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Twisted Cobordism and its Relationship to Equivariant Homotopy Theory

by

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Abstract

This thesis is primarily concerned with developing methods for studying

$$[M, S^V]^G$$

the set of equivariant homotopy classes of equivariant maps from a G-manifold, M, to a representation sphere, S^V, where G is a group. The basic idea is to study a related invariant of the orbit space, M/G, which is called twisted framed cobordism. The study of twisted framed cobordism leads naturally to a formulation of a set of axioms characterizing "twisted generalized cohomology theories". Using spectral sequence arguments, I am able to make some explicit computations of equivariant homotopy sets.

Dedication

To Mam, Dad and Julie

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Abstract

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Chapter 1

Introduction

1.1 Background

If G is a group which acts on two sets X and Y, a map $f : X \to Y$ is said to be equivariant if $f(g.x) = g.f(x)$. In equivariant topology we assume that the actions and maps are continuous with respect to the topologies on X and Y. In this thesis, I will be primarily concerned with (equivariant homotopy classes of) equivariant maps from smooth compact G-manifolds, M, into representation spheres, S^V, (S^V is the 1-point compactification of the linear G-space V). This is a generalization of the classical situation of maps from a compact manifold, M, into a sphere, S^d, studied by Hopf, Pontryagin, Thom and others (see [15] or [19]).

In order to provide some context for the present work, it is worth reviewing briefly some of the ideas of the aforementioned authors. Let $x_0 \in S^d$. Using the smooth approximation theorem and Sard's theorem (see [3]), one can show that any map $M \to S^d$ can be approximated by a smooth map, f, that has x_0 as a regular value. Moreover, one can assume that all homotopies of maps are also smooth and have x_0 as a regular value. This means that $f^{-1}(x_0)$ is a submanifold of M. Now, if U is any disc neighbourhood of x_0, then $S^d - U$ is contractible. So, up to homotopy at least, f is determined by its behaviour on a neighbourhood of the submanifold $f^{-1}(x_0)$. In fact, to determine f up to homotopy, it only necessary to specify $f^{-1}(x_0)$ and to specify the derivative, $Tf|_{f^{-1}(x_0)}$, on $f^{-1}(x_0)$. In the case where M is orientable and $\dim M = d$, $f^{-1}(x_0)$ is a finite set of points and $Tf|_{f^{-1}(x_0)}$ is determined (up to homotopy) by specifying whether f preserves or reverses orientation around each point in $f^{-1}(x_0)$. This leads to the concept of the degree of the map f. If $\dim M > d$, the situation is a bit more complicated. Now, $f^{-1}(x_0)$ is a $(\dim M - d)$-dimensional submanifold of M and determining the behaviour of $Tf|_{f^{-1}(x_0)}$ amounts to specifying a framing of the normal bundle of $f^{-1}(x_0)$ in M. Thus, we are led to the concept of framed cobordism.

In the equivariant setting, one has several complications. Firstly, transversality does not work in general. That is, it is not possible to equivariantly approximate any equivariant map by one for which a given G-fixed point of the codomain is a regular value. However, if we assume that G acts freely on the domain manifold then transversality does indeed work (see chapter 5). Assume that $f : M \to S^V$ is a smooth equivariant map with $0 \in S^V$ a regular value. Then $f^{-1}(0)$ is a G-invariant submanifold of M of dimension $\dim M - \dim V$. Suppose that $\dim G = n$. As in the classical situation, the behaviour of f in a neighbourhood of $f^{-1}(0)$ gives rise to an equivariant framing of the normal bundle of $f^{-1}(0)$ in M. The "equivariant dimension" of $f^{-1}(0)$ is $\dim M - \dim V - n$

(note that the dimension of the orbit space, $f^{-1}(0)/G$ is $\dim M - \dim V - n$) . For this reason, $\dim M - \dim V - n$ is referred to as the "geometric stem". The case of the geometric 0-stem has already been analyzed in [14]. In this case, $f^{-1}(0)$ consists of a finite collection of free G-orbits and by a careful analysis of the local orientation properties of f, one is led to a concept of "equivariant degree". One of the main purposes of this thesis is to develop methods for analyzing higher geometric stems. The basic idea is that rather than looking at the (possibly high dimensional) submanifold, $f^{-1}(0)$, of M, it is advantageous to factor out the action of G and consider the corresponding submanifold of M/G. The equivariant framing of the normal bundle of $f^{-1}(0)$ turns out to correspond to a certain "twisted framing" of the normal bundle of $f^{-1}(0)/G$. From the point of view of making explicit computations, it seems to be easier to work with the orbit space M/G in this manner, rather than directly with M.

1.2 Synopsis of thesis and description of main results

Chapters 2, 3 and 4 contain some background material on fibrewise homotopy theory and equivariant homotopy theory. All of the constructions and results presented here are already known, although some are presented in an unorthodox manner in order to suit our particular needs (e.g. lemma 3.3.5 or the description of the Borel construction in defintion 3.3.1). Lemma 3.3.5 is of particular importance as this underpins the relationship between fibrewise homotopy theory and equivariant homotopy theory. The guiding principle to bear in mind concerning this relationship may be stated as follows: If one wants to examine equivariant properties of a free G-space X, it is often advantageous to consider corresponding fibrewise properties "over" a certain map $X/G \to BG$ (this map classifies $X \to X/G$ in the sense made precise in chapter 3). For example, a "map over $X/G \to BG$" is a commutative diagram

$$\begin{tikzcd} & Y \arrow[d] \\ X/G \arrow[ur] \arrow[r] & BG \end{tikzcd}$$

Chapter 5 presents some basic facts on equivariant cobordism. Again, this material is not new, so the presentation is brief. The most important fact here is theorem 5.0.6 which establishes that equivariant homotopy classes of equivariant maps $M \to S^V$ correspond to cobordism classes of equivariantly V-framed submanifolds of M.

In chapter 6, I present the first new results. The concept of "twisted cobordism" is introduced. Here, the structure on the normal bundle of a submanifold of M is defined with respect to some fibre bundle over M. For example, in classical (untwisted) cobordism, one can think of a framing of the normal bundle as an isomorphism from each normal space to a fixed vector space. In twisted framed cobordism, rather than having a fixed vector space, we associate different vector space to each point of M (i.e. we fix a vector bundle over M). A twisted framing is an isomorphism from each normal space to the vector space associated to that point. The main (new) result is theorem 6.0.12 which demonstrates that the equivariant framed cobordism of M is isomorphic to the twisted framed cobordism of M/G where the twist is described by a specified vector bundle over M/G. One can think of this theorem as being somewhat analagous to Eilenberg's theorem (see [24], chapter VI) which says that ordinary cohomology with twisted coefficients is isomorphic to equivariant ordinary cohomology of the universal covering space.

Before proceeding, it must be noted that during the preparation of this thesis, the author became aware of work by Davis and Lück ([6]) which overlaps with the material presented in chapters 7 and 8. However, the author was not aware of [6] until after these chapters had been completed.

In chapter 7 we introduce the notion of twisted generalized cohomology. We also present the main example - at least for our purposes - of such a cohomology theory, namely, twisted stable cohomotopy theory. Twisted stable cohomotopy theory is the appropriate "stabilization" of twisted framed cobordism. Chapter 8 discusses an Atiyah-Hirzebruch type spectral sequence for twisted generalized cohomology theories. This is an algebraic "machine" which will allow us to make some explicit computations.

In chapter 9 we use the spectral sequence to make the aforementioned computations in the case of the geometric 1-stem. These computations mostly take the form of a short exact sequence

$$0 \to \mathbb{Z}/2 \to \omega \to H \to 0$$

where ω is the object to be computed and H is an ordinary twisted cohomology group which happens to be fairly easily computable. Typically, ω is a twisted stable cohomotopy group or a set of equivariant homotopy classes of equivariant maps. Thus, we have determined ω up to an extension problem. I finish off with an analysis of the equivariant stable homotopy group $\tilde{\omega}^G_{n+1}(EG^+)$. As pointed out in [14] this group may be calculated via a Serre spectral sequence computation. However, the present methods offer some insight into the geometry of the maps which represent elements of $\tilde{\omega}^G_{n+1}(EG^+)$

1.3 Acknowledgements

I would like to thank my supervisor, Professor George Peschke for the many discussions, mathematical and otherwise, that we had during my time at the University of Alberta. Without his encouragement and patience, this work would never have been completed.

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I would also like to thank the Department of Mathematics, University College, Cork for allowing me the time to complete this thesis, while I was working there.

1.4 Notation and conventions

In this section, I will describe some notations and conventions that are used in this thesis.

Throughout this thesis, I will assume that all spaces are compactly generated Hausdorff. This is of particular relevance in Chapter 2 where we consider fibrewise mapping spaces.

G will always denote a group. e is the identity element of G. $[G,G]$ is the commutator subgroup of G and $G^{\mathrm{ab}} := \frac{G}{[G,G]}$ is the abelianization of G. If G is a Lie group, then $\mathrm{ad}(G) := T_eG$ denotes the adjoint representation of G.

$S^V := V \cup \{\infty\}$ will denote the 1-point compactification of the G-representation V. Where the context makes it clear, n will be used to denote the trivial n-dimensional G-representation. For example, $S^{V+n} := S^{V \oplus \mathbb{R}^n}$. In particular, $S^n := S^{\mathbb{R}^n}$.

I denotes the unit interval $[0,1]$

M will always be a compact smooth manifold of dimension m and with boundary ∂M.

EG and BG denote the a universal free G-space and a classifying space for G, respectively.

For $y_0 \in Y$, the map $c_{y_0} : X \to Y$ is the constant map at y_0. That is to say, $c_Y(x) = y_0$ for all $x \in X$.

$\nu_M(N)$ denotes the normal bundle of N in M, where N is a submanifold of M. $\nu_{x_0}(N)$ denotes the normal space to N at x_0 (implicitly N is a submanifold of some manifold M).

TM denotes the tangent bundle of the manifold M. $T_{x_0}M$ denotes the tangent space to M at x_0. If $f : M_1 \to M_2$ is a smooth map then Tf denotes the derivative of f.

Ω will always denote an (unstable) cobordism set of some type. So, for example, Ω_{fr} will denote (twisted) framed cobordism (defined in chapter 6) and Ω^G will denote equivariant cobordism (defined in chapter 5)

In chapter 9, I will use $\triangle_1$ to denote the standard 1-simplex and I will use Whitehead's notation (see [24], chapter VI) for singular chains, cycles etc.

$\pi_n(X, x_0)$ denotes the nth homotopy group (or set), $[(S^n, *); (X, x_0)]$.

If $\mathcal{C}$ is a category, then $|\mathcal{C}|$ denotes the objects of $\mathcal{C}$.

Chapter 2

Introduction to fibrewise homotopy theory

In this chapter I will present some basic results of fibrewise homotopy theory. It is, of course, possible to develop a fibrewise homotopy theory in which both the domain and codomain are nontrivial fibre bundles, see for example [5]. However, I am only interested in the case where the domain fibre bundle is $\mathrm{Id} : X \to X$ and the codomain is a locally trivial fibre bundle, so I only develop the theory for this special case.

2.0.1 Definition Given a space B, we define the category of pairs of spaces over B, denoted T_B^2, as follows: The objects are triples $(X, A; f)$ where (X, A) is a pair of spaces and $f : X \to B$ is a continuous map. A morphisms

$$\varphi : (X, A; f) \to (Y, B; g)$$

consists of a map of pairs $\overline{\phi} : (X, A) \to (Y, B)$ such that $g \circ \overline{\phi} = f$

We shall be particularly interested in the case $B = BG$, where G is a (compact Lie) group, and we shall explore the relationship between T^2_{BG} and the category $\mathrm{fr}G^2$, whose objects are pairs of free G-spaces and whose morphisms are equivariant maps of such pairs.

Let $p : EG \to BG$ be a universal principal G-bundle (i.e. EG is contractible). Then p induces a functor

$$F : T^2_{BG} \to \mathrm{fr}G^2$$

in the following way: Given an object $(X, A; f)$ in T^2_{BG}, let $q : \tilde{X} \to X$ be the induced principal G-bundle over X obtained by taking pullback along f and let $\tilde{A} := q^{-1}(A)$. Then $(\tilde{X}, \tilde{A}) \in |\mathrm{fr}G^2|$. Morphisms are also induced by taking pullback. □

Much of this thesis will be devoted to constructing and studying certain functors on the category T^2_{BG}.

2.0.2 Definition Let $p : E \to B$ be a locally trivial fibre bundle and let E' be a subspace of E such that $p|_{E'} : E' \to B$ is also a locally trivial fibre bundle. I will call the quadruple (E, E', p, B) a relative locally trivial fibre bundle. I will also say (abusing the notation in the process) that $p : (E, E') \to B$ is a relative locally trivial fibre bundle.

2.0.3 Lemma (See [18]) Let (X, A) be a relative CW-complex and let $p : E \to B$ be a fibration. Given a commutative solid diagram

$$\begin{tikzcd} 0 \times X \cup \mathrm{I} \times A \arrow[r] \arrow[d] & E \arrow[d] \\ \mathrm{I} \times X \arrow[r, "F"'] \arrow[ur, dashed, "\overline{F}"] & B \end{tikzcd}$$

there exists $\overline{F} : \mathrm{I} \times X \to E$ which makes the resulting diagram commute.

Proof: First suppose that $(X, A) = (B^n, S^{n-1})$. Then there exists a homeomorphism $h : \mathrm{I} \times B^n \to \mathrm{I} \times B^n$ such that h maps $0 \times B^n \cup \mathrm{I} \times S^{n-1}$ homoeomorphically onto $0 \times B^n$. Thus h allows us to convert our original lifting problem into one of the form

$$\begin{array}{ccc} 0 \times B^n & \longrightarrow & E \\ \downarrow & & \downarrow \\ \mathrm{I} \times B^n & \xrightarrow[F']{} & B \end{array}$$

which can be solved, since p is a fibration.

Now suppose that (X, A) is an arbitrary relative CW-complex. We argue by induction over the skeleta of X. Suppose that $\overline{F}$ has been constructed on $0 \times X \cup \mathrm{I} \times X^n$. Let e^{n+1} be an $(n+1)$-cell of X with characteristic map $l : (B^{n+1}, S^n) \to (X^{n+1}, X^n)$. l induces a lifting problem

$$\begin{array}{ccc} 0 \times B^{n+1} \cup \mathrm{I} \times S^n & \xrightarrow{\overline{F} \circ (\mathrm{id} \times l)} & E \\ \downarrow & & \downarrow \\ \mathrm{I} \times B^{n+1} & \longrightarrow & B \end{array}$$

which has a solution $\overline{G} : \mathrm{I} \times B^{n+1} \to E$. $\overline{G}$ allows us to extend $\overline{F}$ to a lift $0 \times X \cup \mathrm{I} \times (X^n \cup e^{n+1})$. So clearly $\overline{F}$ can be extended to $0 \times X \cup \mathrm{I} \times X^{n+1}$. By induction, we can solve the given lifting problem. □

2.0.4 Definition Let $p : (E, E') \to B$ be a relative locally trivial fibre bundle. Let (X, A) be a pair of spaces. A relative lifting problem is a map $u : 0 \times (X, A) \to (E, E')$ together with a homotopy $F : \mathrm{I} \times X \to B$ such that $p \circ u(0, x) = F(0, x)$.

I will indicate such a lifting problem by a diagram of the type

$$\begin{tikzcd} 0 \times (X, A) \arrow[r, "u"] \arrow[d, "p"] & (E, E') \arrow[d] \\ \mathrm{I} \times (X, A) \arrow[r, "F"'] & B \end{tikzcd}$$

A solution to this lifting problem is a map $\overline{F} : \mathrm{I} \times (X, A) \to (E, E')$ such that $p \circ \overline{F} = F$

2.0.5 Lemma Let $p : (E, E') \to B$ be a relative locally trivial fibre bundle and let (X, A) be a relative CW-complex. Then any relative lifting problem

$$\begin{tikzcd} 0 \times (X, A) \arrow[r, "u"] \arrow[d] & (E, E') \arrow[d] \\ \mathrm{I} \times (X, A) \arrow[r, "F"'] & B \end{tikzcd}$$

has a solution.

Proof: First consider the lifting problem

$$\begin{tikzcd} 0 \times A \arrow[r, "u"] \arrow[d] & E' \arrow[d] \\ \mathrm{I} \times A \arrow[r, "F"'] & B \end{tikzcd}$$

This has a solution $\tilde{F} : \mathrm{I} \times A \to E'$. Now, $\tilde{F}$ and u combine to give a map $\tilde{u} : 0 \times X \cup \mathrm{I} \times A \to E$ and we get a commutative solid diagram

$$\begin{tikzcd} (0 \times X) \cup (\mathrm{I} \times A) \arrow[r, "\tilde{u}"] \arrow[d] & E' \arrow[d] \\ \mathrm{I} \times X \arrow[r, "F"'] \arrow[ur, dashed, "\overline{F}"'] & B \end{tikzcd}$$

By lemma 2.0.3, $\overline{F}$ exists making the resulting diagram commute. $\overline{F}$ is a solution to the given relative lifting problem. $\square$

2.0.6 Definition Let $p : (E, E') \to B$ be a relative locally trivial fibre bundle and let $(X, A; f)$ be an object in T_B^2. Then

$$\mathrm{map}_f((X, A); (E, E')) := \{\phi : (X, A) \to (E, E') \text{ such that } p \circ \phi = f\}$$

This set is topologized as a subspace of $\mathrm{map}(X, E)$.

$$[(X, A); (E, E')]_f := \{\text{path connected components of } \mathrm{map}_f((X, A); (E, E'))\}$$

Let $\phi, \psi \in \mathrm{map}_f((X, A); (E, E'))$. We say that ϕ and ψ are fibrewise homotopic, written $\phi \cong_f \psi$ if there is a homotopy $H : \mathrm{I} \times (X, A) \to (E, E')$ from ϕ to ψ such that

$$p \circ H = f \circ \mathrm{pr}_X$$

where $\mathrm{pr}_X : \mathrm{I} \times X \to X$ is the projection map.

Clearly,

$$[(X, A); (E, E')]_f = \{ \text{ fibrewise homotopy classes of maps } (X, A) \to (E, E')\}$$

.

2.0.7 Remark Note that if $(F, F') = (p^{-1}(b), p^{-1}(b) \cap E')$ for some $b \in B$, and if $c_b : X \to B$ is the constant map that sends everything to b then

$$\mathrm{map}_{c_b}((X, A); (E, E')) = \mathrm{map}((X, A); (F, F'))$$

So we can think of $\mathrm{map}_f((X, A); (E, E'))$ as a "twisted" version of the classical mapping space.

2.0.8 Lemma Let (X, A) be a relative CW-complex with X compactly generated Hausdorff. Suppose that f_1 and f_2 are maps $X \to B$ which are homotopic via a homotopy H. Let $p : (E, E') \to B$ be a relative locally trivial fibre bundle over B. Then H induces a homotopy equivalence

$$H_* : \mathrm{map}_{f_1}((X, A); (E, E')) \to \mathrm{map}_{f_2}((X, A); (E, E'))$$

Proof: First I claim that the induced map

$$p_* : \mathrm{map}((X, A); (E, E')) \to \mathrm{map}(X, B)$$

is a fibration. To see this, consider the lifting problem

$$\begin{tikzcd} 0 \times Y \arrow[r] \arrow[d] & \mathrm{map}((X, A); (E, E')) \arrow[d, "p_*"] \\ \mathrm{I} \times Y \arrow[r, "F"'] & \mathrm{map}(X, B) \end{tikzcd}$$

The adjoint lifting problem is

$$\begin{tikzcd} 0 \times Y \times (X, A) \arrow[r] \arrow[d] & (E, E') \arrow[d, "p"] \\ \mathrm{I} \times Y \times (X, A) \arrow[r, "\underline{F}"'] & B \end{tikzcd}$$

Since $p : (E, E') \to B$ is a relative locally trivial fibre bundle, this has a solution $\tilde{F} : \mathrm{I} \times Y \times (X, A) \to (E, E')$. Let $\overline{F} : \mathrm{I} \times F \to \mathrm{map}((X, A); (E, E'))$ be the adjoint of $\tilde{F}$. Then $\overline{F}$ is a solution to the original lifting problem. Thus p_* is a fibration as claimed. Now, f_1 and f_2 are points in the space $\mathrm{map}(X, B)$ which are joined by the path H. Thus H induces a homotopy equivalence of the fibres of p_* over f_1 and f_2 respectively. But these fibres are precisely $\mathrm{map}_{f_1}((X, A); (E, E'))$ and $\mathrm{map}_{f_2}((X, A); (E, E'))$ respectively. $\square$

2.0.9 Corollary H induces a bijection

$$H_* : [(X, A); (E, E')]_{f_1} \to [(X, A); (E, E')]_{f_2}$$

Proof: Take π_0 of the homotopy equivalence in lemma 2.0.8. $\square$

2.1 The homotopy extension property in T_B^2

In this section, I will formulate the appropriate notion of the homotopy extension property (HEP) for objects in the category T_B^2. I will also show that if (X, A) is a relative CW-complex, then $(X, A; f)$ has HEP. The present exposition is based on that of Husemoller (see chapter 2 in [8]), although he looks at the same problem from the point of view of prolongation of cross sections rather than extensions of lifts.

2.1.1 Definition Let $(X, A; f)$ be an object in T_B^2. Let $p : E \to B$ be a locally trivial fibre bundle. Then we say that $(X, A; f)$ has the homotopy extension property with respect to maps into E if, given a commutative diagram

$$\begin{tikzcd} & E \arrow[d, "p"] \\ 0 \times X \cup I \times A \arrow[ur, "H"] \arrow[r, "f \circ pr_X"'] & B \end{tikzcd}$$

there exists a map

$$\overline{H} : I \times X \to E$$

such that the following diagram commutes:

$$\begin{tikzcd}
& & E \arrow[d, "p"] \\
0 \times X \cup \mathrm{I} \times A \arrow[r] \arrow[urr, "H"] & \mathrm{I} \times X \arrow[r, "f \circ \mathrm{pr}_X"'] \arrow[ur, "\overline{H}"'] & B
\end{tikzcd}$$

2.1.2 Lemma Let $p : E \to B$ be a locally trivial fibre bundle with fibre F. Then $(B^n, S^{n-1}; f)$ has HEP with respect to maps into E.

Proof: $f^*(p)$ is a locally trivial fibre bundle over B^n. Since B^n is contractible $f^*(p)$ is a trivial bundle. Thus the required result follows from the classical homotopy extension property of the pair (B^n, S^{n-1}) with respect to maps into F. □

2.1.3 Theorem Let $p : E \to B$ be a locally trivial fibre bundle. Let (X, A) be relative CW-complex and $f : X \to B$. Then $(X, A; f)$ has HEP with respect to maps into E.

Proof: Let $H : 0 \times X \cup \mathrm{I} \times A \to E$ be given as in definition 2.1.1. We will proceed by induction over the skeleta of X. Suppose that $H^n : 0 \times X \cup \mathrm{I} \times X^n \to E$ has been constructed, extending H. Let e^{n+1} be an $(n+1)$-cell of X with characteristic map $l : (B^{n+1}, S^n) \to (X^{n+1}, X^n)$. We have a map $\bar{l} : 0 \times B^{n+1} \cup \mathrm{I} \times S^n \to 0 \times X \cup \mathrm{I} \times X^n$ that sends $(t, z) \mapsto (t, l(z))$. So $H^n \circ \bar{l} : 0 \times B^{n+1} \cup \mathrm{I} \times S^n \to E$ and $p \circ H^n \circ \bar{l} = f \circ l \mathrm{pr}_2$ where $\mathrm{pr}_2 : \mathrm{I} \times B^{n+1} \to B^{n+1}$ is the projection onto the second factor. By lemma 2.1.2 we may extend $H^n \circ \bar{l}$ to a map $F : \mathrm{I} \times B^{n+1} \to E$ such that $p \circ F = f \circ l \circ \mathrm{pr}_2$. Since l is a characteristic map for the cell e^{n+1}, F induces a map $H^{n*} : 0 \times X \cup \mathrm{I} \times (X^n \cup e^{n+1}) \to E$ such that $p \circ H^{n*} = f \circ \mathrm{pr}_2$. In this way we can extend H^n to $H^{n+1} : 0 \times X \cup \mathrm{I} \times X^{n+1} \to E$ such that $p \circ H^{n+1} = f \circ \mathrm{pr}_2$.

Now, by induction we can extend H to $\overline{H} : \mathrm{I} \times X \to E$ as required. $\square$

Chapter 3

Introduction to equivariant homotopy theory

In this chapter we will introduce some of the basic notions and conventions of equivariant homotopy theory. Many of the results here will be stated without proof. For complete details, the reader is referred to [2] or [21]

Throughout this chapter, G will denote a compact Lie group of dimension n. A left G-space is a space X together with a map $\mu : G \times X \to X$ such that $\mu(g_1, \mu(g_2, x)) = \mu(g_1 g_2, x)$ and $\mu(e, x) = x$. $\mu(g, x)$ will be written $g.x$. A smooth left G-manifold is a manifold M which is a left G-space such that the map μ is smooth. If X and Y are G-spaces, a map $f : X \to Y$ is equivariant if $f(g.x) = g.f(x)$ for all $g \in G$ and $x \in X$. If f_1 and f_2 are equivariant maps, they are equivariantly homotopic if there is a homotopy $H : \mathrm{I} \times X \to Y$ between them which is equivariant. The action of G on $\mathrm{I} \times X$ is given by $g.(t, x) := (t, g.x)$. $[X, Y]^G$ denotes the set of equivariant homotopy classes of equivariant maps from

X to Y.

3.1 Fixed point spaces and orbit spaces

If X is a G-space and H is a subgroup of G, then $X^H := \{x \in X | h.x = x \text{ for all } h \in H\}$. For $x \in X$, $G_x := \{g \in G | g.x = x\}$. X is said to be a free G-space if $G_x = \{e\}$ for all $x \in X$.

3.1.1 Lemma $G_{g.x} = gG_xg^{-1}$

□

3.1.2 Lemma $g.X^H = X^{gHg^{-1}}$

□

X/G denotes space of orbits with the quotient topology.

3.1.3 Theorem (See [7]) If M is a smooth compact free G-manifold, then M/G has a unique smooth structure such that the quotient map $p : M \to M/G$ is a submersion.

□

In fact, in this case $p : M \to M/G$ is an example of a principal G-bundle.

3.1.4 Definition A locally trivial principal G-bundle is a map $p : E \to B$ such that E is a left G-space and for each $x \in B$ there is an open neighbourhood U of x in B and an equivariant homoemorphism $h : p^{-1}(U) \to G \times U$ such that the following diagram commutes

$$\begin{tikzcd}
p^{-1}(U) \arrow[rr, "h"] \arrow[dr, "p"'] & & G \times U \arrow[dl, "\mathrm{pr}_2"] \\
& U &
\end{tikzcd}$$

The action of G on $G \times U$ is given by $g.(g_1, u) := (gg_1, u)$. From now on, I will drop the "locally trivial", and just refer to such bundles as principal G-bundles.

3.1.5 Theorem (See [7] Let M be a compact smooth free G-manifold. Then $p : M \to M/G$ is a principal G-bundle.

□

3.2 The universal principal G-bundle

Recall the following definition (see [21])

3.2.1 Definition Let $(X_j | j \in J)$ be a family of topological spaces. The join $*_{j \in J} X_j$ is defined in the following way: Let

$$\overline{X} = \{(t_j x_j) : j \in J, t_j \in \mathrm{I}, x_j \in X_j, \Sigma t_j = 1, t_j = 0 \text{ for almost all } j\}$$

Then as a set

$$*_{j \in J} X_j = \overline{X} / \sim$$

where $(t_j x_j) \sim (u_j y j)$ if and only if $t_j = u_j$ for all j and if $t_j \neq 0$ then $x_j = y_j$. One has coordinate maps

$$t_i : *_{j \in J} X_j \to \mathrm{I}, (t_j x_j) \mapsto t_i$$

and

$$p_i : t_i^{-1}(]0,1]) \to X_i, (t_j x_j) \mapsto x_i$$

The topology on $*_{j\in J} X_j$ is the coarsest one that makes these maps continuous.

Now we can give Milnor's construction of the universal principal G-bundle.

3.2.2 Definition Given a compact Lie group G, let

$$EG := G * G * \dots$$

and let

$$BG := EG/G$$

Then $q : EG \to BG$ is a principal G-bundle, and it is universal in the following sense.

3.2.3 Theorem Let $p : \tilde{X} \to X$ be a principal G-bundle where X is a paracompact space. Then there exists an equivariant map $\tilde{f} : \tilde{X} \to EG$ and a map $f : X \to BG$ such that the following diagram is a pullback diagram.

$$\begin{tikzcd} \tilde{X} \arrow[r, "\tilde{f}"] \arrow[d, "p"'] & EG \arrow[d, "q"] \\ X \arrow[r, "f"'] & BG \end{tikzcd}$$

Moreover, f and $\tilde{f}$ are unique up to homotopy and equivariant homotopy respectively.

□

This is a standard result, so the proof is omitted. The interested reader can consult [21].

3.2.4 Definition Let $E_1 \to X_1$ and $E_2 \to X_2$ be principal G-bundles. A morphism of principal G-bundles is a commutative diagram

$$\begin{tikzcd} E_1 \arrow[r, "\tilde{f}"] \arrow[d] & E_2 \arrow[d] \\ X_1 \arrow[r, "f"'] & X_2 \end{tikzcd}$$

in which $\tilde{f}$ is equivariant.

Let $k_G(X)$ denote the isomorphism classes of principal G-bundles over X. Applying theorem 3.2.3 we have the following.

3.2.5 Corollary If X is a CW-complex then

$$[X, BG] \cong k_G(X)$$

□

We say that BG is a classifying space for principal G-bundles.

3.2.6 Corollary

$$k_G(S^1) \cong (G/G_0)^{\mathrm{ab}}$$

where G_0 is the component of G containing the identity element.

Proof: $k_G(S^1) \cong [S^1, BG] \cong \pi_1(BG)^{\mathrm{ab}}$. But $\pi_1(BG) \cong \pi_0(G) \cong G/G_0$ □

We will also need the fact that principal G-bundles are fibrations in the category of G-spaces, at least in the case where the domain of the lifting problem is a G-CW-complex (the reader should consult [21] for the definition of a G-CW-complex). More precisely,

3.2.7 Theorem Given a commutative solid diagram

$$\begin{tikzcd} 0 \times Y \arrow[r, "f"] \arrow[d] & \tilde{X} \arrow[d, "p"] \\ I \times Y \arrow[r, "F"'] \arrow[ur, dashed, "\overline{F}"] & X \end{tikzcd}$$

where $p : \tilde{X} \to X$ is a principal G-bundle, Y is a free G-CW-complex and f and F are equivariant, there exists an equivariant $\overline{F}$ which makes the resulting diagram commute.

Proof: First, we will prove the theorem in the case $Y = G \times Y'$ where Y' is a CW-complex and G acts by left multiplication on the first factor. So we have a lifting problem

$$\begin{tikzcd} 0 \times G \times Y' \arrow[r, "f"] \arrow[d] & \tilde{X} \arrow[d, "p"] \\ I \times G \times Y' \arrow[r, "F"'] & X \end{tikzcd}$$

We can restrict this to the nonequivariant lifting problem

$$\begin{tikzcd} 0 \times e \times Y' \arrow[r, "f"] \arrow[d] & \tilde{X} \arrow[d, "p"] \\ I \times e \times Y' \arrow[r, "F"'] & X \end{tikzcd}$$

where e is the identity element of G. This has a solution $\hat{F} : I \times e \times Y' \to \tilde{X}$ (since p is a fibration). Define $\overline{F} : I \times G \times Y'$ by $\overline{F}(t, g, y) := g.\hat{F}(t, e, y)$. $\overline{F}$ is a solution to the equivariant lifting problem.

Now suppose that Y is an arbitrary free G-CW-complex. Y^0 is a set of disjoint free G-orbits, so clearly the lifting problem can be solved over Y^0. Suppose that we have $\overline{F}^n : I \times Y^n \to \tilde{X}$ which solves the lifting problem over Y^n. Let e^{n+1} be an $(n+1)$-cell of Y with characteristic map $l : G \times (B^{n+1}, S^n) \to (Y^{n+1}, Y^n)$. $\overline{F}$, f and l combine to give an equivariant lifting problem

$$\begin{array}{ccc} 0 \times G \times B^{n+1} \cup \mathrm{I} \times G \times S^n & \xrightarrow{f} & \tilde{X} \\ \downarrow & & \downarrow p \\ \mathrm{I} \times G \times B^{n+1} & \xrightarrow[F]{} & X \end{array}$$

Now, there is a homeomorphism $h : \mathrm{I} \times B^{n+1} \to \mathrm{I} \times B^{n+1}$ which maps $0 \times B^{n+1} \cup \mathrm{I} \times S^n$ homeomorphically to $0 \times B^{n+1}$. This allows us to transform the above equivariant lifting problem into one of the form

$$\begin{array}{ccc} 0 \times G \times B^{n+1} & \xrightarrow{f} & \tilde{X} \\ \downarrow & & \downarrow p \\ \mathrm{I} \times G \times B^{n+1} & \xrightarrow[F]{} & X \end{array}$$

which can be solved. Thus, we can extend our partial solution $\overline{F}^n$ to a partial solution $\overline{F}^{n+1} : \mathrm{I} \times Y^{n+1} \to \tilde{X}$. By induction we can find a solution $\overline{F} : \mathrm{I} \times Y \to \tilde{X}$ as required. $\square$

3.3 The Borel construction

3.3.1 Definition Let X and Y be left G-spaces. Then the topological tensor produdct over G is defined by

$$X \times_G Y := (X \times Y)/G$$

Here G acts on $X \times Y$ via the diagonal action, that is, $g.(x, y) = (g.x, g.y)$. The equivalence class of (x, y) will be denoted by $[x, y]$. If (Y, Y') is a pair of G-spaces then

$$X \times_G (Y, Y') := (X \times_G Y, X \times_G Y')$$

3.3.2 Remark Usually $X \times_G Y$ is defined for X a right G-space and Y a left G-space. However, as we will exclusively be dealing with left G-spaces, the above definition is more convenient for us.

3.3.3 Remark $EG \times_G Y$ is often referred to as the "Borel construction"

3.3.4 Theorem Let $p : E \to B$ be a principal G-bundle and let F be a left G-space. Then the map $\phi : E \times_G F \to B$ given by $\phi([x, f]) := p(x)$ is well defined and is a locally trivial fibre bundle with fibre F.

Now let (B, B') be CW-pair. Let $p : E \to B$ be a principal G-bundle and let $E' := p^{-1}(B')$. Suppose that

$$\begin{tikzcd} E \arrow[r, "\bar{f}"] \arrow[d, "p"'] & EG \arrow[d] \\ B \arrow[r, "f"'] & BG \end{tikzcd}$$

is the pullback diagram whose existence is asserted by theorem 3.2.3 - so f classifies $p : E \to B$. Let (Y, Y') be a pair of G-spaces. Let $q : EG \times Y \to EG \times_G Y$ be the principal G-bundle and let $\pi : EG \times_G Y \to BG$ be the induced Y-fibre bundle. The following pair of results are of crucial importance for the rest of the thesis.

3.3.5 Lemma

$$\mathrm{map}_G((E, E'); (Y, Y')) \equiv \mathrm{map}_f((B, B'); EG \times_G (Y, Y'))$$

Proof: Consider the following diagram:

$$\begin{tikzcd} EG \times Y \arrow[r, "pr_1"] \arrow[d] & EG \arrow[d] \\ EG \times_G Y \arrow[r, "\pi"'] & BG \end{tikzcd}$$

This is a pullback diagram. Now consider the following diagram

$$\begin{tikzcd} E \arrow[r, dashed, "\tilde{u}"'] \arrow[d] \arrow[rr, bend left, "\tilde{f}"] & EG \times Y \arrow[r] \arrow[d] & EG \arrow[d] \\ B \arrow[r, dashed, "u"] \arrow[rr, bend right, "f"'] & EG \times_G Y \arrow[r, "\pi"'] & BG \end{tikzcd}$$

The pullback property of the right hand square establishes a bijection between maps $u : B \to EG \times_G Y$ such that $\pi u = f$ and equivariant maps $\tilde{u} : E \to EG \times Y$ whose projection to the first factor is $\tilde{f}$. That is, it establishes a bijection between $\mathrm{map}_f(B; EG \times_G Y)$ and $\mathrm{map}_G(E; Y)$. It is easy to check that maps which send B' to $EG \times_G Y'$ correspond precisely to equivariant maps which send E' to Y'.

The proof of the bicontinuity of this bijection with respect to the mapping space topologies is omitted. □

3.3.6 Remark Note that the space $\mathrm{map}_f((B, B'); EG\times_G(Y, Y'))$ is equivalent to the space of sections of the bundle $E \times_G Y \to B$ which map B' to $E \times_G Y'$. Lemma 3.3.5 is commonly formulated using this space rather than as above.

3.3.7 Corollary

$$[(E, E'); (Y, Y')]^G \cong [(B, B'); EG \times_G (Y, Y')]_f$$

Proof: This follows by taking π_0 of the homeomorphism in lemma 3.3.5. □

3.3.5 and 3.3.7 show the close link between fibrewise homotopy theory and equivariant homotopy theory. We shall mainly work explicity with the former and then use 3.3.5 and 3.3.7 to obtain results about the latter.

Chapter 4

Equivariant stable homotopy

It is not my intention here to present a complete introduction to the subject of equivariant stable homotopy theory - to do justice to such a project would require much more space than I am prepared to devote and besides, it would divert us too far from our main goals. I only wish to present enough of the theory to motivate the later sections of this thesis. In particular. I want to demonstrate the significance of studying the set $\left[(X, A); (S^V, \infty)\right]^G$ where (X, A) is a relatively free G-space. For further details on the material in this section the reader should consult [12] or [21].

Throughout this section X and Y will denote based G-spaces with basepoints x_0 and y_0 respectively. The basepoints are G-fixed points.

4.1 Equivariant stabilization

In classical nonequivariant homotopy, one stabilizes the set $[X, Y]$ by suspending. That is to say, the set of stable homotopy classes of maps from (X, x_0) to (Y, y_0) is defined by

$$\{(X, x_0); (Y, y_0)\} := \varinjlim\{[X; Y]_0 \to [S^1 \wedge X; S^1 \wedge Y]_0 \to \dots$$

Note that $S^1 = \mathbb{R} \cup \infty$ is the 1-pt compactification of a 1-dimensional real vector space. The equivariant analogue of a finite dimensional real vector space is a finite dimensional real representation of G. Suppose that V is such a representation. Let S^V be the 1-pt compactification of V. Given any equivariant map

$$f : (X, x_0) \to (Y, y_0)$$

we define

$$(S^V, \infty) \wedge f : (S^V, \infty) \wedge (X, x_0) \to (S^V, \infty) \wedge (Y, y_0)$$

to be $\mathrm{Id}_{S^V} \wedge f$. This induces a suspension homomorphism (of sets)

$$[(X, x_0); (Y, y_0)]^G \to \left[(S^V, \infty) \wedge (X, x_0); (S^V, \infty) \wedge (Y, y_0)\right]^G$$

We would like to define $\{(X, x_0); (Y, y_0)\}^G$ to be the direct limit over all such suspensions. However, problems arise with this naive approach since the collection of all representations of G is not a set and thus not a very nice thing on which to index a direct limit. I will outline a way to deal with these problems. It is worth noting that these issues have been dealt with much more thoroughly elsewhere. I am only interested in developing the basic notions of equivariant stable homotopy theory as an equivariant homology theory, so I will not deal with equivariant spectra in full generality (see for example [12]).

First, recall that a complete G-universe, $\mathcal{U}$, is an infinite dimensional G-representation such that, for each irreducible representation, V, of G, $\mathcal{U}$ contains countably infinitely many summands isomorphic to V. Let us fix such a universe $\mathcal{U}$. We shall use G-invariant subspaces of $\mathcal{U}$ to index our direct limit. Let U and V be two such G-invariant subspaces. We say that $U \leq V$ if $U \subseteq V$ and if there is a G-invariant subspace W of V such that $V = W \oplus U$ (note that if G is a compact Lie group and V is finite dimensional, then W always exists since V has a G-invariant inner product). Now suppose that $U \leq V$, $V = W \oplus U$ and suppose that we have an equivariant map

$$f : (S^U, \infty) \wedge (X, x_0) \to (S^U, \infty) \wedge (Y, y_0)$$

Then define

$$\sigma_{V-U}(f) : (S^V, \infty) \wedge (X, x_0) \to (S^V, \infty) \wedge (Y, y_0)$$

by the following composite

$$\begin{array}{c}
(S^V, \infty) \wedge (X, x_0) \\
\downarrow \\
(S^W, \infty) \wedge (S^U, \infty) \wedge (X, x_0) \\
\downarrow {\scriptstyle \mathrm{Id} \wedge f} \\
(S^W, \infty) \wedge (S^U, \infty) \wedge (Y, y_0) \\
\downarrow \\
(S^V, \infty) \wedge (Y, y_0)
\end{array}$$

The first and third maps are induced by the canonical G-equivalence

$$(S^{W \oplus U}, \infty) \to (S^W, \infty) \wedge (S^U, \infty)$$

The map $\sigma_{V-U}(f)$ is independent of the choice of W used in its definition. Thus we have defined the suspension

$$\begin{array}{c} \mathrm{map}_G((S^U,\infty)\wedge(X,x_0);(S^U,\infty)\wedge(Y,y_0)) \\ \Big\downarrow{\scriptstyle \sigma_{V-U}} \\ \mathrm{map}_G((S^V,\infty)\wedge(X,x_0);(S^V,\infty)\wedge(Y,y_0)) \end{array}$$

σ_{V-U} induces a homomorphism (of sets)

$$\left[(S^U,\infty)\wedge(X,x_0);(S^U,\infty)\wedge(Y,y_0)\right]^G \to \left[(S^V,\infty)\wedge(X,x_0);(S^V,\infty)\wedge(Y,y_0)\right]^G$$

Now we may define the set of stable equivariant homotopy classes of maps from (X,x_0) to (Y,y_0) as

$$\{(X,x_0);(Y,y_0)\} := \varinjlim \left[(S^V,\infty)\wedge(X,x_0);(S^V,\infty)\wedge(Y,y_0)\right]^G$$

where the direct limit is taken over all finite dimensional G-invariant subspaces V of the G-universe $\mathcal{U}$. Note that if $V\cong\mathbb{R}^2\oplus V'$ then the set $\left[(S^V,\infty)\wedge(X,y_0);(Y,y_0)\right]^G$ has a canonical abelian group structure (as in the nonequivariant setting). Thus $\{(X,x_0);(Y,y_0)\}$ has a canonical abelian group structure.

For the rest of this section we will suppress mention of the basepoints and assume that all maps and homotopies are based. Also by "representation of G" we shall mean finite dimensional G-invariant subspace of $\mathcal{U}$.

4.1.1 Definition Let V_1 and V_2 be representations of G. Then

$$\tilde{\omega}^G_{V_1-V_2}(X;Y) := \{S^{V_1}\wedge X;S^{V_2}\wedge Y\}$$

Also

$$\tilde{\omega}^G_{V_1-V_2}(Y) := \tilde{\omega}^G_{V_1-V_2}(S^0;Y)$$

$\tilde{\omega}^G$ is an example of a (reduced) equivariant homology theory (see [12] or [21]). As we have defined it, $\tilde{\omega}^G$ is indexed on the set of pairs of G representations (finite dimensional G-invariant subspaces of $\mathcal{U}$)

4.1.2 Lemma Let $\phi_1 : V_1 \to V_1'$ and $\phi_2 : V_2 \to V_2'$ be isomorphisms of G-representations. There is a canonical isomorphism

$$\tilde{\omega}^G_{V_1 - V_2}(Y) \to \tilde{\omega}^G_{V_1' - V_2'}(Y)$$

Proof: ϕ_1^{-1} and ϕ_2 induce an isomorphism

$$\left[S^U \wedge S^{V_1} \wedge X; S^U \wedge S^{V_2} \wedge Y;\right] \to \left[S^U \wedge S^{V_1'} \wedge X; S^U \wedge S^{V_2'} \wedge Y;\right]$$

By taking direct limits we get the required isomorphism ∎

The next lemma justifies the use of the notation $V_1 - V_2$ in the subscript to $\tilde{\omega}^G$

4.1.3 Lemma Suppose that V_1 V_1' V_2 and V_2' are G-representations such that

$$V_1 \oplus V_2' = V_1' \oplus V_2$$

then there is a canonical isomorphism

$$\tilde{\omega}^G_{V_1 - V_2}(Y) \to \tilde{\omega}^G_{V_1' - V_2'}(Y)$$

Proof: By assumption we have an isomorphism

$$\begin{gathered}\left[S^U \wedge S^{V_2'} \wedge S^{V_1} \wedge X; S^U \wedge S^{V_2'} \wedge S^{V_2} \wedge Y\right]^G \\ \Big\downarrow \\ \left[S^U \wedge S^{V_2} \wedge S^{V_1'} \wedge X; S^U \wedge S^{V_2} \wedge S^{V_2'} \wedge Y\right]^G\end{gathered}$$

for any G-representation U. The result follows from the fact that the set representations which contain V_2' is cofinal and the set of representations which contain V_2 is cofinal. $\square$

Let $RO(G)$ denote the real character ring of the group G. Then, $RO(G) \cong \{[V] - [U]\}$ where $[V]$ and $[U]$ are isomorphism classes of representations of G. Thus lemmas 4.1.2 and 4.1.3 allow us to think of $\tilde{\omega}^G$ as being indexed by $RO(G)$.

4.1.4 Remark In the context of G-representations, we will often use an integer to stand for trivial representations of that dimension. Thus, for example, $S^{V+n} = S^{V\oplus\mathbb{R}^n}$ or $\tilde{\omega}_n^G(Y) = \tilde{\omega}_{\mathbb{R}^n}^G(Y)$

4.2 The splitting of the stable equivariant homotopy groups

Recall that if H is a subgroup of G then the Weyl group of H is

$$WH := N_G(H)/H$$

where $N_G(H)$ is the normalizer of H in G. Let (H) denote the conjugacy class of the subgroup H. We have the following theorem, due to Segal for finite groups (see [16]) and tom Dieck for compact Lie groups (see [20]).

4.2.1 Theorem

$$\tilde{\omega}_n^G(S^0) \cong \bigoplus_{(H)} \tilde{\omega}_n^{WH}(EWH^+)$$

where the direct sum is taken over all conjugacy classes of subgroups of G $\square$

Thus, in order to undestand the equivariant stable stems we can study them one piece at a time. In other words, try to understand the groups $\tilde{\omega}_n^{WH}(EWH^+)$. We have the following basic result (see [14]).

4.2.2 Theorem

$$\tilde{\omega}_n^G(EG^+) \cong \lim_{\rightarrow} \left[S^{V+n}, A_{S^{V+n}}; S^V, \infty\right]^G$$

where the direct limit is taken over all representations V of G and for each rep U, A_U is the subspace of S^U consisting of all the nonfree orbits.

Proof:

$$\tilde{\omega}_n^G(EG^+) = \lim_{\rightarrow} \left[S^{V+n}, \infty : S^V \wedge EG^+\right]^G$$

Note that if $f : (S^{V+n}, \infty) \to (S^V \wedge EG^+, *)$ then f must send $A_{S^{V+n}}$ to the basepoint $*$, since all the other orbits of $S^V \wedge EG^+$ are free. Let $\mathrm{pr} : S^V \wedge EG^+ \to S^V$ be the projection. Then $\mathrm{pr} \circ f : (S^{V+n}, A_{S^{V+n}}) \to (S^V, \infty)$. On the other hand, suppose that $\phi : (S^{V+n}, A_{S^{V+n}}) \to (S^V, \infty)$. Let $\alpha : S^{V+n} - A \to EG$ be an equivariant map. By theorem 3.2.3 α is unique up to equivariant homotopy. Define $\hat{f} : (S^{V+n}, \infty) \to (S^V \wedge EG^+, *)$ by $\hat{f}(x) = *$ for $x \in A_{S^{V+n}}$ and $\hat{f}(x) = \phi(x) \wedge \alpha(x)$ for $x \notin A_{S^{V+n}}$. These constructions establish a one to one correspondence between $\left[(S^{V+n}, \infty); (S^V \wedge EG^+, *)\right]^G$ and $\left[(S^{V+n}, A_{S^{V+n}}); (S^V, \infty)\right]^G$ which commutes with the suspension maps. $\square$

This theorem illustrates the importance of studying the set $\left[(X, A); (S^V, \infty)\right]^G$ where (X, A) is a relatively free G-manifold. The main purpose of this thesis is to develop the machinery needed to study these sets.

Chapter 5

Equivariant transversality and equivariant framed cobordism

Let G be a compact Lie group of dimension n. Let $\tilde{M}$ be compact free G-manifold with boundary $\partial\tilde{M}$. $M := \tilde{M}/G$. Let V be a real representation of G. If B is a submanifold of A, $\nu_A(B)$ denotes the normal bundle of B in A. Throughout this section all maps are assumed to be equivariant unless otherwise stated.

One of the big problems with trying to generalize classical cobordism theory to the equivariant setting is that, in general, transversality does not work, as the following example demonstrates:

5.0.3 Example Let $G := \mathbb{Z}/2$ and let $X := \{*\}$ with the trivial G action. Let V be the nontrivial 1-dimensional real representation of G and consider the equivariant map $f : X \to S^V$, $f(*) = 0 \in S^V$. f cannot be equivariantly homotoped to any map which has 0 as a regular value, since the only fixed points of S^V are 0 and ∞. (Recall that if $f : X \to Y$ is a smooth map then $y \in Y$ is a regular value if, for all $x \in f^{-1}(y)$, $T_x f : T_x(X) \to T_y(Y)$ is surjective.)

It is clear that the basic problem in this example is that the orbits of the domain manifold are not "free enough" to map equivariantly onto all the orbits of the target manifold. Much work has been done to determine conditions under which equvariant transversality will work. See for example [22] or [4]. I am only interested in the following special case. (My thanks to S.R. Costenoble for communicating the basic argument used in the proof of the following theorem)

5.0.4 Theorem Let G be a compact Lie group, $\tilde{M}$ a compact free G-manifold with boundary $\partial\tilde{M}$ and let V be a real representation of G. Then, any equivariant map $f : (\tilde{M}, \partial\tilde{M}) \to (S^V, \infty)$ is equivariantly homotopic to a map which has 0 as a regular value. Moreover, given two maps g_1 and g_2 which have 0 as a regular value and which are equivariantly homotopic, we can find a homotopy F between them such that 0 is a regular value of F.

Proof: Consider the nonequivariant map $\overline{f} := (\mathrm{Id}, f)/G : (\tilde{M}/G, \partial\tilde{M}/G) \to \tilde{M} \times_G (S^V, \infty)$. By nonequivariant transversality, $\overline{f}$ is homotopic to a map $\overline{g}$ which is transverse to the zero section of the fibre bundle $\tilde{M} \times_G S^V \to M$. Suppose that $\overline{F}$ is a homotopy of $\overline{f}$ to $\overline{g}$. Now, $M \times S^V \to M \times_G S^V$ is a principal G-bundle, so by theorem 3.2.7 we can lift $\overline{F}$ to an equivariant homotopy $\tilde{F} : \mathrm{I} \times (\tilde{M}, \partial\tilde{M}) \to \tilde{M} \times (S^V, \infty)$ such that $\tilde{F}(0, -) = (\mathrm{Id}, f)$. Let $\tilde{g} := \tilde{F}(1, -)$. $\tilde{g}$ is transverse to the zero section of $\tilde{M} \times S^V \to \tilde{M}$. Let $g := \mathrm{pr}_2 \circ \tilde{g}$ where $\mathrm{pr}_2 : \tilde{M} \times S^V \to S^V$ is the projection to the second factor. Then 0 is a regular value of g and $\mathrm{pr}_2 \circ \tilde{F}$ is a homotopy of f to g.

Now suppose that we are given g_1 and g_2 as in the statement of the theorem. Let $H : \mathrm{I} \times \tilde{M} \to S^V$ be an equivariant homotopy from g_1 to g_2. Define maps $\overline{H}$, $\overline{g}_1$ and $\overline{g}_2$ as follows.

$$\begin{aligned} \overline{H} : (\mathrm{I} \times \tilde{M})/G &\to \tilde{M} \times_G S^V \\ [t, m] &\mapsto [m, H(t, m)] \end{aligned}$$

and $\overline{g}_i := (\mathrm{Id} \times g_i)/G$ for $i = 1, 2$. So $\overline{H}$ is a nonequivariant homotopy between nonequivariant maps $\overline{g}_1$ and $\overline{g}_2$. Using nonequivariant transverality, we can find a map $\overline{F} : \mathrm{I} \times \mathrm{I} \times \tilde{M}/G \to (\tilde{M} \times S^V)/G$ such that

- $\overline{F}(0, t, [m]) = \overline{H}(t, [m])$
- $\overline{F}(t, 0, [m]) = \overline{g}_1([m])$ and $\overline{F}(t, 1, [m]) = \overline{g}_2([m])$
- $\overline{F}(1, -, -) : \mathrm{I} \times \tilde{M}/G \to \tilde{M} \times_G S^V$ is transverse to the zero section of $\tilde{M} \times_G S^V \to \tilde{M}/G$.

Now lift $\overline{F}$ to a G-map $\tilde{F} : \mathrm{I} \times \mathrm{I} \times \tilde{M} \to \tilde{M} \times S^V$. Then $F := \mathrm{pr}_{S^V}(\tilde{F}(1, -, -)) : \mathrm{I} \times \tilde{M} \to S^V$ is a homotopy of g_1 to g_2 which has 0 as a regular value. $\square$

5.0.5 Definition Let V be a real representation of G. Then the V-framed cobordism set of V-framed submanifolds of $(\tilde{M}, \partial\tilde{M})$, denoted $\Omega_G^V(\tilde{M}, \partial\tilde{M})$, is defined as follows. Let $S_G^V(\tilde{M}, \partial\tilde{M}) := \{(\tilde{N}, \tilde{\phi})\}$ where $\tilde{N}$ is a G-invariant submanifold of $\tilde{M}$ such that $\tilde{N} \cap \partial\tilde{M}$ is empty and $\tilde{\phi} : \nu_{\tilde{M}}(\tilde{N}) \to \tilde{N} \times V$ is a G-equivariant bundle isomorphism. We define an equivalence relation, called V-framed cobordism, on $S_G^V(\tilde{M}, \partial\tilde{M})$ by the following: $(\tilde{N}_1, \tilde{\phi}_1) \sim (\tilde{N}_2, \tilde{\phi}_2)$ if there exists a G-submanifold W of $\tilde{M} \times \mathrm{I}$ (note that $\tilde{M} \times \mathrm{I}$ is only a manifold after we "straighten the corners") and a G-equivariant bundle isomorphism $\tilde{\Phi} : \nu_{\tilde{M}\times I}(W) \to W \times V$, such that $(W, \tilde{\Phi})|_{\tilde{M}\times\{0\}} = (\tilde{N}_1, \tilde{\phi}_1)$ and $(W, \tilde{\Phi})|_{\tilde{M}\times\{1\}} = (\tilde{N}_2, \tilde{\phi}_2)$. Then

$$\Omega_G^V(\tilde{M}, \partial\tilde{M}) := S_G^V(\tilde{M}, \partial\tilde{M})/\sim$$

5.0.6 Theorem

$$\Big[(\tilde{M}, \partial\tilde{M}), (S^V, \infty)\Big]^G \cong \Omega_G^V(\tilde{M}, \partial\tilde{M})$$

Proof: This is proved exactly as in the classical nonequivariant case, given that we have equivariant transversality (theorem 5.0.4) in this case. I shall only give a sketch of the proof. Given $f : (\tilde{M}, \partial\tilde{M}) \to (S^V, \infty)$, we may assume (because of theorem 5.0.4) that 0 is a regular value. So $f^{-1}(0)$ is a G-invariant submanifold of $\tilde{M}$ that does not intersect $\partial\tilde{M}$. Moreover, Tf (the derivative of f) sends $Tf^{-1}(0)$ to 0, so $Tf|_{Tf^{-1}(0)}$ factors to an equivariant bundle map $Tf| : \nu_{\tilde{M}}(f^{-1}(0)) \to T_0(V) \cong V$. That is, Tf induces a V-framing of $f^{-1}(0)$. So corresponding to f we have $(f^{-1}(0), Tf|) \in S_G^V(\tilde{M}, \partial\tilde{M})$. Moreover, one shows that under this correspondence, homotopies of maps correspond to cobordisms of V-framed submanifolds. Thus, we obtain a homomorphism

$$\Big[(\tilde{M}, \partial\tilde{M}), (S^V, \infty)\Big]^G \to \Omega_G^V(\tilde{M}, \partial\tilde{M}).$$

To invert this homomorphism, we use the equivariant Pontryagin-Thom construction: Given $(\tilde{N}, \phi) \in S_G^V(\tilde{M}, \partial\tilde{M})$, we have an equivariant tube map $\tau : U \to \nu_{\tilde{M}}(\tilde{N})$ where U is some tubular neighbourhood of $\tilde{N}$. Let $\mathrm{pr}_2 : \tilde{N} \times V \to V$ be the projection onto the second factor. Then $\mathrm{pr}_2 \circ \phi \circ \tau : U \to V$ is an equivariant map. Now, if we collapse the complement of U to ∞ we obtain an equivariant map $f : (\tilde{M}, \partial\tilde{M}) \to (S^V, \infty)$. One can show that this construction provides the required inverse

$$\Omega_G^V(\tilde{M}, \partial\tilde{M}) \to \left[(\tilde{M}, \partial\tilde{M}), (S^V, \infty)\right]^G .$$

$\square$

Chapter 6

Twisted Cobordism

In this chapter, I shall adopt the following convention. $p : \tilde{M} \to M$ is a principal G-bundle and if $N \subset M$ then $\tilde{N} = p^{-1}(N)$. More generally, $\tilde{X}$ and X always will bear the relationship that X is $\tilde{X}$ with the G action "modded out". The context should make it clear what "modded out" means in each case.

Now suppose that $\tilde{M}$ is a free G-manifold with boundary $\partial\tilde{M}$. Instead of looking at $\Omega_G^V(\tilde{M}, \partial\tilde{M})$ directly, we shall take the following approach. Let $\left[\tilde{N}, \tilde{\phi}\right] \in \Omega_G^V(\tilde{M}, \partial\tilde{M})$. By modding out the action of G we obtain a submanifold N of M, together with a certain structure on the normal bundle of N. The nature of this structure reflects both the action of G on $\tilde{M}$ and on the representation V. Also, since we understand manifolds of dimensions 1 and 2, this approach offers the possibility of making explicit computations when $\dim M - \dim V - \dim G \leq 2$

6.0.7 Definition Let M be a compact smooth manifold with boundary ∂M. Let ξ be a vector bundle of rank k over M. Then $S_{\mathrm{fr}}^k(M, \partial M; \xi) := \{(N, \phi)\}$ where N is a submanifold of M of codimension k such that $N \cap \partial M$ is empty and $\phi : \nu_M(N) \to \xi|_N$ is a vector bundle isomorphism. We will call $S^k_{\mathrm{fr}}(M, \partial M; \xi)$ the set of "twisted framed submanifolds of M of codimension k". We define an equivalence relation called "twisted ξ-framed cobordism" on $S^k_{\mathrm{fr}}(M, \partial M; \xi)$ as follows: $(N_1, \phi_1) \sim (N_2, \phi_2)$ if there exists a pair (W, Φ) where W is a submanifold of $M \times \mathrm{I}$ of codimension k and $\Phi : \nu_{M\times \mathrm{I}}(W) \to \mathrm{pr}_1^*(\xi)|_W$ is a vector bundle isomorphism (pr_1 is the projection from $M \times \mathrm{I}$ onto the first factor) such that $(W, \Phi)|_{M\times 0} = (N_1, \phi_1)$ and $(W, \Phi)|_{M\times 1} = (N_2, \phi_2)$. Define

$$\Omega^k_{\mathrm{fr}}(M, \partial M; \xi) := S^k_{\mathrm{fr}}(M, \partial M; \xi)/\sim$$

$\Omega^k_{\mathrm{fr}}(M, \partial M; \xi)$ is the set of "twisted ξ-framed cobordism classes of twisted ξ-framed submanifolds of M of codimension k".

6.0.8 Remark If ξ is a trivial bundle of rank k, then

$$\Omega^k_{\mathrm{fr}}(M, \partial M; \xi) \cong \Omega^k_{\mathrm{fr}}(M, \partial M)$$

(the classical framed cobordism set, see, for example, [10]). Thus, $\Omega^k_{\mathrm{fr}}(M, \partial M; \xi)$ is a generalization of classical framed cobordism. It is now apparent why we use the term "twisted" to describe these cobordism sets. The "twist" is introduced by the possible nontriviality of the vector bundle ξ. The situation is analogous to ordinary twisted (co-)homology. In that case, the "twist" is introduced by the possible nontriviality of the local coefficient system (or bundle of abelian groups).

6.0.9 Remark It is possible to define twisted versions of other cobordism theories. For example, let ζ be a line bundle over M. Let $S^k_{\mathrm{or}}(M, \partial M; \zeta) = \{(N, \phi)\}$ where N is a submanifold of codimension k and $\phi : \Lambda^k(\nu_M(N)) \to \zeta|_N$

is a bundle isomorphism (Λ^k denotes the kth exterior power). Cobordisms are defined in the obvious way. One obtains $\Omega^k_{\mathrm{or}}(M, \partial M; \zeta)$ which generalizes the classical oriented cobordism set $\Omega^k_{\mathrm{or}}(M, \partial M)$.

6.0.10 Theorem Let $\tilde{M}$ be a compact free G-manifold with boundary $\partial\tilde{M}$ and let V be a real representation of G of dimension d. Let ξ be the vector bundle

$$\begin{array}{c} \tilde{M} \times_G V \\ \downarrow \\ M \end{array}$$

Then there is a bijective correspondence $S^V_G(\tilde{M}, \partial\tilde{M}) \equiv S^d_{\mathrm{fr}}(M, \partial M; \xi)$

Proof: We will first prove the following:

6.0.11 Lemma Let N be a submanifold of M. Then the following diagram is a pullback

$$\begin{tikzcd} \nu_{\tilde{M}}(\tilde{N}) \arrow[r, "Tp"] \arrow[d] & \nu_M(N) \arrow[d] \\ \tilde{N} \arrow[r, "p"'] & N \end{tikzcd}$$

The vertical arrows are the vector bundle projections. Tp denotes the derivative of p

Proof of lemma: We have the following composite of vector bundle maps

$$\begin{tikzcd} T\tilde{M}|_{\tilde{N}} \arrow[r, "Tp"'] \arrow[rr, bend left, "\theta"] \arrow[d] & TM|_N \arrow[r] \arrow[d] & (TM|_N)/TN \arrow[r, equal] \arrow[d] & \nu_M(N) \\ \tilde{N} \arrow[r, "p"'] & N \arrow[r, "\mathrm{Id}"'] & N & \end{tikzcd}$$

To prove the lemma, it suffices to show that θ is fibrewise surjective and that $\ker(\theta) = T\tilde{N}$. Surjectivity follows from the fact that p is a submersion. On the other hand, if $v \in \ker(\theta)$, then $Tp(v) \in TN$. But $(Tp)^{-1}(TN) = T\tilde{N}$ since p is a submersion. Thus, $v \in T\tilde{N}$. This proves the lemma. $\square$

Now, let $(N, \phi) \in S^d_{\mathrm{fr}}(M; \xi)$. We have the following commutative solid diagram

$$\begin{tikzcd}
\tilde{N} \times V \arrow[r] \arrow[dr, dashed, "\gamma"] \arrow[ddr] & \tilde{N} \times_G V \arrow[dr, "\phi^{-1}", "\cong"'] & \\
& \nu_{\tilde{M}}(\tilde{N}) \arrow[r] \arrow[d] & \nu_M(N) \arrow[d] \\
& \tilde{N} \arrow[r] & N
\end{tikzcd}$$

where the bottom right square is a pullback by the lemma. Hence, we obtain a map $\gamma : \tilde{N} \times V \to \nu_{\tilde{M}}(\tilde{N})$ which is an equivariant bundle isomorphism.

On the other hand, let $(\tilde{A}, \tilde{\psi}) \in S^V_G(\tilde{M}, \partial\tilde{M})$. We have the following composite of vector bundle maps

$$\begin{tikzcd}
\tilde{A} \times V \arrow[r, "\psi^{-1}"] \arrow[d] & \nu_{\tilde{M}}(\tilde{A}) \arrow[r, "Tp"] \arrow[d] & \nu_M(A) \arrow[d] \\
\tilde{A} \arrow[r, equal] & \tilde{A} \arrow[r, "p"'] & A
\end{tikzcd}$$

All these maps are G-equivariant (G acts trivially on the spaces in the right hand column). Therefore, this composite factors to a bundle isomorphism

$$\begin{tikzcd}
\tilde{A} \times_G V \arrow[r, "\psi"] \arrow[d] & \nu_M(A) \arrow[d] \\
A \arrow[r, equal] & A
\end{tikzcd}$$

These constructions establish a bijective correspondence between $S^V_G(\tilde{M}, \partial\tilde{M})$ and $S^d_{\mathrm{fr}}(M, \partial M; \xi)$. $\square$

6.0.12 Theorem Let $\tilde{M}$ be a compact free G-manifold with boundary $\partial\tilde{M}$ and let V be a real representation of G of dimension d. Let ξ be the vector bundle $\tilde{M} \times_G V \to M$. Then

$$\Omega_G^V(\tilde{M}, \partial\tilde{M}) \cong \Omega_{\mathrm{fr}}^d(M; \xi)$$

Proof: We have established a bijection between elements of $S_G^V(\tilde{M}, \partial\tilde{M})$ and elements of $S_{\mathrm{fr}}^d(M, \partial M; \xi)$. To prove the theorem, we must show that the two notions of cobordism also correspond. Suppose that $(\tilde{N}_1, \tilde{\phi}_1)$, $(\tilde{N}_1, \tilde{\phi}_1) \in S_G^V(\tilde{M}, \partial\tilde{M})$ are V-framed cobordant via a cobordism $(\tilde{W}, \tilde{\Phi})$. As in the proof of 6.0.10 we can mod out the G action on $(\tilde{W}, \tilde{\Phi})$ to obtain a twisted framed cobordism (W, Φ) between (N_1, ϕ_1) and (N_2, ϕ_2). Conversely, if (W, Φ) is a twisted framed cobordism between (N_1, ϕ_1) and $(N_2, \phi_2) \in S_{\mathrm{fr}}^d(M, \partial M; \xi)$, then as in 6.0.10 Φ may be pulled back to $\tilde{\Phi} : \nu_{\tilde{M}\times I}(\tilde{W}) \to \tilde{W} \times V$, an equivariant V-framing of $\tilde{W}$. Thus $(\tilde{N}_1, \tilde{\phi}_1)$ is V-framed cobordant to $(\tilde{N}_2, \tilde{\phi}_2)$. $\square$

6.0.13 Corollary Let $\tilde{M}$ be a compact free G-manifold, let V be real G-representation of dimension d and let ξ be the vector bundle $\tilde{M} \times_G V \to M$, then

$$\Big[(\tilde{M}, \partial\tilde{M}); (S^V, \infty)\Big]^G \cong \Omega_{\mathrm{fr}}^d(M, \partial M; \xi).$$

Proof: This follows immediately from theorems 5.0.6 and 6.0.12. $\square$

6.0.14 Corollary If the vector bundle ξ in the previous corollary is trivial then

$$\Big[(\tilde{M}, \partial\tilde{M}); (S^V, \infty)\Big]^G \cong \big[(M, \partial M); (S^d, \infty)\big].$$

Proof: In this case $\Omega_{\mathrm{fr}}^d(M, \partial M; \xi) \cong \Omega_{\mathrm{fr}}^d(M, \partial M) \cong \big[(M, \partial M); (S^d, \infty)\big]$. $\square$

6.0.15 Remark The representation V induces a map $i_V : BG \to BO_d$. Also, the principal G-bundle $\tilde{M} \to M$ induces a map $f : M \to BG$. The condition in the above corollary that ξ is a trivial vector bundle is the same as requiring that the composite $i_V \circ f$ be nullhomotopic. For example, this is the case if V is a trivial representation (i_V is null) or if $\tilde{M} \to M$ is a trivial principal G-bundle (f is nullhomotopic). Corollary 6.0.14 is the first indication that by "modding out" the action of G we can often replace a set of equivariant homotopy classes of maps by something more accessible, in this case a set of (nonequivariant) homotopy classes of maps. Essentially, 6.0.14 gives a condition under which the "twist" introduced by the action of G on $\tilde{M}$ cancels out the "twist" introduced by the action of G on S^V. This process of "modding out" the action of G will allow us to make explicit calculations of equivariant homotopy sets later (see chapter 9).

Chapter 7

Twisted Generalized Cohomology Theories

The constructions that we have seen so far are all unstable. We will now see how Ω^k_{fr} can be stabilized to give a "twisted cohomology theory". First we must say what the appropriate categories are. It turns out that it is more convenient to define our twisted cohomology theories on a certain category of pairs, rather than to talk about "reduced twisted cohomology" theories. The basic reason is that given a pair of free G spaces $(\tilde{X}, \tilde{A})$ there is no canonical way to collapse the subspace $\tilde{A}$ to a single free G orbit. Equivalently, given a map $f : X \to BG$ and a subspace A of X, there is no canonical way to replace f by a map which sends A to a single point.

7.0.16 Definition Let CW^2_{BG} be the full subcategory of T^2_{BG} whose objects are triples $(X, A; f)$ where (X, A) is a relative CW-complex.

7.0.17 Definition Let $(X, A; f)$ be an object in T^2_{BG}. The prism, $\mathrm{I} \times (X, A; f)$, over $(X, A; f)$ is the object $(\mathrm{I} \times X, \mathrm{I} \times A; f \circ \mathrm{pr}_2)$. There are inclusions $i_j : (X, A; f) \to \mathrm{I} \times (X, A; f)$, $j = 0, 1$, given by $i_0(x) := (0, x)$ and $i_1(x) := (1, x)$. Let hT^2_{BG} and hCW^2_{BG} denote the resulting homotopy categories.

7.0.18 Remark Classically (see, for example, [24]) ordinary local cohomology is defined using a category L^* (see chapter VI of [24]) which is related to T^2_{BG} in the following way: Let M be a discrete $\mathbb{Z}G$-module and let $(X, A; f) \in |T^2_{BG}|$. Suppose that $\tilde{X} \to X$ is a principal G-bundle which is classified by f. Then $\tilde{X} \times_G M \to X$ is a bundle of abelian groups over X in the sense of [24]. Also, given $\phi : (X, A; f) \to (Y, B; g)$ we get a commutative diagram

$$\begin{tikzcd} \tilde{X} \arrow[r, "\tilde{\phi}"] \arrow[d] & \tilde{Y} \arrow[d] \\ X \arrow[r, "\phi"'] & Y \end{tikzcd}$$

where $\tilde{\phi}$ is a map of free G-spaces. This induces a bundle map

$$\begin{tikzcd} \tilde{X} \times_G M \arrow[r] \arrow[d] & \tilde{Y} \times_G M \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

So M induces a functor $F_M : T^2_{BG} \to L^*$

We can now give the following:

7.0.19 Definition A cohomology theory on hT^2_{BG} (or hCW^2_{BG}) is a sequence of contravariant functors $h^n : hT^2_{BG} \to$ Abelian groups satisfying the following axioms:

- Exactness: Given $(X, A; f)$ we have an exact sequnce

$$\cdots \longrightarrow h^n(X; f) \longrightarrow h^n(A; f|_A) \xrightarrow{\delta} h^{n+1}(X, A; f) \longrightarrow h^{n+1}(X; f) \longrightarrow \cdots$$

  where δ is a natural transformation.

- Excision: Let $(X; A, B)$ be a triad with $X = A \cup B$ and $\overline{B - A} \subset \mathrm{int}(B)$. Suppose $f : X \to BG$. Then we have an excision isomorphism

$$h^n(X, A; f) \cong h^n(B, A \cap B; f|_B)$$

  induced by the inclusion

$$(B, A \cap B; f|_B) \to (X, A; f)$$

- coefficient homotopy invariance: Given a pair (X, A), $h^n(X, A; -)$ is a functor from the fundamental groupoid $\Pi_1(\mathrm{map}(X, BG))$ to abelian groups and, if F is a homotopy between f_1 and f_2 then the following diagram commutes

$$\begin{tikzcd} h^n(A; f_1) \arrow[r, "F_*"] \arrow[d, "\delta"] & h^n(A; f_2) \arrow[d, "\delta"] \\ h^n(X, A; f_1) \arrow[r, "F_*"'] & h^n(X, A; f_2) \end{tikzcd}$$

  F_* denotes the homomorphism induced by the homotopy F.

I shall refer to the map $f : X \to BG$ as the "coefficient map". We may also require our cohomology theories to satisfy the following axiom:

- Additivity: Let X be a disjoint union of CW-complexes X_α and, for each α suppose that we have $f_\alpha : X_\alpha \to BG$. Let f be the disjoint union of the

f_α. Let A be a subcomplex of X and let $A_\alpha = X_\alpha \cap A$. Then the induced homomorphisms

$$h^n(X, A; f) \to h^n(X_\alpha, A_\alpha; f)$$

represent $h^n(X, A; f)$ as a direct product.

7.0.20 Remark The exactness, excision and additivity axioms in the above definition are fairly straightforward analogues of their untwisted counterparts. The coefficient homotopy invariance axiom has no untwisted counterpart. Roughly speaking, it is there to ensure that h^* does not distinguish between objects $(X, A; f_1)$ and $(X, A; f_2)$ when f_1 is homotopic to f_2.

7.0.21 Example Our first example will be of fundamental importance for the rest of this thesis. Let V be a d-dimensional representation of the group G. Let $(X, A; f)$ be an object in hT^2_{BG} and consider

$$\big[(X, A); (EG \times_G S^V, EG \times_G \infty)\big]_f$$

(Note that we have a sphere bundle $EG \times_G S^V \to BG$.) We think of this set as a twisted version of $\big[(X, A); (S^d, \infty)\big]$. In order to get a cohomology theory we must stabilize this set. We do this as follows: Fix a relative homeomorphism $(\mathrm{I}, \partial\mathrm{I}) \to (S^1, \infty)$. Then, there is a canonical equivariant relative homeomorphism

$$\theta : (S^V, \infty) \times (\mathrm{I}, \partial\mathrm{I}) \to (S^{V+1}, \infty)$$

which collapses $S^V \times \partial\mathrm{I} \cup \infty \times \mathrm{I}$ to ∞. We define the suspension map to be the following composite:

$$\begin{array}{c}
\left[(X,A); EG \times_G (S^V, \infty)\right]_f \\
\big\downarrow \\
\left[(X,A) \times (\mathrm{I}, \partial\mathrm{I}); (EG \times_G (S^V, \infty)) \times (\mathrm{I}, \partial\mathrm{I})\right]_{f \circ \mathrm{pr}_X} \\
\big\downarrow {\scriptstyle \theta_*} \\
\left[(X,A) \times (\mathrm{I}, \partial\mathrm{I}); EG \times_G (S^{V \oplus \mathbb{R}}, \infty)\right]_{f \circ \mathrm{pr}_X}
\end{array}$$

where pr_X is the projection $X \times \mathrm{I} \to X$. We can now define $\omega_V^n(X, A; f)$ to be the direct limit of the following diagram:

$$\begin{array}{c}
\vdots \\
\big\downarrow \\
\left[(X,A) \times (\mathrm{I}, \partial\mathrm{I})^k; EG \times_G (S^{V \oplus \mathbb{R}^{n-d+k}}, \infty)\right]_{f \circ \mathrm{pr}_X} \\
\big\downarrow \\
\left[(X,A) \times (\mathrm{I}, \partial\mathrm{I})^{k+1}; EG \times_G (S^{V \oplus \mathbb{R}^{n-d+k+1}}, \infty)\right]_{f \circ \mathrm{pr}_X} \\
\big\downarrow \\
\vdots
\end{array}$$

Note that if f is a constant map, then $\omega_V^n(X, A; f) = \omega^n(X, A)$, the usual stable cohomotopy of (X, A).

7.0.22 Remark It is worth pointing out the contrast between the stabilization procedure introduced here and the equivariant stabilization described in chapter 4. There, the set $[(X,A); (Y,B)]^G$ was stabilized by suspending with respect to all representations of G. By contrast, the stabilization procedure descibed above may be viewed as stabilizing the set $\left[(\tilde{X}, \tilde{A}); (S^V, \infty)\right]^G$, where $\tilde{X}$

is a free G-space, by suspending only with respect to the trivial representation. It might be possible to introduce a "twisted theory" which corresponds to stabilizing with respect to all representations of G, however it is not necessary for my purposes. It is also not clear yet if anything new would be gained in this way.

7.0.23 Theorem ω_V is an additive cohomology theory on hCW^2_{BG}

Proof: First note that $\left[(X,A)\times(\mathrm{I},\partial I)^i, EG\times_G(S^{V+k},\infty)\right]_{f\circ \mathrm{pr}_1}$ is a group for $i=1$ and an abelian group for $i\geq 2$. This is proved in exactly the same way as the classical untwisted case. So $\omega_V^n(X,A;f)$ is an abelian group.

Exactness: Consider the sequence

$$\begin{array}{c}\left[(X,A), EG\times_G(S^{V+k},\infty)\right]_f\\ \big\downarrow\\ \left[X, EG\times_G(S^{V+k},\infty)\right]_f\\ \big\downarrow\\ \left[A, EG\times_G(S^{V+k},\infty)\right]_{f|_A}\end{array}$$

Clearly, this composition is 0. Conversely, suppose that $f\in \mathrm{map}_f(X, EG\times_G S^{V+k})$ and that $f|_A$ is homotopic to a map that sends A to $EG\times_G\infty$. By 2.1.3 f may be homotoped to a map which sends A to $EG\times_G\infty$. Thus the above sequence is exact. Now, by taking direct limits we see that

$$\omega_V^n(X,A;f)\to\omega_V^n(X;f)\to\omega_V^n(A;f)$$

is exact. To construct $\delta:\omega_V^n(A;f|_A)\to\omega_V^{n+1}(X,A;f)$ we first make the following unstable construction. Let

$$[\phi]\in\left[A\times(\mathrm{I},\partial\mathrm{I}); EG\times_G(S^{V+n-d+1},\infty)\right]_{f\circ \mathrm{pr}_X}$$

So $\phi : A \times \partial \mathrm{I} \to EG \times_G \infty$ We can extend ϕ to a map

$$\overline{\phi} : X \times \{0\} \cup A \times \mathrm{I} \to EG \times_G S^{V+n-d+1}$$

by setting $\overline{\phi}(a,t) = \phi(a,t)$ for $(a,t) \in A \times \mathrm{I}$ and $\overline{\phi}(x,0) = \sigma_\infty f(x)$ where $\sigma_\infty : BG \to EG \times_G S^{V+n-d+1}$ is the ∞-section of $p : EG \times_G S^{V+n-d+1} \to BG$. Now, $p\overline{\phi}(x,t) = f(x)$ for all $(x,t) \in X \times \{0\} \cup A \times \mathrm{I}$ so by the fibrewise homotopy extension property of $(X, A; f)$ (theorem 2.1.3), $\overline{\phi}$ extends to a map $\Phi : X \times \mathrm{I} \to EG \times_G S^{V+n-d+1}$. Let $\psi := \Phi(-,1)$. Clearly, $\psi : (X, A) \to (EG \times_G S^{V+n-d+1}, EG \times_G \infty)$. Also, $p \circ \phi = f \circ \mathrm{pr}_X$. So $[\psi] \in \left[(X, A); EG \times_G (S^{V+n-d+1}, \infty)\right]_{f \mathrm{opr}_X}$. Several choices were made in the construction of ψ, and we must show that the fibrewise homotopy class $[\psi]$ is independent of these choices. For example, if we have two different maps Φ_1 and Φ_2 which extend $\overline{\phi}$, then Φ_1 and Φ_2 will be fibrewise homotopic with respect to the fibration $EG \times_G S^V \to BG$ and thus will lead to the same fibrewise homotopy class $[\psi]$. In this way, we see that $[\psi]$ is indeed independent of the choices made in its construction. Thus we have constructed a map

$$\begin{array}{c} \left[A \times (\mathrm{I}, \partial \mathrm{I}); EG \times_G (S^{V+n-d+1}, \infty)\right]_{f \mathrm{opr}_X} \\ \Big\downarrow \hat{\delta} \\ \left[(X, A); EG \times_G (S^{V+n-d+1}, \infty)\right]_{f \mathrm{opr}_X} \end{array}$$

Now, by taking direct limits, we get the required map

$$\delta : \omega_V^n(A; f|_A) \to \omega_V^{n+1}(X, A; f)$$

By construction

$$\begin{gathered}
[X \times (\mathrm{I}, \partial I); EG \times_G (S^{V+k}, \infty)]_{f \circ \mathrm{pr}_1} \\
\downarrow \\
[A \times (\mathrm{I}, \partial I); EG \times_G (S^{V+k}, \infty)]_{f \circ \mathrm{pr}_1} \\
\downarrow \hat{\delta} \\
[(X, A); EG \times_G (S^{V+k}, \infty)]_f \\
\downarrow \\
[X; EG \times_G (S^{V+k}, \infty)]_f
\end{gathered}$$

is exact, so taking direct limits yields the exactness of

$$\omega_V^n(X; f) \longrightarrow \omega_V^n(A; f) \xrightarrow{\delta} \omega_V^{n+1}(X, A; f) \longrightarrow \omega_V^{n+1}(X; f)$$

Excision: Let $(X; A, B)$ be a triad as in definition 7.0.19 and let $f : X \to BG$. We need to find an inverse to the excision homomorphism

$$\omega_V^n(X, B; f) \to \omega_V^n(A, A \cap B; f|_A)$$

Let $\phi \in \mathrm{map}_{f|\circ \mathrm{pr}_1}((A, A \cap B) \times (I, \partial I)^k; EG \times_G (S^{V+n-d+k}, \infty))$ represent an element of $\omega_V^n(A, A \cap B; f|_A)$. We can extend ϕ to $\overline{\phi} \in \mathrm{map}_{f|\circ \mathrm{pr}_1}((X, B) \times (I, \partial I)^k; EG \times_G (S^{V+n-d+k}, \infty))$ by defining $\overline{\phi}(x, (t_1, \ldots, t_k)) := \sigma_\infty(f(x))$ for $x \in X - A$. Then the assignment $[\phi] \mapsto \boxed{\phi}$ is an inverse to the excision homomorphism.

Coefficient homotopy invariance: Let $F : \mathrm{I} \times X \to BG$ be a homotopy of f_1 to f_2 and let $\phi_1 : (X, A) \times (\mathrm{I}\partial I)^k \to EG \times_G (S^{V+n-d+k}, \infty)$ represent an element of $\omega_V^n(X, A; f_1)$. F may be lifted to a homotopy

$$\overline{F} : \mathrm{I} \times (X, A) \times (I, \partial I)^k \to EG \times_G (S^{V+n-d+k}, \infty)$$

such that $\overline{F}(0,-) = \phi_1$. Let $\phi_2 := \overline{F}(1,-)$ and let $F_*([\phi_1]) := [\phi_2]$. F_* is well defined for if $\overline{F}_1$ and $\overline{F}_2$ are two different lifts of F then they are fibrewise homotopic (see [17]). Thus ϕ_2 is uniquely defined up to fibrewise homotopy. Moreover, F_* clearly satisfies the required naturality properties. See lemma 2.0.8. Additivity: It is clear that the induced morphism

$$\begin{array}{c} \left[(X,A) \times (I,\partial I)^k; EG \times_G (S^{V+n-d+k},\infty)\right]_f \\ \big\downarrow \\ \prod \left[(X_\alpha,A_\alpha) \times (I,\partial I)^k; EG \times_G (S^{V+n-d+k},\infty)\right]_{f_\alpha} \end{array}$$

is a bijection. Now, take direct limits. □

The group $\omega_V^d(X,A;f)$ is a twisted version of the classical stable cohomotopy group $\omega^d(X,A)$. The "twist" depends on two things; the action of the group on the representation V and the nontriviality of the homotopy class of $f : X \to BG$.

Our main motivation for introducing these twisted cohomotopy groups is to provide some method of computing the sets $\left[(\tilde{M},\partial\tilde{M});(S^V,\infty)\right]^G$. So a natural question to ask is: when is $\left[(\tilde{M},\partial\tilde{M});(S^V,\infty)\right]^G \cong \omega_V^d(M,\partial M;f)$ $(d = \dim V)$. We can answer this question using the equivariant Freudenthal suspension theorem (see chapter II in [21]).

7.0.24 Theorem Let $(\tilde{X},\tilde{A})$ be a relative free G CW-complex of cellular dimension m (i.e. $\dim(\tilde{X}/G,\tilde{A}/G) = m$) and suppose that $\dim V = d$. Then the suspension with respect to the trivial representation

$$\left[(\tilde{X},\tilde{A});(S^V,\infty)\right]^G \to \left[(S^1,*)\wedge(\tilde{X},\tilde{A});(S^{V+1},\infty)\right]^G$$

is an isomorphism if $m < 2d-1$ and an epimorphism if $m < 2d$.

Proof: This follows immediately from theorem 2.10 of chapter 2 of [21] □

Now, we can interpret the above theorem as a result about the fibrewise suspension maps described in the definition of $\omega_V^n(X, A; f)$ to get:

7.0.25 Theorem Let (X, A) be a CW pair and let $f : X \to BG$. Suppose that $\dim(X, A) = m$ and that V is a G representation of dimension d. Then

$$\left[(X, A); EG \times_G (S^V, \infty)\right]_f \to \left[(X, A) \times (\mathrm{I}, \partial \mathrm{I}); EG \times_G (S^{V+1}, \infty)\right]_{f\mathrm{opr}}$$

is an isomorphism if $m < 2d - 1$ and an epimorphism if $m < 2d$.

Proof: Let $\tilde{X} \to X$ be the pullback of $EG \to BG$ along f. Then we have a commutative diagram

$$\begin{tikzcd}
\left[(\tilde{X}, \tilde{A}); (S^V, \infty)\right]^G \arrow[r] \arrow[d, "\cong"'] & \left[(\tilde{X}, \tilde{A}) \times (\mathrm{I}, \partial \mathrm{I}); (S^{V+1}, \infty)\right]^G \arrow[d, "\cong"] \\
\left[(X, A); EG \times_G (S^V, \infty)\right]_f \arrow[r] & \left[(X, A) \times (\mathrm{I}, \partial \mathrm{I}); EG \times_G (S^{V+1}, \infty)\right]_{f\mathrm{opr}}
\end{tikzcd}$$

The vertical arrows are isomorphisms by lemma 3.3.7. The top arrow is an isomorphism by theorem 7.0.24 and the theorem follows. □

Thus, under conditions given by the above theorem, computations of $\omega_V^d(X, A; f)$ give us computations of $\left[(\tilde{X}, \tilde{A}); (S^V, \infty)\right]$. In the next section, we will develop a tool that will give us some computational handle on $\omega_V^d(X, A; f)$.

Chapter 8

The Atiyah-Hirzebruch spectral sequence for twisted cohomology theories

Having introduced the concept of a twisted generalized cohomology theory, a natural question to ask is whether or not we can develop an appropriate analogue of the classical Atiyah-Hirzebruch spectral sequence. By analogy with the classical situation, one would expect that such a spectral sequence would have the E_2 term isomorphic to twisted ordinary cohomology with coefficients in some twisted system whose underlying abelian group is the generalized cohomology of a point. This indeed turns out to be the case. Having this spectral sequence will allow us to reduce (in certain cases) computations of twisted stable cohomotopy groups to (easier) computations of twisted ordinary cohomology groups.

Let $(X, A; f)$ be an object in T^2_{BG} and let h be a cohomology theory on T^2_{BG}

as defined in chapter 7. For each integer q we can define a coefficient system, $\mathcal{L}_f^q$, on X as follows: Let $\mathcal{L}_f^q(x) := h^q(*; c_{f(x)})$, where $c_{f(x)} : * \to BG$ sends $*$ to $f(x)$. A path $\alpha : I \to X$ induces a homotopy between $c_{f(\alpha(0))}$ and $c_{f(\alpha(1))}$ and thus α induces a homomorphism $\alpha^* : \mathcal{L}_f^q(\alpha(1)) \to \mathcal{L}_f^q(\alpha(0))$ by the coefficient homotopy invariance axiom. So $\mathcal{L}_f^q$ is a coefficient system on X. The main results of this section are the following two theorems.

8.0.26 Theorem Let h be a cohomology theory on T^2_{BG}. Let (X, A) be a relative CW-complex and let $f : X \to BG$. Then there is a cohomological spectral sequence $E(X, A; f)$ such that $E_2^{p,q}(X, A; f) \cong \mathcal{H}^p(X, A; \mathcal{L}_f^q)$ where $\mathcal{H}$ is ordinary cohomology. If the spectral sequence converges then $E(X, A; f) \Rightarrow h^*(X, A; f)$

8.0.27 Theorem Given a cellular map $\phi : (X, A) \to (Y, B)$ of relative CW-complexes and a map $g : Y \to BG$, ϕ induces a morphism of spectral sequences

$$\phi^* : E(Y, B; g) \to E(X, A; g \circ \phi)$$

Whenever the spectral sequences converge, then this morphism of spectral sequences in turn induces the canonical homomorphism

$$\phi^* : h^*(Y, B; g) \to h^*(X, A; g \circ \phi)$$

The rest of this section will be devoted to proving the two preceding theorems. So for the rest of this section, (X, A) will be a relative CW-complex and $f : X \to BG$ a continuous map.

8.1 Construction of the spectral sequence

8.1.1 Lemma Let X^k denote the k-skeleton of X. Then there is a long exact sequence

$$\cdots \to h^n(X^{k+1}, A; f) \to h^n(X^k, A; f) \to h^{n+1}(X^{k+1}, X^k; f) \to h^{n+1}(X^{k+1}, A; f) \to \ldots$$

Proof: This is a standard cohomology arguement using the long exact cohomology sequences associated to $(X^k, A; f)$, $(X^{k+1}, A; f)$ and $(X^{k+1}, X^k; f)$ (see [24]). □

Let

$$\begin{aligned} D_1^{p,q} &:= h^{p+q}(X^p, A; f) \\ E_1^{p,q} &:= h^{p+q}(X^p, X^{p-1}; f) \\ F^{p,q} &:= \ker(h^{p+q}(X, A; f) \to h^{p+q}(X^p, A; f)) \end{aligned}$$

The $F^{p,q}$'s form a filtration of $h^{p+q}(X, A; f)$. Let

$$G^{p,q} := \frac{F^{p,q}}{F^{p-1,q+1}}$$

We have an exact couple

$$\begin{tikzcd} D_1 \arrow[rr] & & D_1 \arrow[dl] \\ & E_1 \arrow[ul] & \end{tikzcd}$$

arising from the long exact sequences in lemma 8.1.1. If X is finite dimensional, the filtration described above is a finite one, i.e $F^{p,q} = 0$ for $p \gg 0$ and $F^{p,q} = h^{p+q}(X, A; f)$ for $p < 0$. So the resulting spectral sequence converges to $h^*(X, A; f)$. That is

$$E_\infty^{p,q} \cong G^{p,q}$$

The proof of theorem 8.0.27 is now immediate. For, if $\phi : (X, A) \to (Y, B)$ is a cellular map, then clearly ϕ induces a morphism of the corresponding exact couples and thus induces a morphism of the corresponding spectral sequences.

Thus far, the development proceeds exactly as in the classical untwisted case (see [9])

8.2 Identification of the E_2 term

The construction of the spectral sequence parallels closely that of the classical untwisted case. The major difference arises in the identification of the E_2 term. A close examination of the untwisted case reveals that the crucial step in identifying the E_2 term uses the following well known fact:

8.2.1 Lemma Let $f : S^p \to S^p$ be any map. Then there is an integer d_f (the degree of f) such that if h is any generalized cohomology theory, then for all k

$$f^* : h^k(S^p) \to h^k(S^p)$$

is multiplication by d_f □

What is important is that the degree of a map is a concept which is independent of any particular cohomology theory.

Turning to the twisted case the corresponding fact is given below (lemma 8.2.5). First we prove some preliminary results.

Throughout this section c_z will denote the constant map at z

8.2.2 Lemma For $x \in X$ let $\mathcal{L}(x) := h^k(S^n, *; c_{f(x)})$. Then $\mathcal{L}$ is a bundle of abelian groups (local coefficient system) over X

Proof: Certainly, $\mathcal{L}(x)$ is an abelian group for each $x \in X$. Given a path $r : I \to X$ with $r(0) = x_1$ and $r(1) = x_2$, then $f \circ r$ induces a homotopy from $c_{f(x_1)}$ to $c_{f(x_2)}$. Thus, the coefficient homotopy invariance axiom gives us a corresponding homomorphism $\mathcal{L}(x_2) \to \mathcal{L}(x_1)$. □

The next lemma is the twisted version of the suspension isomorphism. It shows that the coefficient system defined in the previous lemma depends only on the difference $k - n$.

8.2.3 Lemma The coefficient systems

$$\mathcal{L}_1 := h^k(S^n, *; c_{f(-)})$$

and

$$\mathcal{L}_2 := h^{k+1}(S^{n+1}, *; c_{f(-)})$$

are canonically isomorphic

Proof: Let $x \in X$. The isomorphism $\mathcal{L}_1(x) \cong \mathcal{L}_2(x)$ is the following composite:

$$\begin{tikzcd}
h^{k+1}(S^{n+1}, *; c_{f(x)}) \arrow[d, "q^*"', "\cong"] \arrow[ddd, bend left=60, "\cong"', "\gamma"] \\
h^{k+1}(D^{n+1}, S^n; c_{f(x)}) \\
h^k(S^n; c_{f(x)}) \arrow[u, "\delta"] \\
h^k(S^n, *; c_{f(x)}) \arrow[u, "i^*"]
\end{tikzcd}$$

Note that since all the coefficient maps are constant maps, we are dealing with untwisted cohomology here. $q : (D^{n+1}, S^n)$ is the quotient map. δ is the connecting homomorphism of the long exact sequence of the pair (D^{n+1}, S^n). $i : S^n \to (S^n, *)$ is the inclusion. All the maps in the above diagram commute with homotopies of the coefficient maps by the coefficient homotopy invariance axiom of defintion 7.0.19, hence, to prove the lemma, it suffices to show that γ is a well defined isomorphism. Consider the following commutative diagram (the coefficient maps are suppressed):

$$\begin{tikzcd}
& h^k(D^{n+1}) \arrow[d, tail] \arrow[dr, "j^*", "\cong"'] & \\
h^k(S^n,*) \arrow[r, tail, "i^*"] \arrow[dr, "\gamma"'] & h^k(S^n) \arrow[r, two heads] \arrow[d, two heads, "\delta"'] & h^k(*) \\
& h^{k+1}(D^{n+1},S^n) &
\end{tikzcd}$$

The horizontal row is part of the long exact sequence of the pair $(S^n,*)$ and the vertical column is part of the long exact sequence of the pair (D^{n+1},S^n). j^* is an isomorphism, since it is induced by the homotopy equivalence $* \to D^{n+1}$. Now, it is a straightforward diagram chase to show that γ is an isomorphism. $\square$

8.2.4 Lemma Let $f : D^p \to BG$. Then there are canonical isomorphisms

$$h^n(D^p,S^{p-1};f) \cong h^n(D^p,S^{p-1};c_{f(0)})$$

and

$$h^n(S^{p-1};f|_{S^{p-1}}) \cong h^n(S^{p-1};c_{f(0)})$$

Proof: This follows immediately from the coefficient homotopy invariance axiom and the fact that $0 \in D^p$ is a deformation retract of D^p. $\square$

Now, we come to the key lemma of this section.

8.2.5 Lemma Let

$$l_\alpha : (D^p,S^{p-1}) \to (X^p,X^{p-1})$$

and

$$l_\beta : (D^{p+1}, S^p) \to (X^{p+1}, X^p)$$

be characteristic maps for a p-cell and a $(p+1)$-cell respectively. Let $x_\alpha := l_\alpha(0)$ and let $x_\beta := l_\beta(0)$. Let

$$X_\alpha^{p-1} := X^p - l_\alpha(\mathrm{int}(D^p))$$

Then, there exist paths $r_i : \mathrm{I} \to X$ for $i = 1, \ldots, d$ with $r_i(0) = x_\beta$ and $r_i(1) = x_\alpha$ and signs $\epsilon_i = \pm 1$ such that the following diagram commutes (in this diagram the unlabelled isomorphisms come from lemma 8.2.4):

$$\begin{tikzcd}
h^{p+q}(D^p, S^{p-1}; f \circ l_\alpha) \arrow[d, "\cong"'] & h^{p+q}(X^p, X_\alpha^{p-1}; f) \arrow[l, "l_\alpha^*"', "\cong"] \arrow[r, "l_\beta|^*"] & h^{p+q}(S^p; f \circ l_\beta|) \arrow[d, "\cong"] \\
h^{p+q}(D^p, S^{p-1}; c_{f(x_\alpha)}) \arrow[d, "\cong"'] & & h^{p+q}(S^p; c_{f(x_\beta)}) \\
h^{p+q}(S^p, *; c_{f(x_\alpha)}) \arrow[d, "\cong"'] & & h^{p+q}(S^p, *; c_{f(x_\beta)}) \arrow[u, "\cong"'] \arrow[d, "\cong"] \\
h^q(*; c_{f(x_\alpha)}) \arrow[rr, "\phi"'] & & h^q(*; c_{f(x_\beta)})
\end{tikzcd}$$

where ϕ is defined as follows. Let $\mathcal{L}_f^q = h^q(*; c_{f(-)})$ as in lemma 8.2.2. Then $\phi : \mathcal{L}_f^q(x_\alpha) \to \mathcal{L}_f^q(x_\beta),\ g \mapsto \sum_{i=1}^{d} \epsilon_i \mathcal{L}_f^q(r_i)(g)$.

The proof of this lemma is deferred till later. For now it suffices to understand its significance. The top row of the diagram in lemma 8.2.5 is a homomorphism that (for our purposes) will arise in two different contexts - the differential of the spectral sequence and the differential of the cellular cochain complex of X with coefficients in a certain local coefficient system. The lemma characterizes these homomorphisms in terms of intrinsic properties of the attaching maps (namely the paths r_i) and the coefficient system $\mathcal{L}^q_f$. Thus, if the two different situations mentioned above give rise to the same coefficient system, then the corresponding homomorphisms are the same.

Now, we will see how lemma 8.2.5 allows us to identify the E_2 term of the twisted Atiyah-Hirzebruch spectral sequence. The differential d_1 is the following composite

$$d_1 : h^{p+q}(X^p, X^{p-1}; f) \longrightarrow h^{p+q}(X^p, A; f) \xrightarrow{\delta} h^{p+q+1}(X^{p+1}, X^p; f)$$

Let Λ_k be an indexing set for the set of k-cells of X. Then we have isomorphisms

$$h^{p+q}(X^p, X^{p-1}; f) \xrightarrow{(l_\alpha^*)} \prod_{\alpha \in \Lambda_p} h^{p+q}(D^p, S^{p-1}; f \circ l_\alpha)$$

and

$$h^{p+q+1}(X^{p+1}, X^p; f) \xrightarrow{(l_\beta^*)} \prod_{\beta \in \Lambda_{p+1}} h^{p+q+1}(D^{p+1}, S^p; f \circ l_\beta)$$

Thus, to identify the differential d_1, we need to identify the composite homomorphism

$$\begin{array}{c} h^{p+q}(D^p, S^{p-1}; f \circ l_{\alpha_0}) \\ \downarrow \\ \prod_{\alpha \in \Lambda_p} h^{p+q}(D^p, S^{p-1}; f \circ l_\alpha) \\ \downarrow{\scriptstyle d_1} \\ \prod_{\beta \in \Lambda_{p+1}} h^{p+q+1}(D^{p+1}, S^p; f \circ l_\beta) \\ \downarrow \\ h^{p+q+1}(D^{p+1}, S^p; f \circ l_{\beta_0}) \end{array}$$

for each pair $(\alpha_0, \beta_0) \in \Lambda_p \times \Lambda_{p+1}$. This suffices since there are only finitely many cells in each dimension (See remark 8.2.6 below). Consider the following commutative diagram

$$\begin{tikzcd}
h^{p+q}(D^p, S^{p-1}; c_{f(x_\alpha)}) \arrow[r, "\cong", "8.2.4"'] & h^{p+q}(D^p, S^{p-1}; f \circ l_\alpha) & \\
h^{p+q}(S^p, *; c_{f(x_\alpha)}) \arrow[u, "\cong"] \arrow[dddd, bend right=60, "\phi"'] & h^{p+q}(X^p, X^{p-1}_\alpha; f) \arrow[u, "\cong"', "l_\alpha^*"] \arrow[r] \arrow[dr] & h^{p+q}(X^p, X^{p-1}; f) \arrow[d] \arrow[dd, bend left=60, "d_1"] \\
& \mathbf{A} & h^{p+q}(X^p, A; f) \arrow[d, "\delta"] \arrow[ddl, "l_\beta^*"'] \\
& & h^{p+q}(X^{p+1}, X^p; f) \arrow[d, "l_\beta^*"] \\
h^{p+q}(S^p, *; f \circ l_\beta) \arrow[r] \arrow[d, "\cong", "8.2.4"'] & h^{p+q}(S^p; f \circ l_\beta) \arrow[r, "\delta"] \arrow[d, "\cong", "8.2.4"'] & h^{p+q}(D^{p+1}, S^p; f \circ l_\beta) \arrow[d, "\cong", "8.2.4"'] \\
h^{p+q}(S^p, *; c_{f(x_\beta)}) \arrow[r] \arrow[rr, bend right=20, "\cong", "\psi"'] & h^{p+q}(S^p; c_{f(x_\beta)}) \arrow[r] & h^{p+q}(D^{p+1}, S^p; c_{f(x_\beta)})
\end{tikzcd}$$

The arrows marked 8.2.4 are isomorphisms by lemma 8.2.4. The region marked **A** commutes by lemma 8.2.5. Also, the composite marked ψ is an isomorphism (this is a simple exercise in untwisted cohomology theory). Thus we have a commutative diagram

$$\begin{tikzcd}
h^{p+q}(D^p, S^{p-1}; f \circ l_\alpha) \arrow[r, "h_1"] \arrow[d, "\cong"'] & E_1^{p,q} \arrow[r, "d_1"] & E_1^{p+1,q} \arrow[r, "h_2"] & h^{p+q+1}(D^{p+1}, S^p; f \circ l_\beta) \arrow[d, "\cong"] \\
h^q(S^p, *; c_{f(x_\alpha)}) \arrow[rrr, "\phi"'] & & & h^q(S^p, *; c_{f(x_\beta)})
\end{tikzcd}$$

The arrow marked h_1 is the inclusion of one of the factors into a direct product and the arrow marked h_2 is the projection from a direct product onto one of the factors.

Now we apply a similar analysis to ordinary cohomology with coefficients in the bundle of abelian groups $\mathcal{L}_f^q$. Recall that $\mathcal{L}_f^q(x) := h^q(*; c_{f(x)})$. Recall (see, for example [24]) that the cellular cochain comples of X with coefficients in $\mathcal{L}$ is the top row if the following commutative diagram:

$$\begin{tikzcd} \cdots \arrow[r] & \mathcal{H}^p(X^p, X^{p-1}; \mathcal{L}) \arrow[rr, "\partial"] \arrow[dr] & & \mathcal{H}^{p+1}(X^{p+1}, X^p; \mathcal{L}) \arrow[r] & \cdots \\ & & \mathcal{H}^p(X^p, A; \mathcal{L}) \arrow[ur] & & \end{tikzcd}$$

In order to identify the differential ∂ we use a similar argument to the one used to identify the differential d_1 of the spectral sequence. One takes the diagram from the previous page and replaces every ocurrence of $h^{p+q}(-,-;f)$ with $\mathcal{H}^p(-,-;\mathcal{L}_f^q)$. The resulting diagram is, once again, commutative by lemma 8.2.5 and therefore for each pair $(\alpha, \beta) \in \Lambda_p \times \Lambda_{p+1}$ we have a commutative diagram

$$\begin{tikzcd} \mathcal{H}^p(X^p, X^{p-1}; \mathcal{L}_f^q) \arrow[r, "\partial"] & \mathcal{H}^{p+1}(X^{p+1}, X^p; \mathcal{L}_f^q) \arrow[d, "h_2"] \\ \mathcal{H}^p(D^p, S^{p-1}; \mathcal{L}_{fol_\alpha}^q) \arrow[u, "h_1"] \arrow[d, "\cong"] & \mathcal{H}^{p+1}(D^{p+1}, S^p; \mathcal{L}_{fol_\beta}^q) \arrow[d, "\cong"] \\ \mathcal{L}_f^q(x_\alpha) \arrow[r, "\phi"'] & \mathcal{L}_f^q(x_\beta) \end{tikzcd}$$

where the arrow marked h_1 is the inclusion of one of the factors into a direct product and the arrow marked h_2 is the projection from a direct product onto one of the factors. Thus we have identified the differential d_1, of the spectral sequence, with the differential ∂ of the cellular cochain complex. Subject to the proof of lemma 8.2.5, this completes the proof of theorem 8.0.26.

8.2.6 Remark I make the assumption that X is compact, since that is the only case that in which I am interested. However, it may be possible to obtain useful information fron the spectral sequence for more general spaces if one makes a more detailed study of the convergence issues. See [23] for a much more detailed discussion of these issues.

8.3 Proof of lemma 8.2.5

When reading this proof, one should bear in mind the analogy with the degree of a map $S^p \to S^p$ in the untwisted situation. It would probably help the reader to review the proof of lemma 8.2.1. The proof of lemma 8.2.5 uses the same ideas, however, we have the added complication of keeping track of all local data via the coefficient system $\mathcal{L}^q_f$. Now for the details.

We may assume that x_α is in general position with respect to $l_\beta|_{S^p}$. Thus

$$l_\beta|_{S^p}^{-1}(x_\alpha) = \{y_1, \ldots, y_d\}$$

Also there is a disc neighbourhood V of x_α in X^p and disjoint disc neighbourhoods $U_1, \ldots, U_d$ of $y_1, \ldots, y_d$ respectively, in S^p such that $l_\beta| : U_i \to V$ is a homeomorphism. Let $\psi_V : D^p \to V$ and $\psi_i : D^p \to U_i$ be charts. Also, we may choose $*$ the basepoint of S^p so that $* \notin U_i$

Fix orientations on D^p and D^{p+1}. The orientation on D^{p+1} determines one on S^p with respect to the inward normal vector. Let

$$q : (D^p, S^{p-1}) \to (S^p, *)$$

be an orientation preserving relative homeomorphism. Let $\hat{r}_i : \mathrm{I} \to D^{p+1}$ be a path such that $\hat{r}_i(0) = 0$ and $\hat{r}_i(1) = y_i$. Let $r_i : \mathrm{I} \to X$ be defined by

$$r_i := l_\beta \circ \hat{r}_i$$

Now consider the following commutative diagram of maps:

$$
\begin{tikzcd}
(S^p,*) \arrow[r, "l_\beta|"] \arrow[dr] & (X^p, X^p - \mathrm{int}(V)) & (D^p, S^{p-1}) \arrow[l, "l_\alpha|"'] \\
& (S^p, S^p - (\mathrm{int}(U_1) \sqcup \cdots \sqcup \mathrm{int}(U_d)) \arrow[u, "l_\beta|"'] & \\
& \displaystyle\bigsqcup_{i=1}^{d} (D^p, S^{p-1}) \arrow[u, "\sqcup_{i=1}^d \psi_i"'] &
\end{tikzcd}
$$

In cohomology this induces the top part of the following diagram:

$$
\begin{tikzcd}
h^{p+q}(S^p,*;f\circ l_\beta) \arrow[ddd, "\cong"'] & h^{p+q}(X^p, X^p - \mathrm{int}(V); f\circ l_\beta) \arrow[l] \arrow[r, "\cong"] \arrow[d] & h^{p+q}(D^p, S^{p-1}; f\circ l_\alpha) \arrow[ddd, "\cong"] \\
& h^{p+q}(S^p, S^p - (\displaystyle\bigsqcup_{i=1}^{d} \mathrm{int} U_i; f\circ l_\beta) \arrow[ul] \arrow[d, "\cong"'] & \\
& \prod_{i=1}^d h^{p+q}(D^p, S^{p-1}; f\circ l_\beta \circ \psi_i) \arrow[d, "\cong"'] & \\
h^{p+q}(S^p,*;c_{f(x_\beta)}) \arrow[d, "q^*"', "\cong"] & \prod_{i=1}^d h^{p+q}(D^p, S^{p-1}; c_{f(x_\alpha)}) \arrow[dl, "\overline{\phi}"'] & h^{p+q}(D^p, S^{p-1}; c_{f(x_\alpha)}) \arrow[l, "\prod_{i=1}^d \epsilon_i"'] \\
h^{p+q}(D^p, S^{p-1}; c_{f(x_\beta)}) & \mathcal{L}_f^q(x_\beta) \arrow[l, "\cong"] & \mathcal{L}_f^q(x_\alpha) \arrow[u, "\cong"']
\end{tikzcd}
$$

$\overline{\phi}$ is defined by the commutativity of the above diagram. Let

$$\overline{\phi}_j := \overline{\phi} \circ i_j$$

where

$$i_j : h^{p+q}(D^p, S^{p-1}; c_{f(x_\alpha)}) \to \prod_{i=1}^{d} h^{p+q}(D^p, S^{p-1}; c_{f(x_\alpha)})$$

is the inclusion of the jth factor in the direct product. To prove the lemma, it suffices to show that the following diagram commutes:

$$\begin{tikzcd}
h^{p+q}(D^p, S^{p-1}; c_{f(x_\alpha)}) \arrow[r, "\bar{\phi}_j"] \arrow[d, "\cong"'] & h^{p+q}(D^p, S^{p-1}; c_{f(x_\beta)}) \arrow[d, "\cong"] \\
\mathcal{L}^q_f(x_\alpha) \arrow[r, "\mathcal{L}^q_f(r_j)"'] & \mathcal{L}^q_f(x_\beta)
\end{tikzcd}$$

This follows from the commutativity of the following diagram:

$$\begin{tikzcd}
\prod_{k=1}^d h^{p+q}(D^p, S^{p-1}; c_{f(x_\alpha)}) \arrow[d, "\cong"'] & h^{p+q}(D^p, S^{p-1}; c_{f(x_\alpha)}) \arrow[l, "i_j"'] \arrow[d, "\cong"] \arrow[dddddl, bend left=40, "\bar{\phi}_j"] \\
\prod_{k=1}^d h^{p+q}(D^p, S^{p-1}; f \circ l_\beta \circ \psi_k) & h^{p+q}(D^p, S^{p-1}; f \circ l_\beta \circ \psi_j) \arrow[l, "i_j"] \\
h^{p+q}(S^p, S^p - \bigsqcup_{i=1}^d U_i; f \circ l_\beta) \arrow[u, "\cong"', "(\psi_k^*)_{k=1}^d"] \arrow[d] & h^{p+q}(S^p, S^p - (\mathrm{int}(U_j)); f \circ l_\beta) \arrow[l] \arrow[u, "\psi_j^*"'] \arrow[d, "\cong"] \arrow[dl] \\
h^{p+q}(S^p, *; f \circ l_\beta) \arrow[d, "\cong"'] & h^{p+q}(S^p, S^p - (\mathrm{int}(U_j)); c_{f(x_\beta)}) \arrow[ddl, "\cong"', "\psi_j^*"] \\
h^{p+q}(S^p, *; c_{f(x_\beta)}) \arrow[d, "\cong"', "q^*"] & \\
h^{p+q}(D^p, S^{p-1}; c_{f(x_\beta)}) &
\end{tikzcd}$$

The unlabelled arrows are induced by the canonical inclusions.

This completes the proof of lemma 8.2.5 and thus completes the proof of theorem 8.0.26.

Chapter 9

Computations and examples

Throughout this chapter, $\mathcal{H}_*(-;\mathcal{L})$ will mean ordinary singular homology with coefficients in $\mathcal{L}$. I shall use the same notation as [24] to describe singular simplices, chains, cycles etc.

9.0.1 Theorem Let ω_V be twisted stable cohomotopy as defined in chapter 7. Let (X, A) be a CW-pair with relative dimension m and let $f : X \to BG$. Then there is an exact sequence

$$\mathcal{H}^{m-2}(X, A; \mathcal{L}_f^0) \xrightarrow{d_2} \mathcal{H}^m(X, A; \mathcal{L}_f^{-1}) \longrightarrow \omega_V^{m-1}(X, A; f) \longrightarrow \mathcal{H}^{m-1}(X, A; \mathcal{L}_f^0)$$

where d_2 is the differential of the Atiyah-Hirzebruch spectral sequence (theorem 8.0.26)

Proof: The E_2 term of the Atiyah-Hirzebruch spectral sequence for $\omega_V^*(X, A; f)$ is given by $E_2^{p,q} \cong \mathcal{H}^{p+q}(X, A; \mathcal{L}_f^q)$. Now, for $p \geq m+1$, $\mathcal{H}^{p+q}(X, A; \mathcal{L}_f^q) = 0$ since $\dim(X, A) = m$. Also, note that the underlying abelian group of the twisted coefficent system $\mathcal{L}_f^q$ is the classical stable cohomotopy group $\omega^q(*)$. Thus, for

$q \geq 1$, $\mathcal{L}_f^q = 0$ since $\omega^q(*) = 0$ for $q \geq 1$. The result now follows from standard spectral sequence arguements. $\square$

Note that the coefficient system $\mathcal{L}_f^{-1}$ has undelying abelian group $\omega^{-1}(*) \cong \mathbb{Z}/2$ and is therefore always a trivial system. Also $\mathcal{L}_f^0$ has underlying group $\omega^0(*) \cong \mathbb{Z}$. Thus $\mathcal{L}_f^0$ is determined by homomorphisms $\pi_1(X_i) \to \mathbb{Z}/2 \cong \mathrm{Aut}(\mathbb{Z})$ where the X_i's are the connected components of X.

We will now specialize to the case $(X, A) = (M, \partial M)$ where M is a connected m-dimensional compact manifold with boundary ∂M. In this case, we can use a version of Poincaré duality to make explicit computations of the cohomology groups in theorem 9.0.1. A first result is:

9.0.2 Theorem Under these hypothesis there is an exact sequence

$$\mathcal{H}^{m-2}(M, \partial M; \mathcal{L}_f^0) \longrightarrow \mathbb{Z}/2 \longrightarrow \omega_V^{m-1}(M, \partial M; f) \longrightarrow \mathcal{H}^{m-1}(M, \partial M; \mathcal{L}_f^0)$$

Proof:

$$\begin{aligned} \mathcal{H}^m(M, \partial M; \mathcal{L}_f^{-1}) &= H^m(M, \partial M; \mathbb{Z}/2) \\ &\cong \mathbb{Z}/2 \end{aligned}$$

by Poincaré duality. Now apply theorem 9.0.1. $\square$

Now I will examine, in more detail, the homomorphism

$$i : \mathbb{Z}/2 \to \omega_V^{m-1}(M, \partial M; f)$$

which occurs in the above exact sequence. The following lemma essentially says that the image of i corresponds to the nonequivariant stable 1-stem of spheres.

9.0.3 Lemma Let U be an open disc neighbourhood in the manifold M. Then

$$\omega_V^{m-1}(M, M - U; f) \cong \mathbb{Z}/2$$

Proof: Let $h : (B^m, S^{m-1}) \to (M, M - U)$ be a relative homeomorphism. By the excision axiom

$$h^* : \omega_V^{m-1}(M, M - U; f) \to \omega_V^{m-1}(B^m, S^{m-1}; f \circ h)$$

is an isomorphism. Now B^m is contractible, so $f \circ h \simeq c$ where $c : B^m \to BG$ is a constant map. Thus

$$\begin{aligned} \omega_V^{m-1}(B^m, S^{m-1}; f \circ h) &\cong \omega_V^{m-1}(B^m, S^{m-1}; c) \\ &\cong \mathbb{Z}/2 \end{aligned}$$

□

Now, recall that by theorem 7.0.25, if $m > 3$,

$$\begin{aligned} \omega_V^{m-1}(B^m, S^{m-1}; f \circ h) &\cong \left[(B^m, S^{m-1}); EG \times_G (S^V, \infty)\right]_{f\circ h} \\ &\cong \Omega_{\mathrm{fr}}^{m-1}(B^m, S^{m-1}; h^*(\xi)) \end{aligned},$$

where ξ is the vector bundle $\tilde{M} \times_G V \to M$. Since $h^*(\xi)$ is a trivial vector bundle, elements of $\Omega_{\mathrm{fr}}^{m-1}(B^m, S^{m-1}; h^*(\xi))$ are represented by framed 1-dimensional submanifolds of B^m and it is well known that the two distinct elements of $\Omega_{\mathrm{fr}}^{m-1}(B^m, S^{m-1}; h^*(\xi))$ correspond to the two different homotopy classes of framings of the trivial rank $m-1$ vector bundle over S^1. In this way, we can explicitly realize the elements of $\omega_V^{m-1}(M, M - U; f)$.

9.0.4 Lemma The homomorphism i factors in the following way:

$$\begin{tikzcd} \omega_V^{m-1}(M, M - U; f) \arrow[r, "j^*"] & \omega_V^{m-1}(M, \partial M; f) \\ \mathbb{Z}/2 \arrow[u, "\cong"] \arrow[ur, "i"'] & \end{tikzcd}$$

where $j : (M, \partial M) \to (M, M - U)$ is the inclusion.

Proof: This follows from theorem 8.0.27 (naturality of the exact sequences).

□

Thus, when $m > 3$, we have an explicit realizations of the image under i of the nontrivial element of $\mathbb{Z}/2$ as follows: Let $l : \mathbb{R}^m \to M$ be a chart for M. Let $\overline{N} := \{(\cos t, \sin t, 0) \in \mathbb{R}^m; 0 \le t < 2\pi\}$. Let $N := l(\overline{N})$. Now, $l^*(\xi)$ is a trivial vector bundle over $\mathbb{R}^m$. Fix a trivialisation ψ of this bundle. The nontrivial framing of the normal bundle of $\overline{N}$ in $\mathbb{R}^m$ induces, via ψ and l, an isomorphism $\nu_M(N) \cong \xi|_N$. Call this isomorphism ϕ. Then $[(N, \phi)]$ is the required element of $\Omega^{m-1}_{\mathrm{fr}}(M, \partial M; \xi)$.

9.0.5 Lemma The element, $[(N, \phi)]$, which we have just constructed, is a nontrivial element of $\Omega^{m-1}_{\mathrm{fr}}(M, \partial M; \xi)$.

Proof: Suppose that (N, ϕ) is bordant to the empty submanifold via a bordism (W, Φ) where W is a 2-dimensional submanifold of $M \times I$ and Φ is a twisted framing of the normal bundle. This leads to the following situation: There is a 2-dimensional manifold W with boundary $\partial W \equiv S^1$, a vector bundle η over W of rank $m - 1$ and a vector bundle automorphism $\theta : \eta \to \eta$ such that the following hold. $\eta|_{\partial W}$ is a trivial bundle over $\partial W \equiv S^1$ and $\theta|_{\partial W}$ is a homotopically nontrivial automorphism of this trivial bundle. W is homotopy equivalent to a wedge of circles, thus $\eta \oplus \eta$ is a trivial bundle of rank $2m - 2$ over W. $\theta \oplus Id_\eta$ is an automorphism of $\eta \oplus \eta$ which restricts to a homotopically nontrival automorphism of the trivial bundle $\eta \oplus \eta|_{\partial W}$. Automorphisms of trivial bundles correspond to maps from the base space into $GL(\mathbb{R}^k)$. So we have a map $\tilde{\theta} : W \to GL(\mathbb{R}^{2m-2})$

such that

$$\tilde{\theta}| : S^1 \equiv \partial W \to GL(\mathbb{R}^{2m-2})$$

represents the nontrivial element of $\pi_1(GL(\mathbb{R}^{2m-2})) \cong \mathbb{Z}/2$. I claim that such a $\tilde{\theta}$ cannot exist. To see this, let $\tau : W \to S^1 \vee \cdots \vee S^1$ be a homotopy equivalence. It is clear (from the classification of 2-dimensional manifolds) that if $p_i : S^1 \vee \cdots \vee S^1 \to S^1$ is the projection onto the ith wedge summand, then the composite

$$S^1 \equiv \partial W \longrightarrow W \xrightarrow{\tau} S^1 \vee \cdots \vee S^1 \xrightarrow{p_i} S^1$$

has even degree for all i. The claim follows easily from this. This completes the proof of lemma 9.0.5. $\square$

9.0.6 Corollary If $m \geq 3$ then there is a short exact sequence

$$0 \to \mathbb{Z}/2 \to \omega_V^{m-1}(M, \partial M; f) \to \mathcal{H}^{m-1}(M, \partial M; \mathcal{L}_f^0) \to 0$$

Proof: Lemma 9.0.5 impies that the map $\mathbb{Z}/2 \to \omega_V^{m-1}(M, \partial M; f)$ is injective. The result follows from theorem 9.0.2. $\square$

To compute the groups $\mathcal{H}^k(M, \partial M; \mathcal{L}_f^0)$ we can use Poincaré duality for (possibly) nonorientable manifolds. Let Θ denote the orientation bundle on M. That is,

$$\Theta(x) := H_m(M, M - \{x\}; \mathbb{Z})$$

Note that

$$\mathcal{H}_m(M, M - \{x\}; \Theta) \cong H_m(M, M - \{x\}; \mathbb{Z}) \otimes H_m(M, M - \{x\}; \mathbb{Z})$$

Let a be any generator of $H_m(M, M - \{x\}; \mathbb{Z})$ and let $\mathcal{I}_x := a \otimes a$. Now we have the following duality theorem:

9.0.7 Theorem (Poincaré Duality) There is a canonical class

$$\mathcal{I} \in \mathcal{H}_m(M, \partial M; \Theta)$$

characterized as follows: For each $x \in M$ the induced homomorphism

$$\mathcal{H}_m(M, \partial M; \Theta) \to \mathcal{H}_m(M, M - \{x\}; \Theta)$$

sends $\mathcal{I} \mapsto \mathcal{I}_x$. Moreover,

$$\mathcal{H}_m(M, \partial M; \Theta) = \mathbb{Z}\mathcal{I}$$

and there is an isomorphism

$$. \cap \mathcal{I} : \mathcal{H}^{m-p}(M, \partial M; \mathcal{L}) \to \mathcal{H}_p(M; \mathcal{L} \odot \Theta).$$

$\mathcal{L} \odot \Theta$ is the coefficient system given by

$$(\mathcal{L} \odot \Theta)(x) := \mathcal{L}(x) \odot \Theta(x).$$

□

9.0.8 Corollary There is a short exact sequence

$$0 \longrightarrow \mathbb{Z}/2 \longrightarrow \omega_V^{m-1}(M, \partial M; f) \longrightarrow \mathcal{H}_1(M, \partial M; \mathcal{L}_f^0 \odot \Theta) \longrightarrow 0$$

□

Now we turn to the question of computing $\mathcal{H}_1(M, \partial M; \mathcal{L}_f^0 \otimes \Theta)$. We have the following result concerning $\mathcal{H}_1(X; \mathcal{L})$. (It is a generalization of the Hurewicz theorem concerning $H_1(X; \mathbb{Z})$)

9.0.9 Theorem Let X be a connected CW-complex with basepoint x_0 and let $\mathcal{L}$ be a local coefficient system on X whose underlying abelian group is $\mathbb{Z}$.

Then $\mathcal{L}$ is determined by a homomorphism $\pi_1(X, x_0) \to \mathbb{Z}/2 \cong \mathrm{Aut}(\mathcal{L}(x_0))$. Let $K_{\mathcal{L}}$ be the kernel of this homomorphism. If we choose a generator, a, of $\mathcal{L}(x_0)$ then there is a corresponding surjective homomorphism

$$\begin{aligned} \beta_a : K_{\mathcal{L}} &\to \mathcal{H}_1(X; \mathcal{L}) \\ [u] &\mapsto [a.u] \end{aligned}$$

where $u : (\triangle_1, \partial\triangle_1) \to (X, x_0)$ represents an element of $K_{\mathcal{L}} < \pi_1(X, x_0)$. The kernel of this homomorphism is $[K_{\mathcal{L}}, K_{\mathcal{L}}]$.

Proof: (Sketch) According to [24], the chain complex of singular simplices with coefficents in $\mathcal{L}$ is chain homotopy equivalent to the chain complex of singular simplices which send all 0-simplices to x_0 (again with coefficients in $\mathcal{L}$). So in computing $\mathcal{H}_1$ we may restrict our attention to the latter chain complex. In this setting is quite clear, that if $u : (\triangle_1, \partial\triangle_1) \to (X, x_0)$ is a singular 1-simplex, then u is a cycle if and only if u representa an element of $\pi_1(X, x_0)$ which is contained in $K_{\mathcal{L}}$. Thus one has an epimorphism

$$K_{\mathcal{L}} \twoheadrightarrow \mathcal{H}_1(X; \mathcal{L}).$$

One checks, as in the proof of the classical Hurewicz theorem (see [24]) that the kernel of this epimorphism is $[K_{\mathcal{L}}, K_{\mathcal{L}}]$. □

9.0.10 Corollary $\mathcal{H}_1(X; \mathcal{L}) \cong \frac{K_{\mathcal{L}}}{[K_{\mathcal{L}}, K_{\mathcal{L}}]} = K_{\mathcal{L}}^{\mathrm{ab}}$

9.0.11 Corollary In the notation of theorem 9.0.9 there is an exact sequence

$$0 \longrightarrow \mathbb{Z}/2 \longrightarrow \omega_V^{m-1}(M, \partial M; f) \longrightarrow K^{\mathrm{ab}}_{\mathcal{L}_f^0 \otimes \Theta} \longrightarrow 0$$

$K_{\mathcal{L}_f^0 \otimes \Theta}$ is often relatively straightforward to compute, so corollary 9.0.11 gives us useful information about $\omega^{m-1}(M, \partial M; f)$. In particular, it tells us the order of $\omega_V^{m-1}(M, \partial M; f)$.

We have already analyzed the geometry underlying the homomorphism

$$\mathbb{Z}/2 \to \omega_V^{m-1}(M, \partial M; f)$$

in lemma 9.0.5. We would also like to understand

$$\omega_V^{m-1}(M, \partial M; f) \to \mathcal{H}_1(M; \mathcal{L}_f^0 \otimes \Theta) \cong K^{\mathrm{ab}}_{\mathcal{L}_f^0 \otimes \Theta}$$

in more detail. First consider the coefficient system $\mathcal{L}_f^0$. Let $\mathcal{L}_V$ denote the bundle of abelian groups $\tilde{M} \times_G [S^d, S^V] \to M$ ($[S^d, S^V]$ is a G-module via the action of G on V). Clearly, $\mathcal{L}_V \cong \mathcal{L}_f^0$. We will assume for the moment, that we are in the stable situation, that is

$$\Omega_{\mathrm{fr}}^{m-1}(M, \partial M; \xi) \cong \omega_V^{m-1}(M, \partial M; f).$$

(We only make this assumption to avoid having to keep mentioning $(I, \partial I)^k$ in what follows.) Let $[N, \phi] \in \Omega_{\mathrm{fr}}^{m-1}(M, \partial M; \xi)$. We can construct the corresponding element of $\mathcal{H}_1(M; \mathcal{L}_V \otimes \Theta)$ as follows: Clearly, we can reduce to the case $N \equiv S^1$. Let $u : \Delta_1 \to N$ be a singular simplex representing a generator of $H_1(N; \mathbb{Z})$. We can choose u so that it is a diffeomorphism relative to the boundary of Δ_1. This choice of u determines an orientation of TN. Now $\phi : \nu_M(N) \to (\tilde{N} \times_G V \to N)$ is a bundle isomorphism. Choose an orientation of $\nu_{u(e_0)}(N)$. Then ϕ determines a corresponding orientation of the fibre of $\tilde{N} \times_G V$ over $u(e_0)$. In turn, this determines a generator, a_V, of $\mathcal{L}_V(u(e_0))$. The aforementioned orientations of

$\nu_{u(e_0)}(N)$ and $T_{u(e_0)}N$ together determine a generator, a_Θ of $\Theta(u(e_0))$.

$$[a_V \otimes a_\Theta.u] \in \mathcal{H}_1(M; \mathcal{L}_V \otimes \Theta)$$

is the required homology class. There were various choices of orientation made in the definition of a_V and a_Θ, however, it is easy to show that the homology class $[a_V \otimes a_\Theta.u]$ is independent of these choices. For example, if we change the choice of orientation of $\nu_{u(e_0)}(N)$, then this will introduce a -1 into a_Θ, but also into a_V, and the resulting homology class remains unchanged.

9.1 $\tilde{\omega}^G_{n+1}(EG^+)$

Throughout this section, let $\mathcal{U}$ be a fixed complete G-universe. By G-representation, we will mean "G-invariant finite dimensional linear subspace of $\mathcal{U}$". For any G-representation V, A_V will denote the subspace of S^V consisting of nonfree orbits. $n = \dim G$

Let G be a compact Lie group. Let $textad(G)$ be the adjoint representation of G. It is possible, as pointed out in [14] to calculate $\tilde{\omega}^G_{n+1}(EG^+)$ abstractly as follows: According to [11] $\tilde{\omega}^G_{n+1}(EG^+) \cong \tilde{\omega}_{n+1}(EG^+ \wedge_G S^{\mathrm{ad}(G)})$. Applying the classical Atiyah-Hirzebruch spectral sequence, we find that there is an exact sequence

$$0 \to \mathbb{Z}/2 \to \tilde{\omega}_{n+1}(EG^+ \wedge_G S^{\mathrm{ad}(G)}) \to H_{n+1}(EG^+ \wedge_G S^{\mathrm{ad}(G)}; \mathbb{Z}) \to 0.$$

To calculate $H_{n+1}(EG^+ \wedge_G S^{\mathrm{ad}(G)}; \mathbb{Z})$ we can apply the Serre spectral sequence to the relative fibration $EG \times_G (S^{\mathrm{As}(G)}, \infty) \to BG$ to get

$$H_{n+1}(EG^+ \wedge_G S^{\mathrm{ad}(G)}; \mathbb{Z}) \cong \mathcal{H}_1(BG; H_n(S^{\mathrm{ad}(G)})) \cong \mathcal{H}_1(BG; \mathcal{L}_{\mathrm{ad}(G)})$$

in the notation of the previous section. Thus, we obtain a short exact sequence

$$0 \to \mathbb{Z}/2 \to \tilde{\omega}^G_{n+1}(EG^+) \to K^{\text{ab}}_{\text{ad}(G)} \to 0.$$

Now, recall from chapter 4 that

$$\tilde{\omega}^G_{n+1}(EG^+) \cong \varinjlim \left[(S^{V+n+1}, A_{V+n+1}); (S^V, \infty)\right]^G.$$

In this section we will use the techniques that we have developed to compute

$$\left[(S^{V+n+1}, A_{V+n+1}); (S^V, \infty)\right]^G$$

and so recover the computation of $\tilde{\omega}^G_{n+1}(EG^+)$. The advantage of this method is that it gives insight into the geometry of the maps which represent elements of $\tilde{\omega}^G_{n+1}(EG^+)$.

Since, we already have a rigorous computation of this group, I will attempt to emphasize the essential geometry in the arguments that follow. I feel that presenting all the details of the cobordism arguments would obscure the essence of the geometry. So the arguments may seem a little sketchy. I refer the reader to Kosinski's excellent book ([10]) for details of some of the framed cobordism arguments. Even though he does not consider twisted framed cobordism, the arguments that he gives can be generalized to our situation.

For the proof of the following lemma, see lemma 2.1 in [14]

9.1.1 Lemma If U is a G-representation of dimension k such that subspace of free orbits, $U - A_U$, is nonempty, then there is a compact k-dimensional G-submanifold $\tilde{M}_U$ of $U - A_U$ with boundary $\partial\tilde{M}$, such that A_U is a G-deformation retract of $S^U - \tilde{M}_U$. Thus, in particular, there is a canonical isomorphism

$$\left[(S^U, A_U); (S^V, \infty)\right]^G \cong \left[(\tilde{M}_U, \partial\tilde{M}_U); (S^V, \infty)\right]^G$$

Let $M_U := \tilde{M}_U/G$

9.1.2 Theorem Let V be a G-representation of dimension $d \geq 3$. Suppose that V has at least one free orbit. Then there is a short exact sequence

$$0 \longrightarrow \mathbb{Z}/2 \longrightarrow \left[(S^{V+n+1}, A_{V+n+1}); (S^V, \infty)\right]^G \longrightarrow \mathcal{H}_1(M; \mathcal{L}_V \otimes \Theta) \longrightarrow 0$$

$n = \dim G$. $M := M_{V+n+1}$ is the submanifold of $(S^{V+n+1} - A_{V+n+1})/G$ whose existence is asserted by lemma 9.1.1 and Θ is the orientation bundle on M_{V+n+1}.

Proof: Clearly, $\tilde{M}$ is an orientable G-manifold of dimension $n+d+1$ with $d \geq 3$. So, we are already in the stable range, and

$$\left[(\tilde{M}, \partial\tilde{M}); (S^V, \infty)\right]^G \cong \omega^d_V(M, \partial M; f)$$

where ω^d_V is twisted cohomotopy with respect to the G-representation V, and $f : M \to BG$ classifies $\tilde{M} \to M$. Now, $\dim M = d+1$, so applying corollary 9.0.8 we obtain a short exact sequence

$$0 \longrightarrow \mathbb{Z}/2 \longrightarrow \omega^d_V(M, \partial M; f) \longrightarrow \mathcal{H}_1(M, \partial M; \mathcal{L}^0_f \otimes \Theta) \longrightarrow 0.$$

But $\mathcal{L}^0_f \cong \mathcal{L}_V$. □

Now, we know that if V is a representation with at least one free G-orbit, then the free part of $V \oplus V$ is connected (see [14]). Moreover, G acts orientably on $V \oplus V$. Thus, the set of representations on which G acts orientably and for which the free part is connected is cofinal in the set of all representations. We shall assume henceforth that any representation has these two properties.

Now, as above, let $\tilde{M} := \tilde{M}_{V+n+1}$ and let $M = \tilde{M}/G$. Our assumption on the orientability of the G-action on V implies that $\mathcal{L}_V \cong \mathbb{Z}$ (the trivial coefficient system over M). Thus we have an exact sequence

$$0 \to \mathbb{Z}/2 \to \left[(S^{V+n+1}, A_{V+n+1}); (S^V, \infty)\right]^G \to K_\Theta^{ab} \to 0.$$

The homotopy exact sequence associated to the fibration $\tilde{M} \to M$ terminates as follows:

$$\pi_1(\tilde{M}, \tilde{m}_0) \longrightarrow \pi_1(M, m_0) \xrightarrow{\partial} \pi_0(G) \cong G/G_0.$$

Let $H_\Theta := \partial(K_\Theta)$. So we have a sequence

$$\begin{array}{c} 0 \\ \downarrow \\ \mathbb{Z}/2 \\ \downarrow{\scriptstyle i} \\ \left[(S^{V+n+1}, A_{V+n+1}); (S^V, \infty)\right]^G \\ \downarrow{\scriptstyle q} \\ H_\Theta^{ab} \\ \downarrow \\ 0 \end{array}$$

which is not necessarily exact but does have the property that $qi = 0$. Now, we have the following lemma whose proof can be found (implicitly) in [14].

9.1.3 Lemma Suppose that G acts orientably on V. Let $\alpha : (\Delta_1, \partial\Delta_1) \to (M, m_0)$ be a loop in M which preserves orientation. Then α lifts to a path $\tilde{\alpha} : \Delta_1 \to \tilde{M}$ and $\tilde{\alpha}(e_1) = g.\tilde{\alpha}(e_0)$ for some $g \in G$ such that g acts orientably on $\mathrm{ad}(G)$.

□

9.1.4 Corollary

$$H_\Theta = K_{\mathrm{ad}(G)}$$

Now, we would like to show that the sequence

$$\begin{array}{c} 0 \\ \downarrow \\ \mathbb{Z}/2 \\ \downarrow{\scriptstyle i} \\ \left[(S^{V+n+1}, A_{V+n+1}); (S^V, \infty)\right]^G \\ \downarrow{\scriptstyle q} \\ K^{\mathrm{ab}}_{\mathrm{ad}(G)} \\ \downarrow \\ 0 \end{array}$$

is "equivariantly stably exact" in the following sense: If $q([f]) = 0$, then there is a representation W such that $\left[f \wedge S^W\right]$ is in the image of

$$i : \mathbb{Z}/2 \longrightarrow \left[(S^{V+W+n+1}, A_{V+W+n+1}); (S^{V+W}, \infty)\right]^G$$

That would be sufficient to recover the calculation of $\tilde{\omega}^G_{n+1}(EG^+)$ that was given at the start of this section. Let

$$\left[\tilde{f}\right] \in \left[(S^{V+n+1}, A_{V+n+1}); (S^V, \infty)\right]^G$$

and let $[N, \phi]$ be the corresponding element of $\Omega^{m-1}_{\mathrm{fr}}(M, \partial M; \xi)$. It can be shown ([10]) that we may assume that N is connected. In order to show that $[f]$ is in the image of

$$i : \mathbb{Z}/2 \to \left[(S^{V+n+1}, A_{V+n+1}); (S^V, \infty)\right]^G$$

it suffices to show that the inclusion $N \to M$ is nullhomotopic. For then, $N \to M$ may be isotoped to an inclusion $N \to M$ whose image is entirely contained within a chart neighbourhood of M. Let $x_0 \in N$ If $\alpha : (\mathrm{I}, \partial \mathrm{I}) \to (N, x_0)$ is a relative homeomorphism, then α lifts to a map $\tilde{\alpha} : \mathrm{I} \to \tilde{M}$ and $\tilde{\alpha}(1) = g.\tilde{\alpha}(0)$ for some g such that $gG_0 \in \left[K_{\mathrm{ad}(G)}, K_{\mathrm{ad}(G)}\right]$. By corollary 3.2.6 there is a section $\sigma : N \to \tilde{M}$ over N of the principal G-bundle $\tilde{M} \to M$. It is possible that the inclusion $\sigma : N \to \tilde{M}$ is not nullhomotopic, however, $\tilde{M} \times V \cup V \times \tilde{M}$ is contained in the free part of $S^{V \oplus V}$ and the inclusion $(\sigma, 0) : N \to \tilde{M} \times V \cup V \times \tilde{M}$, $x \mapsto (\sigma(x), 0)$ certainly is nullhomotopic. Thus $\left[\tilde{f} \wedge S^V\right]$ is in the image of

$$\mathbb{Z}/2 \to \left[(S^{V \oplus V+n+1}, A_{V \oplus V+n+1}); (S^{V \oplus V}, \infty)\right]^G .$$

9.2 Some interesting (unanswered) questions

The next obvious step would be an analagous computation of the geometric 2-stem. Since 2-manifolds are well understood, this should be feasible.

A more difficult question is the following: Using the methods developed in this thesis, we can characterize G-equivariant maps

$$h : (S^{V+k}, A_{V+k}) \to (S^V, \infty)$$

that represent nontrivial elements of $\tilde{\omega}_k^G(EG^+)$ for $k = n, n+1$. Under what conditions do these maps remain stably essential when we restrict symmetry to a subgroup H of G? That is, what maps are in the kernel of the restriction map

$$\tilde{\omega}_k^G(EG^+) \to \tilde{\omega}_k^H(EG^+)$$

In fact, this question was part of the original problem suggested to the author. An analagous problem in equivariant K-theory has been studied in [13]. For equivariant stable homotopy it is possible to deduce some basic results from earlier work (see [25] or [1]). However, it appears to be quite difficult to obtain substantial results in this direction.

As pointed out in the introduction, lemma 3.3.5 underpins the connection between fibrewise homotopy theory and equivariant homotopy theory. From that point of view, it would be interesting to know to what extent the homeomorphism

$$\mathrm{map}_G((E, E'); (Y, Y')) \equiv \mathrm{map}_f((B, B'); EG \times_G (Y, Y'))$$

is natural with repect to homotopies of the map $f : b \to BG$. Note that by the homotopy classification of principal G-bundles, homtopic maps $B \to BG$ induce isomorphic principal G-bundles. However, a specific homotopy does not induce a specific isomorphism of principal G-bundles.

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