A stacky approach to the comparison of axiomatizations of quantum field theory

UNIVERSITÀ DEGLI STUDI DI GENOVA

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Outline

- Axiomatic approaches to Lorentzian QFT:
  Algebraic QFT vs Factorization Algebras

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- Categorical equivalence revisited – stacky approach
  (& how it simplifies the open problem)

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- Open problem: upgrade to *higher* categorical equivalence
  (& motivation)
- Categorical equivalence revisited – stacky approach
  (& how it simplifies the open problem)
- Towards a higher categorical equivalence

Axiomatic Lorentzian QFT

Algebraic QFT (Haag-Kastler, Brunetti-Fredenhagen-Verch, . . .)

- assigns observables to spacetimes,
- encodes pushforward along spacetime embeddings,
- captures multiplication of observables.

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Are these approaches comparable? How?

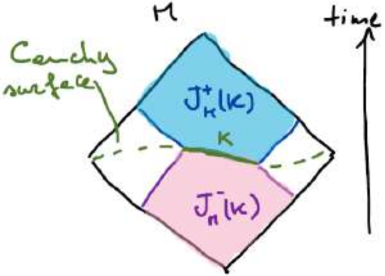
Key: causality and determinism!

Lorentzian geometry

The category Loc consists of

obj: spacetimes

oriented and time-oriented globally hyperbolic Lorentzian manifolds of fixed dimension $m \geq 2$



Lorentzian geometry

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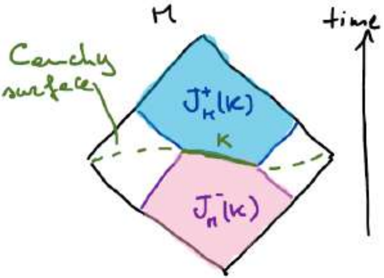
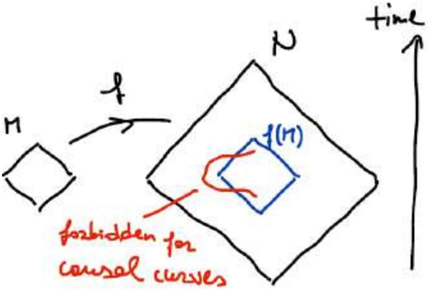
mor: causal embeddings

orientation and time-orientation preserving isometric embeddings with causally convex open image



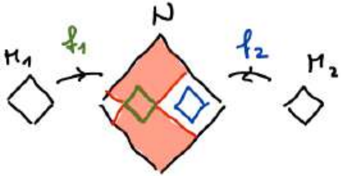


Lorentzian geometry

Causally disjoint pair

$(f_1 : M_1 \to N) \perp (f_2 : M_2 \to N)$

of causal embeddings



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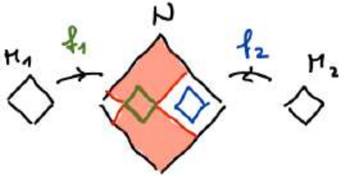

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Time ordered n-tuple $\underline{f} : \underline{M} \to N$

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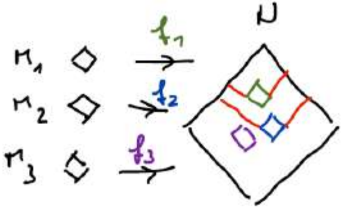

$$J_N^+(f_i(M_i)) \cup f_j(M_j) = \emptyset \ \forall j > i$$

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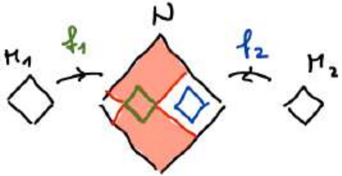

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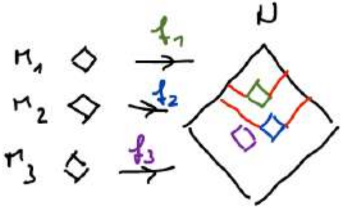

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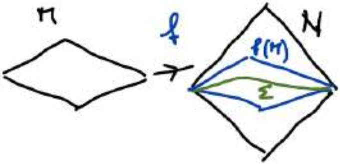
Cauchy embedding $f : M \overset{\sim}{\to} N$



$f(M)$ contains a Cauchy surface Σ

Algebraic quantum field theory (AQFT)

An AQFT $\mathcal{A}$ is a functor $\mathcal{A} : \mathsf{Loc} \to \mathsf{Alg}$[1] such that the diagram

$$\begin{array}{ccc} \mathcal{A}(M_1) \otimes \mathcal{A}(M_2) & \xrightarrow{\mathcal{A}(f_1)\otimes\mathcal{A}(f_2)} & \mathcal{A}(N) \otimes \mathcal{A}(N) \\ {\scriptstyle \mathcal{A}(f_1)\otimes\mathcal{A}(f_2)} \downarrow & \text{(causality)} & \downarrow {\scriptstyle \mu_N^{\mathrm{op}}} \\ \mathcal{A}(N) \otimes \mathcal{A}(N) & \xrightarrow[\mu_N]{} & \mathcal{A}(N) \end{array}$$

commutes for all causally disjoint pairs $(f_1 : M_1 \to N) \perp (f_2 : M_2 \to N)$.

[1]Category of monoids in a (nice) symmetric monoidal category **T**.

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An AQFT $\mathcal{A}$ is Cauchy constant, or satisfies the time-slice axiom, if $\mathcal{A}(f) : \mathcal{A}(M) \stackrel{\cong}{\to} \mathcal{A}(N)$ is an isomorphism (determinism) whenever $f : M \stackrel{\sim}{\to} N$ is a Cauchy embedding.

[1] Category of monoids in a (nice) symmetric monoidal category $\mathbf{T}$.

Time-orderable prefactorization algebras (tPFA)

A tPFA $\mathcal{F}$ consists of

- an object $\mathcal{F}(M) \in \mathbf{T}$ for each spacetime $M \in \mathrm{Loc}$ and
- a time-ordered product

$$\mathcal{F}(\underline{f}) : \mathcal{F}(\underline{M}) := \bigotimes_i \mathcal{F}(M_i) \longrightarrow N \qquad \textit{(causality)}$$

  for each time-orderable[2] tuple $\underline{f} : \underline{M} \to N$,

fulfilling unitality, associativity and permutation equivariance.

[2] Time-orderability = existence of a time-ordering permutation.

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Categorical equivalence

[B–Perin–Schenkel]

$$\begin{array}{ccc} \mathsf{AQFT}^{\mathrm{C},\mathrm{add}} & \overset{\simeq}{\rightleftarrows} & \mathsf{tPFA}^{\mathrm{C},\mathrm{add}} \\ \| & & \| \\ \{\text{AQFTs \& nat. transf.}\}^{\mathrm{C},\mathrm{add}} & & \{\text{tPFAs \& multinat. transf.}\}^{\mathrm{C},\mathrm{add}} \end{array}$$

C = Cauchy constant

add = additive (technical assumption)

Additivity means that observables are exhausted by those supported in relatively compact causally convex opens (rccco) $U \subseteq M$:

$$\mathrm{colim}\Big(\mathcal{F} : \{U \subseteq M \text{ rccco}\} \to \mathbf{T}\Big) \cong \mathcal{F}(M)$$

Categorical equivalence – naive approach

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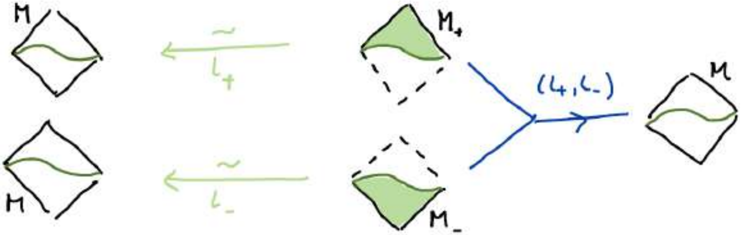
Key: time-ordered products determine spacetime-wise multiplications via Cauchy constancy

$$\mu_M : \mathcal{F}(M) \otimes \mathcal{F}(M) \xleftarrow[\cong]{\mathcal{F}(\iota_+)\otimes\mathcal{F}(\iota_-)} \mathcal{F}(M_+) \otimes \mathcal{F}(M_-) \xrightarrow{\mathcal{F}(\iota_+,\iota_-)} \mathcal{F}(M)$$



Open problem: higher categorical equivalence?

Motivation:

- gauge fields have non-trivial stabilizer groups
  $\rightsquigarrow$ higher homotopy groups
- Batalin-Vilkovisky formalism $\rightsquigarrow$ derived critical loci

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Issues:

- lack of a structural construction of the ordinary equivalence
- ∞-categorical counterpart of additivity

Revisiting the AQFT-vs-tPFA equivalence

Step 1: Replace the additivity property with structure

$$\mathrm{Loc}^{\mathrm{rc}} := \left\{ \begin{array}{rl} \text{obj:} & \text{spacetimes} \\ \text{mor:} & \text{causal embeddings that are Cauchy} \\ & \text{or have relatively compact image} \end{array} \right\} \overset{\text{wide}}{\subseteq} \mathrm{Loc}$$

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Nothing gets lost:

$$\mathrm{AQFT}^{\mathrm{add}} \overset{\text{full}}{\subseteq} \mathrm{AQFT}^{\mathrm{rc}} := \{\text{AQFTs on } \mathrm{Loc}^{\mathrm{rc}}\}$$

$$\mathrm{tPFA}^{\mathrm{add}} \overset{\text{full}}{\subseteq} \mathrm{tPFA}^{\mathrm{rc}} := \{\text{tPFAs on } \mathrm{Loc}^{\mathrm{rc}}\}$$

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Step 2: Reduce the global equivalence problem to a family of spacetime-wise equivalence problems.

AQFT vs tPFA on $\mathbf{Loc}^{\mathrm{rc}}$ $\rightsquigarrow$ AQFT vs tPFA on $\mathbf{Loc}^{\mathrm{rc}}/M$ for all M

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Benefit: localization $\mathcal{O}_{\mathsf{Loc}^{\mathrm{rc}}}[\mathrm{C}^{-1}]$ inexplicit, but each localization $\mathcal{O}_M[\mathrm{C}^{-1}]$ computed via calculus of fractions $\implies$ ∞-localization.

$\mathcal{O}_{\mathsf{Loc}^{\mathrm{rc}}}$: colored operad controlling AQFTs on $\mathsf{Loc}^{\mathrm{rc}}$.

$\mathcal{O}_M$: colored operad controlling AQFTs on M.

Haag-Kastler and Costello-Gwilliam 2-functors

$$\begin{aligned}
\mathsf{HK}^{(\mathrm{C})} : (\mathsf{Loc}^{\mathrm{rc}})^{\mathrm{op}} &\longrightarrow \mathsf{Cat} \\
M &\longmapsto \{\text{AQFTs on } \mathsf{Loc}^{\mathrm{rc}}/M\}^{(\mathrm{C})} \\
(f : M \to N) &\longmapsto (f^* : \mathsf{HK}^{(\mathrm{C})}(N) \longrightarrow \mathsf{HK}^{(\mathrm{C})}(M))
\end{aligned}$$

$$\begin{aligned}
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Remark: $\mathsf{HK}^{(\mathbb{C})}$ closely related to stacks [B–Grant-Stuart–Schenkel].

Decomposition and assembly

To link HK and CG to AQFTs and, respectively, tPFAs on $\mathrm{Loc}^{\mathrm{rc}}$, consider the categories of points

$$\mathsf{HK}^{(\mathrm{C})}(\mathrm{pt}) := \operatorname{bilim} \mathsf{HK}^{(\mathrm{C})} \qquad \ni \Big(\{\mathcal{A}_M\}, \{\phi_f : \mathcal{A}_M \stackrel{\cong}{\to} f^*\mathcal{A}_N\} \Big),$$

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(global data) $\qquad \mathsf{AQFT}^{\mathrm{rc}(,\mathrm{C})} \underset{\mathrm{as}}{\overset{\mathrm{dc}}{\rightleftarrows}} \mathsf{HK}^{(\mathrm{C})}(\mathrm{pt}) \quad (\simeq) \qquad$ (compatible families)

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AQFT-vs-tPFA equivalence revisited

The family $\{\text{AQFTs on } \mathsf{Loc}^{\mathrm{rc}}/M\}^{\mathrm{C}} \simeq \{\text{tPFAs on } \mathsf{Loc}^{\mathrm{rc}}/M\}^{\mathrm{C}}$ forms a 2-natural equivalence

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We rediscover the AQFT-vs-tPFA equivalence out of its spacetime-wise counterpart and the decomposition-assembly equivalence.

Towards a higher AQFT-vs-tPFA equivalence

$\mathbf{T}$ = symm. mon. model category of unbounded cochain complexes

$$\text{Endow} \begin{array}{c} \mathsf{AQFT}^{\mathrm{rc}} \\ \mathsf{tPFA}^{\mathrm{rc}} \\ \mathsf{HK}(M) \\ \mathsf{CG}(M) \end{array} \text{with projective model structures.}$$

[3]Cauchy morphisms are sent to quasi-isomorphisms, instead of isomorphisms.

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Endow $\mathsf{AQFT}^{\mathrm{rc}}$, $\mathsf{tPFA}^{\mathrm{rc}}$, $\mathsf{HK}(M)$, $\mathsf{CG}(M)$ with projective model structures.

Homotopy[3] Cauchy constancy via left Bousfield localization:

$\mathcal{L}_{\hat{\mathrm{C}}}\mathsf{AQFT}^{\mathrm{rc}}$
$\mathcal{L}_{\hat{\mathrm{C}}}\mathsf{tPFA}^{\mathrm{rc}}$
$\mathcal{L}_{\hat{\mathrm{C}}}\mathsf{HK}(M)$
$\mathcal{L}_{\hat{\mathrm{C}}}\mathsf{CG}(M)$

(combinatorial and tractable semimodel categories)

(The projective model structures may not be left proper. This leads to existence of left Bousfield localizations as semimodel categories [Batanin–White].)

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Homotopical decomposition and assembly

HK(pt) (points) $\left(\{\mathcal{A}_M\}, \{\phi_f : \mathcal{A}_M \stackrel{\cong}{\to} f^*\mathcal{A}_N\}\right)$

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and similarly for CG and the left Bousfield localizations $\mathcal{L}_{\hat{\mathrm{C}}}\mathsf{HK}$, $\mathcal{L}_{\hat{\mathrm{C}}}\mathsf{CG}$ [Barwick].

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Proposition [B–Carmona–Grant-Stuart–Schenkel in preparation]
Decoupling and assembly are right Quillen equivalences

$$\mathrm{dc} : (\mathcal{L}_{\hat{\mathrm{C}}})\mathsf{AQFT}^{\mathrm{rc}} \xrightarrow{\sim_Q} (\mathcal{L}_{\hat{\mathrm{C}}})\mathsf{HK}\{\mathrm{pt}\}, \quad \mathrm{as} : (\mathcal{L}_{\hat{\mathrm{C}}})\mathsf{HK}\{\mathrm{pt}\} \xrightarrow{\sim_Q} (\mathcal{L}_{\hat{\mathrm{C}}})\mathsf{AQFT}^{\mathrm{rc}}$$

$$\mathrm{dc} : (\mathcal{L}_{\hat{\mathrm{C}}})\mathsf{tPFA}^{\mathrm{rc}} \xrightarrow{\sim_Q} (\mathcal{L}_{\hat{\mathrm{C}}})\mathsf{CG}\{\mathrm{pt}\}, \quad \mathrm{as} : (\mathcal{L}_{\hat{\mathrm{C}}})\mathsf{CG}\{\mathrm{pt}\} \xrightarrow{\sim_Q} (\mathcal{L}_{\hat{\mathrm{C}}})\mathsf{tPFA}^{\mathrm{rc}}$$

Towards a higher AQFT-vs-tPFA equivalence

Hypothesis: $\mathcal{L}_{\hat{\mathrm{C}}}\mathsf{HK}(M) \overset{\sim_Q}{\to} \mathcal{L}_{\hat{\mathrm{C}}}\mathsf{CG}(M)$ right Quillen equivalences.

This yields a 2-natural right Quillen equivalence:

$$\mathcal{L}_{\hat{\mathrm{C}}}\mathsf{HK} \xrightarrow{\text{2-nat. right Quillen equiv.}} \mathcal{L}_{\hat{\mathrm{C}}}\mathsf{CG}$$

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(pass to categories of homotopical points)

$$\mathcal{L}_{\hat{\mathrm{C}}}\mathsf{HK}\{\mathrm{pt}\} \xrightarrow[\sim_Q \text{ [B–Carmona–Grant-Stuart–Schenkel]}]{} \mathcal{L}_{\hat{\mathrm{C}}}\mathsf{CG}\{\mathrm{pt}\}$$

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(pass to categories of homotopical points)

$$\begin{array}{ccc}
\mathcal{L}_{\hat{\mathrm{C}}}\mathsf{HK}\{\mathrm{pt}\} & \xrightarrow[\sim_Q]{} & \mathcal{L}_{\hat{\mathrm{C}}}\mathsf{CG}\{\mathrm{pt}\} \\
{\scriptstyle \mathrm{dc}}\big\downarrow{\scriptstyle \sim_Q} & & {\scriptstyle \sim_Q}\big\downarrow{\scriptstyle \mathrm{dc}} \\
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\end{array}$$

Towards a higher AQFT-vs-tPFA equivalence

Hypothesis: $\mathcal{L}_{\hat{\mathrm{C}}}\mathsf{HK}(M) \stackrel{\sim_Q}{\to} \mathcal{L}_{\hat{\mathrm{C}}}\mathsf{CG}(M)$ right Quillen equivalences.

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Assuming spacetime-wise higher AQFT-vs-tPFA equivalences, via the higher decomposition-assembly equivalence we deduce the desired higher AQFT-vs-tPFA equivalence.

Towards a higher AQFT-vs-tPFA equivalence

Hypothesis to be checked: for all $M \in \mathsf{Loc}$ the right Quillen functor

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[4] The relative operad $(\mathcal{O}_M, \mathrm{C})$ admits a calculus of left fractions, hence ∞-localization can be modeled by ordinary localization.

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Proposition [B–Carmona–Grant-Stuart–Schenkel in preparation]
Homotopy Cauchy constancy for AQFTs on $\mathsf{Loc}^{\mathrm{rc}}/M$ can be strictified[4], i.e. there is a right Quillen equivalence

$$L^* : \mathsf{HK}(M)^{\mathrm{C}} \xrightarrow{\sim_Q} \mathcal{L}_{\hat{\mathrm{C}}}\mathsf{HK}(M)$$

$\mathsf{HK}(M)^{\mathrm{C}}$ = category of cochain complex valued AQFTs on the localized category $(\mathsf{Loc}^{\mathrm{rc}}/M)[\mathrm{C}^{-1}]$ with projective model structure.

[4] The relative operad $(\mathcal{O}_M, \mathrm{C})$ admits a calculus of left fractions, hence ∞-localization can be modeled by ordinary localization.

Open problem

$$\mathsf{HK}(M)^{\mathrm{C}} \xrightarrow{\sim_Q ???} \mathcal{L}_{\hat{\mathrm{C}}}\mathsf{CG}(M)$$

Algs. over localization $\mathcal{O}_M[\mathrm{C}^{-1}]$ at Cauchy embeddings of AQFT operad over M

Algs. over homotopical localization $\mathrm{L}_{\mathrm{C}}\mathrm{t}\mathcal{P}_M$ at Cauchy embeddings of tPFA operad over M

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Issue: not much is known about homotopical localization of operads.

[Basterra & al]

Attempt to solving the open problem

Pass to categories of operators [Haugseng, Calaque–Carmona] and show

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Hope: modified detection criteria (e.g. allowing for empty homotopy fibers) when the functor already exhibits 1-localization?

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- Solution???: refined detection criteria for ∞-localizations