $S^3$ | 2 | | $\lambda_1$ | $g$ | | | | |
| $S^5$ | 3 | | | $\lambda_2$ | $\mathbb{Z}/2$ | | | |
| $S^7$ | 4 | | | | $\lambda_3$ | $\mathbb{Z}/2$ | | |
| $S^9$ | 5 | | | | | $\lambda_4$ | $\mathbb{Z}/2$ | |
| $S^{11}$ | 6 | | | | | | $\lambda_5$ | $\mathbb{Z}/2$ |

$g$ corresponds to some element in $\pi_2^{st}(S^0)$. The first place I see it, though, is on the 2-sphere. So it's actually in the image of $\pi_4(S^2)$. It's not in the image of $\pi_3(S^1) = 0$. We give a name to that. We say that $S^2$ is the **sphere of origin** of $g$. That means that it comes from the sphere $S^2$ but from no smaller sphere.

Let's imagine we have some element in $\pi_{n+k+1}(S^{n+1})$ and we have the maps

$$\pi_{n+k}(S^n) \to \pi_{n+k+1}(S^{n+1}) \to \pi_k^{st}(S^0).$$

Suppose some element $\overline{x} \in \pi_{n+k+1}(S^{n+1})$ and $n+1$ is the sphere of origin of the stabilization. That means that it's not in the image of $\pi_{n+k}(S^n)$ and in particular it has a **nontrivial Hopf invariant** in $\pi_{n+k+1}(S^{2n+1})$. That's what not desuspending further means.

So what is the Hopf invariant of the class in $\pi_4(S^2)$ that we just called $g$? It is nonzero, since $g$ doesn't desuspend. The Hopf invariant is just ... There's a map from $\pi_4(S^2) \to \pi_4(S^3)$. That comes from the spectral sequence. An element in $\pi_4(S^2)$ defines an element of $\pi_2^s$ and you look at the preimage in the $E_1$ page?? That's also the Hopf invariant.

We're calling the generator of $\pi_1$ $\lambda_1$ or $\lambda_1$ mod 2. In fact you're making an algebra called the $\lambda_1$ algebra. So we write for $g$ $\lambda_1\lambda_1$. When I write down $\lambda_1\lambda_1$, the first thing writes down the sphere of origin and the second thing is the Hopf invariant. I know this is hard to understand. (*At least, the scribe of these notes is completely confused.*) In general, for a sequence $I$,

$$\lambda_k\lambda_I$$

means an element with sphere of origin the $k+1$ sphere and Hopf invariant $\lambda_I$. So $\lambda_1\lambda_1$ means that the sphere of origin is the 2-sphere and the Hopf invariant is $\lambda_1$.

| Abutment: $\pi_k^s(S^0)$ | | $\mathbb{Z}$ | $\mathbb{Z}/2$ | | | | | |
|---|---|---|---|---|---|---|---|---|
| | $n$ | | | | | | | |
| $S^1$ | 1 | $\lambda_0$ | 0 | 0 | 0 | 0 | 0 | |
| $S^3$ | 2 | | $\lambda_1$ | $\lambda_1\lambda_1$ | | | | |
| $S^5$ | 3 | | | $\lambda_2$ | $\lambda_2\lambda_1$ | | | |
| $S^7$ | 4 | | | | $\lambda_3$ | $\lambda_3\lambda_1$ | | |
| $S^9$ | 5 | | | | | $\lambda_4$ | $\lambda_4\lambda_1$ | |
| $S^{11}$ | 6 | | | | | | $\lambda_5$ | $\lambda_5\lambda_1$ |

The generator of $\pi_2^s(S^0)$ has sphere of origin $S^2$ and Hopf invariant $\lambda_1$. Anyway, when we continue this, and assume that the differentials are zero, we learn

$$\pi_2^{st} = \mathbb{Z}/2.$$

And what about the generator? We learn that the sphere of origin is the 2-sphere again. The Hopf invariant is the element we're calling $\lambda_1\lambda_1$. So the next elements are $\lambda_1\lambda_1\lambda_1, \lambda_2\lambda_1\lambda_1, \lambda_3\lambda_1\lambda_1$, and so forth. I claim, once we analyze wrong-way composition, that all the differentials on these triple products are zero. So that means that we get three cyclic groups of order $\mathbb{Z}/2$ in $\pi_3^s$.

Ravenel uses slightly different notation for the $\lambda$-notation, and it actually comes from something called the $\lambda$-algebra. I guess I wanted to tell you something. In fact $\pi_3^s = \mathbb{Z}/8$. Its generator is $\lambda_3$. Twice the generator is $\lambda_2\lambda_1$ and $\lambda_1\lambda_1\lambda_1$ is four times it. That's weird. We can go and feed this back into the calculation, which will tell us about further groups.

Anyway, we learn that $\pi_6(S^2) \simeq \mathbb{Z}/4$.

| Abutment: $\pi_k^s(S^0)$ | | $\mathbb{Z}$ | $\mathbb{Z}/2$ | | | | | |
|---|---|---|---|---|---|---|---|---|
| | $n$ | | | | | | | |
| $S^1$ | 1 | $\lambda_0$ | 0 | 0 | 0 | 0 | 0 | |
| $S^3$ | 2 | | $\lambda_1$ | $\lambda_1\lambda_1$ | $\lambda_1\lambda_1\lambda_1$ | | | |
| $S^5$ | 3 | | | $\lambda_2$ | $\lambda_2\lambda_1$ | $\lambda_2\lambda_1\lambda_1$ | | |
| $S^7$ | 4 | | | | $\lambda_3$ | $\lambda_3\lambda_1$ | $\lambda_3\lambda_1\lambda_1$ | |
| $S^9$ | 5 | | | | | $\lambda_4$ | $\lambda_4\lambda_1$ | |
| $S^{11}$ | 6 | | | | | | $\lambda_5$ | $\lambda_5\lambda_1$ |

Let's turn to other things. If $\lambda_n$ is a permanent cycle of the spectral sequence, then that represents an element with sphere of origin on the $n+1$ sphere and Hopf invariant one. So the question of which $\lambda_n$'s never have a differential come out of them is the *Hopf invariant one problem.* If we're just looking down the diagonals, the question of when the differentials are zero or not is precisely the existence of Hopf invariant one problem.

So we get to the question:

**Question.** When is there an element of Hopf invariant one?

Later in the course we'll see that $\lambda_1, \lambda_3, \lambda_7$ are permanent cycles. For instance, we implicitly used this to see that $\lambda_3$ was a permanent cycle. But look, here's a place where there's definitely not an element of Hopf invariant one: $\pi_1 1(S^6)$. So $\lambda_5$ has to support a differential. If I look it up in my crystal ball, $\lambda_5$ is going to hit $\lambda_3\lambda_1$.

In order to go further in this, we need to understand what the differentials of the $\lambda_i$ are. The amazing thing is that that's a problem you can completely solve.

**Question.** What are the differentials out of the $\lambda_i$?

I want to translate that into a question about the homotopy groups of spheres. $\lambda_n$ wants to be an element of $\pi_{2n+1}(S^{n+1})$ which hits 1 in $\pi_{2n+1}(S^{2n+1})$. If it doesn't hit one, then 1 must go to the Whitehead square in $\pi_{2n-1}(S^n)$ which must be nonzero. What's the differential? We have to find the sphere of origin of the Whitehead square. We want to write this as an element of $\pi_{m+n-1}(S^m)$. And then take the Hopf invariant of that. So anyway, the differentials on the $\lambda$'s, if you think this through, are equivalent to:

**Question.** Differentials on the $\lambda$'s are equivalent to understanding what the sphere of origin of the Whitehead square $[\iota_n, \iota_n]$ is, and understanding what the Hopf invariant is?

The really cool thing is that it's equivalent to answering the vector fields problem. So that's going to be the bulk of what we do over the next month or so—explain how this problem is related to the vector fields problem.

# Lecture 8
# 9/24

We've gotten pretty far. We talked about the EHP sequence and how it could be used to recursively compute the homotopy groups of spheres. I was describing this spectral sequence and I used it to generate a series of questions. I want to talk now how we answer some of these questions.

We drew this spectral sequence:

| Abutment: $\pi_k^s(S^0)$ | | $\mathbb{Z}$ | $\mathbb{Z}/2$ | | | | | |
|---|---|---|---|---|---|---|---|---|
| | $n$ | | | | | | | |
| $S^1$ | 1 | $\lambda_0$ | 0 | 0 | 0 | 0 | 0 | |
| $S^3$ | 2 | | $\lambda_1$ | $\lambda_1\lambda_1$ | | | | |
| $S^5$ | 3 | | | $\lambda_2$ | $\lambda_2\lambda_1$ | | | |
| $S^7$ | 4 | | | | $\lambda_3$ | $\lambda_3\lambda_1$ | | |
| $S^9$ | 5 | | | | | $\lambda_4$ | $\lambda_4\lambda_1$ | |
| $S^{11}$ | 6 | | | | | | $\lambda_5$ | $\lambda_5\lambda_1$ |

We got that the initial differentials were $\lambda_2 \mapsto 2\lambda_1$, $\lambda_4 \mapsto 2\lambda_3$ where the other differentials on the diagonal were zero. We had this recursive method of filling in this table. We could calculate the differentials that $\lambda_2 \to 2\lambda_1$, and we wanted to conclude something about the differential of $\lambda_2\lambda_1$.

**8.1 Example.** So let's review this notation. If $\lambda_3$, an element in $\pi_7(S^4)$ survives this spectral sequence, it represents an element in $\pi_3^s(S^0)$ whose sphere of origin is the 4-sphere. The Hopf invariant lands in $\pi_7 S^7$ and if $\lambda_3$ survived, it would be an element of Hopf invariant one.

This is pretty tedious. You encounter, in the end, problems that you can't solve. Homotopy theory developed over the years by people looking at these tedious calculations and finding something systematic in them. We're going to do that in this semester. I'm going to trace a thread through several aspects of homotopy theory that come out of this tedious calculation. In the end, we still don't know all the homotopy groups of spheres. We know large pieces, but it's still tedious and anecdotal, and there's still room for new people to find new patterns and new conceptual frameworks. But it's amazing how much comes out of really trying to understand this. I'll put some things on the problem set to help you come to grips with how this information is displayed. After this we're going to do stable homotopy for about a month, and then return to this with a lot more information at our disposal. It's a good idea to understand this much of the chart, but we're going to be coming back to this later with a different conceptual framework.

The differentials on $\lambda_i$ are the Hopf invariants of the Whitehead squares. Now we wanted to claim that the $d_1$'s of the $\lambda_k\lambda_1$ are all zero.

**8.2 Theorem.** $d_1(\lambda_k\lambda_1) = 0$ *for all* $k$.

We want to understand what the question to understand this. This differential on, say, $\lambda_2\lambda_1$, goes from the homotopy groups of the 5-sphere to the homotopy groups of the 3-sphere. We have a map

$$\pi_6 S^3 \to \pi_6 S^5 \simeq \mathbb{Z}/2 \simeq \mathbb{Z}/2\,\{\eta\}$$

and that goes around by the $P$ map to $\pi_4(S^2)$, and then that comes around by the $H$ map to $\pi_4(S^2)$ to $\pi_4(S^3) \simeq \mathbb{Z}/2$ and generated by the suspension of $\eta$. So $\eta$ is the generator of $\pi_1^s(S^0) \simeq \pi_{n+1}(S^n), n \geq 3$. What are these maps?

The first thing we need to do is to understand these things as maps between spaces. Let's try to realize this as a map between spaces. We have a map $S^2 \to \Omega S^3 \to \Omega S^5$, and I'm interested in the connecting homomorphism: the map which shifts degrees and goes back to $S^2$. So let's back this fibration one more time, using the Barratt-Puppe sequence to get a fibration sequence

$$\Omega^2 S^5 \to S^2 \to \Omega S^3.$$

The map $\Omega^2 S^5 \to S^2$ is kind of amazing: it's hard to thing about such a map. There are a lot of interesting things happening in this map. That gives us the $P$ map $\pi_6(S^5) \to \pi_4(S^2)$. Then we have the $H$ map $\pi_4(S^2) \to \pi_4(S^3)$. The $H$ map is induced by a map $\Omega S^2 \to \Omega S^3$. In the end, we're looking at some map

$$\Omega^3 S^5 \to \Omega S^2 \to \Omega S^3,$$

and that composite map is the map we want to understand. Specifically, we'd like to understand what this map is doing in $\pi_3$. That is the question of what the first differential $d_1(\lambda_2\lambda_1)$ is. This is a hard thing to understand. These maps are complicated and the spaces are unintuitive. It's difficult to know in general how to calculate the effect of this map on homotopy groups.

Well, let's look at the bottom cell. There is an adjunction map $S^2 \hookrightarrow \Omega^3 S^5$, and there is a suspension map $S^2 \to \Omega S^3$. We have a commutative square:

$$\begin{tikzcd} S^2 \arrow[r] \arrow[d, "2"] & \Omega^3 S^5 \arrow[d] \\ S^2 \arrow[r] & \Omega S^3 \end{tikzcd}.$$

Why is it multiplication by 2? We need to know the effect in $\pi_2$ of $\Omega^3 S^5 \to \Omega S^3$ and the map between the two spaces was $P$ followed by $H$. $P$ sends the generator to the Whitehead square and $H$ sends the thing to the Hopf invariant.

Next, the map $\pi_3(S^2) \twoheadrightarrow \pi_4(S^3) \simeq \pi_6(S^5)$. Our element $\eta$ here actually comes back to the 2-sphere and it's generated by the Hopf map. Let $\eta \in \pi_3(S^2)$ be the Hopf map. And you see, the question of calculating this map is the following question: what is $S^3 \to S^2 \xrightarrow{2} S^2$, and I showed you at the end of this class is multiplication by four. That means in the ss that the differential $d_1$ is multiplication by four, hence zero.

The theme of this lecture is to explore this question:

**Question.** What is $S^3 \to S^2 \xrightarrow{2} S^2$?

Let's ask a more gneeral question.

**Question.** Say I have a map $\alpha : S^{n+k} \to S^n$. Consider multiplication by 2, $S^n \to S^n$. What is $[2] \circ \alpha$? It's not $2\alpha$; so what is that?

I want to solve this by answering an even more general question. How do I get the degree two identity? I take the degree two map as the sum of the identity map with itself. So let's ask this question even more generally. The more general question is this. Let's consider maps between two spaces that always form a group. So let's consider the following situation.

I have a map
$$\alpha : \Sigma A \to \Sigma X,$$
and two maps $f, g : \Sigma X \rightrightarrows Y$. Out of those two maps I can form the sum.

**Question.** How to express $(f+g)_* \alpha$ in terms of $f_* \alpha, g_* \alpha$ and other things?

The answer to this question is only involved in this lecture, but it's worth knowing.

What is $f+g$? You take $\Sigma X \to \Sigma X \vee \Sigma X \xrightarrow{(f,g)} Y$. The comultiplication $\nabla : \Sigma X \to \Sigma X \vee \Sigma X$ is the map which crushes the middle copy of $X$. So in other words, if $\iota_1, \iota_2$ are two inclusion maps $\Sigma X \to \Sigma X \vee \Sigma X$, then

$$\Sigma X \to \Sigma X \vee \Sigma X$$

is the sum $\iota_1 + \iota_2$. In other words:

**Question.** What is $\Sigma A \to \Sigma X \to \Sigma X \vee \Sigma X$?

This is the universal example. Since $\Sigma A \to \Sigma X$ is *not* a suspension map, we can't say that the universal example asks for replacing $\Sigma A$ by $\Sigma X$.

Let's continue with this a little bit. What if $\Sigma A \to \Sigma X$ was a suspension? Then the map is the sum of the two inclusions. So if $\alpha$ is a suspension, then

$$(f+g)_*\alpha = f_*\alpha + g_*\alpha.$$

It looks like it's hard to get somewhere. However, let's adjoint over. If we adjoint over, we can map

$$A \to \Omega\Sigma X \to \Omega(\Sigma X \vee \Sigma X)$$

and I could also ask what the composite is. That'll eventually give us the formula.

The miracle is that $\Omega(\Sigma X \vee \Sigma X)$ is a space you can say something about. There's the *Hilton-Milnor decomposition* of $\Omega(\Sigma X \vee \Sigma Y)$. That will answer this question. My whole goal today was just to explain what its role is in this story. We're not going to meet it again, I don't think, after this lecture. But it's a very important classical piece of homotopy theory. In a way, it's kind of a background motivation for things that are coming.

All right, let's think about this space $\Omega(\Sigma X \vee \Sigma Y)$. The homology of $\Omega\Sigma X$ is, as we saw, a tensor algebra on the homology of $X$. So the homology of $\Omega(\Sigma X \vee \Sigma Y)$ is a tensor algebra on $\widetilde{H}_*(X) \oplus \widetilde{H}_*(Y)$. We'll get some idea of what this looks like if we think about what a tensor algebra does to a direct sum. Let me just start by writing down a few terms. The tensor algebra on $\widetilde{H}_*(X) \oplus \widetilde{H}_*(Y)$ is

$$k+\widetilde{H}_*(X)\oplus\widetilde{H}_*(X)^{\otimes 2}\oplus\widetilde{H}_*(Y)\oplus\widetilde{H}_*(X)\otimes\widetilde{H}_*(Y)\oplus\widetilde{H}_*(Y)\otimes\widetilde{H}_*(Y)\oplus\widetilde{H}_*(Y)\otimes\widetilde{H}_*(X)\dots.$$

I just want to do some algebra. This algebra will point us to an answer. The first bit, the bit that looks like $T(\widetilde{H}_*(X))$, is the homology of $\Omega\Sigma X$. The next thing looks like $T(\widetilde{H}_*(Y))$. Also, there's a piece that starts out as $T(\widetilde{H}_*(X) \otimes \widetilde{H}_*(Y))$. That is, we get a decomposition

$$T(\widetilde{H}_*(X) \oplus \widetilde{H}_*(Y)) \simeq T(\widetilde{H}_*(X)) \otimes T(\widetilde{H}_*(Y)) \otimes T(\widetilde{H}_*(X \wedge Y)) \otimes \dots.$$

This is getting a little hard to tex. This is a piece of algebra. Roughly speaking, we get a decomposition of $T(\widetilde{H}_*(X) \oplus \widetilde{H}_*(Y))$ as a big tensor product of big tensor algebras of things gotten from tensoring things together.

Can I realize that by maps of spaces? In fact I can, because we can certainly map $X \to \Omega(\Sigma X \vee \Sigma Y)$, and certainly map $Y$ in. That extends to a map $\Omega\Sigma X \to \Omega(\Sigma X \vee \Sigma Y)$. If I have two maps $Z_1 \to \Omega(\Sigma X \vee \Sigma Y), Z_2 \to \Omega(\Sigma X \vee \Sigma Y)$, I can get a map fro their product because loop spaces are monoids. So if I want to get tensor algebras going in, I just need to map things like $X^{\wedge s} \wedge Y^{\wedge s}$ in. In fact, $X \wedge Y$ maps into $\Omega(\Sigma X \vee \Sigma Y)$ by taking commutators of the two maps. Part of the Hilton-Milnor theorem is that this winds up continuing.

**8.3 Theorem** (Hilton-Milnor theorem)**.** *There is a decomposition of* $\Omega(\Sigma X \vee \Sigma Y) \simeq \Omega\Sigma Y \times \Omega\Sigma Y \times \Omega\Sigma(X \wedge Y) \times \dots.$

I haven't really given a proof of the Hilton-Milnor theorem, or a statement of it. But I'm going to give you an instruction kit for figuring out the correct statement and proof. The idea is that this is a matter of combinatorics. If I could write $T(V \oplus W)$ as a tensor algebra on tensor algebras on $V, V^{\otimes \bullet}$ and similarly for $W$, then I could realize that easily by taking maps like this easily and forming commutators and stuff. What I hope that this has at least compelled you to see is that it involves two things: making maps and combinatorics of free associative algebras. And that's the thing. That winds up giving you this decomposition.

Let's assume something simple. But this is the way to remember this statement. Let's say I have a tensor algebra $T\{x, y\}$. My goal is to write that as a big tensor product of other algebras. There's a trick here. What do we know about a tensor algebra? A tensor algebra is the universal enveloping algebra of the of a Lie algebra, the free Lie algebra on $x, y$. If you give this a Hopf algebra structure by making $x, y$ primitive, then this Lie algebra is precisely the Lie algebra $L\{x, y\}$ of primitives. What do we know about enveloping algebras of Lie algebras? We have the Poincaré-Birkhoff-Witt theorem.

**8.4 Theorem** (PBW)**.** *If $L$ is a Lie algebra and $U$ is the universal enveloping algebra, then and I take $U \twoheadrightarrow k$ and filter by the augmentation ideal and look at the associated graded, then that's just the symmetric algebra on $L$.*

Up to associated graded, this $U$ is a polynomial algebra. And so that lets me write $U$ as a tensor product, since a symmetric algebra is always a tensor product. Let's take an example.

**8.5 Example.** Let $L$ be a free Lie algebra on two variables. This has a basis $x, y, [x, y], [x, [x, y]], \dots$. You can map the symmetric algebra on all these into the enveloping algebra of $L$: it's not an algebra map though.

So up to associated graded, a tensor product is a tensor algebra is a tensor product of polynomial algebras. I'm just going to say this and I'm going to put it together next time. The Hilton-Milnor theorem says that $\Omega(\Sigma X \vee \Sigma Y)$ is a big product, over a basis for the free Lie algebra on two variables, of copies of $\Omega\Sigma(X^{\wedge i} \wedge Y^{\wedge j})$. To do it properly, I'd have to go into the combinatorics of free Lie algebras. In the beginning of class next time, I'll answer the original question.

# Lecture 9
# 9/26

## §1 Hilton-Milnor again

OK, let's continue. I was sort of telling you about the Hilton-Milnor decomposition. I don't want to dwell on it too long. I just want you to understand that there's a problem that's more or less solvable, at least through a range of dimensions. Let me tell you what the consequence of the Hilton-Milnor theorem is.

Let's suppose that we have a map $\alpha : \Sigma A \to \Sigma X$, and we have two maps

$$f, g : \Sigma X \to Y,$$

and we want a formula for $(f+g)_*\alpha$. Of course, $f+g$ is the composite of the maps

$$\Sigma X \xrightarrow{\nabla} \Sigma X \vee \Sigma X \xrightarrow{f,g} Y,$$

and of course, we could figure it out if we knew this symbol $\nabla_*\alpha$. We decided we would approach this problem by taking the adjoint maps,

$$A \to \Omega\Sigma X \to \Omega(\Sigma X \vee \Sigma X).$$

Hilton-Milnor tells us that the space $\Omega(\Sigma X \vee \Sigma X)$ can be written as an infinite product of things. It starts out looking like

$$\Omega\Sigma X \times \Omega\Sigma X \times \Omega\Sigma(X \wedge X) \times (1 + O(X^{\wedge 3})),$$

as long as we're having fun with terms of analysis. I.e., let's imagine that $X$ is $n-1$-connected. Then $O(X^{\wedge 3})$ means something which is at least $3n-3$ connected. So if $X = S^n$, then this infinite product would be an infinite product of loopspaces of spheres of increasing connectivity.

If

$$\dim A < 2n - 1,$$

then this map only sees the first part, the map into $\Omega\Sigma X \times \Omega\Sigma X$. We get a series of maps from $A$ into each of these terms in the product, you see. If $\dim A < 3n-1$, we see only the first three factors, and so on. What does that tell us about our original map $\Sigma A \to \Sigma X \vee \Sigma X$. Let's remember again how the H-M theorem works. We had an equivalence of $\Omega(\Sigma X \vee \Sigma X)$ with something else.

The maps of the H-M theorem come as follows. $\Sigma X$ goes into $\Sigma X \vee \Sigma X$ into two different ways, and $\Sigma X(\wedge X) \to \Sigma X \vee \Sigma X$ (the Whitehead product of the first two factors). Then we take loops on all these maps and multiply them together. That gives the various factors. My interest in this was just to inform you about how something works without getting bogged down in details. However, the question is something topologists know how to answer, and it's a nice answer.

We had this map

$$\alpha : \Sigma A \to \Sigma X,$$

and we're going over here to $\Sigma X \vee \Sigma X$. Let's suppose that $\dim A < 3n-1$. Then, out of this, we get three maps from $A$ into the first three factors, and if we adjoint them back, we get there maps

$$\Sigma A \to \Sigma X, \Sigma A \to \Sigma X,\ \Sigma A \to \Sigma X \wedge X.$$

By chasing the diagrams around, or by projecting off of each factors, the first two maps are the original inclusion maps, and the third map is the *James-Hopf invariant.* I'm claiming these two things are the same.

Namely, we can get a map $\Sigma A \to \Sigma X \wedge X$ in two different ways. One is via the Hilton-Milnor decomposition, as we've just seen. The other definition of the Hopf invariant of $\alpha : \Sigma A \to \Sigma X$ was

$$A \to \Omega\Sigma X \mapsto \Sigma A \to \Sigma\Omega\Sigma X \to \Sigma \bigvee X^{\wedge n} \implies \Sigma A \to \Sigma X \wedge X.$$

These two give the same thing. I won't get into these things. My goal was to give you the beginning of the answer to this question. The Hopf invariant comes up as the quadratic term that comes up here, and there are "generalized" Hopf invariants which give us higher terms. The other terms can be expressed as James-Hopf maps.

The final answer in this range of dimensions is our final formula:

$$\nabla_* \alpha = (i_1)_* \alpha + (i_2)_* \alpha + [i_1, i_2]_* H(\alpha).$$

So we have to follow the Hopf invariant $\Sigma A \to \Sigma X \wedge X$ followed by the Whitehead product $\Sigma X \wedge X \to \sigma X \vee \Sigma X$. Anyway, notice that this is non-linear. This is kind of the theme of the course. The first two terms are in the stable range, the third is in the "metastable" range. What we're studying in the unstable homotopy theory is focusing on the metastable part. That's ultimately what makes these connections between vector fields on spheres and desuspending the Whitehead product.

If we go back to our original problem here, if we had maps

$$f, g : \Sigma X \to Y,$$

then we find that

$$(f+g)_* \alpha = f_* \alpha + g_* \alpha + [f, g]_* H(\alpha) + \dots.$$

**9.1 Example.** Let's do an example, and reconstruct a formula we already know. Let's take the Hopf map $\eta : S^3 \to S^2$ and compose this with the degree two map $S^2 \to S^2$. What is the composite $S^3 \to S^2$? We calculated using the mapping cone and cup products that it would be $4\eta$. As I pointed out in the last class, $2 = 1 + 1$,

$$2 \circ \eta = (1+1) \circ \eta = \eta + \eta + H(\eta) \circ [\iota, \iota] = \eta + \eta + 2\eta = 4\eta,$$

since $[\iota, \iota] = 2\eta$ and $H(\eta) = 1$. And that's what we knew it to be.

We could do this with other maps as well. The importance of the Hilton-Milnor theorem is to understand the addition in the wrong variable when you're studying maps between suspensions. There's a book on this. Note that when we study the EHP sequence, we discover new elements in the homotopy groups of spheres through their Hopf invariants, which makes this formula very useful.

## §2 Hopf invariant one problem

We know what the Hopf invariant is. The question is:

**Question.** For which $n$ does there exist a map

$$\alpha : S^{2n-1} \to S^n,$$

with Hopf invariant one?

There are lots of definitions. In this case, let's define the Hopf invariant of $\alpha$ by taking the mapping cone $S^n \cup_\alpha e^{2n}$, look at the cohomology of this. If $n \geq 2$, this has a basis $x_n, x_{2n}$ in degrees $n$, and $2n$. Then

$$x_n^2 = H(\alpha) x_{2n}.$$

This problem was solved by Adams. The theorem is:

**9.2 Theorem** (Adams). *Only when $n = 2, 4, 8$ does there exist a map $\alpha$ of Hopf invariant one.*

This would also make sense with mod 2 cohomology, and I could have added $n = 1$ to this list for the degree two map of the sphere $S^1 \to S^1$. But the thing I want to discuss is the case $n \geq 2$.

Now I could just give you the $K$-theory proof, and I will explain that. There's a lot of things to be learned from how one approaches this problem. In a way, Adams's original proof, which I won't talk about, gave a lot more information about what's going on with this Hopf invariant. I'm going to try to give similar information when I discuss the $K$-theory proof.

Adams's theorem uses $K$-theory. I'm going to have to review some things about Adams operations in $K$-theory. So let's discuss the $K$-theoretic proof.

## §3 The $K$-theoretic proof (after Atiyah-Adams)

So let $K$ be $K$-theory. If you don't know about $K$-theory, I don't really have time in this course to really develop it, but you can look at Hatcher's book "Vector bundles and $K$-theory." Let me just tell you some basic facts.

1. To every space $X$, we have a ring $K^0(X)$. There's a graded ring $K^*(X)$ and this is a contravariant functor from spaces to graded rings, and it's a generalized cohomology theory, meaning the Eilenberg-Steenrod axioms are satisfied, except for the dimension axiom. I have a long exact sequence for a pair, I have excision, and so forth, it's just that the cohomology of the point is a little different.

2. If $X$ is a *finite CW complex,* then $K^0(X)$ is the Grothendieck group completion of the set of (complex) vector bundles on $X$. Vector bundles form a monoid under Whitney sum, and the group completion is the smallest group receiving a map out of that monoid. That becomes a ring under tensor product of vector bundles.

3. There's a lot to say. On complex projective space $\mathbb{CP}^n$, we have the tautological line bundle $L$. $\mathbb{CP}^n$ is the space of complex lines in $\mathbb{C}^{n+1}$ and over it you have the tautological line bundle. In particular, an important element is when $n = 1$. Let's look at $\mathbb{CP}^1 = S^2$ and we want to look at a particular element, $1 - L$. So
$$1 - L \in K^0(S^2),$$
which restricts to zero on a point (since $L$ restricted to a point is one), and $1 - L$ really lives in *reduced* $K$-theory.

4. $K$-theory satisfies **Bott periodicity.** And that says that for any $X$, multiplication by the element $1 - L \in S^2$ defines a map
$$\widetilde{K}^0(S^2) \otimes \widetilde{K}^n(X) \to \widetilde{K}^n(S^2 \wedge X) \simeq \widetilde{K}^{n-2}(X)$$
is an isomorphism. (Also, $\widetilde{K}^0(S^2) = \mathbb{Z}$.) The $K$-groups are *periodic* with period 2. In some sense, $\widetilde{K}^n(X)$ only depends on $n \mod 2$, but we're going to introduce some other structure which will break that symmetry.

5. The amazing thing about $K$-theory is that the fact that it is a cohomology theory lets you make a lot of calculations. I'll try to summarize these things as gently as I can.

Let me give you some examples of calculations of $K$-groups. We know that

$$K^0(*) = \mathbb{Z},$$

because a vector bundle over a point is just a vector space. So the monoid of vector bundles over a point is $\mathbb{N}$, and the group completion of that is the integers.

Something that requires proof is that

$$K^{-1}(*) = 0.$$

This, plus Bott periodicity, implies that

$$K^n(*) = \begin{cases} \mathbb{Z} & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}.$$

That tells us the $K$-groups of all spheres. We get

$$\widetilde{K}^0(S^{2n}) = \mathbb{Z}, \quad \widetilde{K}^0(S^{2n+1}) = 0.$$

We also want to know something about the ring structure, but let's leave that for the time being. So I could deduce something from this. Some version of this is going to come up later. I want to work out the ring

$$K^0(\mathbb{CP}^2) =?.$$

Let's imagine that we are going to do this by induction on 2. We're going to study this inductively on 2, so we're going to start the induction when 2 is one. Let's look at $K^0(\mathbb{CP}^1) = K^0(S^2) = \mathbb{Z} \oplus \widetilde{K}^0(S^2) = \mathbb{Z} \oplus \mathbb{Z}$. But we know a little more about it. We know that the generator of the first $\mathbb{Z}$ is 1 and the generator of the second $\mathbb{Z}$ is $1 - L$, by Bott periodicity. So I could also say that

$$K^0(S^2) = \mathbb{Z} \oplus \mathbb{Z},$$

generated by $1, L$. For geometric purposes, if you want to think about $K$-theory as telling something about vector bundles, this is a good basis. However, $1, 1 - L$ is a better basis for homotopy theory.

Let's work out the ring structure. What is $L^2$? Well, $L^2$ has to be a sum of a multiple of 1 and a multiple of $L$, and in the basis $1, L$ it's a little harder to understand. In the second basis, it's easier. Let me point out something.

**9.3 Lemma.** *Let $E$ is a multiplicative cohomology theory and $a, b \in \widetilde{E}^0(\Sigma X)$, then $a \cup b = 0$.*

The reason for that is that we can split our suspension $\Sigma X$ into the union $C_+X \cup C_-X$, with intersection $X$. Then

$$\widetilde{E}^0(\Sigma X) = E^0(\Sigma X, C_+X) = E^0(\Sigma X, C_-X).$$

So $a \cup b \in E^0(\Sigma X, C_+X \cup C_-X) = E^0(\Sigma X, \Sigma X) = 0$. Over here, we can apply $1 - L \in \widetilde{K}^0(S^2)$ has to square to zero. That means

$$1 - 2L + L^2 = 0, \quad \text{so} \quad L^2 = 2L - 1,$$

which is not easy to see (you can find a geometric argument for it, though).

I guess I'm going to wrap this up by talking about the $K$-theory of $\mathbb{CP}^2$. So in the $K$-theory of $\mathbb{CP}^2$, let's look at this sequence

$$\mathbb{CP}^1 \hookrightarrow \mathbb{CP}^2 \twoheadrightarrow S^4,$$

so that we get an exact sequence

$$\cdots \to \widetilde{K}^0(S^4) \to \widetilde{K}^0(\mathbb{CP}^2) \to \widetilde{K}^0(\mathbb{CP}^1) \to \ldots.$$

We've calculated the two extreme groups, and the sequence is actually short exact because $\widetilde{K}^{\text{odd}}(S^4) = \widetilde{K}^{\text{odd}}(S^2) = 0$. We find

$$\widetilde{K}^{\text{odd}}(\mathbb{CP}^2) = 0,$$

and there is a short exact sequence

$$0 \to \mathbb{Z} \to \widetilde{K}^0(\mathbb{CP}^2) \to \mathbb{Z} \to 0,$$

so that $\widetilde{K}^0(\mathbb{CP}^2) = \mathbb{Z} \oplus \mathbb{Z}$. So

$$K^0(\mathbb{CP}^2) = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}.$$

What I'm going to claim is a consequence of the definitions and Bott periodicity. But you have to get yourself organized. The fact is, $K^0(\mathbb{CP}^2)$ has a *basis* given by $1, 1 - L, (1 - L)^2$. The element $(1 - L)^3 = 0$. If you're not familiar with $K$-theory, this would be a good thing to think through. The other reason this identification is important is to take the map

$$\widetilde{K}^0(S^4) \to K^0(\mathbb{CP}^2)$$

and the first thing is $\mathbb{Z}$ generated by the *square* of the Bott periodicity class. It maps over to $(1 - L)^2 \in K^0(\mathbb{CP}^2)$. Note in particular that these generators of $K^0(\mathbb{CP}^2)$ are sums of one-dimensional line bundles. This is an example of something called the *splitting principle* in $K$-theory. We'll come back and use the splitting principle later. It's how you make all calculations with things like Adams operations.

# Lecture 10
# 9/28

## §1 Splitting principle

I'm kind of giving you a crash course in $K$-theory, and describing some of the ideas hat go into the $K$-theory proof of Hopf invariant one. Last time I did a bunch of things very quickly, and there's something in $K$-theory that I'm going to make use of called the *splitting principle.*

**10.1 Proposition.** *If $X$ is a finite CW complex, and $\mathcal{V}$ a vector bundle over $X$. Then there exists a map $p : P \to X$ such that the following happens:*

1. *$K^0(P)$ is a free module of finite rank over $K^0(X)$.*
2. *$\mathcal{V}$ is pulled back under $p$ to a sum of line bundles on $P$.*

So as far as $K$-theory is concerned, if you're willing to pass to finite rank free modules, you can pretend that any vector bundle is a sum of one-dimensional vector bundles. This is proved in two steps.

The first thing is the projective bundle formula. That's the following. Let

$$V \to X$$

be a vector bundle. Let $\mathbb{P}(V) \to X$ be associated projective bundle space. If $V$ is a family of vector spaces parametrized by $X$, then the fiber $\mathbb{P}(V)_x$ is the projective space $\mathbb{P}(V_x)$. If the dimension of $V$ is $n$, then the map $\mathbb{P}(V) \to X$ is a fiber bundle with fiber $\mathbb{CP}^{n-1}$. On $\mathbb{P}(V)$, there is a one-dimensional bundle denoted $L \to \mathbb{P}(V)$. The fiber of $L$ over a point in $\mathbb{P}(V)$ (which is a point in $X$ and a line $\ell \subset V_x$) is precisely the line $\ell$. In other words, all I'm doing is something we talked about for complex projective space, and we're regarding this $\mathbb{P}(V)$ as a family of complex projective spaces.

So anyway, that line bundle $L$ gives an element $z = [1 - L] \in K^0(\mathbb{P}(V)$. What's the projective bundle formula?

$$K^0(\mathbb{P}(V)) \simeq K^0(X)\left\{1, z, z^2, \dots, z^{n-1}\right\}.$$

There are lots of ways of proving this. You could imagine proving it with the Atiyah-Hirzebruch spectral sequence, which we haven't talked about yet. It's easier to prove this directly.

*Proof.* Observe first that there's always a map $K^0(X)\left\{1, z, \dots, z^{n-1}\right\} \to K^0(\mathbb{P}(V))$. I want to regard both sides as functors of $(X, \mathcal{V})$. This map is a natural transformation. If $X$ is a union of two spaces $X = U_1 \cup U_2$, then both sides give a long Mayer-Vietoris sequence. That's just because $K$-theory does, and so this direct sum of $n$ copies of $K$-theory does, and because if $X = U_1 \cup U_2$, then $\mathbb{P}(V) = \mathbb{P}(V|_{U_1}) \cup \mathbb{P}(V|_{U_2})$. So both sides have a Mayer-Vietoris sequence.

Finally, when $X$ is contractible, the map is an isomorphism. That's just the computation we did in the previous class, the computation of $K^0(\mathbb{CP}^{n-1})$. At this point it's a standard argument. You finish by induction through a cell decomposition, or by choosing a nice covering and using a partition of unity or something like that. ▲

When $X$ is not a finite cell complex, you have to use the Milnor exact sequence. The more machinery you use, the easier it is to see this theorem. In equivariant $K$-theory, there's a group acting on everything, I think there isn't really a good proof of the result. There's some little bit of index theory needed to prove this result in equivariant $K$-theory. This is pretty easy, but be warned: in equivariant $K$-theory the analogous theorem is a little harder. The projective bundle formula is very important and lets you do all kinds of things. Note that it was really formal.

Back on the projective bundle, let's call $p : \mathbb{P}(V) \to X$. I can take the original bundle $V \to X$ and pull it back along $p$. Over each fiber in $\mathbb{P}(V)$, I have a specified line. So

$$p^*\mathcal{V} \simeq L \oplus \overline{\mathcal{V}}, \dim \overline{\mathcal{V}} = \dim \mathcal{V} - 1.$$

On the projective space of a vector space, the tautological bundle sits inside the constant vector bundle, and in algebraic topology, short exact sequences of bundles split (e.g. by use of a metric or partition of unity). So by iterating this process and applying it to $\overline{\mathcal{V}}$, we can get the splitting principle.

**Remark.** You could do this in one go by taking the bundle of *flags* of a vector bundle.

There're an awful lot of things that the splitting principle is used for. The use I want to make of it is the following. We're going to use it for computations and other things, but this is a very general construction that always works and for a given $X$ there might be easier splittings sitting around. The consequence I want is the following:

**10.2 Corollary.** *If $T_1, T_2 : K \to K$ are additive natural transformations, and for all line bundles $L$, $T_1([L]) = T_2([L])$, then $T_1 = T_2$.*

Slightly more generally:

**10.3 Corollary.** *If $T_1, T_2 : K \to E$ are additive natural transformations and $E^*(X) \hookrightarrow E^*(\mathbb{P}(V))$ for all $X, V$ as before, and for all line bundles $L$, $T_1([L]) = T_2([L])$, then $T_1 = T_2$.*

That's just a diagram chase. Given a class $[V] \in K(X)$, choose $P \to X$ such that $V$ pulls back to a sum of line bundles. We have a diagram:

$$\begin{tikzcd} K(P) \arrow[r] & E(P) \\ K(X) \arrow[r] \arrow[u] & E(X) \arrow[u] \end{tikzcd},$$

by naturality, which lets us conclude that $T_1([V]) = T_2([V])$ as desired.

**10.4 Example.** Other than $E = K$, we could take $H^*(X; \mathbb{Q}[u^{\pm 1}]) \simeq \bigoplus_n H^{*+2n}(X; \mathbb{Q})$ where $|u| = 2$, so that $E$ is *complex-orientable*. Another class of examples is the *complex-orientable theories* $E$, those multiplicative cohomology theories for which there exists $z \in \widetilde{E}^2(\mathbb{CP}^\infty)$ such that

$$z|_{\widetilde{E}^2(\mathbb{CP}^1)}$$

is the generator of $\widetilde{E}^2(S^2) \simeq E^0(S^0)$. I'll come back in the course and discuss complex-oriented $E$. You'll see why I want to do that in just a second.

The splitting principle is a really nice thing. If I want to check that two natural transformations are equal, then you can just check on the classes of line bundles.

**Remark.** It's even better. You only have to check on the *universal* line bundle on $\mathbb{CP}^\infty$, just by naturality. Every line bundle is pulled back from the universal line bundle on $\mathbb{CP}^\infty$.

That's half of how you use the splitting principle. The other half is the following. I might want to describe a natural transformation by giving its values on line bundles. I'd like a condition that would guarantee that it was defined more generally. I'll just tell you. In fact, $K(X)$ doesn't just inject into $K(P)$, but this map is *faithfully flat*, and we have an equalizer diagram

$$K(X) \to K(P) \rightrightarrows K(P) \otimes_{K(X)} K(P).$$

Anyway, this was supposed to be a crash course in $K$-theory. I want to tell you about two interesting natural transformations from $K$-theory to ordinary cohomology.

## §2 The Chern character

The Chern character goes

$$\mathrm{ch} : K^*(X) \to H^*(X; \mathbb{Q}[u^{\pm 1}]), \quad |u| = -2.$$

It's characterized by the properties:

1. It's a ring homomorphism.
2. The Chern character of a line bundle $L$ is

   $$e^{uc_1} = 1 + uc_1 + \frac{uc_1^2}{2!} + \dots,$$

   where $c_1$ is the first Chern class of $L$. (Objection about the infinite sum.) This is always a little bit of a funny thing, and I never know how to think about it. We have

   $$H^*(\mathbb{CP}^\infty) = \varprojlim H^*(\mathbb{CP}^n) = \varprojlim \mathbb{Z}[x]/(x^{n+1}),$$

   and when you take the inverse limit in the category of graded rings, you get a polynomial rather than a power series ring. So as a graded ring, you can also think of it as a power series ring. Secretly, every time we talk about a cohomology theory, we mean to take the inverse limit over finite subcomplexes (modulo the Milnor sequence). So when I write $H^*(X; \mathbb{Q}[u^{\pm 1}])$, then I really mean an appropriate inverse limit of graded rings.

   If you don't know what the first Chern class is but know that line bundles are pulled back from maps into $\mathbb{CP}^\infty$, then you pull the (well, one of them) generator of $H^2(\mathbb{CP}^\infty)$ back to it.

## §3 The Adams operations

For $k \in \mathbb{Z}$, we have the Adams operation:

1. These are maps $\Psi^k : K(X) \to K(X)$ (really $K^0(X)$) which are additive.
2. They satisfy $\Psi_k(L) = L^{\otimes k}$, and therefore are multiplicative.

We're going to do a lot with Adams operations. They are the first things we are going to use to get new information about homotopy groups. Let me set up how these things are going to be used. I want to put these things together, and to connect these things to the Hopf invariant. We're going to check other properties of the Adams operations in a minute, but let's look at these two properties.

What's the operation $\Psi^2(V)$? If $V$ is a sum of line bundles $L_1 \oplus \cdots \oplus L_n$, then

$$\Psi^2(V) = L_1^2 + \cdots + L_n^2.$$

There are some other things I could form out of $V$. I could take

$$\bigwedge^2 V = \sum_{i<j} L_i L_j.$$

There is also

$$\mathrm{Sym}^2 V = \sum_{i \le j} L_i L_j,$$

and

$$\mathrm{Sym}^2 V \oplus \bigwedge^2 V \simeq V^{\otimes 2}.$$

But notice that if we subtract these things, we get

$$\mathrm{Sym}^2(V) - \bigwedge^2(V) = \Psi_2(V) = \sum L_i^2.$$

Let me just say also that

$$\Psi^2(V) = V^{\otimes 2} \mod 2.$$

On $K$-theory, as follows from all this, we find that:

**10.5 Theorem.** $\Psi^2(x) \equiv x^2 \mod 2$ *for* $x \in K^0(X)$.

That's one fact we get from this.

## §4 Chern character and the Hopf invariant

The other thing has to do with the Chern character. Suppose now that I have a map

$$f : S^{2m-1} \to S^m, \quad m \text{ even},$$

and we can form the mapping cone $C_f = S^m \cup e^{2m}$, which maps to $S^{2m}$ by crushing the bottom cell. We get the Barratt-Puppe sequence

$$S^{2m-1} \to S^m \to S^m \cup e^{2m} \to S^{2m},$$

and for both $K$-theory and cohomology, we get a little short exact sequence of groups

$$0 \to \widetilde{K}^0(S^{2m}) \to \widetilde{K}^0(S^m \cup e^{2m}) \to \widetilde{K}^0(S^m) \to 0.$$

The outside groups are $\mathbb{Z}$. We can also get the same thing if we take the sum of all the even cohomology groups. We get a ses

$$0 \to \widetilde{H}^0(S^{2m}, \mathbb{Q}[u^{\pm 1}]) \to \widetilde{H}^0(S^m \cup e^{2m}, \mathbb{Q}[u^{\pm 1}]) \to \widetilde{H}^0(S^m, \mathbb{Q}[u^{\pm 1}] \to 0,$$

and these fit into a commutative diagram. The claim is that you can use $K$-theory to get the Hopf invariant just as well. We have a commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \widetilde{K}^0(S^{2m}) \arrow[r] \arrow[d] & \widetilde{K}^0(S^m \cup e^{2m}) \arrow[r] \arrow[d] & \widetilde{K}^0(S^m) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \widetilde{H}^0(S^{2m}, \mathbb{Q}[u^{\pm 1}]) \arrow[r] & \widetilde{H}^0(S^m \cup e^{2m}, \mathbb{Q}[u^{\pm 1}]) \arrow[r] & \widetilde{H}^0(S^m, \mathbb{Q}[u^{\pm 1}] \arrow[r] & 0
\end{tikzcd}$$

Now if $x$ is the generator of $K^0(S^{2n})$, then $x$ is pulls back under the crushing map $\mathbb{CP}^n \to S^{2n}$ to $(1-L)^n$. This goes under the Chern character to $(1-e^{uc_1})^n$. Expanding out and using $c_1^{n+1} = 0$, we get for this $(-uc_1)^n$ for the Chern character. So ch sends a generator of $\widetilde{K}^0(S^{2m})$ to the integral generator of $H^{\text{even}}(S^{2m}, \mathbb{Q}[u^{\pm 1}])$. That means if I start with a generator in $\widetilde{K}^0(S^{2m})$ and lift it to some $\widetilde{a} \in \widetilde{K}^0(S^m \cup e^{2m})$ and calculate the Chern character

$$\widetilde{a}^2 = H(f)b$$

(these notes are getting messy; I need to fix them).

Conclusion: the Hopf invariant, at least up to sign, can be calculated either using $K$-theory or cohomology. So that's a rather simple point. It appeared to depend on a lucky definition of the Chern character, but the Chern character is really defined to send generators to generators on spheres.

# Lecture 11
# 8/1

## §1 The $e$-invariant

Today I want to present the Atiyah-Adams proof of Hopf invariant one. I want to make a little more of a story out of it to set us up for things that we're going to do later.

There's a general method of showing that a map of spheres isn't trivial. You start with a map

$$f : S^{n+k-1} \to S^n,$$

and consider the mapping cone $S^n \cup e^{n+k}$. Suppose we have a cohomology theory $E$ and one of two things happens:

1. Either $f$ is induces a nonzero homomorphism in $E^*$-cohomology.
2. Or $E^*(f) = 0$.

If it's nonzero, then $f$ is not null. The classic example is if $E$ is ordinary cohomology and the map is the "$E$-degree." If $E^*(f)$ is zero, then we have a short exact sequence:

$$0 \to E^*(S^{n+k}) \to E^*(S^n \cup e^{n+k}) \to E^*(S^n) \to 0,$$

and if $E$ is a multiplicative cohomology theory, then this is a split exact sequence. The middle term is non-canonically the sum of those two. It's split in the category of $E^*(*)$-modules. But we can often equip $E$ with extra structure. You can look for extra structure on $E^*(X)$ and try to regard this as a sequence in a different category. If you think about this, you already do this in the first semester of an algebraic topology course. You might study the attaching map of a cell in $\mathbb{CP}^2$ and prove that it is not null by remembering that you have a cup product. One of the first things that you do is to introduce a cup product, and you can look at these as algebras rather than just as modules.

This is the beginning of a great idea that was systematically exploited first by Adams, then by many other people. This is the beginning of the information you see in the Adams spectral sequence. This wasn't the first instance of the ASS, but it's the easiest one and you can get a good feeling for what's going on from looking at it.

We're going to take $E$ as $K$-theory and the enhanced structure will be the *Adams operations.* Eventually, we're going to focus on showing that a certain structure that can't exist. First I want to talk about the general algebraic structure.

OK, so since $K$-theory is concentrated in even degrees, life will be easier if $n, k$ are even. Let's change the situation a bit and double all the degrees, and so for a cofiber sequence

$$S^{2(n+k)-1} \to S^{2n} \to S^{2n} \cup e^{2(n+k)} \to S^{2(n+k)},$$

and since $K$-theory is concentrated in even degrees, the degree of $f$ in $K$-theory is automatically zero and we get an exact sequence

$$0 \to \widetilde{K}^0(S^{2(n+k)}) \to \widetilde{K}^0(S^{2n} \cup e^{2(n+k)}) \to \widetilde{K}^0(S^{2n}) \to 0.$$

So we have that short exact sequence. Now the outermost groups are the integers. Also, we know that the Adams operation $\Psi^l$ on $\widetilde{K}^0(S^{2n})$ by $l^n$ and it acts on $\widetilde{K}^0(S^{2(n+k)})$ by $l^{n+k}$. We can regard this as an exact sequence in some category of abelian groups with Adams operations. We'll denote these two objects by $\mathbb{Z}(n)$ and $\mathbb{Z}(n+k)$.

What did we produce? We produced an element $e(f)$ ("e" for *extension*) in $\mathrm{Ext}^1(\mathbb{Z}(n), \mathbb{Z}(n+k))$ where the Ext is in the category of abelian groups together with Adams operations. The $K$-theory functor takes values in this category of "abelian groups with Adams operations" (where "Adams operations" is defined below).

**11.1 Definition.** An **abelian group with Adams operations** is an abelian group $A$ together with morphisms $\Psi^l : A \to A, l \in \mathbb{Z}$ which commute with each other and satisfy $\Psi^l \Psi^k = \Psi^{kl}$.

If we wanted to be really careful, we might restrict the category further. For instance, the $K$-theory functor doesn't give an arbitrary group with Adams operations. Remember the Chern character, which mapped the $K^0(X)$ into $\bigoplus_n H^{2n}(X; \mathbb{Q})$ and was an isomorphism mod torsion.[1] You can check from the splitting principle that the Adams operations on $\bigoplus H^{2n}(X;\mathbb{Q})$ are given by $\Psi^l = l^n$ on $H^{2n}$. In other words, the action of the Adams operations is *semisimple* on rational $K$-theory and the grading of the cohomology can be extracted from them. In other words, if $A$ comes out of the $K$-theory of some space, then $A \otimes \mathbb{Q}$ is a big sum of copies of $\mathbb{Q}(n)$.

There are a lot of situations in which it's easier to define $K$-theory than cohomology, for instance it can be defined for associative algebras. A lot of people have studied this formula in different types of cohomology that come up in algebraic geometry.

If we were really to stop and to be careful about what category this Ext was living in, then we would at least consider abelian groups with the property that when you tensor with the rationals the action becomes semisimple.

## §2 Ext's in the category of groups with Adams operations

How do we calculate this group? What is $\mathrm{Ext}^1(\mathbb{Z}(n), \mathbb{Z}(n+k))$? We're going to do a little with this answer as we move forward. I'm not going to do the entire calculation as we move forward. Let's imbed that group in a short exact sequence.

We have a map

$$\mathbb{Z}(n+k) \to \mathbb{Q}(n+k) \to \mathbb{Q}/\mathbb{Z}(n+k),$$

which gives a long exact sequence of these Ext groups. There is a long exact sequence,

$$\mathrm{Hom}(\mathbb{Z}(n), \mathbb{Q}(n+k)) \to \mathrm{Hom}(\mathbb{Z}(n), \mathbb{Q}/\mathbb{Z}(n+k)) \to \mathrm{Ext}^1(\mathbb{Z}(n), \mathbb{Z}(n+k)) \to \mathrm{Ext}^1(\mathbb{Z}(n), \mathbb{Q}(n+k)).$$

The thing we're interested in is that we're split between two groups which are rational.

**11.2 Lemma.** *When $k \neq 0$, both the outside groups are zero. That is,* $\mathrm{Hom}(\mathbb{Z}(n), \mathbb{Q}(n+k))$ *and* $\mathrm{Ext}^1(\mathbb{Z}(n), \mathbb{Q}(n+k))$ *are zero.*

*Proof.* The hom assertion is kind of obvious. Namely, 1 would have to go to some assertion $a \in \mathbb{Q}(n+k)$, and thus $l^n$ would have to go to $l^{n+k}a$ which is a contradiction. So the Hom group is clearly zero. What about the Ext group? For the Ext group, it's equivalent to take $\mathrm{Ext}^1(\mathbb{Q}(n), \mathbb{Q}(n+k))$. This requires a bit of justification.

Let's not get bogged down. I was deliberately a little vague about the category, and we'll clarify this later on for some of the further calculations. We have a short exact sequence

$$0 \to \mathbb{Q}(n+k) \to E \to \mathbb{Q}(n) \to 0,$$

and let's choose a basis $e_1, e_2$ such that $e_1$ is the image of the basis element of $\mathbb{Q}(n+k)$ and $e_2$ projects to a generator of $\mathbb{Q}(n)$. Then

$$\Psi^l = \begin{bmatrix} l^{n+k} & * \\ 0 & l^n \end{bmatrix},$$

and we can find a *unique* new basis by choosing an eigenvector for $l^{n+k}$ which maps to the generator of $\mathbb{Q}(n)$ and that splits the sequence. So you need a little linear algebra. Note that the $\Psi^l$ commute so can be simultaneously diagonalized. The fact that they commute is important for running this proof. ▲

[1]Because ch is a natural transformation of homology theories which is an isomorphism on the spheres.

I wanted some of these things in our minds for later use. So the question is, we now know that this Ext group can be described as

$$\mathrm{Hom}(\mathbb{Z}(n), \mathbb{Q}/\mathbb{Z}(n+k)) \simeq \mathrm{Ext}^1(\mathbb{Z}(n), \mathbb{Z}(n+k)).$$

That tells us something right away. Whatever this is, this is a subgroup of $\mathbb{Q}/\mathbb{Z}$, consisting of things compatible with the Adams operations. So we find that this group is cyclic.

But what is this group? A homomorphism $\mathbb{Z}(n) \to \mathbb{Q}/\mathbb{Z}(n+k)$ is determined by where it sends 1, to some element $x \in \mathbb{Q}/\mathbb{Z}(n+k)$. What does $x$ have to satisfy in order for the map to be a homomorphism? For all $n$,

$$(l^{n+k} - l^n)x = 0 \in \mathbb{Q}/\mathbb{Z},$$

which is what it takes to make the diagram commute. Or equivalently, the denominator must divide all the numbers $l^{n+k} - l^n$. In other words, we find that this group $\mathrm{Ext}^1(\mathbb{Z}(n), \mathbb{Z}(n+k))$ is cyclic of order

$$\text{g.c.d.}(l^n(l^k - 1)).$$

What is this greatest common divisor?

That turns out to have a rather nice answer. We're going to explore this as time goes on. But I want to get to the Hopf invariant one thing, but I suggest: if you've never studied this stuff, it would do you good if you tried to actually answer this question—before I tell you the answer in the next lecture. It has a beautiful answer.

Let's simplify this a little, though. I might as well work one prime at a time. Let's localize everything at 2 for instance. I want to localize at 2. I just want to take $l = 3^j$ and look at what numbers we get. Now I want to know, what's the largest power of 2 that divides

$$l^n(l^k - 1) = 3^{nj}(3^{kj} - 1).$$

for all $l$? (Here $n, k$ are fixed.) We're going to see that this solves the Hopf invariant problem.

Obviously, this is equivalent to finding the largest power of 2 that divides all the numbers $3^{kj} - 1$. Let's look at numbers of the form $3^m - 1$. If $m$ is odd, then we can write this as $(3-1)(1 + 3 + \cdots + 3^{m-1})$ and the second term has an odd number of terms, so $3^m - 1$ is divisible by one power of 2, at most. So when $m$ is odd, the largest power of 2 dividing $3^m - 1$ is just 2.

What happens when $m$ is even? Let's think about $3^{2m} - 1$ now, and that's the case as

$$9^m - 1 = (1+8)^m - 1 = 8m + \binom{m}{2}8^2 + \dots.$$

There's a little thing to check. You have to check that the remaining terms are highly divisible. But we have to claim that this turns out to be $8m$(odd number) so that

$$(3^{2m} - 1) = 2^{v_2(8m)} \times \text{odd number}.$$

If you go back, you find that if we localize at 2 and consider $l = 3^j$, we find our answer is:

$$\begin{cases} 2 & \text{if } k \text{ is odd} \\ 3 + v_2(j) & \text{if } k \text{ is even.} \end{cases}$$

That's the 2-power part of the g.c.d. of all these numbers $3^{nj}(3^{kj} - 1)$, over all $j$. This is something we need to know for several reasons. So this gives us an upper bound on the size of this Ext group. If I just look at the power of 2 in the denominator, this gives us a bound on that group. I really want to explain this proof of Hopf invariant one, even in five minutes.

# Lecture 12
# 10/3

## §1 Hopf invariant one

I meant to do the non-existence of elements of Hopf invariant one last time. Let's do that now, and I'll discuss it from a couple of different points of view. I'm going to suppose we have a map

$$f : S^{4n-1} \to S^{2n}$$

and we're supposing that the Hopf invariant of $f$ is one. Actually, we just need to suppose that it is odd. I mentioned that using the Chern character, we could calculate the Hopf invariant in $K$-theory and get the same answer. We get a short exact sequence in $K$-theory

$$0 \to \widetilde{K}^0(S^{4n}) \to \widetilde{K}^0(S^{2n} \cup e^{4n}) \to \widetilde{K}^0(S^{2n}) \to 0$$

which was some extension

$$0 \to \mathbb{Z}(2n) \to E \to \mathbb{Z}(n) \to 0$$

in the category of abelian groups with Adams operations.

Call the generator of $\mathbb{Z}(2n)$ $b$, and choose some element $a \in E$ which projects to a generator of $\mathbb{Z}(2n)$. We have

$$a^2 = H(f)b,$$

for $H(f)$ the Hopf invariant of $f$.

We recall also that

$$\Psi^2(a) \equiv a^2 \equiv H(f)b \mod 2.$$

What is $\Psi^2(a)$ really? Well, that's a multiple of $a$ plus a multiple of $b$. That multiple of $a$ has to hit $2^n a$ in $\mathbb{Z}(n)$. So

$$\Psi^2(a) = 2^n a + mb$$

and we know, by reducing mod 2, that $m$ is odd and $m \equiv H(f) \mod 2$.

All these groups are localized at 2, so we might as well *divide by* $m$ and assume $m = 1$. I'm just doing that so I have one fewer symbol in my notation. So we have

$$\Psi^2(a) = 2^n a + b.$$

Here's how the Atiyah-Adams argument goes. I'm going to give it, because it is quite simple, and then I'm going to reflect on it. So we also have an action of $\Psi^3$. So

$$\Psi^3(a) = 3^n a + rb.$$

Now we use the fact that these commute. Let's calculate $\Psi^2(\Psi^3(a))$. We get

$$\Psi^2(\Psi^3(a)) = \Psi^2(3^n a + rb) = 3^n(2^n a + b) + r2^{2n}b.$$

Also,

$$\Psi^3(\Psi^2(a)) = 2^n(3^n a + rb) + 3^{2n}b.$$

The coefficients of $a$ match up (as $6^n$). We're supposed to get

$$6^n a + (3^n + 2^{2n}r)b = 6^n a + (2^n r + 3^{2n})b.$$

I get the following equation. I get that

$$(2^{2n} - 2^n)r = (3^{2n} - 3^n),$$

and again, we're localized at 2. So we get

$$2^n(\text{odd})r = 3^n(3^n - 1),$$

and in order for this to have a solution, $2^n$ must divide $3^n - 1$. And that turns out not to happen very often. We worked out the power of 2 dividing this.

If $n$ is odd, then $3^n - 1 = 2(\text{odd})$ so $n = 1$ is the only possibility. If $n = 2k$ is even, then $3^{2k} - 1 = 8k(\text{odd})$ as we checked. So we have to have that

$$2^{2k} \mid 8k,$$

and that's not going to happen very often—in fact, let's just check when that happens. That happens for $k = 1$, and $k = 2$, but we don't make it for $k \geq 3$. Why is that? Just look at the 2-adic valuation of each side. On the left we get $2k$ and on the right we get $3 + v_2(k)$. So that only happens when $k < 3$.

**12.1 Theorem.** *There can only exist maps of Hopf invariant one in the dimensions* $S^3 \to S^2$, $S^7 \to S^4$, $S^{15} \to S^7$.

That's nice, that's quite, that's clever; if you only want to know the answer to Hopf invariant one this is good. We're going to come back and say more about Hopf invariant one. Let me just tell you how to think a bit more systematically about what's going on here. Remember I talked about this $e$-invariant that we could define. There's something traditionally called the $e$-invariant and I'm slightly modifying it, so let's call it the $\widetilde{e}$-invariant.

**12.2 Definition.** The $\widetilde{e}$-invariant is the $\mathrm{Ext}^1(\mathbb{Z}(n), \mathbb{Z}(2n))$ in the category of groups with Adams operations.

We determined that this $\mathrm{Ext}^1$ was a cyclic group of some order. The Hopf invariant one thing is saying somehow something about the order. I'm just going to reproduce more or less the same calculation but put this in more or less the same context.

I only want to consider $\Psi^2$, $\Psi^3$. These are generating monoids. This is like Ext over the algebra $\mathbb{Z}[t_1, t_2]$. How do I calculate an Ext group like that? In general, I would calculate Ext in this category as the cohomology of a certain complex. In other words, $\mathrm{Ext}(M, N)$ can be calculated by taking the complex

$$\begin{tikzcd} \mathrm{Hom}(M,N) \arrow[r, "\Psi_2^M - \Psi_2^N"] \arrow[d, "\Psi_3^M - \Psi_3^N"] & \mathrm{Hom}(M,N) \arrow[d, "\Psi_3^M - \Psi_3^N"] \\ \mathrm{Hom}(M,N) \arrow[r, "\Psi_2^M - \Psi_2^N"] & \mathrm{Hom}(M,N) \end{tikzcd}$$

(**add this stuff**)

If I have two modules over a polynomial ring $M, N$, and $M$ is free as an abelian group, then you can compute $\mathrm{Ext}^1_{\mathbb{Z}[t]}(M, N)$ by resolving $M$ by $M \otimes \mathbb{Z}[t]$. There's a little bit more of a story here, and I should tell it in a proper context. Let me come back. This was calculating the Ext groups in the category of modules over a polynomial ring in two variables.

Anyway, I could also calculate Ext just with $\Psi^2$, or just with $\Psi^3$.

Alright, let's do the one we're looking at. I want to calculate

$$\mathrm{Ext}^1(\mathbb{Z}(n), \mathbb{Z}(2n)).$$

What does that diagram work out to be? It looks like

$$\begin{tikzcd} \mathbb{Z} \arrow[r, "-(2^{2n}-2^n)"] \arrow[d, "-(3^{2n}-3^n)"] & \mathbb{Z} \arrow[d, "(3^{2n}-3^n)"] \\ \mathbb{Z} \arrow[r, "(2^{2n}-2^n)"] & \mathbb{Z} \end{tikzcd}$$

Everything is implicitly localized at 2.

The Ext group we calculate out by taking the 1st cohomology of this complex. Well, we can compute this by taking the horizontal cokernels. We get the map

$$\mathbb{Z}/2^n \xrightarrow{3^n - 1} \mathbb{Z}/2^n$$

and the kernel of that map is the Ext group. How do I calculate the element that I'm looking at? Well, let's break this into a couple of $e$-invariants.

I'll call this temporarily the $e_2$-invariant, which will be an element of $\mathrm{Ext}^1_{\Psi_2}(\mathbb{Z}(n), \mathbb{Z}(2n))$ and an element $e_3 \in \mathrm{Ext}_{\Psi_3}(\mathbb{Z}(n), \mathbb{Z}(2n))$. How do I calculate these? If you think about it, you calculate $e_2$ of a sequence

$$0 \to \mathbb{Z}(2n) \to E \to \mathbb{Z}(n) \to 0$$

you choose an element $a$ hitting 1 in $\mathbb{Z}(n)$, call the image of 1 in $E$ $b$, and evaluate $\Psi^2(a) = 2^n a + e_2 b$. If you think about the definition, that is the number $e_2$.

Our assumption on Hopf invariant one was that $\Psi_2(a) = 2^n a + (\text{odd})b$. So $e_2$ is a generator of $\text{Ext}^1$. But then—I'm just repeating the same calculation—the generator of this group doesn't go to zero there. That does not extend to an element of $\text{Ext}^1$ over the polynomial ring in two variables. I've just restated this calculation in a slightly richer algebraic context.

There are things to learn from this. This element of Hopf invariant one is trying to live here (*points*) in this $\text{Ext}^1$ term, but it doesn't live here. That's an important thing in homotopy theory. I'll come back and expand on this sometime later. There's something more important to understand here. The picture is that there is an element which is something which wants to be an element of the homotopy groups of spheres, but supports a differential.

What I really wanted to do today—I wanted to expand slightly on the Atiyah-Adams argument and expand on this argument—was to turn to calculating these Ext groups, over *all* the Adams operations. And again, I haven't honestly formulated the correct category for Ext. I want to start doing that now.

Let's go back to the thing I was talking about Monday. We studied

$$\text{Ext}_{\text{Adams/ops}}(\mathbb{Z}(m), \mathbb{Z}(n)).$$

When $m \neq n$, that was the same as

$$\text{Hom}_{\text{Adams}}(\mathbb{Z}(m), \mathbb{Q}/\mathbb{Z}(n))$$

and that's certainly a subgroup of $\mathbb{Q}/\mathbb{Z}$. We were trying to work out the order of that subgroup. So we figured out, by an elementary calculation, that a fraction $p/q$ (reduced) is in this subgroup if and only if

$$q \mid k^m - k^n, \quad \forall k.$$

So we need to figure out the g.c.d. of all these numbers. We need some mechanism for calculating that. That's kind of a cool problem, and I think it's kind of cool that it has a solution. It has a solution that's expressed in two different, very elegant ways. I'm going to talk about one of them today.

So the first thing I want to do is to study this problem one prime at a prime. I might as well try the following: for each prime $l$, let's figure out the largest power of $l$ dividing $k^m - k^n$. Equivalently, we're calculating

$$\text{Hom}_{\text{Adams}}(\mathbb{Z}(m), (\mathbb{Q}/\mathbb{Z})_l(n)).$$

I don't want to over-motivate this. If we sat down and tried to solve this problem, you would discover this next move yourself. But it's useful to know. I want to separate these Adams operations into those relatively prime to $l$ and into $l$ itself.

For $k$ relatively prime to $l$, there's this action $\Psi^k : (\mathbb{Q}/\mathbb{Z})_l \simeq (\mathbb{Q}/\mathbb{Z})_l$. It's an isomorphism. Moreover,

$$\text{Hom}((\mathbb{Q}/\mathbb{Z})_l, (\mathbb{Q}/\mathbb{Z})_l) = \mathbb{Z}_l$$

is the $l$-adic numbers. The number-theory thing you can check is that for $(k, l) = 1$, the action of $k \mapsto \Psi_k$ gives me a function from the integers prime to $l$ into $\text{Aut}((\mathbb{Q}/\mathbb{Z})_l(n))$, and that extends a continuous action of $\mathbb{Z}_l^*$. This is the thing that makes this problem a little easier to solve. So what is the structure of $\mathbb{Z}_l^*$?

The group $\mathbb{Z}_l^*$maps to $(\mathbb{Z}/l)^*$ with kernel the 1-units, $1 + l\mathbb{Z}_l$. Two things happen.

1. If $l$ is odd, then $(\mathbb{Z}/l)^*$ is cyclic of order $l-1$. The kernel $1 + l\mathbb{Z}_l$ is a pro-$l$-group and it's isomorphic for $l$ odd (by the logarithm) to the $\mathbb{Z}_l$. So there's a short exact sequence
$$0 \to \mathbb{Z}_l \to \mathbb{Z}_l^* \to \mathbb{Z}/(l-1) \to 0$$
which splits. The $l-1$st roots of unity are in $\mathbb{Z}_l^*$. There are a lot of ways of making this split. Anyway,
$$\mathbb{Z}_l^* \simeq (\mathbb{Z}/l-1) \times \mathbb{Z}_l.$$

2. When $l = 2$, the structure is a little different. Then the 2-adic units are isomorphic to $\mathbb{Z}/2 \times \mathbb{Z}_2$.

It makes the story a little easier to tell if you work with the $l$-adics. Now, the point is that we can phrase this question somewhat differently. The largest power of $l$ dividing $k^n - k^m, (k, l) = 1$ is also the largest power of $l$ dividing $\lambda^n - \lambda^m$ where $\lambda$ is a topological generator for $\mathbb{Z}_l^*$ (at least when $l$ is odd). Class is almost over, so I just want to look at this. What is that number? So now $\lambda^n - \lambda^m = \lambda^n(1 - \lambda^{m-n})$ and call the number $1 - \lambda^{m-n} = k$.

What happens? I'm just supposed to take $\lambda^k$ and figure out how close to 1 it is. If $l - 1 \nmid k$, then $\lambda^k \equiv 1 \mod l$. Suppose $k = l^r(l-1)$, then the power of $l$ is that one. We're out of time. I'm going to put these ideas together in the next class.

# Lecture 13
# 10/5

Let me just start today by correcting the mistake made yesterday and slightly reexplaining something I did. We were looking at maps $S^{2(n+k)-1} \to S^{2n}$, and we wanted to understand these. The $e$**-invariant of this** was an element of a cyclic group of order $\gcd(m^{n+k} - m^n)_{m\in\mathbb{N}}$. I described how you calculate that, but I think I made it a little complicated.

Let's just look at an example and get some ideas. Let's say we are looking at a map $S^9 \to S^6$. In that case, we want the greatest common divisor of the numbers $m^5 - m^3$, or $m^3(m^2 - 1)$. If you're trying to compute this, you might start writing down some numbers.

1. If $m = 3$, I get $27 \times 8$.

2. If $m = 5$, then we get $5^3(24)$. That tells us that the g.c.d must divide 24.

3. If $m = 7$, then we get $7^3(7^2 - 1)$, and that doesn't improve the g.c.d.

Now we get the idea that the g.c.d. in this case is probably 24. So what are we doing when we check that? Now we have some number $m$ and want to look at $m^3(m^2-1)$ and we want to check that this is divisible by 24. There are two cases. One is where $m$ is prime to $2, 3$, so that $m \in (\mathbb{Z}/24)^*$. So it's equivalent to saying that $m^2 \equiv 1 \mod 24$. That's easy to check. That's also equivalent to saying that $m^2 \equiv 1 \mod 3$ and $\mod 8$. That's equivalent to saying that $(\mathbb{Z}/3)^*$ has exponent two and $(\mathbb{Z}/8)^*$ has exponent two. If you pursue this, and I won't, you can see just by thinking about it naively that this reduces to a question about the structure of the units in $(\mathbb{Z}/p^j)^*$ for some $j$.

Or equivalently, the structure of the group $\mathbb{Z}_p^*$ of $p$-adic units, which is where we arrived last time. What was it that I had said wrong last time? I was writing (for $\lambda$ the topological generator of $1 + p\mathbb{Z}_p$)

$$\mathbb{Z}_p/(\lambda^j - 1), \quad p > 2$$

and the claim is:

1. This is zero if $(p-1) \nmid j$.
2. If $j = p^i(p-1)m$ with $(m, p) = 1$, then this is cyclic of order $\mathbb{Z}/p^{i+1}$.

I left it to you to work this out.

Now notice that I sort of left off something with the 24 business at the start. What about the case of $m$ dividing 2 or 3? When $m = 2$, we get $2^3(2^2-1) = 24$, so we're also good, and if $m = 3$, then that's the one I started with. But those numbers $2, 3$ are slightly different, because they're not units mod 24.

In general, the two factors of the expression

$$m^3(m^2-1)$$

play two different roles. To explain that, let's try to understand what happens under suspension.

## §1 Suspension

Given

$$f : S^{2(n+k)-1} \to S^{2n}$$

I suspend it twice. Then I get $\Sigma^2 f : S^{2(n+1+k)-1} \to S^{2(n+1)}$. Let's ask a question:

What happens to the $e$-invariant of $f$?

To answer the question, recall that the $e$-invariant was defined by looking at the short exact sequence

$$0 \to \mathbb{Z}(n+k) \to \widetilde{K}^0(S^{2n} \cup e^{2(n+k)}) \to \mathbb{Z}(n) \to 0.$$

We take a generator $1 \in \mathbb{Z}(n+k)$ mapping to $b$ in $\widetilde{K}^0(S^{2n} \cup e^{2(n+k)})$, we take $a$ in the middle hitting the generator of $\mathbb{Z}(n+k)$, and we study how $\Psi_m(a) = m^n a + e(f)b$. (**Sort of:** $e(f)$ is really an extension class.)

Now, when we suspend, we get a new exact sequence

$$0 \to \mathbb{Z}(n+1+k) \to \widetilde{K}^0(S^{2(n+1)} \cup e^{2(n+1+k)}) \to \mathbb{Z}(n+1) \to 0,$$

and we can take the generators of the middle group to be the double suspensions $\sigma a, \sigma b$ of $a, b$ previously defined.

Let's make some remarks about double suspension. The double suspension map goes

$$\widetilde{K}^0(X) \to \widetilde{K}^2(S^2 \wedge X) \overset{\text{Bott}}{\simeq} \widetilde{K}^0(S^2 \wedge X)$$

and this map sends a bundle to its multiple by $1 - L$. In other words, $\Psi_m$ of a double suspension $\sigma x$ is $m\sigma\Psi_m x$. That is,

$$\Psi_m(\sigma x) = m\sigma(\Psi_m x).$$

So $\Psi_m$ does not commute with suspension: it does so up to this factor.

So we find from this:

$$\Psi_m(\sigma a) = m\sigma\Psi_m(a) = m\sigma(m^n a + e(f)b) = m^{n+1}(\sigma(a)) + me(f)\sigma(b).$$

(**Note:** $e(f)$ should really be $e_m(f)$ here, and be a function of $m$.)

Anyway, the point is:

**13.1 Proposition.** *$e(\sigma f)$ is the image of $e(f)$ under the map*

$$\mathrm{Ext}^1_{\mathrm{Adams}}(\mathbb{Z}(n), \mathbb{Z}(n+k)) \to \mathrm{Ext}^1_{\mathrm{Adams}}(\mathbb{Z}(n+1), \mathbb{Z}(n+1+k))$$

*given by tensoring an extension with $\mathbb{Z}(1)$.*

Here's what I wanted to say about this. If you play through the calculations, the $e$-invariant is in some cyclic group of order $\mathrm{gcd} m^n(m^k - 1)$. When I suspend it, then we change it to $\mathrm{gcd} m^{n+1}(m^k - 1)$. This number is coming as a product of two relatively prime things. What you're supposed to come away from this saying is that $m^k - 1$ has to do with stable homotopy theory and all the unstable information is contained in the first factor $m^n$. This little discussion was supposed to arrive at this very simple observation.

Let's just illustrate this for a second. We were looking at something in the 3-stem. We were looking at something of the form $S^{2(n+2)-1} \to S^{2n}$ and I was thus looking at the gcd of the numbers $m^n(m^2 - 1)$. I picked $n \geq 3$ and in that case the g.c.d. was 24. But if $n = 2$, then the g.c.d of the numbers $m^n(m^2 - 1)$ for $m$ prime to $2, 3$ is still 24. If I put in $m = 2$, then I get $2^2(3) = 12$. So the $e$-invariant is not the generator.

Conclusion: if I have a map $S^7 \to S^4$, then the $e$-invariant of $f$ must have order 12. If I went down to $S^5 \to S^2$, then the $e$-invariant would have order 6. There are factors of two that we are losing because we are looking at complex and not real $K$-theory. We're going to revisit this in a couple of weeks. The point is that the $m^k - 1$ is telling you stable stuff, and the factor $m^n$ is telling you about the sphere of origin.

Ultimately this is going to feed in and tell us some properties of the EHP sequence.

Here's an interesting reality check. We know for which $k$ a map like this is in the stable range. We know that a stable range $S^{2(n+k)-1} \to S^{2n}$ happens when $k < n$ and that corresponds to something about the g.c.d. of these numbers, which you might try to check.

## §2 The $J$-homomorphism

We need more maps between spheres. The most understandable part of the homotopy groups of spheres is the image of the $J$-homomorphism. I'm going to start with the special orthogonal group $SO(n)$. Given a linear map $\mathbb{R}^n \to \mathbb{R}^n$ which is an isomorphism, I can form a map $S^n \to S^n$ by taking one-point compactifications. This gives a map

$$SO(n) \to \Omega^n S^n.$$

These fit inside each other; there are commutative diagrams:

$$\begin{tikzcd} SO(n) \arrow[r] \arrow[d] & \Omega^n S^n \arrow[d, "E"] \\ SO(n+1) \arrow[r] & \Omega^{n+1} S^{n+1} \end{tikzcd}.$$

We can then go to the limit. We get a map

$$\varinjlim SO(n) = SO \to \varinjlim \Omega^n S^n \stackrel{\text{def}}{=} QS^0.$$

We get a map

$$\pi_j(SO) \to \pi_j(QS^0) = \pi_j^s(S^0).$$

Alternatively, we get a map

$$\widetilde{KO}^0(S^{j+1}) \to \pi_j^s(S^0).$$

We know the $KO$-groups of spheres, and we don't know the homotopy groups of spheres. Amazingly, one knows exactly the image of the map. The image is a summand, it represents the only part of the homotopy groups of spheres which is nontrivial but still really understandable. I want to spend the next week talking about this map and how Adams analyzed it. Then we're going to start talking about vector fields on spheres.

There's another thing about the $J$-homomorphism that we're going to exploit, and that comes back to this picture. We had this map

$$H : \Omega S^{n+1} \to \Omega S^{2n+1}$$

which had the property that if we localize at 2, the homotopy fiber is $S^n$. Looping $n+1$ times gives a map: $\Omega^{n+1} S^{n+1} \to \Omega^{n+1} S^{2n+1}$. We have a commutative diagram,

$$\begin{tikzcd} SO(n) \arrow[r] \arrow[d] & \Omega^n S^n \arrow[d, "E"] \\ SO(n+1) \arrow[r] \arrow[d] & \Omega^{n+1} S^{n+1} \arrow[d, "H"] \\ SO(n+1)/SO(n) \simeq S^n \arrow[r] & \Omega^{n+1} S^{2n+1} \end{tikzcd}.$$

The bottom map is the adjoint to the identity $S^{2n+1} \to S^{2n+1}$.

So there's something better. There's the stable $J$-homomorphism, but there are all these unstable $J$-homomorphisms which fit together with the EHP sequence. The $J$-homomorphism is even better. I get an EHP style spectral sequence for the $SO(n)$ groups mapping to the usual EHP spectral sequence and in the end it converges to the stable $J$-homomorphism. In this course, we're going to use this to get a lot of information. It's going to tell us a lot about the EHP sequence. By the end of the course, we'll have understand Mahowald's picture of how the image of $J$ behaves in the EHP sequence. It's a picture that represents the most complicated calculation that you can do that most people can understand. It makes a pretty picture and it's what inspired the development of chromatic homotopy theory. This will take us about a month to get on board, and we're still going to have time to talk about chromatic stuff.

Anyway, we talked about this $P$ map which connects around. It's a map

$$\Omega^{n+2}S^{2n+1} \to \Omega^n S^n$$

and there's a map

$$\Omega S^n \to SO(n).$$

This fibration $SO(n) \to SO(n+1) \to S^n$ gives a definite map $S^{n-1} \to SO(n-1)$ and it maps to $\Omega^n S^n$ so we get a map $S^{n-1} \to \Omega^n S^n$ which is the Whitehead product. There's a lot of beautiful geometry going on here.

The last thing I'll say about this is, what is this fibration $SO(n) \to SO(n+1) \to S^n$? What's the fiber over a point in $S^n$? Given a point in $S^n$, the fiber over that point is all orthonormal bases of the orthogonal complement, appropriately oriented. That's the same thing as the set of all oriented orthonormal bases of the tangent space of $S^n$ at that point. So this bundle $SO(n) \to SO(n+1) \to S^n$ is the bundle of oriented orthonormal frames of $S^n$. Anything that you do in this kind of homotopy theory can be expressed in terms of the tangent bundle of the sphere or the frame bundle. What it often buys you is that there's some really weird elementary way of describing some question about stable homotopy theory in terms of the geometry of the sphere. Some of these are interesting, some go kind of nowhere. For instance the question of dividing the Whitehead square by 2 on the sphere is equivalent to asking, if I have the bundle of pairs of orthonormal vectors over the sphere, when is there a homotopy of the identity to the self-map which switches the two vectors? Starting in the next lecture, we'll investigate this $J$-homomorphism. Once the tangent bundle to the sphere is in there, lots of questions about homotopy groups of spheres will have formulations in terms of geometry. The vector fields on spheres question is related to the sphere of origin of the Whitehead product.

# Lecture 14
# 10/10

## §1 Vector fields problem

I introduced the $J$-homomorphism last time. There are a number of things to say about it, but I think that if we talk about the vector fields problem first, some of those things might come out more naturally. So we'll start with the vector fields problem. This is the vector fields problem.

**Question.** What is the maximum number of linearly independent vector fields on a sphere $S^{n-1}$?

Let's just think about this. What is a vector field on a sphere? The sphere sits inside euclidean space,

$$S^{n-1} \hookrightarrow \mathbb{R}^n$$

and a vector field is just a continuous way of assigning a tangent vector to the sphere at each point. A **vector field** is a function

$$v : S^{n-1} \to \mathbb{R}^n$$

such that $v(x)$ is tangent to the sphere at $x$ at each $x \in S^{n-1}$. The tangent space at $x$ is the orthogonal complement of $x$ in $\mathbb{R}^n$. So a vector field is a function

$$v : S^{n-1} \to \mathbb{R}^n$$

with the property that

$$v(x) \perp x, \quad \forall x \in S^{n-1}.$$

Of course, a sequence $\{v_1, \ldots, v_k\}$ is **linearly independent** if the sequence you get by evaluating at any $x \in S^{n-1}$ is linearly independent in each tangent space. So for instance, a single vector field is linearly independent if and only if the vector field never vanishes. So just coded in the statement the vector fields have to nowhere vanish.

Now, if I have a linearly independent set $v_1, \ldots, v_k$ of linearly independent vector fields, we can use Gram-Schmidt to make them orthonormal. We can get a new linearly independent set of vector fields $\overline{v_1}, \ldots, \overline{v_k}$ which are orthonormal. We could ask in the vector fields problem for the maximum number of **orthonormal** vector fields on the $n-1$-sphere.

Let's turn this into a homotopy theory question. What are we asking now? Let's look at the orthonormal case. I'm sending

$$x \mapsto v_1, \ldots, v_k$$

such that $v_i \perp x$ and the $v_i$ are orthonormal. In other words, the sequence $x, v_1, \ldots, v_k$ is orthonormal. It's also the same thing as saying that $v_1, \ldots, v_k, x$ is orthonormal. We can phrase that in terms of a mapping problem.

Let me define:

**14.1 Definition.** The **Stiefel manifold** $V_{n,\ell}$ is the space of orthonormal $\ell$-frames in $\mathbb{R}^n$. As a set, it is the set of all sequences $v_1, \ldots, v_\ell \in \mathbb{R}^n$ which are orthonormal. I want to make this into a space, and we can think of it as imbedded in the space of $n \times \ell$ matrices. It is the space of $n \times \ell$ matrices such that the colums are orthonormal.

**14.2 Example.** The Stiefel manifold $V_{n,n}$ is the set of $n$-by-$n$ matrices whose columns are orthonormal, so that's $O(n)$.

There are a lot of other ways of writing this.

**14.3 Example.** $V_{n,\ell}$ is the homogeneous space $O(n)/O(n-\ell)$. That'll play more of a role when we start talking in more detail about Stiefel manifolds. It's a little bit obvious when you think about it, but we'll come back to it in a couple of days.

So this is a space, and I'm going to claim various things about the point-set topology about it. They're all very believable, but the details are important for various claims I'm going to make, in particular for various claims I'm going to make about the $J$-homomorphism. I'm going to give you detailed proofs in the next lecture or the one after that.

So we have a map

$$p : V_{n,k+1} \to S^{n-1}, \quad (v_1, \dots, v_{k+1}) \mapsto v_{k+1}.$$

This map is a fiber bundle. It'll be actually important for us later to really prove that — to write down a local trivialization. What's the fiber? What are we asking over here? We're saying:

**14.4 Proposition.** *$S^{n-1}$ has $k$ linearly independent vector fields if and only if the map $V_{n,k+1} \to S^{n-1}$ has a section.*

We're going to come back and do a lot more with these Stiefel manifolds. But since this is a fiber bundle, having a section is the same is the same as having a section up to homotopy, and that's some statement about $\pi_{n-1}$ of this space. The real story of the vector fields manifold is revealed by the topology of these Stiefel manifolds. But we'll come back and discuss that later.

So let's look at some examples of vector fields.

**14.5 Example.** The circle $S^1$ has one vector field which nowhere vanishes. If I wanted to write down a formula for it, the vector at $x \in S^1$ could be $ix$: we could use the complex numbers to get a perpendicular vector at $x$.

**14.6 Example.** Similarly, $S^{2n-1}$ has a vector field: I get that by thinking of $S^{2n-1} \subset \mathbb{C}^n$ and sending a vector $x$ to the orthogonal vector $ix$ (so we get a 2-frame $x, ix$).

That tells us that odd spheres have nowhere vanishing vector fields. You probably learned in the first semester of algebraic topology that $S^{2n}$ has no nowhere vanishing vector fields. That's usually proved very early in the course, and I want to come back and tell you the real secret of that proof once we understand a little more about Stiefel manifolds.

**14.7 Example.** $S^3 \subset \mathbb{H}$ (the quaternions). I could use quaternionic multiplication to send

$$x \mapsto (ix, jx, kx)$$

which gives *three* vector fields on $S^3$.

**14.8 Example.** More generally, by regarding $S^{4n-1} \subset \mathbb{H}^n$, we find that $S^{4n-1}$ has linearly independent three vector fields.

Finally, there are the octonions $\mathbb{O}$ of **Cayley numbers.** As a vector space,

$$\mathbb{O} \simeq \mathbb{R}^8$$

with the structure of a division algebra by octonionic multiplication. The easiest way to describe the multiplication here is to draw a picture of the Fano plane. You label the vertices of the Fano plane $e_1$ up to $e_7$. And then you have to put a cyclic ordering on the vertices in every one of the lines. A basis for the octonions is $1, e_1, \dots, e_7$ and the rule is that any three in the line multiply like the quaternions, and $e_i^2 = -1$. It's easy then to check that this is a division algebra. If you multiply an element by its conjugate, you get something nonzero. The multiplication is nonassociative, but the table is pretty easy to write down.

We can use this to give $S^7$ seven vector fields.

**14.9 Example.** $S^7$ has seven vector fields sending

$$x \mapsto e_1 x, \dots, e_7 x$$

and more generally $S^{8k-1}$ has seven vector fields, by thinking $S^{8k-1} \subset \mathbb{O}^k$ and sending a vector $x \mapsto (e_1 x, \dots, e_7 x)$.

These are the easy vector fields. There are two things here. Algebra is sort of good. Algebra told us that if we had a bunch of vector fields on one sphere, we get them on lots of spheres. You can prove that in topology and it uses an important map.

**14.10 Proposition.** *If $S^{n-1}$ has (at least) $k$ linearly independent orthonormal vector fields, then $S^{n\ell-1}$ also does.*

*Proof.* The idea is this. Let's induct on $\ell$. Write

$$\mathbb{R}^{n\ell} \simeq \mathbb{R}^n \times \mathbb{R}^{n(\ell-1)}$$

and the unit sphere $S^{n\ell-1} \subset \mathbb{R}^{n\ell}$. That comes to us as the **join** of two other spheres: $S^{n\ell-1}$ and $S^{n-1}$. We can write

$$z = \cos\theta x + \sin\theta y, \quad x \in S^{n-1}, y \in S^{n(\ell-1)-1}.$$

The idea is that if there are $k$ vector fields on each of the two spheres in the join, I can take the linear combination with $\cos\theta$ and $\sin\theta$ of them.

Namely, if $v_1, \dots, v_k$ are vector fields on $S^{n-1}$ and $w_1, \dots, w_k$ are vector fields on $S^{n(\ell-1)-1}$. Then sending

$$(\cos\theta x + \sin\theta y) \mapsto (\cos\theta v_i, \sin\theta w_i)$$

gives $k$ vector fields on $S^{n\ell-1}$.

▲

Ioan James thought about this and realized there was an interesting map going on. I really like this map, although it got subsumed by later technology. You can use this map to give a simple proof of the Adams conjecture. James called this the **intrinsic join.** That's a map

$$V_{n,\ell} * V_{m,\ell} \to V_{n+m,\ell}.$$

It does just what I said—it's just this same formula.

**14.11 Definition.** Recall that $X * Y$ is $X \times Y \times [0,1]/\sim$ where the equivalence relation is that $X \times Y \times \{0\}$ is crushed to $X$ and $X \times Y \times \{1\}$ is crushed to $Y$.

This homeomorphism

$$S^{n-1} * S^{m-1} \simeq S^{m+n-1}$$

sends $(x, y, \theta) \mapsto (x \cos\theta, y \sin\theta)$. Joining with $S^0$ is the unreduced suspension and the join has the homotopy type of the suspension of the smash product. Anyway, it's an important construction. James's intrinsic join construction generalizes the above homeomorphism to the Stiefel manifolds.

Thinking in terms of this map, I could even state a more general theorem.

**14.12 Proposition.** *Suppose $S^{n-1}, S^{m-1}$ have $k$ linearly independent vector fields. Then $S^{m+n-1}$ has $k$ linearly independent vector fields.*

*Proof.* We have sections of the fibrations

$$V_{n,k+1} \to S^{n-1}, \quad V_{m,k+1} \to S^{m-1}$$

by assumption. Then we take the intrinsic join of these two sections. You can check easily that this gives a section of the fibration

$$V_{n+m,k+1} \to S^{m+n-1}.$$

▲

Anyway, the point is: once we got the vector field on the circle, we got it on any odd sphere. Once we had those seven vector fields $S^7$, we got seven vector fields on $S^{8k-1}$. We got this from homotopy theory, and we didn't need algebra.

Now we want to talk about the vector fields problem. We have to construct vector fields, and we have to show that they are no more. The homotopy theory picture does two things. It shows us that there is an upper bound on the number of the vector fields, and it connects the problem to the EHP sequence. There's an awful lot in the story of these Stiefel manifolds. For the rest of the lecture today and in the next lecture, I want to talk about constructing vector fields.

## §2 Constructing vector fields

There's kind of a nice way to motivate this. It takes a lot of leaps of faith, but there's a good lesson in that. If you were faced with the problem of constructing vector fields, you would study the examples we've discussed and try to imitate that construction. But you also can imagine it's hard to think up such algebras. A vector field, you might picture, is something topological. But there's a lesson here: when you're trying to think up an example and you don't know what's going on, imagine that you might be lucky and a lot of convenient accidental things will happen. What are some convenient things that might happen?

To every point in the sphere $S^{n-1}$, we want vector fields $v_1(x), \dots, v_k(x)$ and these are supposed to be orthonormal. In the case of the complex numbers, we used the module structure

$$\mathbb{C} \otimes V \to V$$

to produce vector fields.

Here are two assumptions that happen in the cases we've studied:

1. $x \mapsto v_i(x)$ is actually a *linear* transformation $\mathbb{R}^n \to \mathbb{R}^n$. We're just looking at values on the sphere. This is a good way to try to think of examples. So we want the property that any unit vector gets sent to something perpendicular to it. So we want $k$ such linear transformations.

So we want linear transformations $T : \mathbb{R}^n \to \mathbb{R}^n$ such that $Tx \perp x$ for all $x$. It's also natural to guess that the way this happens is $T^2 = -1$. We want maps

$$e_1, \dots, e_k : \mathbb{R}^n \to \mathbb{R}^n$$

satisfying $e_i^2 = -1$. I guess I'm not going to have time to motivate this. This gives a map from $\mathbb{R}^k \otimes \mathbb{R}^n \to \mathbb{R}^n$ given by taking linear combinations. It's natural to express this condition without referring to a basis. For every $u \in \mathbb{R}^k$, I get a transformation $T_u$ of $\mathbb{R}^n \to \mathbb{R}^n$. It's natural to assume that for every unit vector $u$, $T_u^2 = -I$. I'm running out of time so I'm going to leave this as an exercise, but the exercise is that this implies

$$e_1 e_2 = -e_2 e_1.$$

There's a better way of saying this. These are natural assumptions that you might look for if these vector fields were introduced in an easy way. This motivates introducing the **Clifford algebra.**

**14.13 Definition.** The **Clifford algebra** $\mathrm{Cl}_n(\mathbb{R})$ is the algebra, not necessarily commutative, generated by elements $e_1, \dots, e_n$ subject to the relation

$$e_i^2 = -1, \quad e_i e_j = -e_j e_i, \ i \neq j.$$

The basic fact is that if the Clifford algebra $\mathrm{Cl}_k(\mathbb{R})$ has a representation on $\mathbb{R}^n$, that implies that $S^{n-1}$ has $k$ vector fields. The beautiful thing about this is that you can construct the maximum number of vector fields once you work out the structures of these Clifford algebras. Given $\mathbb{R}^n$, you can find the largest Clifford algebra that acts on it, and then find the maximum number of vector fields.

# Lecture 15
# 10/12

## §1 Clifford algebras

We're continuing to talk about the vector field problem, and just the aspect of constructing vector fields. Last time I discussed a way of constructing vector fields, and I said something a little funny. The basic thing was, I introduced these Clifford algebras $\mathrm{Cl}_k$.

**15.1 Definition.** The **Clifford algebra** $\mathrm{Cl}_k$ is the tensor algebra $T(e_1, \dots, e_k)$ modulo the relations $e_i^2 = -1, e_i e_j + e_j e_i = 0$.

We saw that if $\mathrm{Cl}_k$ acts on $\mathbb{R}^n$, then $S^{n-1}$ has $k$ linearly independent vector fields.

**15.2 Example.** Look at $\mathrm{Cl}_1 = \mathbb{R}[e]/(e^2 = -1) \simeq \mathbb{C}$. If the complex numbers act on a vector space, then multiplication by $i$ gives the sphere one vector field.

I gave some sort of motivation for doing this.

**Remark.** Is there a relationship between the Clifford algebra and the octonions? You might think $\mathrm{Cl}_8$ has something to do with the octonions. But the Clifford algebras are associative and the octonions aren't. But we're going to work out $\mathrm{Cl}_8$ and that has to do with the octonions. It's this thing called trialty. I'll say something about that. $\mathrm{Spin}_8$ has these three irreducible representations and when you tensor two of them, you get the third plus something else. This gives a multiplication law related to the octonions. As far as this story goes, though, the use of the octonions to construct vector fields on the spheres is mostly tangential.

We also saw last time that choosing $k$ vector fields on $S^{n-1}$ was equivalent to choosing a sequence in the fibration

$$V_{k,n-1} \to V_{k+1,n} \to S^{n-1}.$$

Observe that $k$-framings up to homotopy can be classified when we know $\pi_{n-1}(V_{k,n-1})$ if we know that there exists a $k$-frame (that is, a section).

Today, we'll figure out what the number of vector fields that you can get from Clifford algebras, and our job will later be to prove that that is the maximal number. So I motivated this a little bit. I want to generalize this. One way I motivated this was that the action of $\mathrm{Cl}_k$ gave transformations

$$e_1, \dots, e_k : \mathbb{R}^n \to \mathbb{R}^n$$

with $e_i^2 = -1$ and we could extend this linearly and get a map

$$\mathbb{R}^k \times \mathbb{R}^n \to \mathbb{R}^n$$

which is bilinear. The assumption that we made was that there was a coordinate-free way of describing this. For any $v \in \mathbb{R}^k$, we could get a map

$$v : \mathbb{R}^n \to \mathbb{R}^n$$

and we could ask the question whether if it's a unit vector, then $v^2 = -1$. Or more generally, by replacing any $v$ by $v/\|v\|$, we would want

$$v^2 = -\|v\|^2.$$

That's what gave us the relation that the $e_i, e_j$ anticommute: we used the relation $(e_1 + e_2)^2 = -2$.

That motivates a general construction of Clifford algebras, which is worth noting.

**15.3 Definition.** Suppose $V$ is a vector space over a field $k$. Suppose $V$ is equipped with a **quadratic function** $q : V \to k$, i.e. a function with the properties:

1. $q(\lambda x) = \lambda^2 q(x)$ for $x \in V, \lambda \in k$.

2. The function $(x, y) \stackrel{\text{def}}{=} q(x+y) - q(x) - q(y)$ is bilinear.

An example is $v.v$ on euclidean space.

If $V$ is a vector space with a quadratic form $q$, we can define a **Clifford algebra**, which is the free associative algebra on $V$, $T(V)$, modulo the relation $v^2 = -q(v)$ for $v \in V$. That is,

$$\mathrm{Cl}(V, q) \stackrel{\text{def}}{=} T(V)/(v^2 = -q(v)).$$

There are a lot of conventions about what to do. Some people put a factor of two in the relations, which is no harm as long as you're not in characteristic 2. I'm saying this because Clifford algebras are extremely important in all kinds of places in math.

## §2 $\mathbb{Z}/2$-graded algebras

There's really a lot to think about when learning Clifford algebras. They really come up in an amazing number of places. I just wanted to say something a little more general about them. Doing it at this level of generality points out something you'd like to see about them.

**15.4 Example.** Suppose given two quadratic spaces $(V, q), (W, q')$. Then we can form a new quadratic space $(V \oplus W, q'')$ where

$$q''(v, w) \stackrel{\text{def}}{=} q(v) + q'(w).$$

In other words, I'm making the orthogonal sum. You'd like that the Clifford algebras do something nice. You'd *like* to say

$$\mathrm{Cl}(V \oplus W, q'') \simeq \mathrm{Cl}(V, q) \otimes \mathrm{Cl}(W, q').$$

That's *almost* right. There are some indications that this would be right. If $V$ has basis $e_1, \dots, e_k$ then $\mathrm{Cl}(V)$ has basis $e_I$ for $I \subset \{1, 2, \dots, k\}$ and we take for $e_I$ the product in increasing order of elements. That's easy to check from the rules. One has

$$\dim \mathrm{Cl}(V) = 2^k, \quad k = \dim V.$$

So if $V$ has basis $e_1, \dots, e_k$ and $W$ has basis $f_1, \dots, f_l$, then the Clifford algebra on $V \oplus W$ will have basis given by all the products

$$e_I f_J, \quad I \subset \{1, 2, \dots, k\}, J \subset \{1, 2, \dots, l\}$$

and the tensor product $\mathrm{Cl}(V) \otimes \mathrm{Cl}(W)$ will have the same basis. That makes it look like they're the same. But they're not. There's a subtlety here.

They have the same basis, but the multiplication isn't the same. In $\mathrm{Cl}(V \oplus W)$, say $e_1, f_1$ *anticommute* because $e_1 \perp f_1$. But in $\mathrm{Cl}(V) \otimes \mathrm{Cl}(W)$,

$$e_1 \otimes f_1 = (e_1 \otimes 1)(1 \otimes f_1) = (1 \otimes f_1)(e_1 \otimes 1).$$

So that's the thing that goes wrong. Anticommutativity versus commutativity.

There's a nice way to correct this and make the statement true. The way to fix this is to regard $\mathrm{Cl}(V)$ as a $\mathbb{Z}/2$-graded algebra. We let $|e_i| = 1$. That makes sense, because the relation

$$e_i^2 = -q(e_i)$$

is homogeneous for the $\mathbb{Z}/2$-grading. If I do everything in the world of $\mathbb{Z}/2$-graded vector spaces, we can still form the tensor product. Given $\mathbb{Z}/2$ graded vector spaces $X, Y$, we grade

$$X \otimes Y, |x \otimes y| = |x| + |y| \mod 2.$$

However, we want the *symmetry* of the symmetric monoidal structure to have the Milnor structure. The canonical isomorphism

$$X \otimes Y \simeq Y \otimes X$$

sends

$$x \otimes y \mapsto (-1)^{|x||y|} y \otimes x.$$

That affects what you mean by the tensor product of $\mathbb{Z}/2$-graded algebras. If $A, B$ are $\mathbb{Z}/2$-graded algebras, so is $A \otimes B$, but beware. When I multiply

$$(a_1 \otimes b_1)(a_2 \otimes b_2),$$

I have to move things past each other. We set:

$$(a_1 \otimes b_1)(a_2 \otimes b_2) \stackrel{\text{def}}{=} (-1)^{|b_1||a_2|}(a_1 a_2 \otimes b_1 b_2).$$

This is precisely the sign convention we encountered before.

We have, in fact:

**15.5 Proposition.** *As $\mathbb{Z}/2$-graded algebras,*

$$\mathrm{Cl}(V \oplus W) \simeq \mathrm{Cl}(V) \otimes \mathrm{Cl}(W).$$

## §3 Working out Clifford algebras

It's important to know this, and the compelling reason for making them $\mathbb{Z}/2$-graded is to be able to do this. Once you start working with them, you realize that there are all kinds of important reasons for working with the grading. However, I want to identify these Clifford algebras with algebras with we know, in terms of *ordinary* tensor products. Today, though, we'll be mixing both $\mathbb{Z}/2$-graded and ordinary tensor products.

**Temporarily, starting now, I'll write $\hat{\otimes}$ for the graded tensor product, and $\otimes$ for the ungraded one. Both can be used to produce legitimate algebras but $\hat{\otimes}$ is the one that we used above.**

OK, so let's work out these Clifford algebras.

**15.6 Definition.** As before, write $\mathrm{Cl}_n$ for $\mathrm{Cl}(\mathbb{R}^n)$ for the usual norm square quadratic form. We'll write $\mathrm{Cl}'_n$ for $\mathrm{Cl}(\mathbb{R}^n)$ with the quadratic form $q'(v) = -|v|^2$.

Let's make a little table below. For instance, $\mathrm{Cl}_1 \simeq \mathbb{C}$ with a funny $\mathbb{Z}/2$-grading with $i$ in degree 1. $\mathrm{Cl}_1' \simeq \mathbb{R}[e]/(e^2 = 1) \simeq \mathbb{R} \times \mathbb{R}$ where $e = (1, -1)$.

What about the next one, $\mathrm{Cl}_2$? That's

$$\mathrm{Cl}_2 = \mathbb{R}[e_1, e_2]/(e_1^2 = -1, e_2^2 = -1, e_1e_2 = -e_2e_1) \simeq \mathbb{H},$$

where $e_1 = i, e_2 = j, e_1e_2 = k$. Again, there's a funny grading. Next,

$$\mathrm{Cl}_2' \simeq \mathbb{R}[e_1, e_2]/(e_1^2 = 1, e_2^2 = 1, e_1e_2 = -e_1e_2).$$

This takes a little working out. In the ordinary reals, I can solve the quadratic equations $x^2 = 1$. I don't have two anticommuting solutions, though. If I send $e_1$ to the matrix

$$e_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

and these give an isomorphism

$$\mathrm{Cl}_2' \simeq \mathbb{R}(2).$$

**Notation:** If $K = \mathbb{R}, \mathbb{C}, \mathbb{H}$, we let $K(n)$ to be the algebra of $n$-by-$n$ matrices over $K$.

You can keep going, but at this point there's a convenient little trick you can use. If you were sitting down trying to work out these algebras, you'd be able to do it. When you here this argument, it'll go by a little quick. But it's probably worth just playing around with these algebras and discover the rest of the table. Don't be intimidated by my prestidigitious use of clever identities—you too could figure this out.

Here's a very useful lemma.

**15.7 Lemma.** $\mathrm{Cl}_{n+2}' \simeq \mathrm{Cl}_n \otimes \mathrm{Cl}_2'$. *This is the ordinary tensor product. Moreover,* $\mathrm{Cl}_{n+2} \simeq \mathrm{Cl}_n' \otimes \mathrm{Cl}_2$.

*Proof.* We have to use that, and the proof is totally straightforward. I just have to tell you the map. Let's map

$$\mathrm{Cl}_{n+2}' \to \mathrm{Cl}_n \otimes \mathrm{Cl}_2'$$

by sending

$$e_1' \mapsto 1 \otimes e_1', \quad e_2' \mapsto 1 \otimes e_2', \quad e_n' \mapsto e_{n-2} \otimes e_1'e_2' \ (n \geq 2).$$

You have to check that the identities hold: that each square to $-1$ and each anticommutes. It's really important that I **don't** mean $\hat{\otimes}$ here. These are just ordinary tensor products here. You just have to check the relations. You would have figured this out if you were playing around enough and looking for patterns. ▲

That lets me move from one side of this table to the other. We can continue the first four rows from this lemma. But then we need another lemma.

**15.8 Lemma.** $A(n) \otimes A(m) \simeq A(nm)$. *Also,* $\mathbb{H} \otimes \mathbb{C} \simeq M_2(\mathbb{C}) = \mathbb{C}(2)$ *(this is part of knowing about semisimple algebras).*

*Proof.* Here's a way of seeing the last thing. $\mathbb{H}$ acts on itself on the *left*. It acts on $\mathbb{R}^4$, if you like. That commutes with the *right* action. So we can restrict the right action to $\mathbb{C} \subset \mathbb{H}$ and we could restrict the right multiplication to the complex numbers $\mathbb{C}$. That means that $\mathbb{H}$ left acts on $\mathbb{R}^4$ $\mathbb{C}$-linearly. That gives a map

$$\mathbb{H} \to M_2(\mathbb{C})$$

and I can just extend it to a map

$$\mathbb{C} \otimes \mathbb{H} \to M_2(\mathbb{C})$$

and now you just count dimensions and check that it's an isomorphism. This is surprising when you see it, but it's part of a whole story about simple algebras and the Artin-Wedderburn theorem. ▲

Finally, we need to know:

**15.9 Lemma.** *$\mathbb{H} \otimes \mathbb{H} \simeq \mathbb{R}(4)$.*

*Proof.* That's because we have an action of $\mathbb{H} \otimes \mathbb{H}$ on $\mathbb{R}^4$ (given by left and right multiplication) and that gives a map

$$\mathbb{H} \otimes \mathbb{H} \to \mathbb{R}(4)$$

which is an isomorphism. ▲

Finally, we can now fill out the rest of the table.

| $n$ | $\mathrm{Cl}_n$ | $\mathrm{Cl}'_n$ |
|---|---|---|
| 0 | $\mathbb{R}$ | $\mathbb{R}$ |
| 1 | $\mathbb{C}$ | $\mathbb{R} \oplus \mathbb{R}$ |
| 2 | $\mathbb{H}$ | $\mathbb{R}(2)$ |
| 3 | $\mathbb{H} \oplus \mathbb{H}$ | $\mathbb{C}(2)$ |
| 4 | $\mathbb{H}(2)$ | $\mathbb{H}(2)$ |
| 5 | $\mathbb{C}(4)$ | $\mathbb{H}(2) \oplus \mathbb{H}(2)$ |
| 6 | $\mathbb{R}(8)$ | $\mathbb{H}(4)$ |
| 7 | $\mathbb{R}(8) \oplus \mathbb{R}(8)$ | $\mathbb{C}(8)$ |
| 8 | $\mathbb{R}(16)$ | $\mathbb{R}(16)$ |
| 9 | $\mathbb{C}(16)$ | |
| 10 | $\mathbb{H}(16)$ | |
| 11 | $\mathbb{H}(16) \oplus \mathbb{H}(16)$ | |

Let's plug this identity into itself. We get

$$\mathrm{Cl}'_{n+4} \simeq \mathrm{Cl}_{n+2} \otimes \mathrm{Cl}'_2 \simeq \mathrm{Cl}'_n \otimes \mathrm{Cl}_2 \otimes \mathrm{Cl}'_2 \simeq \mathrm{Cl}'_n \otimes \mathrm{Cl}'_4.$$

We get the identities

$$\mathrm{Cl}'_{n+4} \simeq \mathrm{Cl}'_n \otimes \mathrm{Cl}'_4, \quad \mathrm{Cl}_{n+4} \simeq \mathrm{Cl}_n \otimes \mathrm{Cl}_4.$$

That also implies

$$\mathrm{Cl}_{n+8} \simeq \mathrm{Cl}_n \otimes \mathrm{Cl}_8, \quad \mathrm{Cl}'_{n+8} \simeq \mathrm{Cl}'_n \otimes \mathrm{Cl}'_8.$$

But by tensoring with $\mathrm{Cl}_8, \mathrm{Cl}'_8$ just puts 16-by-16 matrices over everything. So the basic structure is eight-fold periodicity. I'm not going to drive this all the way home today, but let's just do an example.

How many vector fields can we get on $S^7$? We need the largest $k$ so that $\mathrm{Cl}_k$ acts on $\mathbb{R}^8$. Well, $\mathbb{H}(2)$ has an eight-dimensional real representation, $\mathbb{C}(4)$ has an eight-dimensional representation, and even $\mathbb{R}(8) \oplus \mathbb{R}(8)$ does. That gives a representation of $\mathrm{Cl}_7$ on $\mathbb{R}^8$ and that's the biggest one that acts. That gives 7 vector fields on $S^7$.

What about 15? How many vector fields do we get on 15? Now I want to look for 16-dimensional representations. Looking at the table, we can get 8 vector fields on $S^{15}$ but can't get any further via Clifford algebras.

# Lecture 16
# 10/15

## §1 Radon-Hurwitz numbers

There are two things I want to do today. First, I'd like to collect this thing about vector fields. Last time we talked about Clifford algebras. We made this table of the Clifford algebras:

| $n$ | $\mathrm{Cl}_n$ | $\mathrm{Cl}'_n$ |
|---|---|---|
| 0 | $\mathbb{R}$ | $\mathbb{R}$ |
| 1 | $\mathbb{C}$ | $\mathbb{R} \oplus \mathbb{R}$ |
| 2 | $\mathbb{H}$ | $\mathbb{R}(2)$ |
| 3 | $\mathbb{H} \oplus \mathbb{H}$ | $\mathbb{C}(2)$ |
| 4 | $\mathbb{H}(2)$ | $\mathbb{H}(2)$ |
| 5 | $\mathbb{C}(4)$ | $\mathbb{H}(2) \oplus \mathbb{H}(2)$ |
| 6 | $\mathbb{R}(8)$ | $\mathbb{H}(4)$ |
| 7 | $\mathbb{R}(8) \oplus \mathbb{R}(8)$ | $\mathbb{C}(8)$ |
| 8 | $\mathbb{R}(16)$ | $\mathbb{R}(16)$ |
| 9 | $\mathbb{C}(16)$ | |
| 10 | $\mathbb{H}(16)$ | |
| 11 | $\mathbb{H}(16) \oplus \mathbb{H}(16)$ | |

That's what they worked out to be. We had the isomorphism

$$\mathrm{Cl}_{n+8} \simeq \mathrm{Cl}_n(16),$$

and then you can work them all out, and find out how many vector fields you can produce on a sphere using them.

For $\mathbb{R}^n$, we wanted to know the largest value of $m$ such that $\mathrm{Cl}_m$ acts on $\mathbb{R}^n$. That gives us $m$ vector fields on $S^{n-1}$. That's the thing we want to figure out. Or in other words, given a Clifford algebra, we'd like to know the smallest vector space it acts on. Let's figure that out. Let's make a table of the smallest representations of the Clifford algebras:

| $n$ | $\mathrm{Cl}_n$ | Dimension of the smallest rep |
|---|---|---|
| 0 | $\mathbb{R}$ | 1 |
| 1 | $\mathbb{C}$ | 2 |
| 2 | $\mathbb{H}$ | 4 |
| 3 | $\mathbb{H} \oplus \mathbb{H}$ | 4 |
| 4 | $\mathbb{H}(2)$ | 8 |
| 5 | $\mathbb{C}(4)$ | 8 |
| 6 | $\mathbb{R}(8)$ | 8 |
| 7 | $\mathbb{R}(8) \oplus \mathbb{R}(8)$ | 8 |
| 8 | $\mathbb{R}(16)$ | 16 |
| 9 | $\mathbb{C}(16)$ | 32 |
| 10 | $\mathbb{H}(16)$ | 64 |

**We want to reverse this information and find the find the largest $\ell$ such that $\mathrm{Cl}_\ell$ acts on $\mathbb{R}^n$.**

Notice that the smallest representation of a Clifford algebra has dimension a power of 2. So we are going to want to write

$$n = 2^j m, m \text{ odd}.$$

The Clifford algebra we want is going to be the largest Clifford algebra acting on a $2^j$-dimensional vector space. We have this basic list of Clifford algebra representations, and each time we move up by eight, then the minimal Clifford algebra representation bumps up by 4.

So let's distinguish $j \mod 4$. Let's write $j = 4r + s, s \in [0, 3]$, so

$$n = 2^{4r+s} m, \quad m \text{ odd}.$$

1. If $s = 0$, then $n$ is a multiple of sixteen. We have $\ell = 8r$ (the best Clifford algebra).
2. If $s = 1$, then it's going to be $\ell = 8r + 1$.
3. If $s = 2$, then it's going to be $\ell = 8r + 3$.
4. If $s = 3$, $\ell = 8r + 7$.

**Summary:** The Clifford algebra construction constructs an action of $\mathrm{Cl}_\ell$ on $\mathbb{R}^n$, or $\ell$ vector fields on $S^{n-1}$, where $\ell$ is as above.

**16.1 Definition.** We write $\rho(n)$ for the number $\ell$ constructed above. These are the **Radon-Hurwitz numbers.** That is, if $n = 2^{4r+s}m$, $m$ odd and $0 \leq s \leq 3$, then

$$\rho(n) = \begin{cases} 8r & s = 0 \\ 8r+1 & s = 1 \\ 8r+3 & s = 2 \\ 8r+7 & s = 3 \end{cases},$$

and we can write this as $8r + 2^s - 1$.

Now we need to figure out how we are going to prove that there are no more vector fields. We have to find these Radon-Hurwitz numbers in algebraic topology. That's kind of a remarkable story.

We've proved:

**16.2 Proposition.** *There are at least $\rho(n)$ linearly independent vector fields on $S^{n-1}$.*

## §2 Algebraic topology of the vector field problem

So we're going to leave these Clifford algebras for now, and return to them later as we get more geometry under our belts. We want to find a way of getting an upper bound on the number of vector fields. We were looking at these Stiefel manifolds $V_{k+1,n+1}$ and the fibration

$$V_{k+1,n+1} \to S^n$$

and we wanted to know if it had a section. A section was equivalent to $S^n$ having $k$ vector fields. So we'd like to understand when a section exists.

One thing we might do is to apply homology and to see if we have a section in homology. If that wasn't enough, we could try to study Steenrod operations. That's what Whitehead and James did, and that gives an upper bound on the number of vector fields. But it's not the best. This method thinks that all the $S^{2^n-1}$-spheres are parallelizable.

We could also study $K$-theory *and* Adams operations and try to understand whether there's a section in $K$-theory. That turns out to give the right answer, the right upper bound. So we want to study this, and to calculate the $K$-theory of these spaces with Adams operations.

**Goal:** Compute

$$KO_*(V_{k+1,n+1}) \to KO_*(S^n)$$

with the action of Adams operations, and see if there's a section. Note that we've written $KO$-homology and used $KO$-theory, not $K$-theory.

## §3 The homology of Stiefel manifolds

The first thing I want to do is to get an idea of this space $V_{k+1,n+1}$. To start with, we want to understand its homology. Let's take the extreme case. Consider $V_{n+1,n+1} = O(n+1)$. As a topological space, that is two copies of $SO(n+1)$, so we might as well try to understand $SO(n+1)$. More generally,

$$V_{k+1,n+1} = O(n+1)/O(n-k) = SO(n+1)/SO(n-k),$$

and so we might as well study the special orthogonal group rather than the orthogonal one.

Let's start with $SO(n+1)$: the group of oriented orthogonal isomorphisms of $\mathbb{R}^n$ with itself. There is a fibration

$$SO(n) \to SO(n+1) \to S^n$$

where the last map sends an orthogonal matrix $T$ to $Te_{n+1}$ (where $e_{n+1}$ is the last basis vector). This is a fiber bundle. Just to get an idea, let's imagine that there is a section. There isn't one, but imagine we had one. Then we could take that section and $SO(n) \to SO(n+1)$ and multiply them to get a map

$$SO(n) \to S^n \to SO(n+1)$$

and that would be a *homeomorphism*. We could repeat with $n$ replaced by $n-1$ and we would find that $SO(n+1) \simeq S^n \times S^{n-1} \times \cdots \times S^1$. (If I had used $O$ instead of $SO$, it would go all the way down till $S^1$.) There's *no reason to think that this is true*, but if it were, we would get

$$H_*(SO(n+1); \mathbb{Z}/2) \simeq H_*(S^n; \mathbb{Z}/2) \otimes \cdots \otimes H_*(S^1; \mathbb{Z}/2).$$

I'm telling you this because if you ever forget what things look like, this is a good way to remember it.

Let's just continue this fantasy world with the Stiefel manifolds. We have that $SO(n+1)$ acts on $V_{k+1,n+1}$ because that's a homogeneous space for that group. That makes $H_*(V_{k+1,n+1}; \mathbb{Z}/2)$ into a *module* over the homology $H_*(SO(n+1); \mathbb{Z}/2)$ (the Pontryagin ring). If you go through this same argument and imagine that all of these fiber bundles had sections, you can get a description of this Stiefel manifold using the group action. What we would get, if we continued this analysis (imagining that we had these sections and that $SO(n+1) = SO(n) \times S^n = SO(n-1) \times S^n \times S^{n-1}$), we would find that the homogeneous space $SO(n+1)/SO(n-k) = S^{n-k} \times \cdots \times S^n$. So we would get

$$H_*(V_{k+1,n+1}; \mathbb{Z}/2) \simeq E[x_n, \ldots, x_{n-k}].$$

**This is in fact true,** and our aim now is to modify the incorrect argument above and to describe some things that actually do work. **We don't actually have these sections.** However, we do have sections like that away from a point. So if I were to take

$$SO(n+1) \to S^n,$$

I don't have a section here, but I do have a section here away from a given point. I do have a section, say, over a disk. How can I make such a section? Well, that's part of just saying it's a fiber bundle which must be trivial over a contractible space. Let's do it explicitly. The map $SO(n+1) \to S^n$ sends a matrix $T \mapsto Te_{n+1}$. We need to find a continuous way of making an orthogonal transformation that takes $e_1$ to a given vector $v$ (away from one point in $S^n$).

Here's one way of doing this. Draw an $n$-sphere $S^n$ with basepoint $e_{n+1}$ and *remove* the point $-e_{n+1}$. Given any other vector $v \in S^n$, we can take the 2-plane spanned by the two vectors $v, e_{n+1}$. Choose a transformation on the plane spanned by $v, e_{n+1}$ which rotates $e_{n+1}$ into $v$, in the plane spanned by these two vectors, and the identity map on the complement. It's not defined at the point $v = -e_{n+1}$. There's another place where they're not linearly independent, where $v = e_{n+1}$, where we just take the identity map. This gives a continuous section

$$S^n \setminus \{-e_{n+1}\} \to SO(n+1).$$

Now I'm going to say something without proof. This tells me that we have the map $p : SO(n+1) \to S^n$ and we can write $S^n = S^n \setminus \{-e_{n+1}\} \cup \{-e_{n+1}\}$. Over the open set $S^n \setminus \{-e_{n+1}\}$, there's a section of $p$ and the fiber bundle looks like $SO(n) \times (S^n \setminus *)$ and over $\{-e_{n+1}\}$ the fiber looks like $SO(n)$. So we can write

$$SO(n+1) = D^n \times SO(n) \sqcup * \times SO(n)$$

as **sets, not as spaces.** This in particular gives a map $D^n \to SO(n+1)$. Now I can iterate this by applying to $SO(n)$.

Or another way of saying this—we don't have sections over spheres, but we do have sections over interiors of disks. We have a whole bunch of maps

$$D_0^1, D_0^2, \dots, \to SO(n+1).$$

For every subset $S \subset \{0, 1, 2, \dots, n\}$, we can define a map

$$\prod_{i \in S} D_0^i \to SO(n+1).$$

The theorem about these is that:

**16.3 Theorem.** *These maps are the interiors of a cell decomposition of $SO(n+1)$.*

If you haven't seen this before, it's probably coming by in a blur. The point is, we were imagining these $SO(n+1)$ as a product of spheres. This isn't true, but the cell decomposition is similar to what it would have been if it were a product of spheres.

I just want to summarize something about these Stiefel manifolds. The last thing is, if we wanted to prove our homology calculation, we'd need to calculate the cellular differential in this cell complex. We need to know something more about how these cells are connected. There's a nice picture here. Let me go back to this little fantasy story. We wrote that if $SO(n+1) \to S^n$ has a section, then $SO(n+1) \simeq SO(n) \times S^n$. But that doesn't happen (except when $n = 1, 3, 7$). However, if it has a **homology section,** then the homology of $SO(n+1)$ is still $H_*(SO(n); \mathbb{Z}/2) \otimes H_*(S^n; \mathbb{Z}/2)$. So it's the same type argument, you could compare it with the Serre spectral sequence for instance.

There's an important map I wanted to get on the board today. Let me put up the map, and then we'll call it a day and answer these questions. As we said, if it had a homology section, I could get the same decomposition of the homology. Let me put up this important map. There's a way of getting a section in homology that tells you a great deal. One way of getting such a section would be to find a space $X \to S^n$ which is an iso on $H_n$ and then find a lift

$$\begin{tikzcd} & SO(n+1) \arrow[d] \\ X \arrow[ur] \arrow[r] & S^n \end{tikzcd}.$$

We will take $X = \mathbb{RP}^n$, where $\mathbb{RP}^n \to S^n$ is the collapse map. This is the key to understanding the topology of these manifolds. The point is that a rotation is the product of two reflections, and a reflection is determined by a line through the origin. That's supposed to motivate the following map:

**16.4 Definition.** The map $\mathbb{RP}^n \to SO(n+1)$ sends a line $\ell$ to reflection through the hyperplane perpendicular to $\ell$, composed with reflection through the hyperplane perpendicular to $e_{n+1}$.

Now, the way I've set it up, the diagram we want commutes. On the interior of the top cell of $\mathbb{RP}^n$, it's actually the map I wrote down earlier. I'll come back to that next time.

# Lecture 17
# 10/17

## §1 The map $\mathbb{RP}^n \to SO(n+1)$

In the last class, I was trying to tell you a little about the topology of $SO(n+1)$. I made a slightly bad convention, which I will change at this point. I told you a way of remembering what its homology looks like and made an argument that its homology actually was of that form. Let me review this now. The important thing about this is the map

$$\mathbb{RP}^n \hookrightarrow SO(n+1).$$

You get this, first, by mapping

$$\mathbb{RP}^n \to O(n+1),$$

by sending a line $\ell$ to the linear operator $R_\ell$ which is $-1$ on $\ell$ and 1 on the orthogonal complement, i.e.

$$R_\ell = (-1)_\ell \oplus 1_{\ell^\perp}.$$

That has determinant $-1$, so the map lands in $O(n+1)$. Then we multiply that with a fixed reflection through another line.

Here's the change of notation.

**17.1 Definition.** Let $\ell_0$ be the line through the first coordinate vector $e_1$.

Therefore, we have:

**17.2 Definition.** The map $\mathbb{RP}^n \to SO(n+1)$ sends a line $\ell$ to $R_\ell \circ R_{\ell_0}$, where $R$ denotes the rotation operators as above.

**Remark.** There are complex and quaternionic analogs of this. In the complex analog, we get a map

$$\mathbb{CP}^n \times S^1 \to U(n+1)$$

sending a line $\ell$ and $\lambda$ to $R_{\ell,\lambda} = \lambda|_\ell \oplus 1|_{\ell^\perp}$ (or rather $R_{\ell,\lambda} \circ R^{-1}_{\ell_0,\lambda}$). So this actually factors through a map

$$\mathbb{CP}^n \wedge S^1 \to U(n+1).$$

With the quaternions, it's more complex as they don't commute.

One of the reasons is that I've switched to the first basis vector, rather than the last basis vector, is that it isn't compatible with the choice of other special orthogonal groups. With the choice that we've now made, we have commutative diagrams:

$$\begin{tikzcd}
\mathbb{RP}^{n-1} \arrow[r] \arrow[d] & SO(n) \arrow[d] \\
\mathbb{RP}^{n} \arrow[r] \arrow[d] & SO(n+1) \arrow[d] \\
\mathbb{RP}^{n}/\mathbb{RP}^{n-1} \arrow[r, "\simeq"] & S^n
\end{tikzcd}$$

and you can check that the bottom map is a homeomorphism. This gives you a map in *homology*, that is, a *homology section* of $SO(n+1) \to S^n$. That lets you write $H_*(SO(n+1);\mathbb{Z}/2)$ as a tensor product of the homology of spheres.

In fact, the argument gives a decomposition of $SO(n+1)$ into cells, and the cells are attached as they are for $\mathbb{RP}^n$. I'm just going to tell you some results. I'm pretty sure something like this is written down in Hatcher's book.

Let's try to say something about $V_{k+1,n+1} = SO(n+1)/SO(n-k)$, the Stiefel manifolds. Note that we also get a commutative diagram:

$$\begin{tikzcd}
\mathbb{RP}^{n-k-1} \arrow[r] \arrow[d] & SO(n-k) \arrow[d] \\
\mathbb{RP}^{n} \arrow[r] \arrow[d] & SO(n+1) \arrow[d] \\
\mathbb{RP}^{n}/\mathbb{RP}^{n-k-1} \arrow[r] & V_{k+1,n+1}
\end{tikzcd}$$

Now $\mathbb{RP}^n$ has a zero-cell, a one-cell, and so on, all the way up to an $n$-cell. $\mathbb{RP}^{n-k-1}$ has all the cells up to $n-k-1$. The quotient space that you get is what you get by crushing those bottom cells: it has a basepoint, and cells from $n-k$ up to $n$. That space has another name.

**17.3 Definition.** $\mathbb{RP}^n/\mathbb{RP}^{n-k-1}$ is written as $\mathbb{RP}^n_{n-k}$ (the subquotient of $\mathbb{RP}^\infty$ with cells in the range of dimensions from $n-k$ to $n$). It is called a **stunted projective space.** This plays an important role with the Stiefel manifolds.

If you go through the inductive argument last time, we get:

**17.4 Theorem.** *The map $\mathbb{RP}^n \to SO(n+1)$ gives an isomorphism of rings,*

$$\bigwedge^{\bullet} H_*(\mathbb{RP}^n;\mathbb{Z}/2) \simeq H_*(SO(n+1);\mathbb{Z}/2).$$

It's also true that you get a cell decomposition of $SO(n+1)$, by induction on $n$. The idea is that $SO(n+1) = SO(n) \times (S^n \setminus *) \sqcup SO(n)$.

**Remark.** There's a pretty easy explanation for why the elements square to zero. We'll come back to this next time.

All we really need to know is that a monomial basis for $\bigwedge H_*(\mathbb{RP}^n; \mathbb{Z}/2)$ goes to a basis for $H_*(SO(n+1))$. Anyway, the same type of reasoning gives a map

$$\boxed{\bigwedge H_*(\mathbb{RP}^n_{n-k}) \to H_*(V_{k+1,n+1}; \mathbb{Z}/2)}$$

by mapping to $H_*(SO(n+1); \mathbb{Z}/2)$ first and mapping down. This is an isomorphism of *modules* over $H_*(SO(n+1); \mathbb{Z}/2)$. There isn't an obvious *ring* structure on the homology of Stiefel manifolds.

**Remark.** Another way to organize this inductive calculation is to use the long exact sequence of $SO(n), SO(n+1)$. Namely, if we know the homology of $SO(n)$, then we can calculate

$$H_*(SO(n+1); SO(n)) = H_*(SO(n) \times D^n, SO(n) \times S^{n-1}).$$

In fact, we have a homeomorphism

$$SO(n+1)/SO(n) \simeq (SO(n) \times D^n)/(SO(n) \times S^{n-1}).$$

More generally, if we have a fiber bundle $p : E \to X$ and $A \hookrightarrow X$ with pull-back $E_A \to A$, then we can calculate $E/E_A$ by the following.

Consider a diagram:

$$\begin{tikzcd} E_A \arrow[r] \arrow[d] & E \arrow[d, "p"] \\ A \arrow[r] & X \\ A' \arrow[r] \arrow[u] & X' \arrow[u] \end{tikzcd}.$$

If $(X', A') \to (X, A)$ is a relative homeomorphism, then $E/E_A \simeq E'/E'_A$. This is a kind of standard trick for analyzing fiber bundles over CW complexes.

Anyway, we can now use the Künneth formula to calculate $H_*(SO(n+1), SO(n)) \simeq H_*(SO(n) \times D^n, SO(n) \times S^{n-1})$ which we can calculate by the Künneth theorem. We get this long exact sequence. One checks that $H_*(SO(n+1)) \to H_*(SO(n+1), SO(n))$ is a surjection and the long exact sequence of $H_*(SO(n))$-modules actually splits. We get

$$H_*(SO(n+1) \simeq H_*(SO(n)) \oplus b_n H_*(SO(n)), \quad |b_n| = n.$$

This gives the desired computation.

## §2 The vector field problem

I want to do something else. Let's go back to the vector fields problem. Consider $S^{15}$ again. We saw, from Clifford algebras, that $S^{15}$ has *eight* vector fields. We aim to prove that there are no more. If we translate that into the theory of Stiefel manifolds, it means that we take the fibration

$$V_{8,16} \to V_{9,16} \to S^{15}$$

and the theory of Clifford algebras produces a section of this fibration. We think that there are not nine though, so $V_{10,16} \to S^{15}$ should not admit a section.

We know something about the homology of these spaces. We know that $H_*(V_{9,16}; \mathbb{Z}/2)$ is an exterior algebra on the homology of the stunted projective space $\mathbb{RP}^{15}_7$. This is slightly inconvenient for reasons you'll see in a moment. I want to use some implications that we had before. I used James's intrinsic join construction to show that $S^{15}$ has eight vector fields, which implies that $S^{16k-1}$ has eight vector fields for any $k$.

I want to take $k = 3$. If $S^{15}$ has nine vector fields, James's intrinsic join construction shows again that $S^{47}$ has nine vector fields.

So I want to look at the map

$$V_{10,48} \to S^{47}$$

and show that it does not have a section.

*Beginning of a proof.* The homology $H_*(V_{9,48}; \mathbb{Z}/2)$ is an exterior algebra $E(H_*(\mathbb{RP}^{47}_{39}))$ and $H_*(V_{10,48}; \mathbb{Z}/2)$ is an exterior algebra $E(H_*(\mathbb{RP}^{47}_{38}))$.

The first thing is the same as $H_*(\mathbb{RP}^{47}_{39})$ and the second is the same as $H_*(\mathbb{RP}^{47}_{38})$, through dimension 79. We're only interested in those spaces through dimension 47. So what we learn is $S^{47}$ has eight vector fields if and only if the map

$$\mathbb{RP}^{47}_{39} \to S^{47}$$

has a section, and $S^{47}$ has nine vector fields if and only if the map

$$\mathbb{RP}^{47}_{38} \to S^{47}$$

has a section. That's a problem you can actually solve. There are a lot of things you can say here. The point is, through a big range of dimensions, a Stiefel manifold is just a stunted projective space. You'll see it come back as telling us a big part of the EHP sequence. ▲

# Lecture 18
# 10/19

## §1 Spheres with one vector field

In the previous lecture, we studied a map $\mathbb{RP}^n \hookrightarrow SO(n+1)$, which was compatible with the inclusions:

$$\begin{tikzcd}
\mathbb{RP}^{n-1} \arrow[r] \arrow[d] & SO(n) \arrow[d] \\
\mathbb{RP}^n \arrow[r] & SO(n+1) \arrow[d] \\
\mathbb{RP}^n/\mathbb{RP}^{n-1} \arrow[r] & S^n
\end{tikzcd}.$$

This also gave us inclusions $\mathbb{RP}^n_{n-k} \stackrel{\text{def}}{=} \mathbb{RP}^n/\mathbb{RP}^{n-k-1} \to V_{k+1,n+1}$. These two maps led to cell decompositions to both spaces, and led to a computation of the homology. Even though there isn't a natural algebra structure, we have

$$H_*(V_{k+1,n+1}; \mathbb{Z}/2) \simeq E(H_*(\mathbb{RP}^n_{n-k}; \mathbb{Z}/2))$$

for $E$ meaning the exterior algebra.

This is kind of a fundamental picture; it's going to be the crux of what we do in class. In this class, we're going to try to see what this has to do with the vector field problem. We did some examples in the last class, and we're going through many more examples today. With solved problems like the vector fields problem, you can just read the solution, but you miss something if you don't put yourself in the position of someone who was faced with solving the problem.

Let's first ask about spheres having *one* vector field. Now that's saying we are looking at $V_{2,n+1} \to S^n$ and we want to know if that has a section. What does this space look like? $V_{2,n+1}$ has an $n$-cell and an $n-1$-cell (which comes from the stunted projective space). Then it has the product of those two cells, which is in dimension $2n-1$.

**Remark.** A more refined statement is that the *cellular chain complex* of $\mathbb{RP}^n_{n-k}$ is the exterior algebra on the cellular chain complex on $V_{k+1,n+1}$. I really want to think of this.

For most values of $n$, we have $2n-1 > n$. As long as $n > 1$, we have this. So what are we asking. We have a map

$$\mathbb{RP}^n_{n-1} \to S^n$$

and that map sends the top cell to the top cell. We're asking whether there exists a map in the opposite direction. As long as $n > 1$, we're asking whether there is a map back. That's because if we had a section $S^n \to V_{k+1,n+1}$, it would be homotopic to a map into the $n$-skeleton, so we only need to pay attention to that skeleton.

I know something about the cellular chains on $\mathbb{RP}^n$. If I draw the cellular chain complex of $\mathbb{RP}^n$, it looks like

$$\mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \dots.$$

If we look at what's happening in homology, the cellular complex of $\mathbb{RP}^n_{n-1}$ is

$$\begin{cases} \mathbb{Z} \xrightarrow{2} \mathbb{Z} & n \text{ even} \\ \mathbb{Z} \xrightarrow{0} \mathbb{Z} & n \text{ odd} \end{cases}.$$

We learn right away that there isn't a section when $n$ is even, because there isn't a section in homology. This is the hairy ball theorem: $S^{2n}$ doesn't have a nowhere vanishing vector field.

We also know that $S^1$ has one, thanks to the complex numbers. Similarly, odd spheres do, thanks to the complex numbers. So $S^{2n+1}$ has one.

Let's say a little more: what is the homotopy type of this stunted projective space? $\mathbb{RP}^n_{n-1}$ is the mapping cone of a map

$$S^{n-1} \to S^{n-1}$$

which is degree two when $n$ is even, and 0 when $n$ is odd. That's from our calculation of the cellular chain complex. We learn:

1. When $n$ is even, $\mathbb{RP}^n_{n-1} = S^{n-1} \cup_2 e^n$.
2. When $n$ is odd, $S^{n-1} \vee S^n$.

So the latter thing alone lets us say that odd spheres have one vector field.

There's one more thing I want to say about this. I'm setting up the trivial case of something that's going to get more sophisticated as we get going. These little stunted projective spaces with two cells come only in two varieties, either the attaching map is 2 or it's zero. A weird corollary:

**18.1 Corollary.** $\Sigma^2 \mathbb{RP}^n_{n-1} \simeq \mathbb{RP}^{n+2}_{n+1}$.

In other words, these stunted projective spaces are *periodic* in $n$.

This is a reason I like to draw real projective spaces, I draw little lines like this which indicate the attaching map on each subquotient (sorry, can't TEX these!).

## §2 Spheres with more than one vector field

Now you have the odd spheres. Let's think about the odd spheres:

$$S^1, S^3, S^5, S^7, S^9,$$

and we know that $S^1$ has one, $S^3$ has three, $S^5$ has at least one, $S^7$ has seven, and $S^9$ has at least one, and $S^{11}$ has (at least) three. The first sphere we don't know anything about is the 5-sphere. One question is whether the five-sphere has two vector fields. Let's consider the map

$$V_{3,6} \to S^5$$

and we want to know whether this has a section. We have

$$\mathbb{RP}^5_3 \hookrightarrow V_{3,6}$$

and $\mathbb{RP}^5_3 \to S^5$ by the collapse map. Then $V_{3,6}$ has a bunch of things in higher dimensions. The Stiefel manifolds start looking like the stunted projective spaces and then have a bunch of much higher-dimension cells.

(It's a little surprising to have a complex with so many high-dimension cells which can map via a degree one map to $S^5$.)

So we have a little copy of $\mathbb{RP}^5_3$ sitting inside $V_{3,6}$. We know that the 4-cell is attached to the 3-cell by the degree 2 map. We know that if we kill the bottom cell, there's a splitting.

Let's think about what this $\mathbb{RP}^5_3$ looks like. I know that there's a map and cofiber sequence

$$S^4 \xrightarrow{f} \mathbb{RP}^4_3 \to \mathbb{RP}^5_3$$

where the map $S^4 \to \mathbb{RP}^4_3$ comes from the double cover $S^4 \to \mathbb{RP}^4$ followed by crushing. So we get an element

$$f \in \pi_4(\mathbb{RP}^4_3).$$

So again, we're interested whether there is a section $S^5 \to \mathbb{RP}^5_3$.

**Claim:** $S^5$ has two vector fields if and only if $f$ is zero.

One direction is easy: if $f = 0$, then the map $f$ is null, and $\mathbb{RP}^5_3 \simeq \mathbb{RP}^4_3 \vee S^5$. So we have a section $S^5 \to \mathbb{RP}^5_3$.

The interesting direction is the other one. Here's a way to argue that. Suppose I had a section $S^5 \to \mathbb{RP}^5_3$. Let's look at the sequence

$$\mathbb{RP}^4_3 \to \mathbb{RP}^5_3 \to S^5$$

and note that we're in the stable range, by the Freudenthal suspension theorem. In a range of dimensions, there's a long exact sequence of homotopy groups:

$$\pi_5(\mathbb{RP}^3_5) \to \pi_5(S^5) \to \pi_4(\mathbb{RP}^4_3) \to \pi_4(\mathbb{RP}^5_3)$$

and if we had two vector fields, the first map in the sequence is a split surjection. This means that $\pi_5(S^5) \to \pi_4(\mathbb{RP}^4_3)$ is zero, and this map sends the generator to $f$. So $f = 0$.

Anyway, let me say it this way. Suppose I have a cofiber sequence

$$S^{n-1} \xrightarrow{f} X \to X \cup e^n \to S^n$$

and suppose I have a section $S^n \to X \cup e^n$. If we're in the stable range (i.e. $X$ is about $n/2$ connected). Then TFAE:

1. The section exists.

2. $f = 0$.

3. $X \cup e^n = X \vee S^n$ in a manner compatible with this cofiber sequence.

If you work with these, you say that the top cell splits off, or the attaching map was zero. You need to be in the stable range to guarantee this.

We want to understand the possibilities of $f$. So we'd like to understand:

**Question.** What is $\pi_4(\mathbb{RP}^4_3)$?

Again, let's use the cofiber sequence

$$S^3 \xrightarrow{2} S^3 \to \mathbb{RP}^4_3$$

and we get a long exact sequence of homotopy groups in the range we care about. We get a map

$$\pi_4(S^3) \xrightarrow{2} \pi_4(S^3) \to \pi_4(\mathbb{RP}^4_3) \to \pi_3(S^3) \xrightarrow{2} \pi_3(S^3).$$

Since we're in the stable range, we can do this. We find that the map $\pi_4(S^3) \to \pi_4(\mathbb{RP}^4_3)$ is an isomorphism, and there are two possibilities for the map $f$ and $\mathbb{RP}^5_3$.

1. Either the attaching map $S^4 \to \mathbb{RP}^4_3$ is nontrivial and factors through the bottom cell.

2. Or the attaching map is zero.

So the question of whether this attaching map is zero is equivalent to whether $S^5$ has two vector fields.

How could we possibly tell what that map might be? To solve the vector field problem, we have to figure out whether this map is not the zero map. Actually, there is an idea that we can try. The nontrivial map $S^4 \to S^3$ is the suspension of the Hopf map, which has Hopf invariant one. So we might try to measure something using the Hopf invariant, except that we've suspended things and we can't yet say anything about the Hopf invariant once we've suspended everything.

I want you to understand what's at stake and what the picture is here. As I said, there are only two possibilities. We'd need to make some calculation about this projective space to decide how the cells were attached. The original method was to use Steenrod operations. You can calculate $\mathrm{Sq}^2$ on the class in $H^3(\mathbb{RP}^5_3; \mathbb{Z}/2)$ and check that it's not zero. Then you could conclude. Alternatively, we can compute the $e$-invariant of this map and find that it's not zero. So in any case, what I learn is that the attaching map is not zero, and there is only one vector field on $S^5$. We'll do this later.

## §3 James periodicity

There's another thing, which has an interesting explanation and which we'll find a better explanation for a little later. So we know from this that $S^5$ does not have two vector fields (I'm telling you this). We noticed last time that there was a kind of periodiicty in the $\mathbb{RP}^n_{n-1}$ was periodic of period 2. There's a generalization of that fact, which we'll have a better understanding of a little bit later. This was originally called **James periodicity.**

Roughly, it states that $\mathbb{RP}^n_{n-k}$ is "periodic" in $n$ with period $2^{\text{somepower}}$ and the power is a little complicated to say. The power is related to the Radon-Hurwitz number. In other words, for some $m$,

$$\mathbb{RP}^{n+m}_{n+m-k} \simeq \Sigma^m \mathbb{RP}^n_{n-k}.$$

For instance,

$$\mathbb{RP}^n_{n-2}$$

is periodic in $n$, with period 4.

Anyway, remember this argument: if the five-sphere had two vector fields, so would $S^{6n-1}$ for all $n$. For instance, $S^{11}$ would. That makes you think that the little piece $\mathbb{RP}^5_3$ would be related to $\mathbb{RP}^{11}_9$. At least, we know that if one has a top cell that splits off, so does the other. I'll explain this.

The picture that we're supposed to get from this: in the stable range, $S^n$ has $k$ vector fields, but not $k+1$, if and only if the attaching map for the top cell in $\mathbb{RP}^n_{n-k-1}$ (i.e., $S^{n-1} \to \mathbb{RP}^{n-1}_{n-k-1}$) factors through the bottom cell, but is not zero.

One thing that's really remarkable is that we're constantly using here the notion of a CW complex, and the notion of skeleta. These discoveries came soon after the discovery of CW complexes and the work on the vector fields problem came soon after Whitehead invented CW complexes. You wouldn't be able to do this at all without this idea.

# Lecture 19
# 10/22

## §1 A loose end

So we're still talking about the algebraic topology of Stiefel manifolds and the vector field problem. There was something I did last time, and I think I made it a little more complicated than it needed to be. We were studying the following: we had a map

$$f : A \to X$$

and a cofiber sequence

$$A \to X \to X \cup CA \to \Sigma A$$

and we supposed that there was a section of $X \cup CA \to \Sigma A$. In this case, we have an equivalence

$$X \cup CA \simeq X \vee \Sigma A.$$

Under convenient conditions (in particular, *in the stable range*), this is equivalent to saying that $f$ is null. I gave a proof of that, but it's a little easier to continue the Puppe sequence one more step

$$A \to X \to X \cup CA \to \Sigma A \stackrel{-f}{\to} \Sigma X$$

and observe that if $\Sigma A$ sits inside as a summand of $X \cup CA$, we get that $\Sigma A \stackrel{-f}{\to} \Sigma X$ is null (namely, in this case we factor $\Sigma A \to \Sigma X$ as $\Sigma A \to X \cup CA \to \Sigma A \to \Sigma X$

and the composite of two maps in a cofibration sequence is null). If $X \to \Omega\Sigma X$ is an equivalence through the dimension of $A$, then we can conclude that $f$ is null. I just wanted to reinforce that this is the better way to organize it.

## §2 Stiefel manifolds and the intrinsic join

Here's the deal. I want to take this argument that if $S^{n-1}$ has $k$ vector fields, then so does $S^{mn-1}$ for each $m$. We had a homotopy-theoretic argument based on the James's *intrinsic join.* This is a map

$$V_{k,n} * V_{k,m} \to V_{k,n+m}$$

(where $*$ means *join*) and the map sends a frame in $\mathbb{R}^m$ and a frame in $\mathbb{R}^n$ and an angle $\theta$ to a frame in $\mathbb{R}^{m+n}$. That is, given a pair of $k$-frames $(f_1, f_2)$ and an angle $\theta$ to $(\cos\theta)f_1 + (\sin\theta)f_2$. So you take the two frames $f_1, f_2$ and rotate them through the angle $\theta$.

These maps are *compatible* when $n, k$ change. The following diagram commutes:

$$\begin{tikzcd} V_{k,n} * V_{k,m} \arrow[r] \arrow[d] & V_{k,n+m} \arrow[d] \\ S^{n-1} * S^{m-1} \arrow[r, "\simeq"] & S^{n+m-1} \end{tikzcd} .$$

More generally, we get a commutative diagram:

$$\begin{tikzcd} V_{k,n} * V_{k,m} \arrow[r] \arrow[d] & V_{k,n+m} \arrow[d] \\ V'_{k,n} * V_{k',m} \arrow[r, "\simeq"] & V_{k',n+m} \end{tikzcd}$$

for $k' \leq k$. We'd like to know what the intrinsic join does in homology. There is a pretty complicated argument in James's paper, but one can use these diagrams to give a simpler one.

To see this, let's review the homology of the join. We have a functorial isomorphism

$$S^{n-1} * S^{m-1} \simeq S^{n+m-1}$$

and we learn that

$$C_*^{\text{cell}}(X * Y) = \Sigma C_*^{\text{cell}}(X) \otimes C_*^{\text{cell}}(Y).$$

In fact, if you think about this, one has a homotopy equivalence

$$X * Y \simeq \Sigma(X \wedge Y).$$

If I use field coefficients, we get

$$\widetilde{H}_*(X * Y) = \Sigma\left(\widetilde{H}_*(X) \otimes \widetilde{H}_*(Y)\right).$$

**Notation:** If $x \in H_k(X), y \in H_l(Y)$, I'll write $x * y$ for the corresponding element in $H_{k+l-1}(X * Y)$.

Here's the thing about the effect in homology of James's map. Let $b_i \in H_i(\mathbb{RP}^b_a)$ be the nonzero homology class of some stunted projective space, $a \le i \le b$. Again, we have a diagram

$$\begin{tikzcd} \mathbb{RP}^{n-1}_{n-k} * \mathbb{RP}^{m-1}_{m-k} \arrow[d] & \\ V_{k,m} * V_{k,m} \arrow[r] & V_{k,m+n} \end{tikzcd}$$

**19.1 Proposition** (To be proved later)**.** *Under the intrinsic join, the James map sends the class $b_{n-1} * b_i$ goes over to the class $b_{i+n}$ (under $\mathbb{RP}^{n+m-1}_{n+m-k} \to V_{k,m+n}$).*

The Stiefel manifold $V_{k,n}$ has classes from $n-k$ to $n-1$ and the manifold $V_{k,m}$ has classes in dimensions from $m-k$ to $m-1$ (and higher product classes). There's the join of these things, which maps from the Stiefel manifold $V_{k,n+m}$ which starts from $n+m-1$ and goes down to $n+m-k$. All of these are strings of exactly $k$ cells. The point is that the top cell joined with one string of $k$ cells matches the $k$ cells in $V_{k,n+m}$. That's what this statement is saying. In fact, it's symmetric. We'll later see that this map looks like the *dual* of the cup product map in cohomology.

It's not as good when we don't use the top cell.

We have this cell decomposition of Stiefel manifolds, and we're just trying to learn about the vector fields problem. This is way more information we need, but it'll serve us well when we get back into the EHP sequence. Sometimes there's a map of a sphere into the Stiefel manifold which hits the top cell.

## §3 James periodicity

*Suppose that $S^{n-1}$ has $k-1$ vector fields.* In that case, we have this map $V_{k,n} \to S^{n-1}$ which has a section. Now that gives a section $S^{n-1} * V_{k,m} \to V_{k,n} * V_{k,m} \to V_{k,n+m}$. At least in homology, we have a commutative diagram:

$$\begin{tikzcd} S^{n-1} * \mathbb{RP}^{m-1}_{m-k} \arrow[r] \arrow[d, "\simeq"] & \mathbb{RP}^{n+m-1}_{n+m-1} \arrow[d] \\ S^{n-1} * V_{k,m} \arrow[r] & V_{k,n+m} \end{tikzcd}.$$

I say in **homology,** because I don't know that there is such a commutative diagram in the homotopy category. However, most of the time, that actually happens in homotopy as well. The dimension of $S^{n-1} * \mathbb{RP}^{m-1}_{m-k}$ has dimension $m+n-1$. This Stiefel manfiold $V_{k,n+m}$ can be described as $\mathbb{RP}^{n+m-1}_{n+m-k} \cup$ cells where the cells have dimension at least $n+m-k+n+m-k+1 = 2n+2m-2k+1$ and higher.

**Remark.** If $Z$ is a CW complex of dimension $\le l$ and $A \to X$ is $l-1$-connected, then we have a surjection $[Z, A] \to [Z, X]$, and if $\dim Z < l$, it's a bijection. Over here, we're saying that the pair $(V_{k,n+m}, \mathbb{RP}^{n+m-1}_{n+m-k})$ is highly connected.

Actually, let's work it out.

What I claim is, most of the time, the map

$$S^{n-1} * \mathbb{RP}^{m-1}_{m-k} \to V_{k,n+m}$$

actually factors through a map into $\mathbb{RP}^{n+m-1}_{n+m-k}$. This is going to happen when

$$2n + 2m - 2k + 1 > n + m - 1.$$

So equivalently, if

$$n + m > 2k - 2.$$

Here $n$ was fixed, and $m$ is larger. So one $m > 2k - 2 - n$ (which happens most of the time: $2k - 2 - n$ is probably negative since $S^{n-1}$ had $k$ vector fields), then the above diagram actually commutes up to homotopy, and in particular we get an equivalence

$$\Sigma^n \mathbb{RP}^{m-1}_{m-k} \simeq \mathbb{RP}^{n+m-1}_{n+m-k}$$

as we talked about earlier. That's **James periodicity.**

You can do better than I've done, and we will do better a little later, but this is something we noticed *just by looking at stunted projective spaces by hand in low dimensions.* Let me just say it in a slightly less cumbersome way: it just says that if $S^{n-1}$ has $k-1$ vector fields, then we get length $k$ stunted projective spaces (i.e., those with $k$ cells) are periodic (with period $n$). This was kind of a miracle in its day. We're soon going to have a much more elegant explanation for James periodicity. This explanation is so geometric, though, that it's worth remembering, and there's something nice that's going to come out of this in a second.

So we've constructed a bunch of vector fields on spheres, so we can prove examples of James periodicity. It's a little bit technical because there are a lot of numbers to process. I could do some actual numerical examples, or I could get on to more conceptual things. Let's do a reality check.

In the last class, we learned that $\mathbb{RP}^n_{n-1}$ is periodic with period 2, that is, we had equivalences

$$\Sigma^2 \mathbb{RP}^{\ell}_{\ell-1} \simeq \mathbb{RP}^{\ell+2}_{\ell+1}.$$

How could we get that here? We could take $n = 2$, $k = 2$ and use the fact that $S^1$ has one vector field. We could use the fact that $S^3$ has three vector fields to get

$$\Sigma^4 \mathbb{RP}^n_{n-4} \simeq \mathbb{RP}^{n+4}_n.$$

The reason that sort of tells us something is that the issue of a sphere having so many vector fields had to do with the top cell of these stunted projective spaces split off. This turns out to be a **stable problem**, because of James periodicity.

**Remark.** We could do that as well for stunted *complex* projective spaces. (To be returned to later.)

There's something even better that comes out. Let's go back to the situation and *reverse* the roles of the two spheres. We had this map

$$S^{n-1} * V_{k,m} \to V_{k,n+m}$$

and sitting inside $V_{k,n+m}$ was this stunted projective space $\mathbb{RP}^{n+m-1}_{n+m-k}$ where this map

$$\mathbb{RP}^{n+m-1}_{n+m-k} \to V_{k,n+m}$$

was an $\ell$-equivalence, for $\ell = 2(n+m-k)$. This is when $S^{n-1}$ has $k-1$ vector fields. Last time, when we thought this through, we just paid attention to the stunted projective space. We note that $S^{n-1} * V_{k,m}$ has dimension $n+d$ where $d$ is *independent of* $n$. On the other hand, we have $2(n+m-k)$-connectivity for the map $\mathbb{RP}^{n+m-1}_{n+m-k} \to V_{k,n+m}$. For a given $k$ and $N$, we can find a sphere $S^{n-1}$ with $n > N$ such that $S^{n-1}$ has $k-1$ vector fields. So playing around with this, we can assume $n \gg 0$.

By taking $n \gg 0$, we may assume that the dimension of that join $S^{n-1} * V_{k,m}$ is smaller than the connectivity of the pair $(V_{k,n+m}, \mathbb{RP}^{n+m-1}_{n+m-k})$. That's because we have a dimension $n + \text{constant}$ versus $2n + \text{constant}$ connectivity. For $n \gg 0$, we produce a map

$$S^{n-1} * V_{k,m} \to \mathbb{RP}^{n+m-1}_{n+m-k}$$

which has the property that the composite

$$S^{n-1} * \mathbb{RP}^{m-1}_{m-k} \to S^n * V_{k,n} \to \mathbb{RP}^{n+m-1}_{n+m-k}$$

is a homotopy equivalence (James periodicity).

The consequence of this is a very beautiful fact:

**19.2 Corollary** (James)**.** *Stably, we have a splitting*

$$S^n \wedge V_{k,m} \simeq S^n \wedge \mathbb{RP}^{m-1}_{m-k} \vee \text{anotherspace}.$$

That stunted projective space, living inside the Stiefel manifold, breaks off after you suspend it a bunch of times. This even applies when $k = m$. When $k = m$, this implies that a big suspension of $SO(m)$ is homotopy equivalent to a big suspension $\mathbb{RP}^{m-1}$ wedge another space. There are some neat uses of this. I was really fascinated by this when I was a graduate student, and there was a lot of speculation about whether there was a further decomposition. The homology of $SO(m)$ is an exterior algebra and as a result, it was believed that the decomposition of $\Sigma^\infty SO(m)$ went for all these exterior pieces. Haynes Miller proved this. "Stable splittings of Stiefel manifolds" is the paper.

That's the end of my little tour of the homotopy theory of Stiefel manifolds, but a lot of arguments—which are not very widely known—can be used to prove things like the **Adams conjecture.** That's the easiest proof of the Adams conjecture I know, but I've never seen it written down anywhere. Some of us who like to think about motivic homotopy theory use analogs of these maps in motivic homotopy theory. They work in many other contexts. You have to be careful where your join coordinate lives. I kind of want to advertise these because they are beautiful applications of the theory of CW complexes. These are some theorems that never became that widely known, but they're extremely useful theorems that work in much broader contexts than this one.

# Lecture 20
# 10/24

I still owe you the computation of the intrinsic join in homology, and I'll say a little about that today. I keep finding myself hampered by the fact that I'm not working in the stable homotopy category. I want to explain today the basics of stable homotopy insofar as I'll use them in the near future.

## §1 Stable homotopy

So we've seen that a lot of our problems that we're looking at don't change if we suspend a few times. We've seen that if $\dim A < 2\text{connectivity}(X)$, then the map

$$[A, X] \to [\Sigma A, \Sigma X]$$

is a bijection. Then $\Sigma A$ has dimension $\dim A + 1$ and the connectivity of $X$ bumps up by 1. So the condition gets easier and easier after suspending. This condition is always eventually met, if $A$ is a finite-dimensional CW complex and $X$ is arbitrary. In other words, we can define:

$$\{A, X\} = \varinjlim_n [\Sigma^n A, \Sigma^n A],$$

and the system actually stabilizes at some finite stage. If $\dim A = d$, then $\dim \Sigma^n A = d + n$, while the connectivity of $\Sigma^n X$ is $n - 1$, so we need $n$ large enough such that

$$d + n < 2(n-1),$$

and that's the same as saying

$$n > d + 2.$$

A lot of times, you're in that range. In this range, good things happen: for instance, cofiber and fiber sequences are the same.

## §2 The Spanier-Whitehead category

This is probably actually good enough for our purposes. Any model for stable homotopy theory that you produce has to contain the Spanier-Whitehead category. This is what you need to start with, and there are a lot of ways of embellishing it to have good properties.

- The objects are finite pointed CW complexes $X$.
- The maps $\{X, Y\} \overset{\text{def}}{=} \varinjlim_n [\Sigma^n X, \Sigma^n Y]$.

Some simple things that happen (I'm assuming you know about the Freudenthal suspension theorem):

1. If $A \subset X$ is a subcomplex and I look at $X/A$, then the following happens. We have two long exact sequences: I can stick this in either variable and get along exact sequence. The easier one is, for any $Y$, I get a long exact sequence

   $$\{X/A, Y\} \to \{X, Y\} \to \{A, Y\},$$

   and this is exact. But it's also true in the other variable. For any $Z$, we have a long exact sequence

   $$\{Z, A\} \to \{Z, X\} \to \{Z, X/A\}.$$

The first statement is true at any stage, and the second is only true after a sufficiently high suspension. The point is that the map

$$\Sigma^n A \to \text{fiber}(\Sigma^n X \to \Sigma^n(X/A))$$

is an equivalence through a large range of dimensions (about $2n$). You can work out that number, say, from the **Serre spectral sequence**. I'm going to leave it to you to do that.

2. In both cases, we can extend the sequence to an *infinite* exact sequence in both directions. So for instance, let's take the first one. It extends in one direction obviously, because we just take the Barratt-Puppe sequence

$$A \to X \to X/A \to \Sigma A \to \Sigma X \to \dots,$$

and map that into $Y$, and even at every stage, I get a long exact sequence. Taking colimits, I get

$$\{A, Y\} \leftarrow \{X, Y\} \leftarrow \{X/A, Y\} \leftarrow \{\Sigma A, Y\} \leftarrow \{\Sigma X, Y\} \leftarrow \dots.$$

We can extend in both dimensions, though, because we've rigged things such that

$$\{A, Y\} \simeq \{\Sigma A, \Sigma Y\},$$

sort of by definition of the colimit. When we move into the right-hand directions, we were suspending the first variable, and when we move in the left-hand directions.

3. The same thing is going to happen with the other sequence. Now these are going to go in the normal direction. In the other sequence, we would have

$$\{Z, A\} \to \{Z, X\} \to \{Z, X/A\} \to \{Z, \Sigma A\} \to \dots$$

and we can extend it in the other direction by suspending $Z$ instead of $A$. For instance, we could continue the sequence:

$$\{\Sigma Z, X\} \to \{\Sigma Z, X/A\} \to \{Z, A\} \to \{Z, X\} \to \{Z, X/A\} \to \{Z, \Sigma A\} \to \dots.$$

This is a little inconvenient to suspend in one variable one way and to suspend in the other variable the other way. This is inconvenient, because you're treating the variables separately, while these long exact sequences are great.

It's therefore nice to be able to add new objects to the Spanier-Whitehead category. We're going to add objects $\Sigma^{-n}A, n > 0$. In order for this to make sense, we have to define maps into and out of it. We set

$$\left\{\Sigma^{-n}A, X\right\} = \{A, \Sigma^n X\}$$

and similarly

$$\left\{Z, \Sigma^{-n}A\right\} = \{\Sigma^n Z, A\}.$$

It's more convenient to add these objects. If I introduce these *formal desuspensions* of objects and define the objects like this, then I can rewrite my sequences like

$$\dots \to \{Z, \Sigma^m A\} \to \{Z, \Sigma^m X\} \to \{Z, \Sigma^m (X/A)\} \to \left\{Z, \Sigma^{m+1} A\right\} \to \dots$$

and that's true for **all integers** $m$**.** The other sequence would have worked out the same. The Spanier-Whitehead category works out a little better when you introduce these formal desuspensions.

## §3 Spanier-Whitehead duality

The reason Spanier-Whitehead introduced the category was because of **Spanier-Whitehead duality.** In the Spanier-Whitehead category, the operation $X \wedge Y$ makes sense: we use the smash product spaces. If you think through the definition, you can arrange things so that

$$\Sigma^{-n} X \wedge \Sigma^{-m} Y = \Sigma^{-n-m} (X \wedge Y).$$

So you can extend to the formal desuspensions. This makes the Spanier-Whitehead category into a **symmetric monoidal category.** The unit is the sphere $S^0$.

In a symmetric monoidal category like this, we say that:

**20.1 Definition.** (This would work in any symmetric monoidal category; I'm in the Spanier-Whitehead category.) A **dual** of $X$ is an object $Y$ equipped with maps

$$X \wedge Y \to S^0, \quad S^0 \to Y \wedge X$$

such that the following composite:

$$X \wedge S^0 \to X \wedge Y \wedge X \to S^0 \wedge X$$

is the tautological equivalence. Similarly, I require that

$$S^0 \wedge Y \to Y \wedge X \wedge Y \to Y \wedge S^0$$

be the tautological equivalence.

If $Y$ is a dual of $X$, then here's a simple proposition:

**20.2 Proposition.** *If $Y$ is a dual of $X$, then I can make the following maps. Given any $W, Z$, I can consider*

$$\{Z \wedge Y, W\} \stackrel{\wedge X}{\to} \{Z \wedge Y \wedge X, W \wedge X\} \to \left\{Z \wedge S^0, W \wedge X\right\}$$

*and this composite map is an isomorphism.*

It's like vector space duality. It's supposed to remind you of what happens in vector spaces. A map $Z \otimes Y$ into $W$ is the same as maps $Z \to W \otimes Y^\vee$, in vector spaces.

Another easy reuslt:

**20.3 Proposition.** *If $Y$ is a dual of $X$, then $X$ is a dual of $Y$.*

That's because it is a **symmetric** monoidal category, and you can commute all of these.

I keep talking about "a" dual, but the dual is unique up to unique isomorphism. This first proposition tells me what functor it represents. We know that

$$\{Y, \cdot\} \simeq \left\{S^0, \cdot \wedge X\right\},$$

and this determines $Y$ in terms of $X$, uniquely up to unique isomorphism in the Spanier-Whitehead category.

I want to write this dual as a functor.

**20.4 Definition.** If $X$ is in the Spanier-Whitehead category, I'm going to write $\mathbb{D}X$ for the Spanier-Whitehead dual of $X$ (assuming it exists).

If I have a map $X_1 \xrightarrow{f} X_2$ and if I know that the duals of both terms exist, then I get a map

$$\mathbb{D}f : \mathbb{D}X_2 \to \mathbb{D}X_1.$$

That just follows because I know what functors are being represented and corepresented by the duals, so it's standard category theory. In fact,

$$\{\mathbb{D}X_2, \mathbb{D}X_1\} \simeq \left\{S^0, X_2 \wedge \mathbb{D}X_1\right\} \simeq \{X_1, X_2\},$$

and we take the map corresponding to $f$ at the end.

So if the dual exists, then $\mathbb{D}$ is a **contravariant functor** from the Spanier-Whitehead category to itself.

**20.5 Definition.** $\mathcal{SW}$ will denote the Spanier-Whitehead category.

In fact, once we've seen that the dual exists, we'll have:

**20.6 Proposition.** *$\mathbb{D} : \mathcal{SW} \to \mathcal{SW}$ is a contravariant functor which squares to the identity.*

Let's note first that the dual of the sphere is the minus sphere, i.e.

$$\mathbb{D}S^n \simeq S^{-n} = \Sigma^{-n}S^0.$$

So every sphere has a dual. If we have a cofiber sequence

$$A \to X \to X/A,$$

and if $A, X$ have duals, then $X/A$ has a dual and it's going to be forced to fit into the cofiber sequence

$$\mathbb{D}A \leftarrow \mathbb{D}X \leftarrow \mathbb{D}(X/A),$$

and that says that $\mathbb{D}(X/A)$ is the desuspension of the cofiber of the map $\mathbb{D}X \to \mathbb{D}A$.

So we have the duals for the sphere, and we get it from things that we can build from spheres and cofiber sequences. We thus get:

**20.7 Proposition.** *Every $X$ has a dual.*

I'm kind of breezing through this. You can read about this in Hatcher or in the exercises in Spanier's book, which have an excellent discussion of duals. Every $X$ has a dual, but this hasn't helped us very much yet. We have another result:

**20.8 Proposition.** *$\mathbb{D}(X \wedge Y) \simeq \mathbb{D}X \wedge \mathbb{D}Y$ for $X, Y \in \mathcal{SW}$.*

## §4 Formulas for $\mathbb{D}X$

This is very useful on its own, but it helps a lot if you know what the dual **is.** You start getting somewhere when you learn what the dual is.

Suppose $X$ is a CW complex and I can imbed $X \hookrightarrow S^n$. Let's even suppose that that this extends to a CW decomposition of the sphere. I'm going to suppose that $S^n \setminus X$ is a finite CW complex (up to deformation retraction). What I'm picturing is, for instance, that $S^{p+q-1} \setminus S^{p-1} \simeq S^{q-1}$. This is a *really important picture,* and I'm going to keep coming back to it in other guises. That's a typical example.

Let $A \subset S^n \setminus X$ be a finite subcomplex which is a deformation retract of $S^n \setminus X$.

You want to place these in the sphere, by possibly moving things up to a homotopy, that:

1. No point of $A$ is antipodal to any point of $X$. Here $A \subset S^n \setminus X$ is homotopy equivalent to it.

2. Then you get a map $A * X \to S^n$ by sending $(a, x, t)$ to the $\gamma_a^x(t)$ where $\gamma_a^x$ is the unique geodesic joining $a$ to $x$.

3. Alexander duality ends up implying that $X$ and $A$ end up becoming Spanier-Whitehead duals. In fact,
$$A * X \simeq \Sigma(A \wedge X),$$
and I have a map from that to $S^n$. That gives a map in the Spanier-Whitehead category,
$$A \wedge \Sigma^{-(N-1)} X \to S^0.$$
This, you can check, makes $\Sigma^{-(N-1)}X$ into the dual of $A$.

4. Notice that you apparently also need to provide a map $A \wedge \Sigma^{-(N-1)}X \to S^0$, but you only really need to provide one map. Given a map $X \wedge Y \to S^0$, that already gives me a transformation
$$\{Z, W \wedge X\} \to \{Z \wedge Y, W \wedge X \wedge Y\} \to \{Z \wedge Y, W\}$$
and if that map is an isomorphism, then $X$ and $Y$ are duals, and the other map (i.e., $S^0 \to Y \wedge X$) is given to us from this isomorphism. That is, the map $S^0 \to Y \wedge X$ is isomorphic under that isomorphism from $\{Y, Y\}$ and we take the one corresponding to the identity.

You can read about this in Hatcher's book. My goal today was to summarize the basic properties of Spanier-Whitehead duality. This last piece isn't formal, but it's a consequence of Alexander duality. In the next class, I'm going to give you Atiyah's formula for the dual, and I want to use that to think about lots of duals. I'm going to use Thom complexes of vector bundles, and I'll introduce it very briefly next time. Spanier-Whitehead duality mixes with the theory of Thom complexes in a very beautiful way, and I'll explain that in the next class.

# Lecture 21
# 10/26

## §1 Thom complexes

We were talking about stable homotopy theory and the **Spanier-Whitehead category.** I want to describe some natural objects in that category, and those are **Thom complexes.**

Suppose I have a space $X$ and a vector bundle $V \to X$. There are a couple of different definitions we can give of the Thom complex.

**21.1 Definition.** Suppose $X$ is compact, e.g., a finite CW complex. Then the **Thom complex** $X^V$ of $V \to X$ is the one-point compactification of $V$ (i.e., of the total space).

For more general $X$, $X^V$ is defined as the direct limit of $X_\alpha^V$ as $X_\alpha \subset X$ runs through the compact subspaces. As a set, it still is the same (one extra point), but the topology is a little different.

We'll use this form of the definition today, but if $X$ is paracompact, you can choose a (positive-definite) metric on $V$, and let $B(V)$ be the unit ball bundle in $V$, and let $S(V)$ be the unit sphere. Then we could define

$$X^V = B(V)/S(V).$$

**Alternative notation.** An alternative notation for the Thom complex, when there are other superscripts, is $\mathrm{Thom}(X, V)$.

**21.2 Example.** Suppose $V = X \times \mathbb{R}^n$ is trivial. Using the second definition, we find that

$$X^V = X \times D^n/(X \times S^{n-1}) = X_+ \wedge S^n.$$

(Here $X_+$ is $X$ with a disjoint basepoint.) Notice that if $X$ is a cell complex, and if I were to draw a "picture" of it, then we notice that the cells of the Thom complex are the same cells as the cells of $X$, but now starting in dimension $n$. There's another thing that's important to think about. We don't imagine that $X$ started out with a basepoint. It can, but you get more uniform statements if you don't regard $X$ as starting out with a basepoint but do regard $X^V$ as having a basepoint.

There's a one-to-one correspondence between the cells of $X_+$ and those of the Thom complex. In particular, we have an isomorphism in cohomology, which is just the suspension isomorphism,

$$H^*(X) \simeq \widetilde{H}^{*+n}(X^V).$$

**Remark.** As said above, $X^V$ has a **canonical basepoint,** for any $V$ (as a one-point compactification or quotient).

Now let's imagine that $X$ is a CW complex, and let's imagine that we attached a cell. So start with some space $A$, which sits inside $X = A \cup_f e^m$. Suppose we have some vector bundle $V \to X$. Let $V_A$ be the restriction to $A$. We have a map of Thom complexes

$$A^{V_A} \to X^V,$$

and we'd like to ask what the quotient space $X^V/A^{V_A}$ is. Notice, however, that we have this pushout diagram:

$$\begin{tikzcd} S^{m-1} \arrow[r] \arrow[d] & D^m \arrow[d] \\ A \arrow[r] & X \end{tikzcd},$$

and I have this vector bundle $V$ sitting over $X$. At this point it's going to be a drag to give $V$ different names, so I'm going to use the same symbol for the pull-backs of $V$.

I'm taking the one-point compactification of $V$ and modding out everything that lives over $A$. That's the same thing as pulling back $V$ to $D^m$, taking the Thom complex, and crushing the sphere. So

$$X^V/A^V = (D^m)^V/(S^{m-1})^V.$$

Now $V|_{D^m} \simeq D^m \times \mathbb{R}^n$, and we already worked out what that Thom complex is. If I further restrict to the boundary sphere, it's still trivial. So we can calculate that

$$(D^m)^V/(S^{m-1})^V = D^m_+ \wedge S^n/S^{m-1}_+ \wedge S^n \simeq (D^m/S^{m-1}) \wedge S^n.$$

Let me make a slightly more general statement:

**21.3 Proposition.** *If I take $V|_{X^{(m)}}$, then $(X^{(m)})^V/(X^{(m-1)})^V$ is a wedge over the m-cells $\bigvee_{\mathrm{m-cells}} S^m \wedge S^n$.*

If you think about this a little more, you learn that:

**21.4 Proposition.** *$(X^{(m)})^V$ is the skeleton filtration of a CW decomposition of $X^V$, which has one $m+n$-cell for every $m$-cell of $X$.*

This argument also shows that we get an isomorphism on the cellular chains:

**21.5 Proposition.** *There is an isomorphism $C_*^{\mathrm{cell}}(X)$ with the $\widetilde{C}_{*+n}(X^V)$ as graded abelian groups.*

**However,** this isomorphism isn't an isomorphism of complexes, as it doesn't generally commute with the differential $d^{\mathrm{cell}}$. The identifications we made depending on trivializations of vector bundles over spheres, and those could be done in different ways. But nevertheless, it leads one to look for an isomorphism,

$$H_*(X) \simeq \widetilde{H}_{*+\dim V}(X^V),$$

or

$$H^*(X) \simeq \widetilde{H}^{*+\dim V}(X^V).$$

Under convenient conditions, there is such an isomorphism.

## §2 The Thom isomorphism

We first have to define something. Suppose $V \to X$ is a vector bundle over $X$ of dimension $n$.

**21.6 Definition.** A **Thom class** for $V$ is an element $U \in H^n(X^V)$ with the property that, for each $x \in X$, the map

$$S^{V_x} \simeq \{x\}^V \to X^V,$$

(where the Thom complex of $V|_{\{x\}}$ is the one-point compactification $S^{V_x}$, for $V_x$ the fiber over $x$), carries $U$ to a generator of $H^n(S^{V_x})$.

The Thom isomorphism theorem, which is quite easy to prove, is that:

**21.7 Proposition.** *If $U$ is a Thom class for $V$, then multiplication by $U$ induces an isomorphism*

$$H^*(X) \simeq \widetilde{H}^{*+\dim V}(X^V) \simeq H^{*+\dim V}(B(V), S(V)),$$

*of modules over $H^*(X)$.*

We note that the cohomology of $(B(V), S(V))$ is a module over $H^*(X)$, because $B(V)$ is homotopy equivalent to $X$. The proof is very easy—here's a sketch.

1. Check for trivial bundles (which is easy).
2. Show by induction on $j$ that if $X = W_1 \cup \cdots \cup W_j$, where each $W_i$ is open and $V|_{W_i}$ is trivial, then it's true for $X$. That's a very simple argument using the Mayer-Vietoris sequence. If you know it for $j-1$, cover it by the union of the first $j-1$ and the last one, and use the Mayer-Vietoris sequence for $X$ and for the Thom complex, and the five-lemma.
3. A limiting argument, if $X$ is paracompact.

You can easily figure this out. But we'll probably come back to the Thom isomorphism for other cohomology theories a little bit later.

## §3 Examples

I wanted to do some examples of Thom complexes, because these are really important. First, I want to generalize one thing. Here's another observation.

**21.8 Proposition.** *Let $V$ be a vector bundle on $X$. $X^{V\oplus\mathbb{R}^m} \simeq S^m \wedge X^V$. More generally, let $V \to X$, and $W \to Y$ be vector bundles. Form the external Whitney sum $V \oplus W \to X \times Y$. Then*

$$(X \times Y)^{V\oplus W} \simeq X^V \wedge Y^W.$$

This is almost immediate from the definition. The first assertion is a special case of the second with $Y = *$.

The point is that adding a trivial bundle just suspends the Thom complex. This means that we can define the Thom complex of a *virtual bundle.* Let $\xi$ be a virtual bundle.

**21.9 Definition.** If $\xi$ is a virtual bundle (in $KO^0(X)$, which is the group completion of the monoid of real vector bundles on $X$) on the finite CW complex $X$, then we can define a **Thom complex** $X^\xi$ in the Spanier-Whitehead category $\mathcal{SW}$ (i.e, the form including the formal desuspensions).

Namely, suppose $\xi = V - W$ where $V, W$ are finite-dimensional bundles over $X$. Choose a vector bundle $W'$ such that $W \oplus W'$ is trivial, so choose an isomorphism $W' \oplus W \simeq \mathbb{R}^N$. Then

$$V + W' - \mathbb{R}^N = V - W = \xi,$$

so we define the Thom complex

$$X^\xi = \Sigma^{-N} X^{V \oplus W'}.$$

I'm going to use this a lot. It's convenient to be able to talk about **Thom spectra,** or Thom complexes of virtual vector bundles, but you have to imagine them in the stable homotopy category.

Here are some examples.

**21.10 Example.** Let $V = kL$ where $L$ is the tautological line bundle $\mathbb{RP}^n$. What is $\mathrm{Thom}(\mathbb{RP}^n, kL)$? There are two ways to work this out, and they correspond to the two different definitions of the Thom complex. One of them is to notice this little fact about projective spaces. Here $\mathbb{RP}^n$ is all the lines through the origin in $\mathbb{R}^{n+1}$.

Let's say I have a line through the origin in $\mathbb{R}^{n+1+k}$ but not a "vertical" line, i.e. not in $\mathbb{R}^k$. In this case, such a line is a point in $\mathbb{RP}^{n+k} \setminus \mathbb{RP}^{k-1}$. A line that's not a vertical line will pass the "vertical line test," and will be the graph of its projection to $\mathbb{R}^{n+1}$. It shows that

$$\mathbb{RP}^{n+k} \setminus \mathbb{RP}^{k-1} \to \mathbb{RP}^n$$

is a bundle, and in fact this bundle is $\bigoplus^k L^*$. Since I'm over the reals, I can write this as $\bigoplus^k L$ (if I were over the complex numbers, I'd have to introduce conjugates). The Thom complex $\mathrm{Thom}(\mathbb{RP}^n, kL)$ is the one-point compactification of the total space. To calculate this, we can imbed the vector bundle (the total space) in a compact space, and then crush the complement. We have

$$kL \hookrightarrow \mathbb{RP}^{n+k} \setminus \mathbb{RP}^{k-1},$$

so that the complement of $kL$ in $\mathbb{RP}^{n+k}$ is $\mathbb{RP}^{k-1}$. So we can assert

$$\mathrm{Thom}(\mathbb{RP}^n, kL) = \mathbb{RP}^{n+k}/\mathbb{RP}^{k-1} = \mathbb{RP}^{n+k}_k,$$

i.e. we get the **stunted projective spaces** studied earlier.

**21.11 Example.** This extends our definition for stunted projective spaces. We can construct $\mathrm{Thom}(\mathbb{RP}^n, kL)$ when $k < 0$, and we can use this to define virtual stunted projective spaces in $\mathcal{SW}$. In other words, we can define $\mathbb{RP}^b_a$ for $a, b \in \mathbb{Z}$, and $b \geq a$.

We're going to build a lot on this Monday, but let me just do one another example which will play an important role.

**21.12 Example.** There is another important vector bundle on $\mathbb{RP}^n$, which is the **tangent bundle.** I'm going to describe this for you. This description also would work for any Grassmannian. Now $\mathbb{RP}^n$ is the space of lines in $\mathbb{R}^{n+1}$. Choose a point, i.e. a line $\ell \subset \mathbb{R}^{n+1}$. A little infinitesimal movement of that line $\ell$ is the graph of a homomorphism $\ell \to \ell^{\perp}$, just as before. This tells you that

$$T\mathbb{RP}^n = \mathrm{Hom}(L, L^{\perp}),$$

where $L \subset \mathbb{R}^{n+1} \times \mathbb{RP}^n$, and the quotient of that imbedding is the quotient $L^{\perp}$. If you hom $L$ into the sequence

$$0 \to L \to \mathbb{R}^{n+1} \times \mathbb{RP}^n \to L^{\perp} \to 0,$$

you get a sequence that goes

$$0 \to \mathrm{Hom}(L, L) \to \mathrm{Hom}(L, \mathbb{R}^{n+1}) \to \mathrm{Hom}(L, L^{\perp}) \to 0.$$

Since $\mathrm{Hom}(L, L)$ is trivial, you get that

$$\mathrm{Hom}(L, L^{\perp}) \simeq L^{n+1}.$$

If you put this together, you find that the tangent bundle to $\mathbb{RP}^n$ plus a trivial bundle is $(n+1)$ copies of $L$, $L^{\oplus(n+1)}$. Equivalently,

$$T\mathbb{RP}^n = (n+1)L - 1.$$

Let's now look at

$$(\mathbb{RP}^n)^{-T\mathbb{RP}^n} = \Sigma(\mathbb{RP}^n)^{-(n+1)L} = \Sigma\mathbb{RP}^{-1}_{-(n+1)}.$$

You're supposed to have this picture of the cells of real projective space, but you're allowed to extend them to negative dimensions. In the next section, we'll identify this with the Spanier-Whitehead dual of $\mathbb{RP}^n_+$, and that gives a useful relationship between stutnted projective spaces and Spanier-Whitehead duality. We'll come back to that in the next lecture.

# Lecture 22
# 10/31

## §1 Spanier-Whitehead duality

In the last class, we talked about **Spanier-Whitehead duality.** Spanier-Whitehead duality has a lot of important aspects to it. It's useful to be able to figure out the Spanier-Whitehead dual of something is. Last class, I showed that if $K \subset S^n \supset L$ are disjoint (with no antipodal points) and the inclusion

$$L \subset S^n \setminus K$$

is a homotopy equivalence, then $L$ and $K$ are Spanier-Whitehead duals. In this situation, you get a map

$$K * L \simeq \Sigma(K \wedge L) \to S^n$$

so you get a map

$$S^{-n} \wedge S^1 K \wedge L \to S^0,$$

which makes $S^{1-n} \wedge L \simeq \mathbb{D}K$. It's of low importance to keep track of the suspensions, at the beginning. If you look at the cells and where they are, you can figure out what the shift had to be. If you're learning about these things for the first time, I would advise ignoring these indices, for now.

I want to tell you a beautiful formula of Atiyah for the Spanier-Whitehead dual of a manifold with boundary. Let $M$ be a smooth, compact manifold with boundary $\partial M$. The boundary might be empty. You'll see that even if we're interested only in closed manifolds, it's important to include the case of a nonempty boundary. We choose an imbedding

$$M \hookrightarrow D^N$$

with the property that $\partial M \subset S^{N-1}$, and nothing other than the boundary goes into $S^{N-1}$. In other words, it's an imbedding of manifolds with boundary $(M, \partial M) \to (D^N, S^{N-1})$. This gives me an imbedding

$$M/\partial M \hookrightarrow D^N/S^{N-1} \simeq S^N.$$

If the boundary happened to be empty, a case which I allowed, then $M/\partial M = M \sqcup *$ (when you mod out by something you add a point and identify everything in that set to the point), and what we have is that

$$M \subset S^N \setminus \{\infty\}, \quad * \to \{\infty\}.$$

So we're going to get a formula for $\mathbb{D}(M/\partial M)$. As we saw, it is $S^N \setminus (M/\partial M)$. What is that? It definitely doesn't contain $\infty$, the point at infinity. In fact,

$$S^N \setminus (M/\partial M) = \mathrm{Int}(D^N) \setminus (M \setminus \partial M).$$

Now the boundary of a manifold has a little collar neighborhood that looks like $\partial M \times [0,1]$, and if I remove the boundary, that's homotopy equivalent to shrinking it down to the edge of that collar neighborhood. So that's homotopy equivalent to $M$, i.e.

$$M \setminus \partial M \sim M.$$

Let's call $M_0 = M \setminus \partial M$.

Choose a tubular neighborhood $V$ of $(M, \partial M) \subset D^N$. We note that $\mathrm{Int}(D^N) \setminus M_0 \simeq \mathrm{Int}(D^N) \setminus V$. Now $\mathrm{Int}(D^N) \simeq \mathbb{R}^N$. So we have a manifold $M^0$ imbedded in $\mathbb{R}^N$ and we're interested in understanding the complement. Now notice that $\mathbb{R}^N/(\mathbb{R}^n \setminus V)$ is the Thom space $\mathrm{Thom}(M^0, V)$. We have a cofiber sequence

$$\mathbb{R}^N \setminus V \to \mathbb{R}^N \to \mathbb{R}^N/\mathbb{R}^N \setminus V$$

so that the suspension of $(\mathbb{R}^N \backslash V)$ is homotopy equivalent to $\mathbb{R}^N/(\mathbb{R}^N \backslash V) = \mathrm{Thom}(M_0, M)$. So

$$\Sigma(\mathbb{R}^N \setminus V) \simeq \mathrm{Thom}(M_0, V).$$

So $V$ is a trivial $N$-dimensional bundle minus $TM$. So

$$\mathrm{Thom}(M, V) = S^N \wedge \mathrm{Thom}(M, -TM).$$

There are a bunch of indices to go get straight here. We figured out the complement of the sphere, which is related to the Spanier-Whitehead dual by some amount of suspension. If you work out all these numbers, we learn that up to suspension,

**22.1 Theorem.** *If $M$ is a compact manifold with boundary $\partial M$, then*

$$\mathbb{D}(M/\partial M) = \mathrm{Thom}(M, -TM).$$

This is a really important fact, and this is used in all kinds of places. For instance, here's a use of it. If you have a Thom class in $TM$, then so does $-TM$, and then I have a Thom isomorphism

$$\widetilde{H}^k(M) \simeq \widetilde{H}^{k-\dim M}(\mathrm{Thom}(M, -TM)) \overset{\mathbb{D}}{\simeq} \widetilde{H}_{d-k}(M, \partial M).$$

This is Poincaré duality, relating the cohomology and homology of $M$. Formulated like this, it tells you that having Poincaré duality is equivalent to having a Thom isomorphism for $-TM$. It also tells you that if you have a transformation of cohomology theories, it'll be as compatible with Poincaré duality as it is with the Thom isomorphism. That's something for which there are good formulas. That's a really useful point. If you're learning about the index theorem, this thing is really useful to internalize. It's part of the story about the index theorem, which tells you that some number computed by analytic means is equal to some number computed by topological means. The topological story is very related to this.

Atiyah formulated this in terms of manifolds with boundary, and it lets you get a slightly more general result. Suppose $M$ is closed, $\partial M = \emptyset$. Suppose $V \to M$ is a vector bundle. Then the *disk bundle* $D(V)$ is a manifold with boundary $S(V)$ (the sphere bundle). What does Atiyah's theorem tell us in this case? In this case, the theorem tells us that

$$\mathbb{D}(\mathrm{Thom}(M, V)) = \mathbb{D}(D(V)/S(V)) = \mathrm{Thom}(D(V), -TD(V)).$$

Let's figure out what the last thing is. In $D(V)$, we have a manifold homotopy equivalent to $M$, and the tangent bundle to the disk bundle corresponds to $TM \oplus V$. So the Thom complex of the disk bundle with coefficients in $-TD(V)$, that's homotopy equivalent to

$$\mathrm{Thom}(M, -TM - V).$$

The conclusion here is:

**22.2 Corollary.**

$$\mathbb{D}(M^V) = M^{-TM-V}.$$

I could have put both of these together and handled the case where $M$ had a boundary as well, but I'll leave that to you. That's a variation on this.

## §2 Application to vector fields

Now I want to bring all this home. I want to apply it to our vector fields problem. There's one more piece of the puzzle that has to come in before we have a really robust tool to work with. Let's do an example.

**22.3 Example.** We checked before that $T\mathbb{RP}^n \oplus \mathbb{R}$ is $(n+1)L$ for $L$ the tautological bundle. Atiyah's formula tells us that

$$\mathbb{D}(\mathbb{RP}^n_+) = \mathrm{Thom}(-nL) = \Sigma\mathbb{RP}^{-1}_{-(n+1)}.$$

We don't know anything about that yet. We could identify this with a different stunted projective space using James periodicity but we don't know that yet. Remember, the last statement is basically a definition.

**22.4 Example.** More generally, suppose I wanted the Spanier-Whitehead dual of $\mathbb{RP}^n_{n-k}$. That's the Spanier-Whitehead dual of $\mathrm{Thom}(\mathbb{RP}^k, (n-k)L)$. What is that? By Atiyah's formula, we get

$$\mathbb{D}\left(\mathrm{Thom}(\mathbb{RP}^k, (n-k)L)\right) = \mathrm{Thom}(\mathbb{RP}^k; 1-(k+1)L-(n-k)L) = \mathrm{Thom}(\mathbb{RP}^k, 1-(n+1)L),$$

which is

$$\Sigma\mathbb{RP}^{-(n-k+1)}_{-(n+1)}.$$

What we're learning is, for all $a, b$,

$$\mathbb{D}\mathbb{RP}^b_a = \Sigma\mathbb{RP}^{-a-1}_{-b-1}.$$

This is sort of a formal statement — we haven't yet identified these things with projective spaces. We're going to use real $K$-theory to get a different proof of James periodicity, and that will give us a way to turn these things into more useful statements for us.

There's one more thing I want to tell you, which has to do with the Spanier-Whitehead dual of a manifold. Let $M$ be a *closed manifold*, sitting inside $\mathbb{R}^N$. We can do the Pontryagin-Thom construction, and we can collapse everything outside a tubular neighborhood to a point, which gives a map

$$S^N \to \mathrm{Thom}(M, \nu),$$

for $\nu$ the normal bundle. Up to a suspension, this is the Spanier-Whitehead dual of $M$. Now I can also map $\mathrm{Thom}(M, \nu)$ **out** to $S^N$. Pick any point in $M$, and collapse everything outside a neighborhood of it to a point. This gives a map $\mathrm{Thom}(M, \nu) \to S^N$ and the composite

$$S^N \to \mathrm{Thom}(M, \nu) \to S^N$$

has degree one. If you want, it's the Pontryagin-Thom collapse map about a **point** in $\mathrm{Thom}(M, \nu)$.

**Remark.** If I have $N \subset M \subset \mathbb{R}^N$, the Pontryagin-Thom collapse goes in the reverse direction,

$$S^N \to \mathrm{Thom}(M, \nu_M) \to \mathrm{Thom}(N, \nu_N).$$

The conclusion of this is, what are these Thom$(M,\nu)$? If $M$ is connected, and $d$-dimensional, then $M$ has a zero-cell, which is the basepoint, and then it has a bunch of cells up to in dimension $d$ and has a single $d$ cell. The Thom complex just shifts the cells, so that has cells in dimension $N-d$ and all the way up to an $N$-cell. These maps show that the **top cell is unattached.** The top cell of Thom$(M,\nu_M)$ splits off, as there's a map from a sphere in, and a sphere out. The Thom complex is $S^N\vee$ something else. I'm going to say this in a more colloquial way because it'll be useful for us in the vector fields problem.

In $\Sigma^\infty_+ M$, the bottom cell splits off. If I take the Spanier-Whitehead dual $\mathbb{D}\Sigma^\infty_+ M$, the **top cell** splits off. That's also Thom$(M,-TM)$, so the top cell of the Thom complex has to split off. This is a really important thing about the Thom complex about $-TM$. It comes with a canonical map from a sphere to it which is the dual of the tautological map from $M$ to a $*$. The relevance of this will become clearer in the next lecture, but I'll remind you that the vector field problem was equivalent to asking that the top cell split off in a stunted projective space. Ultimately this is all going to relate to real $K$-theory and Radon-Hurwitz numbers.

# Lecture 23
# 11/1

We ended last time with the Atiyah duality theorem.

**23.1 Theorem.** *If $M$ is a manifold with boundary, then $\mathbb{D}(M/\partial M)$ is $M^{-TM}$. For a closed manifold $M$, the dual of a Thom complex $M^V$ is $M^{-TM-V}$.*

The second case reduced to the manifold-with-boundary case. If $M$ is a manifold-with-boundary and $V$ is a vector bundle on $M$, we have a more general statement

$$\mathbb{D}(M^V/\partial M^V) = M^{-TM-V}.$$

These are very useful for stunted projective spaces. But they only tell some of the story.

## §1 Real $K$-theory

Let $X$ be a finite CW complex.

**23.2 Definition.** $KO^0(X)$ is the Grothendieck group of real vector bundles on $X$.

You try to make this into a cohomology theory. You can define

$$\widetilde{KO}^{-n}(X) = \widetilde{KO}^0(S^n\wedge X),$$

which gets us negative $KO$-groups. The real version of the periodicity theorem is as follows:

**23.3 Theorem** (Bott periodicity)**.** $\widetilde{KO}^m(X)\simeq \widetilde{KO}^{m-8}(X)$.

The periodicity isomorphism is induced by multiplication by a class $u$ in $\widetilde{KO}^0(S^8) = \widetilde{KO}^{-8}(S^0)$, constructed from Clifford algebras. It's important to understand this class. I'll give you a couple of constructions of it because it's important to understand what it is. But I'd rather make a more systematic discussion.

Here are some other facts about $KO$:

1. $\widetilde{KO}^0(X)$ is homotopy classes of pointed maps $[X, \mathbb{Z}\times BO]$. Here $BO = \varinjlim BO(n)$, and $BO(n)$ can be described as the limit $\varinjlim_{N\to\infty} \mathrm{Gr}_n(\mathbb{R}^N)$. Bott periodicity gives you the homotopy groups of $\mathbb{Z} \times BO$.

2. In fact,
$$\pi_n(\mathbb{Z} \times BO) = \widetilde{KO}^0(S^n)$$
is given by the following table:

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|---|
| $\widetilde{KO}^0(S^n)$ | $\mathbb{Z}$ | $\mathbb{Z}/2$ | $\mathbb{Z}/2$ | 0 | $\mathbb{Z}$ | 0 | 0 | 0 | $\mathbb{Z}$ |

The groups repeat mod 8. There's a song for them. We will write $KO_n$ for these groups, and we can also write them as $\pi_n KO$ if we build a spectrum for $KO$-theory.

We get the **Atiyah-Hirzebruch spectral sequence.** That goes from
$$H^*(X; \pi_* KO) \implies KO^* X.$$
I deliberately didn't write down how the indices work. I could write down the general formula, but I'd rather instead point out the two situations when you use this. You don't have to remember this if you remember how the Serre ss works.

....

Another way of saying this is that $KO^0(X)$ is *built from* $\bigoplus_n H^n(X; \pi_n KO)$. By built from, I mean in the sense of a spectral sequence. $KO^0(X)$ has a filtration whose associated graded is a subquotient of that.

I want to do some examples here, but there's one other thing I want to point out. Suppose $X$ is finite and $V$ is a vector bundle over $X$. Then the Thom complex $X^V$, regarded as a stable object (or an object in $\mathcal{SW}$), depends only on the underlying equivalence class of $V$ in $KO^0(X)$. If I add a bunch of trivial bundles to $V$, that will just suspend this. That is, $X^{V\oplus\mathbb{R}^n} = S^n \wedge X^V$. Also, $V_1 \simeq V_2$ in $KO^0(X)$ if and only if $V_1 \oplus \mathbb{R}^n \simeq V_2 \oplus \mathbb{R}^n$. To say that they're equivalent in $KO$-theory means that I can add *some* vector bundle to them to get them to be isomorphic, i.e. $V_1 \oplus W \simeq V_2 \oplus W$. Now add some other bundle to $W$ that makes it trivial.

If I work in reduced $KO$-theory, then I get the same Thom complex up to suspension. In other words, $X^V$ depends, up to suspension, only on the class of $V - \dim V \in \widetilde{KO}^0(X)$. If $V, W$ are identified in $\widetilde{KO}^0(X)$, then $V \oplus \mathbb{R}^N \simeq W \oplus \mathbb{R}^M$ for some $M, N$.

## §2 Examples

Let's do an example with $X = \mathbb{RP}^8 = \mathbb{RP}^8_0$. Take $X = \mathbb{RP}^8$. We've seen that
$$\mathbb{RP}^{n+8}_n = \mathrm{Thom}(\mathbb{RP}^8, nL).$$

Modulo suspensions, that only depends on $n(L-1) \in \widetilde{KO}^0(\mathbb{RP}^8)$. Now what do we know about $\widetilde{KO}^0(\mathbb{RP}^8)$. You'd use the AHSS. If you wanted the spectral sequence for reduced $KO$-theory, you'd put in reduced cohomology everywhere rather than cohomology.

(**spectral sequence to be filled in later.**)

**Remark.** When you run the spectral sequence for $KO^0(X)$, you only get $\mathbb{Z}/2$'s down the diagonal, never $\mathbb{Z}$'s. When you right this down, you just put in $\mathbb{Z}/2$'s in each of the spots where you have $\mathbb{Z}/2$'s in $KO_n$.

We don't yet know what happens: it might be a bloodbath. There might be a ton of differentials. But we know from this that

$$\widetilde{KO}^0(\mathbb{RP}^8)$$

has order at most 16. If we combine that with this, this implies that

$$\mathbb{RP}_n^{n+8}$$

depends **only on $n$ modulo 16.**

I could replace 8 by any number. For more general values of 8, this gives **James periodicity.** In fact, just saying it this way, "the order of the $KO$-group is at most this number," you get exactly the same James periodicity as you get from Clifford algebras. In fact, you get the same periods from this crude method as the periods that come from Clifford algebras. (Remember, when we had vector fields on spheres, we got a James periodicity result.) I think this is really the way to think about James periodicity. It's sort of the most direct way to think about it. The Thom complex $(\mathbb{RP}^8)^{nL}$ depends only on $n$ modulo the order of this $KO$-group, and that gives you the periodicity.

There's something else that comes out of this. We got vector fields out of Clifford algebras, and we can also get vector fields out of this. By Atiyah's results,

$$\mathbb{D}(\mathbb{RP}^8_+) = \mathbb{RP}^{-1}_{-9},$$

and for this thing, we know that the top cell splits off. That happens in the Spanier-Whitehead dual of any smooth manifold. But this only depends on these numbers mod 16. So we find that the top cell of $\mathbb{RP}_7^{15}$ splits off, stably. That implies that $S^{15}$ has eight vector fields.

**23.4 Corollary.** *$S^{15}$ has eight vector fields.*

**23.5 Example.** This more generally implies that $S^{16n-1}$ has eight vector fields. We might be able to do better when $n$ is even.

We *also* got that out of Clifford algebras.

I'll tell you in a minute what these $KO$-groups work out to be. But we haven't used very much. We've used Bott periodicity, and we've used this stuff about $KO$-theory, but we haven't tried to calculate: we just got an upper bound. And we found a method of constructing vector fields on spheres just from Atiyah duality and $KO$-theory, and we got the same number of vector fields on spheres as we did with Clifford algebras.

Let's just do one more and see if we can guess the general pattern. Let's try $\mathbb{RP}^{10}$. I'm just going to run through the same kind of stuff. $\widetilde{KO}^0(\mathbb{RP}^{10})$ is going to have order at most 64, but the same AHSS argument. That tells me that $\mathbb{RP}_n^{n+10}$ depends only on $n \mod 64$. That gives some sort of James periodicity for 11-cell stunted projective spaces. And what about vector fields?

We know that

$$\mathbb{D}(\mathbb{RP}^{10}) = \mathbb{RP}_{-11}^{-1},$$

and that only depends on the numbers mod 64. Here again the top cell splits off, so we can 64 to each of these numbers. So in $\mathbb{RP}_{53}^{63}$, the top cell splits off. That implies that $S^{63}$ has ten vector fields. That is also the same number that we would get from Clifford algebras. (**Wait, is this right?**)

I actually am going to give you a homework problem right now. So far, I picked projective spaces that ended on a $\mathbb{Z}/2$ in the AHSS. What if I did something like $\mathbb{RP}^5, \mathbb{RP}^6, \mathbb{RP}^7$? If we looked at $\widetilde{KO}^0(\mathbb{RP}^5)$, we'd find that it has at most 8 elements. The same thing is true for $\mathbb{RP}^6$ and the same is true for $\mathbb{RP}^7$.

Here we would learn that

$$\mathbb{RP}_n^{n+5}$$

depends only on $n \mod 8$. Here would learn that $\mathbb{RP}_n^{n+6}$ only depends on $n \mod 8$, and same for $\mathbb{RP}_n^{n+7}$. The last statement implies the ones before it; it is the strongest. The same thing would be true with vector fields. The first one would tell me that $S^{8n-1}$ has five vector fields, the second one would tell me that $S^{8n-1}$ has six vector fields, and the third one would tell me seven vector fields. So we get better results if we take the projective space that ends right when possible.

There are several statements coming together here: we can construct vector fields using Clifford algebras, and using real projective spaces. It turns out that we get the same number. I want to put a couple of statements together. I'll prove this next time.

**23.6 Proposition.** *$\widetilde{KO}^0(\mathbb{RP}^n)$ is cyclic of the order given by the $E_2$ page of the AHSS: there aren't differentials in an out of there. It's generated by $L-1$ for $L$ the tautological bundle.*

If you put this together, we have the following proposition:

**23.7 Proposition.** *If $(L-1)$ has order $m$ in $\widetilde{KO}^0(\mathbb{RP}^n)$, then $S^{m-1}$ has $n$ vector fields.*

We also learn:

**23.8 Proposition.** *If $\mathrm{Cl}_n$ acts on $\mathbb{R}^m$, then $S^{m-1}$ has $n$ vector fields.*

I can rewrite the first proposition by saying that $mL = \mathbb{R}^m$ stably. The question I want to leave you with, which I am going to put on the problem set, is: what is the relationship between these? Can you go from one to the other? What is the relationship between?

- $mL = \mathbb{R}^m$ on $\mathbb{RP}^n$

- $\mathrm{Cl}_n$ acts on $\mathbb{R}^m$.

It's easy to get a number which is off by a factor of two, and there's a trick to improve it. This is kind of miraculous. The reason that $KO$-theory solves the vector fields problem is because they are giving exactly the same number.

# Lecture 24
# 11/5

Today, I want to describe the calculation of the $K$-theory of some stunted projective spaces. I was just rereading Adams's discussion of this—and I recommend going to Adams's paper "Vector fields on spheres" if you want to see some version of these details written up—but I think that the best way to engage with it is to use some of the techniques I'm going to talk about today and to put it together yourself. There are a lot of ways to do this calculation, and it's important to do it from these points of view. So let's begin.

## §1 Outline

Our goal is to calculate $\widetilde{KO}^0(\mathbb{RP}^m_n)$. I'm going to give you some techniques, and we'll put the answer together.

Here are some tools:

- The Atiyah-Hirzebruch spectral sequence
$$H^*(X, KO^*(*)) \implies KO^*(X).$$
There's also an analog for complex $K$-theory, and there are the various maps between them. There is a map
$$K^0(X) \to KO^0(X)$$
which sends a vector bundle $V$ to its realification. Then there's the map $KO(X) \to K(X)$ which complexifies a vector bundle.

- Note that the composite
$$K(X) \to KO(X) \to K(X)$$
sends $V \mapsto V \otimes_{\mathbb{R}} \mathbb{C}$. The complex structure comes from the second factor of $\mathbb{C}$, and this is $V \oplus \overline{V}$.

- The composite $KO(X) \to K(X) \to KO(X)$ is multiplication by 2.

  This is how you make all these calculations in homotopy theory—make maps to things that you know how to calculate.

## §2 $K^*(\mathbb{CP}^n)$

Let's start with some spaces that we know. The first thing is the complex $K$-theory of $\mathbb{CP}^n$. The AHSS for this is rather simple. It's got $\mathbb{Z}$'s in every other degree (**add a drawing**) and zeros everywhere else: there is a checkerboard pattern and consequently no possible differentials. I already made this calculation, and I did something about a projective bundle formula when I talked about Adams's splitting principle.

So the spectral sequence runs

$$H^*(\mathbb{CP}^n; K^*(*)) = H^*(\mathbb{CP}^n, \mathbb{Z}[u^{\pm 1}]) = K_*[u^{\pm 1}] \implies K^*(\mathbb{CP}^n).$$

There is a little thing that we want to check here. We want to check that if we take the class of $1 - L_{\mathbb{C}}$ (where $L_{\mathbb{C}}$ is the tautological bundle) corresponds to $x$. This is kind of an easy statement to believe—what else could it be? But it really is something you need to check, and it gets at the heart of how you use the AHSS. You could prove that by naturality. We could compare the spectral sequence to the case of $\mathbb{CP}^1$, and let's even take reduced cohomology. That looks like $\widetilde{H}^*(\mathbb{CP}^1, K^*(*)) \implies \widetilde{K}^*(\mathbb{CP}^1)$ and the generator in $H^2$ (times $u^{-1}$) has to correspond to the generator in $K^0$.

**Notation:** $L$ is the tautological bundle over $\mathbb{RP}^n$, and $L_{\mathbb{C}}$ is the tautological bundle over $\mathbb{CP}^n$.

What does this mean homotopy-theoretically? Homotopy-theoretically, it means that we have a map $\mathbb{CP}^n \to BU$, and that when we restrict to the 2-skeleton, the class $\mathbb{CP}^1 \to BU$ is the generator of $\pi_2(BU)$. I'll come back and expand on this a little bit later. It's important to remember what it means for a cohomology theory to represent an element in the AHSS. We're going to meet this in a little bit.

So this easily gets that $K^0(\mathbb{CP}^n)$ has a basis $1, x, \ldots, x^n$, and there's one more assertion here: why does $x^{n+1} = 0$? We've talked about this. I'll just remind you: let's consider the pull-back

$$\mathbb{CP}^n \to (\mathbb{CP}^n)^{\wedge(n+1)}$$

and the class $x^{n+1}$ is pulled back by $x \otimes \cdots \otimes x$. However, $\mathbb{CP}^n \to (\mathbb{CP}^n)^{\wedge(n+1)}$ is trivial (because $\mathbb{CP}^n$ is $n$-dimensional and the smash product is $(n+1)$-connected). So that takes care of $K^0(\mathbb{CP}^n)$.

## §3 $K^0(\mathbb{RP}^{2n})$

We have a map $\mathbb{RP}^{2n} \to \mathbb{CP}^n$ (in fact, $\mathbb{RP}^{2n+1} \to \mathbb{CP}^n$), creating a diagram of spectral sequences

$$\begin{tikzcd} H^*(\mathbb{CP}^n, K_*) \arrow[r, "\implies"] \arrow[d] & K^*(\mathbb{CP}^n) \arrow[d] \\ H^*(\mathbb{RP}^{2n}, K_*) \arrow[r, "\implies"] & K^*(\mathbb{RP}^{2n}) \end{tikzcd}.$$

This map is nontrivial in $H^2$. If you like, this map corresponds to the nontrivial cohomology class in $H^2(\mathbb{RP}^{2n}; \mathbb{Z})$, giving a map $\mathbb{RP}^{2n} \to \mathbb{CP}^\infty$ which we restrict to the appropriate skeleton.

What do the two spectral sequences look like? For $\mathbb{CP}^n$, we have $\mathbb{Z}$'s in even degrees and zero everywhere else. For $\mathbb{RP}^{2n}$, we get a bunch of $\mathbb{Z}$'s and $\mathbb{Z}/2$'s in a lot of places.

So we get a similar checkerboard pattern. Once again, there are no possible differentials and the AHSS collapses. So we find that $K(\mathbb{CP}^n) \to K(\mathbb{RP}^{2n})$ is surjective. Moreover, $\widetilde{K}(\mathbb{RP}^{2n})$ has a filtration whose associated graded is this sum of $\mathbb{Z}/2$'s. Let's look at these $\mathbb{Z}/2$'s and try to get some information about them. Notice that under this map $\mathbb{RP}^{2n} \to \mathbb{CP}^n$, the bundle $L_{\mathbb{C}}$ pulls back to $L \otimes \mathbb{C}$.

Let's write $w = 1 - L \otimes \mathbb{C}$. If I look in this spectral sequence here, there's a generator $\alpha$ in this group $H^2(\mathbb{RP}^n; \mathbb{Z}) = \mathbb{Z}/2$. Then the remaining classes are $\alpha, \alpha^2, \dots$. Here $\alpha$ represents $1 - L \otimes \mathbb{C}$, so those $\alpha^m$ represent $(1 - L \otimes \mathbb{C})^m$. What do we learn from that? We learn from that $\widetilde{K}^0(\mathbb{RP}^{2n})$ has a decreasing filtration

$$0 \subset \cdots \subset \widetilde{K}^0(\mathbb{RP}^{2n})$$

whose associated graded is a sum of $\mathbb{Z}/2$'s. Moreover, each of the $\mathbb{Z}/2$'s are generated by the classes of $w^m$.

Now

$$w^2 = 1 - 2L \otimes \mathbb{C} + L^2 \otimes \mathbb{C} = 2 - 2L = 2w.$$

That tells us that $w^2 = 2w$ and tells us how to solve the extension problem.

**24.1 Corollary.** *$\widetilde{K}^0(\mathbb{RP}^{2n}) = \mathbb{Z}/2^n$ generated by $w$.*

## §4 $\widetilde{KO}(\mathbb{RP}^n)$

Now we want to move on discuss $KO$-theory. I'm going to say a few more words about it and leave stunted projective spaces to you. Let's again look at the AHSS for $\mathbb{RP}^8$.

(**draw this**)

This spectral sequence *doesn't* fit the checkerboard pattern — there are possible differentials. If you just read Adams's paper, you'll never know what the differentials are, and it turns out there are differentials. It's worth working this out. If you're just interested in $KO^0$, though, there are no differentials that either come in or leave out the line. All the group extensions turn out to correspond from multiplication by 2, and from there you can work out the $KO$-theory of any stunted projective space.

There are some things that are easy to tell right away. I'm going to change notation and let $\alpha$ be $1 - L$. It's easy to check that the first class is represented by $\alpha$. You have to know something about the $KO$-groups of a point—and the second thing is represented by $\alpha^2$. We need to know the ring structure of $KO^*(*)$ for this.

**24.2 Proposition.** *$KO_*$ is generated by $1, \eta \in \pi_1 KO, h \in \pi_4 KO, \beta \in \pi_8 KO$. Here $\eta^3 = 0, h^2 = 4\beta$, and $\beta$ is invertible.*

Unfortunately we can't use the spectral sequence quite as we did for complex $K$-theory. We know one thing from this. We know that $\widetilde{KO}(\mathbb{RP}^8)$ has at most $2^4 = 16$ elements. We can map that to $\widetilde{K}(\mathbb{RP}^8)$, and we just checked that it was cyclic of order $2^4$ generated by $1 - L \otimes \mathbb{C}$, so $\alpha$ comes down to the generator. It follows that $\widetilde{KO}(\mathbb{RP}^8)$ has to have exactly 16 elements. Therefore

$$\widetilde{KO}(\mathbb{RP}^8) \to \widetilde{K}(\mathbb{RP}^8)$$

is surjective and therefore an isomorphism, by counting. That tells us that

$$\widetilde{KO}^0(\mathbb{RP}^8) = \mathbb{Z}/2^4,$$

generated by this class $\alpha$.

The same argument would have worked for $\widetilde{KO}^0(\mathbb{RP}^{8k})$. This has at most $2^{4k}$ elements and $\widetilde{K}^0(\mathbb{RP}^{8k})$ is cyclic of that order, generated by $1 - L \otimes \mathbb{C}$. Therefore the map is surjective and therefore an isomorphism. This exact same argument will work as long as the $KO$-theory has the same number of $\mathbb{Z}/2$'s as complex $K$-theory. This exact same argument would show, and this is how Adams does it, that the $KO$-theory of $\mathbb{RP}^{8k+1}, \mathbb{RP}^{8k+2}, \mathbb{RP}^{8k}$ is a cyclic group of the same order as the $K$-theory. So we get those real projective spaces.

I'm going to stop at this point and let you think about it. We could also learn from the $\alpha, \alpha^2$ thing that $\widetilde{KO}^0(\mathbb{RP}^2) = \mathbb{Z}/4$. For $\mathbb{RP}^{8k} \subset \mathbb{RP}^{8k+2}$, note that the cofiber is a stunted projective space $\mathbb{RP}^{8k+2}_{8k+1}$ which in turn is an appropriate suspension of $\mathbb{RP}^2$. You can use these sequences to get things about these groups. I'm going to let you play around with it. You'll learn more if you try to get these next groups.

**24.3 Exercise.** Can you compute $\widetilde{KO}^n(\mathbb{RP}^b_a)$ in general?

I concentrated on $\widetilde{KO}^0$, and the reason for that will become clear next lecture. I didn't do any of those stunted ones, but I could those from the long exact sequences and things. I'll just warn you: the answer is a bit unwieldy. It's really doable, though.

Let me just tell you the way to remember how this works.

**24.4 Theorem.** *No differentials affect the part of the AHSS that converges to $\widetilde{KO}^0(\mathbb{RP}^b_a)$. The $\mathbb{Z}/2$'s always assemble into a single cyclic group. The only way you can not get a $\mathbb{Z}/2$ is if you get an $\mathbb{RP}^m_{4k}$, which provides a $\mathbb{Z}$ in $H^{4k}(\cdot, KO_{4k})$.*

That's in $\widetilde{KO}^0$—I haven't said a word about the other $KO$-groups.

I want to go back to one of these examples where we produced these vector fields on spheres. We saw that $\widetilde{KO}^0(\mathbb{RP}^8) = \mathbb{Z}/2^4$. So the dual of $\mathbb{RP}^8$ was $\mathbb{RP}^{-1}_{-9}$ and we can add sixteen to that, so that gives us after suspending $\mathbb{RP}^{15}_7$ whose *top cell splits off.* That gives us that $S^{15}$ has eight vector fields. But we could ask whether it can have nine? I'll pick this up next time.

# Lecture 25
# 11/7

I want to move into the next step of discussing the vector fields problem. We've learned a lot about Stiefel manifolds and Thom complexes, and just to keep the discussion concrete, let's stick with the specific example we've been discussing.

We've seen, in *two ways*, that $S^{15}$ has eight vector fields. The question is, does it have nine? We produced vector fields in two different ways:

1. Clifford algebras.

2. The order of the reduced tautological line bundle on projective space.

The conjecture was that eight is the best possible.

There are a couple of ways in which we can ask this question. We have an equivalent formulation. If we look at $\mathbb{RP}^{15}_7$, then the top cell splits off. The question is, how about $\mathbb{RP}^{15}_6$?

In terms of the cells, $\mathbb{RP}^{15}_6$ has cells from dimension 15 to 6, and the cells are connected by attaching maps. We know that the attaching map from the top cell isn't attached to cells fourteen through seven and the question is whether it's attached to the bottom cell. The question is, what is this attaching map?

To clarify, we have an attaching map $S^{14} \to \mathbb{RP}^{14}_6$ which factors as a map of spaces

$$S^{14} \to \mathbb{RP}^{14} \to \mathbb{RP}^{14}_6$$

where the first is the double cover. The composite to $\mathbb{RP}^{14}_7$ isn't unique, so we can factor the map through $S^6$, which is not unique. The map $S^{14} \to S^6$ is only well-defined modulo something, and when I say whether it's zero, I mean whether it's zero after I take $S^{14} \to \mathbb{RP}^{14}_6$. **There is not a definite canonical map $S^{14} \to S^6$.**

For the EHP sequence, we want our hands on this map $S^{14} \to S^6$. If you go back and think about it, there is extra data specified. We have an explicit construction of eight vector fields on $S^{15}$. This means that **we have an explicit way of splitting of the top cell,** which means we have an explicit splitting off of the top cell, and that means we should get a canonical factorization and a canonical map $S^{14} \to S^6$. Today, we're going to talk about how you get this map.

I'm going to describe this map to you using homotopy theory, but it occurs to me, as I explain it to you, that there's an explicit construction using Clifford algebras, and there must be an explicit factorization using Clifford algebras. It's possibly an interesting thing to think about.

There's another thing that all the classical stuff about vector fields does. There are many places where people convert the problem into the Spanier-Whitehead dual problem. I know one of really good reason to do it, and I'll explain that later. This isn't the point. I'm just going to say that there are a lot of maneuvers where you switch the problem to the Spanier-Whitehead dual, and it makes one calculation much more doable. But in principle, looking at something or its dual is just a formal maneuver and it shouldn't really advance you towards to the solution.

With that said, I want to look at the dual problem.

**Question.** Does the top cell of $\mathbb{RP}^{15}_6$ split off?

That's equivalent to the dual problem:

**Question.** Does the bottom cell of $\mathbb{D}(\mathbb{RP}^{15}_6)$ split off?

The duality stuff (Atiyah duality) tells you that $\mathbb{D}(\mathbb{RP}^{15}_6)$ is, up to a suspension, $\mathbb{RP}^{-7}_{-16}$. In general, we had the formula

$$\mathbb{D}(M^\nu) = M^{-TM-\nu},$$

and we had
$$\mathbb{RP}^b_a = (\mathbb{RP}^{b-a})^{aL},$$
where the tangent bundle of $\mathbb{RP}^{b-a}$ was stably easy to work out.

So we want to know whether the bottom cell of this thing splits off, and this is $\mathrm{Thom}(\mathbb{RP}^9, -16L)$. Let's ask ourselves about this Thom complex, because that's where we are going to learn something.

What do we know about $\widetilde{KO}^0(\mathbb{RP}^9)$? If we remember the AHSS, we got a bunch of $\mathbb{Z}/2$'s lining up, and we found that it was $\mathbb{Z}/32$. Meanwhile $\widetilde{KO}^0(\mathbb{RP}^8) = \mathbb{Z}/16$. So remember that $16L$ is trivial on the 8-skeleton, and it's in the highest filtration of the AHSS. The way I want to use the trivialization of $16L|_{\mathbb{RP}^8}$ is to observe the following.

We have a map
$$\mathbb{RP}^9 \overset{16(1-L)}{\to} BO,$$
and since that is trivial on $\mathbb{RP}^8$, we get a factorization of $S^9$. That has to be the nontrivial element of $\pi_9(BO)$. There's something you need to use about the definition of the spectral sequence. We get a commutative diagram:

$$\begin{tikzcd} \mathbb{RP}^9 \arrow[r, "16(1-L)"] \arrow[d] & BO \\ S^9 \arrow[ur, "\xi"] & \end{tikzcd}$$

This commutative diagram gives us a diagram of Thom complexes,
$$\Sigma^{16}\mathbb{RP}^{-7}_{-16} \to (S^9)^{\xi} = (S^0 \cup_f e^9)$$
We also know that the map $\mathbb{RP}^9 \to S^9$ is nonzero in $H\mathbb{Z}/2$ in dimensions zero and nine. So the above map of Thom complexes is an isomorphism in $H\mathbb{Z}/2$ in the same coefficients in appropriate degrees.

Now, if we dualize everything, and suspend as necessary, we get the dual of the cone on $f$ mapping into the dual of the stunted projective space. So we get a map
$$\mathbb{D}((S^9)^{\xi}) \to \mathbb{RP}^{15}_6$$
up to suspension. This dual, suitably suspended, is $S^6 \cup_{\mathbb{D}f} e^{15}$, mapping into $\mathbb{RP}^{15}_6$.

So we have a map of a two-cell complex in hitting the bottom and top cells. So the (stable) attaching map of the top cell of $\mathbb{RP}^{15}_6$ comes from the attaching map of this two-cell complex followed by the map into $\mathbb{RP}^{14}_6$. In other words, the stable attaching map
$$S^{14} \to \mathbb{RP}^{14}_6$$
factors as
$$S^{14} \overset{\mathbb{D}f}{\to} S^6 \to \mathbb{RP}^{14}_6$$
and this thing with the Atiyah-Hirzebruch SS produces an explicit example of the attaching map.

Adams knew about this, but he doesn't talk about this in the vector fields paper. In that paper, he doesn't discuss $S^{14} \to S^6$, only the composite into $\mathbb{RP}^{14}_6$. He writes that he knows about it and it appears many years later in his $J(X)$ papers.

We have a general question of what this map $\mathbb{D}f$ is. Then we have a dual question of how this works for arbitrary spheres. Let me tell you what the general question you meet is.

**Question.** We have a map $\mathbb{RP}^n \overset{\text{crush}}{\to} S^n \overset{\xi}{\to} BO$, where $\xi : S^n \to BO$ is the generator. (In order for this group to be nonzero, let's assume $n \equiv 0, 1, 2, 4 \mod 8$.) The general thing we would meet is this situation.

The question is about $\xi$. We look at the Thom complex $\Sigma(S^n)^\xi = S^0 \cup_f e^n$ and the question is what $f$ is in terms of $\xi$, and then what is $\mathbb{D}f$ in terms of $\xi$.

Let's first do the question about $\mathbb{D}f$. I think this is quite easy to figure out. The second question applies to any map.

**Question.** For $f : S^{n-1} \to S^0$ any map in $\pi_{n-1}(S^0)$, what is $\mathbb{D}f : S^0 \to S^{n-1}$ (also an element of $\pi_{n-1}(S^0)$)?

I think $\mathbb{D}f = f$. I remember what we used to always say, we used to say "What else could it be?" and that was a good proof back in the day. Of course, it could be $(-1)^n f$ or something like that. It could be something random. But I think it is pretty easy. The thing is, this is a natural transformation. The dual is a functor. Let's come back to that, though.

The real thing about today's lecture is, **what is $f$ in terms of $\xi$?** And this is the important thing. The answer to the first question is that $f$ is the $J$-homomorphism applied to $\xi$ where

$$J : \pi_n(BO) \to \pi_{n-1}(S^0).$$

Let me remind you about the definition of this map. Here $O(n)$ acts on $n$-dimensional euclidean space, so

$$J : O(n) \to \mathrm{LinIso}(\mathbb{R}^n, \mathbb{R}^n) \to \mathrm{Map}_*(S^n, S^n) = \Omega^n S^n$$

That gives me a map $\pi_k O(n) \to \pi_k(\Omega^n S^n) = \pi_{k+m}(S^m)$. It's trivial to check that if I go into $O(n+1)$, this map corresponds to suspension. In the limit, I get a map $\pi_k(O) \to \pi_k(S^0)$. That's the $J$-homomorphism.

**25.1 Proposition.** *If $\xi$ is a vector bundle over $S^n$ classified by some map $S^n \overset{\xi}{\to} BO$, then the Thom complex is $S^0 \cup_{J\xi} e^n$.*

This turns out to be *extremely* important. For a random map of spheres, to show that it's nontrivial, I have to make some computation in the mapping cone. The point is, when a map is in the image of $J$, I can understand the mapping cone and its cohomology in terms of the Thom isomorphism. So this is a really important thing to know.

There's probably just enough time to prove this.

**Remark.** Let me go back and say the previous thing a little more honestly. The problem is, the basepoint of $\Omega^n S^n$ is the *constant map* at the basepoint. Here $\Omega^n S^n$ has many path components for each degree and $O(n)$ goes into the path component of the paths of degree one. So $O(n) \to \Omega^n S^n$ **isn't basepoint-preserving the way I wrote it because I didn't land in the component containing the basepoint.** I landed instead in the component of maps of degree one. I really get a map

$$O(n) \to \Omega^n_{\pm 1} S^n$$

landing in maps of degree $\pm 1$. The basepoint in $O(n)$, the identity, goes to the identity map of $S^n$.

So in reality, we should modify the map by subtracting off (in some group model for $\Omega^n S^n$) the identity map. So let's define $J : O(n) \to \Omega^n S^n$ as **this map**, when you subtract off the identity.

# Lecture 26
# 11/9

A reminder: there's no class next week.

## §1 Thom complexes and the $J$-homomorphism

I ended last class talking about the $J$-homomorphism. We considered the following. There is a map

$$\xi : S^{k+1} \to BO(n)$$

classifying the $n$-dimensional bundle $\xi$ over $S^{k+1}$. We want to form the *Thom complex* $\mathrm{Thom}(S^{k+1}, \xi)$ of that. Now I'm working unstably, so that's of the form $S^n \cup_f e^{n+k+1}$ for some map $f : S^{n+k} \to S^n$. The claim was that $f$ is given by the $J$-homomorphism. We adjoint this over, and $\xi$ gives a map $S^k \to O(n)$ (because $\pi_{k+1}(BO(n)) \simeq \pi_k(O(n))$). Let's call that map

$$\sigma\xi : S^k \to O(n).$$

The claim was that:

**26.1 Proposition.** *$f = J(\sigma\xi)$ where $J$ is the map $\pi_k(O(n)) \to \pi_{n+k}(S^n)$ described in the last class.*

Recall that this comes from the map

$$O(n) \to \Omega^n_1 S^n \stackrel{-1}{\to} \Omega^n_0 S^n.$$

First, I'd like to give you a proof of this.

*Proof.* We have to think about what the relationship between $\sigma\xi$ and the vector bundle $\xi$. If you think this through, I'm just going to tell you the answer. I'm just going to let you check the answer. Given a map

$$\tau : S^k \to O(n),$$

called the **clutching function** or **twisting function**, you make a vector bundle over $S^{k+1}$ as follows: we take two copies of the disk $D_+^{k+1}, D_-^{k+1}$ which fit together along $S^k$ to make the sphere $S^{k+1}$. For instance,

$$D_+^{k+1}$$

could be the upper hemisphere, and $D_-^{k+1}$ is the lower hemisphere. Our vector bundle (from $\tau$) over $S^{k+1}$ is constructed as follows. You take

$$D_+^{k+1} \sqcup D_-^{k+1} \times \mathbb{R}^n / \sim$$

where $\sim$ refers to the equivalence relation $(a_+, v) \simeq (a_-, \tau(a)v)$ for $a \in S^k$. In other words, you fit the two disks along the $S^k$ to make $S^{k+1}$. The vector bundle is trivialized over each top disk, and the gluing along the boundary is done using this twisting function. This construction in general describes a vector bundle over the suspension of any space. Today I'm going to discuss this construction and many different ways of looking at it.

How do we build the Thom complex? The Thom complex of a vector bundle $V$ is $D(V)/S(V)$. I want to write this vector bundle as a pushout because I want a formula for this. Our vector bundle looks like a pushout

$$\begin{tikzcd}
S^k \times \mathbb{R}^n \arrow[r, "(a,v)\mapsto(a,v)"] \arrow[d, "(a,v)\mapsto(a,\tau(a)v)"] & D_+^{k+1} \times \mathbb{R}^n \arrow[d] \\
D_-^{k+1} \times \mathbb{R}^n \arrow[r] & \xi
\end{tikzcd}.$$

I would have a similar diagram for the disk bundle and a similar diagram for the sphere bundle. I would also have a similar diagram for the disk bundle modulo the sphere bundle.

$$\begin{tikzcd}
S^k \times (D(\mathbb{R}^n)/S(\mathbb{R}^n)) \arrow[r, "(a,v)\mapsto(a,v)"] \arrow[d, "(a,v)\mapsto(a,\tau(a)v)"] & D_+^{k+1} \times (D(\mathbb{R}^n)/S(\mathbb{R}^n)) \arrow[d] \\
D_-^{k+1} \times (D(\mathbb{R}^n)/S(\mathbb{R}^n)) \arrow[r] & D(\xi)/S(\xi)
\end{tikzcd}.$$

We'd like to understand the homotopy type of $D(\xi)/S(\xi)$, and in particular it as a CW complex. So this is actually a pushout diagram, but I can also think of it as a double mapping cylinder if I replace the disks by points. I could also write this as a *homotopy pushout* or a *double mapping cylinder*

$$\begin{tikzcd}
(S^k \sqcup *) \wedge S^n \arrow[r, "(a,v)\mapsto(a,v)"] \arrow[d, "(a,v)\mapsto(a,\tau(a)v)"] & S^n \arrow[d] \\
S^n \arrow[r] & \mathrm{Thom}(\xi)
\end{tikzcd}.$$

The horizontal map is projection and the vertical map is derived from $\tau$. We already see our $J$-homomorphism coming into the picture.

Let's extend the columns down and form the Barratt-Puppe sequence. Before I do that, notice that the map $(S^k \sqcup *) \wedge S^n \to S^n$ has a *section.* This kind of gives a cell decomposition of the Thom space. It says I have two $n$-cells and they are glued together. You can derive the proposition from that. I'll leave it as an exercise to finish it from this diagram. It isn't a very hard or deep fact. It's a matter of staring at this. A point is that the two $S^n$'s that come into the cell decomposition get identified. ▲

## §2 The Thom isomorphism in $K$-theory

Where does this place us? We made this long argument, and we eventually came to the following situation. We had some stunted projective space, and we had a map from a Thom complex into a stunted projective space. And what we found was, we had to understand attaching maps of cells in Thom complexes. We're going to understand the attaching maps by calculating $K$-theory and Adams operations. So to go further, we need to understand the $K$-theory (or $KO$-theory) of a Thom complex, and we need to understand the Adams operations.

I haven't even talked about the Thom isomorphism in $KO$-theory, and that's one of the things I want to talk about today. Let's start with the case of $K$-theory.

Let $V$ be a **complex** vector bundle of complex dimension $n$ over the space $X$. We want to construct a natural Thom class in $K$-theory. We'd like to construct

$$U \in \widetilde{K}^0(X^V),$$

and by Thom class, it means that for every $x \in X$, the restriction map

$$\widetilde{K}^0(X^V) \to \widetilde{K}^0(\{x\}_x^V) \simeq \widetilde{K}^0(V_x^+) \simeq \mathbb{Z}$$

restricts to a generator. I want one of these which is natural, and which sends sums to products, and things like that. I probably ought to talk about that. I'm going to summarize most of this without proving things.

There are two really natural ways of making this Thom class. One construction just deduces it from the projective bundle formula. That's because we can identify $\mathrm{Thom}(X, V)$ with $\mathbb{P}(V \oplus 1)/\mathbb{P}(V)$. That's because the lines through the origin in $V \oplus 1$ that do not lie in $V$ determine uniquely a point of $V$. In other words, given a line $\ell$ in $V \oplus 1$, and look at where it intersects the single line 1. It's at a point $(1, v)$. The lines that are in $V$ correspond to the point at $\infty$ in the Thom complex. So anyway,

$$X^V = \mathbb{P}(V \oplus 1)/\mathbb{P}(V).$$

We could make the class there, because we know that we have this tautological line bundle $L$ over $\mathbb{P}(V)$. By the projective bundle formula,

$$\widetilde{K}^0(\mathbb{P}(V \oplus 1)) = K^0(X)\,\{1, x, \ldots, x^n\}$$

where $x = 1 - L$. Similarly,

$$\widetilde{K}^0(\mathbb{P}(V)) = K^0(X)\,\{1, x, \ldots, x^{n-1}\}$$

and the map between them is the obvious map. Therefore,

$$\widetilde{K}^0(\mathbb{P}(V \oplus 1)/\mathbb{P}(V)) \simeq K^0(X)\,\{x^n\}$$

and this *is the Thom isomorphism.* This $x^n$ is the Thom class we want, and we have the Thom isomorphism. Moreover, we get that multiplication by $U = x^n$ gives an isomorphism

$$x^n : K^0(X) \simeq \widetilde{K}^0(X^V).$$

We could do a lot of calculations with this. But if we only work with $K$-theory, we'll miss all those $\mathbb{Z}/2$'s. We'll be off by factors of two if we use complex $K$-theory. So we also need to understand the Thom isomorphism in $KO$-theory. That is different to a different, rather beautiful construction of the Thom class. This is the simplest way, given what we know, to get the Thom isomorphism, and it's really good for making calculations. But we need a little more.

## §3 Difference bundles

There's another description, which Atiyah calls the **difference bundle construction.** Given $A \subset X$, we want to describe classes in $\widetilde{K}^0(X, A)$. To do this, suppose given a vector bundle over $X$, say $V$. It's *often* going to be the trivial bundle. I have a map

$$\tau : V \to V$$

such that over $A$, $\tau$ is an isomorphism. Then this data gives a class in $\widetilde{K}^0(X, A)$.

Intuitively, what I'm looking at is the difference $V - V$. Let me just give you a construction. You're supposed to imagine that you have two copies of $V$ sitting on $X$ and you "glue them together" on $A$, and that's like a clutching function. This is almost in the situation of a clutching construction.

**26.2 Definition.** By excision we have $\widetilde{K}^0(X, A) \simeq \widetilde{K}^0(X \cup_A X, X)$.

First let's make a vector bundle over $X \cup_A X$. To do this, I have these two copies of $X$ glued together at $A$. I glue together two copies of $V$ by $\tau$ on $A$. So if $X_1, X_2$ are the two copies of $X$, then we take

$$V/X_1 \sqcup V/X_2/(v/a_1 = \tau(v)/a_2).$$

So we're gluing two copies of $V$ along $A$ by $\tau$. This gives me a vector bundle $V^\tau$ on $X \sqcup_A X$ which restricts to $V$ on each copy of $X$. So we define the element in $\widetilde{K}^0(X \sqcup_A X, X)$ to be $V^\tau - V$ (where $V$ refers to the bundle on $X \sqcup_A X$ obtained by gluing $V$ by the identity).

Let's do an example. Here's a really good example.

**26.3 Example.** Let's take $X = \mathbb{C}$ (parametrized by $\lambda$), and $V$ trivial. We define the map $\tau$ which sends $(\lambda, v) \mapsto (\lambda, \lambda v)$. That's an isomorphism as long as long as $\lambda \neq 0$. This defines an element in $\widetilde{K}^0(\mathbb{C}, \mathbb{C} \setminus \{0\}) \simeq \widetilde{K}^0(\mathbb{CP}^1)$. Strictly speaking, we need $A$ to be excisive, which $\mathbb{C} \setminus \{0\}$ is not, so we should take $\mathbb{C} \setminus D_1(0)$ or something. We can do this because everything is homotopy invariant.

The question is which vector bundle I have. The answer is, this is a construction of $L - 1$. There are a variety of conventions in here which we'll need to commit to. In particular, this is the generator of $\widetilde{K}^0(\mathbb{CP}^1)$.

Let me just indicate a variation on this, and then I'm going to stop. This thing is supposed to be multiplicative. The generator of $\widetilde{K}^0(\mathbb{C}^2, \mathbb{C}^2 \setminus \{0\})$ is supposed to be the square of the Bott element. What would happen if I actually tried to square this? Then what you really see, if you tensor these two constructions together, is a chain complex. You're supposed to think of this thing over $\mathbb{C}$ defined earlier as a little chain complex $\mathbb{C} \to \mathbb{C}$ and when you multiply them, you're supposed to tensor the chain complex with itself. Then I get a four-term chain complex

$$\mathbb{C} \to \mathbb{C} \oplus \mathbb{C} \to \mathbb{C}$$

over $\mathbb{C}^2$. This four-term chain complex is *acyclic* away from zero. There's a similar construction to get vector bundles from these—I refer you to Atiyah-Bott-Shapiro.

I want to say this in a coordinate-free way. Suppose I have a complex vector space $W$. I'm now going to make a chain complex of trivial bundles. Form a (trivial) vector bundle over $W$ with $W \times \bigwedge^\bullet W$. We make this into a chain complex of vector bundles sending $(w, \eta) \mapsto w \wedge \eta$. That's a generalization of this construction. This defines the generator of the $K$-theory of $W, W \setminus \{0\}$. The nice thing about this description is that it can be applied *fiberwise* in a vector bundle over a space $X$. If $W$ is now a vector bundle over $X$ (for $\pi : W \to X$) with fibers $W_x$, we can do this same construction. We can take $W \times \pi^* \bigwedge^\bullet W$ and make that into a chain complex in the same manner. This gives an element $U \in \widetilde{K}^0(W, W \setminus \{0\})$ which reproduces the previous construction at each fiber.

# Lecture 27
# 11/19

So I realized to some horror that there's only today and four more classes. I think I need to go a little more quickly. Anyway, last time I was talking about the Thom isomorphism in $K$-theory, which I talked about, and I was about to do it in $KO$-theory, and I got into this impromptu lecture about this stuff. It's harder than it looks to pull off. Anyway, I think what I'd be better off doing is just to summarize how the calculations go. I was explaining all that to get the conventions nailed down.

Let me remind you where we were, and describe how the vector field problem gets solved, and discuss the image of $J$, all today.

## §1 Solution of the vector fields problem

I find it easier to go through with a particular numerical example. We could do the general case if we wanted. Let me remind you of the setup. Start with $\mathbb{RP}^8$. This will be a "typical" case. We have

$$\widetilde{KO}(\mathbb{RP}^8) = \mathbb{Z}/2^4,$$

as we know from the Atiyah-Hirzebruch spectral sequence. The dual of $\mathbb{RP}^8_+$ is $\mathbb{RP}^{-1}_{-9}$ and the top cell of this splits off. We can add 16 to everything so that is, up to suspension, $\mathbb{RP}^{15}_7$. This implies that $S^{15}$ has *eight* vector fields. We have to be in the range where this approximates a certain Stiefel manifold, but this is just to illustrate the general method.

We want to show that $S^{15}$ does not have nine vector fields. We try to repeat the argument with $\mathbb{RP}^9$. We know that $\widetilde{KO}(\mathbb{RP}^9) = \mathbb{Z}/32$. You're always doing this. In the vector fields problem, you're *always* looking at $n$ such that $\widetilde{KO}(\mathbb{RP}^n)$ is $\mathbb{Z}/2^a$ and $\widetilde{KO}(\mathbb{RP}^{n+1}) = \mathbb{Z}/2^{a+1}$. You're always looking at that one, because for a fixed power of 2, you're finding the largest value of $n$ with that $KO$-theory. If you go back and think about it, you'll remember it.

Now the dual of $\mathbb{RP}^9_+$ is $\mathbb{RP}^{-1}_{-10}$ and again, the top cell splits off. But we want to show that the top cell of $\mathbb{RP}^{15}_6$ does not split off. There's a lot of monkeying around with Spanier-Whitehead duality at this point. You can show this, but the thing is to make a calculation in $KO$-homology. It's easier to work with $KO$-cohomology for us, though, which takes us to the dual.

I'm not going to use this at the moment, but we also saw that there is a map

$$S^6 \cup_f e^{15} \to \mathbb{RP}^{15}_6$$

which is a monomorphism in homology. The attaching map comes from the generator image of the $J$-homomorphism. The attaching map goes all the way from the top cell to the bottom cell.

This isn't so convenient. The more convenient thing is to work with the dual. This thing is equivalent to showing that the bottom cell of $\mathbb{D}(\mathbb{RP}^{15}_6)$ does not split off. This boils down to something easier. Namely, we want

$$\mathbb{D}(\mathrm{Thom}(\mathbb{RP}^9, -16L) = \mathbb{RP}^{15}_6.$$

Remember the general formula: $\mathbb{D}(\mathbb{RP}^b_a) = \mathbb{RP}^{-a-1}_{-b-1}$, up to suspension. Anyway, let's just use that formula. We get

$$\mathbb{D}(\mathbb{RP}^{15}_6) = \mathbb{RP}^{-7}_{-16},$$

and we can add 32 to everything to get $\mathbb{RP}^{25}_{16}$. This is the Thom complex of $16L$ over $\mathbb{RP}^9$. The general thing that you get when all is said and done is:

We have some $\widetilde{KO}(\mathbb{RP}^n) = \mathbb{Z}/2^a$ and $\widetilde{KO}(\mathbb{RP}^{n+1}) = \mathbb{Z}/2^{a+1}$. We're at an $n$ where when we increase $n$, we increase the size of the $KO$-group. The general case is, we're trying to show that the bottom cell of the Thom complex of $\mathbb{RP}^{2^a+n+1}_{2^a}$ does not split off. For some reason, when you work in $KO$-cohomology, this is the easier problem to work with.

We need to think about what $\widetilde{KO}(\mathbb{RP}^{25}_{16})$ looks like. We have a map

$$\mathbb{RP}^{25} \to \mathbb{RP}^{25}_{16},$$

and $\mathbb{RP}^{25} \hookrightarrow \mathbb{RP}^\infty$. We know the $KO$-theory of the projective spaces and the map and we want to understand the third one. When we look at the AHSS for $\mathbb{RP}^{25}_{16}$, there's nothing until dimension 16, and then ....

Use the fact that there are no differentials in the spectral sequence, as you can see by comparing it with the AHSS for $\mathbb{RP}^{25}$.

We find two things:

- $\widetilde{KO}(\mathbb{RP}^{25}_{16})$ maps to $\widetilde{KO}(S^{16}) = \mathbb{Z}$ and that map is onto, and the kernel, which is $\widetilde{KO}(\mathbb{RP}^{25}_{17})$, is cyclic of order 32.

- If I look at the spectral sequence, I find that the image of $\widetilde{KO}(\mathbb{RP}^{25}_{16}) \to \widetilde{KO}(\mathbb{RP}^{25})$ is cyclic of order 64. Alternatively, use the half-exact sequence that you get from the cofiber sequence
$$\mathbb{RP}^{15} \to \mathbb{RP}^{25} \to \mathbb{RP}^{25}_{16}.$$

Now, if $\mathbb{RP}^{25}_{16} = S^{16} \vee \mathbb{RP}^{25}_{17}$, then the $\mathbb{Z}$ would be generated by a class $x$ with $\Psi_3(x) = 3^8 x$. I didn't get around to this, but the Adams operations also exist in $KO$-theory. There are unique ones which are compatible with complexification. There's a formula for the Adams operations in terms of the exterior powers and operations on vector spaces and you can make those formulas in terms of the reals. Also, $\Psi^k(L) = L^k$ for a line bundle, which is $L$ if $k$ is odd.

In the map $\widetilde{KO}(\mathbb{RP}^{25}_{16}) \to \widetilde{KO}(\mathbb{RP}^{25})$, the generator of the $\mathbb{Z}$ summand hits a generator of the image. So the generator maps to something of order $2^6$. Here's the thing that is surprisingly easy. You might think that all the money is in calculating $\Psi^3$ on this. But the point is that $\widetilde{KO}(\mathbb{RP}^{25})$ is cyclic generated by $L-1$ and $\Psi^3(L-1) = L-1$, so that $\Psi^3$ acts as the **identity** on $\widetilde{KO}(\mathbb{RP}^n)$ for **any** $n$. So over in the image, we find that if the bottom cell splits off, the generator of the image would be killed by $3^8 - 1$, which is $(1+8)^4 - 1 = 32(\text{odd number})$. That's a contradiction because the generator has order 64.

I think the only way to understand this better is to pick another example and play with it. The important things to come to grips with — this argument works in general — I think, is to recognize when the $KO$-groups of real projective spaces change. The thing to do is to think about this argument, but to keep track of when the order of the $KO$-group changes, and in our case, for the existence of the vector fields problem, you want one before it changes, and for the non-existence, you want just after it changes. If you're trying to understand this better, I'd recommend trying to use these issues. But the takeaway is — and this is what Adams did — is that we can use $KO$-theory and Adams operations to show that certain cells don't split off.

In the course of this vector fields problem, we wound up learning (and I'm going to say this in a different way) that the solution to the vector field problem tells us that the stable attaching maps of the $n$-cell in $\mathbb{RP}^n$ "goes down" by the Radon-Hurwitz number and attaches by the generator of the image of $J$, which is nonzero. So, for example, we know that $S^{15}$ has eight vector fields. That's the appropriate Radon-Hurwitz number. It doesn't have nine, though. So that means that the stable attaching map $S^{14} \to \mathbb{RP}^{14} \to \mathbb{RP}^{15}$ factors through $\mathbb{RP}^6$ and no further, and the composite $S^{14} \to \mathbb{RP}^6 \to S^6$ (the obstruction to going down further) is the generator of the image of $J$.

## §2 Adams's work on the image of $J$

If you're not trying to understand all the details of the calculation and you still want to follow the rest of the course, the point is that we understand the stable attaching maps in $\mathbb{RP}^\infty$, and that will be in the next couple of lectures. What I wanted to do, and I only have time to barely start it, is to discuss the image of the $J$-homomorphism. This is another very beautiful thing that Adams solved using essentially these methods.

I'm just going to give you a summary and a guide to how the calculations work after break. We're studying the $J$-homomorphism, which goes

$$\pi_n BO \to \pi^s_{n-1} S^0.$$

What Adams did was to determine the image of this map. Adams showed:

- The image is a summand.
- The $\mathbb{Z}/2$'s (when $n = 8k+1, 8k+2$) go in via a split injection.
- When $n = 4k$, it's cyclic of order the denominator of $B_{2k}/4k$. When I say denominator, I say denominator when you reduce it in lowest terms. Recall that $B_n$ is the $n$th Bernoulli number. It's defined by the identity

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} \frac{B_n x^n}{n!}.$$

I want to show how the Bernoulli numbers come up. This represents the only part of the homotopy groups of spheres that is really computable, and amazingly, it tells us the attaching maps of the cells in $\mathbb{RP}^\infty$.

So there are two parts to this: the Bernoulli number part, and the $\mathbb{Z}/2$ part. We've already shown that the image of the generators are nontrivial—that's what this argument shows. But in fact, the situation when $n = 8k+1, 8k+2$ is in fact kind of interesting. I'm going to leave this as an exercise, although a rather challenging exercise from what we've learned so far, although it's possible to do this exercise given what we've done in class.

**27.1 Exercise** (Challenging exercise)**.** Show that $n = 8k+2$, the following happens: The map $\mathbb{Z}/2 \simeq \pi_n BO \to \pi^s_{n-1} S^0 \to \pi_{n-1} KO \simeq \mathbb{Z}/2$ is an isomorphism.

# Lecture 28
# 11/26

So I actually put a problem set up. It's on the course website. The last two problems are harder than the other ones, and I didn't work out the last one. I'm pretty sure you can't. I saw this problem in one of Adams's $J(X)$-papers and he says "presumably this is true." I kind of like anyone who's an undergraduate or who needs a grade to actually turn in the problem set. In the problem set, I just hit a few things which I think are main topics of the course, although some of the stuff I'm going to do this last week is not there.

## §1 The $e$-invariant

I want to talk about the $e$-invariant and the $J$-homomorphism in this term. So the first thing is the $e$-invariant. There are two convenient ways of defining this, one due to Adams and one due to Toda. Adams says that they're probably equivalent.

**28.1 Definition.** The *e*-**invariant** is a map

$$e : \pi_{2n-1} S^0 \to \mathbb{Q}/\mathbb{Z}$$

from an odd stable homotopy group of spheres to $\mathbb{Q}/\mathbb{Z}$.

Start with a map

$$f : S^{2n-1} \to S^0$$

and form the mapping cone $S^0 \cup_f e^{2n}$ and look at the exact sequence in $K$-theory. You get a short exact sequence

$$0 \to \widetilde{K}^0(S^{2n}) \to \widetilde{K}^0(S^0 \cup_f e^{2n}) \to \widetilde{K}^0(S^0) \to 0$$

and you can also apply the Chern character to go to rational cohomology.

$$\begin{tikzcd}
0 \arrow[r] & \widetilde{K}^0(S^{2n}) \arrow[r] \arrow[d, "\mathrm{ch}"] & \widetilde{K}^0(S^0 \cup_f e^{2n}) \arrow[r] \arrow[d, "\mathrm{ch}"] & \widetilde{K}^0(S^0) \arrow[r] \arrow[d, "\mathrm{ch}"] & 0 \\
0 \arrow[r] & \widetilde{H}^{ev}(S^{2n}) \arrow[r] & \widetilde{H}^{ev}(S^0 \cup e^{2n}) \arrow[r] & \widetilde{H}^{ev}(S^0) \arrow[r] & 0
\end{tikzcd}$$

where the vertical maps are rational isomorphisms. Note that the bottom sequence is canonically split as

$$0 \to \mathbb{Q} \to \mathbb{Q} \oplus \mathbb{Q} \to \mathbb{Q} \to 0.$$

Choose $a \in \widetilde{K}^0(S^0 \cup e^{2n})$ which hits the generator $1 \in \widetilde{K}^0(S^0) = \mathbb{Z}$. We look at the Chern character $\mathrm{ch}(a) \in \widetilde{H}^0(S^0) \oplus \widetilde{H}^{2n}(S^{2n})$. Its component in the first thing is 1 and the component in the second piece is $\widetilde{e}$. The *e*-**invariant** can be defined as

$$\widetilde{e}/\mathbb{Z}$$

or

$$\widetilde{e}/(\mathrm{im}(\widetilde{K}^0(S^{2n}) \overset{\mathrm{ch}}{\to} \widetilde{H}^{even})$$

and it's easy to see that is well-defined, because $a$ is well-defined modulo $\widetilde{K}^0(S^{2n})$. You can also think of this in terms of rational $K$-theory and the splitting in terms of the eigenspaces of the Adams operations (which is the splitting of rational cohomology).

This is one definition of the $e$-invariant. Here's a variation. We could look at the same sequence in $KO$-theory. We could do the same thing, and tensor with the rationals. Then what we get is

$$e_{\mathbb{R}}(f) \in \widetilde{H}^{2n}(S^{2n})/\widetilde{KO}^0(S^{2n}).$$

We want $n$ to be *even here*, see the next paragraph. If $2n \equiv 4 \mod 8$, the group in question is $\mathbb{Q}/2\mathbb{Z}$. If $n \equiv 0 \bmod 8$, the group in question is $\mathbb{Q}/\mathbb{Z}$. That's because the map from $\widetilde{KO}^0(S^{2n}) \to \widetilde{K}^0(S^{2n})$ is an isomorphism if $2n \equiv 0 \mod 8$ while hits $2\mathbb{Z}$ if $2n \equiv 4 \mod 8$.

**Remark.** Recall that

$$\widetilde{KO}(X) \to \widetilde{K}(X) \xrightarrow{\mathrm{ch}} H^{\mathrm{even}}(X;\mathbb{Q})$$

and the $KO$-theory lands in the cohomology of $X$ in degrees a multiple of 4.

So that's the $e$-invariant, and that's the thing which we want to understand how to compute. It's difficult to do this in general. It's difficult to say something about the $K$-theory of a random 2-cell complex. However, when the map is in the image of the $J$-homomorphism, then we can say something about it. There's a complex version and a real version.

The complex version goes

$$\pi_{m-1}U = \pi_m BU \simeq \widetilde{K}^0(S^m) \to \pi_{m-1}(S^0).$$

There's also a real version which goes

$$\pi_{m-1}O \to \pi_m BO \to \widetilde{KO}^0(S^m) \to \pi_{m-1}(S^0).$$

These are related. These maps take vector spaces, real or complex, and turn them into spheres by taking the one-point compactification. Of course, doing that doesn't depend on any complex structure. So we have a factorization of the complex $J$-homomorphism through the real one and the map $U \to O$.

What happens here is that $\pi_{m-1}U$ alternates between being $\mathbb{Z}$ and 0: if $m$ is even, then we get $\mathbb{Z}$. If $m$ is odd, then we get 0. The homotopy groups of $O$ go $\mathbb{Z}/2, 0, \mathbb{Z}, 0, 0, 0, \mathbb{Z}$. The map $U \to O$ is the opposite of the map defined previously, and the composite is multiplication by 2. That is,

$$\mathbb{Z} \simeq \pi_{m-1}U \to \pi_{m-1}O \simeq \mathbb{Z}$$

is an isomorphism when $m \equiv 4 \mod 8$ and is multiplication by 2 when $m \equiv 0 \mod 8$. (You get this by looking at the composite.) It's easier to say everything in terms of complex $K$-theory. When $m \equiv 0 \mod 8$, we'll get something that's off by a factor of two. The cost of working with complex rather than real $K$-theory is an overall factor of two. We'll just work with it, though. That is, we'll compute

$$\pi_{2m-1}U \to \pi_{2m-1}S^0 \xrightarrow{e} \mathbb{Q}/\mathbb{Z}.$$

(Remember this is zero when $m$ is odd.) It's twice the value of the real $e$-invariant $e_{\mathbb{R}}$ on the generator of $KO$ $J$-homomorphism.

Let $x_{2n} \in \pi_{2n}BU$ be the generator, so that we can think of it as a virtual complex vector bundle over $S^{2n}$. If $f = J(x_{2n}) \in \pi_{2n-1}(S^0)$, we want to calculate $e(f)$. In order to do that, we need to know about the $K$-theory of $S^0 \cup_f e^{2n}$. Geometrically, we know that it is

$$\mathrm{Thom}(S^{2n}, x_{2n})$$

where $x_{2n}$ is regarded as a virtual vector bundle. That's the thing about the image of $J$—the mapping cones that you form are Thom complexes. We can take for our class $a \in \widetilde{K}^0(S^0 \cup_f e^{2n})$ the $K$-theory Thom class in this Thom complex. In order to calculate the $e$-invariant, we need to know:

**Question.** What is $\mathrm{ch}(U)$ where $U$ is a Thom class in $K$-theory?

This is part of a very beautiful story that kind of has to do with Grothendieck's original invention of $K$-theory and its relation to cycles, and Atiyah-Hirzebruch's definition of $K$-theory, and the index theorem. The answer to this question was one of the reasons $K$-theory was set up in the first place. I'm going to have to skip some details.

Let's recall the **$K$-theory Thom isomorphism**, which is an isomorphism

$$K^0(X) \simeq \widetilde{K}^0(X^V)$$

where $V \to X$ is a complex vector bundle on $X$. This makes sense for virtual bundles, as well. In fact, it's an isomorphism of modules over $K^0(X)$. The Thom class $U \in \widetilde{K}^0(X^V)$ is a choice of generator. If I have two different Thom classes $U, U'$, they "differ" by a unit in $K^0(X)$—meaning $U = \chi U'$ where $\chi \in K^0(X)^*$. That's true for any cohomology theory. The same remarks apply to the sum $\widetilde{H}^{\mathrm{even}}(\cdot; \mathbb{Q})$ cohomology theory, rational even periodic cohomology.

So take $U_K \in \widetilde{K}^0(X^V)$, and take its Chern character $\mathrm{ch}(U_K) \in \widetilde{H}^{\mathrm{even}}(X^V)$. There's also the rational cohomology Thom class $U_H$. The two of them differ by a unit. There's some unit $\chi = \chi(V) \in H^{\mathrm{even}}(X)^*$ that measures the difference between $\mathrm{ch}(U_K)$ and the rational cohomology Thom class. So

$$\mathrm{ch}(U_K) = \chi(V) U_H.$$

What's $U_H$? If $\dim_{\mathbb{C}} V = n$, then $U_H$ is the ordinary cohomology generator in $H^{2n}(X^V)$. There's a canonical choice of Thom classes in cohomology for complex vector bundles. We also saw earlier that there was a canonical choice of Thom classes in $K$-theory for complex vector bundles — that's $U_K$. So make these canonical choices. That gives meaning to all these symbols and to $\chi(V)$.

Now these Thom classes have some canonical properties. If I have two spaces $X, Y$ and vector bundles $V, W$ over $X, Y$, then the Thom complex of $V \oplus W$ over $X \times Y$ (the "Whitney sum") is the smash product of the Thom complexes $X^V \wedge Y^W$. Under this isomorphism, the homology and $K$-theory Thom classes are *multiplicative.* This means that $\chi$ is **exponential.** It's also a stable, and — this is trivial to check — $\chi(1) = 1$. So

$$\chi : K^0(X) \to H^{\mathrm{even}}(X)$$

and it is called a **stable exponential characteristic class**, because of all this. That is, $\chi(V \oplus W) = \chi(V)\chi(W)$, etc. By the splitting principle, $\chi$ is determined by what it does on line bundles.

So what does $\chi$ do on line bundles? In order to talk about this, we have to get straight some conventions. Let $L$ be the tautological line bundle over $\mathbb{CP}^\infty$. There's the zero section $\mathbb{CP}^\infty \hookrightarrow (\mathbb{CP}^\infty)^L$ which is a homotopy equivalence, since the unit sphere bundle of $\mathbb{CP}^\infty$ is contractible. For various reasons, the convention that works out right is that under this map

$$\widetilde{K}^0((\mathbb{CP}^\infty)^L) \to \widetilde{K}(\mathbb{CP}^\infty)$$

the Thom class $U_K(L)$ goes back to $1 - L$. There's a good way of remembering this. For any vector bundle $V$, the Thom class is supposed to pull back to the total exterior power of $V$. So that's the general formula, and whatever conventions you make, this has to be true, or certain things you want to be multiplicative don't. You don't get to make this up. This forces a whole bunch of other conventions on you.

We want this to map under the Chern character to the generator $x$ of $H^2(S^2)$ under restriction. We need to define

$$\mathrm{ch}(L) = e^{-x} = e^{c_1(L)}, \quad x \in H^2(S^2; \mathbb{Z}).$$

This is all make-work for the sign police, anyway — we're not going to be able to hang on to it. So the Chern character of the Thom class is $1 - e^{-x}$.

What is the ordinary cohomology? We know that $U_H = x$. So we get that

$$\chi(L) = \frac{1 - e^{-x}}{x}.$$

For those of you who are anticipating the appearance of Bernoulli numbers, this is almost there, but not quite. Just hang in there.

Next, we need to calculate what is $\chi(x_{2n})$. There are several ways to do this. I know of two approaches to this. We need to work out the splitting principle. We need to take the virtual bundle $x_{2n}$ on $S^{2n}$ and we need to find a space mapping to this such that this becomes a sum of line bundles.

- You can map $S^2 \times \cdots \times S^2$ to it, where it pulls back to $\prod(1 - L_i)$.
- You can also pull-back to $\mathbb{CP}^n$: there's a map $\mathbb{CP}^n \to S^{2n}$ and it pulls back to $(1 - L)^n$. You can use either of these approaches. I have a particular reason for liking the first reason better, since it generalizes, although it's possible that the algebra works for the second case.

# Lecture 29
# 11/28

The goal is to start with the generator

$$x_{2n} \in \widetilde{K}^0(S^{2n})$$

and we can think of that as a map

$$S^{2n} \to BU$$

and to calculate the Chern character of the Thom class.

Let $U_K \in \widetilde{K}^0(\mathrm{Thom}(S^{2n}, x_{2n}))$ be the Thom class, which we constructed explicitly earlier. We'd like to calculate the Chern character $\mathrm{ch}(U_K)$, because that will give the $e$-invariant

$$e(J(x_{2n})).$$

As we saw,

$$\mathrm{ch}(U_K) = \chi(x_{2n}) U_H$$

where $U_H$ is the homology Thom class and $\chi(x_{2n})$ is the stable exponential characteristic class and was determined by its values on line bundles. So

$$\chi(L) = \frac{1-e^{-x}}{x}, \quad -x = c_1(L)$$

for a line bundle $L$.

To calculate $\chi(x_{2n})$, we need to write $x_{2n}$ as a sum of line bundles, after sum pull-back. To do this, form the pull-back

$$S^2 \times \cdots \times S^2 \to S^2 \wedge \cdots \wedge S^2 \to S^{2n}$$

where there are $n$ factors. Then $x_{2n}$ pulls back to $\prod(1-L_i)$ where $L_i$ is the tautological bundle on the $i$th factor.

I'd like to calculate this. Let's write $g(x) \stackrel{\text{def}}{=} \frac{1-e^{-x}}{x}$. For some low values of $n$, we'd like to calculate this. When $n=2$,

$$\chi(x_2) = \chi(1-L) = \frac{1}{g(x)} \in \mathbb{Q}[x]/x^2 = H^*(S^2;\mathbb{Q}).$$

The expression is $\frac{x}{1-e^{-x}} = 1 + \frac{1}{2}x$. So that's something.

What's the next one going to be? Let's just do one more. What happens to $x_4$? That's $\chi(1-L_1-L_1+L_1L_2)$. This is a sum of line bundles, so that's

$$\frac{g(x_1+x_2)}{g(x_1)g(x_2)}$$

where $x_1 = c_1(L_1), x_2 = c_1(L_2)$. Note that the first Chern class is additive, $c_1(L_1L_2) = x_1 + x_2$. So we can expand this as

$$1 + ex_1x_2 + \text{higher terms}$$

and that $e$ is the $e$-invariant.

It takes a little while to get used to such an expression. If I had to figure out $\chi(x_6)$, I'd have

$$\frac{g(x_1+x_2)g(x_2+x_3)g(x_2+x_3)}{g(x_1)g(x_2)g(x_3)g(x_1+x_2+x_3)} = 1 + ex_1x_2x_3 + \ldots$$

Somehow we have to get used to what this is doing algebraically. If this was plus instead of times, it would be $g(x_1+x_2) - g(x_1) - g(x_2)$, which kills linear functions. The second term would kill quadratic functions. If you want to see one exploitation of it, you can look at my ICM talk on the theorem of the cube in algebraic topology. Anyway, as I said, it's easier to think about if I took the log. So let's take the log and turn it into addition.

So we might as well take the logarithm — the natural log — and compute the $e$-invariant. So

$$\log \chi(x_{2n}) = e_1x_1x_2\ldots x_n.$$

Instead of working with $g$, let's work with the log of $g$. We're trying to figure out what this does. So we have this operator. I'm just going to state something and let you prove it. I want to take the log of $g$, and let

$$f(x) = \log g(x).$$

Now we're interested in the following expression, which we might call $\delta^n$. We're interested in

$$\delta^n f(x_1,\dots,x_n) = 0-(f(x_1)+\dots f(x_n))+(f(x_1+x_2)+\dots)-(f(x_1+x_2+x_3)+\dots)\pm\dots+(-1)^n f(x_1+\dots+$$

We want to figure out $\delta^n f$. It starts out with $x_1\dots x_n$ and we'd like to figure out the coefficient. There's a pretty simple inductive formula for this. It's easy to see that this is some constant times $x_1x_2\dots x_n$ plus higher terms, because if any of the $x_i=0$, then the whole thing is zero.

We have:

$$\delta^n f(x_1,\dots,x_n) = (\delta^{n-1}f)(x_1,\dots,x_{n-1}+x_n)-(\delta^{n-1}f)(x_1,\dots,x_{n-1})-(\delta^{n-1}f)(x_1,\dots,x_{n-2},x_n).$$

If I take

$$\delta f(x,y) = f(x+y)-f(x)-f(y)$$

then I'm iterating it in the last variable, but since it's symmetric I can iterate it in any other variable.

So obviously $\delta^n$ is linear. If $h(x)=x^k$, then you can explicitly what all of these things are. We already know that

$$\delta^n h = 0, \quad k<n.$$

If $k=n$, it's $n!x_1\dots x_n$. That's really trivial to check. If you take the definition, the only place to get an $x_1\dots x_n$ is the last one.

The answer to our question "What is the $e$-invariant $e(x_{2n})$?" is "the number $c_n$ you get by writing $\log g(x)=\sum c_m \frac{x^m}{m!}$." So that's what we have eto figure out. Now this gets fairly easy. So what was our $g(x)$? It was $\frac{1-e^{-x}}{x}$. Therefore

$$d\log g(x) = \frac{e^{-x}}{1-e^{-x}}-\frac{1}{x} = \frac{1}{e^x-1}-\frac{1}{x}.$$

The $\frac{1}{x}$ just cancels. Remember the definition of the **Bernoulli numbers now**,

$$\frac{x}{e^x-1} = \sum B_n\frac{x^n}{n!}$$

and these are the things that go into writing down formulas for $1^k+\dots+n^k$. So this thing $d\log g(x)$ here, if we forget the $\frac{1}{x}$ constant term, that's equal to

$$\sum \frac{B_n}{n}\frac{x^{n-1}}{(n-1)!}$$

so that

$$\log g(x) = \sum\frac{B_n}{n}\frac{x^n}{n!}.$$

If you play around with this, you can see that after the first one, odd Bernoulli numbers are all zero. But that's okay, right. We know that the $J$-homomorphism on half of the classes is zero anyway, because it factors through $KO$-theory.

The conclusion is:

**29.1 Proposition.** *The $e$-invariant of $x_{2n}$ is $\frac{B_n}{n}$.*

So that's pretty good. That gets us somewhere. There's a little bit more to the story though – this isn't quite the expression that comes up in algebraic topology. There are a couple of things we need to modify. So $x_{2n} \in \pi_{2n}BU$ mapping to $\pi_{2n-1}S^0$. That factored through $\pi_{2n}BO$. We're mapping out of $\pi_{2n-1}S^0$ using the $e$-invariant.

So we have a diagram

$$\begin{tikzcd} \pi_{2n}BU \arrow[r] \arrow[d] & \pi_{2n}BO \arrow[dl] \\ \pi_{2n-1}S^0 \arrow[d, "e"] & \\ \mathbb{Q}/\mathbb{Z} & \end{tikzcd}$$

and we observe

$$e(x_{2n}) = 0, \quad n \text{ odd}.$$

In dimension $8k+4$, the map $\pi_{8k+4}BU \to \pi_{8k+4}BO$ is an isomorphism, but in dimensions $8k$, the map is multiplication by 2 $\pi_{8k}BU \to \pi_{2k}BO$. We should really use $KO$-theory rather than $K$. If you follow the definitions, you get another factor of 2. If you put all these together, you learn:

**29.2 Proposition.** *The $KO$-invariant $e_{\mathbb{R}}$ on $J$ of the generator of $\pi_{4k}BO$ is $\frac{B_{2k}}{4k} \in \mathbb{Q}/\mathbb{Z}$.*

That's the magic expression. This number comes up a lot. The denominator of this number comes up in the order of the image of $J$ and the numerator comes up in the number of exotic spheres. So these numbers are very important in topology.

You can do the same thing for $KO$-theory and the associated power series is

$$\frac{e^{x/2} - e^{-x/2}}{x} = e^{1/2x}\left(\frac{1-e^{-x}}{x}\right).$$

There's a lot more to the story. I'm just going to tell you how this winds up working out. So we have this map $\pi_{4k}BO \simeq \widetilde{KO}^0(S^{4k}) \to \pi_{4k-1}S^0$ and the image of the generator was relevant to us: it had to do with the vector field problem and the sphere of origin problem. I just wanted to tell you more about this map. We had maps

$$\pi_{4k}BO \to \pi_{4k-1}S^0 \xrightarrow{e} \mathbb{Q}/\mathbb{Z}$$

and we worked out the image of the composite. Now there are some things to check. We've at least shown that the image of the generator factors through a cyclic group of order the denominator the generator of $B_{2k}/4k$ and in fact the entire stable homotopy groups of spheres map into that cyclic group.

What is the kernel of the $J$-homomorphism? What is the kernel of

$$J : \pi_{4k}BO \to \pi_{4k-1}S^0?$$

The point is, there are two ways to calculate the $e$-invariant. I talked about one earlier. If I have an arbitrary map of an odd sphere $f : S^{4k-1} \to S^0$ and we look at the two cell complex $S^0 \cup_f e^{4k}$. We can look at the short exact sequence

$$0 \to \widetilde{KO}(S^{4k}) \to \widetilde{KO}(S^0 \cup_f e^{4k}) \to \widetilde{KO}(S^0) \to 0$$

and tensored with $\mathbb{Q}$ there was a canonical splitting. The splitting came from the eigenspace decomposition in terms of Adams operations. It splits into the eigenspaces of $\psi_\lambda$, which are $1, \lambda^k$. This implies that the $e$-invariant is killed by

$$\text{g.c.d.}_\lambda \lambda^N(\lambda^k - 1), \quad N \gg 0.$$

This number is also denominator of $B_{2k}/4k$. The proof of this is pretty straightforward and the proof uses the arithmetic interpretation of Bernoulli numbers in terms of power sums. That's what gives you access to this kind of information. You can do that as an exercise or look it up in Adams.

The other part of this is the **Adams conjecture.**

# Lecture 30
# 11/30

All right, I had a whole lot of things I was hoping to do this semester, but I think I should tell you something about the Adams conjecture.

## §1 Adams conjecture

(This isn't standard notation.) Let $X$ be a space. Consider $\widetilde{KO}(X)$, the Grothendieck group of real vector bundles over $X$ (modulo the image of 1). We saw that if we took an automorphism of a vector space and took the one-point compactification of all those spaces, we'd get things in the homotopy groups of spheres. There's a way of formulating this as a map of cohomology theories. Unfortunately I don't know a standard name for this other cohomology theory.

**30.1 Definition.** $\mathcal{G}^0(X)$ is the Grothendieck group of pointed spherical fibrations over $X$. A **pointed spherical fibration** $P \to X$ is a fibration $P \to X$ together with a section $* \to X$ whose fibers are all spheres $S^n$. You made that into a semigroup via the *fiberwise smash product.* The Grothendieck group of that is $\mathcal{G}^0(X)$.

There's a sometimes useful variant of this. You can take *unpointed* spherical fibrations $S^{n-1} \to P \to X$ and you can make that into a semigroup via *fiberwise join.* Those give you the same group completions. As soon as I add a trivial bundle to a spherical fibration, it becomes pointed, and you get the same thing.

Given a vector bundle $V \to X$ over $X$, we can send it to the *fiberwise one-point compactification* $\overline{V} \to X$.

**30.2 Definition.** That gives a map

$$KO(X) \to \mathcal{G}^0(X).$$

When $X = S^n$, then $\widetilde{KO}(X) \simeq \pi_{n-1}O$ because a vector bundle is determined by its clutching function. Similarly, $\mathcal{G}(S^n) = \pi_{n-1}\mathrm{HAut}(S^N), N \gg 0$ where $\mathrm{HAut}(S^N)$ is the space of self-equivalences of the sphere. Observe that $\mathrm{HAut}(S^N)$ is a subspace (not pointed!) of $\Omega^N S^N$ consisting of the identity component. Therefore, if we subtract the identity,

$$\mathcal{G}(S^n) \simeq \pi_{n-1+N}(S^N).$$

So when we take a sphere, this *is* the $J$-homomorphism. So this map

$$KO^0(X) \to \mathcal{G}^0(X)$$

is, for $X = S^n$, the $J$-homomorphism. That's nice, it embeds the $J$-homomorphism into a map of cohomology theories, although it's not quite obvious that $\mathcal{G}^0(X)$ is a cohomology theory. But it is a cohomology theory and this comes from a map of spectra.

Let's call this map

$$J : KO(X) \xrightarrow{J} \mathcal{G}^0(X).$$

**Remark.** More generally, if $X = \Sigma Y$ for $Y$ connected, then $\mathcal{G}^0(X) = [Y, \Omega^N S^N]$ for $N \gg 0$ or stable maps $S^N \wedge Y \to S^N$ — that's the cohomology theory associated to the sphere spectrum.

It's kind of amazing that you can say something of this map. But the Adams conjecture lets you say what the image is. Homotopy theory doesn't want to produce an image — it wants to produce a long exact sequence. In fact, what happens is that

$$\mathcal{G}^0(X)$$

splits into the product of two cohomology theories, one called the image of $J$ and one called the cokernel of $J$.

If $X$ is a finite connected CW complex, then the group $\mathcal{G}^0(X)$ is finite. That follows from the AHSS

$$H^*(X; \mathcal{G}^*(*)) \implies \mathcal{G}^*(X),$$

and since the $\mathcal{G}$-groups of a point — the stable homotopy groups of spheres — are finite.

This space has finite homotopy groups, so we can understand this by localizing at a prime $p$.

**30.3 Theorem** (Adams conjecture)**.** *Fix a finite complex $X$. For every $k$, there exists an $N$ such that for each $x \in KO^0(X)$,*

$$k^N(\psi^k(x) - x) \in \ker J$$

*and these elements (for all $k$) generate the kernel.*

Alternatively, one could formulate this by saying that the kernel of the map

$$KO^0(X)_{(p)} \to \mathcal{G}^0(X)_{(p)}$$

is generated by the elements of the form $\psi^k(x) - x$, for $p \nmid k$.

The Adams conjecture is a theorem. Adams couldn't settle a factor of two, even for the sphere. In Adams's $J(X)$ paper, he makes this conjecture, and he determines the order of the image of $J$. He shows that the order of the im$J$ for

$$\pi_{4k-1}(O) \to \pi^s_{4k-1}(S^0)$$

is the denominator of $\frac{B_{2k}}{4k}$ when $k$ is odd, and when $k$ is even, it's this or twice this. The Adams conjecture implies this answer, that the factor of 2 isn't actually there. Remember, that factor of 2 came from the map $K \to KO$.

This factor of two was settled by Mahowald. The full Adams conjecture was proved by Quillen-Friedlander (using étale homotopy theory), Sullivan (using similar methods), Quillen (a later proof), and a simple proof due to Becker and Gottlieb. I was going to show you a really easy proof in the complex analog using stuff we did in this class, and if there's time and interest I'll talk about that. In a way, the Becker-Gottlieb proof is really easy. The first three proofs are really beautiful and use all this machinery; if this Becker-Gottlieb proof had arrived early enough, we wouldn't have all this mathematics. These are wonderful papers to read, especially the Sullivan one.

Let's do some examples.

**30.4 Example.** Let's take $X = \mathbb{CP}^2$. What is $\widetilde{KO}^0(\mathbb{CP}^2)$? I claim that it's $\mathbb{Z}$ and it's generated by the *real bundle* underlying the tautological bundle. I think I should call it $i_*L - 2$ where $i_* : K \to KO$. If we were to complexify, then $i_*L$ complexifies to $L \oplus L^{-1}$, so the generator becomes $L \oplus L^{-1} - 2$. This follows easily from the AHSS. There's a little to do to solve extension problems, but you can solve it by looking at the $K$-theory spectral sequence. Let's just assume it. We also have to figure out what $\psi^k$ of the generator is. To figure this out, complexify, since complexification is a monomorphism that commutes with the Adams operations. Namely,

$$K(\mathbb{CP}^2) = \mathbb{Z}[x]/x^3, \quad x = L - 1$$

and the generator of $\widetilde{KO}^0(\mathbb{CP}^2)$ goes to $L + L^{-1} - 2$. One finds that

$$\psi_k(y) = k^2 y$$

for $y$ the generator.

Let's localize at 2.

The kernel of $J$ is generated by the element $\psi_3(y) - y = 8y$. At the prime 2, I get $3y$. At other primes, you'll just get zero. All of this implies that the image of $J$ for $\mathbb{CP}^2$ is $\mathbb{Z}/24$.

Recall

$$(\mathbb{CP}^2)^{nL} = \mathbb{CP}^{n+2}_n,$$

and this therefore depends only on the order of $n$ modulo 24. This is a consequence of the fact that the Thom complex of a vector bundle depends only on the associated *stable spherical fibration.* We got statements like this about real projective spaces because the real $KO$-groups had finite order; the complex $K$-groups of $\mathbb{CP}^n$ don't but the $J$-groups have finite order. This is **complex James periodicity.** I just wanted to illustrate that it's pretty easy to do these calculations, though I've never actually worked out James periodicity for all the complex projective spaces. It's a good way of trying to really understand the Adams conjecture.

## §2 A proof of the Adams conjecture

I'll give you a proof of the complex Adams conjecture, which is slightly easier and which doesn't quite settle the factor of two. This is a sketch proof, but I'm going to only use stuff that you can find in Hatcher.

I'm going to do something which was a little sophisticated, in its day. There's a theorem, called the *mod $p$ Dold theorem*:

**30.5 Theorem** (mod $p$ Dold theorem)**.** *Suppose $X$ is a finite CW complex. Suppose I have two spherical fibrations $P_1 \to X, P_2 \to X$ of the same dimension, of the pointed kind.*

*Suppose I have a map between them*

$$\begin{tikzcd} P_1 \arrow[rr] \arrow[dr] & & P_2 \arrow[dl] \\ & X & \end{tikzcd},$$

*where the map on fibers is $S^n \xrightarrow{k} S^n$. Then the class $P_1$ is the class of $P_2$ in $\mathcal{G}^0(X)[k^{-1}]$.*

So if $k$ is prime to $p$, they're equal in $\mathcal{G}^0(X)_{(p)}$. Since the reduced $\mathcal{G}$-theory is finite, it suffices to prove this at every prime. This is something that takes some proof. A nice place to read about this stuff and to get the whole culture is in Sullivan's "Genetics of homotopy theory and the Adams conjecture." This isn't completely obvious. One reason is that $\mathcal{G}^0(X)$ is homotopy classes of maps into $BF$, the classifying space of self homotopy-equivalences into the sphere. This isn't obvious. You need to say that multiplication by $k$ in $BF$ (the infinite loop space) is related to multiplication by $k$ on the sphere.

That lets you redefine the group $\mathcal{G}^0$ when you've localized at a prime $p$. Here's one place you know the Adams conjecture is true. You know that it's true for line bundles. Take the map

$$\mathbb{CP}^\infty \xrightarrow{k} \mathbb{CP}^\infty$$

which classifies $L^{\otimes k}$. That's covered by a map of universal bundles which is fiberwise the degree $k$ map (raise everything to the $k$th power). All I'm doing is taking a line and raising it to the $k$th power, on each fiber it's $z \mapsto z^k$.

By the mod $p$ Dold theorem, this implies that $\psi_k(L) - L$ is in the kernel of $J$, when you've localized at $p$ (for $k$ prime to $p$). This is supposed to be sort of trivial and if you think about it for a while it is.

So you know it for line bundles.

The next step is to try to turn this into a theorem about spaces. So $K$-theory is maps into $BU$, at least for reduced $K$-theory, and $\mathcal{G}$-theory is maps into $B\mathrm{HAut}(S^N)$. So there's a map

$$BU \to B\mathrm{HAut}(S^N)$$

and we're going to localize all these spaces at a prime $p$. The other fact you need, which isn't very hard (I'd have to get into infinite loop spaces) is that this is an infinite loop map between infinite loop spaces. What are we trying to prove? We are trying to show that for $k$ prime to $p$, the map

$$BU_{(p)} \overset{\psi_k - 1}{\to} BU \overset{J}{\to} B\mathrm{HAut}(S^N)_{(p)}$$

is nullhomotopic. We know that it's true for $\mathbb{CP}^\infty$. We want some glorified version of the splitting principle to give it to us for $BU$. All I'm going to use is that these are infinite loop maps. Since these are infinite loop maps, I get a nullhomotopy of the infinite loop map

$$Q\mathbb{CP}^\infty \to BU \overset{\psi_k - 1}{\to} BU \overset{J}{\to} B\mathrm{HAut}(S^N)_{(p)}$$

where $Q\mathbb{CP}^\infty$ is the free infinite loop space generated by $\mathbb{CP}^\infty$.

The key point is that the map $Q\mathbb{CP}^\infty \to BU$ has a section. We're sort of all set up to prove that. I was planning to do that in this course. How do we do that? Well, remember the James splitting. One said that the map

$$\mathbb{RP}^{n-1}_{n-k} \to V_{k,n}$$

and stably this has an inverse. There's also the complex analog. It goes

$$\Sigma\mathbb{CP}^\infty \to (V_{k,n})_{\mathbb{C}} \to \Sigma\mathbb{CP}^{n-1}_{n-k}$$

and this came from constructing vector fields on spheres and James's intrinsic join. When $k = n$, you get that

$$\Sigma\mathbb{CP}^{n-1} \to SU(n)$$

and there's a *stable* map back. In other words, there is a map

$$\Sigma\mathbb{CP}^{n-1} \to SU(n) \to \Omega^N\Sigma^N\Sigma\mathbb{CP}^{n-1}.$$

Now let $N, k \to \infty$. You get a map

$$\Sigma\mathbb{CP}^\infty \to SU \to Q\Sigma\mathbb{CP}^\infty$$

and if you adjoint over, that gives you a diagram

$$\mathbb{CP}^\infty \to \Omega SU \simeq BU \to Q\mathbb{CP}^\infty$$

That gives us a map $BU \to Q\mathbb{CP}^\infty$ and the map $\mathbb{CP}^\infty \to BU$ is the reduced class of the tautological bundle. We also know that all these maps are loop maps.

So now we have

$$\mathbb{CP}^\infty \to BU \to Q\mathbb{CP}^\infty \to BU$$

where $BU \to BU$ is a self-map which is a loop map and which sends $L-1$ to $L-1$. That implies that $BU \to Q\mathbb{CP}^\infty \to BU$ is a homology isomorphism because the homology of $BU$ is the symmetric algebra on the homology on $\mathbb{CP}^\infty$. In fact, this is a homotopy equivalence, and $BU$ is a retract of $\mathbb{CP}^\infty$. This is a proof that uses 1950s era algebraic topology, just using James maps and Bott periodicity. (This argument also works in motivic homotopy theory.) This isn't enough to do the $KO$-Adams conjecture.

# Lecture 31
# 12/3

So there was a calculation I was hoping to go back to and explain this semester. I'll explain how it's set up. We started out talking about the EHP sequence and this thing for calculating the homotopy groups of spheres. We used that to generate a bunch of questions, and one of them the vector field problem. Now that we've analyzed the vector field problem, let me tell you what it means for the EHP sequence.

One thing that plays a role in the vector field problem is the $J$-homomorphism. One might try to set up a $J$-theory analog of the EHP sequence. The $J$-homomorphism goes

$$SO(n) \to \Omega^n S^n,$$

and these are compatible with the suspension map. We have commutative diagrams:

$$\begin{tikzcd} SO(n) \arrow[r] \arrow[d] & \Omega^n S^n \arrow[d] \\ SO(n+1) \arrow[r] & \Omega^{n+1} S^{n+1} \end{tikzcd}$$

and we know the respective homotopy fibers of these things. We know that the homotopy fiber of $SO(n-1) \to SO(n)$ is $\Omega S^{n-1}$, and we get fiber sequences

$$\begin{tikzcd} SO(n) \arrow[r] \arrow[d] & \Omega^n S^n \arrow[d] \\ SO(n+1) \arrow[r] \arrow[d] & \Omega^{n+1} S^{n+1} \arrow[d] \\ S^n \arrow[r] & \Omega^{n+1} S^{2n+1} \end{tikzcd}$$

You get a map from the spectral sequence that would relate the homotpy groups of $SO(n)$ to the EHP sequence and it takes the Hopf invariant map and it desuspends it a few times. One thing that you learn from this is that if $x \in \mathrm{im}(\pi_k SO(n) \to \pi_{n+k}(S^n))$, then the Hopf invariant $H(x)$ desuspends a lot.

The identity element of $S^n$ comes over under the connecting homomorphism to something in $\pi_{n-1}(SO(n-1))$. This map is something that you can work out pretty easily.... This requires a little bit of proof. The formulas work out a bit easier if you use the spin groups. The next thing is that the composite

$$S^{n-1} \to \Omega S^n \to SO(n) \to \Omega^n S^n$$

that map corresponds to the Whitehead square of the identity.

So we get this nice map, which goes from a spectral sequences starting from the homotopy groups of spheres, ending at the homotopy groups of $SO$. What did the vector field problem say? The vector field problem said that $S^n$ has $k$ vector fields if the map $V_{k+1,n+1} \to S^n$ had a section. That was one formulation of the vector field problem, and we approximated these Stiefel manifolds with stunted projective spaces.

Now $V_{k+1,n+1} = SO(n+1)/SO(n-k)$. Let me put down a diagram to make sense of this.

We've got

$$\begin{tikzcd}
SO(n) \arrow[r] \arrow[d] & SO(n)/SO(n-k) \arrow[d] \\
SO(n+1) \arrow[r] \arrow[d] & SO(n+1)/SO(n-k) \arrow[d] \\
S^n \arrow[r] & S^n
\end{tikzcd}$$

and the existence of $k$ vector fields is saying that the identity map $\iota \in \pi_n(S^n)$ goes to zero in the long exact sequence in $\pi_{n-1}SO(n)/SO(n-k)$. That's also equivalent to saying that the image of $\iota$ in $\pi_{n-1}(SO(n)$ lifts to $\pi_{n-1}(SO(n-k))$.

So let's go here. We have $\iota \in \pi_n S^n$ which is going over to $\pi_{n-1}(SO(n))$, which is mapping under this $J$-homomorphism to $\pi_{2n-1}(S^n)$. Then $\iota$ maps to the Whitehead square. The vector field problem says that if the sphere has $k$ vector fields, then this lifts to $\pi_{n-1}(SO(n-k))$. That means that the Whitehead square comees from $\pi_{2n-k-1}S^{n-k}$. We get:

**31.1 Proposition.** *If $S^n$ has $k$ vector fields, then the Whitehead square $[\iota_n, \iota_n]$ desuspends $k$ times.*

So this is an easy diagram chase. We learned exactly what the answer to the vector fields problem was. We learned exactly what the obstruction to desuspending was.

What else did we learn? If $S^n$ has $k$ but not $k+1$ vector fields, then what did we learned? We learned, from all this analysis, that we have $\pi_n S^n$ mapping to $\pi_{n-1}SO(n)$. We've lifted the image of the identity through $\pi_{n-1}SO(n-k)$ but it doesn't lift any further. So that maps down nontrivially to $\pi_{n-1}S^{n-k-1}$.

$$\begin{tikzcd}
\pi_n S^n \arrow[r] & \pi_{n-1}SO(n) & \\
& \pi_{n-1}SO(n-k) \arrow[u] \arrow[r, "\text{generator of im J}"] & \pi_{n-1}S^{n-k-1} \\
& \pi_{n-1}SO(n-k-1) \arrow[u] &
\end{tikzcd} .$$

In fact, the image in $\pi_{n-1}S^{n-k-1}$ is the generator of the image of $J$ there. I've got a bunch of different $J$-homomorphisms here which is rather confusing. Over there, I said that if I have vector fields on spheres, then the Whitehead product desuspends $k$ times. When $k$ is the maximum number of vector fields, then the Whitehead square does not desuspend further and the obstruction to desuspending further —the Hopf invariant — is the generator of the image of $J$.

(Note : everything in the world is localized at 2.)

So this is one part of the story. This relationship between the Whitehead square and the desuspension goes back to James and Toda. Toda understood this relationship between vector fields on spheres and desuspending the Whitehead product. If you remember this picture of the EHP spectral sequence, we had these spheres, and you wrote down the homotopy groups of odd spheres. What the vector fields problem tells you is that there's a differential out of the $\iota$ classes killing the elements in the image of $J$.

In the EHP spectral sequence, this tells you that all the differentials coming out of the diagonal, if you feed this back in. That's a remarkable amount of information. This is important for use in the computational aspects of homotopy theory. Now you might ask what happens to the image of $J$ elements in the EHP sequence. In a given range, you can start working through it. There's a more systematic way to look at it. We had this map

$$\Sigma^N SO(n) \to \Sigma^N \to \mathbb{RP}^{n-1}$$

that is, the map $SO(n) \to \mathbb{RP}^{n-1}$ had a stable retraction via James's "intrinsic join." I can think of that as a map to $\Omega^N\Sigma^N\mathbb{RP}^{n-1}$ and taking the limit, get a map

$$SO(n) \to Q\mathbb{RP}^{n-1} \stackrel{\text{def}}{=} \varinjlim \Omega^N\Sigma^N\mathbb{RP}^{n-1}$$

and these things are all compatible. We have a commutative diagram

$$\begin{tikzcd} SO(n) \arrow[r] \arrow[d] & Q\mathbb{RP}^{n-1} \arrow[d] \\ SO(n+1) \arrow[r] & Q\mathbb{RP}^{n} \end{tikzcd}$$

and there's a cool theorem of Snaith which says that they factor through the maps $\Omega^n S^n \to Q\mathbb{RP}^{n-1}$. This is just a fancy way of producing that Toda produced by different methods. This gives a map from the EHP spectral sequence to the spectral sequence for calculating the stable homotopy groups of $\mathbb{RP}^\infty$: that is, the Atiyah-Hirzebruch ss for $\pi_*^s\mathbb{RP}^\infty$ whose $E_2$-term is $H_*(\mathbb{RP}^\infty, \pi_*^s S^0)$. This seems kind of gargantuan but this is something really understandable – it's something that you can calculate. This diagram ultimately relates all the things we talked about attaching maps about cells and projective spaces and so forth.

See Mahowald's paper "The image of $J$ in the EHP sequence."