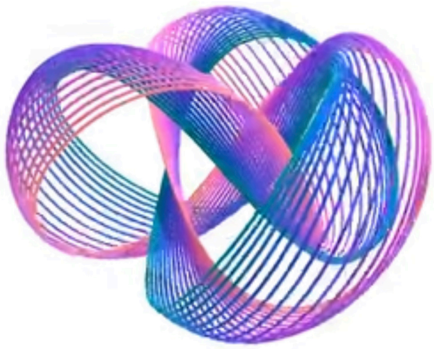
Objective Cohomology

Towards topological quantum computation



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The Plan

-2) This slide

-1) Lightning intro to homotopy type theory

0) H^0

1) H^1

2) H^2

n) H^n

∞) Twisted cohomology of braid groups.

Homotopy Type Theory is

- a logical system for working directly with sheaves of homotopy types.
- a standalone foundation of mathematics
  - Types A of mathematical objects
  - Elements $a : A$ of a given type, "a is an A"
  - Variable Elements $x^2+1 : \mathbb{R}$ (given that $x : \mathbb{R}$)

$$\underbrace{x : \mathbb{R}}_{\text{"Context"}} \vdash x^2+1 : \mathbb{R}$$

  - Variable types $M : \mathrm{Manifold},\ p : M \vdash T_pM : \mathrm{Vect}_{\mathbb{R}}$

$\mathbb{N}$ is the type of natural numbers

$\mathbb{R}$ is the type of real numbers

Set is the type of sets

$\mathrm{Vect}_{\mathbb{R}}$ is the type of real vector spaces

Type is the type of types.

Types of Identifications:

- If x and y are of type A, then

$$x =_A y \quad \text{is the type}$$

of ways to identify x with y as elements of A.

E.g.

- In $\mathrm{Vect}_{\mathbb{R}}$, $e : T_pM = \mathbb{R}^n$ is a linear isomorphism.
- In $\mathrm{Manifold}$, $e : M = N$ is a diffeomorphism.
- In Type, $e : A = B$ is an equivalence.
- In $\mathbb{N}$, $n = m$ has a unique element if and only if n equals m.

"Univalence Axiom" of Voevodsky

[$x : A \vdash b(x) : B(x)$ means "$b(x)$ is a $B(x)$, given that x is an A"]

Pair Types:

$TM :\equiv (p:M)\times T_pM$

- If $B(x)$ is a type for $x:A$, then

$$(x:A)\times B(x) \qquad A\times B$$

is the type of pairs (a,b) with $a:A$ and $b:B(a)$.

Function Types:

$Vec(M) :\equiv (p:M)\longrightarrow T_pM$

- If $B(x)$ is a type for $x:A$, then

$$(x:A)\longrightarrow B(x) \qquad A\longrightarrow B$$

is the type of functions $x\mapsto f(x)$ where $x:A\vdash f(x):B(x)$

Propositions as types

- ⌜X has a unique element⌝ $:\equiv (c:X)\times((x:X)\to(x=c))$
  "$\exists! X$"
- For $f:X\to Y$,
  ⌜f is an equivalence⌝ $:\equiv (y:Y)\to$ ⌜$(x:X)\times(fx=y)$ has a unique element⌝
  "$\forall y:Y, \exists! x:X, fx=y$"
- ⌜P is a proposition⌝ $:\equiv (x,y:P)\to$ ⌜$(x=y)$ has a unique element⌝
  "P has at most one element"

To prove P is to give an element of it. — "the fact that P" $: P$

$P\Rightarrow Q :\equiv P\to Q$, $\quad P\wedge Q :\equiv P\times Q$, $\quad \forall x:X.\, P(x) :\equiv (x:X)\to P(x)$

Propositional truncation:

For any type X, a proposition $\exists X$ and $(-)^{\exists} : X\to\exists X$, initial for such maps

$$\begin{tikzcd} X \arrow[r, "\forall"] \arrow[d, "(-)^{\exists}"'] & P \\ \exists X \arrow[ur, dashed, "\exists!"'] & \end{tikzcd}$$

Types of Structures

- $\ulcorner X \text{ is a set} \urcorner :\equiv (x,y:X) \to \ulcorner (x=y) \text{ is a proposition} \urcorner$

$$\{x:X \mid P(x)\} :\equiv (x:X) \times P(x)$$

- $$\text{Monoid} :\equiv \underbrace{(M:\mathrm{Type}) \times (\cdot : M \times M \to M) \times (1:M)}_{\text{Structure}} \times \underbrace{\ulcorner M \text{ is a set} \urcorner \times ((x:M) \to (1 \cdot x = x)) \times ((x:M) \to (x \cdot 1 = x)) \times ((x\,y\,z:M) \to ((x \cdot y) \cdot z = x \cdot (y \cdot z)))}_{\text{Properties}}$$

- For G a group,

$$\begin{aligned}\mathrm{Tors}_G :\equiv\ & (T:\mathrm{Type}) \times (\alpha : G \times T \to T) \\ & \times \ulcorner T \text{ is a set} \urcorner \\ & \times \forall t:T,\ \alpha(1,t) = t \\ & \times \forall t:T,\ g,h:G,\ \alpha(gh,t) = \alpha(g, \alpha(h,t)) \\ & \times \forall t,t':T,\ \ulcorner \{g:G \mid \alpha(g,t)=t'\} \text{ has a unique element} \urcorner \\ & \times \exists T\end{aligned}$$

Truncated and connected types

- $\ulcorner X \text{ is } {-2}\text{-truncated} \urcorner :\equiv \ulcorner X \text{ has a unique element} \urcorner \equiv: \exists! X$
- $\ulcorner X \text{ is } (n+1)\text{-truncated} \urcorner :\equiv (x,y:X) \to \ulcorner (x=y) \text{ is } n\text{-truncated} \urcorner$

$$\begin{aligned}\ulcorner X \text{ is } {-1}\text{-truncated} \urcorner &\equiv \ulcorner X \text{ is a proposition} \urcorner \\ \ulcorner X \text{ is } 0\text{-truncated} \urcorner &\equiv \ulcorner X \text{ is a set} \urcorner \\ \ulcorner X \text{ is } 1\text{-truncated} \urcorner &\equiv \ulcorner X \text{ is a groupoid} \urcorner\end{aligned}$$

Truncation:

- For any X, an initial map $(\cdot)^{\sqsubset n} : X \to \sqsubset_n X$ from X to an n-connected type.

Def: $\ulcorner X \text{ is } n\text{-connected} \urcorner :\equiv \exists! \sqsubset_n X$.

X is 0-connected iff $\forall x,y:X,\ \exists(x=y)$.

H^0: Functions and groups

- A group is the set of symmetries of some object (of a given type)
- "Objective method": Work with the objects themselves!

Def: G is a higher group if $G = (pt = pt)$ for $pt : BG$ with BG 0-connected.

Eg: $G = (G \underset{Tors_G}{=} G)$, and $Tors_G$ is 0-connected

If $G = (X \underset{T}{=} X)$, then we may define

$$BG :\equiv \{y : T \mid \exists(x = y)\} \equiv (y:T) \times \exists(x=y)$$

makes it 0-connected

Work with BG instead of G

H^0: Functions and groups

- $H^0(X; G) :\equiv \tau_0(X \to G)$
- This is a group with pointwise multiplication if G is.

So what is $H^0(X;G)$ the automorphisms of?

$$\begin{aligned}
\tau_0(X \to G) &= \tau_0(X \to (G \underset{BG}{=} G)) \\
&= \tau_0\big((x \mapsto G) \underset{X \to BG}{=} (x \mapsto G)\big) \\
&= \big((x \mapsto G)^{\tau_1} \underset{\tau_1(X \to BG)}{=} (x \mapsto G)^{\tau_1}\big)
\end{aligned}$$

automorphisms

Notation: If $t : T$, then $\Omega T :\equiv (t \underset{T}{=} t)$

☀ $H^0(X;G)$ is a group when we have $G = \Omega BG$.

H^1: Bundles, homomorphisms, and actions

- $H^1(X;G) :\equiv \pi_0(X \to BG)$

Eg: A map $E : X \to \mathrm{Tors}_G$ is a G-principal **bundle** on X.
E_x is a G-torsor.

Thm: If BG exemplifies G, then the **associated torsor** map $e \mapsto (pt = e) : BG \to \mathrm{Tors}_G$ is an equivalence.

Eg: $V \mapsto \mathrm{Frame}(V) : BGL_n \xrightarrow{\sim} \mathrm{Tors}_{GL_n}$, so that

$H^1(X; GL_n)$ is both iso-classes of n-dim vector bundles on X, and iso-classes of GL_n-principal bundles on X.

H^1: Bundles, homomorphisms, and actions

Thm: If G and H are groups, then

$$\mathrm{Hom}(G,H) = (B\varphi : BG \to BH) \times (pt_{BH} = B\varphi(pt_{BG}))$$

Eg: $\det : GL_n \to GL_1$ corresponds to

$$BGL_n :\equiv \begin{Bmatrix} n\text{-dim} \\ \text{vector} \\ \text{Spaces} \end{Bmatrix} \xrightarrow{B\det} \begin{Bmatrix} 1\text{-dim} \\ \text{vector} \\ \text{Spaces} \end{Bmatrix} \equiv BGL_1$$

$$V \longmapsto \Lambda^n V, \quad pt_{B\det} : \mathbb{R} = \Lambda^n \mathbb{R}^n$$

is $1 \mapsto e_1 \wedge \cdots \wedge e_n$

- $H^1(BG; H) = \mathrm{Hom}(G,H)/\text{conjugacy}$.

Cor: Actions of G are functions $BG \to \mathrm{Set}$.

$$\begin{aligned} \mathrm{Act}_G &\equiv (X : \mathrm{Set}) \times \mathrm{Hom}(G, \mathrm{Aut}(X)) \\ &= (X : \mathrm{Set}) \times (BG \to_{\bullet} B\mathrm{Aut}(X)) \\ &= BG \to \mathrm{Set} \end{aligned}$$

Twisted H^0: Equivariant functions

- If $X, G : B \to \mathrm{Type}$, then

$$H^0_{/B}(X;G) \equiv c_0((b:B) \to X_b \to G_b)$$

Eg: $H^0_{/M}(*;TM) = \mathrm{Vec}(M)$, $H^0_{/M}(*;T^*M) = \Omega^1(M)$

- If $X, G : BK \to \mathrm{Set}$ (actions of K), then

$$H^0_{/BK}(X;G) = \{f : X_{pt} \to G_{pt} \mid \forall k:K,\ f(kx) = k f(x)\}$$
$$\equiv \mathrm{Act}_K(X,G)$$

Eg: $$H^0_{BG}(*;G^{conj}) = \{g:G \mid \forall k:G,\ g = k^{-1} g k\}$$
$$= ZG$$

H^1: Bundles, homomorphisms, and actions

When is $H^1(X;G)$ a group?

When BG is itself a higher group, i.e. $BG = \Omega \overbrace{B^2G}^{\text{a 1-connected type}}$!

But then $G = \Omega^2 B^2 G \ldots$

Thm (Eckman-Hilton): If $G = \Omega^2 T$, then G is abelian.

If G is abelian, then $ZG \xrightarrow{\sim} G$, and
$ZG = H^0_{/BG}(*;G^{conj}) = ((e:BG) \to (e=e)) = (id_{BG} = id_{BG})$

So $ZG = \Omega \mathrm{Aut}(BG) = \Omega^2 B\mathrm{Aut}(BG)$

Def: $$B^2 ZG :\equiv (X : B\mathrm{Aut}(BG)) \times c_0(X = BG)$$
$$= (X : \mathrm{Type}) \times c_0(X = BG).$$

H^2: Central extensions

Let K be abelian, so $B^2K \equiv (X : \mathrm{Type}) \times \tau_0(X = BK)$.

- $H^2(X; K) :\equiv \tau_0(X \to B^2K)$
- Given $c : BG \to B^2K$, get a fiber sequence

$$BK \to BE \to BG \to B^2K$$

$\mathrm{Fib}_c(pt_{B^2K}) \equiv (e : BG) \times (c(e) = (BK, \mathrm{refl}^{\tau_0}))$

This gives a short exact sequence
$$0 \to K \to E \to G \to 0$$

Thm (Eilenberg-MacLane):
$$H^2(BG; K) = \{0 \to K \to E \to G \to 0 \mid K \subseteq Z(E)\}/_{iso}$$

Twisted H^2: <u>All</u> extensions

Recall: $\mathrm{Out}(K) :\equiv \mathrm{Aut}_{Grp}(K)/\text{conjugacy}$

Recall: $H^1(BG; H) \equiv \tau_0(BG \to BH) = \mathrm{Hom}(G, H)/\text{conjugacy}$

So $\tau_0 \mathrm{Aut}(BK) = \mathrm{Out}(K)$, so $\tau_1 B\mathrm{Aut}(BK) = B\mathrm{Out}(K)$

"Higher Schreir theorem"

Thm (M.): Let G and K be higher groups. Then
$$\mathrm{Ext}(G, K) = |BG \to_{\cdot} B\mathrm{Aut}(BK)| = \{\text{actions of } G \text{ on } BK\}$$

Eg: $0 \to \mathbb{R}^4 \to \text{Poincaré} \to \text{Lorentz} \to 0$

$B\text{Lorentz} :\equiv \{\text{4-dim } \mathbb{R}\text{-vector spaces w/ 3,1-inner product}\}$

$B\mathbb{R}^4 :\equiv \{\text{4-dim } \mathbb{R}\text{-affine spaces}\}$

$$\begin{aligned} B\text{Lorentz} &\to B\mathrm{Aut}(B\mathbb{R}^4) \\ V &\mapsto BV \end{aligned}$$

Cor: Let $\varphi : G \to \mathrm{Out}(K)$ be a hom.

Then $H^2_{/B\mathrm{Out}(K)}(BG; K) = \{\text{extensions w/ abstract kernel } \varphi\}/_{iso}$

H^n: Characteristic classes, obstruction theory

- $H^n(X;G) :\equiv \| X \to B^n G \|_0$

Def (Buchholtz): Let G be abelian, then

$$B^{n+1}G :\equiv (X : \mathrm{Type}) \times \| X = B^n G \|_0$$

"gerbe" "band"

We can do this with an action, eg:

$$B\mu_k :\equiv (V : BGL_1(\mathbb{C})) \times (T \subseteq V) \times \ulcorner T \text{ is a } \mu_k\text{-torsor} \urcorner$$

Then $(V,T) \mapsto B^n V : B\mu_k \to \mathrm{Type}$

This gives an action of μ_k on $B^n\mathbb{C}$.

Aside: The shape of a type (Cohesive TT!)

Def: A type X is **discrete** if every map $\gamma : \mathbb{R} \to X$ is constant.

$$\flat X \xrightarrow{(-)_\flat} X \xrightarrow{(-)^{\int}} \int X$$

the "shape" of X

universal map from a discrete type

universal map to a discrete type

"A feature of X is "topological" if it only depends on $\int X$"

Eg: $\theta \mapsto \{t : \mathbb{R} \mid e^{2\pi i t} = \theta\} : \mathbb{S}^1 \to \mathrm{Tors}_{\mathbb{Z}}$ is the shape of $\mathbb{S}^1$.

Eg: $\mathrm{Conf}_n \mathbb{R}^2 :\equiv \mathrm{Injection}(\underline{n}, \mathbb{R}^2)$

$\int \mathrm{Conf}_n \mathbb{R}^2 = BB_n$, where B_n is the **braid group**

Towards Topological Quantum Computation

Given d defects and n particles,

$\mathrm{Conf}_{n+d}\mathbb{R}^2 \xrightarrow[\text{particles}]{\text{forget}} \mathrm{Conf}_d\mathbb{R}^2$ gives $BB_{n+d} \longrightarrow BB_d$

Denote the fiber over $c : BB_d$ by BB_n^{2c}.

Chern-Simons level

Let $\chi : B_{n+d} \to \mu_{K+2}$ be a character which chooses weights for each braid of defects and particles

Then for $c : BB_d$

$$H^{n+d}_{/B\mu_{K+2}}(BB_n^{2c}; \mathbb{C}) \equiv C_0\left(\left((V,T) : B\mu_K\right) \to (e : BB_n^{2c}) \times \mathrm{Hom}_\chi\left((pt = e), T\right) \to B^{n+d}V\right)$$

is the set of conformal blocks.

The action of a braid of defects on this is a computation.