

CLASSIFICATION OF SURFACES

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ABSTRACT. The sphere, torus, Klein bottle, and the projective plane are the classical examples of orientable and non-orientable surfaces. As with much of mathematics, it is natural to ask the question: are these all possible surfaces, or, more generally, can we classify all possible surfaces? In this paper, we examine a result originally due to Seifert and Threlfall that all compact surfaces are homeomorphic to the sphere, the connect sum of tori, or the connect sum of projective planes; for this paper, we follow a modern proof from Lee [2].

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1. INTRODUCTION

In this paper, we prove that all compact surfaces are homeomorphic to the sphere, the connect sum of tori, or the connect sum of projective planes. We develop the notion of Euclidean simplicial complexes to understand the triangulation theorem, and the idea of polygonal presentations as a combinatorial view of a surface. We then prove the classification theorem for surfaces by proving that given any surface, we can get to the polygonal presentation of the sphere, the connect sum of tori, or the connect sum of projective planes via a sequence of elementary transformations which preserve the surface up to homeomorphism.

We assume the reader is comfortable with point-set topology from the basic notions of a topological space and topological continuity to Hausdorffness, compactness, connectedness and constructing new spaces via the subspace, product, and quotient topology. Two results from point-set topology that we will use often

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deserve special mention: *the uniqueness of the quotient topology*, which states that given two quotient maps from the same space, if the maps make the same identifications, then the two resultant quotient spaces are homeomorphic, and the *closed map lemma*, which states that a map from a compact space to a Hausdorff space is a quotient map if it is surjective, and a homeomorphism if it is bijective. We also assume the reader is familiar with linear algebra, in particular affine maps and transformations.

2. SURFACES

We begin by defining our mathematical object of study: the surface.

Definition 2.1. A surface is a 2-manifold, by this we mean a second countable Hausdorff space that is locally homeomorphic to \mathbb{R}^2 .

The classic examples of surfaces are the sphere, the torus, the Klein bottle, and the projective plane.

The torus \mathbb{T}^2 is the subset of \mathbb{R}^3 formed by rotating the circle S^1 of radius 1 centered at 2 in the xz -plane around the z axis.

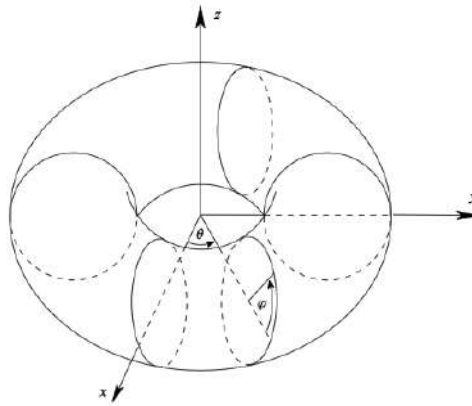


FIGURE 1. A torus as the rotation of a circle around the z -axis.

Equivalently, we see that the torus is homeomorphic to the quotient space of $I \times I$ (where I denotes the closed unit interval) modulo the equivalence relation given by $(x, 0) \sim (x, 1)$ for all $x \in I$ and $(0, y) \sim (1, y)$ for all $y \in I$.

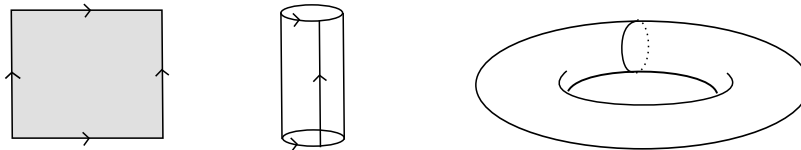


FIGURE 2. The torus as the identification of $I \times I$.

More generally, given an even sided polygon, identifying edges pairwise will always result in a surface. However, is the converse true?

Question 2.2. *Can every surface be constructed from a polygon with the edges identified in an appropriate manner?*

To answer this question, we shall first prove that every surface can be ‘covered’ by finitely many triangles that are connected at an edge, thus by taking the convex polygon spanned by these triangles, we obtain a polygon whose quotient space is the surface. To achieve this goal, we shall rigorously define the idea of a complex of triangles.

3. TRIANGULATION

3.1. Euclidean Simplicial Complex. In this section, we introduce the idea of a simplicial complex, which will serve as the triangular building blocks of manifolds.

Definition 3.1. Given points v_0, \dots, v_k in general position (by which we mean $\{v_1 - v_0, \dots, v_k - v_0\}$ are linearly independent) in \mathbb{R}^n , the *simplex* spanned by them is the set of points

$$\{x \in \mathbb{R}^n \mid x = \sum_{i=0}^k t_i v_i \text{ such that } 0 \leq t_i \leq 1 \text{ and } \sum_{i=0}^k t_i = 1\}$$

with the subspace topology.

Each point v_i is a *vertex* of the simplex and we sometimes denote the simplex spanned by vertices $\{v_0, \dots, v_k\}$ by $\langle v_0, \dots, v_k \rangle$. The dimension of σ is k .

Definition 3.2. Let $\{v_0, \dots, v_k\}$ be vertices of a simplex σ . The simplex spanned by each non-empty subset of $\{v_0, \dots, v_k\}$ is a *face* of σ . The simplex spanned by a proper subset of vertices is a *proper face*. The $(k - 1)$ -dimensional faces are called *boundary faces*.

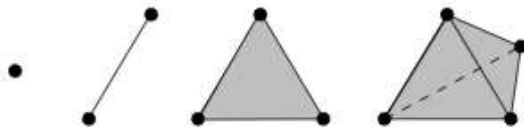


FIGURE 3. From left to right: a 0-simplex, 1-simplex, 2-simplex, 3-simplex.

We can combine simplices together to form a simplicial complex.

Definition 3.3. A *Euclidean simplicial complex* is a collection K of simplices in \mathbb{R}^n satisfying the following conditions:

- (1) If $\sigma \in K$, then every face of σ is in K .
- (2) The intersection of any two simplices in K is either empty or a face of each.
- (3) Every point in a simplex of K has a neighborhood that intersects finitely many simplices of K .

Definition 3.4. The *dimension* of a simplicial complex K is the maximum dimension of any simplex in K .

The following is an example of a valid simplicial complex.

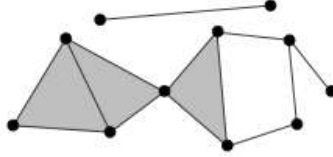


FIGURE 4. A 2-dimensional simplicial complex

For 2-dimensional simplicial complexes, like those pictured above, condition 2 means that simplices intersect at either vertices or edges. The following is an example of condition 2 being broken:

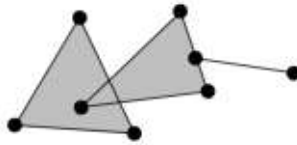


FIGURE 5. Not a simplicial complex

Definition 3.5. Given a Euclidean complex K , the union of all simplices in K is a topological space denoted $|K|$ with the subspace topology from \mathbb{R}^n .

Definition 3.6. Let K be a Euclidean simplicial complex. For any non-negative integer k , the subset $K^{(k)} \subset K$, which is the subset of all simplices with dimension less than or equal to k , is a subcomplex of K called the k -skeleton of K .

Definition 3.7. Further terminology:

- (1) The *boundary* of a simplex is the union of its boundary faces. i.e., the union of all proper faces. We denote the boundary of a simplex σ by $\partial\sigma$.
- (2) The *interior* of a simplex is the simplex minus its boundary. We denote the interior of a simplex σ by $\text{Int } \sigma$.

Whenever we have mathematical objects, a question that naturally arises is: what are the functions that map between these objects. (e.g., group homomorphisms in group theory and linear maps in linear algebra.) In this subsection, we study the maps between Euclidean simplicial complexes. We begin with a motivating proposition.

Proposition 3.8. Let $\sigma = \langle v_0, \dots, v_k \rangle$ be a k -simplex in \mathbb{R}^n . Given $k+1$ points $w_0, \dots, w_k \in \mathbb{R}^m$, there exists a unique map $f : \sigma \rightarrow \mathbb{R}^m$ that is the restriction of an affine map that maps v_i to w_i for each i .

Proof. We may assume that $v_0 = 0$ and $w_0 = 0$, since we can simply apply the invertible affine transformations $x \mapsto x - v_0$ and $y \mapsto y - w_0$. Recall that for a k -simplex, $\{v_1 - v_0, \dots, v_k - v_0\}$ are linearly independent. In our case, we have

$\{v_1, \dots, v_k\}$ as linearly independent. We can let $f : \sigma \rightarrow \mathbb{R}^m$ be the restriction of any linear map such that $v_i \mapsto w_i$ for $1 \leq i \leq k$. To prove that f is uniquely determined by the map of the vertices, observe that

$$f(v) = f\left(\sum_{i=0}^k t_i v_i\right) = \sum_{i=0}^k t_i f(v_i),$$

where $v \in \sigma$ and t_i has the usual conditions. \square

Using this motivating proposition, we define a simplicial map:

Definition 3.9. Let K and L be two Euclidean simplicial complexes. A continuous map $f : |K| \rightarrow |L|$ such that the restriction to each simplex of K maps to some simplex in L via an affine map is a *simplicial map*.

Definition 3.10. The restriction of f (from the previous definition) to $K^{(0)}$ yields a map $f_0 : K^{(0)} \rightarrow L^{(0)}$ called the *vertex map of f* .

Definition 3.11. A simplicial map that is also a homeomorphism (recall that $|K|$ and $|L|$ have topological structure) is a *simplicial isomorphism*.

Lemma 3.12. Let K and L be simplicial complexes. Suppose $f_0 : K^{(0)} \rightarrow L^{(0)}$ is any map satisfying the following: if $\{v_0, \dots, v_k\}$ are vertices of a simplex of K , then $\{f_0(v_0), \dots, f_0(v_k)\}$ are vertices of a simplex of L . Then there is a unique simplicial map $f : |K| \rightarrow |L|$ whose vertex map is f_0 .

Proof. Let $f : |K| \rightarrow |L|$ be a map such that the restriction to each simplex $\sigma = \langle v_0, \dots, v_k \rangle$ maps the vertices of σ to the vertices $\{f_0(v_0), \dots, f_0(v_k)\}$ of a simplex in L via the vertex map f_0 . The convex hull $\langle f_0(v_0), \dots, f_0(v_k) \rangle$ is the simplex in L spanned by $\{f_0(v_0), \dots, f_0(v_k)\}$. Thus, f is a simplicial map.

To show that f is uniquely determined by f_0 , notice that for any point v in each simplex:

$$f(v) = f\left(\sum_{t=0}^k t_t v_t\right) = \sum_{t=0}^k t_t f(v_t) = \sum_{t=0}^k t_t f_0(v_t),$$

where t_i has the usual conditions. \square

Lemma 3.13. Let K and L be simplicial complexes and f_0 and f as above. The function f is a simplicial isomorphism if: i) f_0 is bijective, and ii) $\{v_0, \dots, v_k\}$ are vertices of a simplex of K if and only if $\{f_0(v_0), \dots, f_0(v_k)\}$ are vertices of a simplex of L .

Proof. From the previous lemma, we know that f is a simplicial map, it remains to show that f is a homeomorphism from $|K|$ to $|L|$. Since the vertex map f_0 is bijective, the number of vertices in K equals the number of vertices in L , so $\{v_0, \dots, v_k\}$ are vertices in some simplex of K if and only if $\{f_0(v_0), \dots, f_0(v_k)\} = \{w_0, \dots, w_k\}$ are distinct vertices in L . So $\langle v_0, \dots, v_k \rangle$ and $\langle w_0, \dots, w_k \rangle$ are k -dimensional simplicies in K and L respectively. Thus, σ is a k -simplex in K if and only if $f_0(\sigma)$ (the convex hull of f_0 applied to each vertex point in σ) is a simplex in L , so $|K|$ and $|L|$ with the subspace topology are homeomorphic, so f is a simplicial isomorphism. \square

3.2. Triangulation.

Definition 3.14. A *polyhedron* is a topological space that is homeomorphic to an Euclidean simplicial complex

Definition 3.15. A *triangulation* is a particular homeomorphism between a topological space and a Euclidean simplicial complex.

Notice that there can be multiple different triangulations for a topological space.

Recall that $I \times I / \sim$ with the equivalence relation given by $(x, 0) \sim (x, 1)$ for all $x \in I$ and $(0, y) \sim (1, y)$ for all $y \in I$ is homeomorphic to a torus. We can make $I \times I$ into a simplicial complex K as pictured below:

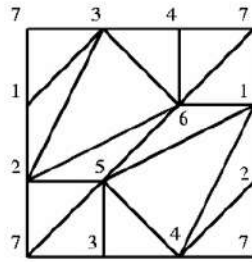


FIGURE 6. The minimal triangulation of the torus.

The homeomorphism between this simplicial complex with the equivalence relation \sim from above and the torus is a triangulation of the torus.

The following is a simple example of an invalid triangulation of the torus:

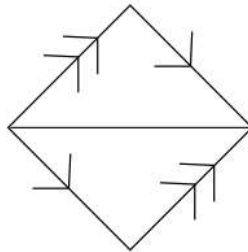


FIGURE 7. Not a triangulation of the torus.

It fails to be a triangulation, because given the identification of the sides of the square region, the two simplexes share 3 edges and 3 vertices, which fails condition 2 of a simplicial complex.

The primary purpose of this section is to prove that all surfaces are triangulable. This result was originally proven by Rado in the 1920's.

Theorem 3.16 (Triangulation Theorem for 2-Manifolds). *Every 2-Manifold is homeomorphic to the polyhedron of a 2-dimensional simplicial complex, in which every 1-simplex is a face of exactly two 2-simplices.*

Proof. The proof of this result is long and intricate, and, thus, we shall not present it here. The basic approach is to cover the manifold with regular coordinate disks and show that each disk can be triangulated compatibly. The main lemma that is needed is the *Schonflies Theorem*, which states that a topological embedding of the circle into \mathbb{R}^2 can be extended to an embedding of the closed disk. A proof of the Schonflies Theorem and the triangulation theorem for surfaces can be obtained in Mohar and Thomassen [1]. \square

4. POLYGONAL PRESENTATION

4.1. **Polygons.** We begin by formally defining a polygon.

Definition 4.1. A subset P of the plane is a *polygonal region* if it is a compact (not necessarily connected) subset whose boundary is a 1-dimensional Euclidean simplicial complex satisfying the following conditions:

- (1) Each point q of an edge that is not a vertex has a neighborhood $U \subset \mathbb{R}^2$ such that $P \cup U$ is equal to the intersection of U with a closed half-plane $\{(x, y) \mid ax + by + c \geq 0\}$.
- (2) Each vertex v has a neighborhood $V \subset \mathbb{R}^2$ such that $P \cup V$ is equal to the intersection of V with two closed half-planes whose boundaries only intersect at v .

Condition 1 and 2, illustrated below, define a subset of \mathbb{R}^2 that is a polygon.

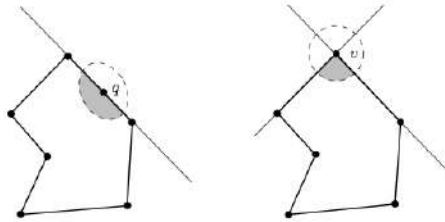


FIGURE 8. Left: Condition 1. Right: Condition 2.

A polygonal can be made into a surface by identifying pairs of edges.

Theorem 4.2. *Let P be a polygonal region in the plane with an even number of edges and suppose we are given an equivalence relation that identifies each edge with exactly one other edge by means of a (Euclidean) simplicial isomorphism. Then the resultant quotient space is a compact surface.*

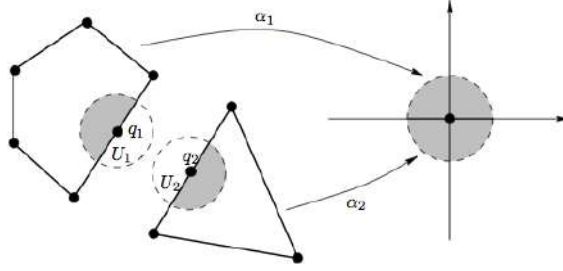
Proof. Let M be the quotient space P / \sim and let $\pi : P \rightarrow M$ denote the quotient map. Since P is compact, $f(P) = M$ is compact. The equivalence relation identifies only edges with edges and vertices with vertices so the points of M are either:

- (1) face points - points whose inverse image in P are in $\text{Int}P$.
- (2) edge points - points whose inverse images are on edges but not vertices.
- (3) vertex points - points whose inverse images are vertices.

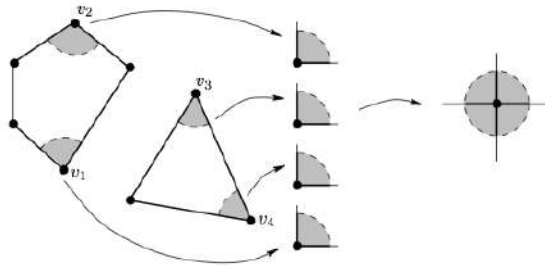
To prove that M is locally Euclidean, it suffices to consider the three types of points.

Face points - Because π is injective on $\text{Int } P$ and π , being a quotient map is surjective, π is bijective on $\text{Int } P$. So by the closed map lemma, π is a homeomorphism on $\text{Int } P$. Since $\text{Int } P \subset \mathbb{R}^2$ is a open set, $\mathbb{R}^2 \cong \text{Int } P \cong \pi(\text{Int } P)$, so every face point is in a locally Euclidean neighborhood, namely $\pi(\text{Int } P)$.

Edge points - For any edge point q , pick a sufficiently small neighborhood such that there are no vertex points in the neighborhood N . By the definition of a polygonal region, q has two inverse images, q_1 and q_2 with neighborhoods U_1 and U_2 such that $V_1 = U_1 \cap P$ and $V_2 = U_2 \cap P$ are disjoint half planes. Furthermore, notice that $\pi|_{V_1 \cup V_2}$ is also a quotient map. We construct affine homeomorphism α_1 and α_2 such that α_1 maps V_1 to a half disk on the upper half plane and α_2 maps V_2 to the lower disk on the lower half plane. We can shrink V_1 and V_2 until they are saturated open sets in P ; i.e., for every boundary point of V_1 , the corresponding boundary point is in V_2 and vice versa. We can now define another quotient map $\varphi : V_1 \cup V_2 \rightarrow \mathbb{R}^2$ such that $\varphi = \alpha_1$ on V_1 and $\varphi = \alpha_2$ on V_2 . Modulus the equivalence relation $r_1 \sim r_2$, where r_1 and r_2 are edge points in V_1 and V_2 respectively, whenever $\varphi(r_1) = \varphi(r_2)$. Notice that φ is a quotient map onto a Euclidean ball centered at the origin and makes the same identifications as π . By the uniqueness of the quotient map, the quotient spaces are homeomorphic, so edge points are locally Euclidean.



Vertex points - Repeat the same process as the edge points, but this time there will be multiple pieces of the polygon that are identified in a fanning manner in \mathbb{R}^2 . The resultant quotient space is homeomorphic to an open ball, so we may conclude by appealing to the uniqueness of the quotient map. Therefore, we know that M is locally Euclidean.



To show that M is Hausdorff, simply pick sufficient small balls. Since M is the quotient space of the quotient map from the polygonal region P , the preimage of any pair of points in M can be separated into disjoint open sets by picking

sufficiently small open balls; the image of these open balls will be open sets in M that separate the two points in M . \square

The converse of this is also true: every compact surface is the quotient space of a polygon with sides pairwise identified, but the proof of this is cleaner after we develop the notion of a polygonal presentation, so we prove this in the next section.

Example 4.3. The sphere $\mathbb{S}^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$ is homeomorphic to the square region $S = \{(x, y) \mid |x| + |y| \leq 1\}$ modulo the equivalence relation $(x, y) \sim (-x, y)$ for $(x, y) \in \partial S$.

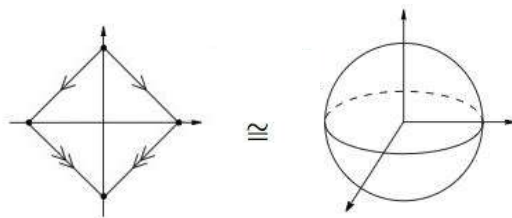


FIGURE 9. Polygon identification homeomorphism to sphere.

Example 4.4. The torus \mathbb{T}^2 is homeomorphic to the square region modulo the equivalence relation $(x, y) \sim (-y, -x)$ for $(x, y) \in \partial S$.

Example 4.5. The Klein bottle \mathbb{K}^2 is homeomorphic to the square region modulo the equivalence relation $(x, y) \sim (-x, -y)$ for $(x, y) \in \partial S$ such that $0 \leq x, y \leq 1$ or $-1 \leq x, y \leq 0$, and another equivalence relation $(x, y) \sim (-y, -x)$ for $(x, y) \in \partial S$ such that $-1 \leq x \leq 0 \leq y \leq 1$ or $-1 \leq y \leq 0 \leq x \leq 1$.

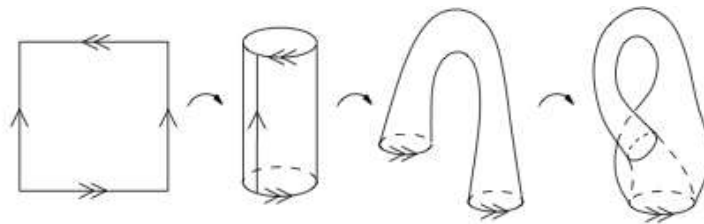


FIGURE 10. Klein Bottle.

Example 4.6. The projective plane \mathbb{P}^2 is homeomorphic to the square region modulo the equivalence relation $(x, y) \sim (-x, -y)$ for $(x, y) \in \partial S$.

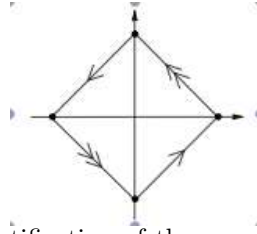


FIGURE 11. The identification of the square region that yields a projective plane.

4.2. The Connect Sum of Surfaces. Given two surfaces, we wish to join them in a natural way such that we end up with another surface. For example, given two tori, the natural gluing process should result in a two hole torus. This operation is called the connect sum.

Definition 4.7. Suppose X and Y are topological spaces, A is a closed subset of Y , and $f : A \rightarrow X$ is a continuous map, then we define an equivalence relation \sim on the disjoint union $X \amalg Y$ such that $a \sim f(a)$ for all $a \in A$. The resulting quotient space $(X \amalg Y)/\sim$, denoted $X \cup_f Y$ is an *adjunction space*.

The connect sum is the adjunction space given a particular choice for the closed set A .

Definition 4.8. Given surfaces M_1 and M_2 and regular coordinate balls $B_i \subset M_i$, the subspace $M'_i := M_i \setminus B_i$ are surfaces with boundaries homeomorphic to S^1 . Let $f : \partial M'_2 \rightarrow \partial M'_1$ be any homeomorphism, then the adjunction space $M'_1 \cup_f M'_2$ is the *connect sum of M_1 and M_2* denoted $M_1 \# M_2$.

The connect sum operation basically involves cutting out open balls from surfaces and gluing points along the S^1 by an equivalence relation, giving a new manifold.

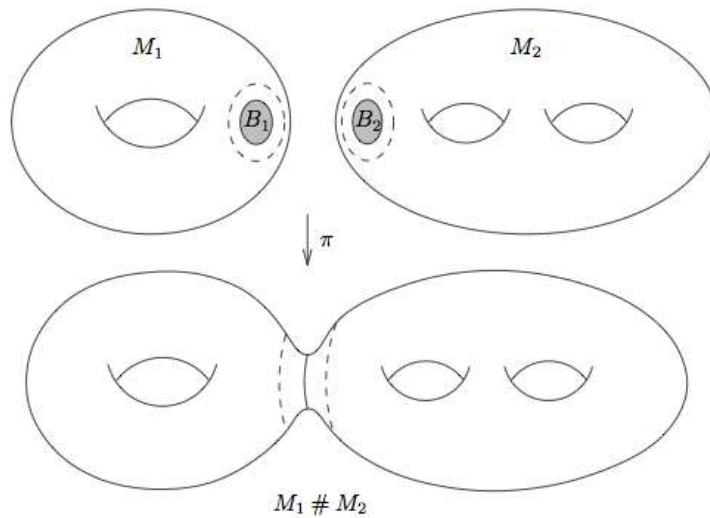


FIGURE 12. connect sum of two Surfaces M_1 and M_2 .

So far, we have assumed that the connect sum of two surfaces indeed results in a surface, now we shall prove it.

Theorem 4.9. *The connect sum $M_1\#M_2$ of two connected surfaces M_1 and M_2 is a connected surface.*

Proof. It suffices to show that $M_1\#M_2$ is locally Euclidean and Hausdorff. The proof of this is similar to Theorem 4.2, so we only provide a sketch of the proof. Let $\pi : M'_1 \amalg M'_2 \rightarrow M_1\#M_2$ be the quotient map. As with Theorem 4.2, there are two types of points: points in $M_1\#M_2$ with preimage in $\text{Int } M'_1$ or $\text{Int } M'_2$, or points with primages $\partial M'_1$ and $\partial M'_2$. As with Theorem 4.2, the first type of points are clearly Euclidean: simply pick a neighborhood small enough such that it is strictly in the interior M'_i , $i = 1, 2$, and homeomorphic to \mathbb{R}^2 . For the second type of points, proceed analogous to the proof for edge points in theorem 4.2. Adjust the neighborhoods such that for each point of $\partial M'_1$ in one neighborhood, the equivalent point in $\partial M'_2$ is in the other neighborhood, and vice versa. Then map the two half planes to \mathbb{R}^2 with the same identification as the original quotient map. By the uniqueness of quotient maps, since the two maps make the same identifications, the two space are homeomorphic, so the second type of points is also locally Euclidean. Hausdorffness follows by picking sufficiently small neighborhoods.

To show that $M_1\#M_2$ is connected, simply note that $M_1\#M_2$ is the union of two connected sets $\pi(M'_1)$ and $\pi(M'_2)$, where $\pi(M'_1) \cap \pi(M'_2) \neq \emptyset$. \square

4.3. Polygonal Presentation. We begin by defining a polygonal presentation:

Definition 4.10. A *polygonal presentation* is a finite set S with finitely many words W_1, \dots, W_k , where W_i is a word in S of length 3 or longer. We denote a polygonal presentation $\mathcal{P} = \langle S \mid W_1, \dots, W_k \rangle$.

To explain this definition, we need two other definitions:

Definition 4.11. Given a set S , a *word* in S is an ordered k -tuple of symbols of the form a or a^{-1} where $a \in S$.

Definition 4.12. The *length* of a word is the number of elements in the word, where a and a^{-1} count as distinct elements.

Notation 4.13. As a matter of notation, we leave out the curly braces when describing the elements of S and denote words by juxtaposition. So if we had, for example, $S = \{a, b\}$ and two words $W_1 = \{aba^{-1}b^{-1}\}$ and $W_2 = \{aa\}$, then $\mathcal{P} = \langle a, b \mid aba^{-1}b^{-1}, aa \rangle$.

As with simplicial complexes, any polygonal presentation determines a topological space $|\mathcal{P}|$ called the geometric realization.

Definition 4.14. The *geometric realization* of a polygonal presentation, denoted $|\mathcal{P}|$ is determined by the following algorithm:

- (1) For each word W_i , let P_i denote the convex k -sided polygonal region in the plane that has its center at the origin, side length 1, equal angles, and one vertex on the positive y axis. (k is the length of the word.)
- (2) Define a bijective function between the symbols of W_i and the edges of P_i in counterclockwise order, starting at the vertex of y -axis.

- (3) Let $|\mathcal{P}|$ denote the quotient space of $\coprod_i P_i$ determined by identifying edges that have the same edge symbol by an affine homeomorphism that matches up the initial vertices and terminal vertices of edges with labels a and a , or a^{-1} and a^{-1} , and initial to terminal vertices for edges labeled a and a^{-1} .

Definition 4.15. In the special case where W_i is a word of length 2, we define P_i to be a sphere if the word is aa^{-1} or $a^{-1}a$ and the projective plane if the word is aa or $a^{-1}a^{-1}$.

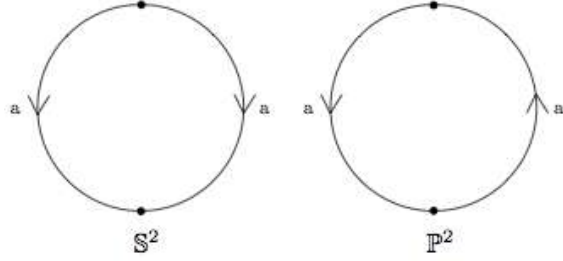


FIGURE 13. Presentation of the two words of length 2.

If we want the geometric realization of a presentation to be a surface, we make the additional stipulation that each symbol $a \in S$ only occurs twice in the presentation \mathcal{P} .

Definition 4.16. A *surface presentation* is a polygonal presentation such that each symbol $a \in S$ occurs only exactly twice in W_1, \dots, W_k , counting each a or a^{-1} as one occurrence.

By theorem 4.2, the geometric realization is a compact surface.

Examples 4.17. The common surfaces \mathbb{S}^2 , \mathbb{T}^2 , K and \mathbb{P}^2 all have presentations:

- (1) The sphere: $\langle a \mid aa^{-1} \rangle$ or $\langle a, b \mid abb^{-1}a^{-1} \rangle$
- (2) The torus: $\langle a, b \mid aba^{-1}b^{-1} \rangle$
- (3) The projective plane: $\langle a \mid aa \rangle$ or $\langle a, b \mid abab \rangle$
- (4) The Klein Bottle: $\langle a, b \mid abab^{-1} \rangle$

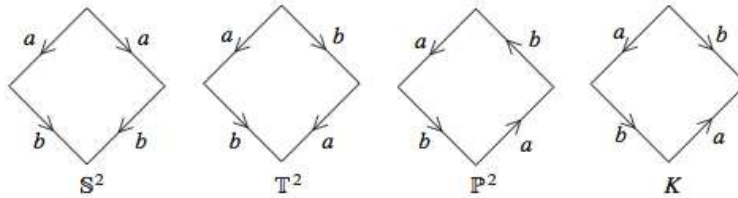


FIGURE 14. Polygonal presentation of \mathbb{S}^2 , \mathbb{T}^2 , \mathbb{P}^2 , and K .

Definition 4.18. If two presentations \mathcal{P}_1 and \mathcal{P}_2 have homeomorphic geometric realizations, we say that they are *topologically equivalent* and write $\mathcal{P}_1 \cong \mathcal{P}_2$.

We are now ready to prove the converse of theorem 4.2.

Theorem 4.19. *Every compact surface admits a polygonal presentation.*

Proof. Let M be a compact surface. By the triangulation theorem, M is homeomorphic to a 2-dimension simplicial complex K , in which each 1-simplex is a face of exactly two 2-simplices.

From this simplicial complex, construct a surface presentation \mathcal{P} such that each 2-simplex is a word of length 3, where edges are labeled with the same letter if they are the same 1-simplex. Thus, we have two quotient maps: $\pi_K : P \rightarrow |K|$ and $\pi_{\mathcal{P}} : P \rightarrow |\mathcal{P}|$, where the domain $P = P_1 \amalg \dots \amalg P_k$. It is sufficient to show that the two quotient maps make the same identifications.

It is clear by construction that the two quotient maps identify the same edges.

Now it remains to show that π_K and $\pi_{\mathcal{P}}$ identify vertices with instructions from the edge identifications. Suppose $v \in K$ is any vertex. v must be in some 1-simplex, otherwise it would be an isolated point. By the triangulation theorem, this edge must be in two 2-simplices σ and σ' . Now we define an equivalence relation on the set of 2-simplices containing v by saying two 2-simplices containing v , σ and σ' , are equivalent if there exists a sequence of 2-simplices $\sigma = \sigma_1, \dots, \sigma_k = \sigma'$ such that σ_i shares an edge with σ_{i+1} for $i = 1, \dots, k - 1$. Thus to prove that the two quotient maps identify the same vertices, it is sufficient to prove that there is only one equivalence class.

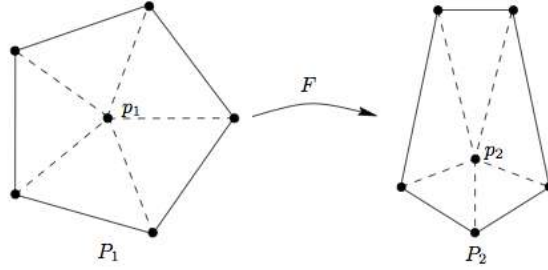
Suppose that there were two equivalence classes $\{\sigma_1, \dots, \sigma_k\}$ and $\{\tau_1, \dots, \tau_m\}$ such that $\sigma_i \sim \sigma_j$ and $\sigma_i \not\sim \tau_j$. Let ϵ be small enough such that $B_\epsilon(v)$ only intersects simplices containing v . $B_\epsilon(v) \cap |K|$ is an open subset of $|K|$, so v has a neighborhood U homeomorphic to \mathbb{R}^2 that is also a subset of $B_\epsilon(v) \cap |K|$. Since this neighborhood is homeomorphic to \mathbb{R}^2 , $U \setminus \{v\}$ is connected. However, if we assume for contradiction that there are two equivalence classes, then $W \cap (\sigma_1 \cup \dots \cup \sigma_k) \setminus \{v\}$ and $W \cap (\tau_1 \cup \dots \cup \tau_m) \setminus \{v\}$ are both open in $|K|$, since their intersection with each simplex is open. Then $W \setminus \{v\} = (W \cap (\sigma_1 \cup \dots \cup \sigma_k) \setminus \{v\}) \cup (W \cap (\tau_1 \cup \dots \cup \tau_m) \setminus \{v\})$ is disconnected, which is a contradiction. \square

The following lemma will provide a simpler method of proving that two polygonal presentations have homeomorphic geometric realizations.

Lemma 4.20. *Let P_1 and P_2 be convex polygons with the same number of edges, and let $f : \partial P_1 \rightarrow \partial P_2$ be a simplicial isomorphism. Then f extends to a homeomorphism $F : P_1 \rightarrow P_2$.*

Proof. Choose any point $p_i \in \text{Int } P_i$, $i = 1, 2$. By convexity, the line segment from p_i to each vertex of P_i lies entirely in P_i . The convex hull spanned by p_i and each pair of adjacent vertices of P_i is a simplex. The disjoint union of these simplices with each inner line segment and their attendant endpoints identified form a simplicial complex whose polyhedron is P_i . Now simply let $F : P_1 \rightarrow P_2$ be the simplicial map whose restriction to ∂P_1 is f and takes p_1 to p_2 . \square

Pictorially, extending the simplicial isomorphism to a homeomorphism F looks like this:



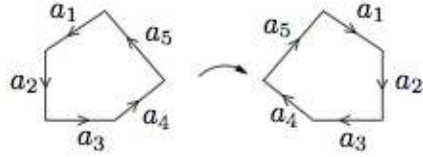
We will now define a series of elementary transformations of polygonal presentations.

Notation 4.21. For the following definitions, we shall adopt the following convention:

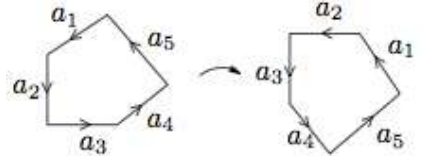
- (1) e denotes any symbol not in S .
- (2) W_1W_2 denotes a word formed by concatenating W_1 and W_2 .
- (3) $(a^{-1})^{-1} = a$

Definition 4.22. The following operations are *elementary transformations* of a polygonal presentation.

- (1) Reflecting: $\langle S \mid a_1 \cdots a_m, W_2, \dots, W_k \rangle \mapsto \langle S \mid a_m^{-1} \cdots a_1^{-1}, W_2, \dots, W_k \rangle$.



- (2) Rotating: $\langle S \mid a_1 \cdots a_m, W_2, \dots, W_k \rangle \mapsto \langle S \mid a_2 \cdots a_m a_1, W_2, \dots, W_k \rangle$.



- (3) Cutting: If W_1 and W_2 both have length at least 2, $\langle S \mid W_1W_2, \dots, W_k \rangle \mapsto \langle S, e \mid W_1e, e^{-1}W_2, \dots, W_k \rangle$.
- (4) Pasting: If W_1 and W_2 both have length at least 2, $\langle S, e \mid W_1e, e^{-1}W_2, \dots, W_k \rangle \mapsto \langle S \mid W_1W_2, \dots, W_k \rangle$.

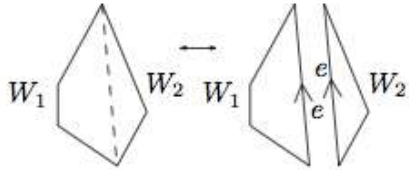


FIGURE 15. Cutting/Pasting.

- (5) Folding: If W_1 has length at least 3, $\langle S, e \mid W_1 e e^{-1}, W_2, \dots, W_k \rangle \mapsto \langle S \mid W_1, W_2, \dots, W_k \rangle$. W_1 can have length 2 if the presentation only has one word.
- (6) Unfolding: $\langle S \mid W_1, W_2, \dots, W_k \rangle \mapsto \langle S, e \mid W_1 e e^{-1}, W - 2, \dots, W_k \rangle$.

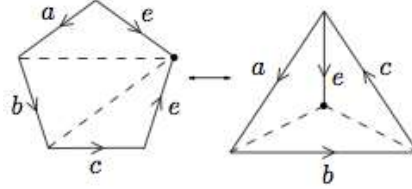


FIGURE 16. Folding/Unfolding.

Theorem 4.23. *Elementary transformations of a polygonal presentation produce a topologically equivalent presentation.*

Proof. Notice that cutting/pasting and folding/unfolding are symmetric, so we only need to prove that one of the pair presents homeomorphic geometric realizations.

- (1) Reflecting: Let P_1 be the geometric realization of a_1, \dots, a_m and P'_1 be the geometric realization of $a_m^{-1}, \dots, a_1^{-1}$. Since reflection is a linear transformation, we choose the reflection matrix to be our homeomorphism; clearly, it is bijective and bicontinuous. We can extend the homeomorphism to W_2, \dots, W_k by the identity map.
- (2) Rotation: Let P_1 be the geometric realization of a_1, \dots, a_m and P'_1 be the geometric realization of a_2, \dots, a_m, a_1 . Similar to reflecting, we choose the rotation matrix to be our homeomorphism. The reflection linear transformation is clearly bijective and bicontinuous. We can similarly extend to homeomorphism to W_2, \dots, W_k by the identity map.
- (3) Cutting: Let P_1 and P_2 be polygons labeled $W_1 e$ and $e^{-1} W_2$ respectively, and let P' be the polygon labeled $W_1 W_2$. Let $\pi : P_1 \amalg P_2 \rightarrow S$ and $\pi' : P' \rightarrow S'$ be the two quotient maps. Let e be the line segment from the terminal to initial vertex of W_1 in P' ; by convexity, the edge is in P' . The continuous map $f : P_1 \amalg P_2 \rightarrow P'$ takes each edge of P_1 or P_2 to its corresponding edge in P' , and identifies e and e^{-1} . Thus, by the closed map lemma, f is a quotient map. So $\pi' \circ f$ and π make the same identifications from the same domain, so by the uniqueness of the quotient map, S and S' are homeomorphic. If there are other words, W_3, \dots, W_k in the polygonal presentation, then extend the homeomorphism by the identity.
- (4) Folding: Assume without loss of generality that the W_1 has at least length 3. (If it has a shorter length, simply introduce a new face, divide an existing face into two parts, labeled with different letters.) First assume that $W_1 = abc$ and let P and P' be polygons of $abcee^{-1}$ and abc respectively. Also, let $\pi : P \rightarrow S$ and $\pi' : P' \rightarrow S'$ be the two quotient maps. Transform P into a simplicial complex by adding edges. The resultant words to represent the simplicial complex are of the form: $e^{-1} ad, d^{-1} bf, f^{-1} ce$, where sides of the same letter are identified and the vertex identification is forced by the edge identification. Let $f : P \rightarrow P'$ be the simplicial map that takes edges in P

to edges with the same label in P' . Then $\pi \circ f$ and π are quotient maps that make the same identifications, so by the uniqueness of the quotient map, S and S' are homeomorphic. We can extend the homeomorphism to the other words W_2, \dots, W_k by the identity. \square

The connect sum of two surfaces can also be expressed as an operation on the polygonal presentation.

Theorem 4.24. *Let M_1 and M_2 be surfaces that admit presentations $\langle S_1 \mid W_1 \rangle$ and $\langle S_2 \mid W_2 \rangle$, in which S_1 and S_2 are disjoint sets and presentation has a single face. Then $\langle S_1, S_2 \mid W_1 W_2 \rangle$ is a presentation of the connect sum of $M_1 \# M_2$.*

Proof. Given the presentation of M_1 as $\langle S_1 \mid W_1 \rangle$, we get $\langle S_1 \mid W_1 \rangle \cong \langle S_1, a, b, c \mid W_1 c^{-1} b^{-1} a^{-1}, abc \rangle$ by cutting 3 times. The word abc represents a polygon and its convex hull is a 2-simplex, which is homeomorphic to \overline{B} . Let B_1 be the interior of the convex hull of the polygon corresponding to the word abc . Thus, the geometric realization of $\langle S_1, a, b, c \mid W_1 c^{-1} b^{-1} a^{-1} \rangle$ is homeomorphic to $M_1 \setminus B_1 := M'_1$. By a similar argument, we get the presentation of M'_2 is $\langle S_2, a, b, c \mid abc W_2 \rangle$. So the presentation $\langle S_1, S_2, a, b, c \mid W_1 c^{-1} b^{-1} a^{-1}, abc W_2 \rangle$, which shows that a, b, c are identified in a complementary manner, is the presentation of $M'_1 \amalg M'_2$ where the ball represented by abc is identified, which is exactly $M'_1 \# M'_2$. Pasting along c and folding a and b gives a homeomorphic presentation $\langle S_1, S_2 \mid W_1 W_2 \rangle$. \square

5. THE CLASSIFICATION THEOREM

We are now ready to prove the main result of this paper. This theorem was first proved in 1907 by Max Dehn and Poul Heegaard.

Theorem 5.1. *Every non-empty, compact, connected 2-manifold is homeomorphic to one of the following:*

- (1) \mathbb{S}^2
- (2) A connect sum of one or more copies \mathbb{T}^2
- (3) A connect sum of one or more copies of \mathbb{P}^2 .

It might appear that some of the surfaces are absent from the list. In particular, the Klein bottle K and any connect sum involving both tori and projective planes, for example $\mathbb{T}^2 \# \mathbb{P}^2$.

Lemma 5.2. *The Klein bottle is homeomorphic to $\mathbb{P}^2 \# \mathbb{P}^2$.*

Proof. The Klein bottle has a presentation: $\langle a, b \mid abab^{-1} \rangle$. By a sequence of elementary transformations, we get

$$\begin{aligned} \langle a, b \mid abab^{-1} \rangle &\cong \langle a, b, c \mid abc, c^{-1} ab^{-1} \rangle \text{ (cut along } c) \\ &\cong \langle a, b, c \mid bca, a^{-1} cb \rangle \text{ (rotate and reflect)} \\ &\cong \langle b, c \mid bbcc \rangle \text{ (paste along } a \text{ and rotate).} \end{aligned}$$

The final presentation is the connect sum of two projective planes. \square

Lemma 5.3. *The connect sum of $\mathbb{T}^2 \# \mathbb{P}^2$ is homeomorphic to $\mathbb{P}^2 \# \mathbb{P}^2 \# \mathbb{P}^2$.*

Proof. By the previous corollary, $\mathbb{P}^2 \# \mathbb{P}^2 \# \mathbb{P}^2 \cong K \# \mathbb{P}^2 = \langle a, b, c \mid abab^{-1}cc \rangle$. By a sequence of elementary transformations,

$$\begin{aligned} \langle a, b, c \mid abab^{-1}cc \rangle &\cong \langle a, b, c, d \mid cabd^{-1}, dab^{-1}c \rangle \text{ (rotate and cut)} \\ &\cong \langle a, b, c, d \mid abd^{-1}c, c^{-1}ba^{-1}d^{-1} \rangle \text{ (rotate and reflect)} \\ &\cong \langle a, b, d, e \mid a^{-1}d^{-1}abe, e^{-1}d^{-1}b \rangle \text{ (paste along } c \text{ and cut along } e) \\ &\cong \langle a, b, d, e \mid ea^{-1}d^{-1}ab, b^{-1}de \rangle \text{ (rotate and reflect)} \\ &\cong \langle a, c, e \mid a^{-1}d^{-1}adee \rangle \text{ (paste along } b, \text{ rotate, and reflect)} \end{aligned}$$

The final presentation is the connect sum of a torus and projective plane. \square

Before we begin, we shall give two preliminary definitions that will make exposition simpler:

Definition 5.4. A pair of edges that are to be identified is *twisted* if they both appear as a, \dots, a or a^{-1}, \dots, a^{-1} .

Definition 5.5. A pair of edges that are to be identified is *complementary* if it appears as a, \dots, a^{-1} or a^{-1}, \dots, a .

Given the two preceding lemmas, we are now ready to prove the classification theorem for compact 2-manifolds.

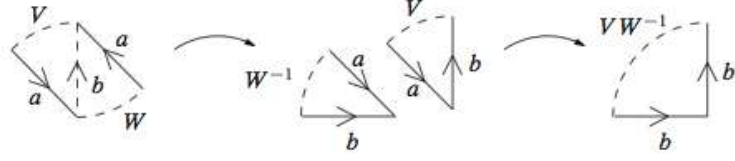
Proof of the Classification Theorem. Given any compact surface, this proof will show that by a sequence of elementary transformations, we get a surface that has a polygonal presentation homeomorphic to the sphere, the connect sum of tori, or the connect sum of projective planes.

Step 1 M admits a presentation that has only one face (only one word). Since M is connected, each word must have a letter in common with another word, so by repeated pasting transformations (with rotations and reflections as necessary), we get a polygonal presentation with only one word, which admits a presentation with one face.

Step 2 M is either a sphere or admits a presentation with no adjacent complementary pairs. If there is an adjacent complementary pair, we may remove it by folding. The only time, when an adjacent complementary pair cannot be removed is if it is the only pair of letters left. i.e., $\langle a, aa^{-1} \rangle$, in which case, we have a sphere. Now we assume that the surface is not a sphere.

Step 3 M admits a presentation in which all twisted pairs are adjacent. Suppose we have a non-adjacent twisted pair. Then the word will take the form $VaWa$, where V and W are non-empty words. By a sequence of elementary transformations:

$$\begin{aligned} \langle a, V, W \mid VaWa \rangle &\cong \langle a, b, V, W \mid Vab, b^{-1}Wa \rangle \\ &\cong \langle a, b, V, W \mid bVa, a^{-1}W^{-1}b \rangle \\ &\cong \langle b, V, W \mid VW^{-1}bb \rangle. \end{aligned}$$

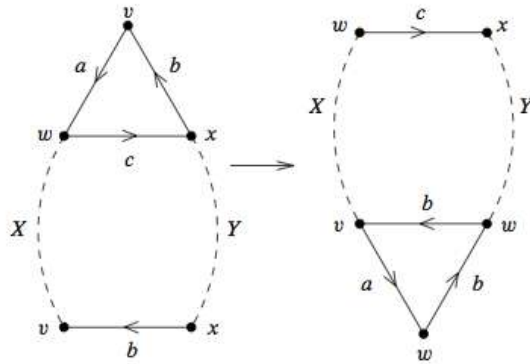


We may have introduced new non-adjacent twisted pairs in the process. However, recall that the set of symbols S is finite, so by repeating the same process, we can transform each non-adjacent twisted pair into adjacent complementary pairs without affecting the bb complementary pair. So after a finite number of iterations, we get a word with no non-adjacent twisted pairs and a string of adjacent complementary pairs. The complementary pairs can be removed by repeating step 2, which does not increase the total number of non-adjacent twisted pairs.

Step 4 M admits a presentation in which all vertices are identified to a single point. Recall that we have an equivalence relation on the set of edges. The identification of the edges, as we have seen before, forces an equivalence relation on the set of vertices; choose some equivalence class of vertices $[v]$. Suppose that there are vertices not in the equivalence class $[v]$. Then there must be some edge a that connects $[v]$ to some other vertex class $[w]$. Since this is a polygonal surface, the other edge that touches a at $[v]$ cannot be a^{-1} , or else we would have got rid of it in step 2. The other edge cannot be a , because, if it were, then the initial and terminal ends would be identified under the quotient map, which is not the case. So we label this other edge b and the other vertex x .

Somewhere else in the polygon, there is another edge labeled either b or b^{-1} . Without loss of generality, assume that it is b^{-1} . The proof if it is b is similar except for an extra reflection. Thus the presentation is of the form $baXb^{-1}Y$. By elementary transformations:

$$\begin{aligned} \langle a, b, X, Y \mid baXb^{-1}Y \rangle &\cong \langle a, b, c, X, Y \mid bac, c^{-1}Xb^{-1}Y \rangle \\ &\cong \langle a, b, c, X, Y \mid acb, b^{-1}Yc^{-1}X \rangle \\ &\cong \langle a, c, X, Y \mid acYc^{-1}X \rangle. \end{aligned}$$

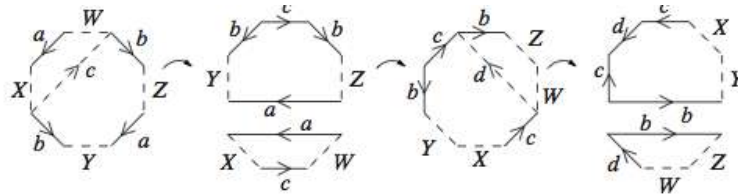


Recall that the $[v]$ referred to the initial vertex of a and the terminal vertex of b , so by pasting the edges labeled b , we have reduced the number of distinct vertices in the polygon labeled v . We may have increased the number of vertices labeled w and we may have introduced new complementary pairs. To repair the latter, perform step 2 again noticing that step 2 does not increase the number of vertices labeled v . Thus, by repeating this process finitely many times, we can eliminate the vertex class $[v]$. Repeating this procedure for each vertex class, we can get the desired result.

Step 5 *If the presentation has any complementary pairs a, a^{-1} , then it has another complementary pair b, b^{-1} that occurs intertwined with the first. i.e., $a, \dots, b, \dots, a^{-1}, \dots, b^{-1}$. Assume that this is not the case, that is, the presentation is of the form $aXa^{-1}Y$, where X and Y only contain matched complementary pairs or adjacent twisted pairs. (By *matched*, we mean that the complementary pairs remain exclusively within X or Y .) Recall that non-adjacent twisted pairs and adjacent complementary pairs are not possible by step 2 and 3. Thus each edge in X is identified with another edge in Y and similarly for Y . This means the terminal vertices of a and a^{-1} both touch vertices in X and the initial vertices are identified with only vertices in Y . This is a contradiction, since all vertices are within one equivalence class by Step 4.*

Step 6 *M admits a presentation in which all intertwined complementary pairs occur together with no other edges in between: $aba^{-1}b^{-1}$. The presentation is given $WaXbYa^{-1}Zb^{-1}$. By elementary transformations:*

$$\begin{aligned}
 & \langle a, b, W, X, Y, Z \mid WaXbYa^{-1}Zb^{-1} \rangle \\
 \cong & \langle a, b, W, X, Y, Z \mid WaXc, c^{-1}bYa^{-1}Zb^{-1} \rangle \\
 \cong & \langle a, b, c, W, X, Y, Z \mid XcWa, a^{-1}Zb^{-1}c^{-1}bY \rangle \\
 \cong & \langle b, c, W, X, Y, Z \mid XcWZb^{-1}c^{-1}bY \rangle \\
 \cong & \langle b, c, W, X, Y, Z \mid c^{-1}bYXcWZb^{-1} \rangle \\
 \cong & \langle b, c, d, W, X, Y, Z \mid c^{-1}bYXcd, d^{-1}WZb^{-1} \rangle \\
 \cong & \langle b, c, d, W, X, Y, Z \mid YXcdc^{-1}b, b^{-1}d^{-1}WZ \rangle \\
 \cong & \langle c, d, W, X, Y, Z \mid YXcdc^{-1}d^{-1}WZ \rangle \\
 \cong & \langle c, d, W, X, Y, Z \mid cdc^{-1}d^{-1}WZYZ \rangle.
 \end{aligned}$$



Notice that this step required no reflection. Repeating this process for each set of intertwined pairs, we get the desired result.

Step 7 *M is homeomorphic to either a connect sum of one or more tori or a connect sum of one or more projective planes. By Steps 1–6, all twisted pairs occur*

adjacent to each other: aa (projective planes) and all complementary pairs occur in intertwined groups $bc^{-1}c^{-1}$ (tori). If the presentation consists exclusively of either case, then we are done, since we would either have the connect sum of tori or connect sum of projective planes. If the presentation contains both twisted and complementary pairs, then the presentation must be one of the following forms: $abc^{-1}c^{-1}X$ or $bc^{-1}c^{-1}aaX$. In either case, by the previous lemma, $\mathbb{T}^2\#\mathbb{P}^2 \cong \mathbb{P}^2\#\mathbb{P}^2\#\mathbb{P}^2$. So, if both cases occur in the presentation, we can eliminate all occurrences of \mathbb{T}^2 by this transformation, and we get the connect sum of \mathbb{P}^2 .

□

6. CONCLUDING REMARKS

In this paper, we proved that all compact surfaces are homeomorphic to the sphere, the connect sum of tori, or the connect sum of projective planes, but the keen reader may have noticed that we have yet to prove that the surfaces are topologically distinct. e.g., a sphere is not homeomorphic to a torus. The answer to this non-trivial question lies with other topological invariants such as the Euler Characteristic and orientability. The interested reader should refer to Lee [2].

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REFERENCES

- [1] B. Mohar and C. Thomassen. Graphs on Surfaces. John Hopkins University Press. 2001.
- [2] J. M. Lee. Introduction to Topological Manifolds. Springer-Verlag. 2000.