Notes on Supersymmetry (following Joseph Bernstein)

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Introduction

These notes are based on the lectures given by Joseph Bernstein in the fall of 1996 at the Institute for Advanced Study. They contain both more and less than those lectures. Less: representations of super Poincaré groups, the Coleman-Mandula theorem and examples of super symmetric Lagrangians that were contained in Bernstein's lectures have been incorporated into other texts of this volume. More: more details have been given, and we have paid special attention to having precise and coherent sign conventions.

The purpose of Bernstein's lectures was to present, in a coherent and systematically mathematical way, material that exists in the physics literature. We have tried to remain faithful to that purpose in these notes. Our approach to super mathematics follows that of Leites [**1980**] and of the Chapters 3 and 4 of Manin [**1988**]. We have tried to avoid duplicating those works. We give complete proofs mainly when those references only sketch them. In our treatment of super algebra (§1), we have emphasized the methods which enable one to hide the signs. In our treatment of super calculus, we have emphasized the use of the functor of points. Our systematic treatment of signs has led us to conventions that at places differ from those used by the physicists. We have tried to point out as they occur those places where our conventions differ from theirs.

Another subtlety arises when one considers $\mathbb{Z}$-graded super objects. There are two fairly natural choices for the sign rules. Suppose x and y are elements with x of $\mathbb{Z}$-degree n and parity p and y of $\mathbb{Z}$-degree m and parity q. When permuting x and y, one may introduce either the sign $(-1)^{(n+p)(m+q)}$, or the sign $(-1)^{nm+pq}$. Both conventions have been used and have their advantages, as is explained in the Appendix to Section 1. Our point of view makes the second rule the natural one, and we adopt it.

We turn next in Section 2 to the notion of super manifolds. We give two basic approaches to these objects. The first uses the point of view of describing supermanifolds by their sheaf of "local functions". This allows us to give a description by coordinate charts and gluing functions. Our second, more geometric, approach uses the functor of points: a supermanifold M is completely determined by giving functorially in S the set of its S-points $M(S) := \mathrm{Hom}(S, M)$. Using this point of view we define super Lie groups. We then discuss in Section 3 the basic objects of differential topology and geometry: vector bundles, the tangent bundle TM and cotangent bundle Ω^1_M. Finally, we discuss densities, i.e. things that can be integrated over a super manifold. We show that they are simply sections of the Berezinian $\mathrm{Ber}(\Omega^1_M)$ twisted by the orientation cover of M. This comes down to showing a change of variable formula for integration over open subsets $U \subset \mathbb{R}^{p|q}$. Namely, that under an isomorphism $\varphi\colon U \to V$ between such open sets we have

$$\int_V f du = \int_U \varphi^*(f) j(\varphi) du$$

where $j(\varphi) = \pm\mathrm{Ber}(d\varphi)$, the sign being the effect of φ on the orientation of the underlying reduced manifold. Then we introduce integral forms which can be integrated over codimension $(p|0)$ submanifolds.

In the fourth section, we discuss real structures and complexifications. When computing, physicists pass freely between real and complex variables. In the simplest cases, this reduces to using a moving frame which is a basis of the *complexified* tangent bundle. The justification can in other cases be more subtle, and one of the main purposes of this section is to give a precise framework in which the computations make sense.

We begin the section with the super analogues of Hilbert spaces and adjoints of operators. Such objects occur naturally in the quantization of symplectic super manifolds, either finite dimensional or, as in field theory, infinite dimensional. Our wish to have a formalism parallel to the classical one, with real functions giving rise to self-adjoint operators, and with the product of functions corresponding to the product of operators, up to terms of higher order in $\hbar$, leads us to a rather strange convention, for which the eigenvalues of an odd "self-adjoint" operator are in $i^{1/2}\mathbb{R}$. Our conventions differ from those used by the physicists.

Our next topic is cs-manifolds: supermanifolds which are complex in the odd direction, and integration on them. Integration on cs-manifolds occurs in manifestly supersymmetric descriptions of some Lagrangians (see e.g. [I-**Supersolutions**, §5.3]), and in some "path integrals".

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CHAPTER 1
Multilinear Algebra

§1.1. The sign rule

The ground rule of superalgebra is that all objects considered should be mod 2 graded, and that in all classical formulas, whenever the order in which two odd quantities appear is changed, a minus sign appears..

Here are some examples. For simplicity, we fix a ground field k of characteristic zero, which the reader may safely assume to be $\mathbb{R}$ or $\mathbb{C}$.

A *super vector space* is a $\mathbb{Z}/2\mathbb{Z}$-graded vector space:

$$V = V_0 \oplus V_1 \ .$$

An element v of V_0 (resp. V_1) is said to be even (resp. odd). Its parity in $\mathbb{Z}/2\mathbb{Z}$ is denoted $p(a)$. A *morphism* from a super vector space V to a super vector space W is a $\mathbb{Z}/2\mathbb{Z}$-degree preserving linear map from V to W. With this notion of morphisms, super vector spaces form an abelian category. The *parity reversing functor* $\prod$ is defined by

$$(\textstyle\prod V)_0 := V_1, \ (\textstyle\prod V)_1 := V_0 \ .$$

If V is finite dimensional, we define its *dimension* to be the pair of integers ($\dim V_0$, $\dim V_1$), usually denoted $m_0|m_1$ for $m_i = \dim(V_i)$.

The *tensor product* of super vector spaces V and W is the tensor product of the underlying vector spaces, with the $\mathbb{Z}/2\mathbb{Z}$-grading

$$(V \otimes W)_k = \bigoplus_{i+j=k} V_i \otimes W_j \ .$$

The tensor product functor is additive and exact in each variable. It has a unit object: if $\underline{1}$ is the vector space k in even degree, $\underline{1} \otimes V$ and $V \otimes \underline{1}$ are canonically isomorphic to V, by $1 \otimes v$, $v \otimes 1 \mapsto v$. It is associative: $(u \otimes v) \otimes w \mapsto u \otimes (v \otimes w)$ is a canonical isomorphism from $(U \otimes V) \otimes W$ to $U \otimes (V \otimes W)$. The sign rule appears in the definition of the *commutativity isomorphism*

$$c_{V,W} \colon V \otimes W \to W \otimes V \tag{1.1.1}$$

which is defined by

$$v \otimes w \longmapsto (-1)^{p(v)p(w)} w \otimes v$$

for v and w homogeneous. From now on homogeneity of the relevant quantities will be tacitly assumed when writing formulas.

Let $(V_i)_{i\in I}$ be a finite family of super vector spaces. An ordering of I can be identified with a bijection σ: $[1,n] \to I$, where $n = |I|$. A tensor product of the V_i is obtained by choosing an ordering σ of I, and by parenthesizing the expression $V_{\sigma(1)} \otimes V_{\sigma(2)} \otimes \ldots \otimes V_{\sigma(n)}$. If $T' = V_{\sigma(1)} \otimes \ldots \otimes V_{\sigma(n)}$ and $T'' = V_{\tau(1)} \otimes \ldots \otimes V_{\tau(n)}$ are two tensor products of the V_i, whichever way one composes associativity and commutativity isomorphisms to obtain an isomorphism from T' to T'', one always obtains the same isomorphism from T' to T''. For v_i homogeneous in V_i, it is given by

(1.1.2) $$v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma(n)} \longmapsto (-1)^N v_{\tau(1)} \otimes \ldots \otimes v_{\tau(n)},$$

where N is the number of pairs of indices $i, j \in I$ such that v_i and v_j are odd and $\sigma^{-1}(i) < \sigma^{-1}(j)$ while $\tau^{-1}(i) > \tau^{-1}(j)$. This is the basic reason why the sign rule works.

Examples. $c_{WV} \circ c_{VW}$ is the identity of $V \otimes W$; the "hexagon" diagram

$$\begin{tikzcd} U \otimes V \otimes W \arrow[rr, "c_{U,V\otimes W}"] \arrow[dr, "c_{U,V}"'] & & V \otimes W \otimes U \\ & V \otimes U \otimes W \arrow[ur, "c_{U,W}"'] & \end{tikzcd}$$

is commutative (if we had not suppressed the associativity isomorphism, this diagram would have had six sides).

A *super algebra* over k is a super vector space A, given with a morphism, called the product: $A \otimes A \to A$. By the definition of morphisms, the parity of the product of homogeneous elements of A is the sum of the parities of the factors.

The super algebra A is *associative* if $(xy)z = x(yz)$. A *unit* is an even element 1 (i.e. a morphism $\underline{1} \to A$) such that $1x = x1 = x$. By "super algebra", we will usually mean an associative super algebra with unit. For such a super algebra A, a left (resp. right) A-module is a super vector space M given with a morphism, also called product: $A \otimes M \to M$ (resp. $M \otimes A \to M$) obeying the usual identities, expressing that M is a module over A considered as an ordinary algebra. The sign rule has not entered so far. It changes the definition of commutativity: the super algebra A is *commutative* if the product of homogeneous elements obeys

$$xy = (-1)^{p(x)p(y)} yx .$$

As classically, if A is commutative, a left A-module is also a right A-module. The passage from left to right uses the sign rule:

(1.1.3) $$m \cdot a := (-1)^{p(m)p(a)} a \cdot m. .$$

We will just say "*module*" for left (or right) module. The *tensor product* of A-modules

$$M \otimes_A N := (M \text{ as right module}) \otimes_A (N \text{ as left module})$$

is again an A-module, and the tensor product functor is associative, commutative and has a unit, the A-module A. The commutativity isomorphism is given by (1.1.1).

If A is an algebra, the *opposite* algebra A^0 is A, with the product

$$x \cdot_{\text{opp}} y := (-1)^{p(x)p(y)} y \cdot x \tag{1.1.4}$$

An element x of A is *central* if its homogeneous components obey $xy = (-1)^{p(x)p(y)} yx$ for all y in A. The *tensor product* of super algebras A, B is $A \otimes B$, with the product

$$(a \otimes b)(c \otimes d) := (-1)^{p(b)p(c)} ac \otimes bd\ . \tag{1.1.5}$$

1.1.6 Example. The algebra D with basis 1, η, with η odd and $\eta^2 = 1$, while classically commutative, is not a commutative super algebra. Its center is reduced to k and the opposite algebra has $\eta^2 = -1$. It is a *super skew field*: nonzero homogeneous elements are invertible.

Many constructions and theorems of algebra extend to the super case. Making the story interesting, there are also new phenomena. Keeping track of signs can be a nuisance. We will explain two methods to hide them, and to extend classical results to the super case without having to work, viz. in 1.2, the categorical method, and in 1.7, the "even rules" method.

§1.2. Categorical approach

Many constructions and theorems of multilinear algebra make sense and remain true in any abelian category $\mathcal{T}$ with a biadditive tensor product which is associative, commutative and with a unit $\underline{1}$ (a *tensor category*). Three examples of tensor categories are (i) vector spaces, (ii) super vector spaces and (iii) modules over a commutative super algebra A. The generalization to the categorical setting can require extra assumptions. It involves no sign. One can then specialize to the case of super vector spaces, or more generally to the case of modules over a commutative super algebra: the signs have been hidden in the commutativity isomorphism (1.1.1).

That $\mathcal{T}$ is a tensor category means that one can speak of the tensor product of a finite family of objects $(V_i)_{i \in I}$ with standard properties. No ordering of I has to be prescribed. For the empty family, one gets the unit object.

For super vector spaces, the tensor product $\bigotimes_{i \in I} V_i$ can be understood as follows. For each ordering σ: $[1, n] \xrightarrow{\sim} I$ of I, one has a tensor product

$$T_\sigma := V_{\sigma(1)} \otimes \ldots \otimes V_{\sigma(n)}\ ,$$

For variable σ, one has a transitive system of isomorphisms among the T_σ, and $\bigotimes_{i \in I} V_i$ is the "common value" of the T_σ. More pedantically, it is the projective limit of the T_σ: an element of $\bigotimes_{i \in I} V_i$ is the data for each σ of t_σ in T_σ, the t_σ corresponding to each other by (1.1.2). An inductive limit would have worked as well. Notice that in the case when all the V_i are the same super vector space, the symmetric group of I acts on the family $(V_i)_{i \in I}$, and hence (by transport of structure) on the tensor product $\otimes_{i \in I} V_i$.

Example. Let $(X_i)_{i \in I}$ be a finite family of nice topological spaces, say finite CW complexes. The Künneth formula in cohomology with coefficients in k says that

$$H^*\Big(\prod_{i \in I} X_i\Big) = \bigotimes_{i \in I} H^*(X_i)\ . \tag{1.2.1}$$

Here, $H^*(X_i)$ is viewed as a super vector space, the $\mathbb{Z}/2\mathbb{Z}$-grading being given by the cohomological degree modulo 2. In (1.2.1), no ordering of I is needed. However, to name decomposable elements of the tensor product, a convenient way is to choose an ordering σ: $[1, n] \to I$ of I and elements $x_i \in H^*(X_i)$. The element $x_{\sigma(1)} \otimes \ldots \otimes x_{\sigma(n)}$ of $\otimes H^*(X_i) = H^*(X_{\sigma(1)}) \otimes \ldots \otimes H^*(X_{\sigma(n)})$ depends on σ.

§1.3. Examples of the categorical approach

We will assume that $\mathcal{T}$ is a k-linear tensor category, i.e. that $k \subset \mathrm{End}(\underline{1})$. We now give examples of how classical constructions can be made categorical, then specialized to the case of super vector spaces.

1.3.1. A *Lie algebra* in $\mathcal{T}$ is an object $\mathcal{L}$ of $\mathcal{T}$, given with a morphism $[\ ,\ ]$: $\mathcal{L} \otimes \mathcal{L} \to \mathcal{L}$ obeying the usual identities. They are interpreted as follows:

Antisymmetry: vanishing of the sum of the bracket $[\ ,\ ]$: $\mathcal{L} \otimes \mathcal{L} \to \mathcal{L}$ and of the opposite bracket, by which we mean the composite $[\ ,\ ] \circ c_{\mathcal{L},\mathcal{L}}$: $\mathcal{L} \otimes \mathcal{L} \to \mathcal{L} \otimes \mathcal{L} \to \mathcal{L}$.

Jacobi identity: vanishing of the sum of three iterated brackets, that is to say the vanishing of the composite of $[\ ,[\ ,\ ]]$: $\mathcal{L} \otimes \mathcal{L} \otimes \mathcal{L} \to \mathcal{L}$ and $1 + \sigma + \sigma^2$, for σ the automorphism of $\mathcal{L} \otimes \mathcal{L} \otimes \mathcal{L}$ associated to the cyclic permutation of the factors (see §1.2).

For super vector spaces, these conditions read

$$[x, y] + (-1)^{p(x)p(y)}[y, x] = 0$$
$$[x, [y, z]] + (-1)^{p(x)p(y)+p(x)p(z)}[y, [z, x]] + (-1)^{p(x)p(z)+p(y)p(z)}[z, [x, y]] = 0\ .$$

1.3.2. An *algebra* in $\mathcal{T}$ is an object A of $\mathcal{T}$, given with a product morphism $A \otimes A \to A$. Associativity, commutativity and unit are expressed by the commutativity of standard diagrams. For instance, a unit element is a morphism $\underline{1} \xrightarrow{u} A$ such that $A = \underline{1} \otimes A \xrightarrow{u} A \otimes A \to A$ and $A = A \otimes \underline{1} \xrightarrow{u} A \otimes A \to A$ are the identity. Commutativity means that the product $A \otimes A \to A$ is equal to the composite $A \otimes A \to A \otimes A \to A$ of the product with $c_{A,A}$.

By "algebra", we will usually mean associative algebra with a unit element. For such an algebra A, a left (resp. right) A-module in $\mathcal{T}$ is an object M of $\mathcal{T}$ given with a product morphism $A \otimes M \to M$ (resp. $M \otimes A \to M$) making standard diagrams commute. The tensor product over A of a right A-module M with a left A-module N is the cokernel of the difference of the two natural maps $M \otimes A \otimes N \to M \otimes N$.

For $\mathcal{T}$ the tensor category of super vector spaces, we recover the super algebras and the super modules of 1.1, and the tensor product over A.

1.3.3. If A is commutative, no distinction is to be made between left and right A-modules, and we will just say A-module. The tensor product of two A-modules is again an A-module. The A-modules form a tensor category $\mathcal{T}_A$. Its unit is the A-module A. An algebra in $\mathcal{T}_A$ is called an A-algebra.

For $\mathcal{T}$ the tensor category of super vector spaces, we recover the rule (1.1.3) to transform a left module structure over a commutative super algebra into a right module structure.

If $A \to A'$ is a morphism of commutative algebras, the extension of scalars $M \mapsto M_{(A')} := A' \otimes_A M \colon \mathcal{T}_A \to \mathcal{T}_{A'}$ is compatible with tensor product. It hence transforms algebras into algebras, modules into modules, and Lie algebras into Lie algebras.

1.3.4. The symmetric power $\mathrm{Sym}^n(V)$ of an object V of $\mathcal{T}$ is a quotient of $V \otimes \ldots \otimes V$ (n factors): it is the coinvariants of the action of the symmetric group S_n described in §1.2. If infinite direct sums exist in $\mathcal{T}$, $\mathrm{Sym}^*(V) := \oplus \mathrm{Sym}^n(V)$ $(n \geq 0)$ is a commutative algebra. It is the commutative algebra freely generated by V: for any commutative algebra A,

$$\mathrm{Hom}_{\mathrm{alg}}(\mathrm{Sym}^*(V), A) \xrightarrow{\sim} \mathrm{Hom}(V, A) .$$

For the case when V is a super vector space, one has

$$\mathrm{Sym}^*(V) = \mathrm{Sym}^*(V_0) \otimes \wedge^*(V_1)$$

where the operations on the right-hand-side are the usual ones in the category of ordinary vector spaces. The parity is the parity of the exterior degree. The degree $*$ is the total degree.

1.3.5. Similarly, we define the exterior power $\wedge^n V$ as the coinvariants of the action of S_n on $V \otimes \ldots \otimes V$ (n factors) by

$$\sigma \longmapsto \varepsilon(\sigma) \cdot \text{action associated to the permutation of the factors,}$$

for $\varepsilon(\sigma) \colon S_n \to \{\pm 1\}$ the sign character.

1.3.6 Graded Objects. Define the tensor category $\mathcal{T}^{\mathrm{gr}}$ of graded objects of $\mathcal{T}$ to be the category of $\mathbb{Z}$-graded objects of $\mathcal{T}$, with the tensor product and associativity isomorphism of $\mathcal{T}$, and the following commutativity isomorphism: for V (resp. W) purely of degree n (resp. m),

$$c_{V,W} \text{ in } \mathcal{T}^{\mathrm{gr}} := (-1)^{nm} c_{V,W} \text{ in } \mathcal{T} \colon V \otimes W \to W \otimes V .$$

For V in $\mathcal{T}$, the exterior algebra $\wedge^* V$ is an object of $\mathcal{T}^{\mathrm{gr}}$, with $\wedge^n V$ in degree n. As object of $\mathcal{T}^{\mathrm{gr}}$, it can be described as the symmetric algebra of the object of $\mathcal{T}^{\mathrm{gr}}$ reduced to V in degree 1.

For $\mathcal{T}$ the category of super vector space, if V is a super vector space, $\wedge^* V$ is with this definition a commutative algebra in the category (super vector spaces)$^{\mathrm{gr}}$ of graded super vector spaces, with the commutativity isomorphism

$$c_{V,W} \colon V \otimes W \to W \otimes V \colon v \otimes w \longmapsto (-1)^{\deg(v)\deg(w) + p(v)p(w)} w \otimes v . \tag{1.3.6.1}$$

For super vector spaces, another definition is sometimes used in the literature: $\wedge^n V := \mathrm{Sym}^n \left(\prod V\right)$. If n is odd, this definition differs by a parity change from the definition we use. For a description of the essential equivalence between the two points of view, we refer to the appendix.

1.3.7 Poincaré-Birkhoff-Witt Theorem. If A is an associative algebra in $\mathcal{T}$, the bracket $[x,y] := xy - yx$: $A \otimes A \to A$ is a Lie bracket. The defining formula for $[\ ,\ ]$ is to be interpreted as follows: it is the difference between the product $A \otimes A \to A$ and its composition with $c_{A,A}$: $A \otimes A \to A \otimes A$. For super algebras, this gives

$$[x,y] := xy - (-1)^{p(x)p(y)} yx \ .$$

Classically, the universal enveloping algebra $\mathcal{UL}$ of a Lie algebra $\mathcal{L}$ is defined by the universal property that for any associative algebra with unit A, one has

$$\mathrm{Hom}_{\mathrm{algebra}}(\mathcal{UL}, A) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Lie}}(\mathcal{L}, A) \ .$$

The algebra $\mathcal{UL}$ is a quotient of the tensor algebra $T\mathcal{L} := \bigoplus_{n \geq 0} \mathcal{L}^{\otimes n}$. One filters $\mathcal{UL}$ by the images of the $\bigoplus_{i \leq n} \mathcal{L}^{\otimes i}$. The Poincaré-Birkhoff-Witt theorem says that there is an isomorphism of graded algebras $\mathrm{Sym}^*(\mathcal{L}) \xrightarrow{\sim} \mathrm{Gr}\, \mathcal{UL}$. The map from $\mathrm{Sym}^*\mathcal{L}$ to $F^n\mathcal{UL}/F^{n-1}\mathcal{UL}$ lifts to $F^n\mathcal{UL}$, by

$$x_1 \cdots x_n \text{ in } \mathrm{Sym}^n\mathcal{L} \longmapsto \frac{1}{n!} \sum_\sigma x_{\sigma 1} \cdots x_{\sigma n} \text{ in } \mathcal{UL} \ . \tag{1.3.7.1}$$

The Poincaré-Birkhoff-Witt theorem is equivalent to the statement that the maps (1.3.7.1) define a vector space isomorphism from $\mathrm{Sym}^*(\mathcal{L})$ to $\mathcal{UL}$, with the product in $\mathcal{UL}$ agreeing with that of $\mathrm{Sym}^*\mathcal{L}$, modulo terms of lower degree. Transporting the product of $\mathcal{UL}$ by this isomorphism, one obtains a product $*$ on $\mathrm{Sym}^*\mathcal{L}$.

Let $\mathcal{L}$ be a Lie algebra in a tensor category $\mathcal{T}$. Our goal is to generalize Poincaré-Birkhoff-Witt in this categorical setting. To do this we will first define the analogue of the product $*$ (with induced multiplication on the associated graded being the usual product in $\mathrm{Sym}^*\mathcal{L}$), then prove that $(\mathrm{Sym}^*\mathcal{L}, *)$ has the universal property required of an universal enveloping algebra.

The components

$$*\colon \mathrm{Sym}^p\mathcal{L} \otimes \mathrm{Sym}^q\mathcal{L} \longrightarrow \bigoplus_{i \leq p+q} \mathrm{Sym}^i\mathcal{L}$$

of the product $*$ are defined by induction on $p+q$. In defining morphisms below we use the following conventions: (a) an iterated product $x_1 * \cdots * x_N$ is defined by induction on N to be $x_1 * (x_2 * \cdots * x_N)$; (b) formulas stand, in a hopefully self evident way, for morphisms from $\mathcal{L}^{\otimes N}$ to $\mathrm{Sym}^*\mathcal{L}$. For instance, $x_{\sigma 1} * \cdots * x_{\sigma N}$ stands for the composite of $\mathcal{L}^{\otimes N} \to \mathrm{Sym}^*\mathcal{L}$ considered in (a) and of the action (1.2) of σ^{-1} on $\mathcal{L}^{\otimes N}$. For $\mathcal{T}$ the tensor category of super vector spaces, before interpreting the given formula as giving images of elements, each term should be multiplied by the sign prescribed by the sign rule. (This enters, of course, in the action of the symmetric group S_N on $\mathcal{L}^{\otimes N}$.) We will also use the fact that defining a map with source $\mathrm{Sym}^p\mathcal{L} \otimes \mathrm{Sym}^q\mathcal{L}$ amounts to the same thing as defining a map with source $\mathcal{L}^{\otimes p} \otimes \mathcal{L}^{\otimes q}$, which is $S_p \times S_q$-invariant. The latter is what is really described.

For $p = 1$, and for $y_1 \cdots y_q$ denoting the projection $\mathcal{L}^{\otimes q} \to \mathrm{Sym}^q(\mathcal{L})$, we define $*$ by

$$\begin{aligned} x * (y_1 \cdots y_q) &:= (xy_1 \cdots y_q \text{ in } \mathrm{Sym}^{q+1}\mathcal{L}) \\ &\quad + \frac{1}{(q+1)!} \sum_\sigma \sum_i (q-i+1) y_{\sigma 1} * \cdots * [x, y_{\sigma i}] * \cdots * y_{\sigma q} \ . \end{aligned} \tag{1.3.7.2}$$

For $p = 0$, $*$ is defined so that $\underline{1} = \mathrm{Sym}^0\mathfrak{L} \hookrightarrow \mathrm{Sym}^*\mathfrak{L}$ is a left unit, and the case $p \geq 2$ is reduced to the case $p = 1$ by

$$(x_1 \cdots x_p) * Y = \frac{1}{p!} \sum_\sigma x_{\sigma 1} * \cdots * x_{\sigma p} * Y \tag{1.3.7.3}$$

for Y in $\mathrm{Sym}^*\mathfrak{L}$. Note that for $q = 0$, (1.3.7.2) is a right unit property for $\underline{1} = \mathrm{Sym}^0\mathfrak{L} \hookrightarrow \mathrm{Sym}^*\mathfrak{L}$. By (1.3.7.3), that $\underline{1} = \mathrm{Sym}^0\mathfrak{L} \hookrightarrow \mathrm{Sym}^*\mathfrak{L}$ is a right unit amounts to having for all p

$$x_1 \cdots x_p \text{ in } \mathrm{Sym}^p\mathfrak{L} = \frac{1}{p!} \sum_\sigma x_{\sigma 1} * \cdots * x_{\sigma p}\ . \tag{1.3.7.4}$$

This identity is proved by induction on p. By induction, the right side of (1.3.7.4) is

$$\frac{1}{p} \sum_i x_i * (x_1 \cdots \hat{x}_i \cdots x_p)\ .$$

If we expand each term by (1.3.7.2), the first summands add up to $x_1 \cdots x_p$ in $\mathrm{Sym}^p\mathfrak{L}$, and the second summands cancel by the antisymmetry of the bracket.

Lemma 1.3.7.5. *The algebra* $(\mathrm{Sym}^*\mathfrak{L}, *)$ *is associative, and the inclusion of* $\mathfrak{L}$ *into* $\mathrm{Sym}^*\mathfrak{L}$ *is a morphism of Lie algebras.*

The second statement results from the definition (the case $q = 1$ of (1.3.7.2))

$$x * y := xy + \frac{1}{2}[x, y]\ ,$$

Indeed, the product xy: $\mathfrak{L} \otimes \mathfrak{L} \to \mathrm{Sym}^2\mathfrak{L}$ is symmetric while the second term $\frac{1}{2}[xy]$ is antisymmetric.

Associativity means that for all p, q and r, the two maps $(x*y)*z$ and $x*(y*z)$ from $\mathrm{Sym}^p\mathfrak{L} \otimes \mathrm{Sym}^q\mathfrak{L} \otimes \mathrm{Sym}^r\mathfrak{L}$ to $\mathrm{Sym}^*\mathfrak{L}$ are equal. We will prove this equality by induction on $p + q + r$. For p, q or $r = 0$, associativity follows from the unit property.

Assuming the induction hypothesis for $p + q + r \leq n$, we first show the vanishing of the map

$$\{x_1, \ldots, x_{n+1}\} := x_1 * x_2 * x_3 * \cdots * x_n - x_2 * x_1 * x_3 \cdots * x_n - [x_1, x_2] * x_3 * \cdots * x_n$$

from $\mathfrak{L}^{\otimes(n+1)}$ to $\mathrm{Sym}^*\mathfrak{L}$. The morphism $\{x_1, \ldots, x_{n+1}\}$ is antisymmetric in the first two variables . It is symmetric in the last $(n-1)$ variables. Indeed, for $3 \leq i \leq n$, the induction hypothesis gives

$$\begin{aligned}\{x_1, \ldots, x_i, x_{i+1}, \ldots, x_{n+1}\} - \{x_1, \ldots, x_{i+1}, x_i, \ldots, x_{n+1}\} &= \\ \{x_1, \ldots, [x_i, x_{i+1}], \ldots, x_{n+1}\} &= 0\ .\end{aligned}$$

We next check that

$$\{x, y, z, \cdots\} + \{y, z, x, \cdots\} + \{z, x, y, \cdots\} = 0\ .$$

This sum is indeed

$$\begin{aligned} &x*[y,z]*\cdots && +y*[z,x]*\cdots && +z*[x,y]*\cdots \\ &-[y,z]*x*\cdots && -[z,x]*y*\cdots && -[x,y]*z*\cdots = \\ &[x[y,z]]*\cdots && +[y,[z,x]]*\cdots && +[z[x,y]]*\cdots = 0 \end{aligned}$$

These properties ensure that for suitable morphisms A_i, one has

(1.3.7.6) $$\{x_i, x_j, x_1, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_{n+1}\} = A_i - A_j \ .$$

For any fixed i, one can for instance take $A_\imath = 0$ and define A_j for $j \neq i$ by the (i,j) instance of (1.3.7.6).

The definition (1.3.7.2) and (1.3.7.4) give

(1.3.7.7) $$\begin{aligned} \frac{1}{n!}x*\sum_\sigma y_{\sigma 1}*\cdots*y_{\sigma n} = &\ \text{symmetrized product of } x, y_1, \ldots, y_n \\ &+\frac{1}{(n+1)!}\sum_\sigma\sum_i (n-i+1)y_{\sigma 1}*\cdots*[x,y_{\sigma i}]*\cdots*y_{\sigma n} \ . \end{aligned}$$

In the last sum, the terms with $i \neq 1$ can be expanded using $[x, y_{\sigma i}] = x*y_{\sigma i} - y_{\sigma i}*x$ and the induction hypothesis. The sum over i becomes

$$n[x,y_{\sigma 1}]*\cdots*y_{\sigma n} + n\,y_{\sigma 1}*x*\cdots*y_{\sigma n} - \sum_i y_{\sigma 1}*\cdots*y_{\sigma i}*x*\cdots*y_{\sigma n} \ ,$$

and (1.3.7.7) gives that

$$\sum_i \{x, y_i, y_1, \ldots, \hat{y}_i, \ldots, y_n\} = 0 \ .$$

In the notation of (1.3.7.6), this means that for any fixed i , $\sum\limits_j (A_j - A_i) = 0$, so that $A_i = \frac{1}{n+1}\sum A_j$ is independent of i and the $\{x_1, \ldots, x_n\}$ vanish.

It remains to deduce associativity for $p, q, r > 0$ with $p+q+r = n+1$ from the induction hypothesis and the vanishing of $\{x_1, \ldots, x_{n+1}\}$. By 1.3.7.4, it suffices to check for Z in $\mathrm{Sym}^r(\mathfrak{L})$ the equality of

$$(x_1*\cdots*x_p)*((y_1*\cdots*y_q)*Z) = (x_1*\cdots*x_p)*(y_1*\cdots*y_q*Z)$$

with

$$((x_1*\cdots*x_p)*(y_1*\cdots*y_q))*Z = (x_1*\cdots*y_q)*Z \ .$$

This results from having for all p

$$(x_1*\cdots*x_p)*Z = x_1*\cdots*x_p*Z$$

(an equality of maps from $\mathfrak{L}^{\otimes p} \otimes \mathrm{Sym}^{n+1-p}\mathfrak{L}$ to $\mathrm{Sym}^*\mathfrak{L}$). After symmetrization in $x_1, \ldots, x_p$, this holds true by the definition (1.3.7.3). It remains to check that

$$\begin{aligned} &(x_1*\cdots x_i*x_{i+1}*\cdots*x_p)*Z - (x_1*\cdots*x_{i+1}*x_i*\cdots*x_p)*Z \\ =&x_1*\cdots*x_i*x_{i+1}*\cdots*x_p*Z - x_1*\cdots*x_{i+1}*x_i*\cdots*x_p*Z \ . \end{aligned}$$

By the induction hypothesis, the left side is

$$(x_1*\cdots*[x_i,x_{i+1}]*\cdots*x_p)*Z = x_1*\cdots*[x_i,x_{i+1}]*\cdots*x_p*Z \ .$$

For the right side, the induction hypothesis (resp. the vanishing of $\{\cdots\}$) gives the same answer if $i \neq 1$ (resp. $i = 1$).

The algebra $(\mathrm{Sym}^*\mathfrak{L}, *)$ deserves to be called the universal enveloping algebra of $\mathfrak{L}$:

Lemma 1.3.7.8. *For A an associative algebra with unit in $\mathcal{T}$, one has*

$$\mathrm{Hom}_{\mathrm{algebra}}((\mathrm{Sym}^*\mathcal{L}, *), A) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Lie}}(\mathcal{L}, A)\ .$$

Proof. Given $f\colon \mathcal{L} \to A$, it results from (1.3.7.4) that the only possible extension of f to an algebra morphism is given by

$$\mathrm{Sym}^n\mathcal{L} \to A\colon y_1 \cdots y_n \longmapsto \frac{1}{n!}\sum_\sigma f(y_{\sigma 1}) \cdots f(y_{\sigma n})\ .$$

We leave it to the reader to check that this map is indeed a morphism of algebras.

Remark. For $\mathcal{T}$ the tensor category of super vector spaces, we get the Poincaré-Birkhoff-Witt theorem for super Lie algebras. Our arguments used characteristic zero in an essential way. For a proof valid for super Lie algebras $\mathcal{L}$ over a super commutative ring A, when $\mathcal{L}$ is free as an A-module and 2 and 3 are invertible in A, we refer to L. Corwin, Y. Ne'eman and S. Sternberg, Graded Lie algebras in mathematics and physics (Bose-Fermi symmetry) Reviews of Modern Physics **47** (1975), p. 573–603.

§1.4. Free modules

A *free module* over a super algebra is a module which is free as an ungraded module, with a homogeneous basis.

Fix a commutative super algebra A. The standard free module $A^{p|q}$ is the module freely generated by even elements $e_1, \ldots, e_p$ and odd elements $e_{p+1}, \ldots, e_{p+q}$. A morphism $T\colon A^{p|q} \to A^{r|s}$ can be represented by a matrix of size $(r+s) \times (p+q)$, with blocks of even and of odd entries as follows:

$$\begin{array}{r} \\ r\{ \\ \\ s\{ \end{array} \begin{array}{c} \overbrace{}^{p}\ \overbrace{}^{q} \\ \begin{pmatrix} \text{even} & \text{odd} \\ & \\ \text{odd} & \text{even} \end{pmatrix} \end{array}$$

We will represent an element x of $A^{p|q}$ by the column vector x^i such that $x = \sum e_i x^i$, and define the entries of the matrix of T by $T(e_j) = \sum e_i t^i_j$. With those conventions, $T(x)$ is given by the matrix product Tx and the composition of morphisms is given by a matrix product.

This description of endomorphisms works as well for right modules over a not necessarily commutative super algebra.

Warning. For M an A-module, the parity changed module $\prod M$ is not the super vector space $\prod M$, with the same right and left A-module structures as M: those A-module structures on $\prod M$ don't correspond to each other by (1.1.3). A correct definition of $\prod M$, for M a left or right module over a not necessarily commutative A is as follows: for right modules, the module structure remains the same; for left modules, the action of $a \in A$ is changed by the sign $(-1)^{p(a)}$. Let $\prod$ be the vector space k put in odd degree. If $\prod M$ is construed as being $\prod \otimes M$, by $1 \otimes m \xrightarrow{\sim} m$, this convention is an application of the sign rule.

With this definition of $\prod M$, one has

$$A^{p|q} \approx A^p \times (\textstyle\prod A)^q\ .$$

§1.5. Free commutative algebras

The commutative A-algebra $A[t_1, \dots, t_p, \theta_1, \dots, \theta_q]$ freely generated by even quantities $t_1 \dots t_p$ and odd quantities $\theta_1, \dots, \theta_q$ is the symmetric algebra $\mathrm{Sym}^*(A^{p|q})$. It can be described as polynomial in the t_i, and exterior in the θ_j, in the usual sense of those words. More precisely, it is the tensor product over k of the following three commutative super algebras: A, $\mathrm{Sym}^*(k^p) \simeq k[t_1, \dots, t_p]$ (purely even), and $\wedge^* k^q$ (with $\wedge^*$ taken in its usual sense, and with the parity defined to be the parity of the exterior degree and the basis for k^q being $\theta_1, \dots, \theta_q$). One should, however, keep in mind that the tensor product of super algebras involves the sign (1.1.5).

§1.6. The trace

Let $T\colon A^{p|q} \to A^{p|q}$ be a morphism given by

$$\begin{array}{c|cc} & \overbrace{}^{p} & \overbrace{}^{q} \\ p\{ & \multicolumn{2}{c}{\left(\begin{matrix} A & B \\ & \\ C & D \end{matrix}\right)} \\ q\{ & & \end{array}.$$

The *trace*, also called, for emphasis, *super trace* of T, is defined by

$$\text{(1.6.1)} \qquad \begin{aligned} \mathrm{Tr}(T) := {} & \text{sum of diagonal entries of } A \\ & - \text{sum of diagonal entries of } D\ . \end{aligned}$$

It is an element of A_0. Other notations: $\mathrm{str}(T)$, or, for endomorphisms of super vector spaces, $\mathrm{Tr}((-1)^F T)$. We now explain how the minus sign in (1.6.1) is forced on us by the categorical point of view. We will see at the same time that the trace is independent of the basis chosen.

The category of A-modules admits an inner Hom functor $\underline{\mathrm{Hom}}_A(M, N)$, characterized by the adjunction formula

$$\text{(1.6.2)} \qquad \mathrm{Hom}_A(M, \underline{\mathrm{Hom}}_A(N, P)) = \mathrm{Hom}_A(M \otimes_A N, P)\ .$$

The even (resp. odd) part of $\underline{\mathrm{Hom}}_A(M, N)$ is the k-vector space of even (resp. odd) maps $f\colon M \to N$ for which

$$f(am) = (-1)^{p(f)p(a)} a f(m)\ , \quad \text{i.e. } f(ma) = f(m)a\ .$$

The A-module structure is given by $(af)(m) = af(m)$, and to $m \mapsto \varphi_m$, the identification in (1.6.2) associates $m \otimes n \mapsto \varphi_m(n)$. The even part of $\underline{\mathrm{Hom}}_A(M, N)$ is $\mathrm{Hom}_A(M, N)$. This is obvious by inspection, as well as a particular case of (1.6.2): if $\underline{1}$ is the A-module A, the unit for $\otimes$, the even part of an A-module H is $\mathrm{Hom}_A(\underline{1}, H)$, and one takes $M = \underline{1}$ in (1.6.2).

For $M = A^{p|q}$ and $N = A^{r|s}$, $\underline{\mathrm{Hom}}_A(M, N)$ can again be viewed as a space of $(r + s) \times (p + q)$ matrices. Entries can now have any parity and $\underline{\mathrm{Hom}}_A(M, N)_i$ consists of the matrices with the $r \times p$ and $s \times q$ blocks of parity i, the other two of parity $i+1$. The map corresponding to a matrix is again given by a matrix product. In the matrix description, the A-module structure of $\underline{\mathrm{Hom}}_A(M,N)$ involves a sign:

$$a \cdot (t^i_j) = (\pm a t^i_j) \tag{1.6.3}$$

with $+$ (resp. $-$) for $1 \leq i \leq r$ (resp. $r+1 \leq i \leq r+s$).

In any tensor category with an inner Hom, the *dual* of M is

$$M^\vee := \underline{\mathrm{Hom}}(M, \underline{1}) \ .$$

Using just (1.6.2), one defines a morphism

$$M^\vee \otimes N \longrightarrow \underline{\mathrm{Hom}}(M,N) \ . \tag{1.6.4}$$

In the special case of A-modules, it is the map sending $\omega \otimes n$ to the map

$$m \longmapsto (-1)^{p(\omega)p(n)} n\omega(m)) \ ,$$

and it is an isomorphism if M is free of finite type.

One has an evaluation map

$$\mathrm{ev}\colon M^\vee \otimes M \to \underline{1}$$

and, if (1.6.4) is an isomorphism, a coevaluation map

$$\delta\colon \underline{1} \to M^\vee \otimes M$$

corresponding by (1.6.4) to the identity map of M. More generally, any morphism f in $\mathrm{Hom}(M,N) = \mathrm{Hom}(\underline{1}, \underline{\mathrm{Hom}}(M,N))$ gives, if (1.6.4) is an isomorphism,

$$\delta(f)\colon \underline{1} \to M^\vee \otimes N \ .$$

The *trace* of an endomorphism f of M is the endomorphism of $\underline{1}$

$$\mathrm{Tr}(f) := \mathrm{ev} \circ \delta(f) \ .$$

For A-modules, ev is $\omega \otimes m \mapsto \omega(m)$. For $M = A^{p|q}$, if e^i in the dual of M is defined by $e^i(e_j) = \delta^i_j$,

$$\delta(f) = \sum f(e_i) \otimes e^i \text{ in } N \otimes M^\vee = \sum (-1)^{p(e_i)} e^i \otimes f(e_i) \text{ in } M^\vee \otimes N \ ,$$

and for an endomorphism f of $A^{p|q}$

$$\mathrm{Tr}(f) = \mathrm{ev} \circ \delta(f) = \sum (-1)^{p(e_i)} e^i(f(e_i)) \ ,$$

which agrees with (1.6.1).

The categorical method can be used to prove that, in a setting where (1.6.4) is an isomorphism, the trace has the usual properties. It is defined not only on morphisms, but on the inner Hom: it is the composite

$$\mathrm{Tr}\colon \underline{\mathrm{Hom}}(M,M) \xleftarrow{\sim} M^\vee \otimes M \xrightarrow{\mathrm{ev}} 1 \ . \tag{1.6.5}$$

The inner Hom $\underline{\mathrm{Hom}}(M, M)$ is an associative algebra, hence also a Lie algebra, and (1.6.5) is a Lie algebra morphism, for the zero bracket on the unit object $\underline{1}$. For T, U in $\underline{\mathrm{Hom}}(M, M)$, this means that

$$\mathrm{Tr}(TU) = (-1)^{p(T)p(U)}\mathrm{Tr}(UT) \ . \tag{1.6.6}$$

For A-modules, and $M = A^{p|q}$, if $\underline{\mathrm{Hom}}(M, M)$ is viewed as a space of matrices, one computes that the trace (1.6.5) is given by

$$\begin{aligned} \mathrm{Tr}\colon T \longmapsto &\text{ sum of diagonal entries of the } p \times p \text{ block} \\ &-(-1)^{p(T)} \text{ sum of diagonal entries of the } q \times q \text{ block} \ , \end{aligned} \tag{1.6.7}$$

generalizing (1.6.1).

From the categorical point of view, the facts that a morphism $u\colon M \to N$ has a transpose $u^t\colon N^\vee \to M^\vee$, that $\mathrm{Tr}(u) = \mathrm{Tr}(u^t)$, and that $u^{tt} = u$ don't pose any sign problems. The meaning of the formula $u^{tt} = u$ is that the natural biduality isomorphisms $M \xrightarrow{\sim} M^{\vee\vee}$ and $N \xrightarrow{\sim} N^{\vee\vee}$ make the following diagram commute:

$$\begin{tikzcd} M \arrow[r, "u"] \arrow[d] & N \arrow[d] \\ M^{\vee\vee} \arrow[r, "u^{tt}"] & N^{\vee\vee} \end{tikzcd}$$

Similar constructions and compatibilities hold for $\underline{\mathrm{Hom}}(M, N)$.

When one translates this into matrices, however, care is required: a basis e_i of a free module M defines a basis e'_i of the dual $M^\vee$, with $e'_i(e_j) = \delta^i_j$. Iterating defines a basis e''_i of the bidual. In the natural isomorphism between M and its bidual, e_i maps not to e''_i, but to $(-1)^{p(e_i)}e''_i$. If $u \in \underline{\mathrm{Hom}}(M, N)$ has the matrix T in the bases $\{e_i\}$ for M and $\{f_j\}$ for N, the *transpose* of T is the matrix of u^t, in the bases $\{f'_j\}, \{e'_i\}$. The sign in the biduality isomorphism explains why, while $u^{tt} = u$ in the sense we explained, the transposition of matrices (for which our definition agrees with that of Manin [**1988**] Ch. 3, §3.1), is an operation of order 4.

§1.7. Even rules

It is often convenient to work with purely even objects. The "even rules" principle shows that if one is willing to extend scalars from k to algebras $k[\theta_1, \dots]$ generated by odd elements one can often reduce to the even context. The price one must pay is to show that things vary functorially under extensions of scalars.

By extension of scalars we have in mind the following. For V a super vector space and B a commutative super algebra, let $V(B)$ be the even part of $V_{(B)} := B \otimes V$ deduced from V by extension of scalars from k to B. It is a module over the even part B_0 of B, and a morphism $B' \to B''$ of super algebras induces a morphism of B'_0-modules from $V(B')$ to $V(B'')$.

Let us see how the "even rules" principle works in defining a morphism f: $\bigotimes_{i \in I} V_i \to V$. Such a morphism induces for each B a B_0-multilinear map $f(B)$ from the $V_i(B)$ to $V(B)$. It is functorial in B. Conversely, one has

Theorem 1.7.1. *Any system of B_0-multilinear maps from the $V_i(B)$ to $V(B)$, which is functorial in B, comes from a unique morphism $f\colon \otimes V_i \to V$.*

Proof. We may and shall assume that $I = [1, n]$ for some n. We first prove uniqueness. Fix $J \subset I$. If $v_i \in V$ is even for $i \notin J$, and odd for $i \in J$, we will recover $f(\otimes v_i)$ from $f(B)$ for $B = k[(\theta_j)_{j \in J}]$. Indeed, let $w_i = v_i$ for $i \notin j$ and $w_i = \theta_i v_i$ for $i \in J$; if J has N elements,

$$f(B)((w_\imath)) = (-1)^{N(N-1)/2} . \prod \theta_\jmath . f(\otimes v_i) \ , \tag{1.7.2}$$

which determines $f(\otimes v_i)$. This proves uniqueness, and we now prove existence.

Let $F(B)$ be a functorial system of multilinear maps. If $(v_i)_{\imath \in I}$, J, B and the $w_\imath$ are as in the proof of uniqueness, each w_i for i in J is annihilated by $B \to B/(\theta_j)$. So is $F(B)((w_i))$ and it follows that $F(B)((w_i))$ in $B \otimes V$ is a multiple of $\prod \theta_j$. Define $f(\otimes v_i)$ by 1.7.2. It remains to check that F is deduced from this f.

Let C be a commutative super algebra. It suffices to check, for any $J \subset [1, n]$, that if $u_i = c_i v_i$ with c_i in C and v_i in V both even for $i \notin J$, and odd for $i \in J$, one has

$$F(C)((u_i)) = f(C)((u_i)) \ .$$

By multilinearity, we reduce to the case where $c_\imath = 1$ for $i \notin J$. The u_i are then the images of the w_i by $B \to C$, $\theta_j \to c_j$ and the claimed equality follows from the definition 1.7.2 of f by functoriality.

Remarks. (i) The same result holds if B runs only over exterior algebras $k[\theta_1, \ldots, \theta_N]$, i.e. commutative super algebras freely generated by odd elements.

(ii) A similar result holds for modules over a fixed commutative superalgebra A, if B runs over commutative A-algebras and $V(B)$ is the B_0-module even part of $B \otimes_A V$.

Corollary 1.7.3. (i) *To give an algebra structure on a super vector space V, it suffices to give a functorial structure of B_0-algebra on the $V(B)$. The algebra will be Lie (resp. associative, commutative,...) if and only if the algebras $V(B)$ are.*

(ii) *For A an algebra, to give a structure of A-module on a super vector space V, it suffices to give a functorial structure of $A(B)$-module on $V(B)$.*

Proof. The algebra and module structures are defined by morphisms of the type considered in the theorem. The algebraic properties listed in the corollary are expressed by the commutativity of diagrams constructed from such maps, or, in the Lie case, by linear relations between composites of such maps. According to the theorem, to give the defining morphisms amounts to giving them functorially on the even part, after extension of scalars. The required commutativity of diagrams, or linear relations, follows from the same on the even part, after extension of scalars, because of the uniqueness statement.

§1.8. Examples of the "even rules" principle

1.8.1. Let A be a super algebra. For each commutative super algebra B, $A(B)$ is an algebra. It is also an algebra for the opposite multiplication: $x \cdot_{\text{opp}} y := y \cdot x$. According to Corollary 1.7.3, this defines a new algebra structure on A: the *opposite* super algebra. To compute $x \cdot_{\text{opp}} y$ for x and y odd, one extends scalars by two odd quantities θ, η, and expresses that

$$\theta x \cdot_{\text{opp}} \eta y = \eta y \cdot \theta x \ .$$

As x is odd, $\theta x \cdot_{\text{opp}} \eta y = -\theta\eta x \cdot_{\text{opp}} y$ while $\eta y \cdot \theta x = \theta\eta y \cdot x$, giving

$$x \cdot_{\text{opp}} y = -y \cdot x \quad \text{for} \quad x, y \quad \text{odd} \ .$$

Thus, we recover 1.1.4. The sign has been hidden in the compatibility between extension of scalars and tensor products:

$$(ax) \otimes (by) = (-1)^{p(x)p(b)} ab(x \otimes y) \ .$$

1.8.2. The signs in the definition of a super Lie algebra are similarly characterized by the requirement that, after any extension of scalars, the even part be an ordinary Lie algebra. For A an associative super algebra, the sign in the definition of the bracket is to get the usual bracket on the even part, after any extension of scalars. That this bracket is Lie follows from the same statement in the even case.

1.8.3. For M and N two A-modules, the formation of $\underline{\mathrm{Hom}}_A(M, N)$ is compatible with extensions of scalars $k \mapsto B$, for B an exterior algebra $k[\theta_1, \ldots, \theta_N]$. The even part of $\underline{\mathrm{Hom}}_A(M, N)$ is $\mathrm{Hom}_A(M, N)$. If M is a free module of finite type, the restriction to the even part of the trace (1.6.7) on $\underline{\mathrm{Hom}}_A(M, M)$ is the trace (1.6.1) on $\mathrm{Hom}_A(M, M)$. That (1.6.7) is a morphism of Lie algebras results from the fact that for the trace (1.6.1), and endomorphisms f, g of M, one has

$$\mathrm{Tr}(fg) = \mathrm{Tr}(gf) \ .$$

The sign $(-1)^{p(T)}$ in the matrix description of the trace (1.6.7) is then a consequence of the sign in (1.6.3).

1.8.4. An (even) *derivation* of a super algebra A is a morphism of super vector spaces D: $A \to A$ such that

$$D(ab) = Da \cdot b + a \cdot Db \ . \tag{1.8.4.1}$$

If we extend scalars from k to $k[\varepsilon]/(\varepsilon^2)$, with ε even, the derivations of A are identified with the automorphisms of the $k[\varepsilon]$-algebra $A \otimes k[\varepsilon]$, which are the identity modulo ε, by

$$D \longmapsto (\text{automorphism } a \mapsto a + \varepsilon Da)$$

The derivations form the even part of the even or odd derivations, for which (1.8.4.1) is replaced by

$$D(ab) = Da \cdot b + (-1)^{p(D)p(a)} a \cdot Db \ . \tag{1.8.4.2}$$

The sign in (1.8.4.2) ensures that if D is odd, and that we extend scalars from k to $k[\theta]$ with θ odd, then θD is an even derivation.

If $\mathcal{D} \subset \underline{\mathrm{Hom}}_k(A, A)$ is the super vector space of the even or odd derivations of A, and if we extend scalars to $B = k[\theta^1, \ldots, \theta^q]$, $\mathcal{D}_{(B)} := B \otimes \mathcal{D}$ is the B-module of even or odd B-linear derivations of $A_{(B)} = B \otimes A$. The even part of $\mathcal{D}_{(B)}$ are even derivations, classically known to form a Lie algebra. By Corollary 1.7.3, $\mathcal{D}$ is a super Lie algebra.

§1.9. Alternate description of super Lie algebras

Even if §1.7 gives a systematic way of considering only even elements, the use of odd elements can lead to more transparent formulations. I particularly like the following description (cf. Leites [**1980**] 1.9.4) of what a super Lie algebra $\mathfrak{L}$ over A is. It is the data of a quadratic map

$$x^2\colon \mathfrak{L}'_1 \to \mathfrak{L}'_0$$

of A'_0-modules, functorial in the extension of scalars A', obeying (1.9.1) below. The bilinear form $[x,y] = (x+y)^2 - x^2 - y^2$ associated to x^2 extends (by 1.7 applied to bilinear maps from $(\prod\mathfrak{L}, \prod\mathfrak{L})$ to $\mathfrak{L}$) to a bracket $\mathfrak{L}\otimes\mathfrak{L} \to \mathfrak{L}$. The axiom is

(1.9.1) $$[x, x^2] = 0\ ,$$

a form of the Jacobi identity reminiscent of the Bianchi identity.

Example. The Lie algebra freely generated by an odd element D has the basis $\{D, D^2\}$, with the only non-trivial bracket being $[D,D] = 2D^2$. The universal enveloping algebra is the free associative algebra generated by D. As required by the Poincaré-Birkhoff-Witt theorem, it admits the basis $\{(D^2)^n D^m | n \geq 0,\ m = 0 \text{ or } 1\}$.

§1.10. The Berezinian of an automorphism

Let L be a free module of finite type over a commutative super algebra A. We write $\mathrm{GL}(L)$ for the group of automorphisms of the A-module of L, and define $\mathrm{GL}_{p|q}(A) = \mathrm{GL}(A^{p|q})$. the *Berezinian* is a homomorphism

$$\mathrm{Ber}\colon \mathrm{GL}(L) \to \mathrm{GL}_{1|0} = A_0^*$$

which generalizes the determinant. We give three descriptions of it.

(A) The usual determinant det: $\mathrm{GL}(n) \to \mathrm{GL}(1)$ induces the trace on the Lie algebras, and this characterizes it. Similarly, if ε is even of square zero and T an (even) endomorphism,

(1.10.1) $$\mathrm{Ber}(1+\varepsilon T) = 1 + \varepsilon\,\mathrm{Tr}(T)$$

and, with enough preliminaries on super algebraic groups, (1.10.1) can be turned into a definition (cf. 2.10). For an automorphism $T = \begin{pmatrix} T' & 0 \\ 0 & T'' \end{pmatrix}$ of $A^{p|q}$ with no nonzero odd entries, the sign in the definition 1.6.1 of the trace gives

(1.10.2) $$\mathrm{Ber}(T) = \det(T')\det(T'')^{-1}\ .$$

Generalizing 1.10.1, for T nilpotent, or in settings where the exponential series converges, one has

(1.10.3) $$\mathrm{Ber}\exp(T) = \exp\,\mathrm{Tr}(T)\ .$$

(B) For ordinary free modules, the usual determinant of $T\colon L \to L$ is the scalar by which T acts on the top exterior power $\wedge^{\dim L}L$. This definition cannot be generalized to the super case; for $q \neq 0$, $A^{p|q}$ has no top exterior power. Another, a priori, reason is that the Berezinian makes sense only for automorphisms, not for endomorphisms.

However, in the classical case, computing the Ext by a Koszul complex, one finds a canonical isomorphism

$$\operatorname{Ext}^{\dim(L)}_{\operatorname{Sym}^*(L^\vee)}(A, \operatorname{Sym}^*(L^\vee)) = \wedge^{\dim(L)} L \; . \tag{1.10.4}$$

In (1.10.4), A is viewed as a $\operatorname{Sym}^*(L^\vee)$-module by the augmentation map. For A a commutative super algebra, and L a free module of dimension $p|q$, the $\underline{\operatorname{Ext}}$ group $\underline{\operatorname{Ext}}^n_{\operatorname{Sym}^*(L^\vee)}(A, \operatorname{Sym}^*(L^\vee))$ vanishes for $n \neq p$. For $n = p$, it is free of rank $1|0$ if q is even, and of rank $0|1$ if q is odd. One defines

$$\operatorname{Ber}(T) = \text{ action of } T \text{ on } \underline{\operatorname{Ext}}^p_{\operatorname{Sym}^*(L^\vee)}(A, \operatorname{Sym}^*(L^\vee)) \; . \tag{1.10.5}$$

Example. For $L = A^{0|q}$, $\operatorname{Sym}^*(L^\vee)$ is isomorphic to $A[\theta_1, \ldots, \theta_q]$. The Ext group $\underline{\operatorname{Ext}}^i_{A[\theta_1,\ldots,\theta_q]}(A, A[\theta_1, \ldots, \theta_q])$ vanishes for $i \neq 0$. For $A = k$, this results from $k[\theta_1, \ldots, \theta_q]$ being an injective module over itself. In general, it results from the $A[\theta_1, \ldots, \theta_q]$-module $A[\theta_1, \ldots, \theta_q]$ being isomorphic, up to a parity change, to $\underline{\operatorname{Hom}}_A(A[\theta_1, \ldots, \theta_q], A)$ (Frobenius property). The $\underline{\operatorname{Ext}}^0$ is

$$\underline{\operatorname{Hom}}_{A[\theta_1,\ldots,\theta_q]}(A, A[\theta_1, \ldots, \theta_q]) \; .$$

It admits the basis consisting of the one element $a \mapsto \theta_1 \ldots \theta_q a$, hence is of dimension $1|0$ for q even, and $0|1$ for q odd. An automorphism T of L acts on $\operatorname{Sym}^*(L^\vee) \simeq A[\theta_1, \ldots, \theta_q]$; it maps $\theta_1 \ldots \theta_q$ to $\operatorname{Ber}(T)\theta_1 \ldots \theta_q$.

(C) Let T be an automorphism of $A^{p|q}$, with matrix $\left(\begin{smallmatrix} K & L \\ M & N \end{smallmatrix}\right)$ as given in (1.4). Let B be the quotient of A by the ideal generated by the odd elements. It is the quotient of A_0 by a nilpotent ideal I. After extension of scalars to B, the matrix of T takes the form

$$\begin{pmatrix} K \bmod I & 0 \\ 0 & N \bmod I \end{pmatrix} .$$

It follows that the matrices with even entries K and N are invertible modulo the nilpotent ideal I. They are hence invertible, and one defines

$$\operatorname{Ber}(T) := \det(K - LN^{-1}M)\det(N)^{-1} \; , \tag{1.10.6}$$

a formula suggested by (1.10.2) and the decomposition

$$\begin{pmatrix} K & L \\ M & N \end{pmatrix} = \begin{pmatrix} 1 & LN^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} K - LN^{-1}M & 0 \\ M & N \end{pmatrix} .$$

The matrices K, $LN^{-1}M$ and N have entries in the commutative ring A_0, so that the determinants in (1.10.6) make sense. It is a slightly nontrivial exercise to prove that 1.10.6 is multiplicative. To compare with the point of view (A), one checks (1.10.1). To compare with (B), one checks that with both definitions one has the property (1.10.2), multiplicativity, and that for an automorphism of a short exact sequence $0 \to L' \to L \to L'' \to 0$,

$$\operatorname{Ber}(T) = \operatorname{Ber}(T').\operatorname{Ber}(T'') \; . \tag{1.10.7}$$

Those properties uniquely characterize the Berezinian.

§1.11. The Berezinian of a free module

Classically, one defines the determinant of a free module of rank n by $\det(L) := \wedge^n L$. Similarly, one could define the Berezinian of a free module of rank $p|q$ over a commutative super algebra A by

$$\mathrm{Ber}(L) := \underline{\mathrm{Ext}}^{\,p}_{\mathrm{Sym}^*(L^\vee)}(A, \mathrm{Sym}^*(L^\vee)) \tag{1.11.1}$$

(cf. 1.10 (B)). Another definition is obtained by turning around the definition 1.10 (B) of the Berezinian of an automorphism: a basis $(e_1, \ldots, e_p,\ e_{p+1}, \ldots, e_{p+q})$ of L defines a one-element basis $[e_1, \ldots, e_p,\ e_{p+1}, \ldots, e_{p+q}]$ of $\mathrm{Ber}\, L$, and for any automorphism T,

$$[Te_1, \ldots, Te_{p+q}] = \mathrm{Ber}(T)[e_1 \ldots, e_{p+q}]\ . \tag{1.11.2}$$

To keep track of signs, we will consider $\mathrm{Ber}(L)$ as an object of the tensor category (1.3.6) of graded super modules. The grading will be called grading by cohomological degree, and $[e_1, \ldots, e_{p+q}]$ is of parity $q \bmod 2$ and cohomological degree p (as suggested by (1.11.1). For

$$0 \to L' \to L \to L'' \to 0 \tag{1.11.3}$$

a short exact sequence of free modules, if $e'_1, \ldots, e'_p,\ e'_{p+1}, \ldots, e'_{p+q}$ is a basis of L', and if $\tilde{e}''_1, \ldots, \tilde{e}''_r,\ \tilde{e}''_{r+1}, \ldots, \tilde{e}''_{r+s}$ lifts to L the basis $e''_1, \ldots, e''_r,\ e''_{r+1}, \ldots, e''_{r+s}$ of L'', we define an isomorphism

$$\mathrm{Ber}(L') \otimes \mathrm{Ber}(L'') \to \mathrm{Ber}(L) \tag{1.11.4}$$

by

$$\left[e'_1 \ldots e'_{p+q}\right] \otimes \left[e''_1 \ldots e''_{r+s}\right] \longmapsto \left[e'_1 \ldots e'_p \tilde{e}''_1 \ldots \tilde{e}''_r e'_{p+1} \ldots e'_{p+q} \tilde{e}''_{r+1} \ldots \tilde{e}''_{r+s}\right]\ .$$

With the sign rule (1.3.6.1), for any exact square

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & L^1_1 \arrow[r] \arrow[d] & L^1 \arrow[r] \arrow[d] & L^1_2 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & L_1 \arrow[r] \arrow[d] & L \arrow[r] \arrow[d] & L_2 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & L^2_1 \arrow[r] \arrow[d] & L^2 \arrow[r] \arrow[d] & L^2_2 \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

the diagram

$$\begin{tikzcd}
\otimes \mathrm{Ber}(L^i_j) \arrow[r] \arrow[d] & \otimes \mathrm{Ber}(L^i) \arrow[d] \\
\otimes \mathrm{Ber}(L_j) \arrow[r] & \mathrm{Ber}(L)
\end{tikzcd}$$

is commutative.

Appendix to §1: Graded super vector spaces

Some classical constructions give objects which are graded and are subject to Koszul's sign rule. Examples: cohomology groups of all kinds (H^*, Tor_*, $\mathrm{Ext}^* \dots$), the de Rham complex and other standard complexes.

To handle their analogue in the super world, two points of view have been used. As we will see, they are basically equivalent. However, they lead to different sign conventions.

Point of View I

One considers such objects as graded objects in the super category, i.e., graded mod 2 graded objects. The grading which is the analogue of the grading in the classical case will be called the cohomological grading. The commutativity isomorphism for the tensor product of graded mod 2 graded vector spaces is defined as follows: for v of bidegree (p, n), viz. parity p and cohomological degree n, and w of bidegree (q, m),

$$c^{\mathbf{I}}_{V,W}\colon V \otimes W \to W \otimes V \quad \text{is} \quad v \otimes w \longmapsto (-1)^{pq+nm} w \otimes v.$$

This is the point of view adopted in §1.3.6.

Point of View II

One considers that the classical construction already lives in the super world, with parity being cohomological degree modulo 2. In the super world, one continues to obtain super objects; they also have a cohomological degree, with no influence on signs: the commutativity isomorphism of graded mod 2 graded vector spaces is defined as

$$c^{\mathbf{II}}_{V,W}\colon V \otimes W \to W \otimes V\colon v \otimes w \to (-1)^{pq} w \otimes v$$

for v of parity p and w of parity q. Of course, one could consider modules instead of vector spaces.

Comparison of I and II

In his lectures, Bernstein used the point of view **II**, which he prefers. In our rendition of it here, we have used the point of view **I**. Here are our reasons for doing so.

(A) In classical occurrences of super objects, the sign rule is violated by some of the standard conventions. For instance, for V and $V^\vee$ vector spaces in duality it is usual to define the duality between $\wedge^p V$ and $\wedge^p V^\vee$ by

$$\langle \omega_1 \wedge \dots \wedge \omega_p,\ v_1 \wedge \dots \wedge v_p \rangle = \det\left(\langle \omega_i, v_j \rangle\right) .$$

In the determinant, the diagonal term is $\langle \omega_1, v_1 \rangle \cdots \langle \omega_p, v_p \rangle$, with no sign, despite the fact that v_i passed over ω_j for $i < j$. Using the point of view **I** allows us to keep the inherited classical conventions, while consistently using the sign rule as far as parity is concerned.

(B) The point of view **I** is forced on us by the categorical method 1.2 to hide signs. Indeed, when mimicking in a tensor category $\mathcal{T}$ a classical construction which could be construed as graded (or super), we are led to consider the category of graded (or just mod 2 graded) objects of $\mathcal{T}$, with the commutativity of tensor product being given by modifying the one induced from $\mathcal{T}$ by the Koszul sign rule. When $\mathcal{T}$ is the category of super vector spaces, this gives $c^{\mathbf{I}}$ of **I**.

(C) One does not have to decide early on whether an object should be seen as having a cohomological degree.

Example. Let A be a commutative super k-algebra. The point of view **I** suggests defining the module Ω^1_A of Kähler differentials as being an A-module (hence bimodule) Ω, provided with a morphism of super k-vector spaces d: $A \to \Omega$ such that

$$d(ab) = a \cdot db + da \cdot b, \tag{C.1}$$

which is universal. Later, when considering the de Rham complex, one may decide that Ω^1_A is of cohomological degree 1. In the point of view **II**, deciding that in the classical case (purely even A) Ω^1_A is odd requires d to be an odd map, and that (C.1) be replaced by

$$d(ab) = (-1)^{p(a)} a \cdot db + da \cdot b \tag{C.2}$$

(D) The point of view **I** minimizes the use of the parity change functor Π. It is replaced by the imposition of an odd cohomological degree. Using the functor Π leads to nightmares of signs, for the following reasons.

(i) Let $\Pi\, k$ be k, viewed as an odd k-vector space. The functor Π is best viewed as being the tensor product with $\Pi\, k$. One has to decide whether it is $V \mapsto \Pi\, k \otimes V$ or $V \mapsto V \otimes \Pi\, k$. The two are canonically isomorphic, but lead to different sign conventions.

(ii) One has natural isomorphisms $(\Pi V) \otimes W \xrightarrow{\sim} \Pi(V \otimes W)$ and $(V \otimes \Pi W) \xrightarrow{\sim} \Pi(V \otimes W)$, exchanged by the commutativity of $\otimes$. The diagram

$$\begin{array}{ccc}
\Pi V \otimes \Pi W & \longrightarrow & \Pi(V \otimes \Pi W) \\
\downarrow & & \downarrow \\
\Pi(\Pi V \otimes W) & \longrightarrow & \Pi\,\Pi(V \otimes W)
\end{array}$$

is anticommutative, rather than commutative.

The point of view **II** has advantages too: one has only one parity to consider, for applying the sign rule, and some constructions are more natural. For example, if D^- is the standard odd line (coordinate ring $k[\theta]$, θ odd, $\theta^2 = 0$), the de Rham complex of a super manifold M is (up to a completion) the space of functions on the super manifold

$$\underline{\mathrm{Hom}}(D^-, M) \ ,$$

where the $\underline{\mathrm{Hom}}$ is defined by $\mathrm{Hom}(S, \underline{\mathrm{Hom}}(D^-, M)) = \mathrm{Hom}(S \times D^-, M)$ functorially in S. In the point of view **I**, one has to apply to the de Rham complex the functor "associated simply graded object" explained below.

The points of view **I** and **II** are equivalent, in the sense that the tensor categories $\mathcal{T}_{\mathbf{I}}$ and $\mathcal{T}_{\mathbf{II}}$ of graded mod 2 graded vector spaces introduced in **I** and **II** are equivalent. That is to say, there is an equivalence of categories

$$\widetilde{\mathrm{s}}\colon \mathcal{T}_{\mathbf{I}} \longrightarrow \mathcal{T}_{\mathbf{II}}$$

and an isomorphism of functors

$$\alpha\colon \widetilde{\mathrm{s}}(V \otimes_{\mathbf{I}} W) \xrightarrow{\sim} \widetilde{\mathrm{s}}(V) \otimes_{\mathbf{II}} \widetilde{\mathrm{s}}(W)$$

compatible with the associativity and commutativity isomorphisms for $\otimes_{\mathbf{I}}$ and $\otimes_{\mathbf{II}}$.

The functor $\widetilde{\mathrm{s}}$ is a regrading functor: $\widetilde{\mathrm{s}}(V)$ has the same underlying vector space as V, with v of parity p and cohomological degree n acquiring the parity $p+n$, and keeping its cohomological degree.

On the underlying vector space, $\otimes_{\mathbf{I}}$ and $\otimes_{\mathbf{II}}$ are both the usual tensor product. For v in $V^{p,n}$ and w in $W^{q,m}$, α is defined as

$$\alpha\colon v \otimes w \longmapsto (-1)^{nq} v \otimes w.$$

The compatibility with the commutativity isomorphisms is the commutativity of the diagram

$$\begin{array}{ccc}
V^{p,n} \otimes W^{q,m} & \xrightarrow{(-1)^{nq}} & V^{p,n} \otimes W^{q,m} \\
{\scriptstyle (-1)^{pq+nm}}\big\downarrow & & \big\downarrow{\scriptstyle (-1)^{(p+n)(q+m)}} \\
W^{q,m} \otimes V^{p,n} & \xrightarrow{(-1)^{mp}} & W^{q,m} \otimes V^{p,n}
\end{array} .$$

The compatibility with the associativity isomorphisms comes from the identity

$$n_1(p_2 + p_3) + n_2 p_3 = n_1 p_2 + (n_1 + n_2) p_3 \ .$$

The functor "*associated simply* (mod 2) *graded*" is the composite of $\widetilde{\mathrm{s}}$ with the functor "forgetting the cohomological degree" to super vector spaces. We denote this composite by s.

Example. Let A be a graded mod 2 graded algebra (point of view **I**). It is given by a multiplication

$$\cdot : A \otimes A \to A.$$

Applying s, we obtain a super algebra, with a of parity p and cohomological degree n becoming of parity $p+n$. The new product

$$*\colon \mathrm{s}(A) \otimes \mathrm{s}(A) \xleftarrow{\sim} \mathrm{s}(A \otimes A) \xrightarrow{\mathrm{s}(\cdot)} \mathrm{s}(A)$$

is

$$x * y = (-1)^{nq} x \cdot y$$

for x in $A^{p,n}$ and y in $A^{q,m}$. If A is commutative, meaning that for x of bidegree (p,n) and y of bidegree (q,m) one has $xy = (-1)^{pq+nm} yx$, then the superalgebra $(\mathrm{s}(A), *)$ is a commutative super algebra.

CHAPTER 2
Super Manifolds: Definitions

§§2.1–2.7. Super manifolds as ringed spaces

2.1. Many kinds of spaces can be defined as topological spaces endowed with a sheaf of algebras. This is the case for C^∞ or C^r manifolds, complex manifolds, complex analytic spaces or, in algebraic geometry, schemes. Super manifolds can be defined in this way, too. Let $\mathcal{C}^\infty$ be the sheaf of C^∞-functions on $\mathbb{R}^p$. The space $\mathbb{R}^{p|q}$ is the topological space $\mathbb{R}^p$, endowed with the sheaf $\mathcal{C}^\infty[\theta^1, \ldots, \theta^q]$ of commutative super $\mathbb{R}$-algebras, freely generated over $\mathcal{C}^\infty$ by odd quantities $\theta^1, \ldots, \theta^q$. The coordinate functions t^i of $\mathbb{R}^p$, and the θ^j are the *coordinates* of $\mathbb{R}^{p|q}$. A *super manifold* M of dimension $p|q$ is a topological space $|M|$ endowed with a sheaf of super $\mathbb{R}$-algebras, which is locally isomorphic to $\mathbb{R}^{p|q}$. Its *structural sheaf* is denoted $\mathcal{O}_M$, or simply $\mathcal{O}$. By abuse of language, the sections of $\mathcal{O}_M$ are sometimes called *functions* on M. The odd functions generate a nilpotent ideal J of $\mathcal{O}_M$, and $(|M|, \mathcal{O}_M/J)$ is a C^∞-manifold of dimension p, i.e. is locally isomorphic to $(\mathbb{R}^p, \mathcal{C}^\infty)$. By the definition of a super manifold, it suffices to check this for $\mathbb{R}^{p|q}$, in which case it is clear by inspection. One calls $(|M|, \mathcal{O}_M/J)$ the *reduced* manifold M_{red} of M.

In the case of C^∞ manifolds, the sheaf $\mathcal{O}$, even if initially given just as a sheaf of $\mathbb{R}$-algebras, can be viewed as a sheaf of actual functions: the *value* of $f \in \Gamma(U, \mathcal{O})$ at $x \in U$ is the unique real number λ such that $f - \lambda$ is not invertible in any neighborhood V of x. The association $f \mapsto$ function "value of f at variable x" embeds $\mathcal{O}$ into the sheaf of continuous functions.

For M a super manifold, a section f of $\mathcal{O}_M$ has an image f_{red} in $\mathcal{O}_{M_{\text{red}}}$, which can be viewed as a C^∞ function on $|M|$. In particular, one can still speak of the *value* of f at x in $|M|$; it is the only real number λ such that $f - \lambda$ is not invertible in any neighborhood of x. However, we will have to live with the fact that sections of $\mathcal{O}_M$, even if called "functions on M", are not determined by their values at each point. For instance, an odd function takes the value 0 at each point $x \in |M|$.

An *open submanifold* U of M is an open subset $|U|$ of $|M|$, endowed with the restriction to $|U|$ of $\mathcal{O}_M$.

Morphisms between super manifolds are defined to be morphisms of ringed spaces: a morphism $f\colon M \to N$ is a continuous map $|f|\colon |M| \to |N|$ given with a morphism of sheaves of super $\mathbb{R}$-algebras from $|f|^*\mathcal{O}_N$ to $\mathcal{O}_M$. If V is an open subset of $|N|$ and if v in $\mathcal{O}_N(V)$, its image in $\mathcal{O}_M(f^{-1}V)$ is called the pullback of v by f and denoted $v \circ f$, $v(f)$, or f^*v.

2.2 Remarks. (i) In the purely even case of ordinary C^∞-manifolds, this notion of morphism agrees with the usual one. The point is that for f a morphism from M to N, and v a section of $\mathcal{O}_N$, if $f(x) = y$, the value of $f^*(v)$ at x is necessarily $v(y)$. Indeed, for $\lambda \neq v(y)$, $f^*(v) - \lambda = f^*(v - \lambda)$ is invertible in a neighborhood of x, with inverse $f^*((v - \lambda)^{-1})$.

(ii) The structural sheaf $\mathcal{O}$ of a super manifold M is a sheaf of local rings: its stalk $\mathcal{O}_x$ at a point $x \in |M|$ is a local ring, with maximal ideal the kernel of the evaluation map $f \mapsto f(x)$. A morphism $f\colon M \to N$ induces a morphism from M_{red} to N_{red} and, by (i), is a morphism of locally ringed spaces: the induced homomorphism, from $\mathcal{O}_{f(x)}$ to $\mathcal{O}_x$, maps the maximal ideal to the maximal ideal. Our definition hence agrees with that of Manin [**1988**] Ch. 4, §1.2.

2.3 Examples. (i) The quotient map from $\mathcal{O}_M$ to $\mathcal{O}_M/J$ defines a morphism $M_{\mathrm{red}} \hookrightarrow M$.

(ii) The inclusion of an open submanifold U of M is a morphism $U \hookrightarrow M$.

(iii) Let M_0 be a C^∞ manifold, and let V be a locally free module over the sheaf $\mathcal{C}^\infty$ of C^∞ functions on M_0. The space M_0, endowed with the sheaf $\wedge^* V$ of $\mathbb{R}$-algebras graded by the exterior degree modulo 2, is a super manifold M, with $M_{\mathrm{red}} = M_0$. The inclusion of $\mathcal{C}^\infty$ into $\wedge^* V$ defines a morphism $M \to M_{\mathrm{red}}$, which is a retraction to the embedding of M_{red} in M. Any super manifold is isomorphic to one constructed in this way, but not canonically so: if M is of dimension $p|q$ with $p \geq 1$ and $q \geq 2$, no retraction $M \to M_{\mathrm{red}}$ is invariant under the automorphisms of M.

If $f\colon M \to \mathbb{R}^{p|q}$ is a morphism of super manifolds, its *coordinates* f_i $(1 \leq i \leq p)$ and φ_j $(1 \leq j \leq q)$ are the pullbacks of the coordinates of $\mathbb{R}^{p|q}$. The definitions of super manifolds and of morphisms are reasonable only because one has:

Proposition 2.4. *Let $U \subset \mathbb{R}^{p|q}$ be an open submanifold of $\mathbb{R}^{p|q}$. The map $f \mapsto$ coordinates (f_i, φ_j) of f is a bijection from*

(i) *the set of morphisms $f\colon M \to U$, to*

(ii) *the set of systems of p even functions f_i and q odd functions φ_j on M, such that the values $(f_1(x), \dots, f_p(x))$ for x in $|M|$ are in $|U| \subset \mathbb{R}^p$.*

This is proved in Leites [**1980**] 2.1.7, or Manin [**1988**] Ch. 4, §1.8. The main point is that if $f_1, \dots, f_p$ and $\varphi_1, \dots, \varphi_q$ are respectively even and odd functions on M, and if F is a function on $\mathbb{R}^{p|q}$, one can give a natural meaning to $F(f_1, \dots, f_p, \varphi_1, \dots, \varphi_q)$.

Suppose first that there are no φ. This means that $q = 0$ and F is a C^∞-function on $\mathbb{R}^p$. Suppose also that M is an open submanifold of $\mathbb{R}^{n|m}$, with coordinates $x_1, \dots, x_n, \chi_1, \dots, \chi_m$. A function u on M can then be expressed in a unique way as $u = \sum u_J \chi^J$, where the u_J are C^∞ functions on $|M|$ and where, for $J \subset [1, m]$, χ^J is the product of the χ_j for j in J, taken in order of increasing j. In particular, each of the even functions f_i can be decomposed as $f_i = f_i^0 + r_i$, where f_i^0 is a C^∞-function on $|M|$ and where r_i is even and nilpotent. One defines $F(f_1, \dots, f_p)$ by Taylor's formula:

$$F(f_1, \dots, f_p) := \sum \partial^{\mathbf{k}} F(f_1^0, \dots, f_p^0) r^{\mathbf{k}} / \mathbf{k}! \,. \tag{2.4.1}$$

In the sum, only finitely many terms don't vanish. A general super manifold M is locally isomorphic to an open submanifold of some $\mathbb{R}^{n|m}$, and one should check that (2.4.1) gives, on M, a result independent of the local isomorphism used. In general, a function F on $\mathbb{R}^{p|q}$ (coordinates t^i, θ^j) can be uniquely written as $F = \sum F_I \theta^I$, with each F_I a C^∞ function on $\mathbb{R}^p$, and

$$F(f_1, \ldots, f_p, \varphi_1, \ldots, \varphi_q) := \sum F_I(f_1, \ldots, f_p)\varphi^I . \tag{2.4.2}$$

The morphism to $\mathbb{R}^{p|q}$ defined by $f_1, \ldots, f_p, \varphi_1, \ldots, \varphi_q$ is given by the continuous map $(f_{1\,\mathrm{red}}, \ldots, f_{p\,\mathrm{red}})$ and the pullback of functions $F \mapsto F(f_1, \ldots, f_p, \varphi_1, \ldots, \varphi_q)$.

2.5. Thanks to Proposition 2.4, the definition 2.1 of super manifolds is equivalent to a definition in terms of an atlas. Local charts are open subsets of $\mathbb{R}^{p|q}$. Gluing maps are isomorphisms $f\colon U \to V$, for U and V open subsets of $\mathbb{R}^{p|q}$. They are given by even and odd functions $f_1, \ldots, f_p, \varphi_1, \ldots, \varphi_q$, where each of those functions, say, u, has a decomposition $\sum u_I \theta^I$, with u_I a C^∞ function on the open subset U of $\mathbb{R}^n$. To be an isomorphism, f must admit an inverse $V \to U$.

2.6 Remarks. (i) The formula (2.4.1) involves an arbitrarily large number of derivatives. For this reason, C^r super manifolds don't make sense.

(ii) In infinite dimension, one should not try to define super manifolds as being ringed spaces. This has nothing to do with "super." Already for infinite dimensional (complex) spaces modeled on a Banach space, Douady (le problème des modules pour les sous-espaces analytiques compacts d'un espace analytique donné, Ann. Inst. Fourier tXVI 1 (1966) p. 1-95) observed that the useful notion is defined in terms of local charts; the sheaf of functions is insufficient information.

(iii) The infinite dimensional super manifolds M we will meet will be spaces of "fields." We will not even try to define them as topological spaces endowed with additional structures. In the spirit of the functor of points, explained in 2.8, 2.9, we only care to know what is a morphism from a finite dimensional super manifold S to M, and only the functor $S \mapsto \mathrm{Hom}(S, M)$ will be defined.

(iv) It follows from Proposition 2.4 that finite products exist in the category of super manifolds. For U and V open submanifolds of $\mathbb{R}^{p|q}$ and $\mathbb{R}^{r|s}$, $U \times V$ is the open submanifold of $\mathbb{R}^{p+r|q+s}$ with $|U \times V| = |U| \times |V|$. By 2.4, the defining property $\mathrm{Hom}(M, U \times V) \xrightarrow{\sim} \mathrm{Hom}(M, U) \times \mathrm{Hom}(M, V)$ holds. The general case follows by gluing.

(v) More generally, if $f\colon M \to S$ is locally (on M) isomorphic to $\mathrm{pr}_2\colon \mathbb{R}^{p|q} \times S \to S$, then, for any morphism $u\colon S' \to S$, the fiber product $M \times_S S'$ exists. Its defining property is that for any T, $\mathrm{Hom}(T, M \times_S S') \xrightarrow{\sim} \mathrm{Hom}(T, M) \times_{\mathrm{Hom}(T,S)} \mathrm{Hom}(T, S')$. The projection $f'\colon M' := M \times_S S' \to S'$ is said to be deduced from f by *base change* from S to S'.

(vi) Complex super manifolds can similarly be defined as topological spaces endowed with a sheaf of $\mathbb{C}$-algebras, locally isomorphic to some $(\mathbb{C}^p, \mathcal{O}[\theta^1, \ldots, \theta^q])$, for $\mathcal{O}$ the sheaf of holomorphic functions on $\mathbb{C}^p$. Same definition for "real analytic", with $\mathbb{C}^p$ replaced by $\mathbb{R}^p$ and $\mathcal{O}$ by the sheaf of real analytic functions.

(vii) In Manin [**1980**] Ch. 4, §1.6, analytic super spaces and super schemes are similarly defined as ringed spaces.

(viii) The reader may prefer to add to the definition of super manifold the condition that the underlying topological space be Hausdorff and σ-compact.

2.7. The difficulty that functions are not determined by their values is familiar in algebraic geometry. There, as here, it is due to the fact that the structural sheaf $\mathcal{O}$ can have nilpotent elements.

In complex algebraic geometry, the simplest example is afforded by the affine algebraic variety $\mathrm{Spec}(D)$ with algebra of functions the algebra of dual numbers $D := \mathbb{C}[\varepsilon]/(\varepsilon^2)$. A morphism from $\mathrm{Spec}(D)$ to the affine space A^n is simply a morphism of $\mathbb{C}$-algebras from $\mathbb{C}[t_1, \ldots, t_n]$ to D. It is the same as the data of a point $z = (z_1, \ldots, z_n)$ of A^n, and of a tangent vector v at z. The corresponding morphism of $\mathbb{C}$-algebras is $P \mapsto P(z) + \varepsilon\partial_v P(z)$.

Similarly, a morphism from $\mathbb{R}^{0|1}$, with ring of functions $\mathbb{R}[\theta]$, to $\mathbb{R}^{p|q}$ (coordinates $t_1, \ldots, t_p$, $\theta_1, \ldots, \theta_q$) is the data of a point $(x_1, \ldots, x_p)$ of $\mathbb{R}^p$, and of an odd morphism ∂ from the local ring at x to $\mathbb{R}$, which obeys the Leibniz identity $\partial(fg) = \partial f.g(x) + (-1)^{p(f)} f(x).\partial g$. The corresponding morphism of rings is $f \mapsto f(x) + \theta\partial f$. (In §3.3 we shall see that such morphisms are odd tangent vectors to $\mathbb{R}^{p|q}$.)

In both cases, a space M can be thought of as a reduced space M_{red}, surrounded by a nilpotent fuzz. An automorphism $f\colon M \to M$ can be the identity on M_{red}, and shear the fuzz. Example: the automorphism $(t, \theta_1, \theta_2) \mapsto (t + \theta_1\theta_2, \theta_1, \theta_2)$ of $\mathbb{R}^{1|2}$, i.e. the automorphism with coordinates $(t + \theta_1\theta_2, \theta_1, \theta_2)$ (cf. 2.4) does not respect the projection $t\colon \mathbb{R}^{1|2} \to \mathbb{R}^{1|0}$.

In algebraic geometry, nonreduced space are defined to be singular. Not so here. Another, related, difference is that $\mathbb{R}^{0|1}$, and more generally $\mathbb{R}^{p|q}$, admits the group law

$$+\colon \mathbb{R}^{p|q} \times \mathbb{R}^{p|q} \to \mathbb{R}^{p|q}\,, \tag{2.7.1}$$

with coordinates by $(t'_1 + t''_1, \ldots, t'_p + t''_p, \theta'_1 + \theta''_1, \ldots, \theta'_q + \theta''_q)$. Not so for $\mathrm{Spec}(D)$: there is no morphism from D to $D \otimes D = \mathbb{C}[\varepsilon', \varepsilon'']/(\varepsilon'^2, \varepsilon''^2)$ mapping ε to $\varepsilon' + \varepsilon''$. Indeed, $(\varepsilon' + \varepsilon'')^2 = \varepsilon'\varepsilon'' + \varepsilon''\varepsilon' = 2\varepsilon'\varepsilon'' \neq 0$. For ε' and ε'' odd, the mixed terms would cancel, and we obtain a group law.

§§2.8–2.9. The functor of points approach to super manifolds

2.8. The language of ringed spaces can be cumbersome. As in algebraic geometry, an approach closer to the geometric intuition is the language of the functor of points. This approach is also closer to the way physicists make computations.

Let S be a super manifold. An *S-point* of a super manifold M is a morphism $S \to M$. It should be thought of as a family of points of M parametrized by S. The set $M(S)$ of S-points of S is functorial in S: a morphism $u\colon T \to S$ induces the map $m \mapsto m \circ u$ from $M(S)$ to $M(T)$. An S-point of M can be identified with a section of the projection $M \times S \to S$. From this point of view, the functoriality in S is given by the base change by $T \to S$.

A morphism $f\colon M \to N$ induces for each S a map f_S from $M(S)$ to $N(S)$, and this map is functorial in S. This construction is a bijection from the set of morphisms $f\colon M \to N$ to the set of systems of maps $f_S\colon M(S) \to N(S)$, functorial in S. This fact is called Yoneda's lemma. Its proof is deceptively trivial. The inverse construction attaches to (f_S) the image by $f_M\colon \mathrm{Hom}(M, M) \to \mathrm{Hom}(M, N)$ of the identity map of M.

Examples. (i) For S reduced to a point, i.e. to $\mathbb{R}^{0|0}$, an S-point of M is simply a point of $|M|$.

(ii) By 2.4, an S-point of $\mathbb{R}^{p|q}$ can be identified with a system of p even functions and q odd functions on S.

2.9. To construct a super manifold M, it is often best to first construct the corresponding functor of points, and to prove that this functor is representable, i.e. corresponds to a super manifold. To construct a morphism $f\colon M \to N$ it is often best to construct the corresponding functorial map $M(S) \to N(S)$. This requires systematically working "over a base S". Instead of super manifolds, one should consider *families of super manifolds parametrized by* S, i.e. morphisms $M \to S$ which are isomorphic locally on M to a projection $U \times S \to S$. Together with φ: $M \to S$, one should systematically consider the families deduced from it by a base change g: $S' \to S$, i.e. the fiber products $M_{(S')} := M \times_S S'$, see (2.6(v), and one should use only constructions which are *geometric*, i.e. compatible with any base change. The construction $M \mapsto M_{\text{red}}$ is not geometric, as in general $M_{\text{red}} \times_S S' \neq (M \times_S S')_{\text{red}}$.

We will need the relative version of Yoneda's lemma. Let φ: $M \to S$ be a family of super manifolds parametrized by S. For g: $T \to S$, a T-point of M/S is a section of $M_{(T)}$. If ψ: $N \to S$ is another family, giving an S-map f: $M \to N$, i.e. a map f such that $\psi f = \varphi$, is equivalent to giving functorially in T, a map f_T: (T-points of M/S) $\to$ (T-points of N/S)).

Functions can be defined by their values at each point too, if "point" is taken to mean "T-point": for M/S, giving a function f on M is equivalent to giving functorially in $T \to S$, for each T-point t of M/S, the value of f at t (the function $t^* f$ on T). First proof: as in the proof of Yoneda's lemma, take $T = M$, t =identity. Second proof: reduce to Yoneda's lemma, identifying even (resp.odd) functions to maps to $\mathbb{R}^{1|0}$ (resp. $\mathbb{R}^{0|1}$).

§2.10. Super Lie groups

A *super Lie group* is a group object in the category of super manifolds: it is a super manifold G given with a product law μ: $G \times G \to G$ such that there exists a unit $e\colon (\mathbb{R}^{0|0}) \to G$ and an inverse map: $G \to G$ making standard diagrams commute. By Yoneda's lemma, the data of a product law μ amounts to that of a composition law μ_S on the set of S-points $G(S)$, functorial in S, and the condition that G be a super Lie group is that each $G(S)$ be a group.

A convenient way to define a super Lie group is to first define the group-valued functor $G(S)$, and then to check that it is representable by a super manifold, which is then automatically a super Lie group. See the examples in 2.11 below. A useful representability criterion is the following. Suppose that a super Lie group H acts on a super manifold M, and that m is a point of M. For each S, $H(S)$ acts on $M(S)$, and m defines m_S in $M(S)$. Define $G(S)$ to be the subgroup of $H(S)$ which fixes m_S. Then $S \mapsto G(S)$ is representable by a submanifold of H, the stabilizer of m in G.

One has the usual relation between super Lie groups and super Lie algebras. For more details, see §3.3.5 and §3.8. The topological subtleties concern only the even part. For instance: If G is a super Lie group, with Lie algebra $\mathfrak{g}$, G_{red} is a Lie subgroup, with Lie algebra the even part $\mathfrak{g}_0$ of $\mathfrak{g}$. If G and H are connected super Lie groups, a morphism f of Lie algebras from $\mathfrak{g} = \operatorname{Lie} G$ to $\mathfrak{h} = \operatorname{Lie} H$ extends to a (unique) morphism of groups if and only if $f_0\colon \mathfrak{g}_0 \to \mathfrak{h}_0$ extends to a morphism of ordinary Lie groups from G_{red} to H_{red}.

§2.11. Classical series of super Lie groups

Here are descriptions by their functors of points of the linear groups, the orthosymplectic groups, and the groups of the P and Q series (cf. Manin [**1988**], Ch. 4 §10).

$\mathrm{GL}(p|q)$. The group of S-points is the group of automorphisms of the sheaf of $\mathcal{O}_S$-modules $\mathcal{O}_S^{p|q}$. The description by matrices shows that it is an open submanifold of some affine superspace.

For each S, the Berezinian (1.10) provides a morphism from the group of S-points of $\mathrm{GL}(p,q)$ to the group of S-points of $\mathrm{GL}(1|0)$. Those morphisms define a morphism of super Lie groups

$$\mathrm{Ber}\colon \mathrm{GL}(p|q) \to \mathrm{GL}(1|0)\ . \tag{2.11.1}$$

The kernel of the (2.11.1) is the special linear group $\mathrm{SL}(p|q)$.

$\mathrm{OSp}(n|2m)$: Fix an even supersymmetric nondegenerate bilinear form Φ on the super vector space $\mathbb{R}^{n|2m}$ with values in $\mathbb{R}^{1|0}$. In non-super terms, Φ is given by a nondegenerate symmetric bilinear form on $\mathbb{R}^n$, and a nondegenerate alternating form on $\mathbb{R}^{2m}$. The corresponding group $\mathrm{O}(\Phi)$ is the subgroup of $\mathrm{GL}(n|2m)$ preserving Φ: the group of S-points is the group of automorphisms of $\mathcal{O}^{n|2m} = \mathcal{O} \otimes_{\mathbb{R}} \mathbb{R}^{n|2m}$ preserving Φ.

$\pi\,\mathrm{Sp}(n|n)$: Same as above with Φ odd, and antisymmetric in the super sense. The groups of the P series are the kernels, in $\pi\,\mathrm{Sp}$, of the Berezinian.

Q series: Let D be the super division algebra $\mathbb{R}[\eta]$ with η odd and $\eta^2 = -1$ (cf. 1.1.6). The S-points of the Lie group $\mathrm{GL}(n, D)$ are the automorphisms of the sheaf of right $\mathcal{O} \otimes D$-modules $(\mathcal{O} \otimes D)^n$.

For $n = 1$, this is $(\mathcal{O} \otimes D)_0^*$: the invertible $a + \alpha\eta$, with a even and α odd, with the multiplication law

$$(a + \alpha\eta)(b + \beta\eta) = (ab + \alpha\beta) + (a\beta + \alpha b)\eta\ .$$

The Lie algebra is the free Lie algebra in one odd generator.

The map $a + \alpha\eta \mapsto a + \alpha$ identifies this group with the multiplicative group of $\mathcal{O} = \mathcal{O}_0 \oplus \mathcal{O}_1$. The map

$$a + \alpha\eta \longmapsto \alpha/a$$

is a morphism from $\mathrm{GL}(1, D)$ to $\mathbb{R}^{0|1}$. Indeed,

$$(a\beta + \alpha b)/(ab + \alpha b) = (\beta/b + \alpha/a)/(1 - (\alpha/a)\cdot(\beta/b)) = \beta/b + \alpha/a$$

as $\alpha^2 = \beta^2 = 0$. As in the story of Dieudonné's determinants (Dieudonné, les déterminants sur un corps non commutatif, Bull. S.M.F. 71 (1943), p. 27-45) for linear groups over skew fields, this morphism extends to a morphism of Lie groups, called the odd determinant

$$\mathrm{o\,det}\colon \mathrm{GL}(n, D) \to \mathbb{R}^{0|1}.$$

For $n \geq 2$, the kernels of the odd determinant, divided by the diagonal subgroup $\mathrm{GL}(1|0)$, are the Lie groups of the Q series.

CHAPTER 3
Differential Geometry of Super Manifolds

§3.1. Introduction

Many notions and results from ordinary differential geometry extend to super manifolds in a straightforward way. Vector bundles on M can be interpreted as sheaves of locally free super modules over $\mathcal{O}_M$. A super manifold M has a tangent bundle T_M. If M is of dimension $p|q$, the vector bundle T_M is a locally free super $\mathcal{O}_M$-module of dimension $p|q$. A morphism $f\colon M \to N$ of super manifolds induces a morphism $df\colon T_M \to f^*T_N$. Analogues of the inverse and of the implicit function theorem, as well as of the Frobenius theorem, hold. There is the usual relationship between actions of Lie algebras and local actions of Lie groups, and in particular between even vector fields and flows. From this follows a definition of the Lie derivative $\mathcal{L}_X$ of a section of any canonical vector bundle with respect to a vector field X.

The dual of the tangent bundle is the cotangent bundle Ω^1, and $d\colon \mathcal{O}_M \to \Omega^1_M$ is a universal derivation. For $\Omega^*_M = \wedge^*\Omega^1_M$, with $\wedge^*$ understood as in 1.3.5, 1.3.6, d extends uniquely to a square zero derivation, the *exterior derivative* $d\colon \Omega^*_M \to \Omega^{*+1}_M$. The Poincaré lemma holds. As a consequence, the cohomology of the topological space $|M|$ is $H^*(\Gamma(M, \Omega^*_M), d)$.

A connection on a vector bundle $\mathcal{V}$ is $\nabla\colon \mathcal{V} \to \Omega^1 \otimes \mathcal{V}$ obeying the Leibniz identity. It extends to $\nabla\colon \Omega^* \otimes \mathcal{V} \to \Omega^{*+1} \otimes \mathcal{V}$ and the curvature F in $\Omega^2 \otimes \underline{\mathrm{End}}\,(\mathcal{V})$ describes ∇^2. It is the obstruction to $\mathcal{V}$ being flat, that is to $\mathcal{V}$ admitting a local basis (e_i) with $\nabla e_i = 0$. As classically, one can consider more generally connections on G-torsors (a.k.a. principal G-bundles) with G being a super Lie group, the case of vector bundles corresponding to $G = \mathrm{GL}(r|s)$.

For M of dimension $p|q$ with $q \neq 0$, there are forms of all degrees $n \geq 0$: there are no top forms. An n-form can be integrated on a $n|0$-dimensional submanifold. What can be integrated on M are densities, which can be interpreted as sections of $\mathrm{Ber}(\Omega^1_M)$, tensored with the orientation sheaf of M. One can also define *integral forms*, which can be integrated on submanifolds of codimension $p|0$ and give rise to a super version of Stokes' theorem. Densities are the top integral forms.

§3.2. Vector bundles

In classical differential geometry, vector bundles over a manifold M can be viewed in two ways.

(i) As a fiber bundle V over M, with typical fiber $\mathbb{R}^n$ and structural group $\mathrm{GL}(n)$, for some n. It is a manifold V mapping to M, with an additional structure.

(ii) As a sheaf $\mathcal{V}$ of locally free modules of finite type over $\mathcal{C}^\infty_M$.

The same two points of view can be used in supergeometry. A vector bundle of rank $p|q$ over M is

(i′) a fiber bundle V over M, with typical fiber $\mathbb{R}^{p|q}$ and structural group $\mathrm{GL}(p,q)$;

(ii′) a sheaf of $\mathcal{O}_M$-super modules $\mathcal{V}$, locally free of dimension $p|q$.

The super manifold V corresponding to the super module $\mathcal{V}$ is most easily defined by its functor of points: after any base change, a section of V over M is an even section of $\mathcal{V}$. In other words, for any super manifold S, the set of S-points of V can be identified with the set of pairs (f, v), where f: $S \to M$ is an S-point of M and v is an even section of the pullback $f^*\mathcal{V}$ of $\mathcal{V}$ to S. To check that the functor $S \to \{$ set of pairs $(f, v)\}$ is representable can be reduced to the same question locally on M. Locally, $\mathcal{V}$ is free and we can choose a basis $e_1, \dots, e_p$, $f_1, \dots, f_q$ of $\mathcal{V}$. Proposition 2.4 then shows that the functor considered is represented by $\mathbb{R}^{p|q} \times M$. Another basis will define another isomorphism of V with $\mathbb{R}^{p|q} \times M$, deduced from the first one by g: $M \to \mathrm{GL}(p|q)$. This gives the fiber space structure of V.

Warnings. (i) It is dangerous to call "even" a vector bundle of dimension $(r|0)$: if M is of dimension $p|q$ with $q \neq 0$, the odd part $\mathcal{V}_1$ is not zero. Indeed, $\mathcal{V}$ is locally isomorphic to $\mathcal{O}^r_M$ and $\mathcal{O}_M$ has a non-trivial odd part.

(ii) Similarly, if M is not purely even, the mod 2 grading of a vector bundle $\mathcal{V}$ does not decompose it into the sum of an even and an odd bundle.

(iii) The parity changed $\mathcal{V}$, denoted $\prod \mathcal{V}$, has been defined in (1.4, Warning). The spaces V and $\prod V$ corresponding to $\mathcal{V}$ and $\prod \mathcal{V}$ don't have the same underlying topological space. If $\mathcal{V}_{\mathrm{red}}$ is the restriction of $\mathcal{V}$ to M_{red}, $|V|$ (resp. $|\prod V|$) is the vector bundle over $|M| = M_{\mathrm{red}}$ corresponding to $(\mathcal{V}_{\mathrm{red}})_0$ (resp. $(\mathcal{V}_{\mathrm{red}})_1$).

§3.3. The tangent bundle, the cotangent bundle and the de Rham complex

On $\mathbb{R}^{p|q}$ (coordinates t^i and θ^j), the even derivations $\partial/\partial t^i$ and the odd derivations $\partial/\partial\theta^j$ of the structural sheaf $\mathcal{O}$ are defined as follows. For $f = \sum f_I \theta^I$, with each f_I a C^∞ function on $\mathbb{R}^p$,

$$\partial/\partial t^i(f) := \sum \partial/\partial t^i(f_I)\theta^I \ .$$

For f written as $\sum\limits_{j \notin I}(f_I\theta^I + f_{j,I}\theta^j\theta^I)$,

$$\partial/\partial\theta^j(f) := \sum f_{j,I}\theta^I \ .$$

As proved in Leites [**1980**] 2.2.3, one has

3.3.1 Proposition. *The $\mathcal{O}$-module of $\mathbb{R}$-linear derivations of $\mathcal{O}$ is free of dimension $p|q$, with basis the $\partial/\partial t^i$ and the $\partial/\partial\theta^j$.*

A super manifold M of dimension $p|q$ being locally isomorphic to $\mathbb{R}^{p|q}$, it follows that the $\mathcal{O}$-module of derivations of $\mathcal{O}_M$ is a vector bundle of dimension $p|q$. It is the *tangent bundle* T_M of M. The sections of T_M are called *vector fields.* These are the derivations on $\mathcal{O}_M$. The super vector space $\mathrm{Vect}(M)$ of all vector fields is a super Lie algebra, see (1.8.4).

The *cotangent bundle* of a super manifold M is the dual Ω^1_M of T_M. We will systematically write the duality pairing between the tangent and cotangent bundle as

$$\langle\ ,\ \rangle\colon T_M \otimes \Omega^1_M \to \mathcal{O}_M\ ,$$

with $\langle uD, v\omega\rangle = (-1)^{p(D)p(v)}uv\,\langle D,\omega\rangle$ for $u, v \in \mathcal{O}_M$. We will avoid the notation $\omega(D)$ which by the sign rule should be taken to mean $(-1)^{p(D)p(\omega)}\,\langle D,\omega\rangle$.

One defines $d\colon \mathcal{O} \to \Omega^1_M$ by

$$\langle D, df\rangle = Df\ .$$

This is the universal derivation of even parity with values in an $\mathcal{O}$-module. As classically, d extends uniquely to a square zero derivation d of the commutative graded super algebra $\Omega^*_M := \wedge^*\Omega^1_M$ (1.3.5):

$$d^2 = 0 \tag{3.3.2}$$

$$d(\alpha\beta) = d\alpha\cdot\beta + (-1)^p\alpha\cdot d\beta \quad \text{for} \quad \alpha \quad \text{in} \quad \Omega^p_M\ . \tag{3.3.3}$$

This defines the differential graded algebra of differential forms on a super manifold. If M is of dimension $p|q$ with $q \neq 0$, $\Omega^n_M \neq 0$ for all $n \geq 0$: there are no "top forms". A p-form can be restricted to a $p|0$ dimensional submanifold, and integrated if the submanifold is oriented (this is especially meaningful over a base S). Objects of a different kind, the integral forms explained in 3.12 below, can be integrated on submanifolds of codimension $p|0$.

3.3.4 Poincaré lemma. *The complex Ω^*_M is a resolution of the constant sheaf $\mathbb{R}$ on $|M|$.*

Proof. On $\mathbb{R}^{p|q}$, the de Rham complex Ω^*_M is the algebraic tensor product of the pullback of the de Rham complexes on the factors $\mathbb{R}^{p|0}$ and $\mathbb{R}^{0|q}$. This reduces the Poincaré lemma to the classical case of $\mathbb{R}^p$, and to the case of $\mathbb{R}^{0|q}$. The latter can further be reduced to the case of $\mathbb{R}^{0|1}$. For $\mathbb{R}^{0|1}$, Ω^n is $\mathbb{R}[\theta]\cdot(d\theta)^n$, with $d[(a+b\theta)(d\theta)^n] = b(d\theta)^{n+1}$. That Ω^* is a resolution of $\mathbb{R}$ follows by inspection.

The sheaves Ω^i_M are soft. It hence follows from the Poincaré lemma that the cohomology $H^*(|M|,\mathbb{R})$ of the topological space $|M|$ can be computed by the de Rham complex Ω^*_M:

$$H^*(|M|,\mathbb{R}) = H^*(\Gamma(M,\Omega^*_M))\ . \tag{3.3.5}$$

Remark. The algebra Ω^*_M is a commutative graded super algebra, in the sense of 1.3.6. Its relation with the parity changed tangent bundle is as follows. With the notation of the Appendix to §1, $\mathrm{s}(\Omega^*_M)$ is a commutative super algebra of functions on the parity changed tangent bundle, viewed as a fiber bundle over M. It consists of those functions which are fiberwise polynomial.

The parity changed tangent bundle ΠT represents the functor $\underline{\mathrm{Hom}}(\mathbb{R}^{0|1}, M)$ of maps from $\mathbb{R}^{0|1}$ to M: for any S, its S-points are the S-morphisms from $\mathbb{R}^{0|1} \times S$ to $M \times S$. As $\mathbb{R}^{0|1}$ is a group, it acts on $\underline{\mathrm{Hom}}(\mathbb{R}^{0|1}, M)$ by translations on $\mathbb{R}^{0|1}$. This defines an action of $\mathbb{R}^{0|1}$ on ΠT. The Lie algebra of $\mathbb{R}^{0|1}$ is of dimension $0|1$, with a basis reduced to one odd element D, with $[D, D] = 0$. It acts on ΠT. Possibly up to a sign, D acts on $\mathrm{s}(\Omega^*_M)$ as $\mathrm{s}(d)$: $\alpha \mapsto (-1)^q d\alpha$ for α of parity q.

3.3.6 The super Lie algebra of a super Lie group

A vector field X on a super Lie group G is said to be *left invariant* if for any S, and any S-point g of G, the corresponding vertical vector field on $G_S := G \times S \to S$ is invariant under the left translation by g. As usual, it is enough to consider the universal case $S = G$ and g the identity map: the vector field X_1 deduced from X, vertical relatively to pr_2: $G \times G \to G$, should be invariant by $(g, h) \to (hg, h)$.

By transport of structure, diffeomorphisms, for instance left translations, preserve brackets. It follows that the left invariant vector fields on G form a sub super Lie algebra of the algebra of all vector fields on G. By evaluation at the unit element e of G, the space of left invariant vector fields maps bijectively to the tangent space T_eG of G at e. Transporting to T_eG the bracket of left invariant vector fields, one obtains the *Lie algebra* of G.

Left invariant vector fields can be viewed as infinitesimal right translations: the left invariant vector field with value X at e is the corresponding derivative in h of gh: $G \times G \to G$.

Right invariant vector fields are defined similarly. The involution $g \mapsto g^{-1}$ of G exchanges left and right translations. It hence maps left (resp. right) invariant to right (resp. left) invariant vector fields. If by evaluation at e one identifies T_eG with the space of right invariant vector fields, the resulting bracket $[\ ,\ ]_r$ is the opposite of the Lie algebra bracket. Indeed, $g \mapsto g^{-1}$ induces $x \mapsto -x$ on T_eG, hence $[-x, -y] = -[x, y]_r$.

Example. Consider the super Lie group structure on $\mathbb{R}^{1|1}$ with product law

$$(t^1, \theta^1) \cdot (t^2, \theta^2) = (t^1 + t^2 + \theta^1\theta^2, \theta^1 + \theta^2) \ .$$

The left invariant vector fields are obtained by derivation in (t^2, θ^2), at $(t^2, \theta^2) = 0$, of the product law. At $(0, 0)$, one has

$$\begin{aligned} &\partial_t(t^1 + t + \theta^1\theta) &&= 1 \qquad &&\partial_\theta(t^1 + t + \theta^1\theta) &&= -\theta^1 \\ &\partial_t(\theta^1 + \theta) &&= 0 &&\partial_\theta(\theta^1 + \theta) &&= 1, \end{aligned}$$

and the left invariant vector fields are hence spanned by ∂_t and $-\theta\partial_t + \partial_\theta$. A similar computation shows that the right invariant vector fields are spanned by ∂_t and $\theta\partial_t + \partial_\theta$.

§3.4. The inverse and implicit function theorems

Results which classically are proved by using a sequence of successive approximations can often, in the super case, be proved by first using the classical result on the reduced variety, then finding "successive approximations" by working modulo higher and higher power of the ideal generated by the odd functions. As this ideal is nilpotent, the process terminates. An example is the inverse function theorem, as proved by Leites [**1980**] 2.3.1.

3.4.1. Theorem. *If, at one point x of M, the morphism $f: M \to N$ induces an isomorphism from the tangent space of M at x to the tangent space of N at $f(x)$, then f induces an isomorphism from a neighborhood of x to one of $f(x)$.*

In local coordinates, the condition on f is that its Jacobian matrix be invertible.

This theorem has the usual corollaries: implicit function theorem, Jacobian criterion for $f: M \to N$ to be a submersion (= family of manifolds parametrized by N, see 2.7) or an immersion, structure of morphisms of "constant rank". See Leites (1980) 2.3. One also has relative variants for families of manifolds.

§3.5. Distributions

Let M be a super manifold with tangent bundle T. A *distribution* on M is an $\mathcal{O}_M$-submodule $\mathcal{D}$ of T which is locally a direct factor. The bracket of vector fields, i.e. of derivations, induces an $\mathcal{O}$-linear map

$$\mathcal{D} \otimes_{\mathcal{O}} \mathcal{D} \to T/\mathcal{D}$$

the *Frobenius bilinear form.* A distribution $\mathcal{D}$ is said to be *integrable* if its Frobenius bilinear form vanishes, i.e. if $\mathcal{D}$ is stable under bracket. As classically, one has the

3.5.1. Theorem. *If the distribution $\mathcal{D}$ is integrable, then the super manifold M admits locally a product structure $M = M' \times M''$, giving a decomposition $T = T' \oplus T''$ of the tangent bundle, with $\mathcal{D} = T'$.*

In other words, there exists locally a submersion $M \to M/\mathcal{D}$, for which $\mathcal{D}$ is the relative tangent bundle.

For $\mathcal{D}$ of dimension $1|0$, the theorem results from the

Lemma 3.5.2. *If an even vector field D does not vanish at the point x, then a neighborhood of x admits a coordinate system $(x^1, \ldots, \theta^q)$ for which $D = \partial/\partial x^1$.*

Proof. Let $\mathcal{J} \subset \mathcal{O}$ be the ideal generated by the odd functions. As D is even, it maps $\mathcal{J}$ to $\mathcal{J}$ and induces a vector field D_{red} on M_{red}. On the vector bundle $\mathcal{J}/\mathcal{J}^2$ on M_{red}, it also induces a connection in the direction of D_{red}, i.e. an operator ∂: $\mathcal{J}/\mathcal{J}^2 \to \mathcal{J}/\mathcal{J}^2$ such that $\partial(fx) = D_{\mathrm{red}}(f)x + f\partial x$. By classical results, one can find a local coordinate system on M_{red} and a local basis of $\mathcal{J}/\mathcal{J}^2$ for which D_{red} (resp. the connection in the direction of D_{red}) are given by $\partial/\partial x_1$.

By (3.4.1), if we lift the coordinates of M_{red} to even functions on M, and the basis of $\mathcal{J}/\mathcal{J}^2$ to sections of $\mathcal{J}$, we obtain a local coordinate system on M. In it, we will have

$$D = \partial/\partial x^1 \bmod \mathcal{J}^2 T \ .$$

This is the beginning of a sequence of successive approximations in which, starting from a local coordinate system in which $D = \partial/\partial x^1 \bmod \mathcal{J}^k T$ $(k \geq 2)$, one constructs a new one in which $D = \partial/\partial x^1 \bmod \mathcal{J}^{k+1}T$. To do this, one looks for a diffeomorphism

$$(x^i, \theta^j) \to (x^i + X^i, \theta^j + \chi^j) \tag{3.5.3}$$

with X^i, χ^j in $\bar{\mathcal{J}}^k$, which transforms D into $\frac{\partial}{\partial x^1} \bmod \mathcal{J}^{k+1}T$. [Formula (3.5.3) describes, for any S, the coordinates of the image of the S-point with coordinates (x^i, θ^j), or equivalently the pullback of the coordinate functions.] Let X be the vector field $\sum X^i \partial/\partial x^i + \sum \chi^j \partial/\partial \theta^j$. One has

$$\text{transform of } D \equiv D - [X, D] \equiv D + [\partial/\partial x_1, X] \quad (\text{mod } \mathcal{J}^{k+1}T)\,.$$

One observes that $\partial/\partial x_1$ is onto and hence the inductive step can be carried out. As explained in 3.4, this process terminates.

Proof of 3.5.1. Suppose $\mathcal{D}$ is of dimension $r|s$. As long as $r > 0$, one can choose in $\mathcal{D}$ an even vector field D, generating a direct factor. Applying 3.5.2, we may and shall assume that D is $\partial/\partial x^1$. Applying the assumption that $[\partial/\partial x^1, \mathcal{D}] \subset \mathcal{D}$, one checks that $\mathcal{D}$ comes from a distribution of dimension $(r-1, s)$ on the quotient space with coordinates $x^2, \ldots, \theta^s$. This distribution is also stable under bracket and, iterating that construction, one reduces to the case where $r = 0$.

We now suppose that $r = 0$. Let $D_1, \ldots, D_s$ be odd derivations forming a basis of $\mathcal{D}$. For $f_1, \ldots, f_s$ odd functions, $D := \sum f_i D_i$ is an even derivation. The exponential series $\exp(D) := \sum D^n/n!$ terminates and $\exp(D)$: $\mathcal{O} \to \mathcal{O}$ is an automorphism. There is a unique diffeomorphism of M, still noted $\exp(D)$, so that *pullback* of functions by the diffeomorphism is the map $\exp(D)$ on $\mathcal{O}$. If we perform this construction on $\mathbb{R}^{0|s} \times M$, taking the f_i to be the coordinate functions θ^i of $\mathbb{R}^{0|s}$, we obtain an automorphism of $\mathbb{R}^{0|s} \times M$, compatible with the projection to $\mathbb{R}^{0|s}$. Projecting to M, we obtain a morphism

$$\exp\left(\sum \theta^i D_i\right) : \mathbb{R}^{0|s} \times M \to M\,. \tag{3.5.4}$$

If $T \subset M$ is of codimension $0|s$, and transversal to $\mathcal{D}$, the induced morphism

$$\exp\left(\sum \theta^i D_i\right) : \mathbb{R}^{0|s} \times T \to M \tag{3.5.5}$$

is a diffeomorphism. We identify $\mathbb{R}^{0|s} \times T$ and M by this diffeomorphism. By construction the distribution $\mathcal{D}$ is tangent to $\mathbb{R}^{0|s}$-factors along $\{0\} \times T$. We need to show that the same is true everywhere, or in other words that (3.5.4) preserves the projection to T. The Campbell-Hausdorff formula for $\exp\left(\sum \alpha^i D_i\right) \exp\left(\sum \theta^i D_i\right)$ terminates. This shows that the diffeomorphisms $\exp\left(\sum \alpha^i D_i\right)$ respect the projection to T, which is the required submersion.

§3.6. Connections on vector bundles

The formalism of connections on vector bundles gives rise to no surprise, but some questions of signs and of notations have to be treated carefully. We concentrate on them.

A connection on $\mathcal{V}$ is a morphism in the category of sheaves of super vector spaces ∇: $\mathcal{V} \to \Omega^1 \otimes \mathcal{V}$ obeying the Leibniz identity

$$\nabla(fv) = df \otimes v + f\nabla v. \tag{3.6.1}$$

For a vector field X, one then defines a section $\nabla_X v$ of $\mathcal{V}$ by

$$\nabla_X v = \langle X, \nabla v\rangle \, , \tag{3.6.2}$$

with $\langle X, \alpha \otimes v\rangle := \langle X, \alpha\rangle \, v$. From (3.6.1), it follows that

$$\nabla_X(fv) = Xf \cdot v + (-1)^{p(X)p(f)} f\nabla_X v \, . \tag{3.6.3}$$

One extends ∇ to an endomorphism of $\Omega^* \otimes \mathcal{V}$ by requiring that for α in Ω^p, one has

$$\nabla(\alpha x) = d\alpha \cdot x + (-1)^p \alpha \cdot \nabla x \, , \tag{3.6.4}$$

a formula which generalizes (3.3.3). With our convention of separating parity and cohomological degree, the sign in (3.6.4) does not depend on the parity of α. Only its cohomological degree matters. The operator ∇, of cohomological degree 1, is of parity 0.

The curvature, in $\Omega^2 \otimes \underline{\mathrm{End}}\,(\mathcal{V})$, is defined by

$$\nabla^2 x = F \cdot x \tag{3.6.5}$$

for x in $\Omega^* \otimes \mathcal{V}$. The curvature is of even parity. The product $F \cdot x$ has the following meaning: for T an even or odd endomorphism of $\mathcal{V}$, α in Ω^2, β in Ω^p and v in $\mathcal{V}$,

$$(\alpha \otimes T) \cdot (\beta \otimes v) = (-1)^{p(T)p(\beta)} (\alpha \wedge \beta) \otimes T(v) \, .$$

A vector field X defines a contraction operator ι_X on Ω^* of the same parity as X and of cohomological degree -1. It is characterized by the properties that $\iota_X f = 0$ for f in Ω^0, that $\iota_X \alpha = \langle X, \alpha\rangle$ for α in Ω^1 and that it is a derivation of the commutative graded super algebra Ω^*: for α in Ω^k,

$$\iota_X(\alpha \wedge \beta) = \iota_X(\alpha) \wedge \beta + (-1)^k (-1)^{p(X)p(\alpha)} \alpha \wedge \iota_X(\beta) \, .$$

Classically, for α a k-form and X_i vector fields, one defines $\alpha(X_1, \ldots, X_k)$ as follows: if $\alpha = \alpha_1 \wedge \ldots \wedge \alpha_k$ for 1-forms α_i, then

$$\alpha(X_1, \ldots, X_k) = \det(\alpha_i(X_j)) \, . \tag{3.6.6}$$

This definition does not follow the Koszul sign rule, but this is no obstruction to extending it to the super case, thanks to the separation of the roles of cohomological degree and of parity. Using the "even rules" principle, one defines $\alpha(X_1, \ldots, X_k)$ by requiring (3.6.6) to hold when α and the X_i are of even parity.

For a k-form with values in a vector bundle $\mathcal{E}$, i.e. for α in $\Omega^k \otimes \mathcal{E}$, the sign rule requires in addition that $(\alpha \otimes e)(X_1, \ldots, X_k) = (-1)^{p(e) \sum p(X_i)} \alpha(X_1, \ldots, X_k) e$. One should also remember that $\alpha(X) = (-1)^{p(\alpha)p(X)} \langle X, \alpha \rangle$.

With these definitions, ∇_X acting on $\mathcal{V}$ gives rise to the usual formula

$$F(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]} \ .$$

Since F is even, one has

$$F(X,Y) = -\iota_X \iota_Y F \ .$$

A possible convention for defining the components of F in a moving frame D_α is

$$F_{ab} = -\iota_{D_a} \iota_{D_b} F \ .$$

With this convention,

$$F_{ab} = [\nabla_{D_a}, \nabla_{D_b}] - \nabla_{[D_a, D_b]}$$

and if ω^a is the coframe for which $\langle D_a, \omega^b \rangle = \delta_a^b$,

$$F = \frac{1}{2} \omega^a \wedge \omega^b F_{ab} \qquad \text{(sum on } a,\ b) \ .$$

§3.7. Actions of super Lie algebras; vector fields and flows; Lie derivative

Suppose a super Lie group G acts on a super manifold M. Differentiating at the identity e of G the map $G \times M \to M$, we obtain a map ρ from the tangent space T_eG to vector fields on M. For X in T_eG, differentiating in g the identity $(gh)m = g(hm)$, one sees that $G \times M \to M$ maps the right invariant vector fields on G corresponding to X to $\rho(X)$. By 3.3.6, it follows that ρ is a morphism from the Lie aglebra of G to the opposite of the Lie algebra of vector fields on M. From Theorem 3.5.1, one deduces the converse relation, at least locally: Namely, given a morphism Φ from the super Lie algebra of G to the opposite of the super Lie algebra of vector fields on M there is a morphism from a neighborhood of $\{e\} \times M$ in $G \times M$ to M which is a local action whose differential induces Φ.

In particular, if $\mathcal{L}$ is the super Lie algebra of a super Lie group G, any super Lie algebra morphism from $\mathcal{L}$ to the tangent bundle T of M, with $\mathcal{O} \otimes \mathcal{L} \to T$ the embedding of a vector bundle direct factor, is locally isomorphic to a standard $\mathcal{L} \to T_{N \times G}$, with $\mathcal{L}$ mapping to the left invariant vector fields on G.

Examples. (i) The super Lie algebra $\mathbb{R}^{0|1}$ is generated by a single odd element D with $D^2 = \frac{1}{2}[D,D] = 0$. A non-vanishing odd vector field D on a supermanifold, with $D^2 = 0$, is, in a suitable local coordinate system, the vector field $\partial/\partial\theta^1$.

(ii) The super Lie algebra of the super Lie group $\mathbb{R}^{1|1}$ of 3.3.6 is generated by a single odd element D with $D^2 = 2[D,D] \neq 0$. If D is an odd vector field on a super manifold and D^2 is nonvanishing, then, in a suitable local coordinate system, D is $\partial/\partial\theta^1 + \theta^1 \partial/\partial x^1$.

(iii) An even vector field X on a supermanifold M defines a flow $\exp(tX)$: $\mathbb{R} \times M \to M$. For M noncompact, flow lines can escape to infinity in finite time and the flow is defined only in some neighborhood of $\{0\} \times M$. For X an even vector field, and for $\mathcal{V}$ a bundle canonically associated to a supermanifold, the *Lie derivative* of a section of $\mathcal{V}$ is defined by

$$\mathcal{L}_X(s) := \lim_{t\to 0} \tfrac{1}{t}(\exp(tX)^* s - s) \ . \tag{3.7.1}$$

Example (iii) holds as well for a family $M \to S$ of supermanifolds and a vertical even vector field. The induced flow commutes with the projection to S. For odd vector fields, following the "even rules" principle of (1.7), $\mathcal{L}_X$ is defined so that it is compatible with changes of base and that for ε an odd function on the base, $\mathcal{L}_{\varepsilon X} = \varepsilon\mathcal{L}_X$. The Lie derivative $\mathcal{L}_X$ is $\mathcal{O}_S$-linear, of the same parity as X.

Proposition 3.7.2. *For any vector fields X, Y on a supermanifold M and any function f we have*

$$\mathcal{L}_X(f) = X(f) \quad \text{and} \quad \mathcal{L}_X(Y) = [X,Y].$$

Proof. The first equation is clear from the definition. To prove the second notice that the Lie deriviative satisfies the Leibniz rule in the sense that for any bundles $\mathcal{V}_1, \mathcal{V}_2, \mathcal{V}_3$ canonically associated to a supermanifold and for any canonical pairing $\mu\colon \mathcal{V}_1 \otimes \mathcal{V}_2 \to \mathcal{V}_3$ we have

$$\mathcal{L}_X(\mu(v_1 \otimes v_2)) = \mu(\mathcal{L}_X(v_1) \otimes v_2) + (-1)^{p(X)p(v_1)}\mu(v_1 \otimes \mathcal{L}_X(v_2)).$$

In particular,

$$\begin{aligned} X(Y(f)) = \mathcal{L}_X(Y(f)) &= \mathcal{L}_X(Y)(f) + (-1)^{p(X)p(Y)}Y(\mathcal{L}_X(f)) \\ &= \mathcal{L}_X(Y)(f) + (-1)^{p(X)p(Y)}Y(X(f)). \end{aligned}$$

We also have the super version of the Cartan formula.

Proposition 3.7.3: Cartan Formula. *For any vector field X let the contraction i_X be as defined in (3.6). Then we have the following equality of derivations on the de Rham complex Ω^*_M:*

$$\mathcal{L}_X = [d, i_X] = di_X + i_X d.$$

Proof. The action of $\mathcal{L}_X$ on the de Rham complex Ω^*_M is determined by the following properties, all deduced from (3.7.1) and (3.7.2): it is a derivation of the commutative graded super algebra Ω^*_M (cf. §1, App.) of cohomological degree 0 and of parity that of X; it commutes with d; on functions, $\mathcal{L}_X f = Xf = \langle X, df\rangle$. The Cartan formula then results from the fact that the derivations on each side of the equality agree on $\mathcal{O}$ and Ω^1.

§3.8. Super Lie groups and Harish-Chandra pairs

The following construction reduces the relation between super Lie groups and Lie algebras to its classical analog. Define a *Harish-Chandra pair* to be a system consisting of a Lie group K, a Lie algebra $\mathfrak{g}$, acted on by K, and a K-equivariant embedding ε: $\mathrm{Lie}(K) \hookrightarrow \mathfrak{g}$, such that

$$\mathrm{ad}\,\varepsilon = d \text{ (action of } K \text{ on } \mathfrak{g}) \ .$$

A super Lie group G defines a Harish-Chandra pair with $K = G_{\mathrm{red}}$, $\mathfrak{g} = \mathrm{Lie}\,G$ and ε the embedding of the even part $\mathfrak{g}_0$ in $\mathfrak{g}$. This construction is an equivalence of categories from super Lie groups to Harish-Chandra pairs with K an ordinary Lie group and $\mathrm{Lie}\,K$ the even part of $\mathfrak{g}$.

(3.8.1)

An action of a super Lie group G on a super manifold M is equivalent to an action of the underlying reduced group on M and a morphism from the super Lie algebra $\mathfrak{g}$ of G to the opposite to the super Lie algebra of vector fields on M which are compatible in the sense that the differential of the action of the reduced group at the identity agrees with the restriction to the even sub Lie algebra $\mathfrak{g}_0$.

§3.9. Densities

A *density* on an open submanifold U of $\mathbb{R}^{p|q}$ is an $\mathbb{R}$-linear $\mathbb{R}$-valued form on the space of functions with compact support on U, of the form

$$g = \sum g_I(\mathbf{t})\theta^I \longmapsto \sum_I \int d_I(\mathbf{t})g_I(\mathbf{t})dt^1 \dots dt^p \ , \tag{3.9.1}$$

for C^∞-functions d_I on $\mathbb{R}^p$. For μ a density, the linear form μ is usually written $g \mapsto \int \mu g$. For $V \subset U$, a density on U induces one on V, and densities form a sheaf on $\mathbb{R}^{p|q}$. It is a sheaf of super vector spaces, and it has a natural structure of $\mathcal{O}$-module, defined by

$$\int (\mu u)g = \int \mu(ug) \quad \text{for } u \text{ a local section of } \mathcal{O} \ . \tag{3.9.2}$$

Proposition 3.9.3. *The $\mathcal{O}$-module $\mathcal{D}$ of densities on $\mathbb{R}^{p|q}$ is free of rank* $(1|0)$ *if q is even, of rank* $(0|1)$ *if q is odd.*

Proof. We leave it to the reader to check that the following density $[dt^1 \cdots dt^p\, d\theta^1 \cdots d\theta^q]$ is a basis of $\mathcal{D}$: to $g = \sum g_I\theta^I$, it associates the integral $\int dt^1 \cdots dt^p h$ of the coefficient h of $\theta^q \cdots \theta^1$.

§3.10. Change of variables formula for densities

If φ: $U \to V$ is an isomorphism between open submanifolds of $\mathbb{R}^{p|q}$, φ transforms densities on U into densities on V. To check this, one may observe that the densities are the linear functionals which are continuous for the distribution topology on the components g_I of g, and that this topology is respected by the change of variables φ. Indeed, in components, $g \mapsto g \circ \varphi$ involves only change of variables, C^∞ derivations and multiplication by C^∞ functions.

This allows us to generalize 3.9 to a definition of a density on any supermanifold: it is a linear functional on the space of functions with compact support whose restriction to any local coordinate chart is of the form (3.9.1).

From 3.9.3, it follows that if φ: $U \to V$ is an isomorphism between open submanifolds of $\mathbb{R}^{p|q}$, then for some even function $j(\varphi)$, one has

$$\int_V [dt^1 \cdots dt^p\, d\theta^1 \cdots d\theta^q]g = \int_U [dt^1, \dots, dt^p\, , d\theta^1 \dots , d\theta^q]\, j(\varphi) g \circ \varphi \tag{3.10.1}$$

(change of variables formula).

Using the basis $\partial/\partial t^1, \dots, \partial/\partial t^p$, $\partial/\partial\theta^1, \dots, \partial/\partial\theta^q$ of the tangent bundle T of $\mathbb{R}^{p|q}$, one identifies it with $\mathcal{O}^{p|q}$, and $d\varphi$: $TU \to \varphi^*TV$ becomes an automorphism of $\mathcal{O}_U^{p|q}$. We denote by $\mathrm{Ber}(d\varphi)$ the Berezinian of this automorphism. We write $\mathrm{or}(\varphi)$ for the sign ± 1, equal to $+1$ where φ: $|U| \xrightarrow{\sim} |V|$ preserves the orientation of $\mathbb{R}^p$, and to -1 where it doesn't.

Proposition 3.10.2. *The change of variables factor $j(\varphi)$ in 3.10.1 is given by*

$$j(\varphi) = \mathrm{or}(\varphi)\mathrm{Ber}(d\varphi)\ . \tag{3.10.3}$$

A local coordinate system $t^1, \dots, \theta^q$ defines an orientation of $|M|$, a section $[dt^1, \dots, d\theta^q]$ of $\mathrm{Ber}(\Omega^1)$ and a density which, by anticipation, we denoted $[dt^1 \cdots d\theta^q]$ as well (see 3.9.3). For u with compact support, one defines

$$\int (\text{section } [dt^1, \dots, d\theta^q] \text{ of } \mathrm{Ber}(\Omega^1))\, u := \int (\text{density } [dt^1 \cdots d\theta^q])u\ . \tag{3.10.4}$$

Proposition 3.10.5. *Let M be a super manifold. Let $\mathrm{or}_{|M|}$ denote the orientation sheaf of its reduced submanifold. Then there is a unique $\mathbb{R}$-linear map*

$$\int_M : \Gamma_0(\mathrm{Ber}(\Omega^1_M) \otimes \mathrm{or}_{|M|}) \to \mathbb{R}$$

from the super vector space of compactly supported sections of the Berezinian line bundle of M twisted by the orientation sheaf for $|M|$ with the following property. Let $U \subset M$ be an open subset with coordinates $t^1, \dots, \theta^q$. Let φ be a section of $\mathrm{Ber}(\Omega^1_M)$ with compact support contained in U which written in these coordinates is given by $[dt^1, \dots, d\theta^q]u$. Let τ be the section of $\mathrm{or}_{|U|}$ determined by these coordinates. Then $\int_M \varphi \otimes \tau$ agrees with

$$\int [dt^1 \cdots d\theta^q]u$$

as given in 3.10.4.

Proof. This is proved using a partition of unity to decompose any compactly supported section as a sum of sections supported in coordinate charts. The independence of the resulting sum on the chosen decomposition uses the change of variables formula (3.10.3).

Thus, what can be canonically integrated over a super manifold M is a compactly supported section of $\mathrm{Ber}(\Omega^1_M)$, twisted by the orientation bundle of $|M|$. When $|M|$ is oriented, we can integrate compactly supported sections of $\mathrm{Ber}(\Omega^1_M)$.

We will sketch the proof of a relative statement generalizing (3.10.2). For $M \to S$ a family of supermanifolds, a *relative density* on M is a suitably continuous $\mathcal{O}_S$-linear morphism $f_!: \mathcal{O}_M \to \mathcal{O}_S$, where $f_!$ is the direct image with compact supports. Notation: $\int_{M/S} \mu g$. In local coordinates, "suitably continuous" means that for $U \times V \to V$ with $U \subset \mathbb{R}^{p|q}$ and $V \subset \mathbb{R}^{r|s}$, the components of $\int_{U\times V/V} \mu g$ should be obtained from the components of g by integration along the fibers of $|U| \times |V| \to |V|$, relatively to some C^∞ relative densities on $|U| \times |V|$.

If M'/S' is deduced from M/S by a base change u: $S' \to S$:

$$\begin{array}{ccc} M' & \xrightarrow{u_M} & M \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ S' & \xrightarrow{u} & S\,, \end{array}$$

a relative density μ on M has a pullback μ' to M', characterized by

$$\int \mu' \, u_M^*(g) f'^*(h) = u^* \left(\int \mu \, g \right) h$$

for g on M and h on S'. If M/S is of relative dimension $p|q$, the relative densities on M form a $\mathcal{O}_M$-module locally free of rank $1|0$ if q is even, $0|1$ if q is odd. In local coordinates, for M an open subspace of $\mathbb{R}^{p|q} \times S$, the relative density $[dt^1 \cdots d\theta^q]$ (the pullback of the density $[dt^1 \cdots d\theta^q]$ of $\mathbb{R}^{p|q}$) is a basis.

Let φ be a S-isomorphism between open subsets U and V of $\mathbb{R}^{p|q} \times S$. The module $\mathcal{D}$ of relative densities being of rank $1|0$ or $0|1$ (according to the parity of q), we know a priori that for some even function $j(\varphi)$, one has

$$\varphi^*[dt^1, \dots, d\theta^q] = [dt^1, \dots, d\theta^q] j(\varphi)$$

on U, i.e. that for g on V,

$$\int_{V/S} [dt^1, \dots, d\theta^q] g = \int_{U/S} [dt^1, \dots, d\theta^q] j(\varphi) g \circ \varphi \, . \tag{3.10.6}$$

We identify the relative tangent bundle T of $\mathbb{R}^{p|q} \times S$ with the pullback of the tangent bundle $\mathcal{O}^{p|q}$ of $\mathbb{R}^{p|q}$, and write $\mathrm{Ber}(d\varphi)$ for the Berezinian of $d\varphi$: $TU \to \varphi^* TV$, identified with an automorphism of $\mathcal{O}_U^{p|q}$. We write $\mathrm{or}(\varphi)$ for the sign ± 1, equal to $+1$ where φ: $|U| \to |V|$, restricted to each fiber over $|S|$, preserves the orientation of $\mathbb{R}^p$, and to -1 where it doesn't. The generalization of (3.10.2) to the relative case is:

Proposition 3.10.7. *With notation as above, one has again*

$$j(\varphi) = \mathrm{or}(\varphi) \mathrm{Ber}(d\varphi) \, .$$

Both sides of (3.10.7) are compatible with a base change $S' \to S$, and at the cost of considering not just φ but also any morphism deduced from φ by base change (this changes nothing in the setting), it suffices to check (3.10.7) at each point, in the sense of 2.9: for u a section of U/S, one should check (3.10.7) along u (i.e. on S, after pullback by u). By translation invariance, we may and shall assume that u and $\varphi(u)$ are the section 0 of $\mathbb{R}^{p|q} \times S/S$.

Lemma 3.10.8. *Proposition 3.10.7 holds for $U = V = \mathbb{R}^{p|q} \times S$, and for φ a family of linear maps from $\mathbb{R}^{p|q}$ to $\mathbb{R}^{p|q}$.*

Proof. By translation invariance, if φ is linear, $j(\varphi)$ or (φ) as well as $\mathrm{Ber}(d\varphi)$ are pullbacks of functions on S. These functions are multiplicative: they define morphisms

$$j\,, \mathrm{or} \text{ and } \mathrm{Ber}\colon \ \mathrm{GL}(p|q) \to \mathrm{GL}(1|0) \, .$$

Matrices of the form $\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$ or $\begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix}$ being commutators, the morphisms j, or and Ber are determined by their restriction to $\mathrm{GL}(p) \times \mathrm{GL}(q) \subset \mathrm{GL}(p|q)$. This group being an ordinary manifold, it suffices to prove (3.10.8) for φ any point of $\mathrm{GL}(p) \times \mathrm{GL}(q)$, i.e. for S a point and for φ: $\mathbb{R}^{p|q} \to \mathbb{R}^{p|q}$ linear, of the form $\begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}$.

By a compatibility with products, one can even consider just the cases of $\mathbb{R}^{p|0}$ and $\mathbb{R}^{0|q}$. In the first case, one has just the classical change of variables formula on $\mathbb{R}^p$, for a linear change of variable, leading to a factor $|\det(\varphi)| = \mathrm{or}(\varphi)\det(d\varphi)$. In the case of $\mathbb{R}^{0|q}$, one observes that for a linear map $\varphi\colon \mathbb{R}^{0|q} \to \mathbb{R}^{0|q}$, the degree in θ is preserved by φ^* and that

$$\varphi^*(\theta^1 \cdots \theta^q) = \det(\text{matrix of } \varphi) \cdot \theta^1 \cdots \theta^q \ .$$

This completes the proof of Lemma 3.10.8.

Proof of 3.10.7. Suppose that $\varphi\colon U \to V$ maps 0 to 0. We deform φ to its linearization by the standard operation of zooming in at 0. For any λ, let $R(\lambda)$ be the homothety of ratio λ. Define

$$\varphi_\lambda := R(\lambda^{-1})\varphi R(\lambda) \ .$$

By multiplicativity of $\varphi \mapsto j(\varphi)$ for a composite map, $j(\varphi_\lambda)$ at 0 is independent of λ. The family φ_λ extends to $\lambda = 0$, with $\varphi_0 =$ linear part $d\varphi$ of φ. By constancy, it suffices to prove 3.10.7 for φ_0, to which 3.10.8 applies.

§3.11. The Lie derivative of sections of $\mathrm{Ber}(\Omega^1_M)$

The Lie derivative has been defined in 3.7. One has:

Proposition. *For v a section of $\mathrm{Ber}(\Omega^1_M)$ one has*

$$\mathcal{L}_{fX}(v) = (-1)^{p(f)p(X)}\mathcal{L}_X(fv) \ . \tag{3.11.1}$$

By the sign rule for modules over commutative super algebras, $vf := (-1)^{p(f)p(v)} fv$. If we define $vX := (-1)^{p(v)p(X)}\mathcal{L}_X v$, (3.11.1) can be rewritten

$$(vf)X = v(fX) \ , \tag{3.11.2}$$

an identity which allows us to define

$$d\colon \mathrm{Ber}(\Omega^1_M) \otimes_{\mathcal{O}_M} T \to \mathrm{Ber}(\Omega^1_M) \tag{3.11.3}$$

by $d(v \otimes X) = vX$.

Proof. We will prove (3.11.1) in a relative setting as well. The question being local and compatible with changes of base, applying the "even rules" principle, we may and shall suppose X and f even. We may and shall assume in addition that the reduced fibers of M/S are oriented, and that v has compact support.

By transport of structures, integration of sections with compact support of $\mathrm{Ber}(\Omega^1_{M/S})$ is invariant by orientation preserving diffeomorphisms, and in particular

$$\int_{M/S} \exp(tX)^* v = \int_{M/S} v \ ,$$

hence by passage to the limit for $t \to 0$

$$\int_{M/S} \mathcal{L}_X v = 0 \,. \tag{3.11.4}$$

For any function u, $\mathcal{L}_X(uv) = \mathcal{L}_X u \cdot v + u \cdot \mathcal{L}_X v = Xu \cdot v + u \cdot \mathcal{L}_X v$. Applying (3.11.4) to uv, we obtain

$$\int u\mathcal{L}_X v = -\int Xu \cdot v$$

and, for f even,

$$\int u\mathcal{L}_{fX} v = -\int fXu \cdot v = -\int Xu \cdot (fv) = \int u\mathcal{L}_X (fv) \,.$$

This holding for any u, (3.11.1) follows.

§3.12. Integral forms

For an ordinary manifold M of dimension n, one has on Ω^*_M the operations $\alpha\wedge$ for α in Ω^1_M and i_X for X in the tangent bundle T. They turn Ω^*_X into a module over the Clifford algebra generated by $\Omega^1_M \oplus T$, meaning that $(\alpha\wedge)^2 = (i_X)^2 = 0$ and that $i_X \circ \alpha\wedge + \alpha\wedge \circ i_X = \langle X, \alpha \rangle$. One can define Ω^* as being the Clifford module freely generated by $\mathcal{O}$, with the relations $i_X f = 0$ for f in $\mathcal{O}$. One could also define it as the Clifford module freely generated by $\omega := \Omega^n$, with the relations $\alpha\wedge v = 0$ for v in ω.

From the second point of view, the exterior derivative d can be characterized by the properties that $d = 0$ on ω and that $di_X + i_X d = \mathcal{L}_X$. Indeed, the formula

$$d(i_{X_1} \cdots i_{X_p} v) = \mathcal{L}_{X_1}(i_{X_2} \cdots i_{X_p} v) - i_{X_1} d(i_{X_2} \cdots i_{X_p} v) \tag{3.12.1}$$

gives inductively a way to compute d. As $d = 0$ on ω, Cartan's formula reduces for v in ω to

$$di_X v = \mathcal{L}_X v \ ,$$

which shows that $\mathcal{L}_{fX} v = \mathcal{L}_X fv$. One can check using just this property that (3.12.1), used as a definition of d, will not lead to contradiction.

In the case of a supermanifold M of dimension $p|q$, one can use the second construction to construct a complex I^*_M, with top component $\omega := \mathrm{Ber}(\Omega^1_M)$. If $q \neq 0$, this complex is not the de Rham complex. The de Rham complex is bounded from below, while I^*_M, the complex of *integral forms*, is bounded from above.

More precisely, I^*_M is defined as follows.

(a) As a graded $\mathcal{O}$-module, it is freely generated from $\omega := \mathrm{Ber}(\Omega^1)$, put in cohomological degree p, by $\mathcal{O}$-linear operations i_X of cohomological degree -1 obeying $[i_X, i_Y] = 0$:

$$i_X i_Y + (-1)^{p(X)p(Y)} i_Y i_X = 0 \ ,$$

and $i_{fX} = fi_X$. Iterated "contractions" i_X induce isomorphisms

$$\wedge^i T \otimes \omega \xrightarrow{\sim} I^{p-i} \,. \tag{3.12.2}$$

(b) Operations $\alpha\wedge$ of comological degree 1, for α in Ω^1, are defined by $\alpha\wedge v = 0$ for v in ω and $[i_X, \alpha\wedge] = \langle X, \alpha\rangle$, i.e. $i_X(\alpha\wedge\eta) - (-1)^{p(X)p(\alpha)}\alpha\wedge i_X\eta = \langle X, \alpha\rangle\,\eta$. The operations $\alpha\wedge$ give I^* the structure of a module over Ω^*.

(c) An *exterior derivative* d is defined. It is characterized as being of cohomological degree 1 (implying that $d = 0$ on ω) and $[d, i_X] = \mathfrak{L}_X$.

Exercise. Check that this is a valid definition, and that for α in Ω^i and $\eta \in I^*$,

$$d(\alpha\wedge\eta) = d\alpha\wedge\eta + (-1)^i\alpha\wedge d\eta\ .$$

Proposition 3.12.3. *Let M be a supermanifold of dimension $p|q$ with $|M|$ oriented. Let η be a compactly supported section of I^{p-1}. Then*

$$\int_M d\eta = 0.$$

Proof. By (3.12.2) every $\eta \in I^{p-1}$ is a linear combination of elements of the form $i_{X_a}(\alpha_a)$ for X_a a vector field and $\alpha_a \in \omega$. If η has compact support, we can take the α_a to have their support in a slightly larger compact set. Thus, it suffices to prove this result for $\eta = i_X(\alpha) \in I^{p-1}$. But, by (3.10.5) and (3.11.4) we have

$$0 = \int_M \mathfrak{L}_X(\alpha) = \int_M di_X(\alpha) + i_X d\alpha = \int_M di_X(\alpha).$$

§3.13. A second definition of integral forms

Another way to define the complex I^*_M is by duality, repeating the definition we gave for densities: once an orientation of M_{red} has been chosen, and for a suitable topology on Ω^i, the space of sections with compact support of I^{p-i} is the dual of the space of sections of Ω^i. In the approach of 3.12, the duality is

$$\alpha, \eta \longmapsto \int_M \alpha\wedge\eta\ .$$

Contractions, products with forms and d are then defined as transposes of the same operations for Ω^*:

$$\int \alpha\wedge(\beta\wedge\eta) = \int(\alpha\wedge\beta)\wedge\eta\ ,$$
$$\int \alpha\wedge i_X\eta = -(-1)^{p(\alpha)p(X)}\int i_X(\alpha)\wedge\eta\ ,$$
$$\int \alpha\wedge d\eta = -(-1)^{\deg(\alpha)}\int d\alpha\wedge\eta\ .$$

From the point of view of 3.12, the last identity results from $\int d\eta = 0$, as established in (3.12.3).

The virtue of integral forms, explaining their name, and the reason Bernstein and Leites [**1977**] introduced them, is that an integral form of dimension $p - i$ can be integrated on submanifolds of codimension $i|0$. We will give two explanations of what happens, leaving proofs as exercises for the reader.

§3.14. Generalized functions

Generalized functions on a super manifold M can be defined as the elements either of a completion of the space $\Gamma(M, \mathcal{O})$ of functions, or as the dual of the space of compactly supported densities with the C^∞ topology. In a local coordinate system $x^1, \ldots, x^p$, $\theta^1, \ldots, \theta^q$, they are written $\sum T_I \theta^I$, with the T_I generalized functions, a.k.a. distributions, on $\mathbb{R}^p$. Pullbacks by submersive maps are defined. The generalized functions form an $\mathcal{O}$-module $\mathcal{O}^\frown$, and for any vector bundle $\mathcal{V}$, the *generalized sections* of $\mathcal{V}$ are the sections of $\mathcal{O}^\frown \otimes_{\mathcal{O}} \mathcal{V}$.

Example. On $\mathbb{R}$, let Y be the generalized (L^1) function defined by $Y(x) = 0$ for $x < 0$, $Y(x) = 1$ for $x > 0$. One has

$$dY = \delta(x) dx \tag{3.14.1}$$

For f: $M \to \mathbb{R}^{1|0}$ an even function on a super manifold M, submersive where it vanishes, $\int_{f \geq 0} \eta$ is defined to be $\int Y(f)\eta$. This integral depends not only on the region $f \geq 0$ of M_{red}, but also on the chosen "boundary", the submanifold of codimension $1|0$ given by the equation $f = 0$. For instance, on $\mathbb{R}^{1|2}$,

$$Y(x + \theta^2\theta^1) = Y(x) + \delta(x)\theta^2\theta^1$$

and for the standard orientation of $\mathbb{R}$

$$\int_{x+\theta^2\theta^1 \geq 0} [dx\, d\theta^1 d\theta^2] u(x) = u(0) + \int_{x \geq 0} [dx\, d\theta^1 d\theta^2] u(x) \ .$$

If $N \subset M$ is a submanifold of codimenson $i|0$, a normal orientation of N defines a generalized section δ_N of Ω^i_M: if $f_1, \ldots, f_\imath$ is a system of equations, compatible with the normal orientation,

$$\delta_N := \delta(f_1) \cdots \delta(f_\imath) df_1 \wedge \cdots \wedge df_i$$

does not depend on the choice of equations. If η is an integral form in $I^{p-\imath}_M$, its restriction to I^{p-i}_N is defined by

$$\int_N f(\eta|N) = \int_M f\delta_N \wedge \eta$$

for f with compact support on M.

A Stokes formula (Bernstein and Leites (1977)) for integrals $\int_{f \geq 0}$ over domains with boundary then results from the formula $\int d\eta = 0$.

§3.15. Integral forms as functions of infinitesimal submanifold elements

Let V be a vector space of dimension N. Fix $p \leq N$ and let G be the Grassmannian variety of subspaces F of V of dimension p. A p-form $\alpha \in \wedge^p V^*$ induces on each $F \in G$ an element of $\det(F^*) = \wedge^p F^*$. If $\det(F^*)$ is the line bundle on G with fiber $\det(F^*)$ at F, one has in algebraic geometry, i.e. for "polynomial" sections of $\det(F^*)$,

$$\wedge^p V^* \xrightarrow{\sim} H^0(G, \det(F^*)) \ .$$

It follows that to give a p-form on a manifold M is the most general way to give a form α_N of maximal degree on all submanifolds N of M of dimension p, if one insists that (a) at $n \in N$, the value of α_N at n depends only on the tangent space $T_{N,n} \subset T_{M,n}$ of N at n; (b) at fixed $n \in M$, α_N on $T \subset T_{M,n}$ is "polynomial" in T.

In the case of super manifolds, the appearance of the integral forms is similarly made natural by the following

Lemma 3.15.1. *Let V be a super vector space of dimension $N|M$. Let G be the super scheme of subspaces F of dimension $(N-p)|M$. On G, let ω be the line bundle* $\mathrm{Ber}(F^\vee)$. *Then,*

$$H^0(G, \omega) = \wedge^p V \otimes \mathrm{Ber}(V^*) \ .$$

For submanifolds of dimension $i|j$ of a super manifold of dimension $N|M$, such a game cannnot be played if $j \neq 0$ or M:

Lemma 3.15.2. *Let V be a super vector space of dimension $N|M$ and G be the super scheme of subspaces of codimension $i|j$, with $j \neq 0$, M. On G, let ω be the line bundle* $\mathrm{Ber}(F^\vee)$. *Then $H^0(G, \omega) = 0$.*

Sketch of proof of 3.15.2. We first consider the case when $i = 0$. The reduced scheme G_{red} is then the ordinary grassmannian of j-dimensonal subspaces F of the M-dimensional space V_1. On G_{red}, the restriction of ω is the line bundle $\det(F)$. The normal bundle of G_{red} in G is the odd bundle $\mathrm{Hom}(F, V_0)$. Working modulo successive powers of the ideal of G_{red} in G, this reduces us to checking, in ordinary algeraic geometry, the vanishing of the

$$H^0(G_{\mathrm{red}}, \det(F) \otimes \wedge^a(\mathrm{Hom}(F, V_0)^\vee)) \ .$$

Now we consider the case $i > 0$. Let $\widetilde{G}$ be the super scheme of partial flags of type $i|0 \subset i|j$, and H be the super scheme of subspaces of dimension $i|0$. One has forgetful maps

$$G \xleftarrow{p} \widetilde{G} \xrightarrow{q} H \ .$$

As p is smooth, it suffices to check that $H^0(\widetilde{G}, p^*\omega) = 0$. It then suffices to check such a vanishing for the fibers of q, which are super schemes of subspaces of dimension $0|j$ in a space of dimension $(N-i)|M$, and these have vanishing cohomology.

CHAPTER 4
Real Structures

§§4.1–4.3. Real structures and $*$-operations

4.1. A *real structure* on a complex super algebra A is the data of a real super algebra $A_{\mathbb{R}}$ and of an isomorphism of complex super algebras from$A_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ to A. This amounts to giving an $\mathbb{C}$-antilinear involutive automorphism σ: $A \to A$: take for $A_{\mathbb{R}}$ the fixed points of σ.

Another kind of real structure is the data of an involutive $\mathbb{C}$-antilinear isomorphism from A to the opposite algebra: a $*$-operation, obeying

$$(xy)^* = (-1)^{p(x)p(y)} y^* x^* \tag{4.1.1}$$

For a commutative super algebra, such a $*$-operation is the same thing as a real structure.

In the classical (non super) setting, algebras with a $*$-operation are a natural "noncommutative" analogue of topological spaces, when topological spaces are viewed as encoded by the commutative algebra of continuous functions on them.

The formula (4.1.1) is the natural extension of the formula $(xy)^* = y^* x^*$ to the case of super algebras, and super algebras with a $*$-operation do occur as quantizations of super symplectic manifolds.

4.2. The notions 4.1 of real structures depend, from the categorical point of view, on the fact that the complex tensor category $\mathcal{T}_{\mathbb{C}}$ of complex super vector spaces is deduced by extension of scalars from $\mathbb{R}$ to $\mathbb{C}$ from the real tensor category $\mathcal{T}_{\mathbb{R}}$ of real super vector spaces. The complex tensor category $\mathcal{T}_{\mathbb{C}}$ is also deduced by extension of scalars from the following "quaternionic" real tensor category $\mathcal{T}'_{\mathbb{R}}$. The objects of $\mathcal{T}'_{\mathbb{R}}$ are the complex super vector spaces V, given with a $\mathbb{C}$-antilinear automorphism σ: $V \to V$ for which

$$\sigma^2 x = (-1)^{p(x)} x\ . \tag{4.2.1}$$

The tensor product is that of $\mathcal{T}_{\mathbb{C}}$, with $\sigma(x \otimes y) = \sigma(x) \otimes \sigma(y)$. The extension of scalars from $\mathcal{T}'_{\mathbb{R}}$ to $\mathcal{T}_{\mathbb{C}}$ is "forgetting σ".

To the "real form" $\mathcal{T}'_{\mathbb{R}}$ of $\mathcal{T}_{\mathbb{C}}$ correspond new notions of real structures.

Examples. (i) A purely odd object of $\mathcal{T}'_{\mathbb{R}}$ can be identified with a vector space over the field $\mathbb{H}$ of quaternions, with σ being the multiplication by j.

(ii) In $\mathcal{T}'_{\mathbb{R}}$, one has a notion of "positive definite" symmetric bilinear form on an object V. Concretely, a symmetric bilinear form ψ is the data of a symmetric form ψ_0 on V_0 and of an alternating form ψ_1 on V_1, with $\psi_i(\sigma x, \sigma y) = \overline{\psi_i(x,y)}$.

From this reality condition it follows that $\psi_i(x, \sigma x)$ is real:

$$\overline{\psi_i(x,\sigma x)} = \psi_i(\sigma x, \sigma^2 x) = (-1)^i \psi_i(\sigma x, x) = \psi_i(x, \sigma x) \ ,$$

and the positivity condition is

$$\psi_i(x,\sigma x) > 0 \qquad \text{for} \qquad x \neq 0 \ .$$

One should beware that the tensor product of two forms which are positive in this sense is not positive, except when one of the underlying super vector spaces is purely even.

(iii) A $\mathcal{T}'_{\mathbb{R}}$-real structure σ (resp. $*$-operation $*$) on a complex super algebra A is σ (resp. $*$): $A \to A$, $\mathbb{C}$-antilinear, obeying (4.2.1) and $\sigma(xy) = \sigma x \cdot \sigma y$ (resp. $(xy)^* = (-1)^{p(x)p(y)} y^* x^*$).

4.3. Manin (1988), Ch. 3, §6 considers additional notions of reality, with no categorical underpinning. A typical one is $*$-operations ρ: $A \to A$, antilinear, involutive, with

$$(ab)^\rho = b^\rho a^\rho \ . \tag{4.3.1}$$

Our understanding is that such objects arose only because physicists did not consistently apply the sign rule when considering super Hilbert spaces. They can be reduced to $*$-operations. Indeed, if ρ: $A \to A$ is antilinear, define

$$a^* = \begin{cases} a^\rho & \text{for } a \text{ even;} \\ ia^\rho & \text{for } a \text{ odd .} \end{cases} \tag{4.3.2}$$

Then, ρ is involutive if and only if $*$ is, and (4.3.1) holds for ρ if and only if (4.4.1) holds for $*$. In particular, for A commutative, (4.3.1) gives $(ab)^* = a^* b^*$.

§4.4. Super Hilbert spaces

From now on, our setting for reality will be 4.1. According to the sign rule, a *Hermitian form* on a complex super vector space H is a morphism ψ: $H \otimes_{\mathbb{R}} H \to \mathbb{C}$ of $\mathbb{R}$-super vector spaces, $\mathbb{C}$-antilinear in the first variable and $\mathbb{C}$-linear in the second, with

$$\overline{\psi(x,y)} = (-1)^{p(x)p(y)} \psi(y,x) \ . \tag{4.4.1}$$

It follows from (4.4.1) that $\psi(x,x)$ is real for x even, and purely imaginary for x odd. As ψ is assumed even, $\psi(x,y) = 0$ for x and y of distinct parities.

The tensor product of Hermitian forms ψ and ψ' on H and H' is defined using the sign rule:

$$\psi \otimes \psi'(x \otimes x', y \otimes y') = (-1)^{p(x')p(y)} \psi(x,y)\psi'(x',y') \ .$$

In finite dimensions, a *super Hilbert space* is an Hermitian super space $(H, \langle\ \ \rangle)$ for which

$$\begin{aligned} \langle x,x\rangle &> 0 \qquad \text{for } x\neq 0 \text{ even;} \\ i^{-1}\langle x,x\rangle &> 0 \qquad \text{for } x\neq 0 \text{ odd .} \end{aligned} \tag{4.4.2}$$

The *adjoint* T^* of $T\colon H\to H$ (homogeneous but not necessarily even) is defined by

$$\langle x, Ty\rangle = (-1)^{p(x)p(T)}\langle T^*x, y\rangle \tag{4.4.3}$$

Categorical reasoning, or a direct argument, shows that $T\mapsto T^*$ is a $*$-operation: it is $\mathbb{C}$-antilinear, $T^{**}=T$ and

$$(TU)^* = (-1)^{p(T)p(U)}U^*T^* \ . \tag{4.4.4}$$

Let H be a super Hilbert space. On the ordinary vector space $H_0\oplus H_1$, we define an ordinary Hilbert space structure $(\ ,\ )$ by

$$\begin{aligned} (x,y) &= 0 && \text{if } x \text{ and } y \text{ are not of the same parity} \\ (x,y) &= \langle x,y\rangle && \text{if } x \text{ and } y \text{ are even} \\ i(x,y) &= \langle x,y\rangle && \text{if } x \text{ and } y \text{ are odd .} \end{aligned} \tag{4.4.5}$$

The adjoint † for $(\ ,\ )$, and the adjoint (4.4.3) are related by

$$T^* = \begin{cases} T^\dagger & \text{for } T \text{ even;} \\ iT^\dagger & \text{for } T \text{ odd .} \end{cases} \tag{4.4.6}$$

(cf. (4.3.2)).

The positivity condition (4.4.2) is stable under tensor product: as $H\otimes H'$ is the orthogonal direct sum of the $H_i\otimes H'_j$, it suffices to check (4.4.2) for x in each of those four summands. The least obvious case is that of the component $H_1\otimes H'_1$, for which we have

$$\begin{aligned} \langle x\otimes x', y\otimes y'\rangle &= -\langle x,y\rangle\langle x',y'\rangle = -i^2(x,y)(x',y') \\ &= (x,y)(x',y') \ . \end{aligned}$$

This formula shows that the Hermitian form $\langle\ ,\ \rangle$ on $H_1\otimes H'_1$ is the ordinary tensor product of the positive Hermitian forms $(\ ,\ )$ on the vector spaces H_1 and H'_1. It is hence positive definite. More generally, the ordinary Hilbert space structure $(\ ,\ )$ of $H\otimes H'$ is the ordinary tensor product of the ordinary Hilbert space structures $(\ ,\ )$ of H and H'.

Passing to infinite dimensions creates the usual analytic difficulties, unchanged in the super setting.

§4.5. SUSY quantum mechanics

In relativistic SUSY quantum mechanics, the Hilbert space H of states of a physical system $\mathcal{S}$ is a super Hilbert space. It carries an action of the *Poincaré* group and the parity is given by the action of its order 2 central element (spin-statistics relation). The Hilbert space corresponding to n indistinguishable copies of $\mathcal{S}$ is $\mathrm{Sym}^n(H)$, where the symmetric power is taken in the super sense. In the non-relativistic setting, one similarly has a projective action of the galilean group. Its restriction to the special orthogonal group defines a linear action of the spin group, and the action of $\mathrm{Ker}(\mathrm{Spin} \to \mathrm{SO})$ defines parity.

The definitions (4.4.2) of super Hilbert space and (4.4.3) of super adjoint disagree with those of the physicists and have the unsettling consequence that if T is odd and self-adjoint, its eigenvalues are in $i^{1/2}\mathbb{R}$. However, the corresponding eigenvectors don't correspond to physical states, being neither even nor odd. This saves our conventions from being physically absurd.

It would have been as consistent as (4.4.2) to impose as positivity condition $i\langle x, x\rangle > 0$ for x odd. For a discussion of related sign conventions, we refer to [I-**Signs**].

A representation ρ of a real super Lie algebra $\mathfrak{g}$ on H is *unitary* if the $\rho(X)$ for X in $\mathfrak{g}$ are anti-Hermitian: $\rho(X) = -\rho(X)^*$. For X odd, this requires $\rho([X, X])$ to have a spectrum in $-i\mathbb{R}^+$. Indeed, the spectrum of $\rho(X)$ is in $i \cdot i^{1/2}\mathbb{R} = i^{3/2}\mathbb{R}$, and $\rho([X, X]) = 2\rho(X)^2$.

§4.6. Real and complex super manifolds

In the classical (non super) setting, two points of view are useful in relating real and complex analytic manifolds. In the first, starting from a complex analytic manifold M, one defines a *real structure* on M as being an involution ρ of M, antilinear in the sense that for f a holomorphic function on $U \subset M$, $\rho^*(f)$ is antiholomorphic, that is $\rho^*(f)^-$ is again holomorphic. The fixed locus M^ρ of ρ, together with the sheaf of sections of $\mathcal{O}|M^\rho$ fixed by $f \mapsto \overline{\rho^*(f)}$, is then a real analytic manifold. One says that M is a *complexification* of M^ρ. Any real analytic manifold N can be obtained in this way, and the germ of M along $N = M^\rho$ is uniquely determined (up to unique isomorphism) by N.

In the second point of view, a complex analytic structure on M is an additional structure on the underlying real analytic manifold $M_\mathbb{R}$. The additional structure is that of an integrable complex structure on the tangent bundle.

The conjugate $\bar{M}$ of the complex analytic manifold M is the space M, with the sheaf $\mathcal{O}_{\bar{M}}$ of antiholomorphic functions on M. The morphism from $\mathcal{O}_M$ to $\mathcal{O}_{\bar{M}}$: $f \mapsto \bar{f}$, is a $\mathbb{C}$-antilinear isomorphism of sheaf of rings. It is convenient to write $m \mapsto \bar{m}$ for the identity map $M \to \bar{M}$: if f is holomorphic on M, then so is $\overline{f(\bar{m})}$ on $\bar{M}$.

From the first point of view, the real analytic variety $M_\mathbb{R}$ *underlying* the complex analytic variety M is to be described as the fixed point set of a real structure on a complexification. One can take as complexification $M \times \bar{M}$, the real structure being $(x, y) \mapsto (\bar{y}, \bar{x})$. The underlying real analytic variety $M_\mathbb{R}$ is embedded in $M \times M^-$ diagonally. In local coordinates: complex-valued real analytic functions on $U \subset \mathbb{C}^N$ can be identified with complex analytic functions of the z_i and $\bar{z}_i$, defined in a neighborhood of $\{(z, \bar{z}) | z \in U\}$. The physicists often use this identification and, with less justification, write $f(z, \bar{z})$ for a C^∞-function on U.

The same two points of view can be used to relate real and complex super manifolds. A *complex conjugate* of a complex analytic super manifold $(M, \mathcal{O}_M)$ is a complex analytic super manifold $(\bar{M}, \mathcal{O}_{\bar{M}})$, provided with a $\mathbb{C}$-antilinear isomorphism $m \to \bar{m}$ of ringed spaces. For instance; $|\bar{M}|$ is $|M|$, and $\mathcal{O}_{\bar{M}}$ is $\mathcal{O}_M$, with the complex conjugate $\mathbb{C}$-algebra structure, and the structural isomorphism is the identity. In an obvious sense, M is the complex conjugate of $\bar{M}$. It is convenient to write $f \mapsto \bar{f}$ for the isomorphism of $\mathcal{O}_M$ with $\mathcal{O}_{\bar{M}}$: for f holomorphic on U, $\bar{f}$ is holomorphic on $\bar{U}$, and $\overline{(\lambda f)} = \bar{\lambda}\bar{f}$.

A *real structure* on a complex super manifold M is an involutive isomorphism $\rho: M \to \bar{M}$. In other words, it is an involutive automorphism ρ of the topological space $|M|$, given with an $\mathbb{C}$-antilinear isomorphism $\rho^*\mathcal{O}_M \to \mathcal{O}_M$, usually written $f \mapsto \bar{f}$. The fixed point set M^ρ of M, provided with the sheaf $\mathcal{O}^\rho$ of complex conjugation invariant sections of $\mathcal{O}|M^\rho$, is then a real analytic super manifold. One again says that M is a complexification of M^ρ, and a real analytic manifold N determines the germ (along N) of its complexifications.

The underlying real analytic super manifold $M_\mathbb{R}$ of a complex analytic super manifold M is the real locus in $M \times M^-$ of the real structure $(x, y) \to (\bar{y}, \bar{x})$. Example: for $\mathbb{C}^{0|1}$, with coordinate θ, the ring of functions on $(\mathbb{C}^{0|1})_\mathbb{R}$ is the fixed points of complex conjugation in $\mathbb{C}[\theta, \bar{\theta}]$. It is $\mathbb{R}[\theta + \bar{\theta}, (\theta - \bar{\theta})/i]$.

Let $z^1, \ldots, z^p, \theta^1, \ldots, \theta^q$ be a local complex coordinate system on M. By abuse of notation, physicists will often write $f(z^1, \ldots, \theta^q, \bar{z}^1, \ldots, \bar{\theta}^q)$ for a C^∞-function on $M_\mathbb{R}$.

§4.7. Complexifications, in infinite dimensions

When dealing with function spaces it can also be convenient to work in complexified spaces, with reality conditions coming in as an afterthought, if needed. If X and Y have complexifications $X_\mathbb{C}$ and $Y_\mathbb{C}$, the complexification of $\underline{\mathrm{Hom}}(X, Y)$ is the space $\underline{\mathrm{Hom}}(X_\mathbb{C}, Y_\mathbb{C})$ of holomorphic maps f from $X_\mathbb{C}$ to $Y_\mathbb{C}$. One requires f to be defined only in a suitable neighborhood of $X \subset X_\mathbb{C}$, and that $f(X)$ be close to $Y \subset Y_\mathbb{C}$.

Example. (i) If v is a section of the complexified tangent bundle of X, with X compact and X and v real analytic, $\exp(tv)$ (t small) belongs to $\mathrm{Diff}(X)_\mathbb{C}$.

(ii) (After G. Segal), Let S^1 be the unit circle in $\mathbb{C}$. A point of $\mathrm{Diff}(S^1)_\mathbb{C}$ is a holomorphic map from an annulus $1-\eta < r < 1+\eta$ to an annulus $1-\varepsilon < r < 1+\varepsilon$. It should have a restriction to S^1 close to a diffeomorphism of S^1. Composition is only partially defined. However, if f maps S^1 to the interior of the unit disc, one can associate to f the annulus $[f]$ between $f(S^1)$ and S^1 in $\mathbb{C}$. Both boundary components of $[f]$ are parametrized by S^1, using the identity map, resp. f. To the composition of maps corresponds the (always defined) gluing of annuli. Because of this, a substitute for $\mathrm{Diff}(S^1)_\mathbb{C}$ is the monoid of isomorphism classes of annuli with parametrized boundaries.

In conformal field theory, one meets complex (projective) representations of the real semigroup underlying the complex semigroup $\mathrm{Diff}(S^1)_\mathbb{C}$. At the Lie algebra level, if W is the Lie algebra of vector fields on S^1, with complexification $W_\mathbb{C}$, a complex (projective) representation of the real Lie algebra $r(W_\mathbb{C})$ underlying $W_\mathbb{C}$ is the same as a complex (projective) representation of $r(W_\mathbb{C}) \otimes \mathbb{C} = W_\mathbb{C} \otimes_\mathbb{R} \mathbb{C} = W_\mathbb{C} \times W_\mathbb{C}$.

§4.8. cs manifolds

Ordinary C^∞-manifolds are ringed spaces locally isomorphic to $(\mathbb{R}^p, \mathcal{C}^\infty)$. It would make no difference to define them as ringed spaces locally isomorphic to ($\mathbb{R}^p$, sheaf of $\mathbb{C}$-algebras $\mathcal{C}^\infty \otimes \mathbb{C}$), i.e. to take as structural sheaf the sheaf of complex-valued C^∞-functions. From the sheaf of $\mathbb{C}$-algebras of complex-valued C^∞-functions one indeed recovers the real-valued functions as those which take real values at every point. In the context of super manifolds, there is no such equivalence, and the following notion is of interest.

Definition 4.8.1. A cs *manifold* (c for "complex", s for "super") of dimension $p|q$ is a topological space X with a sheaf of $\mathbb{C}$-algebras $\mathcal{O}$, such that locally $(X, \mathcal{O})$ is isomorphic to $(\mathbb{R}^p, \mathcal{C}^\infty[\theta^1, \ldots, \theta^q] \otimes \mathbb{C})$.

As we explained, a cs manifold of dimension $p|0$ can be identified with an ordinary C^∞-manifold of dimension p. A cs manifold of dimension $0|q$ is the same thing as a complex super manifold of dimension $0|q$. If M is a super manifold, the sheaf $\mathcal{O}_\mathbb{C} = \mathcal{O} \otimes \mathbb{C}$ of complex functions on M is the structural sheaf of a cs manifold M^{cs}. In general, there are more morphisms from M^{cs} to N^{cs} than from M to N.

Variant. Replacing in 4.7 the sheaf of C^∞-functions by the sheaf of real analytic functions, one obtains the *analytic* cs *manifolds*. By a completion for a suitable topology of the structural sheaf, an analytic cs manifold defines a cs manifold.

Example. If M is a complex analytic super manifold, and if ρ is a real structure on M_{red}, the space $|M|^\rho$, with the restriction to $|M|^\rho$ of the structural sheaf of M, is an analytic cs manifold (of which M is said to be a *complexification*).

§4.9. Integration on cs manifolds; examples

In the classical situation, the real locus M^ρ of a real structure ρ on a complex manifold M can serve as a cycle of integration: once it is oriented, and provided one has suitable control at infinity, one can integrate on M^ρ a holomorphic differential form of maximal degree. The integral depends on M^ρ only up to suitable homologies.

Example 4.9.1. To compute $\int e^{-(x^2/2)+iax}dx$, it is better to go over to the holomorphic picture, where the integration cycle $\mathbb{R} \subset \mathbb{C}$ is needed only up to homologies given by 2-chains on which the integrand decays fast enough. In this setting, the completing the square change of variables $y = x - ia$ makes good sense, giving the answer $\sqrt{2\pi}\, e^{-a^2/2}$.

At the other extreme of $0|q$ manifolds, integration is a purely algebraic operation. For $M^{0|q}$ complex, integration of a section of the Berezinian line bundle makes sense. No real structure, and no orientation, are needed.

For a general complex super manifold M, integration of a holomorphic section of the Berezinian line bundle $\mathrm{Ber}(\Omega^1_M)$ requires only an integration cycle of half the real dimension on M_{red}, and suitable control at infinity. A variant of this principle is that for cs manifolds, with M_{red} oriented, integration of sections with compact support of $\mathrm{Ber}(\Omega^1_M)$ makes sense. Similar statements hold in a relative setting.

Example 4.9.2. Let M be a super manifold of dimension $p|q$. In the complexified tangent bundle $T_\mathbb{C}$, fix a locally direct factor τ of dimension $0|s$ for some s. If τ is integrable, i.e. stable under bracket, $|M|$ endowed with the sheaf $\mathcal{O}^\tau_\mathbb{C}$ of complex functions on M, annihilated by τ, is a cs manifold of dimension $p|(q-s)$. We denote it M/τ. Some supersymmetric Lagrangians involve integration on such cs spaces ("integration of chiral functions"); see [I-**Supersolutions**, §5.3].

Example 4.9.3. If one tries by a "Wick rotation" $t \mapsto it$ to convert a super Minkowski space (see [I-**Supersolutions**, §1.1]) into a Euclidean analogue, one often runs into trouble, in that the real spin representation of $\mathrm{Spin}(1,n)$ used to define the super Minkowski space is not a real representation of $\mathrm{Spin}(n+1)$. In such cases, the Euclidean analogue of super Minkowski space is only a cs manifold whose underlying reduced space is Euclidean space $\mathbb{R}^{n+1}$. Corresponding Euclidean field theories involve integration on this cs manifold.

Consider for instance the case of the super Minkowski space of dimension $4|4$. Its construction ([I-**Supersolutions**, §2.4]) starts with the unique irreducible spinorial real representation S of the double covering $\mathrm{SL}(2,\mathbb{C})$ of the Lorentz group $\mathrm{SO}(1,3)$. As a representation of $\mathrm{SL}(2,\mathbb{C})$ (which is viewed as a real Lie group), S is the 4-dimensional real vector space underlying the complex 2-dimensional defining representation of $\mathrm{SL}(2,\mathbb{C})$. The complexification of $\mathrm{SL}(2,\mathbb{C})$ is $\mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C})$, with the real locus $\mathrm{SL}(2,\mathbb{C})$ embedded by $g \mapsto (g, \bar{g})$; the complexification $S_{\mathbb{C}}$ of S is the sum of the defining representations of the two factors $\mathrm{SL}(2,\mathbb{C})$. The real form of $\mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C})$ corresponding to the real form $\mathrm{SO}(4)$ of $\mathrm{SO}(1,3)_{\mathbb{C}}$ is $\mathrm{SU}(2)\times\mathrm{SU}(2)$. There is no real form of $S_{\mathbb{C}}$ stable under $\mathrm{SU}(2)\times\mathrm{SU}(2)$. Indeed, the defining complex representation W of $\mathrm{SU}(2)$ is not real, but rather quaternionic: it admits an intertwining operator $j\colon W \to W$, $\mathbb{C}$-antilinear of square -1.

Example 4.9.4. There is a far-reaching analogy between Riemann surfaces, i.e. complex curves, and complex super manifolds of dimension $1|1$, provided with a $0|1$ distribution τ "as non-integrable as possible": for T the tangent bundle, the Frobenius pairing $\tau \otimes \tau \to T/\tau$ should be an isomorphism. Such objects are called SUSY curves. Let $\mathcal{M}$ be the moduli space of SUSY curves of genus g. Let us ignore the point that because SUSY curves can have nontrivial automorphism, $\mathcal{M}$ is a stack, rather than a complex super manifold.

The moduli space $\mathcal{M}$ has a natural real structure. This expresses the fact that the complex conjugate of a SUSY curve is again a SUSY curve, and more generally that if $p : X \to S$ is a complex analytic family of SUSY curves parametrized by S, then $\bar{p}\colon \bar{X} \to \bar{S}$ is again a family of SUSY curves.

The complexification $\mathcal{M} \times \bar{\mathcal{M}}$ of $\mathcal{M}_{\mathbb{R}}$, the real analytic space underlying $\mathcal{M}$, can hence be rewritten as being $\mathcal{M}\times\mathcal{M}$, with real structure $(x,y) \mapsto (\bar{y},\bar{x})$. The reduced space $(\mathcal{M} \times \mathcal{M})_{\mathrm{red}}$ is the space of pairs of Riemann surfaces Σ_1, Σ_2, each provided with a square root of the canonical bundle. In it, the locus where Σ_1 and Σ_2 are complex conjugate defines a cs manifold. It is to integration on this cs manifold that one refers when one speaks of "summing independently over spin structures for left and right movers" ([II-**Strings**, §9.2]).

Example 4.9.5. If X is a super manifold of dimension $0|q$, for any super manifold Y, the mapping space $\underline{\mathrm{Hom}}(X,Y)$ is again a finite dimensional super manifold.

For simplicity, let us assume that X is connected, hence isomorphic to $\mathbb{R}^{0|q}$. Let $X_{\mathbb{C}}$ be the complexification of X. If Y is purely even and $q = 1$, the cs manifold $\underline{\mathrm{Hom}}(X,Y)^{\mathrm{cs}}$ depends only on $X_{\mathbb{C}}$ and Y. If Y is real analytic, this can be thought of as follows. Let $Y_{\mathbb{C}}$ be a complexification of Y. Then $\underline{\mathrm{Hom}}(X_{\mathbb{C}}, Y_{\mathbb{C}})$, computed in the holomorphic world, is a complexification of $\underline{\mathrm{Hom}}(X,Y)$. The corresponding reduced space is $Y_{\mathbb{C}}$, and $\underline{\mathrm{Hom}}(X,Y)^{\mathrm{cs}}$ is the cs space with complexification $\underline{\mathrm{Hom}}(X_{\mathbb{C}},Y_{\mathbb{C}})$, defined by the real structure Y of $Y_{\mathbb{C}} = \underline{\mathrm{Hom}}(X_{\mathbb{C}},Y_{\mathbb{C}})_{\mathrm{red}}$.

The same does not hold for $q > 1$. For simplicity, take $q = 2$. Fix coordinates θ_1, θ_2 on $X_{\mathbb{C}}$. Take $Y = \mathbb{R}$, of dimension $1|0$, and $Y_{\mathbb{C}} = \mathbb{C}$. A holomorphic map from $X_{\mathbb{C}}$ to $Y_{\mathbb{C}}$ (rather, a family of such maps, parametrized by some S) can be written

$$\theta_1, \theta_2 \longmapsto a + \alpha_1\theta_1 + \alpha_2\theta_2 + b\theta_1\theta_2 \ .$$

The space $\underline{\mathrm{Hom}}(X_{\mathbb{C}},Y_{\mathbb{C}})$ is hence $\mathbb{C}^{2|2}$, with coordinates $(a, b, \alpha_1\alpha_2)$. The reduced space is $\mathbb{C}^2$, with coordinates (a, b). The real structure of Y tells us what it means for a to be real. For b, we need a real structure on X as well. To get a cs space, we would need to divide out by translations in b.

Similar troubles occur in infinite dimensional map spaces. To obtain a cs structure, one sometimes needs to "eliminate auxiliary fields".

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