EXPLICIT QUILLEN MODELS FOR CARTESIAN PRODUCTS

Rational Homotopy Theory

Explicit Quillen model of products

Sectional category *à la* Quillen

Rational Homotopy Theory

If $f : X \longrightarrow Y$ is a continuous map between simply connected CW-complexes, the following properties are equivalent:

1. $\pi_n(f)\otimes\mathbb{Q} : \pi_n(X)\otimes\mathbb{Q} \xrightarrow{\cong} \pi_n(Y)\otimes\mathbb{Q}$, $n \geq 2.$

2. $H_n(f)\otimes\mathbb{Q} : H_n(X\,;\mathbb{Q}) \xrightarrow{\cong} H_n(Y\,;\mathbb{Q})$, $n \geq 2.$

Such a map is called a <u>rational homotopy equivalence</u>.

Rational Homotopy Theory

Definition

X is <u>rational</u> if its homotopy groups are $\mathbb{Q}$-vector spaces.

A <u>rationalisation</u> of X is a pair $(X_{\mathbb{Q}},\ e)$, with $X_{\mathbb{Q}}$ a rational space and and $e : X \longrightarrow X_{\mathbb{Q}}$ a rational homotopy equivalence.

The study of the rational homotopy type of X is the study of the homotopy type of its rationalisation $X_{\mathbb{Q}}$.

Rational Homotopy Theory

Then, if $\boldsymbol{X}$ is a finite simply connected CW-complex

$$\pi_n(\boldsymbol{X}) = \bigoplus_r \mathbb{Z} \oplus \mathbb{Z}_{{p_1}^{r_1}} \oplus \cdots \oplus \mathbb{Z}_{{p_m}^{r_m}}$$

$$\pi_n(\boldsymbol{X}_{\mathbb{Q}}) \cong \pi_n(\boldsymbol{X}) \otimes \mathbb{Q} = \bigoplus_r \mathbb{Q}$$

The study of the rational homotopy type of $\boldsymbol{X}$ is the study of the homotopy type of its rationalisation $\boldsymbol{X}_{\mathbb{Q}}$.

Algebraic models

The rational homotopy type of X is completely determined in algebraic terms.

DGL	CDGA

Algebraic models

Definition

A <u>differential graded Lie algebra</u> is a graded vector space $L = \oplus_{p \in \mathbb{Z}} L_p$ with:

A bilinear operation $[\,.\,,.\,] : L \times L \longrightarrow L$ such that $[L_p, L_q] \subset L_{p+q}$ satisfying:

a) $[a,b] = -(-1)^{pq}[b,a]$, $a \in L_p$, $b \in L_q$

b) $[a,[b,c]] = [[a,b],c] + (-1)^{pq}[b,[a,c]]$,

$a \in L_p$, $b \in L_q$, $c \in L$

DGL

Algebraic models

Definition

A <u>differential graded Lie algebra</u> is a graded vector space $L = \oplus_{p \epsilon \mathbb{Z}} L_p$ with:

A linear map $\partial : L \longrightarrow L$ such that $\partial L_p \subset L_{p-1}$ satisfying:

a) $\partial \circ \partial = 0$

b) $\partial[a, b] = [\partial a, b] + (-1)^p [a, \partial b]$,

$a \in L_p, b \in L$

DGL

Algebraic models

$$\left\{ \begin{array}{c} \textbf{Simply connected} \\ \textbf{CW}-\textbf{complexes} \end{array} \right\} \underset{<\cdot>_Q}{\overset{\lambda}{\rightleftarrows}} \mathbf{DGL}_+$$

Rational homotopy equivalences

Quasi-isomorphisms

$$<\lambda(\boldsymbol{X})>_Q \simeq \boldsymbol{X}_{\mathbb{Q}}$$

$(L, [\cdot,\cdot], \partial)$ is a DGL-**model** of $\boldsymbol{X}$ if

$$\lambda(\boldsymbol{X}) \xrightarrow{\simeq} \bullet \xleftarrow{\simeq} \cdots \xrightarrow{\simeq} \bullet \xleftarrow{\simeq} L$$

DGL

Algebraic models

$$\left\{ \begin{array}{c} \textbf{Simply connected} \\ \textbf{CW}-\textbf{complexes} \end{array} \right\} \underset{<\cdot>_Q}{\overset{\lambda}{\rightleftarrows}} \mathbf{DGL}_+$$

Rational homotopy equivalences

Quasi-isomorphisms

$$<\lambda(\boldsymbol{X})>_Q \simeq \boldsymbol{X}_{\mathbb{Q}}$$

$(L,[\cdot,\cdot],\partial)$ is a DGL-**model** of $\boldsymbol{X}$ if

DGL

Minimal Quillen model

$$(\mathbb{L}(W),d) \xrightarrow{\simeq} (L,[\cdot,\cdot],\partial)$$

Algebraic models

Rational homotopy groups

$$H_n(L) \cong \pi_{n+1}(\boldsymbol{X}) \otimes \mathbb{Q}$$

Rational homology groups

$$(\mathbb{L}(W), \partial) \xrightarrow{\simeq} L$$

Minimal Quillen model

$$s(W \oplus \mathbb{Q}) \cong H_*(X;\ \mathbb{Q})$$

DGL

Algebraic models

Spheres

$(\mathbb{L}(v), 0), \qquad |v| = n - 1$

is a model for the n-dimensional sphere S^n.

Products

If (L, ∂) and (L', ∂') are DGL-models for X and Y respectively, then $(L \times L', \partial \times \partial')$ is a DGL-model of $X \times Y$.

Wedge products

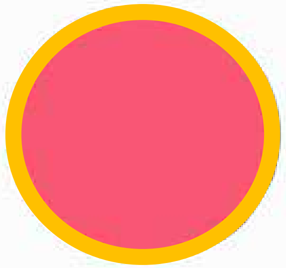
If $(\mathbb{L}(V), d)$ and $(\mathbb{L}(W), d')$ are minimal DGL models for X and Y respectively, then $(\mathbb{L}(V \oplus W), D)$ is a DGL-model of $X \vee Y$.

DGL

Algebraic models

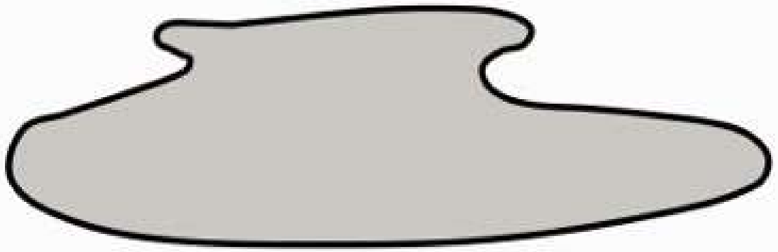
CW-decomposition

A based topological space $(\boldsymbol{X},\ *)$



$= e_\alpha^{n+1}$

$f_\alpha : S^n \to \boldsymbol{X}_n\ ;\quad [f_\alpha] \in \Pi_n(\boldsymbol{X}_n)$



$\boldsymbol{X}_n$ n-skeleton

$(\,\mathbb{L}(V)\,, \partial\,)$

$v_\alpha \in V_n$

$\partial v_\alpha \in H_{n-1}(\mathbb{L}(V_{<n}))$

$\cong$

$\Pi_n(\boldsymbol{X}_n) \otimes \mathbb{Q}$

$(\,\mathbb{L}(V_{<n}), \partial\,)$

Explicit Quillen models of products

Let (L, ∂) and (L', ∂') be DGL models of X and Y.

Then $(L \times L', \partial \times \partial')$ is DGL model of $X \times Y$.

Let $(\mathbb{L}(V), \partial_V) \xrightarrow{\simeq} (L, \partial)$ and $(\mathbb{L}(W), \partial_W) \xrightarrow{\simeq} (L', \partial')$ be Quillen minimal models of X and Y.

$$\Phi\colon \big(\mathbb{L}(V \oplus W \oplus s(V \otimes W)),\ D\,\big) \xrightarrow{\simeq} (\mathbb{L}(V) \times \mathbb{L}(W), \partial_V \times \partial_W)$$

$$\Phi(v) = v\ ;\quad \Phi(w) = w\ ;\quad \Phi(s(v \otimes w)) = 0\ ;$$

$$D(v) = \partial_V v\ ;\quad D(w) = \partial_W w\ ;$$

$$D(s(v \otimes w)) = [v, w] + \beta(v, w)$$

Daniel Tanré,
Homotopie Rationelle: Modeles de Chen, Quillen, Sullivan
1983

Explicit Quillen models of products

Example

Recall that $(\mathbb{L}(v), 0)$ and $(\mathbb{L}(w), 0)$ with $|v| = n-1$; $|w| = m-1$ are Quillen minimal models of S^n and S^m.

A Quillen minimal models of $S^n \times S^m$ is given by:

$$(\mathbb{L}(v \oplus w \oplus s(v \otimes w)),\ D\ )$$

$$D(v) = 0\ ;\ D(w) = 0\ ;\ \ D(s(v \otimes w)) = [\, v\, , w\,]$$

Explicit Quillen models of products

A Quillen minimal model of $S^n \times \mathbb{C}P^2$ is given by:

$$(\mathbb{L}(v, x, y, s(v\otimes x), s(v\otimes y),)\ D)$$

$$D(v) = 0\,;\ D(x) = 0\,;\quad Ds(v\otimes x) = [v, x]$$

$$D(y) = [x, x]\,;\qquad Ds(v\otimes y) = [v, y] + 2\,[x, s(v\otimes x)]$$

$\beta(v, w) \in Ker\ \Phi$

Explicit Quillen models of products

Example

Let $(\mathbb{L}(V), 0)$ be a model of a co-H-space X and $(\mathbb{L}(W), \partial)$ a model of any space Y.

A Quillen minimal model of $X \times Y$ is given by:

$$(\mathbb{L}(V \oplus W \oplus s(V \otimes W)), D) \qquad D(v) = 0\,; \quad D(w) = \partial w\,;$$

$$D(s(v \otimes w)) = [v, w] - (-1)^{|v|} \sigma_v(\partial w)$$

$$\sigma_v(w') = s(v \otimes w')$$

Derivation

Greg Lupton, Sam Smith
J. Pure and Applied Algebra
2007

Explicit Quillen models of products

$$\big(\mathbb{L}(V \oplus W \oplus s(V \otimes W)),\ D\big) \qquad D(v) = 0\,; \quad D(w) = \partial w\,;$$

$$D\sigma_v(w) = ad_v(w) - (-1)^{|v|}\sigma_v \partial(w)$$

$$\sigma_v(w') = s(v \otimes w')$$

$$D^2 s(v \otimes w) \overset{?}{=} 0$$

$$L = \big(\mathbb{L}(V \oplus W \oplus s(V \otimes W))\big)$$

$$[D, \sigma_v],\ \ ad_v,\ \sigma_v,\ D \in Der(L)$$

Explicit Quillen models of products

$$\big(\mathbb{L}(V \oplus W \oplus s(V \otimes W)),\ D\big) \qquad D(v) = 0\,; \quad D(w) = \partial w\,;$$

$$D\sigma_v(w) = ad_v(w) - (-1)^{|v|}\sigma_v \partial(w)$$

$$\sigma_v(w') = s(v \otimes w')$$

$$[\,D, \sigma_v\,]\,(w) = ad_v(w) \qquad \text{for any } \ w \in W$$

Explicit Quillen models of products

$$\big(\mathbb{L}(V \oplus W \oplus s(V \otimes W)),\ D\,\big) \qquad D(v) = 0\,; \quad D(w) = \partial w\,;$$

$$D\sigma_v(w) = ad_v(w) - (-1)^{|v|}\sigma_v\partial(w)$$

$$\sigma_v(w') = s(v \otimes w')$$

$$[\,D, \sigma_v\,]\,(w) = ad_v(w) \qquad \text{for any} \quad w \in W$$

$$[\,D, \sigma_v\,]\,(w) = D\sigma_v(w) - (-1)^{|v|+1}\sigma_v D(w)$$

Explicit Quillen models of products

$$\left(\mathbb{L}(V \oplus W \oplus s(V \otimes W)),\ D\right) \qquad D(v) = 0\,; \quad D(w) = \partial w\,;$$

$$D\sigma_v(w) = ad_v(w) - (-1)^{|v|}\sigma_v\partial(w)$$

$$\sigma_v(w') = s(v \otimes w')$$

$$[\,D, \sigma_v\,]\,(w) = ad_v(w) \qquad \text{for any} \quad w \in W$$

$$[\,D, \sigma_v\,]\,(w) = ad_v(w) - (-1)^{|v|}\sigma_v\partial(w) - (-1)^{|v|+1}\sigma_v D(w)$$

Explicit Quillen models of products

$$(\mathbb{L}(V \oplus W \oplus s(V \otimes W)),\ D) \qquad D(v) = 0\,; \quad D(w) = \partial w\,;$$

$$D\sigma_v(w) = ad_v(w) - (-1)^{|v|}\sigma_v \partial(w)$$

$$\sigma_v(w') = s(v \otimes w')$$

$[D, \sigma_v](w) = ad_v(w)$ for any $w \in W$

$[D, \sigma_v](A) = ad_v(A)$ for any $A \in \mathbb{L}(W) \subseteq L$

Explicit Quillen models of products

$$\left(\mathbb{L}(V \oplus W \oplus s(V \otimes W)),\ D\right) \qquad D(v) = 0\,; \quad D(w) = \partial w\,;$$

$$D\sigma_v(w) = ad_v(w) - (-1)^{|v|}\sigma_v \partial(w)$$

$$\sigma_v(w') = s(v \otimes w')$$

$$[\,D, \sigma_v\,]\,(A) = ad_v(A) \quad \text{for any} \quad A \in \mathbb{L}(W) \subseteq L$$

$$D^2 s(v \otimes w)$$

Explicit Quillen models of products

$$\big(\mathbb{L}(V \oplus W \oplus s(V \otimes W)),\ D\big) \qquad D(v) = 0\,; \quad D(w) = \partial w\,;$$

$$D\sigma_v(w) = ad_v(w) - (-1)^{|v|}\sigma_v \partial(w)$$

$$\sigma_v(w') = s(v \otimes w')$$

$$[\,D, \sigma_v\,]\,(A) = ad_v(A) \qquad \text{for any} \quad A \in \mathbb{L}(W) \subseteq L$$

$$D\left(ad_v(w) - (-1)^{|v|}\sigma_v \partial(w)\right)$$

Explicit Quillen models of products

$$\left(\mathbb{L}(V \oplus W \oplus s(V \otimes W)),\ D\right) \qquad D(v) = 0\,; \quad D(w) = \partial w\,;$$

$$D\sigma_v(w) = ad_v(w) - (-1)^{|v|}\sigma_v \partial(w)$$

$$\sigma_v(w') = s(v \otimes w')$$

$$[\,D, \sigma_v\,]\,(A) = ad_v(A) \quad \text{for any} \quad A \in \mathbb{L}(W) \subseteq L$$

$$(-1)^{|v|} ad_v(\partial w\,) - (-1)^{|v|}\ D\,\sigma_v \partial(w)$$

Explicit Quillen models of products

$$\left(\mathbb{L}(V \oplus W \oplus s(V \otimes W)),\ D\right) \qquad D(v) = 0\,; \quad D(w) = \partial w\,;$$

$$D\sigma_v(w) = ad_v(w) - (-1)^{|v|}\sigma_v\partial(w)$$

$$\sigma_v(w') = s(v \otimes w')$$

$$[\,D, \sigma_v\,]\,(A) = ad_v(A) \qquad \text{for any} \quad A \in \mathbb{L}(W) \subseteq L$$

$$(-1)^{|v|}\,[\,D, \sigma_v\,](\partial w\,) \; - (-1)^{|v|}\; D\,\sigma_v\partial(w)$$

Explicit Quillen models of products

Let X and Y be 2-cones (or 2 stage spaces).

Then $(\mathbb{L}(V_0 \oplus V_1), \partial_V)$ and $(\mathbb{L}(W_0 \oplus W_1), \partial_W)$ are Quillen minimal models of X and Y respectively, where

$$\partial_V(V_0) = 0 \quad ; \quad \partial_V(V_1) \subseteq \mathbb{L}(V_0)$$

$$\partial_W(W_0) = 0 \quad ; \quad \partial_W(W_1) \subseteq \mathbb{L}(W_0)$$

Explicit Quillen models of products

$$(\mathbb{L}(V_0 \oplus V_1), \partial_V) \qquad \partial_V(V_0) = 0 \;\; ; \;\; \partial_V(V_1) \subseteq \mathbb{L}(V_0)$$

$$(\mathbb{L}(W_0 \oplus W_1), \partial_W) \qquad \partial_W(W_0) = 0 \;\; ; \;\; \partial_W(W_1) \subseteq \mathbb{L}(W_0)$$

Then a Quillen model of $X \times Y$ is:

$$(\mathbb{L}(V \oplus W \oplus s(V_0 \otimes W_0) \oplus s(V_0 \otimes W_1) \oplus s(V_1 \otimes W_0) \oplus s(V_1 \otimes W_1)), D)$$

$D(v) = \partial_V v$;

$D(w) = \partial_W w$;

$D(s(v \otimes w)) = [v, w]$

$s(v \otimes w) \in s(V_0 \otimes W_0)$

$D(s(v \otimes w)) = [v, w] - (-1)^{|v|} \sigma_v(\partial w)$

$s(v \otimes w) \in s(V_0 \otimes W_1)$

$D(s(v \otimes w)) = [v, w] - (-1)^{(|v|+1)|w|} \sigma_w(\partial v)$

$s(v \otimes w) \in s(V_0 \otimes W_1)$

Explicit Quillen models of products

$$(\mathbb{L}(V_0 \oplus V_1), \partial_V) \qquad \partial_V(V_0) = 0 \;\; ; \;\; \partial_V(V_1) \subseteq \mathbb{L}(V_0)$$

$$(\mathbb{L}(W_0 \oplus W_1), \partial_W) \qquad \partial_W(W_0) = 0 \;\; ; \;\; \partial_W(W_1) \subseteq \mathbb{L}(W_0)$$

Then a Quillen model of $X \times Y$ is:

$$(\mathbb{L}(V \oplus W \oplus s(V_0 \otimes W_0) \oplus s(V_0 \otimes W_1) \oplus s(V_1 \otimes W_0) \oplus s(V_1 \otimes W_1)), D)$$

$$D(s(v \otimes w)) = [v, w] - (-1)^{|v|}\sigma_v(\partial w) - (-1)^{(|v|+1)|w|}\sigma_w(\partial v)$$

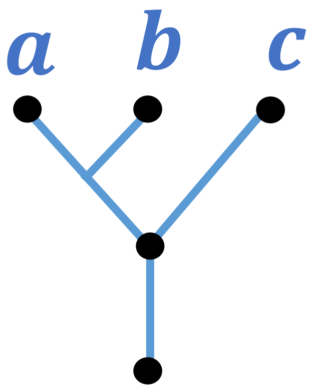
$$+(-1)^{|v|}(\partial v) * (\partial w)$$

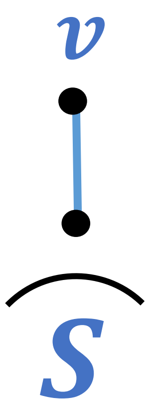
U. Buijs, J.G. Carrasquel, L. Vandembroucq 2024

Explicit Quillen models of products

$$T,\ S \in \left(\mathbb{L}(V \oplus W \oplus s(V \otimes W)),\ D\right) \qquad T * S$$

$T =$





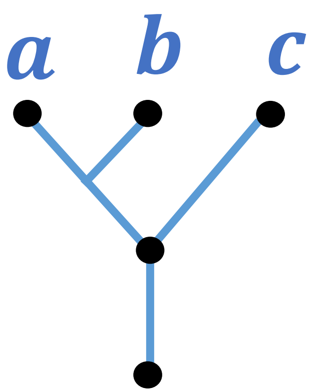
$= \sigma_v(S)$

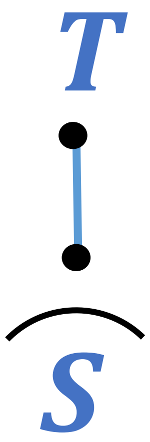
$$\sigma_v(w') = s(v \otimes w')$$

$$T = [[a, b], c]$$

Explicit Quillen models of products

$$T,\ S \in \big(\mathbb{L}(V \oplus W \oplus s(V \otimes W)),\ D\big) \qquad\qquad T * S$$

$T =$ 



$= \sigma_T(S)$

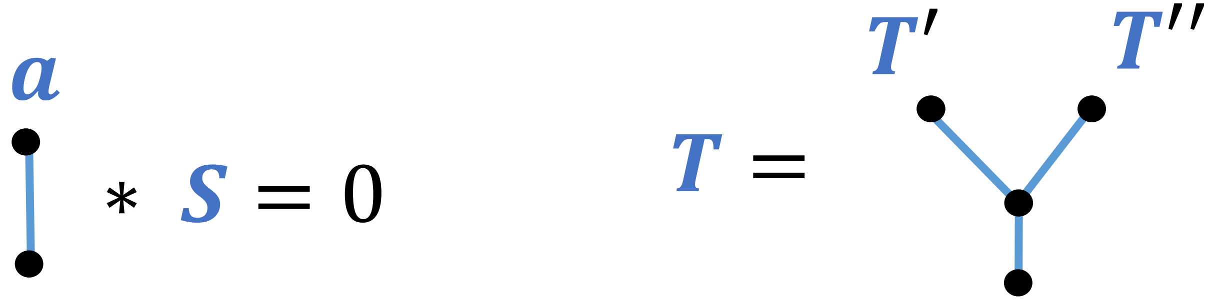
$$T = [[a,b],c]$$

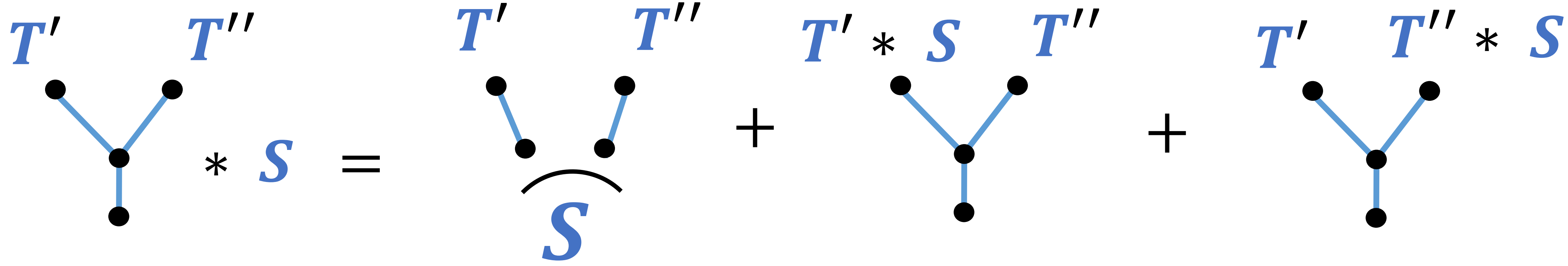
$$\sigma_T(w') = (-1)^{|T||w'|}\sigma_{w'}(T)$$

$$\sigma_{w'}(v) = (-1)^{|v||w'|}s(v \otimes w')$$

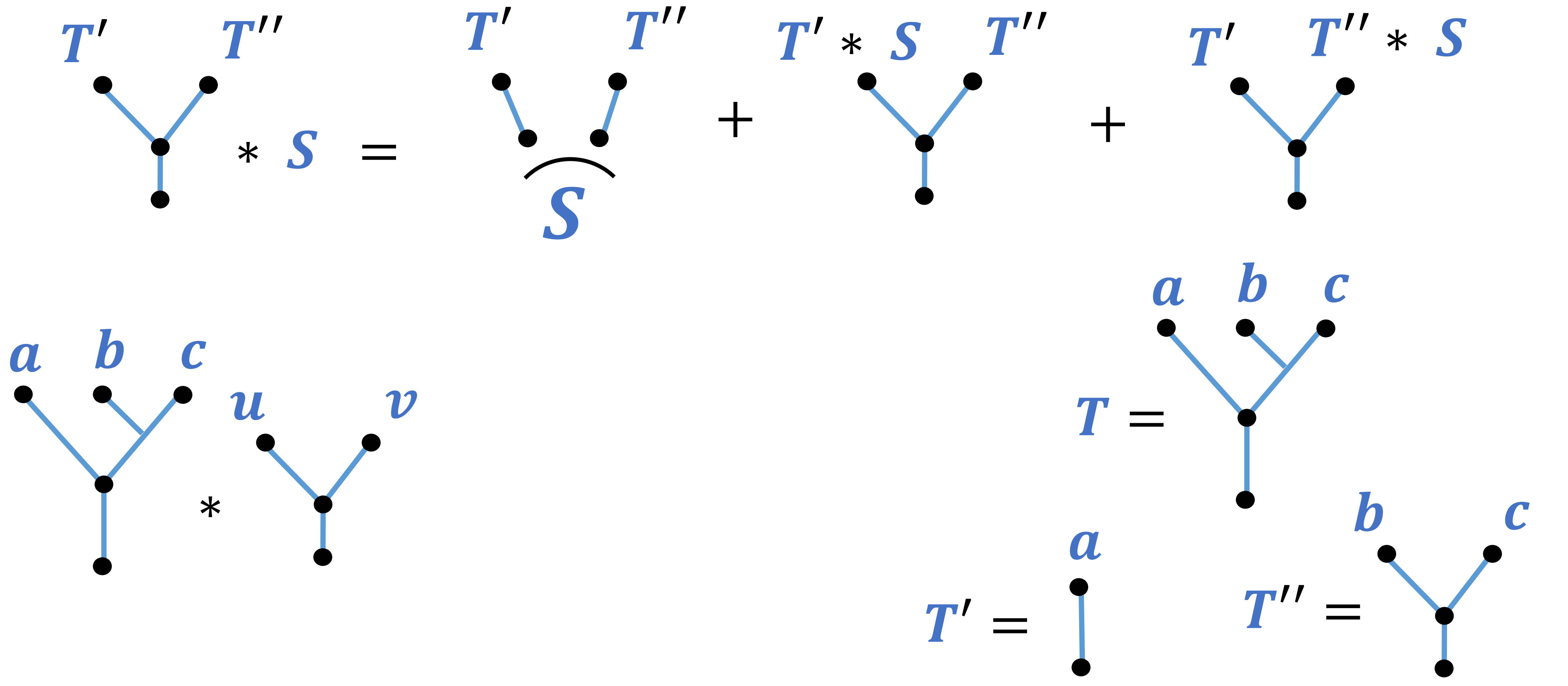
Explicit Quillen models of products

$$T,\ S \in \big(\mathbb{L}(V \oplus W \oplus s(V \otimes W)),\ D\,\big) \qquad T * S$$

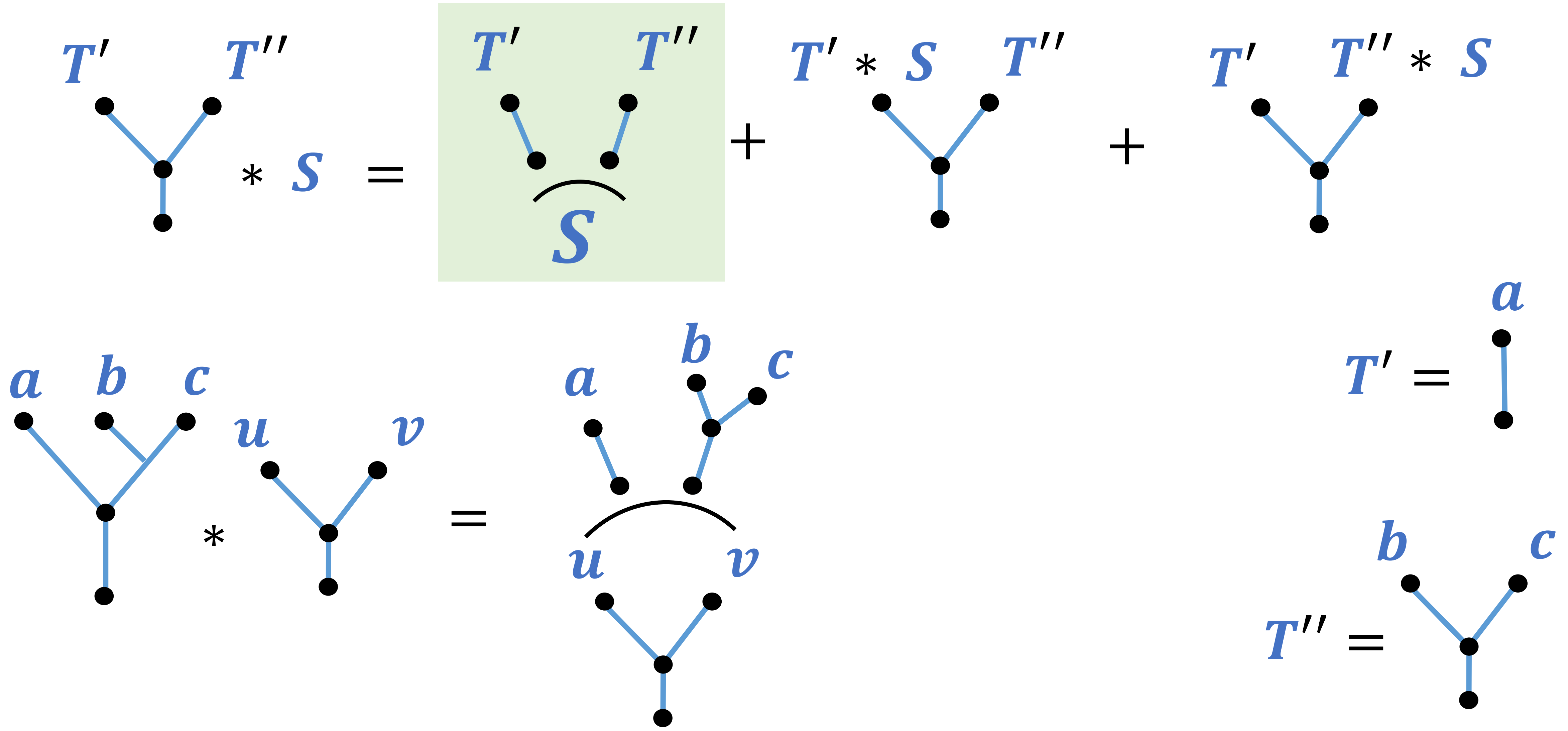




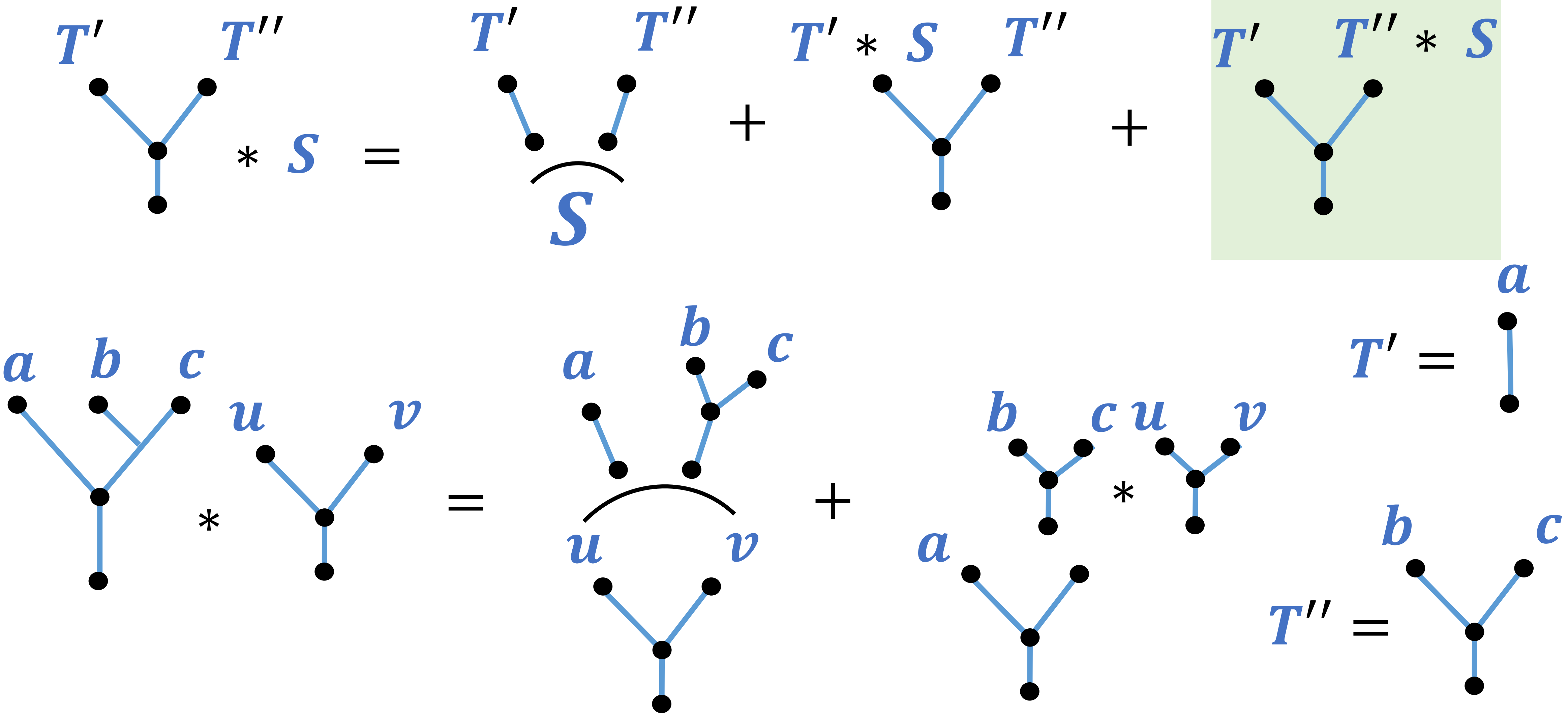
Explicit Quillen models of products



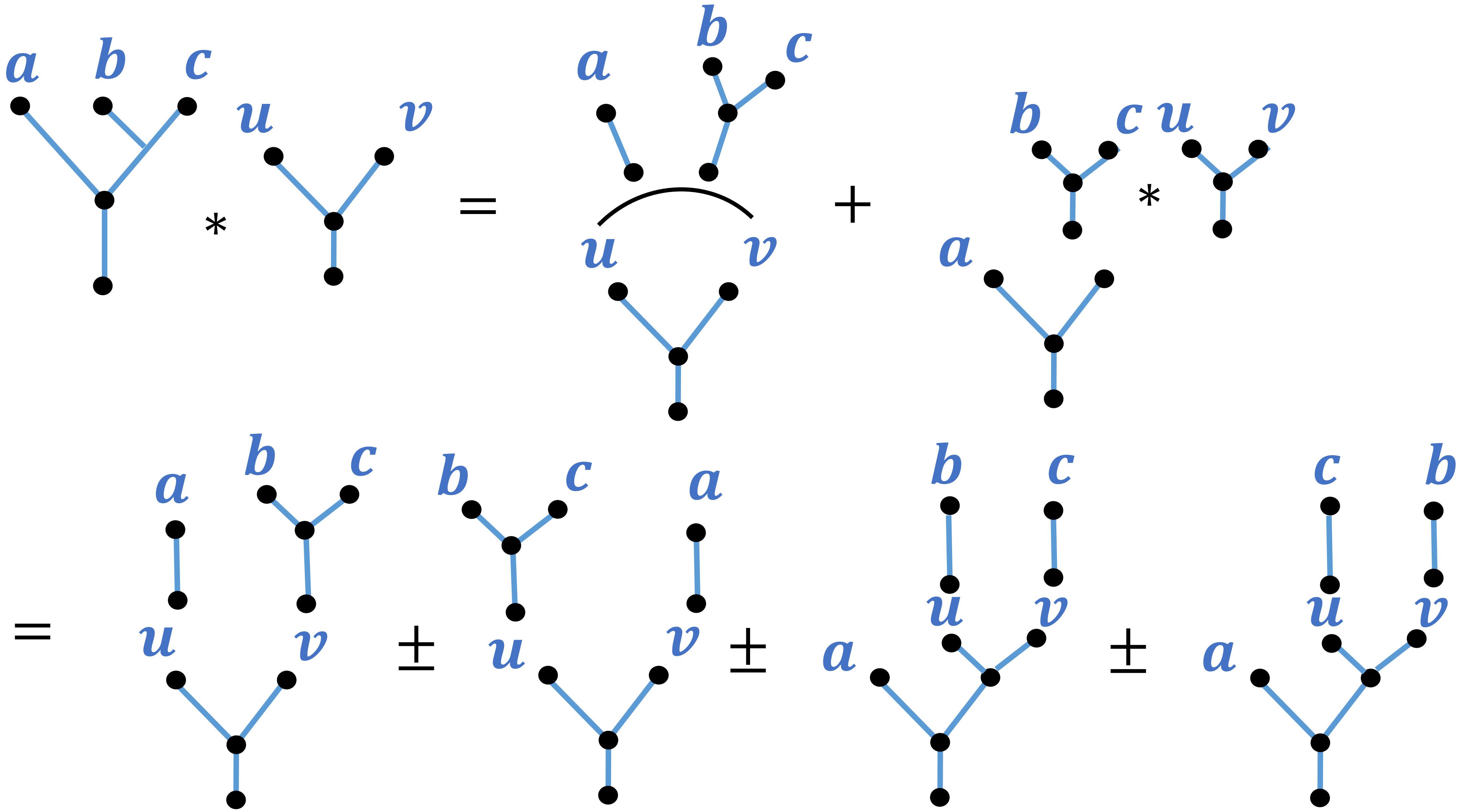
Explicit Quillen models of products



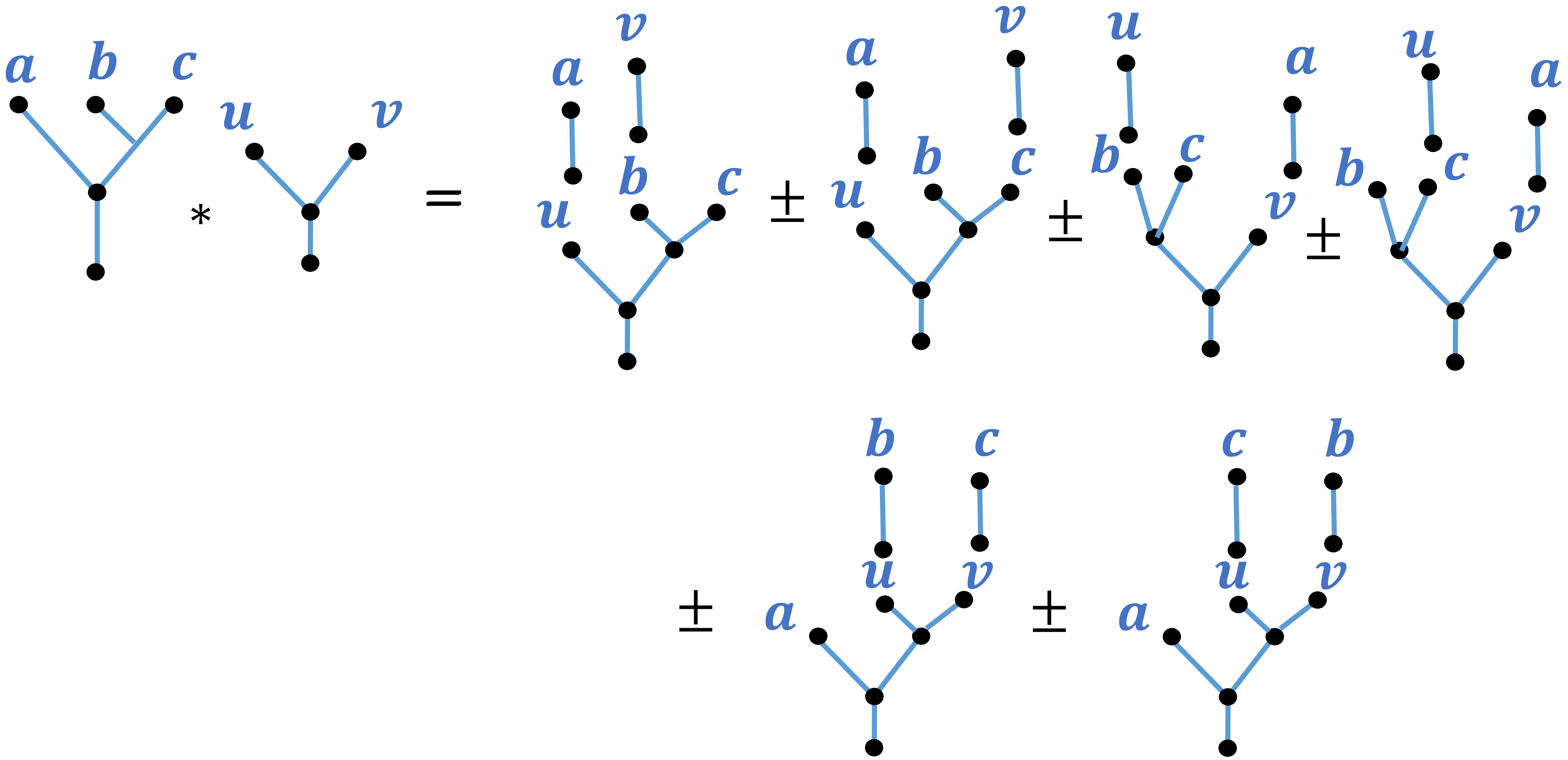
Explicit Quillen models of products



Explicit Quillen models of products



Explicit Quillen models of products



Explicit Quillen models of products

$$[a,[b,c]]*[u,v] = \pm[s(a\otimes u),[s(b\otimes v),c]] \pm[[s(b\otimes u),c],s(a\otimes v)]$$
$$\pm[s(a\otimes u),[b,s(c\otimes v)]] \pm[[b,s(c\otimes u)],s(a\otimes v)]$$
$$\pm[a,[s(b\otimes u),s(c\otimes v)]] \pm[a,[s(c\otimes u),s(b\otimes v)]]$$

Sectional category
à la Quillen

Definition

The ***Lusternik-Schnirelmann category*** of $\boldsymbol{X}$, denoted by cat $\boldsymbol{X}$, is the least integer m (or ∞) such that $\boldsymbol{X}$ is the unión of $m+1$ open subsets U_i each contractible in $\boldsymbol{X}$.

Sectional category
à la Quillen

Definition

Consider a based topological space $(\boldsymbol{X}, *)$

$$\boldsymbol{X}^{m+1} = \underbrace{\boldsymbol{X} \times \cdots \times \boldsymbol{X}}_{m+1 \text{ times}}$$

The ***fat wedge***, $T^m\boldsymbol{X} \subseteq \boldsymbol{X}^{m+1}$, is the subspace given by

$$T^m(\boldsymbol{X}) = \{(\boldsymbol{x_0}, \cdots, \boldsymbol{x_m}) \in \boldsymbol{X}^{m+1} \mid \boldsymbol{x_i} = * \text{ for at least one } \boldsymbol{i}\}$$

Sectional category *à la* Quillen

The ***diagonal map*** $\Delta_{\boldsymbol{X}} : \boldsymbol{X} \to \boldsymbol{X}^{m+1}$ is the continuous map given by

$$\Delta_X(x) = (x, \ldots, x)$$

Definition

The ***Whitehead category*** of $\boldsymbol{X}$, denoted $\mathrm{Whcat}\,\boldsymbol{X}$, is the least integer m (or ∞) such that $\Delta_{\boldsymbol{X}}$ is homotopic to a map

$$\Delta_{\boldsymbol{X}} : \boldsymbol{X} \to T^m(\boldsymbol{X}) \subseteq \boldsymbol{X}^{m+1}.$$

Sectional category *à la* Quillen

Proposition

Consider a path-connected based space $(\boldsymbol{X}, \ *)$

(i) If $\boldsymbol{X}$ is normal, then $\mathrm{Wh\ cat}\,\boldsymbol{X} \ \leq \mathrm{cat}\,\boldsymbol{X}$.

(ii) If $*$ is contained in a subspace U that is open an contractible in $\boldsymbol{X}$, then $\mathrm{cat}\,\boldsymbol{X} \leq \mathrm{Wh\ cat}\,\boldsymbol{X}$.

Sectional category *à la* Quillen

Definition

Let $\boldsymbol{X}$ be a simply connected topological space. The ***rational LS category***, $\mathrm{cat}_0\, \boldsymbol{X}$ is the least integer m such that $\boldsymbol{X} \simeq_{\mathbb{Q}} \boldsymbol{Y}$ and $\mathrm{cat}\, \boldsymbol{Y} = m$.

In fact, if $\boldsymbol{X}$ is a simply connected CW complex, we have

$$\mathrm{cat}_0 \boldsymbol{X} = \mathrm{cat}\, \boldsymbol{X}_{\mathbb{Q}}$$

Sectional category *à la* Quillen

In their celebrated paper ‘*Rational LS category and its applications*’ **Yves Félix** and **Steve Halperin** characterized algebraically rational LS-category in terms of Sullivan models.

$$\boldsymbol{X} \quad \leftrightsquigarrow \quad (\Lambda V, d)$$

Sectional category
à la Quillen

Theorem

Let X be a space and $(\Lambda V, d)$ be a model for X. Then the rational LS category of X is the least m for which the CDGA projection

$$\rho_m\colon (\Lambda V, d) \to \left(\frac{\Lambda V}{\Lambda^{>m} V}, \bar{d}\right)$$

admits a homotopy retraction.

Sectional category
à la Quillen

Why not using Quillen models to algebraically describe rational LS category?

Sectional category
à la Quillen

A ***based topological space*** $(\boldsymbol{X},\ *)$

The ***cartesian product*** $\boldsymbol{X}^{m+1} = \underbrace{\boldsymbol{X} \times \cdots \times \boldsymbol{X}}_{m+1 \text{ times}}$

The ***diagonal map*** $\Delta_{\boldsymbol{X}} : \boldsymbol{X} \to \boldsymbol{X}^{m+1}$; $\Delta_{\boldsymbol{X}}(x) = (x, \ldots, x)$

The ***fat wedge***, $T^m(\boldsymbol{X}) \subseteq \boldsymbol{X}^{m+1}$.

$$\Delta_{\boldsymbol{X}} : \boldsymbol{X} \to T^m(\boldsymbol{X}) \subseteq \boldsymbol{X}^{m+1}.$$

Sectional category *à la* Quillen

A ***based topological space*** $(\boldsymbol{X},\ *)$ $\qquad$ $(\,\mathbb{L}(V)\,,\partial\,)$

The ***cartesian product*** $\quad \boldsymbol{X}^{m+1} = \underbrace{\boldsymbol{X} \times \cdots \times \boldsymbol{X}}_{m+1 \text{ times}}$

$$\boldsymbol{X}^{m+1} \quad \leftrightsquigarrow \quad (\,\mathbb{L}(\bigoplus V_i \oplus V_{i,j} \oplus \cdots \oplus V_{1,2,\cdots,m+1})\,, D\,)$$

A ***based topological space*** $(\boldsymbol{X}, *)$ $\quad\quad (\mathbb{L}(V), \partial)$

The ***cartesian product*** $\boldsymbol{X}^{m+1} = \underbrace{\boldsymbol{X} \times \cdots \times \boldsymbol{X}}_{m+1 \text{ times}}$

$$\boldsymbol{X}^{m+1} \quad \leftrightsquigarrow \quad (\mathbb{L}(\bigoplus V_i \oplus V_{i,j} \oplus \cdots \oplus V_{1,2,\cdots,m+1}), D)$$

The ***diagonal map*** $\Delta_X : \boldsymbol{X} \to \boldsymbol{X}^{m+1}$; $\quad \Delta_{\boldsymbol{X}}(x) = (x, \cdots, x)$

$$\Delta_{\mathbb{L}(V)} : (\mathbb{L}(V), \partial) \longrightarrow (\mathbb{L}(\bigoplus V_i \oplus \underbrace{V_{i,j} \oplus \cdots \oplus V_{1,2,\cdots,m+1}}), D)$$

$$v \quad \longmapsto \quad v_1 + v_2 + \cdots + v_{m+1} + \Psi$$

The ***cartesian product*** $\boldsymbol{X}^{m+1} = \underbrace{\boldsymbol{X} \times \cdots \times \boldsymbol{X}}_{m+1 \text{ times}}$

$$\boldsymbol{X}^{m+1} \quad \leftrightsquigarrow \quad (\,\mathbb{L}(\bigoplus V_i \oplus V_{i,j} \oplus \cdots \oplus V_{1,2,\cdots,m+1}), D\,)$$

The ***diagonal map*** $\Delta_X : \boldsymbol{X} \to \boldsymbol{X}^{m+1}$; $\quad \Delta_X(x) = (x, \cdots, x)$

$$\Delta_{\mathbb{L}(V)} : (\,\mathbb{L}(V), \partial\,) \longrightarrow (\,\mathbb{L}(\bigoplus V_i \oplus \underbrace{V_{i,j} \oplus \cdots \oplus V_{1,2,\cdots,m+1}}), D\,)$$

$$v \quad \longmapsto \quad v_1 + v_2 + \cdots + v_{m+1} + \Psi$$

The ***fat wedge***, $T^m(\boldsymbol{X}) \subseteq \boldsymbol{X}^{m+1}$.

$$(\,\mathbb{L}(\overbrace{\bigoplus V_i \oplus V_{i,j} \oplus \cdots \oplus}\ \cancel{V_{1,2,\cdots,m+1}}), D\,)$$

The ***diagonal map*** $\Delta_{\boldsymbol{X}} : \boldsymbol{X} \to \boldsymbol{X}^{m+1}$; $\Delta_{\boldsymbol{X}}(x) = (x, \cdots, x)$

$$\Delta_{\mathbb{L}(V)} : \ (\mathbb{L}(V), \partial) \longrightarrow (\mathbb{L}(\bigoplus V_i \bigoplus V_{i,j} \bigoplus \cdots \bigoplus V_{1,2,\cdots,m+1}), D)$$

$$v \longmapsto v_1 + v_2 + \cdots + v_{m+1} + \Psi$$

The ***fat wedge***, $T^m(\boldsymbol{X}) \subseteq \boldsymbol{X}^{m+1}$.

$$(\mathbb{L}(\bigoplus V_i \bigoplus V_{i,j} \bigoplus \cdots \bigoplus \sout{V_{1,2,\cdots,m+1}}), D)$$

Theorem Let $\boldsymbol{X}$ be a space and $(\mathbb{L}(V), \partial)$ a Quillen model of $\boldsymbol{X}$. Then, the rational LS category of $\boldsymbol{X}$ is the least m for which the morphism $\Delta_{\mathbb{L}(V)}$ does not require the vectors of $V_{1,2,\cdots,m+1}$.

Sectional category
à la Quillen

Definition

The ***sectional category*** of a continuous map $f: X \longrightarrow Y$ is the least integer m for which there are $m+1$ local homotopy sections for f whose domains form an open cover of Y.

Sectional category *à la* Quillen

If X is path-connected, then

Example 1

$$\text{cat}\,(X) = \text{secat}\,(* \hookrightarrow X)$$

Example 2

$$\text{TC}\,(X) = \text{secat}\,(\Delta : X \longrightarrow X \times X)$$

Sectional category *à la* Quillen

There exists also a **Whitehead** characterization of sectional category of a map $f: X \longrightarrow Y$.

Take $i: A \hookrightarrow Y$ a cofibration replacement for f.

We define the m-th **fat wedge of i** as

$$T^m(i) = \{(y_0, \cdots, y_m) \in Y^{m+1} \mid y_i \in i(A) \text{ for at least one } i\}$$

Sectional category *à la* Quillen

The sectional category $\mathrm{secat}(\not{f}) = \mathrm{secat}(\imath)$ is the least m for which the diagonal map $\Delta_{\boldsymbol{Y}} : \boldsymbol{Y} \to \boldsymbol{Y}^{m+1}$ is homotopic to a map

$$\Delta_{\boldsymbol{Y}} : \boldsymbol{Y} \to T^m(\imath) \subseteq \boldsymbol{Y}^{m+1}.$$

Why not using Quillen models to algebraically describe rational sectional category?

Sectional category
à la Quillen

We need Quillen models for

- A cofibration $i\colon \boldsymbol{A} \hookrightarrow \boldsymbol{Y}$
- The fat wedge $T^m(i)$

$$\boldsymbol{A} \qquad\qquad \boldsymbol{Y}$$

$$i\colon (\mathbb{L}(A), \partial) \longrightarrow (\mathbb{L}(A \oplus W) D)$$

where $Da = \partial a$ if $a \in A$

$$\boldsymbol{Y}^{m+1}$$

$$(\mathbb{L}(A \oplus W) D) \longrightarrow (\mathbb{L}(\oplus (A \oplus W)_i \oplus (A \oplus W)_{i,j} \oplus \cdots \oplus (A \oplus W)_{1,2,\cdots,m+1})$$

$$v \mapsto v_1 + \cdots + v_k + \Psi$$

$$T^m(i)$$

$$(\mathbb{L}(\oplus (A \oplus W)_i \oplus (A \oplus W)_{i,j} \oplus \cdots \oplus (A \oplus W)_{1,2,\cdots,m+1})$$