THE MOTION OF AN EXTENDED PARTICLE IN THE GRAVITATIONAL FIELD

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THE MODEL

With the Einstein theory of gravitation there is a minimum size for a particle of given mass m, namely the radius $\varrho = 2m$ in the Schwarzschild system of coordinates. The gravitational field can be extended to smaller radii consistent with Einstein's field equations for empty space, but the region $r < 2m$ is then physically inaccessible (it would need an infinite time to send in a signal and get it out again), so it cannot be allowed in a physical theory.

Thus, to get a precise theory of the motion of a particle in the gravitational field, one cannot take the particle to be a point singularity. One must take it to have a finite size ϱ, such that Einstein's equations for empty space hold only for $r > \varrho$ and ϱ must be $\geqslant 2m$. It is awkward to work with the case $\varrho = 2m$, because of the singular character of space-time at this radius. We shall here consider the case of $\varrho > 2m$.

One cannot very well take the particle to be a rigid sphere, because of the ambiguity in the definition of a sphere in curved space-time. We therefore assume the surface of the particle to be flxeible, so that the shape and size can vary. The simplest assumptions will be made that lead to definite equations of motion for such a particle, with stable equilibrium states.

In choosing these assumptions one can be guided by analogy with the electromagnetic field. One can get a reasonable theory of a charged particle of finite size in the electromagnetic field by assuming that the surface of the particle is a perfect conductor carrying a distribution of electric charge, and that there is a surface tension which counterbalances the electrostatic repulsion [1]. There is then no electromagnetic field inside the particle, and the electromagnetic potentials are continuous at the surface while their first derivatives are not.

We shall make analogous assumptions for our gravitational particle. We assume it carries a surface distribution of mass, which adjusts itself so that there is no gravitational field inside, i.e. space-time is flat inside. We assume also a surface pressure to counterbalance the mutual attraction of the surface distribution of mass. If these are the only forces, one can have a particle at rest in equilibrium, but it is unstable, in contradistinction to the electromagnetic case. To bring in stability we need some further force, and the simplest assumption is to take an additional energy term proportional to the total volume inside the particle.

An extended particle in the combined gravitational and electromagnetic fields has been considered by Lees [2]. His model differs from the present one through having constraints on the size and shape of the particle.

THE ACTION PRINCIPLE

A comprehensive action principle will be set up, giving both the field equations and the equations of motion of the particle. It will determine the motion of each element of the surface, so it will give the motion of the particle as a whole as well as the changes in its size and shape. The total action is of the form

$$I = I_O + I_S + I_I$$

where I_O is the action for the space outside the particle, I_S is the surface action and I_I is the action for the space inside.

We take I_O to be the usual action for the Einstein field, namely the integral of the total curvature density

$$I_O = \int \mathcal{G} g^{\mu\nu} \{(\Gamma^{\alpha}_{\mu\sigma}\Gamma^{\sigma}_{\alpha\nu} - \Gamma^{\alpha}_{\mu\nu}\Gamma^{\sigma}_{\alpha\sigma}) + \Gamma^{\sigma}_{\mu\sigma,\nu} - \Gamma^{\alpha}_{\mu\nu,\alpha}\}\, d^4x$$

taken over the region outside the particle, where $-\mathcal{G}^2$ is the determinant of the $g_{\mu\nu}$. We may write it as

$$I_O = \int (\mathcal{L} + m^{\alpha}{}_{,\alpha})\, d^4x$$

where $\mathcal{L}$ does not involve any second derivatives of the $g_{\mu\nu}$. We have then

$$m^{\alpha} = \mathcal{G}(g^{\mu\alpha}\Gamma^{\sigma}_{\mu\sigma} - g^{\mu\nu}\Gamma^{\alpha}_{\mu\nu}) \tag{1}$$

$$\begin{aligned} \mathcal{L} &= \mathcal{G} g^{\mu\nu}(\Gamma^{\alpha}_{\mu\sigma}\Gamma^{\sigma}_{\alpha\nu} - \Gamma^{\alpha}_{\mu\nu}\Gamma^{\sigma}_{\alpha\sigma}) - (\mathcal{G} g^{\mu\nu})_{/\nu}\Gamma^{\sigma}_{\mu\sigma} + (\mathcal{G} g^{\mu\nu})_{/\alpha}\Gamma^{\alpha}_{\mu\nu} \\ &= \tfrac{1}{4}\,\mathcal{G} g_{\mu\nu,\varrho}\, g_{\alpha\beta,\sigma}\{g^{\varrho\sigma}(g^{\mu\alpha}g^{\nu\beta} - g^{\mu\nu}g^{\alpha\beta}) + 2g^{\nu\sigma}(g^{\mu\varrho}g^{\alpha\beta} - g^{\mu\alpha}g^{\varrho\beta})\}. \end{aligned} \tag{2}$$

We can now transform I_O to

$$I_O = \int \mathcal{L} d^4x - \int m^{\alpha} dS_{\alpha}, \tag{3}$$

where dS_{α} is an element of the surface of the particle. In this form it does not involve any second derivatives of the $g_{\mu\nu}$. We shall assume that I_S and I_S likewise involve only the $g_{\mu\nu}$ and their first derivatives.

The condition that space-time is flat inside the particle is assumed as a constraint of the action principle. We require δI to be zero only for variations of the $g_{\mu\nu}$ that preserve this flatness.

Let the equation of the surface of the particle be

$$f(x) = 0.$$

This equation must not be varied in the variational procedure, because if it were varied, δI would not depend linearly on the parameters that specify the variation of f, on account of the gravitational field being different just outside the surface and just inside. Thus $f(x)$ is kept fixed all through the calculation. For convenience we take it to be

$$x^1 = 0, \quad \text{with } x^1 > 0 \text{ outside.} \tag{4}$$

We suppose a continuous system of coordinates inside and outside the particle, and we use the suffixes $a, b, c, \ldots$ to take on the values $0, 2, 3$ only. Then the g_{ab} are continuous, and also their tangential derivatives $g_{ab,c}$, but the derivatives $g_{ab,1}$ need not be continuous. Also the $g_{1\mu}$ need not be continuous, and can be varied independently on both sides of the surface. Let

$$c^{\mu\nu} = g^{\mu\nu} - \frac{g^{1\mu}g^{1\nu}}{g^{11}} \tag{5}$$

so that $c^{\mu\nu} = 0$ if μ or ν is one and c^{ab} is the reciprocal matrix to g_{ab}.

I_S is an integral over the surface of the particle,

$$I_S = \int \boldsymbol{n}^a dS_a = \int \boldsymbol{n}^1 dx^0 dx^2 dx^3$$

with the equation (4) for the surface. We must choose $\boldsymbol{n}^1$ to be a three-dimensional scalar density with respect to the coordinates x^0, x^2, x^3 of the surface, and to be invariant under any transformation of coordinates which does not alter the surface $x^1 = 0$ and the coordinates x^0, x^2, x^3 in it. The basic quantities that have this invariance property, and can, therefore, enter into $\boldsymbol{n}^1$, are the g_{ab} and their tangential derivatives $g_{ab,c}$, and also the quantities $\mathcal{G}\Gamma^1_{ab}$. The latter have different values just outside the surface and just inside, on account of the discontinuity in $g_{1\mu}$ and $g_{ab,1}$. Either value for $\mathcal{G}\Gamma^1_{ab}$ has the necessary invariance property and can enter into $\boldsymbol{n}^1$. To distinguish the two values, we shall denote the inside one by $\mathcal{G}^*\Gamma^{*1}_{ab}$.

We shall now assume

$$\boldsymbol{n}^1 = -2\mathcal{G}c^{ab}\Gamma^1_{ab} + 2\omega\mathcal{M}, \tag{6}$$

where ω is a constant and $\mathcal{M}^2$ is the determinant of the g_{ab}. The first term in (6) is connected with the outside gravitational field, and is needed for a purpose that will become clear later. The second term gives the surface pressure.

Finally, we assume the stabilizing term in the action

$$I_I = 2\lambda \int \mathcal{G} d^4x$$

taken over the space inside the particle, λ being a constant.

THE VARIATION OF I_o+I_s

We have from (3)

$$I_o+I_s = \int \mathcal{L} d^4x + \int (n^1-m^1) dx^0 dx^2 dx^3,$$

the four-dimensional integral being taken over the region $x^1 > 0$ and the three-dimensional integral over the surface $x^1 = 0$. Hence,

$$\delta(I_o+I_s) = \int \left\{ \frac{\partial \mathcal{L}}{\partial g_{\alpha\beta}} - \left(\frac{\partial \mathcal{L}}{\partial g_{\alpha\beta,\varrho}} \right)_{|\varrho} \right\} \delta g_{\alpha\beta} d^4x +$$
$$+ \int \left\{ \left[-\frac{\partial \mathcal{L}}{\partial g_{\alpha\beta,1}} + \frac{\partial(n^1-m^1)}{\partial g_{\alpha\beta}} - \left(\frac{\partial(n^1-m^1)}{\partial g_{\alpha\beta,c}} \right)_{|c} \right] \delta g_{\alpha\beta} + \right.$$
$$\left. + \frac{\partial(n^1-m^1)}{\partial g_{\alpha\beta,1}} \delta g_{\alpha\beta,1} \right\} dx^0 dx^2 dx^3. \tag{7}$$

In the region $x^1 > 0$, $\delta g_{\alpha\beta}$ is arbitrary, so the coefficient of $\delta g_{\alpha\beta}$ in the first term of (7) must vanish. This gives Einstein's equations for empty space, holding in the region outside the particle.

The $\delta g_{\alpha\beta,1}$ in the second term of (7) means the value of this field quantity ust outside the surface, and this value is arbitrary. Hence, its coefficient in (7) must vanish. In order not to have too many equations of motion coming from the action principle, we must arrange that this coefficient shall vanish identically. The first term for n^1 in (6) produces the desired effect, since from (1)

$$m^1 + 2\mathcal{G} c^{ab} \Gamma^1_{ab} = \mathcal{G} g^{1\varrho} \{ \Gamma^\sigma_{\varrho\sigma} + (2c^{\mu\nu} - g^{\mu\nu}) \Gamma_{\mu\nu\sigma} \}$$
$$= \mathcal{G} g^{1\varrho} \left\{ \frac{1}{2} g^{\mu\nu} g_{\mu\nu,\varrho} + \left(g^{\mu\nu} - \frac{2g^{1\mu} g^{1\nu}}{g^{11}} \right) \left(g_{\mu\varrho,\nu} - \frac{1}{2} g_{\mu\nu,\varrho} \right) \right\}$$
$$= \mathcal{G} g^{1\varrho} \left(g^{\mu\nu} - \frac{g^{1\mu} g^{1\nu}}{g^{11}} \right) g_{\mu\varrho,\nu}$$
$$= \mathcal{G} g^{1\varrho} c^{ab} g_{a\varrho,b}, \tag{8}$$

which does not involve any derivatives $g_{\alpha\beta,1}$.

We are left with

$$\delta(I_o+I_s) = \int \mathcal{Y}^{\alpha\beta} \delta g_{\alpha\beta} dx^0 dx^2 dx^3 \tag{9}$$

where

$$\mathcal{Y}^{\alpha\beta} = -\frac{\partial \mathcal{L}}{\partial g_{\alpha\beta,1}} + \frac{\partial(n^1-m^1)}{\partial g_{\alpha\beta}} - \left(\frac{\partial(n^1-m^1)}{\partial g_{\alpha\beta,c}} \right)_{|c}, \tag{10}$$

From (2), the first term of $\mathcal{Y}^{\alpha\beta}$ has the value

$$-\frac{1}{2} \mathcal{G} g_{\mu\nu,\varrho} \{ (g^{\mu\alpha} g^{\nu\beta} - g^{\mu\nu} g^{\alpha\beta}) g^{1\varrho} + g^{\mu\varrho} g^{\alpha\beta} g^{1\nu} +$$
$$+ g^{1\alpha} g^{\mu\nu} g^{\beta\varrho} - 2 g^{\mu\alpha} g^{\beta\varrho} g^{1\nu} \} \tag{11}$$

This expression is written in a form not symmetrical between α and β, for brevity, but it should be understood as symmetrized. The same applies to the following expression. We have from (8)

$$\frac{\partial(m^1+2\mathcal{G}c^{ab}\Gamma^1_{ab})}{\partial g_{\alpha\beta}}-\left(\frac{\partial(m^1+2\mathcal{G}c^{ab}\Gamma^1_{ab})}{\partial g_{\alpha\beta,c}}\right)_{|c}=$$

$$\mathcal{G}g_{\mu\varrho,\nu}\left(\frac{1}{2}\,g^{\alpha\beta}g^{1\varrho}c^{\mu\nu}-g^{1\alpha}g^{\varrho\beta}c^{\mu\nu}-g^{1\varrho}c^{\mu\alpha}c^{\nu\beta}\right)-(\mathcal{G}g^{1\alpha}c^{\beta c})_{|c}. \tag{12}$$

Subtracting (12), from (11), we get after some reduction

$$(c^{\mu\alpha}c^{\nu\beta}-c^{\mu\nu}c^{\alpha\beta})\mathcal{G}\Gamma^1_{\mu\nu}.$$

The surface pressure term $2\omega\,\mathcal{M}$ in $\boldsymbol{n}^1$ gives as its contribution to $\mathcal{Y}^{\alpha\beta}$

$$2\omega\frac{\partial\mathcal{M}}{\partial g_{\alpha\beta}}=\omega\mathcal{M}c^{\alpha\beta}.$$

So altogether we get for $\mathcal{Y}^{\alpha\beta}$

$$\mathcal{Y}^{\alpha\beta}=(c^{\mu\alpha}c^{\nu\beta}-c^{\mu\nu}c^{\alpha\beta})\mathcal{G}\Gamma^1_{\mu\nu}+\omega\mathcal{M}c^{\alpha\beta}. \tag{13}$$

We see that $\mathcal{Y}^{\alpha\beta}$ vanishes when α or β is one, which expresses that the surface part of I_o+I_s is independent of the $g_{1\mu}$. Thus we may write (9) as

$$\delta(I_o+I_s)=\int\mathcal{Y}^{ab}\delta g_{ab}dx^0dx^2dx^3. \tag{14}$$

THE EQUATIONS OF MOTION

The space inside the particle has to be flat, so the g_{ab} at the surface are not arbitrary. They must specify a three-dimensional surface that can be embedded in a four-dimensional flat space. Any variation of the $g_{\mu\nu}$ inside the particle and of the g_{ab} on the surface must be of the kind that comes merely from a change of the coordinate system. Thus for $x^1\leqslant 0$,

$$\delta g_{\mu\nu}=g_{\mu\nu,\varrho}\xi^\varrho+g_{\mu\varrho}\xi^\varrho{}_{|\nu}+g_{\nu\varrho}\xi^\varrho{}_{|\mu}, \tag{15}$$

where ξ^ϱ is infinitesimal and gives the change in the coordinates.

We now get from (14)

$$\delta(I_o+I_s)=\int\mathcal{Y}^{ab}(g^*_{ab,\varrho}\xi^\varrho+2g^*_{a\varrho}\xi^\varrho{}_{|b})dx^0dx^2dx^3$$

$$=-2\int(\mathcal{Y}^{ab}\Gamma^*_{ab\varrho}+\mathcal{Y}^{ab}_{|b}g^*_{a\varrho})\xi^\varrho dx^0dx^2dx^3.$$

The * is attached to field quantities at points just inside the surface when there would otherwise be ambiguity through the corresponding field quantities just outside being different.

The variation of I_I gives

$$\delta I_I=\lambda\int\mathcal{G}g^{\mu\nu}\delta g_{\mu\nu}d^4x$$

$$= 2\lambda \int \mathcal{G} g^{\mu\nu}\Big(\frac{1}{2}\, g_{\mu\nu,\varrho}\xi_\varrho + g_{\mu\varrho}\xi^\varrho{}_{|\nu}\Big)\, d^4x$$

$$= 2\lambda \int (\mathcal{G}_{|\varrho}\xi^\varrho + \mathcal{G}\xi^\varrho{}_{|\varrho}) d^4x = 2\lambda \int \mathcal{G}^* \xi^1 dx^0 dx^2 dx^3.$$

The integral here is, of course, merely the change of the four-dimensional volume inside the particle produced by the change ξ^ϱ in the coordinate system, with the equation of the surface maintained as $x^1 = 0$.

The ξ^ϱ can be arbitrary at each point of the surface, so the action principle $\delta(I_0 + I_S + I_I) = 0$ gives

$$\mathcal{Y}^{ab}\Gamma^*_{ab\varrho} + \mathcal{Y}^{ab}|_b g^*_{a\varrho} - \lambda\mathcal{G}^* g^1_\varrho = 0. \tag{16}$$

There are four equations here. For three of them, those with $\varrho = 0,2$ or 3, we may drop the *'s, so that they appear as

$$\mathcal{Y}^{ab}\Gamma_{abc} + \mathcal{Y}^{ab}|_b g_{ac} = 0. \tag{17}$$

These three must hold identically, as they merely express that the action is invariant under a change in the coordinates x^0, x^2, x^3 in the surface. One can easily check the identities with the help of Einstein's field equations just outside the surface.

We are left with just one equation, which we have in its most convenient form if we multiply (16) by $g^{*1\varrho}$; thus,

$$\mathcal{Y}^{ab}\Gamma^{*1}_{ab} - \lambda\mathcal{G}^* g^{*11} = 0. \tag{18}$$

We may also write it

$$\mathcal{Y}^{ab}\mathcal{G}^*\Gamma^{*1}_{ab} - \lambda\mathcal{M}^2 = 0 \tag{19}$$

when it is expressed in terms of invariants with respect to coordinate transformations which do not alter the surface $x^1 = 0$ and the coordinates x^0, x^2, x^3 in it.

Equation (18) or (19), with $\mathcal{Y}^{ab}$ given by (13), is the equation of motion for the surface. It is produced, together with the field equations for the outside space, by the action principle.

THE SPHERICALLY SYMMETRIC SOLUTION

We shall apply the theory to a spherically symmetric particle with its centre at rest and its radius varying with the time. The field outside the particle is then just the Schwarzschild solution of the Einstein equations, there being no possibility of gravitational waves consistent with spherical symmetry.

Let ϱ be the radius of the particle, a function of the time t. In terms of the Schwarzschild coordinates r, θ, φ, t, we take $x^1 = r - \varrho$, $x^2 = \theta$, $x^3 = \varphi$, $x^0 = t$, so that the equation of the surface is $x^1 = 0$. Then for $x^1 \geqslant 0$,

$$ds^2 = \gamma dt^2 - \frac{(dx^1 + \dot{\varrho}dt)^2}{\gamma} - r^2 d\theta^2 - r^2 \sin^2\theta d\varphi^2, \tag{20}$$

where $\gamma = 1-2m/r$. Thus,

$$g_{00} = \gamma - \frac{\dot{\varrho}^2}{\gamma}, \quad g_{10} = -\frac{\dot{\varrho}}{\gamma}, \quad g_{11} = -\frac{1}{\gamma}$$

$$g_{22} = -r^2, \quad g_{33} = -r^2 \sin^2\theta$$

with the other components of $g_{\mu\nu}$ vanishing.

We find

$$\mathcal{G} = r^2 \sin\theta, \quad \mathcal{M} = \left(\gamma - \frac{\dot{\varrho}^2}{\gamma}\right)^{1/2} r^2 \sin\theta,$$

$$c^{00} = \left(\gamma - \frac{\dot{\varrho}^2}{\gamma}\right)^{-1}, \quad c^{22} = -\frac{1}{\gamma^2}, \quad c^{33} = -\frac{1}{r^2 \sin^2\theta},$$

with c^{ab} vanishing for $a \neq b$. We find further

$$\Gamma^1_{00} = \ddot{\varrho} + m\gamma r^{-2}(1-3\dot{\varrho}^2\gamma^{-2}),$$

$$\Gamma^1_{22} = -\gamma r, \quad \Gamma^1_{33} = -\gamma r \sin^2\theta$$

with Γ^1_{ab} vanishing for $a \neq b$. From (13) we now get, making the approximation of neglecting $\dot{\varrho}^2$ but not $\ddot{\varrho}$, and using γ_0 to denote the value of γ when $x^1 = 0$ $\left(\text{namely } \gamma_0 = 1 - \frac{2m}{\varrho}\right)$:

$$\begin{aligned}
\mathcal{Y}^{00} &= -c^{00}(c^{22}\mathcal{G}\Gamma^1_{22} + c^{33}\mathcal{G}\Gamma^1_{33} - \omega\mathcal{M}) \\
&= -\sin\theta(2\varrho - \omega\gamma_0^{-1/2}\varrho^2), \\
\mathcal{Y}^{22} &= -c^{22}(c^{00}\mathcal{G}\Gamma^1_{00} + c^{33}\mathcal{G}\Gamma^1_{33} - \omega\mathcal{M}) \\
&= \sin\theta(\ddot{\varrho}\gamma_0^{-1} + \varrho^{-1} - m\varrho^{-2}\omega\gamma_0^{1/2}), \\
\mathcal{Y}^{33} &= -c^{33}(c^{00}\mathcal{G}\Gamma^1_{00} + c^{22}\mathcal{G}\Gamma^1_{22} - \omega\mathcal{M}) \\
&= \frac{Y^{22}}{\sin^2\theta},
\end{aligned}$$

with $\mathcal{Y}^{ab}$ vanishing for $a \neq b$.

The metric inside the particle must be chosen so that it describes a flat space with the same g_{ab} as (20) at $x^1 = 0$. The solution is easily seen to be

$$ds^2 = \{\gamma_0 + (1-\gamma_0^{-1})\dot{\varrho}^2\}dt^2 - (dx^1 + \dot{\varrho}dt)^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\varphi^2 \tag{21}$$

with $r = \varrho + x^1$ as before and now $-\varrho \leqslant x^1 \leqslant 0$. The metric (21) gives

$$g_{00} = \gamma_0 - \frac{\varrho^2}{\gamma_0}, \quad g_{10} = -\dot{\varrho}, \quad g_{11} = -1,$$

$$g_{22} = -r^2, \quad g_{33} = -r^2 \sin^2\theta$$

with the other components of $g_{\mu\nu}$ vanishing.

We find for $x^1 = 0$, with neglect of $\dot{\varrho}^2$ but not $\ddot{\varrho}$,

$$\mathcal{G}^* = \gamma_0^{1/2}\varrho^2 \sin\theta, \quad g^{*11} = -1,$$

$$\Gamma^{*1}_{00} = \ddot{\varrho}, \quad \Gamma^{*1}_{22} = -\varrho, \quad \Gamma^{*1}_{33} = -\varrho \sin^2\theta.$$

Substituting these results into (18), we get as the equation of motion for small $\dot{\varrho}$,

$$(2\varrho - \omega\gamma_0^{-1/2}\varrho^2)\ddot{\varrho} + 2(\ddot{\varrho}\gamma_0^{-1} + \varrho^{-1} - m\varrho^{-2} - \omega\gamma_0^{1/2})\varrho - \lambda\gamma_0^{1/2}\varrho^2 = 0. \tag{22}$$

It is of the form

$$A(\varrho)\ddot{\varrho} + B(\varrho) = 0$$

with

$$A(\varrho) = 2\gamma_0^{-3/2}(\varrho^{-1} - m\varrho^{-2}) - \frac{1}{2}\omega\gamma_0^{-1},$$

$$B(\varrho) = \gamma_0^{-1/2}(\varrho^{-2} - m\varrho^{-3}) - \omega\varrho^{-1} - \frac{1}{2}\lambda.$$

The equilibrium radius $\varrho = R$ is given by

$$B(R) = 0.$$

We may choose any value for R greater than $2m$ and any ω, and then choose λ to fit this equation.

The equilibrium is stable if

$$A\frac{dB}{d\varrho} > 0$$

for $\varrho = R$. This leads to

$$\left\{\frac{4}{\left(1-\frac{2m}{R}\right)^{1/2}}\left(\frac{1}{R} - \frac{m}{R^2}\right) - \omega\right\}\left\{\omega - \frac{1}{\left(1-\frac{2m}{R}\right)^{3/2}}\left(\frac{2}{R} - \frac{6m}{R^2} + \frac{5m^2}{R^3}\right)\right\} > 0.$$

We can satisfy this condition by choosing ω to lie between the two quantities

$$\frac{4}{\left(1-\frac{2m}{R}\right)^{1/2}}\left(\frac{1}{R} - \frac{m}{R^2}\right), \qquad \frac{1}{\left(1-\frac{2m}{R}\right)^{3/2}}\left(\frac{2}{R} - \frac{6m}{R^2} + \frac{5m^2}{R^3}\right),$$

except in the case when the two quantities coincide. This case occurs when

$$\frac{R}{m} = \frac{1}{2}(3+\sqrt{3}), \tag{24}$$

which gives a value for R just a little greater than the Schwarzschild radius $2m$.

CONCLUSION

We may choose any value for R greater than $2m$, excluding the value (24), and then choose ω and λ to fit the conditions. We then have a theory for the motion of a particle with the radius R. The particle is stable for small disturbances that preserve its spherical symmetry. Further work would be needed to check whether it is still stable if the spherical symmetry is disturbed.

REFERENCES

[1] P. A. M. Dirac, *Proc. Roy. Soc.* **A** (in the press.)
[2] A. Lees, *Phil. Mag.* **28**, 385 (1939).

DISCUSSION

A. Schild:

Would you wish to put conditions on the particle that make the mass surface density always positive?

P. A. M. Dirac:

I would like to have a Hamiltonian which is positive definite. That would be the natural way of securing that the motion is always stable and that you don't get runaway solutions. I have big doubts as to whether it is possible to have a positive definite Hamiltonian, because the Newtonian energy is negative; but one should have the aim of getting a positive definite Hamiltonian. If that cannot be satisfied, then I would like to have, at any rate, a positive definite surface energy.

H. Bondi:

I'll abuse my position as chairman to ask a question myself. I am rather worried about the assumption of the vanishing field inside. When one considers any extended body, then clearly the motion of that body will depend on the equation of state one assumes for the material. We know this in fact from the effect of the tidal friction of the earth on the motion of the moon. Now, the type of equation of state one wants should definitely be what in electrical network theory is called passive, that is to say, that no energy from other sources of energy is fed in, but that it is a natural response, purely reactive, or, perhaps, dissipative, as in the case of tidal friction. Now in electrical theory we know that a purely reactive network, namely a perfectly conducting shell, will so distribute its charges as to give zero field inside.

But in gravitation, precisely the opposite is the case. If we assume, for example, the earth to be a shell, moving in the field of the sun, the earth falling freely; then, of course, the residual forces on the earth are the tidal forces. And if this were a shell with particles in it, that could move freely, then these particles which congregate here would make tides, which would increase the field inside, and would not abolish it. So, in the natural motion, and, probably, in any passive situation, we would get something much closer to paramagnetic behaviour, where we get an increase of the field inside, than to shielding which corresponds to electrostatics.

P. A. M. DIRAC:

Those remarks of yours would rather suggest that my particle would not be stable for disturbances which are not spherically symmetrical. I think that would be the natural interpretation of your remarks.

H. BONDI:

Actual instability would depend on the properties of the material, but an enhancement of the inhomogeneity of the field would certainly occur.

P. A. M. DIRAC:

But in the way that the theory is at present formulated there has to be no field inside, no matter what disturbances occur outside.

H. BONDI:

Well, I fear that this is in some sense unphysical.

P. A. M. DIRAC:

Yes. You could get a more physical theory by bringing in an action for the internal region corresponding to some physical conditions. That would complicate the theory, but make it more physical.

V. A. FOCK:

I should like to ask the following question. You are considering very small particles, because they are nearly point particles and their mass m is very small.

P. A. M. DIRAC:

They need not be very small.

V. A. FOCK:

What kind of particles are these? Are they quantum particles or classical particles? I cannot imagine particles of such small size that are not quantum particles. And if so, how are quantum-mechanical considerations to be introduced in your theory?

P. A. M. DIRAC:

I would like to answer first that these particles do not have to be small. I have not anywhere made the assumption that they are small. We have exact equations of motion which would also apply if the particles are very far from being small. If they are small, I would agree that we ought to bring in quantum theory; and that brings in very many new problems.

V. A. FOCK:

But in the case that they are not small, they must have a very large density; so large that perhaps the notion of density is no more applicable in this case.

P. A. M. DIRAC:

I do not think the density would have to be large. The density could very well be small also. With large particles you get into the difficulty that two of them may collide, and then you would need some new equations of motion to describe that situation.

J. A. WHEELER:

The work here is in line with the old Lorentz model of the electron which, however, ran into difficulty. And it's very nice to see that if one goes into general relativity one has possibilities to construct objects that do not have that difficulty. In this connection it might be mentioned that there are also two other kinds of objects that one can construct within the framework of general relativity, namely geons and topological objects—handles or wormholes. In those two cases one has examined the question of stability. And the questions which you have brought up in such an interesting way here have also been looked at in those two cases; namely the scattering of radiation by such objects; and the interaction of such objects with other fields. However, of course, for all three objects (the problem that you speak of here, and in the case of the geon and in the case of the wormhole) one is talking of things that have not the slightest connection with particles of the real physical world, but speaking rather of *models* which are of great interest in understanding the nature and implications of relativity. However, it is a bit puzzling to me to understand why one would introduce a *new* physical phenomenon like surface tension here, when already in the framework of well-established general relativity and electromagnetism one has the tools at hand with which to construct model objects of considerable interest in their own right. I raise this issue of this new physical term particularly because I myself do not understand what governs the law of aggregation of this substance that causes the surface tension. What decides into how many spheres it will collect? What decides — if it wants to break up into pieces — whether this is allowed or forbidden? This is why it would seem to me simpler not to bring in the surface tension to construct such models.

P. A. M. DIRAC:

You made reference to the difficulties of Lorentz. He could have avoided these difficulties if he had used an action principle. And the only way that I have succeeded in avoiding these Lorentz difficulties is by using an action principle, all the way through. Now, this introduction of the surface tension, or rather surface pressure, enables one to have particles of any size. They could be small particles, not much bigger than the Schwarzschild particle, or they could be very large. I don't think you could have the small particles without bringing in something like this surface pressure, or some other non-Einstein terms. I suppose your geons are extremely large, aren't they?

J. A. WHEELER:

Comparable in size to the sun or larger *if* they are to be analyzed without getting into the important problems of *quantizing* general relativity.

P. A. M. DIRAC:

Yes. I would agree that this model is rather remote from physical reality, but I wish to say again that we are working in a new field and we make the simplest assumptions which lead to a physically sensible theory. We can add on further terms to the action later on if we want them.

B. S. DEWITT:

I should like to make a comment about a matter of principle, and this is also in answer to Prof. Wheeler. I think the example Prof. Dirac has shown us is a very excellent example of the intimate relation which exists between the physical description of the geometry of space-time and the dynamical behaviour of bodies which occupy space-time. I should like to suggest that instead of pushing 100% in the direction of examining only empty space, we should study more, perhaps, the actual description of material objects which occupy space-time. This, for example, I found very useful in the analysis of the Bohr–Rosenfeld problem: to really describe the elastic test bodies that one uses. One learns very interesting new things this way. After all, our original ideas of distance and Riemannian geometry, are based on our experience with physical objects like rods and clocks. And, the effort to describe these objects in manifestly covariant language and to learn how they behave is worth it, I think.

P. A. M. DIRAC:

I agree completely.

C. Møller:

May I ask if there is in this model a definite relation between the mass, the constant in the Schwarzshild solution outside, and the radius of this object.

P. A. M. Dirac:

No. You can have it independent by suitably choosing the surface pressure.

A. Lichnerowicz:

En 1946 ou 1947 j'ai fabriqué un modèle de l'électron avec de la matière à l'intérieur et une densité superficielle correspondante. Il y a certains rapports avec ce modèle; mais dans mon modèle il y a de la matière fluide à l'intérieur au lieu du vide.

P. A. M. Dirac:

Do you have the gravitational field taken into account?

A. Lichnerowicz:

Yes, in the interior. But also a tensor, a repulsive tensor on the surface.

P. A. M. Dirac:

Do you have an action principle?

A. Lichnerowicz:

Yes, it consists of two parts.