Functorial languages in Homological algebra

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Plan

- Higher (co)limit formulas
- Functorial languages on groups. **fr**-language
- $\mathbf{fr}_\infty$-language
- **FR**-language on Lie algebras
- Lift to spectra
- Functorial surfaces spanned by functorial languages
- Flux-spectra

Introduction. First traces of the higher limit approach

- In [Qui89] Quillen derived formulas of the following kind:

$$HC_{2n}(A) = \lim F/(I^{n+1} + [F, F]),$$

where A is an algebra over k and $HC_{2n}(A)$ is its Hochschild homology. The limit is computed over the category of free extensions of an algebra A consisting of short exact sequences of the form $0 \to I \to F \to A \to 0$.

- In our talk, we work in the framework of the **Tarski-Grothendieck** axioms and therefore, when we use the categories of free extensions Pres, Pres(G), we extend our universe so that we can consider these categories as small in this universe. Then the categories Fun(Pres(G), Mod(k)) are well defined in the extended universe.

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Higher (co)limits

- Given a (small) category $\mathcal{E}$, we will use the categories $\mathrm{Fun}(\mathcal{E}, \mathrm{Mod}(k))$ of functors (*representations* of $\mathcal{E}$ in k-modules). There are left-exact and right-exact functors of limit and colimit respectively:

$$\lim, \mathrm{colim}\colon \mathrm{Fun}(\mathcal{E}, \mathrm{Mod}(k)) \to \mathrm{Mod}(\mathrm{k}).$$

- The categories of representations have enough injective and projective objects (see [Wei], for example), and so we can define the left and right derived functors of limit and colimit, which we shall denote as $\mathrm{colim}_n, \lim^i\colon \mathrm{Fun}(\mathcal{E}, \mathrm{Mod}(\mathrm{k})) \to \mathrm{Mod}(\mathrm{k})$. These are called *higher colimits* and *higher limits*, respectively.

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- Similar to the Quillen's formulas were computed in the context of group homology in [EM08], [MP09], and finally in [SOI15], [IMS19], Roman Mikhailov, Sergey Ivanov obtained the higher (co)limit formulas for group homology (see [SOI15, Th. 5.1.] and [IMS19, Th. 4.3.]):

$$\lim^i \left(R_{ab} \otimes M\right)_G \simeq H_{2n-i}(G;M), i < n \tag{1}$$
$$\operatorname{colim}_n H_1(F;M) \simeq H_{n+1}(G;M),$$

  where higher limits and colimits are computed over the category Pres(G) of free presentations of G whose objects are short exact sequences $0 \to R \to F \to G \to 0$ with F being free, M is any G-module, R_{ab} is a relation module of a free presentation of G defined as $R/[R,R]$ with a G-action by conjugation.
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- Homology groups $H_*(G; S^n(\mathfrak{g}))$ have been studied in literature extensively in 90-s. For instance, they are shown to be periodic (for certain classes of groups and $*$) with bounded exponent. This implies:
- For a group G with no torsion up to rp, where $1 \le r < p$ and p is some prime, there are isomorphisms for $i = 0, 1$:

$$\lim{}^i(\Lambda^{rp} R_{ab})_G \otimes \mathbb{Z}_{(p)} \simeq H_{r(p+2)-i}(G; \mathbb{Z}/p\mathbb{Z}).$$

- For the case $r = p$ we have the following:

$$\lim(\Lambda^{p^2} R_{ab})_G \otimes \mathbb{Z}_{(p)} \simeq H_{p^2+2}(G; \mathbb{Z}/p\mathbb{Z}) \oplus H_{p^2+2p}(G; \mathbb{Z}/p\mathbb{Z}).$$

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- Given a group H the lower central series is the series of subgroups $\dots \subset \gamma_{n+1} \subset \gamma_n H \subset \dots H$ defined by induction $\gamma_{n+1} H := [\gamma_n H, H]$, $\gamma_1 R := R$.
- We studied higher limits of functors taking a free extension $0 \to R \to F \to G \to 0$ of group G and sending it to the homology of $F/\gamma_n R$, where $\gamma_n R$ is an n-term of the lower central series of R.
- $$\lim H_2(F/\gamma_5 R) \simeq \lim \frac{\gamma_5(R)}{[\gamma_5(R), F]} = \lim \frac{[R,R,R,R,R]}{[R,R,R,R,R,F]} \simeq H_4(G; \mathbb{Z}/5\mathbb{Z}),$$

  where for the first isomorphism we apply the Hopf formula.
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fr-language

- In [IM15] and [IMP19], the authors proposed yet simple but deep construction that describes many different homological functors Gr → Ab from the category of groups to the category of abelian groups.
- Consider $\mathbb{Z}F$: Pres → Ring, a functor of rings on the category of all free extensions of the form $0 \to R \to F \to G \to 0$, which takes a free extension and sends it to the group ring $\mathbb{Z}F$. There are two functorial ideals $\mathbf{f}$ and $\mathbf{r}$ in the (functorial) ring $\mathbb{Z}F$ that defined as follows:

$$\mathbf{f} = \ker(\mathbb{Z}F \to \mathbb{Z}),$$
$$\mathbf{r} = \ker(\mathbb{Z}F \to \mathbb{Z}G).$$

That is, $\mathbf{f}$ is simply the augmentation ideal of group F, and it is generated by expressions of the form $w - 1$ where $w \in F$, and $\mathbf{r}$ is a subideal of $\mathbf{f}$ generated by expressions of the form $r - 1$ where $\mathbf{r} \in R$.

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- Since $\mathbf{f}, \mathbf{r}$ are ideals of the functor of rings $\mathbb{Z}F$, we can form sums and intersections of monomials:

$$\mathbf{rf} \cap \mathbf{fr}, \mathbf{r} + \mathbf{ff}, \mathbf{r}^{k+1} + \mathbf{fr}^k\mathbf{f}, \dots$$

  These are functors on the category of free extensions Pres with values in abelian groups.

- All such combinations form a lattice $\mathrm{ML}(\mathbf{f}, \mathbf{r})$. Its elements we may call **fr**-*codes*.
- Given a functorial ideal $w(\mathbf{f}, \mathbf{r}) \in \mathrm{ML}(\mathbf{f}, \mathbf{r})$ and a group G, one can define (see [IM15, Def. 6.1.])

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- Given a functorial ideal $w(\mathbf{f}, \mathbf{r}) \in \mathrm{ML}(\mathbf{f}, \mathbf{r})$ and a group G, one can define (see [IM15, Def. 6.1.])

$${}^{i}[w(\mathbf{f}, \mathbf{r})](G) = \lim{}^{i}(w(\mathbf{f}, \mathbf{r})|_{\mathrm{Pres(G)}}).$$

fr-language

- It turns out that by exploiting some features of the category Pres(G) this construction can be made functorial in group G.
- The first such feature is that it has all *binary coproducts* (in particular, its classifying space is contractible). That feature is used extensively, since it ensures triviality of higher limits of constant functors.
- Secondly, this category is *strongly connected*, in that the hom-set $\hom(c, c')$ is not empty for any pair of objects c and c'.
- Hence, with each **fr** expression $w(\mathbf{f}, \mathbf{r})$ we associate a graded functor

$$^{i}[w(\mathbf{f}, \mathbf{r})] : \mathrm{Gr} \to \mathrm{Ab}.$$

- In the end we shall show how to extend the construction to spectra, in that we define $[w(\mathbf{f}, \mathbf{r})] : \mathrm{Gr} \to \mathrm{Spectra}$ such that $\pi_{-i}[w(\mathbf{f}, \mathbf{r})](G) \simeq {}^{i}[w(\mathbf{f}, \mathbf{r})](G)$.

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- Such constructions we shall refer to as *functorial languages*. In fact, in [Gol24] we suggest a formal definition of such constructions using the Quillen cohomology of ∞-categories.
- An **fr**-*code* of some functor $\mathcal{F}\colon \mathrm{Gr} \to \mathrm{Ab}$ is a functorial ideal $w(\mathbf{f},\mathbf{r}) \in \mathrm{ML}(\mathbf{f},\mathbf{r})$ and an isomorphism ${}^{i}[w(\mathbf{f},\mathbf{r})] \simeq \mathcal{F}$ for some integer i. From [IMP19] we borrow the table of functors and their codes:

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fr-language

fr-code	$\lim^1$	$\lim^2$	$\lim^3$
r	$\mathbf{g}$	0	0
rr	0	$\mathbf{g} \otimes \mathbf{g}$	0
rrr	0	0	$\mathbf{g} \otimes \mathbf{g} \otimes \mathbf{g}$
fr+rf	$\mathbf{g} \otimes_{\mathbb{Z}[G]} \mathbf{g}$	0	0
ffr+frf+rff	$\mathbf{g} \otimes_{\mathbb{Z}[G]} \mathbf{g} \otimes_{\mathbb{Z}[G]} \mathbf{g}$	0	0
r+ff	G_{ab}	0	0
r+fff	$\mathbf{g}/\mathbf{g}^3$	0	0
rf+ffr	$\mathbf{g}^2 \otimes_G \mathbf{g}$	0	0
fr+rf+fff	$G_{ab} \otimes G_{ab}$	0	0
rr+fff	$\mathsf{Tor}(G_{ab}, G_{ab})$	$G_{ab} \otimes G_{ab}$	0
rr+frf	$H_3(G)$	$\mathbf{g} \otimes_G \mathbf{g}$	0
rrf+frr	$H_4(G)$	$(\mathbf{g} \otimes \mathbf{g} \otimes \mathbf{g})_G$	0
rfr+frf	$\mathsf{coker}\{H_3(G) \otimes G_{ab} \to H_2\left(G, \mathbf{g} \otimes_{\mathbb{Z}[G]} \mathbf{g}\right)\}$	$\mathsf{im}\{H_2(G) \otimes G_{ab} \to H_1\left(G, \mathbf{g} \otimes_{\mathbb{Z}[G]} \mathbf{g}\right)\}" \oplus "\mathbf{g}^2 \otimes_{\mathbb{Z}[G]} \mathbf{g}$	0
rff+ffr	$\frac{\mathbf{g}^2 \otimes_{\mathbb{Z}[G]} \mathbf{g}^2}{\sim}$	0	0
rr+frf+rff	$H_2(G, G_{ab})$	$G_{ab} \otimes G_{ab}$	0
rr+ffr	0	$G_{ab} \otimes \mathbf{g}$	0
rfr+frr	0	$(\mathbf{g} \otimes_{\mathbb{Z}[G]} \mathbf{g}) \otimes \mathbf{g}$	0

rr+ffr+rff	$\frac{\mathbf{g}^2\otimes_{\mathbb{Z}[G]}\mathbf{g}^2}{\approx}$ " $\oplus$ " $\mathsf{Tor}(G_{ab}, G_{ab})$	$G_{ab} \otimes G_{ab}$	0
rr+ffr+frf+rff	$\mathbf{g} \otimes_{\mathbb{Z}[G]} \mathbf{g} \otimes_{\mathbb{Z}[G]} \mathbf{g}$ " $\oplus$ " $\mathsf{Tor}(G_{ab}, G_{ab})$	$G_{ab} \otimes G_{ab}$	0
rff+frr	$\mathsf{Tor}(H_2(G), G_{ab})$	$H_2(G) \otimes G_{ab}$ " $\oplus$ " $\mathsf{Tor}(G_{ab}, G_{ab})$ " $\oplus$ " $\ker\{\mathbf{g} \otimes G_{ab} \twoheadrightarrow G_{ab} \otimes G_{ab}\}$	0
rrf+rfr+frr	$H_2(G, H_2(G))$ " $\oplus$ " $\mathsf{Tor}(G_{ab}, H_2(G))$	$H_2(G) \otimes G_{ab}$ " $\oplus$ " $H_2(G, G_{ab})$ " $\oplus$ " $\ker\{\mathbf{g} \otimes \mathbf{g} \to G_{ab} \otimes G_{ab}\}$	0

- Here by $A" \oplus " B$ an extension of B by A is denoted.
- One may notice that the **fr**-language paramatrizes a "neigbourhood" of functors Gr → Ab such that it contains new functors that relate in a nontrivial way the old ones. For instance we have a short exact sequence:
- For functors admitting **fr**-codes $\mathcal{F}, \mathcal{H}$ one may find nontrivial natural transformations which are immediately verified within the functorial language. Indeed, assume $\mathcal{F} \simeq {}^i[w(\mathbf{f},\mathbf{r})]$, $\mathcal{H} \simeq {}^i[w'(\mathbf{f},\mathbf{r})]$ and if we have an inclusion of ideals $w(\mathbf{f},\mathbf{r}) \subset w'(\mathbf{f},\mathbf{r})$ then it yields a natural transformation ${}^i[w(\mathbf{f},\mathbf{r})] \to {}^i[w'(\mathbf{f},\mathbf{r})]$

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rrf+rfr+frr	$H_2(G, H_2(G))$ " $\oplus$ " $\mathsf{Tor}(G_{ab}, H_2(G))$	$H_2(G) \otimes G_{ab}$ " $\oplus$ " $H_2(G, G_{ab})$ " $\oplus$ " $\mathsf{ker}\{\mathbf{g} \otimes \mathbf{g} \to G_{ab} \otimes G_{ab}\}$	0

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Why functorial languages?

- These higher limit formulas are quite interesting since they show that different functors for which we need certain resolutions and additional constructions (compare for instance $H_i(G)$ and $\mathfrak{g}/\mathfrak{g}^3$) to be defined admit descriptions on the same footing - **they are cohomology groups of categories of free extensions only that they have different codes!**
- Simple codes describe very interesting functors, for instance:

$$L_{n-i} \overset{n}{\otimes} (G_{ab}) \simeq {}^{i}[\mathbf{r^n} + \mathbf{f^{n+1}}](G)$$

where $L_i \overset{n}{\otimes}$ is the derived functor of a nonadditive functor of the tensor power (these are also known as derived functors by Dold-Puppe), (see [HAZ97])
- A functorial language is a source of new functors (so new invariants) that turn out to be related to the known ones, for example we have such extensions: $0 \to H_2(G; H_2(G)) \to {}^{1}[\mathbf{rrf} + \mathbf{rfr} + \mathbf{frr}](G) \to \mathrm{Tor}(G_{ab}, H_2(G)) \to 0$

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Combinatorics

- My friend Vasily Ionin and others obtained some interesting properties of the **fr**-language indicating that this is a somewhat fundamental, free construction.
- Consider an inclusion of all polynomial **fr**-ideals (all finite combinations of the form $\mathbf{rrr} + \mathbf{ff}$, $\mathbf{rfr} + \mathbf{frf} + \mathbf{ffff} \ldots$) into the latice Ideals($\mathbb{Z}F$) of all functorial ideals of $\mathbb{Z}F$.
- **This inclusion is split!** The retraction takes a functorial ideal $\mathfrak{a}$ and returns a maximal polynomial **fr**-code contained in $\mathfrak{a}$. It exists since we may generate an ideal spanned by all polynomial **fr**-codes contained in it and by the next observation it is a polynomial **fr**-code.
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fr-language. Mysterious gaps

- We see that even with such a scarce, simple construction as the **fr**-language we may describe many homological functors.
- Nonetheless, there are some unexpected gaps consisting of functors which do not seemingly admit an **fr**-code. For instance, the functor of the second homology only fits into a short exact sequence:

$$0 \to H_2(G) \to {}^1[\mathbf{fr} + \mathbf{rf}](G) \to {}^2[\mathbf{fr} \cap \mathbf{rf}](G) \to 0.$$

- Or, in terms of spectra we have fiber sequence of functors $\mathrm{Gr} \to \mathrm{Spectra}$:

$$[\mathbf{rf} + \mathbf{fr}]_G \to \Sigma[\mathbf{r} \cap \mathbf{ff}]_G \to \mathrm{H}(H_2(G)),$$

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- It is has not yet been proved that $H_2(G)$ does not admit an **fr**-code.
- This hints us that such functors (obtained by extensions of this form) should be regarded on the same footing as those admitting codes.
- Let ξ again be a subcategory of the category of groups and a functorial language $\mathfrak{F}$. We want to study categories of functors of all such possible extensions which we call *functorial surfaces* $\mathrm{surf}(\xi, \mathfrak{F})$ spanned by a functorial language $\mathfrak{F}$.
- We expect there to exist methods to qualitatively study these categories. The universal method for this is the Algebraic K-theory. For formal reasons this hints us to extend this whole construction from abelian to stable ∞-categories.
- For in the end we shall discuss how we can extend this whole theory to spectra.

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Functorial languages as functions on groups

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- Why restricting languages to categories ξ? A language acquires new relations between functors, for instance for perfect groups we have

$$ {}^{i}[\mathbf{r} + \mathbf{ff}](-) \simeq 0 $$

  and so on. We suggest an idea that the functorial languages should reflect properties of categories of groups and their constituents!
- This is somewhat resembling to a widely known hypothesis in Linguistic relativity known as the *Sapir-Whorf Hypothesis*. It states that the structure of a language determines a native speaker's perception and categorization of experience.
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fr_∞-language

- Let k be a ring and define $\mathbf{f}$ as a functorial ideal of $k[-]$ functor of rings: $\mathbf{f} := ker(kF \to k)$ and given a free extension $0 \to R \to F \to G \to 0$ we consider the lower central series of the group of relations $\gamma_n R$. Using this we define $\mathbf{r}_n = ker(kF \to k(F/\gamma_n R))$, so we have a chain $\mathbf{r}_{n+1} \subset \mathbf{r}_n \subset \dots \mathbf{f}$ of functorial ideals of $k[-]$ which spans a lattice $\mathrm{ML}(\mathbf{f}, \mathbf{r}, \dots, \mathbf{r}_n, \dots)$ of all possible intersections, sums of monomials constructed with these ideals.
- For $c \geq 1$ we have natural isomorphisms ${}^1[\mathbf{r}_m](G) \simeq \mathrm{inv} \bigcap_{i \geq 1} \Delta(F/\gamma_c R)^i$ and $\lim^i \Delta(F/\gamma_c R) \simeq {}^{i+1}[\mathbf{r}_m](G)$
- Assume G is residually torsion-free nilpotent, then ${}^1[\mathbf{r}_2]_G \simeq 0$.
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- Assume G is 3-torsionless. Then, there is a short exact sequence:

$$H_4(G;\mathbb{Z}/3\mathbb{Z}) \to {}^1[\mathbf{r}_3\mathbf{f} + \mathbf{f}\mathbf{r}_3]_G \to \lim(\Delta(F/\gamma_3 R)^\omega) \equiv \lim(\cap_i \Delta(F/\gamma_3 R)^i)$$

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FR-language for Lie algebras

- Here we consider categories of free extensions of Lie algebras A (over $\mathbb{Z}$) of the form $0 \to R \to F \to A \to 0$ and we have simple functors $R \subset F : \mathrm{Pres}(A) \to \mathrm{Lie}$. We may as well form the lattice of FR-code (only that instead of products we form commutators $[F, R, R, F]$...)s.
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Lift to spectra

One may prove the following simple:

- (Lemma) Let $\mathcal{F} : C \to \text{Ab}$ be a functor on a strongly connected category C with binary coproducts. For any object $c \in C$ there is a natural in $\mathcal{F}$ lift to $\hat{\mathcal{F}} : C \to \text{Spectra}$ such that

$$\lim^i(\mathcal{F}) \simeq \pi_{-i}\hat{\mathcal{F}}(c), \forall i$$

This is independent up to equivalence of choice of c in that for any $c \to c'$ we have equivalence of spectra $\mathcal{F}(c) \to \mathcal{F}(c')$. Moreover, this lift satisfies the condition that for any short exact sequence of functors $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ the sequence $\hat{\mathcal{F}}_1(c) \to \hat{\mathcal{F}}_2(c) \to \hat{\mathcal{F}}_3(c)$ is a fiber sequence in the stable ∞-category $\text{Fun}(C, \text{Spectra})$.

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Proof

- We recall the standard complex from [IMP19]. For a category with binary coproducts C there is a functor $\mathbf{B} : C \to C^{\Delta}$: $\mathbf{B}(c)^n := \coprod_{i=0}^{n} c$, boundaries and degeneracies can be found in [IMP19, Def. 2.4.].
- Given $\mathcal{F} : C \to \mathrm{Ab}$ we have $C \to C^{\Delta} \to \mathrm{Ab}^{\Delta}$ which takes $c \in C$ and sends it to $\mathcal{F}\mathbf{B}(c)$. Further we make a use of the *Moore complex functor* $Q : \mathrm{Ab}^{\Delta} \to \mathrm{Ch}(\mathbb{Z})$. This functor is described in [IMP19, Def. 2.6.]. This functor is known to be exact in that it takes a short exact sequence of cosimplicial abelian groups and sends it to a short exact sequence of chain complexes.
- The category of chain complexes we consider to be equipped with a canonical projective model structure where fibrations are precisely the surjective chain morphisms (see [HOV]), so fixing an object $c \in C$ we have a functor

$$\mathcal{F} \in \mathrm{Fun}(C, \mathrm{Ab}) \mapsto Q\mathcal{F}\mathrm{B}(c) \in \mathrm{Ch}(\mathbb{Z})$$

  which sends short exact sequences to homotopy fiber sequences in $\mathrm{Ch}(\mathbb{Z})_{proj}$. Let $\mathrm{Ch}(\mathbb{Z})$ further denote an ∞-category of chain complexes which is a localisation of the category of chain complexes by quasi-isomoprhisms.

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- We recall the explicit construction of the Eilenberg-Maclane functor $\mathcal{H} : \mathrm{Ch}(\mathbb{Z}) \to \mathrm{Spectra}$ (see [GRA19, pp.19]). Given a chain complex A the spectrum $\mathcal{H}A$ is the following spectrum:

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- This functor is known to preserve fiber sequences. Moreover, the homotopy groups satisfy: $\pi_i(\mathcal{H}A) \simeq \mathrm{colim}_k H_{i+k}(A)$ (see, for instance, [GRA19, pp. 20]). One may notice that when a chain complex A is concentrated in negative degrees the homotopy groups $\pi_{-i}A$ are isomorphic with $H^i(A)$ for any i.
- In particular for $A \equiv Q\mathcal{F}\mathbf{B}(c)$ (which is concentrated in negative degrees) a $(-i)$-th homotopy group of $\mathcal{H}Q\mathcal{F}\mathbf{B}(c)$ is isomorphic with $H^i(Q\mathcal{F}\mathbf{B}(c))$ and since the Moore chain complex is chain homotopy equivalent to the alternate sum complex these are isomorphic with higher limits $\lim^i(\mathcal{F})$ [IMP19, Cor. 2.9.].

Proof

- This functor is known to preserve fiber sequences. Moreover, the homotopy groups satisfy: $\pi_i(\mathcal{H}A) \simeq \operatorname{colim}_k H_{i+k}(A)$ (see, for instance, [GRA19, pp. 20]). One may notice that when a chain complex A is concentrated in negative degrees the homotopy groups $\pi_{-i}A$ are isomorphic with $H^i(A)$ for any i.
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Lift to spectra

- Now consider the case when $\mathcal{F}$ is a functor on a category $\mathrm{Pres}(G)$. Let $G \mapsto (F[G] \to G) \equiv c(G) \in \mathrm{Pres}(G)$ be a canonical free presentation of a group G. This is a functor $c : \mathrm{Gr} \to \mathrm{Pres}$. The spectrum $\mathcal{HQFB}(c(G)) \equiv [w(\mathbf{f},\mathbf{r})](G)$ now has the form:

(4.2.a)

$$\mathrm{DK}(\cdots \to 0 \to Z_0),$$

(4.2.b)

$$\mathrm{DK}(\cdots \to 0 \to \mathcal{F}(c) \simeq \mathcal{HQFB}(c)_0 \to Z^1),$$

(4.2.c)

$$\mathrm{DK}(0 \to \mathcal{F}(c) \underset{[\delta^0]}{\to} \mathrm{Coker}(\mathcal{F}(c) \underset{\delta^1}{\to} \mathcal{F}(c \coprod c)) \to Z^2),$$

(4.2.d)

$$\mathrm{DK}(0 \to \mathcal{F}(c) \underset{[\delta^0]}{\to} \mathrm{Coker}(\mathcal{F}(c) \underset{\delta^1}{\to} \mathcal{F}(c \coprod c)) \to \mathrm{Coker}(\mathcal{F}(c \coprod c) \oplus \mathcal{F}(c \coprod c) \underset{-\delta^1+\delta^2}{\to} \mathcal{F}(\coprod_{i=1}^{3} c)) \underset{[\delta^0]}{\to} Z^3),$$

(4.2.e)

$$\cdots$$

Functorial surfaces

- Finally we have established the desired lift $\pi_{-i}[w(\mathbf{f},\mathbf{r})](G) \simeq {}^{i}[w(\mathbf{f},\mathbf{r})](G)$
- Now we are ready to define the functorial surfaces using the construction of the following:
- (Lemma) Let C be a stable ∞-category with a full subcategory $S \subset C$ containing a zero object. Then we define a full subcategory $\bar{S} := \cup_n < S >_n$ where $< S >_{n+1}$ consists of such Z that are extensions $X \to Z \to Y$ of some objects X, Y from $< S >_n$ and $< S >_0 := \cup_i S[i]$ is the union of all shifts of S. $\bar{S}$ is a stable ∞-category which stable under forming extensions in C.

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Functorial surfaces

- Now given a category/variety of groups $\xi \subset \mathrm{Gr}$, a functorial language $\mathfrak{F}$ (e.g. $\mathbf{fr}$, $\mathbf{fr}_\infty(k)$ and so on) we consider a full subcategory of $\mathrm{Fun}(\xi, \mathrm{Spectra})$ spanned by functors of the form $[w(\mathbf{f}, \mathbf{r})](-)$ [Gol24, Definition 4.4.]. Then by setting $S \equiv [w(\mathbf{f}, \mathbf{r})](-)$ and by applying the above construction we construct the stable ∞-category $\mathrm{surf}(\xi, \mathfrak{F})$.
- Its objects are such functors $\xi \to \mathrm{Spectra}$ which are extensions of extensions of extensions... of $\Sigma^m[w(\mathbf{f}, \mathbf{r})](-)$ functors.

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$$
\begin{array}{ccccccc}
\Sigma^m[w(\mathbf{f},\mathbf{r})] & \longrightarrow & X_1 & \longrightarrow & \Sigma^k[w'(\mathbf{f},\mathbf{r})] & & \\
 & & \downarrow & & & & \\
 & & X_2 & \longrightarrow & X_3 & \longrightarrow & Z_2 \\
 & & \downarrow & & & & \\
\Sigma^i[\hat{w}(\mathbf{f},\mathbf{r})] & \longrightarrow & Z_1 & \longrightarrow & \Sigma^j[\hat{w}'(\mathbf{f},\mathbf{r})] & &
\end{array}
$$

- We call this a functorial surface over ξ spanned by the functorial language $\mathfrak{F}$.

Functorial surfaces

- Let $\mathcal{K}$ denote the functor of the Algebraic K-theory (see [Bar16]) and the K-spectrum of a functorial surface on ξ spanned by a functorial language $\mathfrak{F}$ we call a *flux-spectrum* of $\mathfrak{F}$ on ξ and denote as $\mathrm{flux}(\xi, \mathfrak{F}) \equiv K\mathrm{surf}(\xi, \mathfrak{F}))$.
- (Gol24, Proposition 4.8.) Let ${}_{pol}\mathbf{fr}, \mathbf{fr}_\infty(k)$ be a functorial language defined only by using the polynomial $\mathbf{fr}_\infty$ codes. Then the functorial surfaces on any $\xi \subset \mathrm{Gr}$ spanned by ${}_{pol}\mathbf{fr}, \mathbf{fr}_\infty(k)$ are equivalent and so as the flux spectra.
- The homotopy groups $\pi_i \mathrm{flux}(\xi, \mathfrak{F}) \equiv \mathrm{flux}_i(\xi, \mathfrak{F})$ - these are interesting invariants of ξ. We see that functorial languages may be considered as some coefficients for the exotic homology theory defined on categories of groups/Lie algebras (or more generally on functors $\xi \to \mathrm{Gr}, \mathrm{Lie}$)...

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Summarise

- Let $C \subset G$ be a subcategory (or more generally $C \to G$) of a category G with a functorial language.
- Functorial languages is a way to parametrize certain families of functors from C to Spectra.
- Such families of functors form functorial surfaces - stable ∞-subcategories of $\mathrm{Fun}(C, \mathrm{Spectra})$
- Algebraic K-theory spectra of a functorial surface if the flux-spectrum.
- Thus, a functorial language may be thought of as a "function" which we may "integrate" (applying the algebraic K-theory) over the category C and the result of such "integration" is the flux-spectrum.
- Properties of C and its objects + combinatorial properties of a functorial language yield implications on flux-spectra.

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Thank you for your attention

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