STABLE HOMOTOPY VERSION OF SEIBERG-WITTEN INVARIANT

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Abstract. In this paper we construct stable homotopy version of Seiberg-Witten invariant. The construction is given for families of any closed oriented 4-manifolds parameterized by compact spaces. As an application we show a divisibility of Seiberg-Witten invariant for non-simple type 4-manifolds.

§1. Introduction.

Let X be a 4-dimensional oriented closed manifold with a spin^c-structure. For simplicity we assume that the first Betti number b_1 of X is 0. The Seiberg-Witten invariant is defined as the fundamental homology class of the moduli space of the monopoles associated with the spin^c-structure. Strictly speaking the invariant is well defined when (1) the rank b_+ of a maximal positive definite subspace $H^+(X)$ of $H^2(X, \mathbf{R})$ is odd and larger than 1, and (2) one of the two orientations of $H^+(X)$ is fixed. We denote b_+ as $2p+1$. The formal dimension of the moduli space is even when $b_1 = 0$ and b_+ is odd. If we write $2d$ for the formal dimension, then the Seiberg-Witten invariant is valued in $H_{2d}(\mathbf{CP}^\infty, \mathbf{Z})$, which is isomorphic to $\mathbf{Z}$ when $d \geq 0$. The invariant is zero when $d < 0$. The following problem is due to E. Witten [7].

Problem. *Is the Seiberg-Witten invariant zero when $d > 0$?*

A 4-manifold satisfying this property is called a *simple type* manifold. In this paper we show that when $d > 0$, there is a restriction on the possible values of the Seiberg-Witten invariant. To state our result, we need to introduce the coefficients of the following Taylor expansion:

$$\left(\frac{\log(1-x)}{x}\right)^p = \sum_{l=0}^{\infty} a_{p,l} x^l.$$

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Theorem(1.1). *When $d > 0$, the Seiberg-Witten invariant is divisible by the denominators of $a_{p,1}, a_{p,2}, \cdots$, and $a_{p,d}$.*

Note that an integer is zero if and only if it is divisible by every nonzero integers. So integrality theorems like Theorem(1.1) could be regarded as a small step for the problem.

On the other hand, when $b_+ = 1$ the Seiberg-Witten invariant depends on chambers. It is known, from the wall crossing formula, that there are examples which have non-zero Seiberg-Witten invariant for some positive d. The above divisibility still holds in this case for any chamber.

D. Ruberman pointed out to the author that the integrality of L-genus of the moduli space of monopoles implies a divisibility of the Seiberg-Witten invariant.

What we actually do is to give a refinement of the Seiberg-Witten invariant

as a certain stable homotopy class, which is constructed by using a finite dimensional approximation of the map defined by the monopole equation.

When X is spin and the spin^c-structure is the one derived from a spin structure of X, then the construction of the stable homotopy class is given in [5]. One purpose of the present paper is to show that the construction is extended to any spin^c-structure.

The second purpose of this paper is to give a definition of the refinement for a general setting. We consider the case when b_1 is not necessarily zero. We also extend the construction to families of oriented closed 4-manifolds parameterized by compact spaces. The stable homotopy class is well defined for any such family. For example, it is defined for a family of homotopy 4-spheres.

The well-definedness of the stable homotopy class could be regarded as a topological version of an argument showing that renormalization groups preserve expectation values.

We obtain the original Seiberg-Witten invariant when we detect the stable homotopy class by using the ordinary cohomology theory. If we use K-theory instead of the ordinary cohomology theory, then we obtain a K-theory version of Seiberg-Witten invariant. The K-theory version is related to the original invariant through a map defined by Chern character with Todd class as correction term. Abstractly, the Chern character gives rise to two lattice structures on a certain single vector space over $\mathbf{Q}$, and the two invariants, each of which sits on the corresponding lattice are identified with each other. Then measuring the difference of the two lattices implies a property of an integrality of the identified invariants. The difference is described by the Todd class which appears when the Thom classes in the two cohomology theories are compared. The formal power series mentioned above is essentially equal to the Todd class of a vector bundle over $\mathbf{CP}^\infty$. This kind of argument is standard in applications of K-theory ([1],[2]).

In [3] S. Bauer independently defined stable homotopy version of the Seiberg-Witten invariant and gave applications to topology of algebraic surfaces.

In Section 2, We recall the monopole equation. In Section 3, we give the definition of the refinement of the Seiberg-Witten invariant in Section 3 by using a finite dimensional approximation of the monopole equation. In Section 4, we use cohomology theories to explicitly detect the refined invariant and see that it is in fact a refinement. In Section 5, we compare the ordinary cohomology with the K-theory. We use the assumptions, $b_1(X) = 0$ and $b_+(X) > 1$, only in Sections 4 and 5.

§2. Monopole equation.

In this section we formulate the monopole equation for a spin^c-structure of X in a line following [5]. Let $\mathbf{H}$ be the quaternion numbers, Sp_1 the group of the quaternions with norm 1 and S^1 the intersection of Sp_1 with $\mathbf{C}$ in $\mathbf{H}$.

We define five $Spin^c_4$-modules ${}_-\mathbf{H}_+$, ${}_+\mathbf{H}$, ${}_-\mathbf{H}$, ${}_+\mathbf{H}_+$ and $\tilde{\mathbf{C}}$ as follows. As real vector spaces, the first four modules are just four copies of $\mathbf{H}$. The actions of $(q_-, q_+, z) \in Spin^c_4 = (Sp_1 \times Sp_1 \times S^1)/\{(1,1,1,)(-1,-1,-1)\}$ on $a \in {}_-\mathbf{H}_+$, $\phi \in {}_+\mathbf{H}$, $\psi \in {}_-\mathbf{H}$ and $\omega \in {}_+\mathbf{H}_+$ are defined by $q_- a {q_+}^{-1}$, $q_+ \phi z^{-1}$, $q_- \psi z^{-1}$ and $q_+ \omega {q_+}^{-1}$ respectively. The last one is a complex one-dimensional representation $\tilde{\mathbf{C}}$ defined by the multiplication of z^2.

Let X be a closed 4-manifold. For a principal $Spin^c_4$-bundle P on X, we have five associated vector bundles T, S^+, S^-, Λ and L from the $Spin^c_4$-modules ${}_-\mathbf{H}_+$, ${}_+\mathbf{H}$, ${}_-\mathbf{H}$, ${}_+\mathbf{H}_+$ and $\tilde{\mathbf{C}}$.

Suppose we are given a pair of a principal $Spin^c_4$-bundle P and an isomorphism $TX \cong T$. Then, since T has a natural orientation and a natural Riemannian metric, the pair induces an orientation and a Riemannian metric of X. We call the homotopy class of $(P, TX \cong T)$ a spin^c-structure.

The $Spin^c_4$-equivariant map ${}_-\mathbf{H}_+ \times {}_+\mathbf{H} \to {}_-\mathbf{H}$ defined by $(a, \phi) \mapsto a\phi$ induces the Clifford multiplication $C\colon T \otimes S^+ \to S^-$. Similarly the $Spin^c_4$-equivariant map ${}_-\mathbf{H}_+ \times {}_-\mathbf{H}_+ \to {}_+\mathbf{H}_+$ defined by $(a, b) \mapsto \bar{a}b$ induces a twisted Clifford multiplication $\bar{C}\colon T \otimes T \to \Lambda$.

Since ${}_-\mathbf{H}_+ \oplus \tilde{\mathbf{C}}$ is a faithful representation of the Lie algebra of $Spin^c_4$, a pair of metric connections on T and on L induces a principal connection on P. We use the Riemannian connection on $T = TX$ and a fixed connection A_0 on L. Then we have the covariant derivatives ∇_1 on $\Gamma(S^+)$ and ∇_2 on $\Gamma(T)$. Let D_1 and D_2 be the twisted Dirac operators

$$D_1 = C\nabla_1\colon \Gamma(S^+) \to \Gamma(S^-) \quad \text{and} \quad D_2 = \bar{C}\nabla_2\colon \Gamma(T) \to \Gamma(\Lambda).$$

Let D be the direct sum of D_1 and D_2:

$$D = D_1 \oplus D_2\colon \Gamma(S^+ \oplus T) \to \Gamma(S^- \oplus \Lambda).$$

Let Q be a quadratic map from $S^+ \oplus T$ to $S^- \oplus \Lambda$ induced from the $Spin^c_4$-equivariant map

$${}_+\mathbf{H} \times {}_-\mathbf{H}_+ \to {}_-\mathbf{H} \times {}_+\mathbf{H}_+, \qquad (\phi, a) \mapsto (a\phi i, \phi i \bar{\phi}).$$

We shall consider the nonlinear map

$$D + Q\colon V \to W,$$

where V is the L^2_4-completion of $\Gamma(S^+ \oplus T)$ and W is the L^2_3-completion of $\Gamma(S^- \oplus \tilde{\Lambda})$.

Remark. We can identify the imaginary part of $\Gamma(\Lambda)$ with the self-dual 2-forms ([5] Section 2 Remark(1)). Let $F^+_{A_0}$ be the self-dual part of the curvature of A_0. Then an element v of V is called *monopole* when $(D+Q)v + F^+_{A_0} = 0$.

Next, we consider the symmetry of the map $D+Q$ under a group action of $\mathrm{Harm}(X, S^1)$ which is defined as the kernel of the composition of the exterior derivative $d\colon \mathrm{Map}(X, S^1) \to \Gamma(T)$ and $D_2\colon \Gamma(T) \to \Gamma(\Lambda)$. Here we identify T with its dual T^* by using the Riemannian metric and we regard $\mathrm{Map}(X, S^1)$ as a group by using the multiplication of S^1. Then $\mathrm{Harm}(X, S^1)$ consists of the harmonic maps from X to S^1. (See [5] Section 2 Remark(2).) Note that the center of $Spin^c_4$ is $S^1 = \{1\} \times \{1\} \times S^1$ and it naturally acts on each fiber of S^+ and S^-. We can identify the connected component of $\mathrm{Harm}(X, S^1)$ containing 1 with the center of $Spin^c_4$. Then $\mathrm{Harm}(X, S^1)$ acts from the right on S^+ and S^- by the right multiplication. We want to define an action of $\mathrm{Harm}(X, S^1)$ on T and Λ so that $D+Q$ is $\mathrm{Harm}(X, S^1)$-equivariant. When we locally write e^{if} for an element of $\mathrm{Harm}(X, S^1)$, we have:

$$D_1(\phi e^{if}) = C\nabla_1(\phi e^{if}) = C((\nabla_1 \phi)e^{if} + df \otimes \phi e^{if} i) = (D_1\phi)e^{if} + df\,\phi e^{if} i,$$

$$D_2(a - df) = D_2 a - D_2 df = D_2 a,$$

$$Q(\phi e^{if}, a - df) = (a - df)\phi e^{if} i \oplus \phi e^{if} i e^{-if} \bar{\phi} = a\phi e^{if} i - df\,\phi e^{if} i \oplus \phi i \bar{\phi}$$

and hence

$$(D+Q)(\phi e^{if}, a - df) = (D_1\phi + a\phi i)e^{if} \oplus D_2 a + \phi i \bar{\phi}$$

Now we define the action of $\mathrm{Harm}(X, S^1)$ on Λ as trivial action. The action on T is defined by looking at the above relation. Note that we have the exact sequence

$$1 \longrightarrow S^1 \longrightarrow \mathrm{Harm}(X, S^1) \longrightarrow H^1(X, \mathbf{Z}) \to 0,$$

and df is the image of e^{if} written in terms of harmonic 1-form. The action on T is defined through the additive action of $H^1(X, \mathbf{Z})$ if $H^1(X, \mathbf{Z})$ is identified with the harmonic 1-forms with integral periods.

On the other hand, it is easy to check that D_1, D_2, and Q commute with the S^1-actions.

We decompose V and W into L^2-direct sums:

$$V = H^1(X, \mathbf{R}) \oplus \bar{V}, \qquad W = H^0(X, \mathbf{R}) \oplus \bar{W},$$

where $H^i(X, \mathbf{R})$ is the space of harmonic i-forms. Then the image of $D+Q$ is contained in $\bar{W}$ ([5]). We regard $D+Q$ as a family of maps from $\bar{V}$ to $\bar{W}$ parameterized by $H^1(X, \mathbf{R})$. The action of $\mathrm{Harm}(X, S^1)$ on V preserves the direct sum. We decomposed this action into two parts corresponding to the direct summands. First, fix a splitting of the above exact sequence and identify $\mathrm{Harm}(X, S^1)$ with the product $H^1(X, \mathbf{Z}) \times S^1$. The action of $H^1(X, \mathbf{Z})$ on V preserves the orthogonal decomposition and the action is free on $H^1(X, \mathbf{R})$ since it is just given by translation.

Now we have a family of S^1-equivariant maps from $\bar{V}$ to $\bar{W}$ parameterized by the torus $H^1(X,\mathbf{R})/H^1(X,\mathbf{Z})$. For this parameter space we use the notation T_0. The actions of S^1 on $\bar{V}$ and $\bar{W}$ are just the restrictions of its actions on V and W respectively.

We shall construct a finite dimensional approximation of this family. The Seiberg-Witten invariant is defined by using the approximation.

Remark. We can identify $\Gamma(T)$ with the space of connections on L by the correspondence $a \mapsto A_0 + ai$. The moduli space of S^1-connections on L which has the same curvature with A_0 is parameterized by $T_0 = H^1(X,\mathbf{R})/H^1(X,\mathbf{Z})$. The space of all the gauge-equivalence classes of connections on L is a product of T_0 and an infinite dimensional vector space. The $\Gamma(T)$-part of the above decomposition of V is identified with this product structure if we consider only the kernel of D_2 in order to take a slice for the gauge group action.

§3. Finite dimensional approximation.

We have a family of maps $D+Q : \bar{V} \to \bar{W}$. Here everything is parameterized continuously by the finite dimensional torus T_0, but we suppress the notation for its parameter.

Since the zero set of $D+Q$ is compact ([6]), we can take a large R so that $D+Q$ does not have zero on the sphere of radius R in $\bar{V}$. Since the parameter space is compact, we can take R uniformly.

We construct a finite dimensional approximation of this family. It is a non-linear analogue of the construction of index for a family of Fredholm operators [2].

step 1

Let $\bar{W}_\lambda$ be the subspace of $\bar{W}$ spanned by eigenspaces of DD^* with eigenvalues less than or equal to λ. Similarly we define $\bar{V}_\lambda$ by using the eigen-decomposition with respect to D^*D. Let $p_\lambda : \bar{W} \to \bar{W}_\lambda$ be the orthogonal projection.

In [5], it is shown that for large enough λ, $D + p_\lambda Q$ is a good finite dimensional approximation of $D+Q$ in that it does not vanish on the finite dimensional sphere in $\bar{V}_\lambda$ of radius R centered in 0 while the image of this sphere is contained in the finite dimensional vector space $\bar{W}_\lambda$. We denote this sphere as $S_R(\bar{V}_\lambda)$.

The proof of [5] implies that we can take λ uniformly again. However the orthogonal projection $p_\lambda : \bar{W} \to \bar{W}_\lambda$ does not vary continuously with respect to the parameter. Actually the space $\bar{W}_\lambda$ itself may jump. It is necessary to modify the projection so that we have a continuous family of maps.

Let $\beta : (-1,0) \to [0,\infty)$ be a compact-supported smooth non-negative cut-off function whose integral over $(-1,0)$ is 1. For each $\lambda > 1$, let $\tilde{p}_\lambda : \bar{W} \to \bar{W}_\lambda$ be the smoothing of the projection defined by

$$\int_{-1}^{0} \beta(\lambda+t) p_{\lambda+t} dt$$

The composition of $\tilde{p}_\lambda$ with the inclusion $\bar{W}_\lambda \to \bar{W}$ varies continuously.

Then the proof in [5] can immediately be extended to obtain:

Lemma(3.1). *For large λ, $D + \tilde{p}_\lambda Q$ does not vanish on the sphere $S_R(\bar{V}_\lambda)$. Here λ can be taken uniformly with respect to the family.*

step 2

Since $\bar{W}_\lambda$ does not vary continuously with respect to the parameter, we want to replace $\bar{W}_\lambda$ with a vector bundle $\bar{W}^{\mathrm{f}}$. We follow the procedure to define index for a family of Fredholm operators ([2]). We modify the argument slightly in order to consider a nonlinear term in Step 3.

Lemma(3.2). *There is an S^1-equivariant vector bundle $\bar{W}^{\mathrm{f}}$ over the parameter space T_0 and an S^1-equivariant bundle homomorphism $\chi : \bar{W}^{\mathrm{f}} \to \bar{W}$ which have the following properties.*

(1) For each parameter, the image of χ contains $\bar{W}_\lambda$.

(2) There is an S^1-equivariant homomorphism $s : \bar{W} \to \bar{W}^{\mathrm{f}}$ for each parameter, and the restriction of the composition χs on $\bar{W}_\lambda$ is the identity.

(3) There is an S^1-equivariant isomorphism from $\bar{W}^{\mathrm{f}}$ to the product bundle $T_0 \times (\mathbf{C}^b \oplus \mathbf{R}^c)$ for some b and c.

Note that (1) is an immediate consequence of (2).

Proof. Take an open covering U_i of the parameter space T_0 so that there is $\lambda_i > \lambda$ which is not equal to the eigenvalues of DD^* or D^*D for every parameter in U_i. Then $\bar{W}_{\lambda_i}$ varies continuously for parameters in U_i. (This continuity is shown by using a min-max principle to characterize eigenspaces [4].) When we replace the open covering with a finer one, if necessary, we can assume that the family $\bar{W}_{\lambda_i}$ makes a trivial S^1-equivariant vector bundle over U_i. Fix a trivialization and let $\bar{W}^{\mathrm{f}}_i$ be the obvious extension of this bundle over the whole parameter space T_0.

Now the construction of $\bar{W}^{\mathrm{f}}$, χ and s is as follows. We take the direct sum $\oplus_i \bar{W}^{\mathrm{f}}_i$ for $\bar{W}^{\mathrm{f}}$. For parameters in U_i, let χ_i be the inclusion $\bar{W}_{\lambda_i} \to \bar{W}$ and let s_i be the orthogonal projection $\bar{W} \to \bar{W}_{\lambda_i}$. They are defined only on U_i. Take a partition of unity $\{\rho_i\}$ for the open covering and define χ and s as

$$\chi = \sum_i \rho_i \chi_i, \qquad s = \sum_i \rho_i s_i.$$

The right-hand-sides are well defined and they satisfy (1) and (2).

Consider the kernel of the surjective map

$$D + \chi : \bar{V} \oplus \bar{W}^{\mathrm{f}} \to \bar{W}.$$

¿From (1) in Lemma (3.2) we can show that this map is always surjective. Hence the kernel has a constant dimension given by

$$\dim \bar{V}_\lambda - \dim \bar{W}_\lambda + \operatorname{rank} \bar{W}^{\mathrm{f}} = \operatorname{index}(D : \bar{V} \to \bar{W}) + \operatorname{rank} \bar{W}^{\mathrm{f}}.$$

Hence $\bar{V}^{\mathrm{f}} := \mathrm{Ker}(D + \phi)$ is an S^1-equivariant (finite rank) vector bundle over the parameter space.

Now we can replace the family of linear map $D : \bar{V}_\lambda \to \bar{W}_\lambda$ with the following continuous family D^{f}.

Lemma(3.3). *The family*

$$D^{\mathrm{f}} : \bar{V}^{\mathrm{f}} \to \bar{W}^{\mathrm{f}}, \qquad (v, e) \mapsto e$$

depends continuously on the parameter space T_0.

The formal difference $[\bar{V}^{\mathrm{f}}] - [\bar{W}^{\mathrm{f}}]$ gives the index of the family $D : \bar{V}_\lambda \to \bar{W}_\lambda$ ([2]).

step 3

Now we have the continuous family of linear maps D^{f} between finite dimensional vector spaces and the continuous family of nonlinear maps $\tilde{p}_\lambda Q$ between infinite dimensional vector spaces. Let us define a continuous family of nonlinear maps Q^{f} between finite dimensional vector spaces as

$$Q^{\mathrm{f}} : \bar{V}^{\mathrm{f}} \to \bar{W}^{\mathrm{f}}, \qquad (v, e) \mapsto -s\tilde{p}_\lambda Q.$$

Then a good finite dimensional approximation of $D + Q$ is given by $D^{\mathrm{f}} + Q^{\mathrm{f}}$ in the following sense. Fix an S^1-invariant metric on $\bar{W}^{\mathrm{f}}$. Let $S_{(R,R_1)}(\bar{V}^{\mathrm{f}})$ be the topological sphere bundle over T_0 defined as the boundary of the topological disk bundle

$$\bar{V}^{\mathrm{f}} \cap (B_R(\bar{V}) \times B_{R_1}(\bar{W}^{\mathrm{f}})).$$

Lemma(3.4). *For large* R_1, $D^{\mathrm{f}}+Q^{\mathrm{f}}$ *does not vanish on the sphere bundle* $S_{(R,R_1)}(\bar{V}^{\mathrm{f}})$.

Proof. Take (v, e) in the intersection of $\bar{V}^{\mathrm{f}}$ and $B_R(\bar{V})$. Assume that $(D^{\mathrm{f}}+Q^{\mathrm{f}})(v, e) = 0$. It suffices to show that (1) v does not lie on the boundary sphere $S_R(\bar{V})$ and that (2) e is bounded. ¿From the definition of $\bar{V}^{\mathrm{f}}$, we have $Dv + \chi e = 0$. ¿From the definition of D^{f} and Q^{f}, we have $e - s\tilde{p}_\lambda Q(v) = 0$. Since the image of $\tilde{p}_\lambda$ is contained in $\bar{W}_\lambda$, we obtain the following equation.

$$-Dv = \chi e = \chi s\tilde{p}_\lambda Q(v) = \tilde{p}_\lambda Q(v).$$

¿From Lemma(3.1), we can show that v is contained inside the disk $B_R(\bar{V})$ and does not lie on the boundary sphere $S_R(\bar{V})$. Moreover, since v is bounded, $e = s\chi\tilde{p}_\lambda Q(v)$ is also bounded.

In the above argument we did not use any particular property of the parameter space T_0 except its compactness. For a family of 4-manifolds $\mathbf{X}$ parameterized by a compact space $\mathbf{K}$, we can obtain a finite approximation of the family of monopole equation, where the total parameter space is a bundle over $\mathbf{K}$ with fiber T_0. Strictly speaking, we have to fix a family of spin^c-structure. (Later we shall consider all the spin^c-structures.) Let us denote the family of spin^c-structures as $\mathbf{c}$.

Now we define a Seiberg-Witten invariant as a certain stable homotopy class of a map. We define it for family of 4-manifolds. We do not assume any inequality for $b_+(X)$ in this definition. For example it gives an invariant of homotopy 4-spheres, though we do not know how to detect it explicitly. Later on, when we detect this invariant by using cohomology theories, we will use an assumption for $b_+(X)$.

Fix a family of Riemannian metric of the fiber which varies continuously in C^∞-topology. Let $\mathbf{T}_0$ be the total space of the fiber bundle over $\mathbf{K}$ with fiber $T_0 = H^1(X, \mathbf{R})/H^1(X, \mathbf{Z})$. This is a fiber bundle over $\mathbf{K}$.

¿From the above argument, we can construct a finite approximation $D^{\mathrm{f}} + Q^{\mathrm{f}} : \bar{V}^{\mathrm{f}} \to \bar{W}^{\mathrm{f}}$ parameterized by $\mathbf{T}_0$ since the lemmas can be extended to any family of 4-manifolds parameterized by a compact set.

Let $\mathbf{R}$ be the trivial real 1-dimensional representation space of S^1, and $\mathbf{C}$ be the standard complex 1-dimensional representation space.

Definition(3.2). Let $M(\mathbf{X})$ be the set of the isomorphism classes of the triples (E, F, f), where

(1) E is a trivial S^1-equivariant vector bundle over $\mathbf{T}_0$ whose fiber is a direct sum of finitely many $\mathbf{R}$'s and $\mathbf{C}$'s,

(2) F is an S^1-equivariant real finite-rank vector bundle over $\mathbf{T}_0$ and

(3) f is an S^1-equivariant bundle map from $S(E)$ to $S(F)$.

By using the extended lemmas and the finite approximation we can give the element $(\bar{V}^{\mathrm{f}}, \bar{W}^{\mathrm{f}}, S(D^{\mathrm{f}} + Q^{\mathrm{f}}))$ in $M(\mathbf{X})$. To define a topological invariant, which should be independent of the choices to construct the finite dimensional approximation, we need to take a quotient of $M(\mathbf{X})$ by an equivalence relation. Before that, we recall the definition of *join*. Suppose S_0 and S_1 are some subsets of vector spaces V_0 and V_1 respectively. Assume that S_0 and S_1 does not contain any real line passing the origin. Then the join of S_0 and S_1 is defined to be the set of the points in the direct sum $V_0 \oplus V_1$ of the form $(1-t)a_0 \oplus ta_1$ for some $a_0 \in S_0$, $a_1 \in S_1$ and $t \in [0, 1]$. Note that the fiber-wise join of two sphere bundles is topologically the sphere bundle of the direct sum of the associated vector bundles. We call it the join of the sphere bundles. Then, for two maps between sphere bundles, we can naturally construct the *join* of the maps between the joins of the sphere bundles.

Definition(3.3). Two elements (E_0, F_0, f_0) and (E_1, F_1, f_1) of $M(\mathbf{X})$ are *stable homotopic* to each other if and only if there are two finite dimensional representation spaces G_0 and G_1 of S^1 satisfying the following conditions.

(1) The two representations are direct sums of finitely many $\mathbf{R}$'s and $\mathbf{C}$'s.

(2) We regard G_0 and G_1 as trivial vector bundles over $\mathbf{K}$. Then $E_0 \oplus G_0$ is isomorphic to $E_1 \oplus G_1$ and $F_0 \oplus G_0$ is isomorphic to $F_1 \oplus G_1$.

(3) The join of f_0 and the identity on $S(G_0)$ is an S^1-equivariant bundle map from $S(E_0 \oplus G_0)$ to $S(F_0 \oplus G_0)$. Similarly we have an S^1-equivariant map from $S(E_1 \oplus G_1)$ to $S(F_1 \oplus G_1)$. Then, through the isomorphism in (2), the two joins are homotopic to each other.

Definition(3.4)(stable homotopy version of Seiberg-Witten invariant)**.**

(1) Let $\mathbf{M}(\mathbf{X})$ be the set of all the stable homotopy classes

of $M(\mathbf{X})$.

(2) Define $SW(\mathbf{X}, \mathbf{c})$ to be the stable homotopy class of $(\bar{V}^{\mathrm{f}}, \bar{W}^{\mathrm{f}}, S(D^{\mathrm{f}} + Q^{\mathrm{f}}))$.

Remark. The set $\mathbf{M}(\mathbf{X})$ has a natural structure of Abelian semi-group.

One of the purposes of this paper is to show the well-definedness of $SW(\mathbf{X}, \mathbf{c})$. The proof is parallel to that of the well-definedness of the index of family of Fredholm operators.

Theorem(3.5). *The stable homotopy class $SW(\mathbf{X}, \mathbf{c}) \in \mathbf{M}(\mathbf{X})$ is*

independent of the family of Riemannian metric and other choices necessary to define it, and hence it gives a topological invariant of the pair $(\mathbf{X}, \mathbf{c})$.

Proof.

Step 1

First we fix the family of Riemannian metrics and show that the stable homotopy class does not depend on other choices. Let λ_0 and λ_1 be two large real numbers. Suppose $\chi_j : \bar{W}^{\mathrm{f}}_j \to \bar{W}$ and $s_j : \bar{W} \to \bar{W}^{\mathrm{f}}_j$ $(j = 0, 1)$ are the maps satisfying the condition of Lemma(3.2). We compare these two and assume for simplicity that the other data are the same for these two cases. (The other data can be treated using an argument similar to that in Step 2.) We have two finite dimensional approximations $D^{\mathrm{f}}_j + Q^{\mathrm{f}}_j : \bar{V}^{\mathrm{f}}_j \to \bar{W}^{\mathrm{f}}_j$ $(j = 0, 1)$. For $j = 0, 1$ we denote the triple $(\bar{V}^{\mathrm{f}}_j, \bar{W}^{\mathrm{f}}_j, S(D^{\mathrm{f}}_j + Q^{\mathrm{f}}_j))$ by (E_j, F_j, f_j). Recall that $F_j = \bar{W}^{\mathrm{f}}_j$ $(j = 0, 1)$ are trivial vector bundles. Take F_1 and F_0 for G_0 and G_1 respectively.

We consider a finite dimensional approximation parameterized by $t \in [0, 1]$ defined as follows. For each $t \in [0, 1]$, we use

$$\chi(t) = t\chi_0 + (1 - t)\chi_1 : \bar{W}^{\mathrm{f}}_0 \oplus \bar{W}^{\mathrm{f}}_1 \to \bar{W},$$

and

$$s(t) = ts_0 + (1 - t)s_1 : \bar{W} \to \bar{W}^{\mathrm{f}}_0 \oplus \bar{W}^{\mathrm{f}}_1.$$

to obtain the finite approximation $D^{\mathrm{f}}(t) + Q^{\mathrm{f}}(t) : \bar{V}^{\mathrm{f}}(t) \to \bar{W}^{\mathrm{f}}(t)$. From this construction, $D^{\mathrm{f}}(j) + Q^{\mathrm{f}}(j)$ is equal to the direct sum of $D^{\mathrm{f}}_j + Q^{\mathrm{f}}_j$ and the identity of G_j. This implies that the joins of f_j and the identity of $S(G_j)$ for $j = 0, 1$ are homotopic to each other, and hence the two stable homotopy classes are the same.

Step 2

Suppose we have two choices for the family of Riemannian metrics. Then We can connect them continuously as a family parameterized by $\mathbf{T}_0 \times [0, 1]$. Since the construction of finite dimensional approximation works for any family parameterized by a compact space, we can construct a family of finite dimensional approximations. Hence the homotopy classes of the finite dimensional approximations for $t = 0$ and $t = 1$ are the same.

We can easily extend the above construction for all the spin^c-structures at once.

So far, the symmetry of all the spaces and maps has been S^1. If we collect all the spin^c structures, however, then we have a symmetry of Pin_2, where Pin_2 is the normalizer of S^1 in Sp_1 and is generated by S^1 and j.

Let ι be the involution of $Spin^c_4 = (Sp_1 \times Sp_1 \times S^1)/\{\pm 1\}$ defined by $\iota(q_-, q_+, z) = (q_-, q_+, z^{-1})$. Then any representation of $Spin^c_4$ can be twisted by ι. Let ${}_-\mathbf{H}'_+$, ${}_+\mathbf{H}'$, ${}_-\mathbf{H}'$, ${}_+\mathbf{H}'_+$ and $\tilde{\mathbf{C}}'$ be the the twisting of ${}_-\mathbf{H}_+$, ${}_+\mathbf{H}$, ${}_-\mathbf{H}$, ${}_+\mathbf{H}_+$ and $\tilde{\mathbf{C}}$. Since the S^1-component of $Spin^c_4$ acts trivially on ${}_-\mathbf{H}_+$ and $\tilde{\mathbf{C}}$, the $Spin^c_4$-modules ${}_-\mathbf{H}'_+$ and $\tilde{\mathbf{C}}'$ are canonically isomorphic to ${}_-\mathbf{H}_+$ and $\tilde{\mathbf{C}}$ respectively. However we introduce another isomorphisms below and it is convenient to distinguish them.

Let ${}_-j_+$, ${}_+j$, ${}_-j$, ${}_+j_+$ and $\tilde{j}$ be the $Spin^c_4$-equivariant homomorphisms defined by:

$$
\begin{aligned}
{}_-j_+ &: {}_-\mathbf{H}_+ \to {}_-\mathbf{H}'_+, \qquad a \mapsto -a,\\
{}_+j &: {}_+\mathbf{H} \to {}_+\mathbf{H}', \qquad \phi \mapsto \phi j\\
{}_-j &: {}_-\mathbf{H} \to {}_-\mathbf{H}', \qquad \psi \mapsto \psi j\\
{}_+j_+ &: {}_+\mathbf{H}_+ \to {}_+\mathbf{H}'_+, \qquad \omega \mapsto -\omega\\
\tilde{j} &: \tilde{\mathbf{C}} \to \tilde{\mathbf{C}}', \qquad t \mapsto \bar{t}.
\end{aligned}
$$

Let $Spin^c(X)$ be the set of spin^c structures on X, i.e., $Spin^c_4(X)$ is the set of all the equivalent classes of pair of an $Spin^c_4$-bundle P and an isomorphism $T = P \times_{Spin^c_4} {}_-\mathbf{H}_+ \cong TX$. It is well known, from a simple argument of obstruction theory, that $Spin^c_4(X)$ is an affine space over $H^2(X, \mathbf{Z})$. The twisting of ι induces an involution on $Spin^c(X)$. For an $Spin^c_4$-bundle P, let $P' := P \times_\iota Spin^c_4$ be the twisting of P. We use the notation $S^{+'}$ etc. to denote the vector bundles associated with P'. Then the five homomorphisms above give five bundle homomorphisms ${}_+j : S^+ \to S^{+'}$ etc. Note that the connections on L correspond to the connections on $L' = L^{-1}$ bijectively through ι. Take a pair of connections A_0 and A'_0 on P and P' corresponding to each other by this bijection. We use the notations D' and Q' etc. to denote the object associated with P'. It is easy to see the following:

Lemma(3.6). *The two maps $D + Q : V \to W$ and $D' + Q : V' \to W'$ correspond to each other via the maps ${}_+j$ etc.*

For each spin^c structure, we want to fix a connection A_0 so that they are compatible with the involution. One minor problem occurs when the involution on $Spin^c(X)$ has a fixed point. In that case we have to take A_0 so that it is compatible with the involution. We need the next lemma.

Lemma(3.7). *If a spin^c-structure is preserved by the involution, then the spin^c-structure is reduced to a spin structure and hence L is topologically trivial. Then we can take a trivial connection for A_0.*

Proof. Suppose we have an isomorphism $f : P \to P'$. Since P' is identified with P as a set, we can consider the fixed point P_0 of P. The fixed-point set of the action of ι on $Spin^c_4$ is the subgroup $Spin_4$. This implies that P_0 is a principal $Spin_4$-bundle and hence the structure group of P is reduced to $Spin_4$. The rest of the statement is obtained immediately.

We take the trivial flat connection as A_0 for the spin^c-structure which can be reduced to a spin structure. Then the map $D+Q : V \to W$ and its finite dimensional approximation has a natural Pin_2 symmetry [5]. We choose A_0 for each spin^c-structure and assume that they are compatible with the involution in the following sense: (1) if the spin^c-structure comes from an spin structure, then A_0 is a trivial connection and (2) if not, then A_0 and A_0' correspond to each other by the involution. By using this choice we have a family $D^{\mathrm{f}} : \bar{V}^{\mathrm{f}} \to \bar{W}^{\mathrm{f}}$ parameterized by $Spin^c(X) \times T_0$. ¿From the bundle homomorphisms $_+j$ etc we define self maps J on the families $\{\bar{V}\}$ and $\{\bar{W}\}$. This action is not an involution, but its order is 4. This map together with the obvious S^1-action gives rise to a Pin_2 action on the families.

Let $\tilde{\mathbf{R}}$ and $\mathbf{H}$ be the following real representation space of Pin_2: $\tilde{\mathbf{R}}$ is the unique non-trivial 1-dimensional representation and $\mathbf{H}$ is the space of quaternions on which Pin_2 acts as right multiplication.

Proposition(3.8). *We can take a finite approximation of $D : \bar{V} \to \bar{W}$ so that $\bar{W}^{\mathrm{f}}$ is a trivial vector bundle whose fiber is a direct sum of finitely many $\tilde{\mathbf{R}}$'s and $\mathbf{H}$, and the family $D^{\mathrm{f}} : \bar{V}^{\mathrm{f}} \to \bar{W}^{\mathrm{f}}$ parameterized by $Spin^c(X) \times T_0$ has an Pin_2 symmetry.*

Proof. We can just repeat the construction of the finite dimensional approximation with the action of Pin_2. Let $\tilde{\mathbf{T}}_0$ be the total space of the fiber bundle over $\mathbf{K}$ with fiber $Spin^c(X) \times T_0$. Since the argument is almost the same except for replacement of S^1, $\mathbf{R}$, $\mathbf{C}$ and $\mathbf{T}_0$ by Pin_2, $\tilde{\mathbf{R}}$, $\mathbf{H}$ and $\tilde{\mathbf{T}}_0$ respectively, we omit the details. Lemma(3.7) implies that this family is preserved by the action of J. Since this family already has the S^1-symmetry, we totally have a symmetry of Pin_2 which is generated by S^1 and J.

Remark. The above argument using Lemma(3.7) is not actually necessary to define the action of Pin_2. If we use the original monopole equation $D+Q+F_{A_0} = 0$, then it does not depend on the choice of A_0 and we can easily define the Pin_2-action which preserves this equation.

We now formulate the invariant without fixing a spin^c-structure. Suppose $\mathbf{X}$ is a fiber bundle with fiber X over $\mathbf{K}$, where X is a closed oriented 4-manifold. Fix a family of Riemannian metric of the fiber which varies continuously in C^∞-topology. Recall that $\tilde{\mathbf{T}}_0$ is the total space of the fiber bundle over $\mathbf{K}$ with fiber $Spin^c(X) \times T_0$. We regard it as a Pin_2-equivariant fiber bundle over $\mathbf{K}$, where Pin_2-action on $T_0 = H^1(X, \mathbf{R})/H^1(X, \mathbf{Z})$ factors through multiplication of $\{\pm 1\} = \langle J \rangle$ while its action on $Spin^c(X)$ also factors through the involution.

¿From the above argument we can construct a finite dimensional approximation

$D^{\mathrm{f}} + Q^{\mathrm{f}} : \bar{V}^{\mathrm{f}} \to \bar{W}^{\mathrm{f}}$ parameterized by $\tilde{\mathbf{T}}_0$, which now has a Pin_2-symmetry.

Parallel to Definition(3.2), we use the notation $\tilde{M}(\mathbf{X})$ to denote the set of the isomorphism classes of the triples (E, F, f), where

(1) E is a trivial Pin_2-equivariant vector bundle over $\tilde{\mathbf{T}}_0$ whose fiber is a direct sum of finitely many $\tilde{\mathbf{R}}$'s and $\mathbf{H}$'s,

(2) F is a Pin_2-equivariant real finite-rank vector bundles over $\tilde{\mathbf{T}}_0$ and

(3) f is a Pin_2-equivariant bundle map from $S(E)$ to $S(F)$.

Similarly we define the set of stable homotopy classes $\tilde{\mathbf{M}}(\mathbf{X})$ by using the obvious equivalence relation similar to Definition(3.3).

The rest of the argument is also quite parallel to the previous argument. We define $SW(\mathbf{X})$ to be the stable homotopy class of $(\bar{V}^{\mathrm{f}}, \bar{W}^{\mathrm{f}}, S(D^{\mathrm{f}} + Q^{\mathrm{f}}))$. Then we obtain:

Theorem(3.9). *The stable homotopy class $SW(\mathbf{X}) \in \tilde{\mathbf{M}}(\mathbf{X})$ is independent of the family of Riemannian metric and other choices to define it, and hence it gives a topological invariant of* $\mathbf{X}$

§4. Evaluation of the Seiberg-Witten invariant using cohomology theory

In this section we show how to detect the stable homotopy version of the Seiberg-Witten invariant in the simplest case. We consider a single oriented closed 4-manifold X satisfying $b_1(X) = 0$ and $b_+(X) > 1$. We also assume that $b_+(X)$ is odd and write $b_+(X)$ as $2p+1$. Moreover we fix a spin^c-structure c and consider the stable homotopy class $SW(X, c)$. In this case the finite dimensional approximation is just an S^1-equivariant map between two representation spaces of S^1. ¿From an index calculation, the map is defined as a map from $\mathbf{C}^{a+m} \oplus \mathbf{R}^n$ to $\mathbf{C}^m \oplus \mathbf{R}^{2p+1+n}$, where $a = (c_1(L)^2 - \mathrm{sign}(X))/8$, and m and n are some non-negative integers. When p is larger than or equal to a, it is easy to see that there is only one homotopy class of S^1-equivariant map from $S(\mathbf{C}^{a+m} \oplus \mathbf{R}^n)$ to $S(\mathbf{C}^m \oplus \mathbf{R}^{2p+1+n})$. In the rest of this section we assume that $p < a$.

Let f_0 be an S^1-equivariant map between spheres representing the class $SW(X)$. By taking suspension, we can extend f_0 to be an S^1-equivariant map from the pair $(B(\mathbf{C}^{a+m} \oplus \mathbf{R}^n), S(\mathbf{C}^{a+m} \oplus \mathbf{R}^n))$ to the pair $(B(\mathbf{C}^m \oplus \mathbf{R}^{2p+1+n}), S(\mathbf{C}^m \oplus \mathbf{R}^{2p+1+n}))$. The S^1-equivariant map f preserves the fixed point sets of the S^1-action. The restriction of f_0 on the fixed point sets is a map from the pair $(B(\mathbf{R}^n), S(\mathbf{R}^n))$ to the pair $(B(\mathbf{R}^{2p+1+n}), S(\mathbf{R}^{2p+1+n}))$. If we perturb this map slightly, the origin of $B(\mathbf{R}^{2p+1+n})$ does not lie on the image of $B(\mathbf{R}^n)$. The perturbation is not unique. However, since we are assuming that $2p + 1 > 1$, the relative homotopy class of the perturbed map from the pair $(B(\mathbf{R}^n), S(\mathbf{R}^n))$ to the pair $(B(\mathbf{R}^{2p+1+n}) \setminus \{0\}, S(\mathbf{R}^{2p+1+n}))$ is uniquely determined. Since $S(\mathbf{R}^{2p+1+n})$ is a retraction of $B(\mathbf{R}^{2p+1+n}) \setminus \{0\}$, the map can be perturbed further to a map from the pair $(B(\mathbf{R}^n), S(\mathbf{R}^n))$ to the pair $(S(\mathbf{R}^{2p+1+n}), S(\mathbf{R}^{2p+1+n}))$. ¿From homotopy extension property we can extend the perturbation to a perturbation of f_0 supported in the interior of $B(\mathbf{C}^{a+m} \oplus \mathbf{R}^n)$. Average it linearly by using the S^1-action, then we can assume that the perturbed map f_1 is still S^1-equivariant.

Now f_1 is an S^1-equivariant map from the pair $(B(\mathbf{C}^{a+m} \oplus \mathbf{R}^n), S(\mathbf{C}^{a+m} \oplus \mathbf{R}^n) \cup B(\mathbf{R}^n))$ to the pair $(B(\mathbf{C}^m \oplus \mathbf{R}^{2p+1+n}), S(\mathbf{C}^m \oplus \mathbf{R}^{2p+1+n}))$. We replace these data in the following way just for convenience.

(1) As for the first pair of the spaces $(B(\mathbf{C}^{a+m}\oplus\mathbf{R}^n), S(\mathbf{C}^{a+m}\oplus\mathbf{R}^n)\cup B(\mathbf{R}^n))$, this is homotopically equivalent to the pair of a disk bundle and its sphere bundle. In fact if we write $S(\mathbf{C}^{a+m})$ for the sphere in the disk $B(\mathbf{C}^{a+m})$ centered in 0 with half radius, then the complement of $S(\mathbf{C}^{a+m}\oplus\mathbf{R}^n)\cup B(\mathbf{R}^n)$ in $B(\mathbf{C}^{a+m}\oplus\mathbf{R}^n)$ is a trivial disk bundle over $S(\mathbf{C}^{a+m})$ with fiber $\mathbf{R}^{n+1}$. We use the notation S to denote $S(\mathbf{C}^{a+m})$ and E to denote this trivial bundle.

(2) As for the second pair of the spaces $(B(\mathbf{C}^m\oplus\mathbf{R}^{2p+1+n}), S(\mathbf{C}^m\oplus\mathbf{R}^{2p+1+n}))$, we multiply it by S in order to regard every space as an S^1-equivariant bundle over S. We use the notation F to denote the trivial vector bundle over S whose fiber is $\mathbf{C}^m\oplus\mathbf{R}^{2p+1+n}$.

(3) As for the S^1-equivariant map between the pairs, we use, instead of f_1, the bundle map f over S induced from f_1. We need to use homotopy extension property to construct f.

The geometric data we have now is the S^1-equivariant homotopy class of the S^1-equivariant bundle map f from the disk bundle of E to the disk bundle of F which preserves their boundaries. Since the S^1-action is free on S, we can divide everything by S^1 to get a bundle map $\bar{f}$ between disk bundles associated with $S\times_{S^1}E$ and $S\times_{S^1}F$ over $\bar{S}=S/S^1$:

$$\bar{f}:(B(\bar{E}),S(\bar{E}))\to(B(\bar{F}),S(\bar{F})).$$

This is the final geometric data we shall use. Instead of using maps between pairs, we could formulate everything by using maps between Thom spaces.

It is easy to show that a certain stable homotopy class of this map is well defined for X with $b_1=0$ and $b_+>1$. Since we will not use this well-definedness, we omit the details.

Suppose h is a multiplicative generalized cohomology theory for which $\bar{E}$ and $\bar{F}$ are orientable. We shall use this h to detect the stable homotopy class.

First we need to fix the orientations of $\bar{E}$ and $\bar{F}$. We shall explain it later for the ordinary cohomology and the K-theory. We shall need some extra geometric data to define the orientations. Fixing the orientations of $\bar{E}$ and $\bar{F}$ implies, by definition, that $h^*(B(\bar{E}),S(\bar{E}))$ $h^*(B(\bar{F}),S(\bar{F}))$ are free $h^*(\bar{S})$-modules generated by given classes $\tau^h_{\bar{E}}$ and $\tau^h_{\bar{F}}$ respectively.

Definition(4.1). Suppose X satisfies $b_1=0$ and $b_+>0$. Let c be a spin^c-structure of X. (We also assume that certain data necessary to define the orientations are given.) Then the *h-version of Seiberg-Witten invariant* $k^h(X,c)$ of (X,c) is defined by using the following relation:

$$k^h(X,c)\tau^h_{\bar{E}}=\bar{f}^*\tau^h_{\bar{F}}.$$

Remark. The above $k^h(X,c)$ is defined as an element of $h^*(\bar{S})$. However this cohomology group itself depend on various choices to define f. Strictly speaking, we need to construct some inverse system of cohomology groups. It would be more systematic to use homology rather than cohomology. We shall discuss this point later for the ordinary cohomology.

Ordinary cohomology

Let $k^H(X,c)$ be the Seiberg-Witten invariant for ordinary cohomology. To define it, we need the orientations of $\bar{E}$ and $\bar{F}$. If we change both of these orientations, the invariant does not change. We only need the orientation of the formal difference $[E]-[F]$ which is an element of $K(point)$. Since complex vector spaces have natural orientations, we obtain:

Lemma(4.2). *Each choice of the orientation of $[H^0(X,\mathbf{R})]-[H^1(X,\mathbf{R})]+[H^+(X,\mathbf{R})]$ gives the required orientation to define $k^H(X,c)$.*

Proof. The formal difference $[E]-[F]$ is equal to $[H^1(X,\mathbf{R})]-[H^+(X,\mathbf{R})]$ modulo some formal difference of complex vector spaces.

Since we assumed that $b_1=0$, it is not necessary to put $H^1(X,\mathbf{R})$. However we put it so that the statement is generalized to any X or any family of 4-manifolds. We fix one of the choices of the orientations.

The degree of $k^H(X,c)$ is given by

$$\deg k^H(X,c) = \deg \tau^H_{\bar{F}} - \deg \tau^H_{\bar{E}} = \dim F - \dim E = (2m+2p+1+n)-(n+1) = 2m+2p.$$

Since $\bar{S}$ is a complex projective space of complex dimension $a+m-1$, its cohomology ring is given by

$$H^*(\bar{S},\mathbf{Z}) = \mathbf{Z}[\alpha], \qquad \alpha^{a+m}=0$$

where α is the first Chern class of hyperplane line bundle. The above relation is the only one satisfied by α.

Recall that we are assuming that $a>p$. The relation between $k^H(X,c)$ and the usual Seiberg-Witten invariant is given by the following definition and lemma.

Definition(4.3). Define $k(X,c)\in\mathbf{Z}$ by

$$k^H(X,c) = k(X,c)\alpha^{m+p}.$$

Lemma(4.4). *Perturb f if necessary and assume that the equation $f(v)=0$ is transversal. Then the quotient of the space of the solutions $\{v|f(v)=0\}$ divided by S^1 is an oriented closed submanifold of $\bar{S}\times E$ and its fundamental class is equal to $k(X,c)\times[generator]$ in $H_{2d}(\bar{S}\times E)=\mathbf{Z}$ for $d=a-p-1$.*

It is not hard to see that the characterization of $k(X,c)$ in the above lemma implies that $k(X,c)$ is equal to the usual Seiberg-Witten invariant.

The dimension $2d$ of $\{v|f(v)=0\}/S^1$ is calculated as follows.

$$2d = \dim\bar{S}+\dim E-\dim F = 2(a+m-1)+(n+1)-(2m+2p+1+n) = 2(a-p-1).$$

K-cohomology

Let $k^K(X,c)$ be the Seiberg-Witten invariant for K-cohomology. To define it, we need the orientations of $\bar{E}$ and $\bar{F}$. It suffices to give S^1-invariant weak complex structures on E and F. Here a weak-complex structure of a real vector space is an obvious equivalence class of the complex structure of the direct sum of the vector space and $\mathbf{R}^l$ for some l. ("Weak-spin^c-structure" would be enough.) It is necessary to give weak-complex structures only on the S^1-invariant parts. Then we obtains

Lemma(4.5). *Each choice of the homotopy class of weak-complex structures on $H^1(X,\mathbf{R})$ and $H^+(X,\mathbf{R})$ gives the required orientation to define $k^K(X,c)$.*

Since we assumed that $b_1 = 0$, it is not necessary to give a weak-complex structure of $H^1(X,\mathbf{R})$. A choice of homotopy class of the weak-complex structure of a real vector space is equivalent to a choice of orientation of the space. However if we consider family of 4-manifolds, we would need to fix more than the the choice of orientation to define the K-theory version of Seiberg-Witten invariant, though we do not deal with that case in this paper.

We fix one of the choices of the orientations of $H^+(X,\mathbf{R})$ and hence a homotopy class of weak-complex structure on it. For simplicity we assume that n is odd so that the S^1-invariant parts have a complex structure which is compatible with the given choice. We can always choose such a representative in the stable homotopy class.

Let L be the hyperplane line bundle on the complex projective space $\bar{S}$. Then $K^*(\bar{S})$ $(* = 0,1)$ is given by

$$K^*(\bar{S}) = \mathbf{Z}[\xi], \qquad \xi^{a+m} = 0$$

where ξ is the K-theoretic Euler class of hyperplane line bundle L. More explicitly $\xi = 1 - [L^*]$. The degree of ξ is 0. The above relation is the only one satisfied by ξ. (Here we use the above convention of K-theoretic Euler class so that we have $ch_1(\xi) = \alpha$.)

The K-theory Seiberg-Witten invariant is a polynomial in ξ with integral coefficient.

$$k^K(X,c) \in \mathbf{Z}[\xi].$$

In the next section we show that the integrality of each coefficient gives a divisibility of $k(X,c)$.

§5. Divisibility.

Theorem(1.1) is immediately shown from the next theorem.

Theorem(5.1). *Let $a_{p,l}$ be the coefficient of x^l in the Taylor expansion of $(\frac{\log(1-x)}{x})^p$. Then we have the following relation in $K(\bar{S}) \otimes \mathbf{Q}$.*

$$k^K(X,c) = k(X,c)\xi^{m+p}\sum_l a_{p,l}\xi^l.$$

Proof of Theorem(1.1) admitting Theorem(5.1). Since $k^K(X,c)$ is a polynomial in ξ with integral coefficient, $k(X,c)a_{p,n}$ is an integer as long as ξ^{m+p+n} is not zero, i.e., $m+p+n < m+a$. This inequality is equivalent to $n \le a-p-1 = d$.

To show Theorem we compare the K-cohomology and the ordinary cohomology by using the Chern character. We use the following lemma.

Lemma(5.2). *Suppose a complex vector bundle G over $\bar{S}$ has a splitting into a direct sum of line bundles:$G = \oplus_l L_l$. We use the notation α_i to denote $c_1(L_i)$ and ξ_i to denote $1-[L_i^*]$. Then we have*

$$\mathrm{ch}(\xi_i) = 1 - e^{-x_i}, \qquad \mathrm{ch}(\log(1-\xi_i)) = x_i,$$

$$\mathrm{ch}(\tau_G^K)\prod_i \frac{x_i}{1-e^{-x_i}} = \tau_G^H$$

and

$$\mathrm{ch}(\tau_G^K \prod_i \frac{\log(1-\xi_i)}{\xi_i}) = \tau_G^H.$$

Note that x_i and ξ_i are nilpotent in $H^*(\bar{S}, \mathbf{Z})$ and in $K^*(\bar{S})$ respectively. However the expressions of the terms in the above equalities are well defined by using Taylor expansion. All the equalities should be understood in $H^*(\bar{S}, \mathbf{Q})$. The first two equalities follows from the definition. The third one is the well-known relation that gives a characterization of the Todd genus. The fourth equality is equivalent to the third one.

Proof of Theorem(5.1). ¿From Lemma we have

$$\mathrm{ch}(\tau_{\bar{E}}^K) = \tau_{\bar{E}}^H$$

and

$$\mathrm{ch}(\tau_{\bar{F}}^K(\frac{\log(1-\xi)}{\xi})^m) = \tau_{\bar{F}}^H.$$

Hence we have

$$\mathrm{ch}(k^K(X,c)(\frac{\log(1-\xi)}{\xi})^m) = k^H(X,c).$$

On the other hand, from $k^H(X,c) = k(X,c)\alpha^{p+m}$, we have

$$\mathrm{ch}(k(X,c)(\log(1-\xi))^{p+m}) = k^H(X,c).$$

Since the Chern character is injective on $\bar{S}$, we obtain

$$k^K(X,c)(\frac{\log(1-\xi)}{\xi})^m = k(X,c)(\log(1-\xi))^{p+m},$$

which implies Theorem 5.1.

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References

[1] M. F. Atiyah, *K-theory*, Benjamin, New York, 1967.

[2] M. F. Atiyah and I. M. Singer, *The index of elliptic operators IV*, Ann. of Math. **93** (1971), 119–138.

[3] S. Bauer, *Connected sums of 4-manifolds*, in preparation.

[4] R. Courant and D. Hilbert, *Methods of mathematical physics I*, John Wiley & Sons, New York, 1962.

[5] M. Furuta, *Monopole equation and the* 11/8*-conjecture*, preprint.

[6] P. B. Kronheimer and T. S. Mrowka, *The genus of embedded surfaces in the projective plane*, Math. Res. Letters **1** (1994), 797–808.

[7] E. Witten, *Monopoles and four-manifolds*, Math. Res. Letters **1** (1994), 769–796.

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