Cyclic cohomology of etale groupoids

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Abstract

We extend Connes's computation of the cyclic cohomology groups of smooth algebras arising from foliations with separated graphs. We find that the characteristic classes of foliations factor through these groups. Our results also explain some results of Atiyah and Segal on orbifold Euler characteristic in the setting of cyclic homology.

Contents

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1 Introduction

This paper is part of a program whose purpose is to understand the connections between Connes's index theorem for foliations [12] and Bismut's local index theorem for families [3]. We are particularly interested in relating this with the characteristic classes of foliations [6], and computing the bivariant Chern-Connes character [31, 32].

A very general index theorem for foliations was proved in [17] (Theorem 5, page 67; an english translation is soon due). This theorem gives a geometric description of the index associated to any even cyclic cocycle for foliation algebra (i.e. the convolution algebra of smooth compactly-supported functions on the graph of the foliation). An important ingredient in the proof of this theorem is the construction of a map from the cyclic cohomology of the foliation algebra to the de Rham cohomology of the underlying manifold (ibidem, corollary 2, page 64).

Bismut's approach is to extend the heat kernel proof of the Atiyah-Singer index theorem using Quillen's theory of superconnections [36]. Specifically he proves that the rescaled curvature of a superconnection he associates to the bundle of L^2 sections along the fibers, is convergent to the curvature of the index bundle [2].

One can see that there are both conceptual and technical difficulties in extending the local index theorem for families to foliations. There are (as we understand the problem now) three major steps to be undertaken. The first step is to make sense of the curvature of the index. As in the family index case a solution to this problem is known: the curvature is an element of a cyclic homology group [15, 26]. What is not known in general, however is what is the form of these homology groups. The second step is to prove that the superconnection formalism remains valid for operators "with connected spectrum", operators that in particular have neither closed range, nor finite dimensional kernel. Of course in this case the index, an element of a K_0-group, is to be defined by the connecting morphism in algebraic K-theory. The solution to this problem depends essentially upon cyclic cohomology. The last step is to carry out the local computations.

In this paper we undertake the first step. We make a the following assumptions. First we use the (slightly simplified) setting of etale groupoids. It is known that these groupoids are Morita equivalent to the groupoids obtained from foliations [24]. Since Morita equivalent algebras have the same cyclic cohomology groups one would only need to establish that Morita equivalent groupoids have Morita equivalent algebras. This principle is definitely true for the corresponding noncommutative topological spaces (i.e. $C^\star$-algebras) [24, 34], and is also true for the case we are interested in. We will establish this principle for the smooth case in a forthcoming paper [29]. Then we make the far less trivial assumption that our groupoids are Hausdorff. This is definitely true in many interesting cases coming from foliations [44] but not in general. To give the reader a sense of the difficulties we will only mention that there are fairly simple examples of nonseparated groupoids for which smooth forms make sense (they always do using linear combinations of forms supported in coordinate patches) but the de Rham differential does not make sense! We do obtain results valid without any separability assumption, but they are not complete.

The final result identifies the periodic cyclic cohomology groups with the cohomology of some "fiber bundles" over the classifying space G of the given groupoid. This extends a result of Connes [16]. These spaces are obtained from the universal principal G-bundle. We identify the part corresponding to the units with the result predicted by Connes [16]. The result generalizes the case of group algebras [11] and that of crossed products of smooth commutative algebras by discrete groups [19, 30, 42]. Moreover since the classifying space construction of Morita equivalent groupoids gives homotopy equivalent spaces [23, 35], we obtain that the characteristic classes of a foliation $(V, \mathcal{F})$ factor through $HC^{per}(C_c^\infty(V, \mathcal{F}))$, see also [16, 17].

2 Smooth groupoids and their convolution algebras

We are going to work with smooth groupoids. We first recall some definitions.

Definition. 2.1 *A groupoid G is a small category in which every morphism is invertible.*

This is the shortest but at the same time the least explicit definition. We are going to make this definition more explicit bellow. Also recall that a category is called *small* if the class of its objects, and hence also the morphisms, form a set. The objects $X = G^{(0)}$ of G will be called *units* and the morphisms $G^{(1)}$ of G will be called *arrows*. In the next section we will identify G to $G^{(1)}$. The set of composable pairs of arrows will be denoted by $G^{(2)}$:

$$G^{(2)} = G^{(1)} \times_X G^{(1)} = \{(g, h) : d(g) = r(h)\} \tag{1}$$

Here $d(g)$ and $r(g)$ denote the *domain* and, respectively, the *range* of the arrow g.

Definition. 2.2 *A smooth groupoid G is a groupoid such that the spaces G^0, $G^{(1)}$ and $G^{(2)}$ are smooth manifolds $G^{(0)}$ is a Husdorff space, the maps $d, r, \circ, (\ )^{-1}$ are smooth, d, r are submersions, and $G^{(0)} \to G^{(1)}$ is an embedding.*

We see that a smooth groupoid G is given as

$$G = (G^{(1)}, G^{(0)}, d, r, \circ, (\ )^{-1}, u)$$

where:

(i) $G^{(0)}$, $G^{(1)}$ and $G^{(2)} = \{(g, h) : d(g) = r(h)\}$ are smooth manifolds, $G^{(0)}$ being separated (i.e. Hausdorff);

(ii) The "range" and "domain" maps $r, d : G^{(1)} \to G^{(0)}$, the partially defined multiplication $\circ : G^{(2)} \to G^{(1)}$, $g \circ h = gh$, the "inverse" map $(\ )^{-1} : G^{(1)} \to G^{(1)}$, and the "unit" map $u : G^{(0)} \to G^{(1)}$ are smooth maps satisfying:

1. $r(gh) = r(g)$, $d(gh) = d(h)$ for any pair $(g, h) \in G^{(2)}$, and the partial multiplication $g \circ h = gh$ is associative.

2. $d(u(x)) = r(u(x)) = x \ \forall x \in G^{(0)}$, $u(r(g))g = g$ and $gu(d(g)) = g$ $\forall g \in G^{(1)}$ and u is a smooth embedding.

3. $r(g^{-1}) = s(g)$, $d(g^{-1}) = r(g)$, $gg^{-1} = u(r(g))$ and $g^{-1}g = u(d(g))$.

Here are some examples of smooth groupoids:

1. *Lie groups.*

2. *Principal groupoids:* Let X be a smooth manifold and define $G^{(0)} = X$ and $G^{(1)} = X \times X$ such that a pair (x, y) is composable with the pair (x', y') if and only if $y = x'$ and then $(x, y)(y, y') = (x, y')$.

3. *Group actions:* If H is a Lie group acting smoothly on the left on a manifold X, then the crossed product groupoid is defined by $G^{(0)} = X$, $G^{(1)} = H \times X$, $d(h,x) = x$, $r(h,x) = hx$ and $(h_1, hx)(h,x) = (h_1 h, x)$.

Definition. 2.3 *A topological groupoid $G = (G^{(1)}, G^{(0)}, r, s, \circ, (\ )^{-1}, u)$ is a groupoid such that the sets G^0 and $G^{(1)}$ are topological spaces, the maps $d, r, \circ, (\ )^{-1}$ are continuous, d, r are open maps, and $G^{(0)} \to G^{(1)}$ is a homeomorphism onto its image.*

In [12, 13, 37] it is shown how to associate to a locally compact groupoid endowed with a Haar measure a convolution algebra. These definitions and constructions easely generalize to the smooth case.

Definition. 2.4 *A smooth Haar system on a smooth groupoid G is a family of positive Radon measures $(\lambda^x)_{x \in G^{(0)}}$ on $G^{(1)}$ satisfying the following conditions:*

1. *The support of λ^x is $G^x = \{g \in G^{(1)} : r(g) = x\}$ and λ^x is a smooth measure on G^x.*

2. *(Left invariance) For any continuous function $f : G^x \to [0, \infty)$ and any $g \in G^{(1)}$ such that $r(g) = x$ and $d(g) = y$ we have*

$$\int_{G^x} f(t) d\lambda^x(t) = \int_{G^y} f(gt) d\lambda^y(t)$$

   (i.e. $g_\star(\lambda^{d(g)}) = \lambda^{r(g)}$).

3. *(Smoothness) If $U \subset G^{(1)}$ is a coordinate neighborhood in $G^{(1)}$ (so in particular it is separated) and φ is a smooth compactly supported function on U, then the function*

$$G^{(0)} \ni x \to \int_{G^x} \varphi(t) d\lambda^x(t)$$

   is a smooth function on $G^{(0)}$ (φ is extended to be 0 outside U).

We shall denote by $C_c^\infty(X)$, where X is a smooth but not necessarily separated manifold, the complex linear subspace of the space of all functions on $G^{(1)}$ generated by $C_c^\infty(U)$ for all coordinate domains U. In general it is not closed under pointwise multiplication. If we denote by $C^\infty(X; sep)$ the space of *continuous* functions that are smooth in every coordinate domain then $C_c^\infty(X)$ is a module over $C^\infty(X; sep)$ for the pointwise multiplication. We agree to denote $C_c^\infty(G^{(1)})$ by $C_c^\infty(G)$.

A smooth Haar system $(\lambda^x)_{x\in G^{(0)}}$ on G defines a convolution product on $C_c^\infty(G)$

$$(ff_0)(g) = \int_{G^{r(g)}} f(\gamma)f_0(\gamma^{-1}g)d\lambda^{r(g)}(\gamma)$$

We denote by $A(G)$ the smallest subalgebra of compactly supported Borel functions on G which is closed under the convolution product and contains $C_c^\infty(G)$.

In the following we are going to be interested in etale groupoids.

Definition. 2.5 *A smooth etale groupoid is a smooth groupoid G such that the domain and range maps are local diffeomorphisms.*

Etale groupoids are sometimes called discrete groupoids.

Fix $g \in G^{(1)}$ and choose a neighborhood U of g in $G^{(1)}$ such that the domain and range maps define diffeomorphisms $d : U \to U_0 \subset G^{(0)}$ and $r : U \to U_1 \subset G^{(0)}$. We will call a neighborhood with this property a *defining neighborhood*. A defining neighborhood U gives rise to a diffeomorphism $\varphi_U : U_0 \to U_1$ whose germ does not depend on the choice of U, because the intersection of two defining neighborhoods is also a defining neighborhood. For an element $g \in G^{(1)}$ we will denote by φ_g the germ of φ_U.

It is easy to observe that for a smooth etale groupoid the counting measure on each G^x, $x \in G^{(0)}$ defines a smooth Haar system.

In order to describe the multiplication on $C_c^\infty(G)$ for this Haar system let us make the following observations.

If U is a defining neighborhood of g then the smooth compactly supported functions on U are in one-to-one correspondence with smooth compactly supported functions on $d(U)$, and similarly for $r(U)$.

Let f, f' be two smooth functions whose supports are compact and contained in defining neighborhoods U, U' of g and, respectively, g'. If UU' is empty we have $ff' = 0$. So we will assume that UU' is not empty. Using the fact that U and U' are defining neighborhoods, we get smooth compactly supported functions f_1 on $d(U)$ and, respectively, f_1' on $r(U')$. Their pointwise product is a smooth compactly supported function on $d(U)$ which then gives, using φ_U, a smooth function on $r(U)$, whose support is compact and contained in $r(UU')$. Since UU' is a defining neighborhood for any of its elements, we finally obtain a smooth compactly supported function h on UU'. Then

$$h = ff' \tag{2}$$

This shows that for smooth etale groupoids $A(G) = C_c^\infty(G)$.

The main examples of etale smooth groupoids come from the action by diffeomorphisms of discrete groups and from smooth foliations.

The following Proposition is proved in [28].

Proposition. 2.6 *Let g, g' be arrows in $G^{(1)}$. If g and g' can not be separated by two open sets then $d(g) = d(g')$, $r(g) = r(g')$, and there exists an open set $V \in G^{(0)}$ whose closure contains $d(g)$ such that $\varphi_{g|V} = \varphi_{g'|V}$.*

The converse is also true if $g \to \varphi_g$ is one-to-one.

We agree to identify $G^{(0)}$ with $u(G^{(0)}) \subset G^{(1)}$.

Proposition. 2.7 *If G is a smooth etale groupoid then, using the above identification, $G^{(0)}$ is an open subset of $G^{(1)}$ which is closed if and only if G is separated.*

Proof. By definition u is an immersion. For a smooth etale groupoid G $G^{(0)}$ and $G^{(1)}$ have the same dimension. Then u is also a submersion, so it is an open map.

If $G^{(1)}$ is not separated, let $g, \gamma \in G$ which cannot be separated. We have $d(g) = d(\gamma)$ and $r(g) = r(\gamma)$ since d and r are continuous and $G^{(0)}$ is separated. Let U and W be defining neighborhoods of g and γ such that $r(U) = r(W)$. We show that $d(g)$ and $g^{-1}\gamma$ can not be separated. Indeed consider the neighborhoods $d(U)$ and $f(W)$ of $d(g)$ and $g^{-1}\gamma$, where $f(\gamma')$ is defined as $g'^{-1}\gamma'$ with g' the unique element in U satisfying $r(g') = r(\gamma')$. By the definition of a etale groupoid f is a continuous map $W \to G^{(1)}$. We see that since g and γ can not be separated the intersection $V = U \cap W$ is a nonempty set containing g and γ in its closure and such that g and γ are limits of a sequence $x_n \in V$. This shows that $d(g)$ and $g^{-1}\gamma$ are limits of the sequence $d(x_n) = f(x_n)$ so $G^{(0)}$ is not closed.

Conversely, if $G^{(1)}$ is separated a convergent sequence has a unique limit. Let x_n be a sequence of elements of $G^{(0)}$ which is convergent to $g \in G^{(1)}$.

Then $x_n = d(x_n)$ converges also to $d(g)$. This shows that $g = d(g)$ so it belongs to $G^{(0)}$, and hence $G^{(0)}$ is closed. $\square$

Proposition. 2.8 *Let G be a smooth etale groupoid. Assume $G^{(1)}$ is separated and denote by $B^{(0)}_{\text{fin}}$ the elements of finite order in $G^{(1)}$:*

$$B^{(0)}_{\text{fin}} = \{g : \exists\, n > 0 \text{ such that } g^n \in G^{(0)}\}$$

Then $B^{(0)}_{\text{fin}}$ is a smooth submanifold of $G^{(1)}$ and is an open and closed subset of $B^{(0)} = \{g \in G^{(1)} : r(g) = d(g)\}$. Moreover the function $o : B^{(0)}_{\text{fin}} \to \mathbb{N}$ assigning to an element of finite order its order is continuous.

Proof. Let $g \in B^{(0)}_{\text{fin}}$ be an element of order n. Then φ_g has finite order so we can assume that it comes from a diffeomorphism φ_W of an open neighborhood V of $r(g) = d(g)$ for some defining neighborhood W of g. Choosing a metric on V invariant for the action of φ_W we see that the set of fixed points of φ_W is a submanifold of V. Since for $h \in W$ we have $r(h) = \varphi_W(d(h))$ we obtain that $B^{(0)} \cap W$ is a submanifold of W and hence also of $G^{(1)}$.

In order to prove now that o is continuous it is enough to show that all elements in a given path component of $B^{(0)} \cap W$ have the same order. Let $h^m \in G^{(0)}$ for some natural number m. If $f : [0,1] \to B^{(0)} \cap W$ is a path starting at h we obtain that $f(t)^m$ is a path starting at a unit and hence it is completely contained in $G^{(0)}$ due to the previous proposition. $\square$

3 A reduction to loops

In this section we give a preliminary description of the Hochschild and cyclic homology of the algebra of functions on a smooth etale groupoid. More precise results will be obtained in the next sections for separated groupoids.

We will identify $G^{(1)}$, the space of arrows of a groupoid G, to G itself. This is justified by the fact that $G^{(1)}$ and the partial multiplication determine the rest of the structure of G. This will simplify notation without leading to confusions.

Recall that for a locally convex algebra A the Hochschild homology of A with coefficients in a bimodule M, denoted here $\mathrm{H}_n(A, M)$, is the homology of $Bar(A, M)$, the Bar complex of the bimodule M:

$$M \xleftarrow{b} M \otimes A \xleftarrow{b} \dots \xleftarrow{b} M \otimes A^{\otimes n} \xleftarrow{b} M \otimes A^{\otimes (n+1)} \xleftarrow{b} \dots \qquad (3)$$

Here $\otimes$ is the projective tensor product [22] and b, the Hochschild boundary, is given by the usual formula:

$$
\begin{aligned}
b(m \otimes a_1 \otimes \ldots \otimes a_n) &= ma_1 \otimes \ldots \otimes a_n - m \otimes a_1 a_2 \otimes \ldots \otimes a_n \\
+m \otimes a_1 \otimes a_2 a_3 \otimes \ldots \otimes a_n &+ \ldots + (-1)^n a_n m \otimes a_1 \otimes \ldots \otimes a_{n-1}
\end{aligned} \tag{4}
$$

In case $M = A$ with the usual A bimodule structure we denote $\mathrm{H}_n(A, A) = \mathrm{HH}_n(A)$ and call it simply the Hochschild homology of A.

For a separated smooth etale groupoid G and $A = C_c^\infty(G)$ we have that there exist linear surjection $\Phi : A^{\otimes(n+1)} \to C_c^\infty(G^{n+1})$ defined by

$$
\Phi(f_0 \otimes \ldots \otimes f_n)(g_0, \ldots, g_n) = f_0(g_0) f_1(g_1) \cdots f_n(g_n)
$$

One can prove this map to be injective too, but for our purpose this is irelevant. In any case it is not a homeomorphism unless G is compact [22], see also [4]. With respect to Φ the action of the differential b becomes:

$$
\begin{aligned}
(bf)(g_0, \ldots, g_{n-1}) &= \sum_{i=0}^{n-1} (-1)^i \sum_{r(\gamma)=r(g_i)} f(g_0, \ldots, g_{i-1}, \gamma, \gamma^{-1} g_i, \ldots, g_{n-1}) \\
&+ (-1)^n \sum_{r(\gamma)=r(g_0)} f(\gamma^{-1} g_0, g_1 \ldots, g_{n-1}, \gamma)
\end{aligned} \tag{5}
$$

In order to include the nonseparated case too we let $A^{\otimes(n+1)} = C_c^\infty(G^{n+1})$, the boundary being defined by the previous formula.

The cyclic homology of $C_c^\infty(G)$ [15, 26, 41] is the homology of the complex

$$
\begin{aligned}
A \xleftarrow{b} A \otimes A/(1-\tau) A \otimes A \xleftarrow{b} \ldots \xleftarrow{b} A^{\otimes n}/(1-\tau) A^{\otimes n} \\
\xleftarrow{b} A^{\otimes(n+1)}/(1-\tau) A^{\otimes(n+1)} \xleftarrow{b} \ldots
\end{aligned} \tag{6}
$$

where $\tau(a_0 \otimes \ldots \otimes a_n) = (-1)^n a_n \otimes a_0 \otimes a_1 \otimes \ldots \otimes a_{n-1}$.

It can be defined also as the homology of the bicomplex [26]

$$
\begin{array}{ccccccc}
\cdots & & \cdots & & \cdots & & \\
b\downarrow & & \downarrow -b' & & \downarrow b & & \\
A^{\otimes(n+2)} & \xleftarrow{\epsilon} & A^{\otimes(n+2)} & \xleftarrow{N} & A^{\otimes(n+2)} & \xleftarrow{\epsilon} & \cdots \\
b\downarrow & & \downarrow -b' & & \downarrow b & & \\
A^{\otimes(n+1)} & \xleftarrow{\epsilon} & A^{\otimes(n+1)} & \xleftarrow{N} & A^{\otimes(n+1)} & \xleftarrow{\epsilon} & \cdots \\
b\downarrow & & \downarrow -b' & & \downarrow b & & \\
\cdots & & \cdots & & \cdots & & \\
b\downarrow & & \downarrow -b' & & \downarrow b & & \\
A & \xleftarrow{\epsilon} & A & \xleftarrow{N} & A & \xleftarrow{\epsilon} & \cdots
\end{array}
$$

where we have denoted $\epsilon = 1 - \tau$, $N = 1 + \tau + \ldots \tau^n$, $b'(a_0 \otimes \ldots \otimes a_n) = b(a_0 \otimes \ldots \otimes a_n) - (-1)^n a_n a_0 \otimes a_1 \otimes \ldots \otimes a_{n-1}$.

To define what we mean by "reduction to closed loops" consider for $0 \leq i \leq n$ the open set $U_{n+1,i} = \{(g_0, \ldots, g_n) \in G^{n+1} : r(g_{i+1}) \neq d(g_i)\}$ with the convention that $g_{n+1} = g_0$. This sequence of open sets defines an increasing sequence of subcomplexes of the Hochshild complex by

$$F_i A^{\otimes n} = \sum_{j=0}^{i} \{f \in C_c^\infty(G^n) : \mathrm{supp}(f) \subset U_{n,j}\} \tag{7}$$

We let as usual $F_{-1}A^{\otimes n} = 0$ and $F_\infty A^{\otimes n} = F_m A^{\otimes n}$ for any $m \geq n-1$.

Lemma. 3.1 *Let $F_i A^{\otimes n}$ be defined by the above equation for any n. Then we have $b(F_i A^{\otimes(n+1)}) \subset F_i A^{\otimes n}$ and the induced complexes $(F_i A^{\otimes n}/F_{i-1}A^{\otimes n}, b)$ are acyclic (i.e. their homology vanishes). The same is true of the b'-complex.*

Proof. The first part follows from the definition. In order to prove the second part it is enough to find for any n a family of linear maps $\chi_m^{(i)} : F_i A^{\otimes n} \to A^{\otimes(n+1)}$, $m \in M$, with the property that for any $a \in F_i A^{\otimes n}$ we can find an m such that $\chi_m^{(i)}(a)$ and $\chi_m^{(i)}(b(a))$ are in $F_i A^{\otimes(n+1)}$, respectively in $F_iA^{\otimes n}$, and $(b\chi_m^{(i)} + \chi_m^{(i)}b)a = a \ (mod\ F_{i-1}A^{\otimes n})$. Indeed let M be the set of pairs $(K, (\varphi_j)_{1\leq j\leq N})$ where K is a compact subset of $G^{(0)}$ and φ_j are smooth compactly supported functions on $G^{(0)}$ such that $\sum \varphi_j^2 = 1$ on K. By definition $\varphi_j \in C_c^\infty(G)$.

For any smooth compactly supported functions ψ_0, ψ_1 we define the linear map $l^{(i)}_{\psi_0,\psi_1} : A^{\otimes(n+1)} \to A^{\otimes(n+2)}$ by the formula

$$l^{(i)}_{\psi_0,\psi_1}(a_0 \otimes \ldots \otimes a_n) = a_0 \otimes \ldots a_i \otimes \psi_0 \otimes \psi_1 a_{i+1} \otimes \ldots \otimes a_n$$

if $0 \leq i \leq n-1$,

$$l^{(n)}_{\psi_0,\psi_1}(a_0 \otimes \ldots \otimes a_n) = \psi_1 a_0 \otimes \ldots \otimes \ldots \otimes a_n \otimes \psi_0$$

and $l^{(i)}_{\psi_0,\psi_1} = 0$ if $i > n$. Observe that $l^{(i)}_{\psi_0,\psi_1}(F_{i-1}A^{\otimes n}) \subset F_{i-1}A^{\otimes(n+1)}$. We define for $m = (K, (\varphi_j)_{1\leq j\leq N})$

$$\chi_m^{(i)} = \sum_j l^{(i)}_{\varphi_j,\varphi_j}$$

Consider now an arbitrary $a \in F_iA^{\otimes n}$. By the definition of $F_{i-1}A^{\otimes n}$ and $F_iA^{\otimes n}$ we know that a can be written as $a_1 + a_2$ where $a_1 \in F_{i-1}A^{\otimes n}$ and $L = \mathrm{supp}(a_2) \subset U_{n,i}$. If $m = (K, (\varphi_j)_{1\leq j\leq N})$ where K is big enough, and the supports of φ_j are small enough (i.e. if $L \subset K^n$ and the support of each φ_j has diameter $< \epsilon/3$ for $\epsilon = \inf\{\mathrm{d}(d(g_i), r(g_{i+1})) : (g_0, \ldots, g_n) \in \mathrm{supp}(a_2)\}$, for a metric d on K which defines the topology on K) we find out that $\chi_m^{(i)}(a)$ and $\chi_m^{(i)}(b(a))$ are in $F_iA^{\otimes(n+1)}$, respectively in $F_iA^{\otimes n}$. An easy computation also shows that $b(\chi_m^{(i)}(a_2)) + \chi_m^{(i)}(b(a_2)) + (-1)^i a_2$ has support contained in $U_{n,i-1}$. This proves the Lemma.

The proof for b' goes without change using the same maps $\chi_m^{(i)}$. $\square$

The above Lemma immediately implies the following Proposition which we see as a "reduction to loops".

Proposition. 3.2 *Let G be a smooth etale groupoid. The maps of complexes*

$$(A^{\otimes(n+1)}, b) \to (A^{\otimes(n+1)}/F_\infty A^{\otimes(n+1)}, b)$$
$$(A^{\otimes(n+1)}, b') \to (A^{\otimes(n+1)}/F_\infty A^{\otimes(n+1)}, b')$$

and

$$(A^{\otimes(n+1)}/(1-\tau)A^{\otimes(n+1)}, b) \to (A^{\otimes(n+1)}/F_\infty A^{\otimes(n+1)} + (1-\tau)A^{\otimes(n+1)}, b)$$

are quasi-isomorphisms (i.e. induce isomorphisms in homology).

Proof. The previous Lemma implies that the projection

$$A^{\otimes(n+1)} \to A^{\otimes(n+1)}/F_\infty A^{\otimes(n+1)}$$

is a quasi-isomorphism for the b and b' complexes. Using the double complex definition of cyclic cohomology it follows that the projection map is a quasi-isomorphism also for the cyclic complex. $\square$

Let R_{inv} denote the ring of locally constant functions defined in a neighborhood of $B^{(0)}$ in G and satisfying $f(x) = f(\gamma^{-1}x\gamma)$. We have an R_{inv}-module structure given nn $A^{\otimes(n+1)}/F_\infty A^{\otimes(n+1)}$by

$$(\rho f)(g_0, \dots, g_n) = \rho(g_0 \dots g_n) f(g_0, \dots, g_n)$$

for $\rho \in R_{inv}$. Moreover b, b' are compatible with this structure.

We now consider an analog of the decomposition of the cyclic cohomology of a group algebra according to conjugacy classes [11].

Let $\mathcal{O} \subset B^{(0)}$ be an open-closed set invariant under the action of G, that is $g\gamma g^{-1} \in \mathcal{O}$ for any $\gamma \in \mathcal{O}$ and $g \in G$ such that $d(g) = r(\gamma)$. To this open set there corresponds an idempotent $e_{\mathcal{O}} \in R$. Define $HH_n(C_c^\infty(G))_{\mathcal{O}}$ to be the homology of $e_{\mathcal{O}}(A^{\otimes(n+1)}/F_\infty A^{\otimes(n+1)})$ and define similarly $HC_n(C_c^\infty(G))_{\mathcal{O}}$.

The following Proposition follows from definitions.

Proposition. 3.3 *Let $B^{(0)} = \cup_\alpha \mathcal{O}_\alpha$ be a partition of $B^{(0)}$ into disjoint invariant open sets. Then*

$$HH_n(C_c^\infty(G)) \simeq \bigoplus_\alpha HH_n(C_c^\infty(G))_{\mathcal{O}_\alpha}$$

$$HC_n(C_c^\infty(G)) \simeq \bigoplus_\alpha HC_n(C_c^\infty(G))_{\mathcal{O}_\alpha}$$

4 A simplicial resolution

Convention. In this sections we will assume that G is a separated manifold.

In what follows it will be convenient for us to use the language of sheaves. Standard references for this are [20, 40], see also [39].

Recall ([39], page 323) that a sheaf $\mathcal{S}$ of vector spaces on a topological space Z is a contravariant functor from the category of open subsets U of

Z and inclusion maps to vector spaces satisfying certain conditions. This implies that to any open set $U \subset Z$ there is associated a vector space $\mathcal{S}(U)$ and that for any $V \subset U$ there exists a restriction morphism $r_{VU} : \mathcal{S}(U) \to \mathcal{S}(V)$, $\mathcal{S}(\emptyset) = 0$, satisfying $r_{WU} r_{UV} = r_{WV}$, and $r_{UU} = id$. This data defines a *presheaf*. A *sheaf* is a presheaf such that if $U = \cup V_i$ the map $\mathcal{S}(U) \to \prod \mathcal{S}(V_i)$ is injective and its image consists of the kernel of a suitable defined map $\prod \mathcal{S}(V_i) \to \prod \mathcal{S}(V_i \cap V_j)$. To any presheaf there is associated a canonical sheaf. Define the *stalk* $\mathcal{S}_x$ of a sheaf $\mathcal{S}$ at $x \in Z$ by $\mathcal{S}_x = \varinjlim \mathcal{S}(U)$ where the direct limit is taken over the set of all open neighborhoods U of x.

If $f : Z_0 \to Z$ is a continuous map and $\mathcal{S}$ is a sheaf on Z then there is a canonical pull-back sheaf $f^{-1}\mathcal{S}$ on Z_0 see [20]. We have $f^{-1}\mathcal{S}_y = \mathcal{S}_{f(y)}$.

If Z is a separated (i.e. Hausdorff) smooth manifold, then $U \to C^\infty(U)$, or more generally $U \to \Omega^p(U)$ define sheaves, the sheaf of smooth functions and, respectively, the sheaf of smooth p-forms on Z. The assignment $U \to C^\infty(U^n)$ is only a presheaf. The associated sheaf is the space of germs at the diagonal of smooth functions on G^n, it coincides with $i^{-1}(\mathcal{C}^\infty(G^n))$, if $i : G \to G^n$ is the diagonal embedding and $\mathcal{C}^\infty(G^n)$ is the sheaf of smooth functions on G^n.

In case f is a local homeomorphism, the case we are going to be mostly interested in, $f^{-1}\mathcal{S}(U)$ is canonically isomorphic to $\mathcal{S}(f(U))$, whenever f is a homeomorphism from U to an open set in Z.

A section of $\mathcal{S}$ over U is simply an element of $\mathcal{S}(U)$. Such a section is said to have compact support if its restriction to the complement of a compact set vanishes. We will denote by $H^0_c(Z, \mathcal{S})$ the space of compactly supported sections of $\mathcal{S}$ over Z.

If $f : Z_0 \to Z$ is a local homeomorphism and $\mathcal{S}$ is a sheaf on Z then we have a map $f_\star : H^0_c(Z_0, f^{-1}\mathcal{S}) \to H^0_c(Z, \mathcal{S})$ which we shall sometimes refer to as *integration along the fibers of* f. If f is a homeomorphism from an open set U to its image, then the restriction of $f_\star$ to $H^0_c(U, f^{-1}\mathcal{S})$ is the isomorphism onto $H^0_c(f(U), \mathcal{S})$ which defines $f^{-1}\mathcal{S}$.

We now proceed to identify $A^{\otimes(n+1)}/F_\infty A^{\otimes(n+1)}$ with sections of a sheaf.

Notations. In the following we shall denote for any defining neighborhood $U \subset G$ by φ_U the difeomorphism defined by U, that is $\varphi_U : d(U) \to r(U)$, $\varphi_U(g) = UgU^{-1}$. (Observe that for a defining neighborhood U the set UgU^{-1} consists of at most one element.)

Consider the Burghelea spaces

$$B^{(n)} = \{(g_0, g_1, \ldots, g_n) \in G^{n+1} : d(g_i) = r(g_{i+1})\}$$

By definition $B^{(n)}$ is the the complement of $\cup U_{n+1,i}$ in G^{n+1}.

We see that $B^{(0)} = \{g \in G : d(g) = r(g)\}$ is the space of loops in G.

The composition defines a continuous map

$$\pi : B^{(n)} \to B^{(0)},\ \pi(g_0, g_1, \ldots, g_n) = g_0 g_1 \ldots g_n$$

The spaces $B^{(n)}$ have a natural simplicial structure whose face maps are given by $d_i(g_0, g_1, \ldots, g_n) = (g_0, \ldots, g_i g_{i+1}, \ldots, g_n)$ for $0 \le i \le n-1$ and $d_n(g_0, g_1, \ldots, g_n) = (g_n g_0, \ldots, g_{n-1})$. The maps d_i for $0 \le i \le n-1$ satisfy $\pi \circ d_i = \pi$. Denote by $t : B_n \to B_n$ the cyclic permutation map $t(g_0, g_1, \ldots, g_n) = (g_n, g_0, \ldots, g_{n-1})$, then $\pi \circ d_n = \pi \circ t$.

Any point $x = (g_0, g_1, \ldots, g_n) \in B^{(n)}$ has a neighborhood in G^{n+1} homeomorphic to U^{n+1} for some small neighborhood U of $d(g_n)$ in $G^{(0)}$. This is due to the fact that G is etale. We specify this isomorphism in the following way. Choose $V_0, V_1, \ldots, V_n$ defining neighborhoods of $g_0, g_1, \ldots, g_n$ in G, satisfying $d(V_k) = r(V_{k+1})$ for $k = 0, \ldots, n-1$. (Recall that a defining neighborhood is a neighborhood on which d and r are local homeomorphisms.) Let $V = d(V_n)$ and $W_i = V_i V_{i+1} \ldots V_n V$. The map $\eta_i : V \to W_i$, $\eta_i(g) = V_i V_{i+1} \ldots V_n g (V_{i+1} \ldots V_n)^{-1}$, is a diffeomorphism. Denote by $\Psi : C_c^\infty(V^{n+1}) \to C_c^\infty(W_0 \times \ldots \times W_n)$ the induced map.

$$\Psi(f_0 \otimes \ldots \otimes f_n) = (f_0 \circ \eta_0^{-1}) \otimes (f_1 \circ \eta_1^{-1}) \otimes \ldots \otimes (f_n \circ \eta_n^{-1}) \tag{8}$$

The above observation suggests the following definition. Let $i_n : B^{(0)} \to G^{n+1}$ be the map $\gamma \to (d(\gamma), d(\gamma), \ldots, d(\gamma))$ and $\mathcal{C}^\infty(G^{n+1})$ be the sheaf of smooth functions on G^{n+1}. Define $\mathcal{F}_n^{(0)} = i_n^{-1}\mathcal{C}^\infty(G^{n+1})$.

The sequence of sheaves $\mathcal{F}_n^{(0)}$ has a simplicial structure which is a perturbation of the usual simplicial structure of the Bar complex of $\mathcal{C}^\infty(G)$ (see [14, 26, 27]). It is defined as follows.

The vector space $(\mathcal{C}^\infty(G)_v)^{\otimes(n+1)}$ is a subspace of the stalk $(\mathcal{F}_n^{(0)})_g$ of $\mathcal{F}_n^{(0)}$ at g, here $v = d(g)$. The simplicial structure has face morphisms d_i, $0 \le i \le n$ given by the usual formulae for $0 \le i \le n-1$:

$$d_i(a_0 \otimes \ldots \otimes a_n) = a_0 \otimes \ldots \otimes a_{i-1} \otimes a_i a_{i+1} \otimes a_{i+1} \otimes \ldots \otimes a_n \tag{9}$$

d_n is replaced by:

$$d_n(a_0 \otimes \ldots \otimes a_n) = \varphi_g^{-1}(a_n)a_0 \otimes \ldots \otimes a_{n-1} = (a_n \circ \varphi_g)a_0 \otimes \ldots \otimes a_{n-1} \quad (10)$$

The degeneration morphisms are unchanged, see [15, 30]. These formulae are direct generalizations of the corresponding formulae for crossed products see [19, 30]. Let $d = \sum(-1)^i d_i$ be the simplicial boundary.

The sheaves $\mathcal{F}_n^{(0)}$ are the pull-backs of some sheaves on $G^{(0)}$. The morphism d_n, however, is not a pull-back, and can only be defined on $B^{(0)}$.

Proposition. 4.1 *Define on $B^{(n)}$ the sheaves $\mathcal{F}_m^{(n)} = \pi^{-1}\mathcal{F}_m^{(0)}$. The map Ψ defined in equation* (8) *defines a vector space isomorphism*

$$\Psi : H_c^0(B^{(n)}, \mathcal{F}_n^{(n)}) \simeq A^{\otimes(n+1)}/F_\infty A^{\otimes(n+1)}.$$

Proof. This is a standard fact about sheaves which we review for the convenience of the reader.

Let f be a global section of $\mathcal{F}_n^{(n)}$ which vanishes outside a compact set K. For any $x = (g_0, \ldots, g_n) \in K$ there is a neighborhood V of $g = \pi(x) = g_0 \ldots g_n$ and a function $h \in C^\infty(V^{n+1})$ such that $f(x)$ is the germ of h at g. Replacing V, if necessary, with a smaller neighborhood, we can assume that we can find $W_0, \ldots, W_n$ as in the definition of Ψ (see equation 8 and above for definitions and notations) and hence $\Psi(h)$ is a smooth function in a neighborhood $W_0 \times \ldots \times W_n$ of x. We can find $V_{0,g} \subset V$ a relatively compact open neighborhood of g such that $f(v)$ is the germ of h at v for all $v \in V_{0,g}$. We can also assume that the closure of $V_{0,g}$ is contained in V. Since K is compact we can find a finite cover of it with open sets of the form $V_{0,g}$: $K \subset \cup_{i=1}^N V_{0,\gamma_i}$. Denote the corresponding functions on V_{0,γ_i}^{n+1} by h_i. The functions $\Psi(h_i)$ coincide on a common domain so they define a smooth function on a neighborhood of K in G^{n+1}. It coincides on a smaller neighborhood of K with the restriction of a compactly supported function on G^{n+1}. This shows that Ψ gives a well defined morphism $H_c^0(B^{(n)}, \mathcal{F}_n^{(n)}) \to A^{\otimes(n+1)}/F_\infty A^{\otimes(n+1)}$, denoted also Ψ, and that this morphism is surjective. It is obvious that it is injective. $\square$

The previous Proposition also explains the twisting in the definition of d_n for the sheaves $\mathcal{F}_n^{(0)}$. In order to separate the effect of the cohomology of the simplicial set $B^{(n)}$ and that of the cohomology in the sheaf direction we will need new constructions.

Consider for $n, m \geq 0$ the spaces

$$\begin{aligned} Z_{n,m} = \{&(k_0, \dots, k_n, \gamma, h_1, \dots, h_m) \in G^{n+m+2} : \\ &d(k_0) = \dots = d(k_n) = d(\gamma) = r(\gamma) = r(h_1), \\ &d(h_1) = r(h_2), \dots, d(h_{m-1}) = r(h_m)\} \end{aligned} \tag{11}$$

The spaces $Z_{n,m}$ have the structure of a bisimplicial set. The face maps for the first simplicial structure are given by

$$d_0^I(k_0, \dots, k_n, \gamma, h_1, h_2, \dots, h_m) = (k_0 h_1, \dots, k_n h_1, h_1^{-1}\gamma h_1, h_2, \dots, h_m) \tag{12}$$

$$d_i^I(k_0, \dots, k_n, \gamma, h_1, \dots, h_m) = (k_0, \dots, k_n, \gamma, h_1, h_2, \dots, h_i h_{i+1}, \dots, h_m) \tag{13}$$

for $1 \leq i \leq m-1$, and

$$d_m^I(k_0, \dots, k_n, \gamma, h_1, h_2, \dots, h_m) = (k_0, \dots, k_n, \gamma, h_1, h_2, \dots, h_{m-1}) \tag{14}$$

The face maps for the second simplicial structure delete arrows:

$$d_i^{II}(k_0, \dots, k_n, \gamma, h_1, \dots, h_m) = (k_0, \dots, \hat{k}_i, \dots, k_n, \gamma, h_1, \dots, h_m) \tag{15}$$

Define $p_1 : Z_{n,m} \to B^{(0)}$ by

$$p_1(k_0, \dots, k_n, \gamma, h_1, \dots, h_m) = \gamma \tag{16}$$

and consider on $Z_{n,m}$ the sheaves $\mathcal{A}_n = p_1^{-1}\mathcal{F}_n^{(0)}$. The above equations show that integration along the fibers of the structural face morphisms of the first simplicial structure induces a differential

$$d^I = \sum_{i=0}^{m} (-1)^i d_{i\star}^I : H_c^0(Z_{n,m}, \mathcal{A}_n) \to H_c^0(Z_{n,m-1}, \mathcal{A}_n) \tag{17}$$

$$\begin{aligned} d^I(a)(k_0, \dots, k_n, \gamma, h_1, h_2, \dots, h_{m-1}) = \\ &\sum_h \varphi_{h^{-1}}^{\otimes(n+1)}(a)(k_0 h^{-1}, \dots, k_n h^{-1}, h\gamma h^{-1}, h, h_1, h_2, \dots, h_{m-1}) \\ + &\sum_h \sum_{i=1}^{m-1} (-1)^i a(k_0, \dots, k_n, \gamma, h_1, h_2, \dots, h_i h^{-1}, h, h_{i+1}, \dots, h_{m-1}) \\ + &(-1)^m \sum_h a(k_0, \dots, k_n, \gamma, h_1, h_2, \dots, h_{m-1}, h) \end{aligned} \tag{18}$$

The define a second differential as follows. Consider the composition δ_i of the following morphisms

$$H_c^0(Z_{n,m},\mathcal{A}_n) \xrightarrow{d_{i\star}^{II}} H_c^0(Z_{n-1,m},\mathcal{A}_n) \xrightarrow{d_i} H_c^0(Z_{n-1,m},\mathcal{A}_{n-1})$$

the first one being an integration along the fibers and the second one is induced by the morphism of sheaves $d_i : \mathcal{F}_n^{(0)} \to \mathcal{F}_{n-1}^{(0)}$. Let $d^{II} = \sum_{i=0}^{n}(-1)^i\delta_i$. An explicit formula is given by

$$\begin{aligned} & d^{II}(a)(k_0,\ldots,k_{n-1},\gamma,h_1,\ldots,h_m) = \\ & \qquad \sum_k \sum_{i=0}^{n}(-1)^i d_i a(k_0,\ldots,k_{i-1},k,k_i,\ldots,k_{n-1},\gamma,h_1,\ldots,h_m) \end{aligned} \tag{19}$$

Consider the maps $\nu_{n,m} : Z_{n,m} \to B^{(n)}$ defined by (compare to [30])

$$\nu_{n,m}(k_0,\ldots,k_n,\gamma,h_1,h_2,\ldots,h_m) = (k_n\gamma k_0^{-1}, k_0k_1^{-1}, k_1k_2^{-1},\ldots,k_{n-1}k_n^{-1}) \tag{20}$$

These maps satisfy

$$\nu_{n,m}\circ d_i^I = \nu_{n,m+1} \text{ for } 0\le i\le n \tag{21}$$

and

$$\nu_{n,m}\circ d_i^{II} = \nu_{n+1,m} \text{ for } 0\le i< n \tag{22}$$

The following Proposition summarizes the properties of the previous constructions.

Proposition. 4.2 *The sheaves and maps defined above satisfy the following properties:*

(1) $(d^I)^2=0$, $(d^{II})^2=0$ *and* $d^Id^{II}=d^{II}d^I$.

(2) There exists an augmentation morphism

$$\epsilon : H_c^0(Z_{n,0},\mathcal{A}_n) \to A^{\otimes(n+1)}/F_\infty A^{\otimes(n+1)}$$

defined using integration along the fibers of $\nu_{n,0}$ *such that the augmented complex*

$$A^{\otimes(n+1)}/F_\infty A^{\otimes(n+1)} \xleftarrow{\epsilon} H_c^0(Z_{n,0},\mathcal{A}_n) \xleftarrow{d^I} H_c^0(Z_{n,1},\mathcal{A}_n) \xleftarrow{d^I} \ldots \tag{23}$$

is acyclic.

(3) The morphism ϵ *commutes with the differentials, that is* $\epsilon d^{II} = b\epsilon$, *so* ϵ *induces a morphism of complexes from the total complex defined by the double complex* $(H_c^0(Z_{n,m},\mathcal{A}_n),d^I,d^{II})$ *to the complex* $(A^{\otimes(n+1)}/F_\infty A^{\otimes(n+1)},b)$.

Proof. The proof of (1) is a straightforward computation based on the definitions and equations (12)-(19).

Let a be a smooth function in a neighborhood of v and $d(g) = d(g') = v$, we shall denote by gag'^{-1} the germ at gg'^{-1} of the function $b(h) = a(V^{-1}hV')$ where V and V' are defining neighborhoods of g and, respectively, g'. We define $\epsilon_0 : (\mathcal{A}_n)_z \to \mathcal{C}^\infty_{\nu_{n,0}(z)}$ by

$$\epsilon_0(a_0 \otimes \ldots \otimes a_n) = (k_n\gamma a_0 k_0^{-1}) \otimes (k_0 a_1 k_1^{-1}) \otimes \ldots \otimes (k_{n-1} a_n k_n^{-1}) \tag{24}$$

where $z = (k_0, k_1, \ldots, k_n)$. The final formula for ϵ is obtained by integration along the fibers of $\nu_{n,0}$. This definition immediately gives $\epsilon d^{II} = b\epsilon$.

Define the sheaves $\mathcal{A}'_n = \nu^{-1}_{n,m}\mathcal{F}^{(n)}_n$ on $Z_{n,m}$. Let W be a defining neighborhood of k_n^{-1} for $z = (k_0, \ldots, k_n, \gamma, h_1, \ldots, h_m)$, then $a \to a \circ (\varphi_W \times \ldots \times \varphi_W)$ defines an isomorphism of stalks $(\mathcal{A}_n)_z \simeq (\mathcal{A}'_n)_z$. This shows that $(\mathcal{A}_n) \simeq \mathcal{A}'_n$. We use this isomorphism to define a differential $d^I : H^0_c(Z_{n,m}, \mathcal{A}'_n) \to H^0_c(Z_{n,m-1}, \mathcal{A}'_n)$.

There exists a section s of $\nu_{n,0}$ (i.e. $\nu_{n,0} \circ s = id$) defined by

$$s(g_0, \ldots, g_n) = (g_1 \ldots g_n, g_2 \ldots g_n, \ldots, g_{n-1}g_n, g_n, d(g_n), g_0 \ldots g_n)$$

Since $s_\star$ satisfyies $\nu_{n,0\star}s_\star = id$ we see that $s_\star$ is well defined (i.e. $s^{-1}\mathcal{A}'_n = \mathcal{F}^{(n)}_n$) and ϵ is onto. In order to prove that the augmented d^I complex is acyclic it is enough to find $\sigma_m : H^0_c(Z_{n,m}, \mathcal{A}'_n) \to H^0_c(Z_{n,m+1}, \mathcal{A}'_n)$ satisfying $d^I\sigma_m + \sigma_{m-1}d^I = 1$ for any $m \geq 0$ where $\sigma_{-1} = s_\star$. Indeed $\sigma_m = s_{m\star}$ where

$$\begin{aligned} &s_m(k_0, k_1, \ldots, k_n, \gamma, h_1, h_2, \ldots, h_m) = \\ &\quad (k_0k_n^{-1}, k_1k_n^{-1}, \ldots, k_{n-1}k_n^{-1}, r(k_n), k_n\gamma k_n^{-1}, k_n, h_1, h_2, \ldots, h_m) \end{aligned}$$

satisfies this condition. We observe that $s_{m\star}$ is well defined since we have $s_m^{-1}\mathcal{A}'_n = \mathcal{A}'_n$. $\square$

Next we proceed to replace the spaces $Z_{n,m}$ with other spaces on which there is no twisting necessary in the definition of the differentials.

Let Y_m be defined by $Y_0 = B^{(0)}$ and

$$\begin{aligned} Y_m = \{(\gamma, h_1, h_2, \ldots, h_m) : d(\gamma) = r(\gamma) = r(h_1), \\ d(h_1) = r(h_2), \ldots, d(h_{m-1}) = r(h_m)\}, \quad m > 0 \end{aligned} \tag{25}$$

The spaces Y_m have the structure of a simplicial space with faces defined such that the projection

$$p_2 : Z_{n,m} \to Y_m, \;\; p_2(k_0, \ldots, k_n, \gamma, h_1, \ldots, h_m) = (\gamma, h_1, \ldots, h_m) \tag{26}$$

is a simplicial map for the first simplicial structure on $Z_{n,m}$'s. (This gives $d_0(\gamma, h_1, \ldots, h_m) = (h_0^{-1}\gamma h_0, h_1, \ldots, h_m)$, etc.) The degeneracies insert units.

Consider the map

$$p : Y_m \to B^{(0)}, \quad p(\gamma, h_1, \ldots, h_m) = \gamma \tag{27}$$

so $p \circ p_2 = p_1$, see equations (16) and (26). Define a sequence of sheaves on Y_m by $\mathcal{B}_n = p^{-1}\mathcal{F}_n^{(0)}$.

The vector spaces $H_c^0(Y_m, \mathcal{B}_n)$ form a double complex with respect to the differentials d^I induced by integration along the fibers of the face maps, and d^{II} induced by the simplicial structure of the sheaves $\mathcal{F}_n^{(0)}$.

$$\begin{aligned}
d^I(a)(\gamma, h_1, h_2, \ldots, h_{m-1}) = \\
&\sum_h \varphi_{h^{-1}}^{\otimes(n+1)}(a)(h\gamma h^{-1}, h, h_1, h_2, \ldots, h_{m-1}) \\
+ \quad &\sum_h \sum_{i=1}^{m-1} (-1)^i a(\gamma, h_1, h_2, \ldots, h_i h^{-1}, h, h_{i+1}, \ldots, h_{m-1}) \\
+ \quad &(-1)^m \sum_h a(\gamma, h_1, h_2, \ldots, h_{m-1}, h)
\end{aligned} \tag{28}$$

$$d^{II}(a)(\gamma, h_1, \ldots, h_m) = \sum_{i=0}^{n} (-1)^i d_i a(\gamma, h_1, \ldots, h_m) \tag{29}$$

This double complex corresponds to a bisimplicial structure on the vector spaces $H_c^0(Y_m, \mathcal{B}_n)$.

The definitions of the two differential d^I and d^{II} above are very closely related to the differentials of the bicomplex $(H_c^0(Z_{n,m}, \mathcal{A}_n), d^I, d^{II})$. In fact the only difference is that for the first complex the second differential d^{II} is defined *only* in terms of morphisms of sheaves, whereas for the second complex in order to define d^{II} we need to integrate along fibers.

The following Proposition explains the relation between these two bicomplexes and gives a complex better suited for the computation of the Hochschild homology of $C_c^\infty(G)$.

Proposition. 4.3 *We have $\mathcal{A}_n = p_2^{-1}\mathcal{B}_n$ so there are natural maps*

$$p_{2\star} : H_c^0(Z_{n,m}, \mathcal{A}_n) \to H_c^0(Y_m, \mathcal{B}_n) \tag{30}$$

which commute with d^I and d^{II}. The map $p_{2\star}$ defines a quasi-isomorphism for the d^{II} differential, and hence also for the total complexes.

We obtain that the homology of the total complex associated to the bicomplex $(H_c^0(Y_m, \mathcal{B}_n), d^I, d^{II})$ is naturally isomorphic to $HH_\star(C_c^\infty(G))$.

Proof. The first statement was explained in our discussion about integration along fibers.

The relations $p_{2\star}d^I = d^I p_{2\star}$ and $p_{2\star}d^{II} = d^{II}p_{2\star}$ follow from the definitions. We proceed now to prove that the morphisms $p_{2\star}$ induce an isomorphism for the d^{II} homology. Define on Y_m the sheaf $\mathcal{B}'_n$ whose stalk at $y \in Y_m$ is the vector space generated by $p_2^{-1}(y)$, where p_2 is the projection $p_2 : Z_{n,m} \to Y_m$. The locally constant sections correspond to locally constant functions f on open subsets of $Z_{n,m}$ such that $\mathrm{supp}(f)$ intersects the fibers of p_2 in compact sets. The main feature of $\mathcal{B}'_n$ is that we have an isomorphism $\alpha : H_c^0(Z_{n,m}, \mathcal{A}_n) \simeq H_c^0(Y_m, \mathcal{B}_n \otimes \mathcal{B}'_n)$. Both $\mathcal{B}_n$ and $\mathcal{B}'_n$ have natural simplicial structures which induce a simplicial structure on $\mathcal{B}_n \otimes \mathcal{B}'_n$. This simplicial structure defines a complex $(H_c^0(Y_m, \mathcal{B}_n \otimes \mathcal{B}'_n), d)$ such that α is a morphism of complexes.

Let $\underline{\mathbb{C}}$ denote the constant sheaf of complex numbers. The map of sheaves $\mathcal{B}'_n \to \underline{\mathbb{C}}$ which sends the canonical basis of each stalk to $1 \in \mathbb{C}$ defines a morphism $\beta : H_c^0(Y_m, \mathcal{B}_n \otimes \mathcal{B}'_n) \to H_c^0(Y_m, \mathcal{B}_n \otimes \underline{\mathbb{C}}) \simeq H_c^0(Y_m, \mathcal{B}_n)$. Using the definitions we see that $p_{2\star} = \beta \circ \alpha$. This shows that it is enough to prove that β is a quasi-isomorphim.

Both $\mathcal{B}_n \otimes \mathcal{B}'_n$ and $\mathcal{B}_n$ are fine sheaves ([39], page 330) so in order to prove that β induces a quasi-isomorphism of complexes it is enough to prove that we have a quasi-isomorphism of the complexes of stalks $(\mathcal{B}_n \otimes \mathcal{B}'_n)_y \to (\mathcal{B}_n)_y$ (loc. cit. Theorem 9, page 335). This follows from the fact that the homology of the complex of sheaves $\mathcal{B}'$ is $\underline{\mathbb{C}}$ and from the Eilenberg-Zilber theorem [27]. □

5 The main results

In this section we study the homology of the complex of stalks $(\mathcal{F}_n^{(0)}, d)$ defined in the previous section (equations (9), (10) and below). It turns out that the homology of this complex is the homology of a Koszul complex, which we review below.

Let R be a commutative complex algebra and $f_1, \ldots, f_n \in R$. Fix a basis

$e_1, \dots, e_n$ of $\mathbb{C}^n$ and define $K_j = R \otimes \wedge^j \mathbb{C}^n$, and $\partial : K_j \to K_{j-1}$ by

$$\partial(a \otimes e_{i_1} \wedge \dots \wedge e_{i_j}) = \sum_{k=1}^{j} (-1)^k f_{i_k} a \otimes e_{i_1} \wedge \dots \wedge \hat{e}_{i_k} \wedge \dots \wedge e_{i_j}$$

The complex $\mathrm{K}_\star(f_1, \dots, f_n) = (K_j, \partial)$ is called the Koszul complex associated to the sequence $(f_1, \dots, f_n)$.

Consider now a diffeomorphism $\varphi : U \to V$, $U, V \subset G$, such that $\varphi(v) = v$. It induces an automorphism of the ring $\mathcal{C}_v^\infty$ of germs of smooth functions on G at v: $\varphi(a) = a \circ \varphi^{-1}$. Choose coordinate functions $x_1, \dots, x_n$ on U ($n = \dim U$), and let f_i denote the germ of $x_i - x_i \circ \varphi$.

Lemma. 5.1 *Denote by S_p the group of permutations of the set $\{e_1, \dots, e_p\}$, and let $\kappa : K_p(f_1, \dots, f_p) \to (\mathcal{F}_p^{(0)})_\gamma$ be defined by*

$$\begin{aligned}\kappa(a \otimes e_{i_1} \wedge \dots \wedge e_{i_p})(g_0, g_1, \dots, g_p) = \\ a(g_0) \sum_{\sigma \in S_p} \mathrm{sign}(\sigma) x_{i_{\sigma(1)}}(g_1) x_{i_{\sigma(2)}}(g_2) \dots x_{i_{\sigma(p)}}(g_p)\end{aligned} \tag{31}$$

where x_i and f_i are defined as above, $\varphi = \varphi_W$ for some defining neighborhood W of $\gamma \in G$, and $\varphi_\gamma =$ the germ of the diffeomorphism φ_W. Then κ is a quasi-isomorphism.

Proof. The proof is a standard argument in homological algebra. We review this argument for the convenience of the reader.

We can assume that U is the open unit ball in $\mathbb{R}^n$. Let $R = C^\infty(\overline{U})$, then $\mathrm{K}_\star(x_1 \otimes 1 - 1 \otimes x_1, \dots, x_n \otimes 1 - 1 \otimes x_n)$ is a resolution of R with projective $R \otimes R$ modules (everything is in the topological sense, in particular $R \otimes R$ is the projective tensor product, and a projective module is a direct summand of the module $R \otimes E \otimes R$ for some complete locally convex space E). The Bar complex $(R^{\otimes(n+2)}, b')$ is also a strong projective resolution of R ("strong" means that b' is chain homotopic to 0, an assumption needed in the topological case since not every subspace is complemented) so the two complexes are chain equivalent as complexes of $R \otimes R$-modules [27]. This property is preserved by localization.

The statement then follows using Lemma 3.1, localization at the maximal ideal corresponding to $v = d(\gamma) = r(\gamma)$ and tensoring with $M(\gamma)$. Here $M(\gamma)$ is the twisted bimodule which has $\mathcal{C}^\infty(G)_v$ as underlying space but the action of $\mathcal{C}^\infty(G)_v$ is changed to $a \cdot m \cdot a_1 = \varphi_\gamma^{-1}(a) m a_1 = (a \circ \varphi_\gamma) m a_1$. $\square$

Denote by $\Omega^p(X)_g$ the germs of smooth p-forms on a manifold X at g.

Lemma. 5.2 *Let $\gamma, M(\gamma), v = d(\gamma) = r(\gamma), \varphi : U \to V, \varphi_g$ be as above. Suppose that $X = \{x \in G^{(0)} : \varphi(x) = x\}$ is a smooth submanifold of V, and that*

$$\cup_{p>0}\{\xi \in T_xG^{(0)} : (d\varphi_g - 1)^p\xi = 0\} = T_xX \tag{32}$$

(1) Consider the restriction $r_X : \mathcal{C}^\infty(G)_v \to \mathcal{C}^\infty(X)_v$. Then r_X induces a quasi-isomorphism $Bar(\mathcal{C}^\infty(G)_v, M(\gamma)) \to Bar(\mathcal{C}^\infty(X)_v, \mathcal{C}^\infty(X)_v)$.

(2) We have $H_p((\mathcal{F}_\star^{(0)})_\gamma, d) \simeq H_p(\mathcal{C}^\infty(G)_v, M(\gamma)) \simeq \Omega^p(X)_\gamma$, and the isomorphism is induced by

$$a_0 \otimes \ldots \otimes a_p \to i_X^\star(a_0 da_1 \ldots da_p) \tag{33}$$

where d is the DeRham differential and $i_X : X \to G^{(0)}$ is the inclusion.

(3) The condition (32) *is satisfied for any γ of finite order, and more generally if γ preserves a metric.*

Proof. Let $x_1, \ldots, x_k$ be a normal coordinate system on a neighborhood of v defined using a connection on V and such that $x_1, \ldots, x_l$ correspond to the eigenvalue 1 on $d\varphi$ and the other ones correspond to a $d\varphi$ invariant complement of T_vX in T_vV. This implies using (32) that $\varphi(x_j) - x_j$ for $j = l+1, \ldots, k$ form a regular sequence [21] and hence the homology of the Koszul complex of the sequence $\varphi(x_j) - x_j$ for $1 \le j \le k$ in the ring $\mathcal{C}^\infty(G)_v$ is the same as the homology of the Koszul complex of the sequence $(\varphi(x_j) - x_j) + I$ for $1 \le j \le l$ in the ring $\mathcal{C}^\infty(G)_v/I$ where I is the ideal generated by the sequence $\varphi(x_j) - x_j$ for $j = l+1, \ldots, k$ (see [21]). Since the first complex computes the twisted Hochschild homoology $H_\star(\mathcal{C}^\infty(G)_v, M(\gamma))$ and the second complex computes the Hochschild homology of $\mathcal{C}^\infty(X)_v \simeq \mathcal{C}^\infty(G)_v/I$ this proves part (1).

The first isomorphism in part (2) follows from Lemma 5.1, and the second isomorphism follows from part (1).

The last statement is a well known fact which is proved using the Levi-Civita connection for an invariant metric. □

Let $\mathcal{O}$ be a component of $B^{(0)}$ and define $Y_m(\mathcal{O}) = p^{-1}(\mathcal{O})$ where p is as in equation 27. The spaces $Y_m(\mathcal{O})$ form a simplicial space $Y(\mathcal{O})$. We define $\Omega_c^{p,\star}(Y(\mathcal{O}))$, the *compactly supported p-forms* on $Y(\mathcal{O})$, as the homology of the complex $(\Omega_c^p(Y_q(\mathcal{O})), d)$ were d is the simplicial differential and is defined in the usual way in terms of the face maps: $d = \sum(-1)^i d_{i\star}$.

Theorem. 5.3 *Let $\mathcal{O}$ be an open-closed invariant subset of $B^{(0)}$. Assume that any $g \in \mathcal{O}$ satisfies the condition (32) of the previous Lemma, then*

$$HH_n(C_c^\infty(G))_{\mathcal{O}} \simeq \bigoplus_{p+q=n} \Omega_c^{p,q}(Y(\mathcal{O}))$$

Proof. We use Lemma 5.1 and the fact that the restriction map (33) $H_c^0(Y_m, \mathcal{B}_n) \to \Omega_c^n(Y_m)$ defines a quasi-isomorphism $(H_c^0(Y_m, \mathcal{B}_n), d^I, d^{II}) \to (\Omega_c^n(Y_m), d^I, 0)$. $\square$

In order to get a similar description of the cyclic cohomology we have to introduce new versions of the spaces $Z_{n,m}$ and Y_m. The problem is as usual that the action of the cyclic group in the cyclic structure on $A^{\otimes(n+1)}$ does not lift to an action on the sheaves on $Z_{n,m}$ [33, 30]. As in the case of group algebras and crossed products [11, 33, 30] we have to treat the torsion and torsion free elements separately.

Recall the map $p_1 : Z_{n,m} \to B^{(0)}$, $p_1(k_0, \dots, k_n, \gamma, h_1, \dots, h_m) = \gamma$, and that we have defined $B_{fin}^{(0)}$ as the subset of finite order elements of G (see Proposition 2.8). We consider on $p_1^{-1}(B_{fin}^{(0)})$ the equivalence relation

$$(k_0, \dots, k_n, \gamma, h_1, \dots, h_m) \equiv (k_0\gamma^l, \dots, k_n\gamma^l, \gamma, h_1\gamma_1^{l_1}, \dots, h_m\gamma_m^{l_m})$$

where $\gamma_j = (h_1 \dots h_j)^{-1}\gamma(h_1 \dots h_j)$ and $l, l_1, \dots, l_m$ are arbitrary nonnegative integers. The space $\tilde{Z}_{n,m}$ is defined as the quotient of $p_1^{-1}(B_{fin}^{(0)})$ with respect to this equivalence relation. Define $\tilde{Y}_m$ similarly to be the set of equivalence classes of elements of $Y_m(B_{fin}^{(0)})$ with respect to the action of $\mathbb{Z}^m$ defined by the action of γ on each component. We get a projection map $\tilde{p}_2 : \tilde{Z}_{n,m} \to \tilde{Y}_m$.

Observe first that the cyclic group generated by φ_γ acts on the stalks at γ of the complex $(\mathcal{F}_n^{(0)}, d)$ by the formula

$$a_0 \otimes \dots \otimes a_n \to \varphi_\gamma(a_0) \otimes \varphi_\gamma(a_1) \otimes \dots \otimes \varphi_\gamma(a_n)$$

and that this action induces a trivial action on homology due to Lemma 5.2 (at least in the case γ a torsion element). The action of this group lifts to the pull-back sheaves $\mathcal{A}_n$ and $\mathcal{B}_n$. Define $\mathcal{A}_n^{inv}$ and $\mathcal{B}_n^{inv}$ to be the elements invariant under these actions (on each stalk acts a different group).

The formulae (12)-(15) immediately generalize to $\tilde{Z}_{n,m}$. In order to obtain formulae generalizing d^I and d^{II} defined in equations (17) and (18) we have to replace $\mathcal{A}_n$ by $\mathcal{A}_n^{inv}$ and to let h and k in the sums defining d^I and d^{II} run through a complete set of representatives of the cosets with respect to the action of the cyclic group generated by γ or its conjugates.

Recall that $B_{fin}^{(0)}$ is an invariant closed-open set, Proposition 2.8. The idempotent $e_{B_{fin}^{(0)}}$ is defined before Proposition 3.3.

The map $\nu_{n,0} : Z_{n,m} \to B^{(n)}$ defined in equation (20) factors to a map $\tilde{\nu}_{n,0} : \tilde{Z}_{n,m} \to B^{(n)}$.

Propositions 4.2 and 4.3 and their proofs generalize in the following way.

Proposition. 5.4 *The sheaves $\mathcal{A}_n^{inv}$ and the maps defined above satisfy:*

(1) $(d^I)^2 = 0$, $(d^{II})^2 = 0$ and $d^I d^{II} = d^{II} d^I$.

(2) The map $\tilde{p}_{2\star} : H_c^0(\tilde{Z}_{n,m}, \mathcal{A}_n^{inv}) \to H_c^0(\tilde{Y}_m, \mathcal{B}_n^{inv})$ commutes with d^I and d^{II} and defines a quasi-isomorphism.

(3) There exists an edge morphism

$$\tilde{\epsilon} : H_c^0(\tilde{Z}_{n,0}, \mathcal{A}_n^{inv}) \to e_{B_{fin}^{(0)}}(A^{\otimes(n+1)}/F_\infty A^{\otimes(n+1)})$$

which is defined using integration along the fibers of $\tilde{\nu}_{n,0}$ and which induces a quasi-isomorphism from the total complex of $(H_c^0(\tilde{Z}_{n,0}, \mathcal{A}_n^{inv}), d^I, d^{II})$ to the complex $(e_{B_{fin}^{(0)}}(A^{\otimes(n+1)}/F_\infty A^{\otimes(n+1)}), b)$. In particular $\epsilon d^{II} = b\epsilon$.

Proof. The proof of (1) is a straightforward computation.

For (2) the only difference with the proof of Proposition 4.3 is that this time the stalks $(\mathcal{B}_n')_y$ have the same homology as that of a finite cyclic group (of order = the order of $p(y)$). Since we are working over a field of characteristic 0 the homology of the stalks $(\mathcal{B}_n')_y$ vanishes [27] in positive degree, and hence $\mathcal{B}_n' \to \underline{\mathbb{C}}$ is a quasi-isomorphism.

In order to define $\tilde{\epsilon}$ we first observe that in the formula (24) defining ϵ we can replace $Z_{n,m}$ with $\tilde{Z}_{n,m}$ and $\mathcal{A}_n$ with $\mathcal{A}_n^{inv}$. The Proposition 2.8 shows that there exists a decomposition of $B_{fin}^{(0)} = \cup \mathcal{O}_k$ where $\mathcal{O}_k$ is the open-closed invariant set of elements of order k. Let $e_k = e_{\mathcal{O}_k}$ be the idempotents defined before Proposition 3.3.

For each k the map $e_k H^0_c(Y_m, \mathcal{B}_n) = H^0_c(Y_m(\mathcal{O}_k), \mathcal{B}_n) \to e_k H^0_c(Y_m, \mathcal{B}^{inv}_n)$ is the projection map onto the invariants of the cyclic group of order k (in each stalk). Let $e_{fin} = e_{B^{(0)}_{fin}} = \sum_{k\geq 0} e_k$. Since, by Lemma 5.2 the action of this group on the homology of the stalks is trivial and all the involved sheaves are fine we get that the projection map on the space of invariants $(e_{fin}H^0_c(Y_m, \mathcal{B}_n), d^I, d^{II}) \to (e_{fin}H^0_c(Y_m, \mathcal{B}^{inv}_n), d^I, d^{II})$ is a quasi-isomorphism which commutes with $p_{2\star}$. This shows that we also have a quasi-isomorphism $(e_{fin}H^0_c(Z_{n,m}, \mathcal{A}_n), d^I, d^{II}) \to (e_{fin}H^0_c(Z_{n,m}, \mathcal{A}^{inv}_n), d^I, d^{II})$, and hence the homology of $(e_{fin}(A^{\otimes(n+1)}/F_\infty A^{\otimes(n+1)}), b)$ is the same as that of the subcomplex $\epsilon(e_{fin}H^0_c(Z_{n,0}, \mathcal{A}^{inv}_n))$. Since ϵ and $\tilde{\epsilon}$ have the same image and the complex $(H^0_c(\tilde{Z}_{n,m}, \mathcal{A}^{inv}_n), d^I)$ is a resolution of $\epsilon(H^0_c(Z_{n,0}, \mathcal{A}^{inv}_n))$ we get (3). $\square$

The extra piece of information needed to generalize theorem 5.3 to cyclic cohomology is contained in the following Lemma.

Lemma. 5.5 *There exists an action t of the generator of $\mathbb{Z}_{n+1}$ on the vector spaces $H^0_c(\tilde{Z}_{n,m}, \mathcal{A}^{inv}_n)$ and $H^0_c(\tilde{Y}_m, \mathcal{B}^{inv}_n)$ which is compatible with the maps ϵ, d^I and $p_2\star$. Moreover it defines a structure of cyclic vector spaces on these spaces which commutes with d^I for the last two complexes. The cyclic structure on $A^{\otimes(n+1)}/F_\infty A^{\otimes(n+1)}$ is obtained using the usual cyclic permutation and the simplicial structure of the sheaves $\mathcal{F}^{(0)}_n$.*

Cyclic vector spaces were introduced by Connes [14]. The sheaves $(\mathcal{F}^{(0)}_n)^{inv}$ have a cyclic structure very similar to the usual structure [30], see also below.

This Lemma is very closely related to the treatment of crossed products by compact Lie groups in [33].

Proof. The action of t on $A^{\otimes(n+1)}/F_\infty A^{\otimes(n+1)}$ is defined by the usual formula $t(a_0 \otimes \ldots \otimes a_n) = (-1)^n a_n \otimes a_0 \otimes \ldots \otimes a_{n-1}$ and by a similar but twisted formula on $(\mathcal{F}^{(0)}_n)_\gamma$ [30]: $t_\gamma(a_0 \otimes \ldots \otimes a_n) = (-1)^n (a_n \circ \varphi_\gamma) \otimes a_0 \otimes \ldots \otimes a_{n-1}$. On $H^0_c(\tilde{Z}_{n,m}, \mathcal{A}^{inv}_n)$ it is given by

$$t(a)(k_0, \ldots, k_n, \gamma, h_1, \ldots, h_m) = t_\gamma(a(k_n\gamma, k_0, \ldots, k_{n-1}, \gamma, h_1, \ldots, h_m))$$

and similarly on $H^0_c(\tilde{Y}_m, \mathcal{B}^{inv}_n)$.

The compatibility with d^I and $p_{2\star}$ follows from the above formulae. The compatibility with ϵ is the same computation as in [30], actually it is for this reason that we introduced the twisted action t_γ. $\square$

Observe that the above action on $\mathcal{A}_n^{inv}$ does extend to an action on $\mathcal{A}_n$ which is however not cyclic. Also it is only on $A^{\otimes(n+1)}/F_\infty A^{\otimes(n+1)}$ that a natural cyclic structure can be defined and not on $A^{\otimes(n+1)}$, in general. This shows the power of the sheaf approach.

Let $\mathcal{O}$ be an invariant open-closed subset of $B_{fin}^{(0)}$. Define for any n the double complex $\mathcal{D}^{(n)}(\mathcal{O})$ by $\mathcal{D}_{p,q}^{(n)} = (\Omega_c^{n-q}(\tilde{Y}_p(\mathcal{O})), d^I, d_{DR})$, where d_{DR} is the DeRham differential, $n, p, q \geq 0$. There exists a morphism σ of bicomplexes $\mathcal{D}^{(n+1)}[0,1] \to \mathcal{D}^{(n)}$ which identifies $\mathcal{D}_{p,q+1}^{(n+1)}$ to $\mathcal{D}_{p,q}^{(n)}$.

Recall [15, 26] that there is a periodicity morphism $S : HC_n \to HC_{n-2}$, defined at the level of complexes. The homology of the inverse limit complex is denoted HC^{per} and called *periodic cyclic* homology, it is a $\mathbb{Z}_2$-graded theory. The groups HC^{per} fit into a $\lim^1$ exact sequence:

$$0 \to \lim_{\leftarrow}{}^1 HC_{i+2m+1} \to HC_i^{per} \to \lim_{\leftarrow} HC_{i+2m} \to 0$$

([25], page 160).

Theorem. 5.6 *Suppose G is a smooth etale separated groupoid. Let $\mathcal{O}$ be an invariant closed-open subset of $B_{fin}^{(0)}$, and $\mathcal{D}^{(n)}(\mathcal{O})$ be the double complex defined above. We have using the notations introduced above*

(1) $e_{\mathcal{O}} HC_n(C_c^\infty(G)) \simeq \oplus_{m=0}^n H_{n-m}(\mathcal{D}^{(m)}(\mathcal{O}))$.

(2) The periodicity morphism $S : e_{\mathcal{O}} HC_n(C_c^\infty(G)) \to e_{\mathcal{O}} HC_{n-2}(C_c^\infty(G))$ is induced by $\sigma : H_{n-m+1}(\mathcal{D}^{(m+1)}(\mathcal{O})) \to H_{n-m}(\mathcal{D}^{(m)}(\mathcal{O}))$ defined above.

(3) $e_{\mathcal{O}} HC_n^{per}(C_c^\infty(G)) \simeq \prod_{m \in n+2\mathbb{Z}} H_m(\mathcal{D}^{(N)})$ if N is even $\geq \dim G$.

Proof. Let $V = \mathbb{C}[u, u^{-1}]/u^{-1}\mathbb{C}[u^{-1}]$, u being considered of degree 2. Using Proposition 5.4 and Lemma 5.5 we see that the cyclic homology group $e_{\mathcal{O}} HC_n(C_c^\infty(G))$ is computed by the triple complex

$$(H_c^0(\tilde{Y}_m(\mathcal{O}), \mathcal{B}_n^{inv}) \otimes V, d^I, d^{II}, Bu^{-1})$$

We use the formalism in [26]. Also recall that d^I is induced by integration along the face maps, d^{II} is the Hochschild differential and acts along the stalks, and B is as defined in [15, 26] by getting rid of the acyclic columns involving b'. Then Lemma 5.2 shows that there exists a (degree preserving) quasi-isomorphism of triple complexes where the total degree is $n + p + q$:

$$(H_c^0(\tilde{Y}_p, \mathcal{B}_{n-q}^{inv}) u^q, d^I, d^{II}, Bu^{-1}) \to (\mathcal{D}_{p,q}^{(n)}, d^I, 0, d_{DR})$$

from which (1) and (2) follow. We also obtain that $\mathcal{D}^{(n)}(\mathcal{O}) = \mathcal{D}^{(N)}(\mathcal{O})$ (up to a shift) for $n \geq N$, and hence

$$e_{\mathcal{O}} HC_n^{per}(C_c^\infty(G)) \simeq \lim_{\leftarrow i} \bigoplus_{m=-i}^{n+i} HC_{n+N+2m}(\mathcal{D}^{(N)}(\mathcal{O}))$$

which proves (3). $\square$

For an arbitrary manifold M we denote by $\mathfrak{o}$ the orientation local coefficient system (sheaf) [43], and by $H_p(M, \mathfrak{o})$ the homology groups with values in this coefficient system. We consider complex orientations, so the stalk of $\mathfrak{o}$ at any point is $\mathbb{C}$. This sheaf is functorial for local homeomorphisms, and lifts to the geometric realization $\mathcal{Y}(\mathcal{O})$ of the simplicial space $Y_m(\mathcal{O})$.

Consider for each m the sphere bundle $SY_m(\mathcal{O})$ and the ball bundle $BY_m(\mathcal{O})$ defined such that $BY_m(\mathcal{O}) - SY_m(\mathcal{O}) = TY_m(\mathcal{O})$. The simplicial realization of these spaces will be denoted by $S\mathcal{Y}(\mathcal{O})$ and $B\mathcal{Y}(\mathcal{O})$ respectively.

Remark For a groupoid G which is Morita equivalent to a manifold the isomorphism in (3) is the Generalized Mayer-Vietoris principle in [8]. In particular it gives the equality between the de Rham and the Cech cohomology groups with values in the complex locally constant sheaf $\underline{\mathbb{C}}$.

The following theorem gives a more concrete description of the periodic cyclic homology of $C_c^\infty(G)$.

Theorem. 5.7 *(1) $e_{\mathcal{O}} HC_n^{per}(C_c^\infty(G)) \simeq \prod_{m \in n+2\mathbb{Z}} H_m(B\mathcal{Y}(\mathcal{O}), S\mathcal{Y}(\mathcal{O})) \otimes \mathbb{C}$, where the last groups denote the usual (simplicial) homology.*

(2) Suppose all the components of $\mathcal{O}$ have the same dimension N (mod 2), then $e_{\mathcal{O}} HC_n^{per}(C_c^\infty(G)) \simeq \prod_{m \in n+N+2\mathbb{Z}} H_m(\mathcal{Y}(\mathcal{O}), \mathfrak{o})$.

Proof. The proof is to adapt the Poincaré duality to the case of simplicial manifolds and their geometric realization.

Denote, for a dimension N manifold M, by $\Omega_q(M, \mathfrak{o})$ the space of compactly supported q-currents with values in $\mathfrak{o}$. By definition $\Omega_q(M, \mathfrak{o})$ is the dual of the space $\Omega^q(M, \mathfrak{o})$ of smooth q-forms with coefficients in the canonically flat bundle $\mathfrak{o}$ and with arbitrary support. The space $\Omega_q(M, \mathfrak{o})$ consists of compactly supported distributional sections of $\wedge^q TM \otimes \mathfrak{o} \otimes |\Lambda|$, where $|\Lambda|$

is the bundle of 1-densities. The integration of densities defines a diffeo-invariant map $\Omega_c^q(M) \to \Omega_{N-q}(M, \mathfrak{o})$, commuting with the natural de Rham differential on currents. This gives a chain morphism $(\mathcal{D}_{p,q}^{(N)}, d^I, d_{DR}) \to (\Omega_q(Y_p, \mathfrak{o}), d^I, d_{DR})$. Moreover this map is a quasi-isomorphism due to the ordinary Poincaré isomorphism. The rest of (2) follows from a theorem of Bott [6]. (1) follows from (2). $\square$

We give some applications.

Corollary. 5.8 (Connes) *The periodic cyclic homology $HC_n^{per}(C_c^\infty(G))$ of algebras associated to smooth etale Hausdorff groupoids contain as a direct factor the twisted cohomology of BG.* $\square$

This is obtained from the above theorem by taking $\mathcal{O}$ to be the set of units which is closed and open since G is assumed to be Hausdorff and etale. Compare to [16].

Let now G be a transformation groupoid obtained from a smooth action of a discrete group Γ on a manifold X. Explicitly $G^{(0)} = X$ and $G^{(1)} = X \times \Gamma$ with $d(x, \gamma) = x$, $r(x, \gamma) = \gamma x$ and $(\gamma x, \gamma')(x, \gamma) = (x, \gamma' \gamma)$.

Fix a torsion conjugacy class a in Γ and denote by X^γ the set of fixed points of γ. Then $\mathcal{O}_a = \{(x, \gamma) : \gamma \in a, x \in X^\gamma\}$ is an invariant open-closed subset of $B^{(0)}$. Let Γ_γ denote the centralizer of γ and $E = E\Gamma_\gamma$ be a contractible space equiped with a free and proper action of Γ_γ. Then, using the notations in Theorem 5.7, we have that $\mathcal{Y}(\mathcal{O})$ is homotopy equivalent to the homotopy quotient $(X^\gamma \times E)/\Gamma_\gamma = X^\gamma // \Gamma_\gamma$.

Corollary. 5.9 *Let G be the transformation groupoid defined above. Using the notation we have defined we have the following isomorphism*

$$e_{\mathcal{O}_\gamma} HC_n^{per}(C_c^\infty(G)) = \prod_{m \in n+N+2\mathbb{Z}} H_m(X^\gamma // \Gamma_\gamma, \mathfrak{o})$$

where $N = \dim X^\gamma$. $\square$

Suppose that in the previous remark the group is finite or that it is compact with discrete stabilizers. Then we obtain (up to Morita equivalence of groupoids, see [1, 23, 34, 37]) that G is the groupoid associated to an orbifold. Since the stabilizers in the above discussion are compact we can get rid of them (use the Cartan-Leray spectral sequence) and using the Poincaré duality we obtain

Corollary. 5.10 *(i)* $e_{\mathcal{O}_\gamma} HC_n^{per}(C_c^\infty(G)) = \prod_{m \in n+2\mathbb{Z}} H^m(X^\gamma/\Gamma_\gamma)$

(ii) $HC_n^{per}(C_c^\infty(G)) = \prod_{m \in n+2\mathbb{Z}} H^m(\hat{X}/\Gamma)$ *where* $\hat{X} = \{(x,\gamma) : \gamma x = x\} \subset X \times \Gamma$.

(iii) If the group Γ *is finite and* $\chi(X,\Gamma)$ *denotes the orbifold Euler characteristic of* X/Γ *[1, 18] we have*

$$\chi(X,\Gamma) = \dim HC_0^{per}(C_c^\infty(G)) - \dim HC_1^{per}(C_c^\infty(G))$$

□

See also [1].

This gives an interpretation in cyclic homology of the orbifold Euler characteristic, and a way to define it in general, that is when the orbifold is not a quotient by a finite group. (We warn the reader that this orbifold Euler characteristic does not coincide with the one defined by Satake [38]).

For the proof of (iii) one can either proceed directly or use the results of [5, 10, 9] stating that for finite groups the Connes-Chern character gives an isomorphism $K_\Gamma^\star(X) \otimes \mathbb{C} \simeq HC_\star^{per}(C_c^\infty(G))$.

It is also interesting to mention that in the orbifold case the complex $\mathcal{D}^{(N)}$ of theorem 5.6 is quasi-isomorphic to the de Rham complex of the union of the strata corresponding to $\mathcal{O}$.

Let X be a smooth manifold, assumed to be compact for simplicity. Fix a covering (U_α) of X with contractible open domains of some charts. Let ξ be a rank n complex vector bundle on X and $g_{\alpha,\beta} : U_\alpha \cap U_\beta \to GL(n)$ be the transition functions of ξ. The covering U_α defines a groupoid G by $G^{(0)} = \cup_\alpha (U_\alpha \times \{\alpha\})$ and $G^{(1)} = \cup_{\alpha,\beta}(U_\alpha \cap U_\beta) \times \{(\alpha,\beta)\}$ (i.e. we consider disjoint unions). This groupoid is Morita equivalent to X, and hence the space of orbits of G is X. Choose a partition of unity (φ_α^2) of X subordinated to (U_α). Then if we define $e = \varphi_\alpha g_{\alpha,\beta} \varphi_\beta$ on $(U_\alpha \cap U_\beta) \times \{(\alpha,\beta)\}$

we obtain an idempotent $e \in M_n(C_c^\infty(G))$. Using the Chern character in cyclic cohomology and the quasi-isomorphism mentioned above we get the following expression for the Chern character of the vector bundle ξ in terms of transition functions and partitions of unity:

$$Ch(\xi) = Tr(\sum (n!)^{-1}(edede/2\pi\imath)^n) \tag{34}$$

Here Tr is the canonical map $\Omega^p(G^{(1)}) \otimes M_n \to \Omega^p(X)$. Compare to [7].

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