HOMOTOPY TYPES

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CONTENTS

The theory of homotopy types is one of the most basic parts of topology and geometry. At the centre of this theory stands the concept of algebraic invariants. In what follows we give a general introduction to this subject including recent results and explicit examples. There are three main topics:

Homotopy types with nontrivial fundamental group
(§ 2, § 3, § 4, § 5)

Homotopy types with trivial fundamental group
(§ 6, § 7, § 8, § 9, § 12)

Stable homotopy types
(§ 10, § 11)

Almost all definitions and notations below are explicitly described and statements of results are complete. Prerequisites are elementary topology, elementary algebra and some basic notions from category theory.

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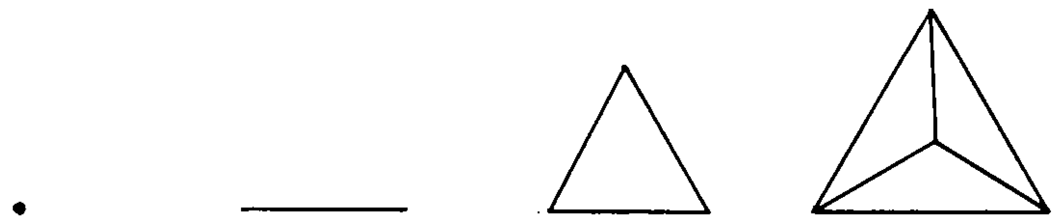
§ 1 What are homotopy types

For each number $n = 0, 1, 2, \ldots$ one has the simplex Δ^n which is the convex hull of the unit vectors $e_0, e_1, \ldots e_n$ in the Euclidean $(n+1)$ -space $\mathbb{R}^{n+1}$. Hence Δ^0 is a point, Δ^1 an interval, Δ^2 a triangle, Δ^3 a tetrahedron, and so on:



Δ^0 Δ^1 Δ^2 Δ^3

The dimension of Δ^n is n. A point $x \in \Delta^n$ is given by barycentric coordinates,

$$x = \sum_{i=0}^{n} t_i e_i \quad \text{with} \quad \sum_{i=0}^{n} t_i = 1 \quad \text{and} \quad t_i \geq 0.$$

The name simplex describes an object which is supposed to be very simple; indeed, natural numbers and simplexes both have the same kind of innocence. Yet once the simplex was created, algebraic topology had to emerge:

For each subset $a \subset \{0, 1, \ldots, n\}$ with $a = \{a_0 < \ldots < a_r\}$ one has the r-dimensional face $\Delta_a \subset \Delta^n$ which is the convex hull of the set of vertices $e_{a_0}, \ldots, e_{a_r}$. Hence the set of all subsets of the set $[n] = \{0, 1, \ldots, n\}$ can be identified with the set of faces of the simplex Δ^n. There are "substructures" S of the simplex obtained by the union of several faces, that is,

$$S = \Delta_{a_1} \cup \Delta_{a_2} \cup \ldots \cup \Delta_{a_k} \subset \Delta^n.$$

Finite polyhedra are topological spaces X homeomorphic to such substructures S of simplexes Δ^n, $n \geq 0$. A homeomorphism $S \approx X$ is called a triangulation of X. Hence a polyhedron X is just a topological space in which we do not see any simplexes. We can introduce simplexes via a triangulation, but this must be seen as an artifact similar to the choice of coordinates in a vector space or manifold (compare H. Weyl, Philosophy of Mathematics and Natural Science, 1949: "The introduction of numbers as coordinates ... is an act of violence ..."). Finite polyhedra form a large universe of objects. One is not interested in a particular individual object of the universe but in the classification of species. A system of such species and subspecies is obtained by the equivalence classes

homotopy types and homeomorphism types.

Recall that two spaces X, Y are homeomorphic, $X \approx Y$, if there are continuous maps $f : X \to Y$ and $g : Y \to X$ such that the composites $fg = 1_Y$ and $gf = 1_X$ are the identity maps. A class of homeomorphic spaces is called a homeomorphism type. The initial problem of algebraic topology - Seifert and Threlfall [LT] called it the main problem - was the classification of homeomorphism types of finite polyhedra. Up to now such a classification was possible only in a very small number of special cases. One might compare this problem with the problem of classifying all knots and links. Indeed the initial datum for a finite polyhedron is just a set $\{a_1, \dots, a_k\}$ of subsets $a_i \subset [n]$ as above and the initial datum to describe a link, namely a finite sequence of neighbouring pairs $(i, i+1)$ or $(i+1, i)$ in $[n]$ (specifying the crossings of $n+1$ strands) is of similar or even higher complexity. But we must emphasize that such a description of an object like a polyhedron or a link cannot be identified with the object itself: there are in general many different ways to describe the same object, and we care only about the equivalence classes of objects, not about the choice of description.

Homotopy types are equivalence classes of spaces which are considerably larger than homeomorphism types. To this end we use the notion of deformation or homotopy. The principal idea is to consider 'nearby' objects (that is, objects, which are 'deformed' or 'perturbed' continuously a little bit) as being similar. This idea of perturbation is a common one in mathematics and science; properties which remain valid under small perturbations are considered to be the stable and essential features of an obejct. The equivalence relation generated by 'slight continuous perturbations' has its precise definition by the notion of homotopy equivalence: Two spaces X and Y are homotopy equivalent, $X \simeq Y$, if there are continuous maps $f : X \to Y$ and $g : Y \to X$ such that the composites fg and gf are homotopic to the identity maps, $fg \simeq 1_Y$ and $gf \simeq 1_X$. (Two maps $f, g : X \to Y$ are homotopic, $f \simeq g$, if there is a family of maps $f_t : X \to Y$, $0 \leq t \leq 1$, with $f_0 = f$, $f_1 = g$ such that the map $(x, t) \longmapsto f_t(x)$ is continuous as a function of two variables.) A class of homotopy equivalent spaces is called a homotopy type.

Using a category $\underline{\underline{C}}$ in the sense of S. Eilenberg and Saunders Mac Lane [GT] one has the general notion of isomorphism type. Two objects X, Y in $\underline{\underline{C}}$ are called equivalent or isomorphic if there are morphisms $f : X \to Y$, $g : Y \to X$ in $\underline{\underline{C}}$ such that $fg = 1_Y$ and $gf = 1_X$. An isomorphism type is a class of isomorphic objects in $\underline{\underline{C}}$. We may consider isomorphism types as being special entities: for example, the isomorphism types in the category of finite sets are the numbers. A homeomorphism type is then an isomorphism type in the category $\underline{\underline{Top}}$ of topological spaces and continuous maps, whereas a homotopy type is an isomorphism type in the homotopy category $\underline{\underline{Top}}/\simeq$ in which the objects are topological spaces and the morphisms are not individual maps but homotopy classes of ordinary continuous maps.

The Euclidean spaces $\mathbb{R}^n$ and the simplexes Δ^n, $n \geq 1$, all represent different homeomorphism types but they are contractible, i.e. homotopy equivalent to a point. As a further example, the homeomorphism types of connected 1-dimensional polyhedra are the graphs which form a world of their own, but the homotopy types of such polyhedra correspond only to numbers since each graph is homotopy equivalent to the one point union of a certain number of circles S^1.

Homotopy types of polyhedra are archetypes underlying most geometric structures. This is demonstrated by the following table which describes a hierarchy of structures based on homotopy types of polyhedra. The arrows indicate the forgetful functors.

$$\begin{tikzcd}[column sep=small]
\textbf{real algebraic sets} \arrow[d] & \textbf{Kaehler manifolds} \arrow[d] & & \\
\begin{array}{c}\textbf{semi analytic sets}\\ \textbf{(analytic isomorphism)}\end{array} \arrow[dd] & \begin{array}{c}\textbf{complex manifolds}\\ \textbf{(complex isomorphism)}\end{array} \arrow[dr] & & \begin{array}{c}\textbf{Riemannian manifolds}\\ \textbf{(isometry)}\end{array} \arrow[dl] \\
& & \begin{array}{c}\textbf{differentiable manifolds}\\ \textbf{(diffeomorphism)}\end{array} \arrow[dll] \arrow[dr] & \\
\begin{array}{c}\textbf{polyhedra}\\ \textbf{(homeomorphism)}\end{array} \arrow[dd] & & & \begin{array}{c}\textbf{topological manifolds}\\ \textbf{(homeomorphism)}\end{array} \arrow[d] \\
& & & \begin{array}{c}\textbf{locally finite polyhedra}\\ \textbf{(proper homotopy equivalence)}\end{array} \arrow[dlll] \\
\begin{array}{c}\textbf{polyhedra}\\ \textbf{(homotopy equivalence)}\end{array} & & &
\end{tikzcd}$$

This hierarchy can be extended in many ways by further structures. Each kind of object in the table has its own notion of isomorphism; again as in the case of polyhedra not the individual object but its isomorphism type is of main interest. We only sample a few properties of these objects.

Some of the arrows in the table correspond to results in the literature. For example, every differentiable manifold is a polyhedron, see J.H.C. Whitehead [OC] or Munkres [EDT]. Any (metrizable) topological manifold is proper homotopy equivalent to a locally finite polyhedron though a topological manifold needs not to be a polyhedron, see Kirby-Siebenmann [FE]. Any semi-analytic set is a polyhedron, see Lojasiewicz [TS]. There are also connections between the objects in the table in terms of realizability. For example, each differentiable manifold admits the structure of a Riemannian manifold, or each closed differentiable manifold has the structure of an irreducible real algebraic set (in fact, infinitely many birationally non isomorphic structures), see Bochnak-Kucharz [AM].

The famous Poincaré conjecture states that the homotopy type of a 3-sphere contains only one homeomorphism type of a topological manifold. Clearly not every finite polyhedron is homotopy equivalent to a closed topological manifold. For this the polyhedron has to be, at least, a Poincaré complex; yet there are also many Poincaré complexes which are not homotopy equivalent to topological manifolds.

By the result of M.H. Freedman [TF] all simply connected 4-dimensional Poincaré complexes have the homotopy type of closed topological manifolds, they do not in general have the structure of a differentiable manifold by the work of Donaldson [AG]. Homotopy types of Kähler manifolds are very much restricted by the fact that their (real) homotopy type is 'formal', see Deligne-Griffiths-Morgan-Sullivan [RH].

Now one might argue that the set given by diffeomorphism types of closed differentiable manifolds is more suitable and restricted than the vast variety of homotopy types of finite polyhedra. This, however, turned out not to be true. Surgery theory showed that homotopy types of arbitrary simply connected finite polyhedra play an essential role for the understanding of differentiable manifolds. In particular, one has the following embedding of a set of homotopy types into the set of diffeomorphism types: Let X be a finite simply connected n-dimensional polyhedron, $n > 2$. Embed X into an Euclidean space $\mathbb{R}^{k+1}$, $k \geq 2n$, and let $N(X)$ be the boundary of a regular neighbourhood of $X \subset \mathbb{R}^{k+1}$. This construction yields a well defined function $\{X\} \mapsto \{N(X)\}$ which carries homotopy types of simply connected n-dimensional finite polyhedra to diffeomorphism types of k-dimensional manifolds. Moreover for $k = 2n + 1$ this function is injective, see Kreck-Schafer [CS]. Hence the set of simply connected diffeomorphism types is at least as complicated as the set of homotopy types of simply connected finite polyhedra.

In dimension ≥ 5 the classification of simply connected diffeomorphism types (up to connected sum with homotopy spheres) is reduced via surgery to problems in homotopy theory which form the unsolved hard core of the question. This kind of reduction of geometric questions to problems in homotopy theory is an old and standard operating procedure. Further examples are the classification of fibre bundles and the determination of the ring of cobordism classes of manifolds.

All this underlines the fundamental importance of homotopy types of polyhedra. There is no good intuition what they actually are, but they appear to be entities as genuine and basic as numbers or knots. In my book [AH] I suggested an axiomatic background for the theory of homotopy types; A. Grothendieck [PS] commented:

> "Such suggestion was of course quite interesting for my present reflections, as I do have the hope indeed that there exists a 'universe' of schematic homotopy types..."

Moreover J.H.C. Whitehead [AH] in his talk at the International Congress of Mathematicians 1950 in Harvard said with respect to homotopy types and the homotopy category of polyhedra:

> "The ultimate object of algebraic homotopy is to construct a purely algebraic theory, which is equivalent to homotopy theory in the same sort of way that 'analytic' is equivalent to 'pure' projective geometry".

Today, 45 years later, this idea still remains a dream which has not yet come true. The full realization seems far beyond the reach of existing knowledge and techniques. Some progress in several directions will be described below.

§ 2 How to build homotopy types

There are many different topological and combinatorial devices which can be used to construct the homotopy types of connected polyhedra, for example, simplicial complexes, simplicial sets, CW-complexes, topological spaces, simplicial groups, small categories, and partially ordered sets.

Up to now we have worked with finite polyhedra by viewing them as substructures of a simplex. One needs also polyhedra which are not finite since for example the universal covering space of a finite polyehedron, in general, is not finite, also the Euclidean spaces $\mathbb{R}^n$, $n \geq 1$, are non-finite polyhedra. Infinite polyhedra are defined by 'simplicial complexes'. The following abstract notion of a simplicial complex is just a recipe for joining many simplexes together to obatin a space which is called the 'realization' of the simplicial complex.

(2.1) Definition. A simplicial complex X is a set of finite sets closed under formation of subsets. Equivalently X is a set of finite subsets of a set U such that U is the union of all sets in X and for $a \in X$, $b \subset a$ also $b \in X$. The set $U = X^0$ is called the set of vertices of X. The simplicial complex X is a partially ordered set by inclusion.

We obtain the realization of a simplicial complex X by associating with each element $a \in X$ a simplex Δ_a which is the convex hull of the set a in the real vector space with basis X^0. The vertices of Δ_a are elements of a. For $b \subset a$ the simplex $\Delta_b \subset \Delta_a$ is a face of Δ_a. The realization of X is the union of sets

$$|X| = \bigcup_{a \in X} \Delta_a \tag{2.2}$$

with the topology induced by the topology of the simplexes. That is, a subset in $|X|$ is open if and only if the intersection with all simplexes is open. If X is finite we can choose a bijection $X^0 \approx \{0, 1, \ldots, N\}$ such that $|X|$ coincides with the substructure $\cup\{\Delta_{j(a)},\ a \in X\}$ in the simplex Δ^N. The realization $|X|$ is compact if and only if X is finite.

(2.3) Definition. A polyhedron is a topological space homeomorphic to the realization of a simplicial complex.

Simplicial complexes have the disadvantage that for a subcomplex $Y \subset X$ the quotient space $|X| \setminus |Y|$ is not the realization of a simplicial complex. This is one of the reasons to introduce 'simplicial sets' which are considerably more flexible than simplicial complexes. Again a simplicial set X is a combinatorial affair, i.e. a family of sets and maps between them from which again may be deduced a topological space $|X|$. There is a more general notion of a 'simplicial object' which actually became one of the most influential notions of algebraic topology.

(2.4) Definition. The simplicial category $\underline{\underline{\Delta}}$ is the following subcategory of the category of sets. The objects are the finite sets $[n] = \{0, 1, \ldots, n\}$, $n \geq 0$, and the morphisms $\alpha : [n] \to [m]$ are the order preserving functions, i.e. $x \leq y$ implies

$\alpha(x) \leq \alpha(y)$. A simplicial object X in a category $\underline{\underline{C}}$ is a contravariant functor from $\underline{\underline{\Delta}}$ to the category $\underline{\underline{C}}$; we also write

$$X : \underline{\underline{\Delta}}^{op} \to \underline{\underline{C}}$$

where $\underline{\underline{\Delta}}^{op}$ is the oppossite category of $\underline{\underline{\Delta}}$. Hence X is determined by objects $X[n]$, $n \geq 0$, in $\underline{\underline{C}}$ and by morphisms $\alpha^* : X[m] \to X[n]$ one for each order preserving function $\alpha : [n] \to [m]$. Morphisms in the category $\underline{\underline{sC}}$ of simplicial objects are the natural transformations.

Hence simplicial sets, simplicial groups and simplicial spaces are the simplicial objects in the category of sets, $\underline{\underline{Set}}$, groups $\underline{\underline{Gr}}$, and topological spaces $\underline{\underline{Top}}$, respectively. A simplicial set is also a simplicial space by using the discrete topology functor $\underline{\underline{Set}} \subset \underline{\underline{Top}}$. A simplicial space X is good if every surjective map α in $\underline{\underline{\Delta}}$ induces a 'cofibration' $\alpha^* : X[m] \to X[n]$. For example the inclusion $|B| \subset |A|$ given by a simplicial subcomplex B of a simplicial complex A is a cofibration. We define the realization of a good simplicial space X by the following quotient of the disjoint union of products $X[n] \times \Delta^n$ in $\underline{\underline{Top}}$,

$$|X| = \left(\bigcup_{n \geq 0} X[n] \times \Delta^n \right) / \sim$$

Here the equivalence relation is generated by $(a, \alpha_* x) \sim (\alpha^* a, x)$ for $\alpha : [n] \to [m]$, $a \in X[m]$, $x \in \Delta^n$ where $\alpha_* : \Delta^n \to \Delta^m$ is the restriction of the linear map given on vertices by α. For different realizations of simplicial spaces compare the Appendix of Segal [CC].

There are the following basic examples of simplicial sets. For any topological space X we obtain the simplicial set

$$(2.6) \qquad SX : \underline{\underline{\Delta}}^{op} \to \underline{\underline{Set}}, \begin{cases} (SX)[n] = \{a : \Delta^n \to X \in \underline{\underline{Top}}\} \\ \alpha^*(a) = a \circ \alpha_* \end{cases}$$

which is called the singular set of X. One has the canonical map

$$T : |SX| \to X,\ T(a, x) = a(x)$$

which is a homotopy equivalence if X is a polyhedron. Moreover T is a weak homotopy equivalence for any space X, (that is, T induces isomorphisms of homotopy groups with respect to all base points). Clearly the singular set SX is very large. This, however, has the advantage that SX is a 'Kan set'; for such Kan sets it is possible to describe homotopy theory purely combinatorially, see Curtis [SH] and May [SO].

In the next example we use the morphisms d_i, s_i which generate the category $\underline{\underline{\Delta}}$ multiplicatively. The maps d_i are the unique injective maps $d_i : [n-1] \to$

$[n] - \{i\} \subset [n]$, and the maps s_i are the unique surjective maps $s_i : [n] \to [n-1]$ with $s_i(i) = s_i(i+1) = i \in [n-1]$.

For any small category $\underline{\underline{X}}$ we obtain the simplicial set

$$(2.7) \qquad Nerve(\underline{\underline{X}}) : \underline{\underline{\Delta}}^{op} \to \underline{\underline{Set}}$$

which is called the <u>nerve</u> of $\underline{\underline{X}}$. Here $Nerve(\underline{\underline{X}})[n]$, $n \geq 1$, is the set of all sequences $(\lambda_1, \ldots, \lambda_n)$ of n composable morphisms

$$X_0 \xleftarrow{\lambda_1} X_1 \longleftarrow \ldots \xleftarrow{\lambda_n} X_n$$

in $\underline{\underline{X}}$. For $n = 0$ let $Nerve(\underline{\underline{X}})[0]$ be the set of objects of $\underline{\underline{X}}$. The functor $Nerve(\underline{\underline{X}})$ is defined on generating morphisms of $\underline{\underline{\Delta}}$ by $s_0^*(A) = 1_A$ for $A \in Nerve(\underline{\underline{X}})[0]$ and

$$s_i^*(\lambda_1, \ldots, \lambda_n) = (\lambda_1, \ldots, \lambda_{i-1}, 1, \lambda_i, \ldots, \lambda_n)$$

where 1 is the appropriate identity. Moreover

$$d_i^*(\lambda) = \begin{cases} A & i = 0 \\ B & i = 1 \end{cases}$$

for $\lambda : A \to B \in Nerve(\underline{\underline{X}})[1]$ and for $n \geq 2$

$$d_i^*(\lambda_1, \ldots, \lambda_n) = \begin{cases} (\lambda_2, \ldots, \lambda_n) & \text{for } \; i = 0 \\ (\lambda_1, \ldots, \lambda_i\lambda_{i+1}, \ldots, \lambda_n) & \text{for } \; i = 1, \ldots, n-1 \\ (\lambda_1, \ldots, \lambda_{n-1}) & \text{for } \; i = n \end{cases}$$

There is a more formal way to define the simplicial set $Nerve(\underline{\underline{X}})$ as follows. For this recall that any <u>partially ordered set</u> has the structure of a small category: objects are the elements of the set and there is a unique morphism $a \to b$ iff $a \geq b$. This way one obtains a functor $H : \underline{\underline{\Delta}} \to \underline{\underline{Cat}}$ where $\underline{\underline{Cat}}$ is the category of small categories and functors. The functor H carries the object $[n]$ to the category $H[n]$ given by the ordered set $[n]$. Using H we define the functor

$$Nerve(\underline{\underline{X}}) : \underline{\underline{\Delta}}^{op} \to \underline{\underline{Set}}, \qquad \begin{cases} Nerve(\underline{\underline{X}})[n] = \{a : H[n] \to \underline{\underline{X}} \in \underline{\underline{Cat}}\} \\ \alpha^*(a) = a \circ \alpha_* \quad \text{with} \quad \alpha_* = H(\alpha) \end{cases}$$

which coincides with the definition above, compare Gabriel-Zisman [CF]. The realization $|Nerve\underline{\underline{X}}|$ is also called the classifying space of $\underline{\underline{X}}$.

Since products in the category $\underline{\underline{Top}}$ of topological spaces do not behave well with respect to quotient maps we shall use in the next definition the full subcategory $\underline{\underline{Top}}(cg)$ of spaces whose topology is compactly generated. The product $X \times Y$ in $\underline{\underline{Top}}(cg)$ yields the structure of a monoidal category. The usefulness of compactly generated spaces was observed by Brown [CC] and Steenrod [CC].

If $\underline{\underline{X}}$ is a small topological category, i.e. a category enriched over the monoidal category $\underline{\underline{Top}}(cg)$, then $Nerve(\underline{\underline{X}})$ is a simplicial space given by $X_n = Nerve(\underline{\underline{X}})[n]$ above. For $n = 0$ the set X_0 is discrete and X_n, $n \geq 1$, is the product $\underline{\underline{X}}(X_1, X_0) \times \ldots \times \underline{\underline{X}}(X_n, X_{n-1})$ where $\underline{\underline{X}}(A, B) \in \underline{\underline{Top}}(cg)$ is the space of morphisms $A \to B$ in $\underline{\underline{X}}$. In particular, if $H \in \underline{\underline{Top}}(cg)$ is a topological monoid, i.e. a topological category with a single object, then the simplicial space $Nerve(H)$ is the geometric bar construction of H, see for example Baues [GL]. This is a good simplicial space if the inclusion of the neutral element $\{1\} \subset H$ is a closed cofibration, (i.e. H is well pointed). For a well pointed topological group $G \in \underline{\underline{Top}}(cg)$ the realization

$$B(G) = |Nerve(G)| \tag{2.8}$$

is the classifying space of G which is the Eilenberg-Mac Lane space $K(G, 1)$ if G is discrete, see Milgram [BC]. This classifying space is homeomorphic to the infinite projective space $\mathbb{R}P_\infty, \mathbb{C}P_\infty$ and $\mathbb{H}P_\infty$ in case the topological group G is $\mathbb{Z}/2$, S^1 and S^3 respectively.

A simplicial complex X is a partially ordered set and hence also a small category and we can form the simplicial set $Nerve(X)$. The realizations

$$|X| \approx |Nerve(X)| \tag{2.9}$$

are homeomorphic. In fact, $|Nerve(X)|$ can be identified with the barycentric subdivision of $|X|$.

Simplicial complexes and simplicial sets both are of combinatorial nature, but they tend to be very large objects even if one wants to describe simple spaces like products of spheres. J.H.C. Whitehead observed that for many purposes only the 'cell structure' of spaces is needed. In some sense 'cells' play a role in topology which is similar to the role of 'generators' in algebra. Let

$$\begin{aligned} (2.10) \qquad & D^n = \{x \in \mathbb{R}^n, \| x \| \leq 1\} \\ & \overset{\circ}{D}{}^n = \{x \in \mathbb{R}^n, \| x \| < 1\}, \partial D^n = D^n - \overset{\circ}{D}{}^n = S^{n-1} \end{aligned}$$

be the closed and open n-dimensional disk. An (open) n -cell e, $n \geq 1$, in a space X is a homeomorphic image of the open disk $\overset{\circ}{D}{}^n$ in X, a 0-cell is a point in X. As a set a'CW-complex' is the disjoint union of such cells. A CW-complex is not just a combinatorial affair since the 'attaching maps' in general may have very complicated topological descriptions.

(2.11) Definition. A CW-complex X with skeleta $X^0 \subset X^1 \subset X^2 \subset \ldots \subset X$ is a topological space constructed inductively as follows:

(a) X^0 is a discrete space whose elements are the 0-cells of X.

(b) X^n is obtained by attaching to X^{n-1} a disjoint union of n-disks D_i^n via continuous functions $\varphi_i : \partial(D_i^n) \to X^{n-1}$, i.e. take the disjoint union $X^{n-1} \smile \cup D_i^n$ and pass to the quotient space given by the identifications $x \sim \varphi_i(x)$, $x \in \partial D_i^n$. Each $\overset{\circ}{D}{}_i^n$ then projects homeomorphically to an n-cell e_i^n of X. The map φ_i is called the attaching map of e_i^n.

(c) X has the weak topology with respect to the filtration of skeleta.

The realization $|X|$ of a simplicial complex is a CW-complex with the n-cells given by elements $a \in X$ with $dim(\Delta_a) = n$. Also the realization $|X|$ of a simplicial set is a CW-complex with the n-cells given by 'non-degenerate' elements in $X[n]$. Here an element is degenerate if and only if it is in the image if one of the functions $s_i^* : X[n-1] \to X[n], i \in [n-1]$. A CW-complex, however, need not be a polyhedron, see Metzler [BU], but a CW-complex is always homotopy eqivalent to a polyhedron. A CW-space is a topological space homotopy equivalent to a CW-complex. We now describe some of the many ways to create homotopy types of polyhedra.

(2.12) Theorem. *Homotopy types of polyhedra are the same as the homotopy types of the spaces in (a) ... (f) respectively:*

(a) *realizations $|X|$ of simplicial complexes X,*

(b) *realizations $|X|$ of simplicial sets X,*

(c) *realizations $|SX|$ of singular sets of topological spaces X,*

(d) *classifying spaces $|Nerve(\underline{\underline{X}})|$ of small categories $\underline{\underline{X}}$,*

(e) *classifying spaces $|Nerve(X, \leq)|$ of partially ordered sets $(X, \leq)$,*

(f) *CW-complexes.*

CW-complexes X, Y have a compactly generated topology and the product $X \times Y$ in $\underline{\underline{Top}}(cg)$ is again a CW-complex (this does not hold for the product in $\underline{\underline{Top}}$). A CW-monoid is a CW-complex X which is also a monoid in $\underline{\underline{Top}}(cg)$ such that the neutral element is a 0-cell and such that the multiplication is cellular. For example a simplicial group G yields the realization $|G|$ which is a CW-monoid. Here G, considered as a simplicial set, is a group object in $\underline{\underline{sSet}}$ with a multiplication $G \times G \to G$ in $\underline{\underline{sSet}}$ inducing the multiplication $|G| \times |G| = |G \times G| \to |G|$ in

$\underline{\underline{Top}}(cg)$.

A simplicial group F is called a free simplicial group if for each $n \geq 0$ the group $F[n]$ is a free group with a given basis and if all s_i^* carry basis elements to basis elements, compare Curtis [SH].

(2.13) Theorem. *Homotopy types of connected polyhedra are the same as the homotopy types of the spaces (a) and (b) respectively:*

(a) *classifying spaces $B(H) = |Nerve(H)|$ of CW-monoids H for which the set $\pi_0(H)$ of path components is a group,*

(b) *classifying spaces $B(|G|)$ where $|G|$ is the realization of a free simplicial group.*

Hence free simplicial groups suffice to describe all homotopy types of connected polyhedra. This yields a very significant algebraic tool to construct such homotopy types. Computations in free simplicial groups, however, are still extremely complicated. It is shown in Baues [CH] that the complexity of simplicial groups can be reduced considerably in case one studies homotopy types of connected 4-dimensional polyhedra. The connection of free simplicial groups and CW-complexes was described by Kan [CW]:

(2.14) Theorem. *Let X be a CW-complex with trivial 0-skeleton $X^0 = *$. Then there is a free simplicial group G with $X \simeq B(|G|)$ such that the set of non-degenerate generators in $G[n]$ coincides with the set of $(n+1)$ -cells in X, $n \geq 0$.*

This illuminates the role of cells as generators in topology. Unfortunately the free group $G[n]$ has also all the degenerate generators coming from cells in dimension $\leq n$. Therefore the free group $G[n]$ is very large already for CW-complexes with a few cells. We call G a free simplicial group associated to X if $X \simeq B(|G|)$ as in the theorem. There is, in fact, an algebraic homotopy theory of free simplicial groups which via the functors $G \longmapsto B(|G|)$ is equivalent to the homotopy theory of connected polyhedra (compare Curtis [SH] and Quillen [HA]).

(2.15) Remark. Further methods of representing homotopy types were introduced by Smirnov [HT] (compare also Smith [IC]) and Kapranov-Voevodskii [GH].

§ 3 Whitehead's realization problem

The main problem and the hard core of algebraic topology is the 'classification' of homotopy types of polyhedra. Here the general idea of classification is to attach to each polyhedron 'invariants', which may be numbers, or objects endowed with algebaric structures (such as groups, rings, modules, etc.) in such a way that homotopy equivalent polyhedra have the same invariants (up to isomorphism in the case of algebraic structures). Such invariants are called homotopy invariants. The ideal would be to have an algebraic invariant which actually characterizes a homotopy type completely. The fascinating task of homotopy theory is thus the investigation of 'algebraic principles' hidden in homotopy types. We may be confident that such principles are of importance in mathematics far beyond the scope of topology as for example shown by the development of 'homological algebra' which now plays a role in ring theory, algebraic geometry, number theory and many other fields. A further very recent example is the use of 'operads' outside topology; compare for example Getzler-Jones [OH] and Ginzberg-Kapranov [KO].

The main numerical invariants of a homotopy type are 'dimension' and 'degree of connectedness'.

(3.1) Definition. The dimension $\mathrm{Dim}(X) \leq \infty$ of a CW-complex is defined by $\mathrm{Dim}(X) \leq n$ if $X = X^n$ is the n-skeleton. The dimension $\dim(X)$ of the homotopy type $\{X\}$ is defined by $\dim(X) \leq \mathrm{Dim}(Y)$ for all CW-complexes Y homotopy equivalent to X.

(3.2) Definition. A space X is (path) connected or 0-connected if any two points in X can be joined by a path in X, this is the same as saying that any map $\partial D^1 \to X$ can be extended to a map $D^1 \to X$ where D^1 is the 1-dimensional disc. This notion has an obvious generalization: A space X is k-connected if for all $n \leq k+1$ any map $\partial D^n \to X$ can be extended to a map $D^n \to X$ where D^n is the n-dimensiomal disc. The 1-connected spaces are also called simply connected.

The dimension is related to homology since all homology groups above the dimension are trivial, whereas the degree of connectedness is related to homotopy since below this degree all homotopy groups vanish. It took a long time in the development of algebraic topology to establish homology and homotopy groups as the main invariants of a homotopy type. For completeness we recall the definitions of these groups.

(3.3) Definition. Let $\underline{\underline{Top}}^*$ be the category of topological spaces with basepoint $*$ and basepoint preserving maps. The set $[X, Y]$ denotes the set of homotopy classes of maps $X \to Y$ in $\underline{\underline{Top}}^*$. Choosing a basepoint in the sphere S^n we obtain the homotopy groups

$$\pi_n(X) = [S^n, X]$$

This is a set for $n = 0$ and a group for $n \geq 1$, abelian for $n \geq 2$. The group structure is induced by the map $\mu : S^n \to S^n \vee S^n$ obtained by identifying the equator of S^n

to a point, that is for $\alpha, \beta \in \pi_n(X)$ we define $\alpha + \beta = (\alpha, \beta) \circ \mu$. The set $\pi_0(X)$ is the set of path components of X and $\pi_1(X)$ is called the fundamental group of X. An element $f \in [X, Y]$ induces $f_* : \pi_n X \to \pi_n Y$ by $f_* \alpha = f \circ \alpha$ so that π_n is a functor on the category $\underline{\underline{Top}}^* / \simeq$.

(3.4) Definition. For a simplicial set X let $C_n X$ be the free abelian group generated by the set $X[n]$ and let

$$\partial_n : C_n X \to C_{n-1} X$$

be the homomorphism defined on basis elements $x \in X[n]$ by

$$\partial_n(x) = \sum_{i=0}^{n} (-1)^i d_i^*(x)$$

Then one can check that $\partial_n \partial_{n+1} = 0$ so that the quotient group

$$H_n X = \text{kernel } \partial_n / \text{image } \partial_{n+1}$$

is defined. This is the n-th homology group of X. For a topological space X we define the homology $H_n X = H_n SX$ by use of the singular set. The homology H_n yields a functor from the homotopy category $\underline{\underline{Top}} / \simeq$ to the category of abelian groups.

The crucial importance of homotopy groups and homology groups relies on the following results due to J.H.C. Whitehead.

(3.5) Theorem. *A) A connected CW-space X is contractible if and only if for a basepoint in X all homotopy groups $\pi_n(X)$, $n \geq 1$, are trivial.*
B) A simply connected CW-space X is contractible if and only if all homology groups $H_n(X)$, $n \geq 2$, are trivial.

The theorem shows that homotopy groups and in the simply connected case also homology groups are able to detect the trivial homotopy type. In fact, homotopy groups and homology groups are able to decide whether two spaces have the same homotopy type:

(3.6) Whitehead theorem. *Let X and Y be connected CW-spaces and let $f : X \to Y$ be a map. Then f is a homotopy equivalence in $\underline{\underline{Top}} / \simeq$ if and only if, for a basepoint in X, condition A) or equivalently B) holds.*

A) *The map f induces an isomorphism between homotopy groups, $f_* : \pi_n X \cong \pi_n Y$, $n \geq 1$.*

B) *The map f induces an isomorphism between fundamental groups, $f_* : \pi_1 X \cong \pi_1 Y$, and the induced map $\tilde{f} : \tilde{X} \to \tilde{Y}$ between universal coverings induces an isomorphism between homology groups, $\tilde{f}_* : H_n \tilde{X} \cong H_n \tilde{Y}$, $n \geq 2$.*

Hence homotopy groups constitute a system of algebraic invariants which, in a certain sense, are sufficiently powerful to characterize the homotopy type of a CW-space. This does not mean that $X \simeq Y$ just because there exist isomorphisms $\pi_n X \cong \pi_n Y$ for every $n = 1, 2, \dots$. The crux of the matter is not merely that $\pi_n X \cong \pi_n Y$, but that a certain family of isomorphisms, $\phi_n : \pi_n X \cong \pi_n Y$, has a <u>geometrical realization</u> $f : X \to Y$. That is to say, the latter map f induces all isomorphisms ϕ_n via the functor π_n, namely $\phi_n = \pi_n(f)$ for $n \geq 1$. Therefore the emphasis is shifted to the following problem; compare Whitehead [AH].

(3.7) <u>Realization problem of Whitehead</u>. Find necessary and sufficient conditions in order that a given set of isomorphisms or, more generally, homomorphisms, $\phi_n : \pi_n X \to \pi_n Y$, have a geometrical realization $X \to Y$.

The Whitehead theorem shows that also the invariants $\pi_1 X$, $H_n \tilde{X}$, are sufficiently powerful to detect homotopy types. Therefore there is a realization problem for these invariants in a similar way. In particular, within the category of simply connected CW-spaces the functors π_n could be replaced by H_n. The realization problem of Whitehead above is highly unsolved, and is indeed one of the hardest problems of algebraic topology. We shall describe below solutions for some special cases; see (10.11). Using simplicial groups Kan gave a purely combinatorial description of Whitehead's realization problem. For this we need the following Moore chain complex of a simplicial group.

(3.8) <u>Definition</u>. A <u>chain complex</u> (C, ∂) of groups is a sequence of homomorphisms

$$\dots \longrightarrow C_n \xrightarrow{\partial_n} C_{n-1} \longrightarrow \dots, \; n \in \mathbb{Z},$$

in the category of groups with image ∂_{n+1} a normal subgroup of kernel ∂_n. For each n, the homology $H_n(C, \partial)$ is defined to be the quotient group kernel (∂_n)/image (∂_{n+1}). For each simplicial group G one has the <u>Moore chain complex</u>, NG, with

$$\begin{aligned} N_n(G) &= \bigcap_{i \neq 0} \text{kernel}\,(d_i^*) \\ \partial_n &= d_0^* \quad (\text{restricted to} \quad N_n G) \end{aligned}$$

We define <u>homotopy groups</u> of G by $\pi_n G = H_n(NG)$.

A basic theorem of Kan [CD] shows that homotopy groups of simplicial groups, in fact, correspond exactly to homotopy groups of connected CW spaces:

(3.9) <u>Theorem</u>. *Let G be a simplicial group. Then there is a natural isomorphism* $(n \geq 0)$

$$\pi_n(G) = \pi_n|G| = \pi_{n+1} B(|G|)$$

Hence if G_X is associated to the connected CW-space X, that is $X \cong B(|G_X|)$, we can compute $\pi_{n+1}(X) = \pi_n(G_X)$ by the Moore chain complex $N(G_X)$. For example let $G_{S^{n+1}}$ be the free simplicial group with only one non-degenerate generator in degree n, then $G_{S^{n+1}}$ is associated to the sphere S^{n+1} and

$$\pi_{n+k}(G_{S^{n+1}}) = H_{n+k}N(G_{S^{n+1}}) = \pi_{n+k+1}S^{n+1}$$

gives us a purely combinatorial description of homotopy groups of spheres. This way Kan gave a new proof of Hopf's result $\pi_3 S^2 = \mathbb{Z}$. In general, however, free simplicial groups are so complicated that this formula was not suitable for computing homotopy groups of spheres. Theorem (3.9) leads to the following interpretation of Whitehead's realization problem.

(3.10) Theorem. *Let X, Y be connected CW-spaces and let G_X, G_Y be free simplicial groups associated to X and Y respectively. Then a set of homomorphisms $\phi_n : \pi_n X \to \pi_n Y$ is realizable by a map $X \to Y$ if and only if there is a map $f : G_X \to G_Y$ in $\underline{\underline{sGr}}$ inducing for $n \geq 0$ the homomorphism*

$$\phi_{n+1} : \pi_{n+1}X = \pi_n G_X \xrightarrow{f_*} \pi_n G_Y = \pi_{n+1}Y.$$

We say that two simplicial groups G, G' are <u>weakly equivalent</u> if there is a map $f : G \to G'$ in $\underline{\underline{sGr}}$ inducing isomorphisms $f_* : \pi_n G \cong \pi_n G'$. This yields actually an equivalence relation for free simplicial groups. As usual a 1-1 <u>correspondence</u> is a function which is injective and surjective. The next result is a consequence of (3.10) and (2.13).

(3.11) Corollary. *There is a 1-1 correspondence between homotopy types of connected CW-spaces and weak equivalence classes of free simplicial groups. The correspondence is given by $X \mapsto G_X$ with the inverse $G \mapsto B(|G|)$.*

We point out that 'weak equivalence' generates an equivalence relation for all simplicial groups and that weak equivalence classes of all simplicial groups are the same as weak equivalence classes of free simplicial groups. In fact, for any simplicial group G' there is a free simplicial group G and a weak equivalence $G \to G'$ which is called a <u>free model</u> of G'.

(3.12) Definition. Let $\underline{\underline{C}}$ be a category with a given class of morphisms called weak equivalences. Then the <u>localization</u> or <u>homotopy category</u> of $\underline{\underline{C}}$ is the category $Ho(\underline{\underline{C}})$ together with a functor $q : \underline{\underline{C}} \to Ho(\underline{\underline{C}})$ having the following universal property: For every weak equivalence f the morphism $q(f)$ is an isomorphism; given any functor $F : \underline{\underline{C}} \to \underline{\underline{B}}$ with $F(f)$ an isomorphism for all weak equivalences f, there is a unique functor $\Theta : Ho(\underline{\underline{C}}) \to B$ such that $\Theta q = F$. Except for set theoretic difficulties the category $Ho(\underline{\underline{C}})$ exists, see Gabriel-Zisman [CF].

(3.13) Theorem. *Let $\underline{\underline{spaces}}$ be the category of connected CW-spaces with basepoint and let $\underline{\underline{spaces}}/\simeq$ be the corresponding homotopy category. Then there is an equivalence of categories*

$$Ho(\underline{\underline{sGr}}) \xrightarrow{\sim} \underline{\underline{spaces}}/\simeq$$

which carries a simplicial group G to the classifying space $B(|G|)$.

The results (3.9) ... (3.13) are due to Kan, see Curtis [SH] and Quillen [HA].

§ 4 Algebraic models of n-types

When studying a CW-complex or a polyhedron X it is natural to consider in succession the skeleta $X^1, X^2, \dots$, where X^n consists of all the cells in X of at most n-dimensions. Now the homotopy type of X^n is not an invariant of the homotopy type of X. Therefore J.H.C. Whitehead introduced the n-type, this being a homotopy invariant of X, which depends only on X^{n+1}. There are two ways to present n-types. On the one hand they are certain equivalence classes of $(n+1)$ -dimensional CW-complexes, on the other hand they are homotopy types of certain spaces.

(4.1) Definition. Let $\underline{\underline{CW}}$ be the category of connected CW-complexes X with basepoint $* \in X^0$ and of basepoint preserving cellular maps. Let $\underline{\underline{CW}}^{n+1}$ be the full subcategory of $\underline{\underline{CW}}$ consisting of $(n+1)$ -dimensional objects. For maps $F, G : X^{n+1} \to Y^{n+1}$ in $\underline{\underline{CW}}^{n+1}$ let $F|X^n$, $G|X^n : X^n \to Y^{n+1}$ be the restrictions. Then we obtain an equivalence relation $\sim$ by setting $F \sim G$ iff there is a homotopy $F|X^n \simeq G|X^n$ in $\underline{\underline{Top}}^*$. Let $\underline{\underline{CW}}^{n+1}/\sim$ be the quotient category. Now an n-type in the sense of J.H.C. Whitehead is an isomorphism type in the category $\underline{\underline{CW}}^{n+1}/\sim$.

(4.2) Definition. Recall that $\underline{\underline{spaces}}$ is the category of connected CW-spaces with basepoint and pointed maps. Let

$$n - \underline{\underline{types}} \subset \underline{\underline{spaces}}/\simeq$$

be the full subcategory consisting of spaces X with $\pi_i(X) = 0$ for $i > n$. Such spaces or their homotopy types are also called n-types.

The two definition of n-types are compatible since there is an equivalence of categories

$$P_n : \underline{\underline{CW}}^{n+1}/\sim \xrightarrow{\sim} n - \underline{\underline{types}} \tag{4.3}$$

We define the functor P_n by use of the following $n-th$ Postnikov functor

$$P_n : \underline{\underline{CW}}/\simeq \to n - \underline{\underline{types}}$$

For X in $\underline{\underline{CW}}$ we obatin P_nX by 'killing homotopy groups', that is, we choose a CW-complex P_nX with $(n+1)$ -skeleton

$$(P_nX)^{n+1} = X^{n+1}$$

and with $\pi_i(P_nX) = 0$ for $i > n$. For a cellular map $F : X \to Y$ in $\underline{\underline{CW}}$ we choose a map $PF^{n+1} : P_nX \to P_nY$ which extends the restriction $F^{n+1} : X^{n+1} \to Y^{n+1}$ of F. This is possible since $\pi_iP_nY = 0$ for $i > n$. The functor P_n in (4.4) and (4.3) carries X to P_nX and carries F to the homotopy class of P_nF. Different choices for P_nX yield canonically isomorphic functors P_n.

Isomorphism types in $\underline{\underline{CW}}^{n+1}/\sim$ were originally called '$(n+1)$ -types', they are now called n-types since they correspond to homotopy types for which only $\pi_1, \dots, \pi_n$ might be non trivial.

There is an important relationship between n-types and homotopy types of $(n+1)$ -dimensional CW-spaces. Two $(n+1)$ -dimensional connected CW-spaces X^{n+1}, Y^{n+1} have the same n-type iff one of the following conditions (A) and (B) is satiesfied:

(A) There is a map $F: X^{n+1} \to Y^{n+1}$ which induces isomorphisms $\pi_i(F)$ for $i \leq n$.

(B) There is a homotopy equivalence $P_n X^{n+1} \simeq P_n Y^{n+1}$

(4.4) Theorem. *(J.H.C. Whitehead [SH]): Let X^{n+1}, Y^{n+1} be two finite $(n+1)$ - dimensional CW-complexes which have the same n-type. Then there exist $a, b < \infty$ such that the one point unions*

$$X^{n+1} \vee \bigvee_a S^{n+1} \simeq Y^{n+1} \vee \bigvee_b S^{n+1}$$

are homotopy equivalent.

The theorem shows that each n-type Q determines a connected tree $HT(Q, n+1)$ which we call the tree of homotopy types for $(Q, n+1)$. The vertices of this tree are the homotopy types $\{X^{n+1}\}$ of finite $(n+1)$ -dimensional CW-complexes with $P_n X^{n+1} \simeq Q$. The vertex $\{X^{n+1}\}$ is connected by an edge to the vertex $\{Y^{n+1}\}$ if Y^{n+1} has the homotopy types of $X^{n+1} \vee S^{n+1}$. The roots of this tree are the homotopy types $\{Y^{n+1}\}$ which do not admit a homotopy equivalence $Y^{n+1} \simeq X^{n+1} \vee S^{n+1}$. Theorem (4.4) shows that the tree $HT(Q, n+1)$ is connected. For a proof of theorem (4.4) see II.§6 in Baues [CH].

Remark. There are various results on the tree $HT(Q, n+1)$ in case $Q = K(\pi, 1)$ is an Eilenberg-Mac Lane space of degree 1. In this case the tree is determined by the group π. Results of Metzler [HZ], Sieradski [SS] and Sieradski-Dyer [DA] show that for $n \geq 1$ there exist trees $HT(K(\pi,1), n+1)$ with at least two roots.

As pointed out by Whitehead [CHI] one has to consider the hierarchy of categories and functors

$$(4.5) \qquad 1-\underline{\underline{types}} \xleftarrow{P} 2-\underline{\underline{types}} \xleftarrow{P} 3-\underline{\underline{types}} \xleftarrow{P} \dots$$

where the functor P is given by the Postnikov functor above. Since 1-types are the same as Eilenberg-Mac Lane spaces $K(\pi, 1)$ we can identify a 1-type with an abstract group. In fact, the fundamental group π_1 gives us the equivalence of categories

$$(4.6) \qquad \pi_1 : 1-\underline{\underline{types}} \xrightarrow{\sim} \underline{\underline{Gr}}$$

From this point of view n-types are natural objects of higher complexity extending abstract groups. Following up on this idea Whitehead looked for a purely algebraic equivalent of an n-type, $n \geq 2$. An important requirement for such an algebraic system is "realizability", in two senses. In the first sense this means that there is an n-type which is in the appropriate relation to a given one of these algebraic systems, just as there is a 1-type whose fundamental group is isomorphic to a given group. The second sense is the 'realizability' of homomorphisms between such algebraic systems by maps of the corresponding n-types.

Mac Lane-Whitehead [TC] showed that a 'crossed module' is a purely algebraic equivalent of a 2-type:

(4.7) Definition. An N-group or an action of a group N on a group M is a homomorphism f from N to the group of automorphisms of M. For $x \in M$, $\alpha \in N$ we denote the action by $x^\alpha = f(\beta)(x)$ where β is the inverse of α. Then a pre-crossed module $\partial : M \to N$ is a group homomorphism together with an action of N on M such that

$$\partial(x^\alpha) = \alpha^{-1}\partial(x)\alpha$$

that is, ∂ is equivariant with respect to the action of N on N by inner automorphisms. A Peiffer commutator in M is the element

$$\langle x, y\rangle = x^{-1}y^{-1}x(y^{\partial x}) \quad \text{for} \quad x, y \in M.$$

Now ∂ is a crossed module if all Peiffer commutators are trivial. A morphism between crossed modules (or pre crossed modules) is a commutative diagram in $\underline{\underline{Gr}}$

$$\begin{array}{ccc} M & \xrightarrow{g} & M' \\ {\scriptstyle\partial}\downarrow & & \downarrow{\scriptstyle\partial'} \\ N & \xrightarrow{f} & N' \end{array}$$

where g is f-equivariant, that is $g(x^\alpha) = (gx)^{(f\alpha)}$. This is a weak equivalence if (f, g) induces isomorphisms $\pi_i(\partial) \cong \pi_i(\partial')$ for $i = 1, 2$ where $\pi_1(\partial) = \text{cokernel}(\partial)$ and $\pi_2(\partial) = \text{kernel}(\partial)$.

(4.8) Theorem. *Let $\underline{cross}$ be the category of crossed modules and let $Ho(\underline{cross})$ be the localizations with respect to weak equivalences. Then there is an equivalence of categories*

$$2-\underline{\underline{types}} \xrightarrow{\sim} Ho(\underline{cross})$$

For a proof of this result compare (III.8.2) in Baues [CH]. Many further properties of crossed modules are described in this book, in particular, crossed modules lead to algebraic models which determine the homotopy types of connected 3-dimensional polyhedra.

Using Kan's result (3.13) also a simplicial group G with $\pi_i(G) = 0$ for $i \geq 2$ is an algebaric model of a 2-type. The crossed module ∂_G associated to G is obtained by the Moore chain complex $N(G)$ in (3.8). We have

$$\text{(4.9)} \qquad \partial_G : N_1(G)/dN_2(G) \to N_0(G)$$

with $N_0(G) = G[0]$. Here $G[0]$ acts on $N_1(G)$ by $x^\alpha = s_0^*(\alpha)^{-1} \cdot x \cdot s_0^*(\alpha)$ so that $d : N_1(G) \to N_0(G)$ is a pre-crossed module. The normal subgroup $dN_2(G)$ of $N_1(G)$ contains all Peiffer commutators so that ∂_G induced by d is a well defined crossed module. Hence ∂_G reduces the complexity of the simplicial group G considerably, so that a crossed module describes the algebra behind a 2-type more precisely and simpler than a simplicial group.

After Step Two in the hierarchy of n-types was achieved by Mac Lane-Whitehead in 1950 one had to consider Step Three. The solution for Step Three was obtained recently in Baues [CH] where 'quadratic modules' are shown to be the appropriate algebraic models of 3-types.

(4.10) Definition. A quadratic module $\sigma = (w, \delta, \partial)$ is a diagram of N-groups and N-equivariant homomorphisms

$$C \otimes C \xrightarrow{w} L \xrightarrow{\delta} M \xrightarrow{\partial} N$$

satisfying the equations

$$\begin{cases} \partial\delta = 0 \\ x^{-1}y^{-1}x(y^{\partial x}) = \delta w(\{x\} \otimes \{y\}) \\ a^{-1}b^{-1}ab = w(\{\delta a\} \otimes \{\delta b\}) \\ a^{\partial x} = a \cdot w(\{\delta a\} \otimes \{x\} + \{x\} \otimes \{\delta a\}) \end{cases}$$

for $a, b \in L$ and $x, y \in M$. Here C is the abelianization of the quotient group $M/P_2(\partial)$ where $P_2(\partial)$ is the subgroup of M generated by Peiffer commutators $\langle x, y\rangle$ in the pre-crossed module ∂. The element $\{x\} \in C$ is represented by $x \in M$ and the action of $\alpha \in N$ on the $\mathbb{Z}$-tensor product $C \otimes C$ is given by $(\{x\} \otimes \{y\})^\alpha = \{x^\alpha\} \otimes \{y^\alpha\}$. A morphism

$$\varphi : \sigma = (w, \delta, \partial) \to \sigma' = (w', \delta', \partial')$$

between quadratic modules with $\varphi = (l, m, n)$ is given by a commutative diagram in $\underline{\underline{Gr}}$

$$\begin{tikzcd} C \otimes C \arrow[r, "w"] \arrow[d, "\varphi_* \otimes \varphi_*"] & L \arrow[r, "\delta"] \arrow[d, "l"] & M \arrow[r, "\partial"] \arrow[d, "m"] & N \arrow[d, "n"] \\ C' \otimes C' \arrow[r, "w'"'] & L' \arrow[r, "\delta'"'] & M' \arrow[r, "\partial'"'] & N' \end{tikzcd}$$

where (m, n) is a map between pre-crossed modules which induces $\varphi_* : C \to C'$ and where l is n-equivariant. This is a weak equivalence if φ induces isomorphisms $\varphi_* : \pi_i(\sigma) \cong \pi_i(\sigma')$ for $i = 1, 2, 3$ where

$$\pi_1(\sigma) = \text{cokernel}\,\partial$$
$$\pi_2(\sigma) = \text{kernel}\,\partial/\text{image}\,\delta$$
$$\pi_3(\sigma) = \text{kernel}\,\delta$$

(4.11) <u>Theorem</u>. *Let $\underline{\underline{quad}}$ be the category of crossed modules and let $Ho(\underline{\underline{quad}})$ be the localization with respect to weak equivalences. Then there is an equivalence of categories*

$$3-\underline{\underline{types}} \xrightarrow{\sim} Ho(\underline{\underline{quad}})$$

Compare (IV.§ 10) in Baues [CH]. In this book many further properties and examples of quadratic modules are described, in particular quadratic modules lead to algebraic models which determine homotopy types of connected 4-dimensional polyhedra. One can deduce from a simplicial group G with $\pi_i(G) = 0$ for $i \geq 3$ the associated quadratic module σ_G as follows: We derive from the Moore chain complex $N(G)$ in (3.8) the quadratic module $\sigma_G = (w, \delta, \partial)$ with

$$(4.12) \qquad C \otimes C \xrightarrow{w} N_2(G)/U \xrightarrow{\delta} N_1(G)/P_3(\partial) \xrightarrow{\partial} N_0(G)$$

Here the action of $N_0(G) = G[0]$ is obtained by s_0^* and $s_1^* s_0^*$ as in (4.9) and δ and ∂ are induced by the boundary maps in $N(G)$. Moreover $P_3(\partial)$ is the subgroup of $N_1(G)$ generated by triple Peiffer commutators $\langle x, \langle y, z\rangle\rangle$ and $\langle\langle x, y\rangle, z\rangle$ in the pre crossed module $\partial = d_1^*$, see (4.9). We define for $x, y \in N_1(G)$ the formal Peiffer bracket $\langle x, y\rangle \in N_2(G)$ by

$$\langle x, y\rangle = s_1^*(x^{-1}y^{-1}x)(s_0^*x)^{-1}(s_1^*y)(s_0^*x).$$

Then $d_2\langle x, y\rangle = \langle x, y\rangle$ holds. Now U is the subgroup of $N_2(G)$ generated by formal triple brackets $\langle x, \langle y, z\rangle\rangle$, $\langle\langle x, y\rangle, z\rangle$ and by elements $d_3(u)$ with $u \in N_3(G)$. Finally the function w is defined by $w(\{x\} \otimes \{y\}) = \{\langle x, y\rangle\}$ where $\langle x, y\rangle$ is the formal Peiffer bracket. See also (IV. B. 11) in Baues [CH].

Again a quadratic module is a considerable simplification of a simplicial group G representing a 3-type. In fact, we restrict G to degrees ≤ 2 and we are even allowed to divide out triple Peiffer commutators and formal triple Peiffer commutators in the Moore chain complex. We therefore say that a quadratic module has 'nilpotency degree two', a crossed module has 'nilpotency degree one'.

<u>Remark</u>. Theorem (4.8) goes back to the work of J.H.C. Whitehead [CHII] and Mac Lane-Whitehead [TC] though they do not formulate the result as an equivalence of categories. In the literature there are two ways to generalize crossed modules in order to obtain models of n-types, $n \geq 2$. On the one hand Loday [SF] defines algebraic systems called 'cat^n-groups', (see also Porter [TS] and Bullejos-Cegarra-Duskin [CG]) on the other hand Conduché [MC] considers 'crossed modules of length 2' representing 3-types which were generalized by Carrasco [CH] and

Carrasco-Cegarra [GT] for n-types; this approach of Conduché and Carrasco describes additional structure for the Moore chain complex $N(G)$ which is sufficient to determine the simplicial group G. Moreover Brown-Gilbert [AM] and Joyal-Tierney obtained further algebraic models of 3-types. But the quadratic modules above are the only models of 3-types which have nilpotency degree 2.

A 'nilpotent' algebraic model for 4-types is not known. For simply connected n-types, however, we can use the work of Curtis [LC] for the construction of nilpotent models.

(4.13) Definition. For a group G let $\Gamma_{m+1}G$ be the subgroup of all iterated commutators of length $m+1$. Then G has nilpotency degree m or equivalently is a nil(m)-group if $\Gamma_{m+1}G$ is trivial. Let $\underline{\underline{nil}}(m)$ be the full subcategory in $\underline{\underline{Gr}}$ consisting of nil(m) -groups. A free nil(m) -group, i.e. a free object in $\underline{\underline{nil}}(m)$, is the same as the quotient $F/\Gamma_{m+1}F$ where F is a free group. Let $\underline{\underline{snil}}(m)$ be the category of simplicial nil(m) -groups with weak equivalences defined as in $\underline{\underline{sGr}}$. A free simplicial nil($m$) -group is defined in a similar way as a free simplicial group, see § 2.

Let $\{a\}$ be the least integer $\geq a$.

(4.14) Theorem. *For $2 \leq n \leq 1 + \{\log_2(m)\}$ let $\underline{\underline{T}}(n,m)$ be the full subcategory of $\underline{\underline{snil}}(m)$ consisting of objects G with $\pi_i G = 0$ for $i = 0$ and $i \geq n$. Then there exists an equivalence of categories*

$$n - \underline{\underline{types}}_2 \xrightarrow{\sim} Ho\,\underline{\underline{T}}(n,m)$$

Here the left hand side denotes the full homotopy catgeory of simply connected n-types.

For $m = 2$ and $n = 3$ the result is also a consequence of (4.11). This indicates that there might be a suitable generalization of both, theorem (4.11) and (4.14), available for n-types which are not simply connected.

Theorem (4.14), as it stands, is not contained in the work of Curtis. The equivalence in the theorem carries the n-type X to a free simplicial nil(m) -group $\overline{G}_X$ with $\pi_i\overline{G}_x = 0$ for $i \geq n$ and for which

$$\overline{G}_X^n = (G_X/\Gamma_{m+1}G_X)^n.$$

Here both sides denote the corresponding subobjects generated by basis elements in degree $\leq n$. Hence $\overline{G}_X$ is the 'n-type' of $G_X/\Gamma_{m+1}G_X$ in the category $\underline{\underline{snil}}(m)$, compare the construction of the Postnikov section in (4.4). The result of Curtis [LC] implies that there is a natural isomorphism ($i \geq 0$)

$$\pi_i(\overline{G}_X) = \pi_{i+1}(X)$$

for all simply connected n-types X. The inverse of the functor $X \longmapsto \overline{G}_X$ carries the simplicial group G in $\underline{\underline{T}}(n,m)$ to the classifying space $B(|G|)$.

We have seen that the category 2-$\underline{\underline{types}}$ has the algebraic model category $\underline{\underline{cross}}$ in (4.8). This generalizes as follows.

(4.15) Definition. A crossed complex ρ is a sequence

$$\dots \xrightarrow{d_4} \rho_3 \xrightarrow{d_3} \rho_2 \xrightarrow{d_2} \rho_1$$

of homomorphisms between ρ_1 -groups where d_2 is a crossed module and ρ_n, $n \geq 3$, is abelian and a π_1 -module via the action of ρ_1 where $\pi_1 = \text{cokernel}(d_2)$. Moreover $d_{n-1}d_n = 0$ for $n \geq 3$. A morphism $f : \rho \to \rho'$ is a sequence of homomorphisms $f_n : \rho_n \to \rho'_n$ which commute with d_n and are f_1 -equivariant. Let $\pi_n(\rho) = \text{kernel}(d_n)/\text{image}(d_{n+1})$ be the homology of ρ. Then f is a weak equivalence if $\pi_n(f)$ is an isomorphism for all n. Let $\underline{\underline{cross}}^n$ be the category of crossed chain complexes ρ with $\rho_i = 0$ for $i > n$ and $\pi_i(\rho) = 0$ for $1 < i < n$ so that $\underline{\underline{cross}}^2 = \underline{\underline{cross}}$.

The next result is a consequence of the work of Brown-Higgins [CS].

(4.16) Theorem. *Let $\underline{\underline{K}}_1^n \subset n - \underline{\underline{types}}$ be the full homotopy category of n-types X with $\pi_i X = 0$ for $1 < i < n$, $n \geq 2$. Then there is an equivalence of categories*

$$\underline{\underline{K}}_1^n \xrightarrow{\sim} Ho(\underline{\underline{cross}}^n)$$

For $n = 2$ this is exactly the result in (4.8). The objects in $\underline{\underline{cross}}^n$ which are by (4.16) models of special n-types have only nilpotency degree 1. In particular 3-types X with $\pi_2 X$ have a model in cross^3 so that in this case a quadratic module σ as in (4.10) is not needed to determine the homotopy type. We can associate with σ the crossed chain complex $\rho(\sigma)$,

$$L/w(C \otimes C) \xrightarrow{\delta} M/\delta w(C \otimes C) \xrightarrow{\partial} N, \tag{4.17}$$

obtained by dividing out the 'quadratic part'. If $\pi_2(\sigma) = 0$ then $\rho(\sigma)$ determines the homotopy type of σ. Therefore the quadratic structure w of σ is only relevant if $\pi_2 \neq 0$. In the next section we study the category $\underline{\underline{K}}_1^n$ from a different point of view.

§ 5 Cohomology of groups and cohomology of categories

We show that the classical cohomology of groups is related to special homotopy types. We also introduce the cohomology of categories with coefficients in a natural system, which generalizes the cohomology of groups and which turned out to have deep impact on homotopy classification. We shall need the cohomology of categories in particular for the comparison of Postnikov invariants and boundary invariants; see (8.11) below.

Let π be a group. A (right) π-module M, also denoted by the pair (π, M), is an abelian group M together with an action of π on M. As usual the homotopy group $\pi_n(X)$, $n \geq 2$, are actually $\pi_1(X)$ -modules. Let $\underline{\underline{Mod}}$ and $\underline{\underline{mod}}$ be the following categories. Objects in both are the modules (π, M) as above. Morphisms $(\pi, M) \to (\pi', M')$ are pairs

$$(a, f) = (a : \pi \to \pi',\ f : M \to M') \in \underline{\underline{Mod}}$$
$$(e, g) = (b : \pi' \to \pi,\ g : M \to M') \in \underline{\underline{mod}}$$

where a, b are maps between groups and f, g are maps between abelian groups such that $f(x^\alpha) = f(x)^{a(\alpha)}$ and $g(x^{b(\beta)}) = g(x)^\beta$ for $x \in M$, $\alpha \in \pi$, $\beta \in \pi'$. Using homotopy groups one has a functor $(n \geq 2)$

$$(5.1) \qquad (\pi_1, \pi_n) : \underline{\underline{Top}}^* \to \underline{\underline{Mod}}$$

The cohomology of groups is a functor (see K.S. Brown [CG] and (5.12) below)

$$(5.2) \qquad H^n : \underline{\underline{mod}} \to \underline{\underline{Ab}}$$

which carries (π, M) to $H^n(\pi, M)$. Let b^*M be the π' -module M given by $x^\beta = x^{b(\beta)}$. Then $(b, 1) : (\pi, M) \to (\pi', b^*M)$ is a morphism in $\underline{\underline{mod}}$ which induces $b^* = H^n(b, 1)$,

$$b^* : H^n(\pi, M) \to H^n(\pi', b^*M).$$

On the other hand $(1, f) : (\pi, M) \to (\pi, a^*M')$ in $\underline{\underline{mod}}$ induces $f_* = H^n(1, f)$,

$$f_* : H^n(\pi, M) \to H^n(\pi, a^*M').$$

We use the cohomology of groups for the definition of the following category, which is the 'Grothendieck construction' of the functor H^n in (5.2).

(5.3) Definition. *The objects in the category $\underline{\underline{Gro}}(H^n)$ are triple (π, M, k) where (π, M) is a π -module and $k \in H^n(\pi, M)$. Morphisms $(\pi, M, k) \to (\pi', M', k')$ are maps $(a, f) : (\pi, M) \to (\pi', M')$ in $\underline{\underline{Mod}}$ which satisfy the equation*

$$a^*(k') = f_*(k) \in H^n(\pi, a^*M')$$

Composition is defined as in $\underline{\underline{Mod}}$; the forgetful functor $\underline{\underline{Gro}}(H^n) \to \underline{\underline{Mod}}$ is faithful.

The objects in $\underline{\underline{Gro}}(H^{n+1})$ are in fact algebraic models of special n-types.

(5.4) Theorem. *For the full homotopy category $\underline{\underline{K}}_1^n \subset \underline{\underline{n - types}}$ of n-types X with $\pi_i X = 0$ for $1 < i < n$ there is a functor*

$$T^n : \underline{\underline{K}}_1^n \to \underline{\underline{Gro}}(H^{n+1})$$

with the following properties: The functor T^n is full and reflects isomorphisms and for each object (π, M, k) in $\underline{\underline{Gro}}(H^{n+1})$ there is X in $\underline{\underline{K}}_1^n$ and an isomorphism $(\pi, M, k) \cong T^n(X)$ in $\underline{\underline{Gro}}(H^{n+1})$. The functor T^n is defined by $T^n(X) = (\pi_1(X), \pi_n(X), k(X))$ where $k(X)$ is the k-invariant.

In consequence of these properties of the functor T^n an object in $\underline{\underline{Gro}}(H^{n+1})$ may be described as an algebraic equivalent of a n-type in $\underline{\underline{K}}_1^n$; that is, T^n induces a 1-1 correspondence between homotopy types in $\underline{\underline{K}}_1^n$ and isomorphism types in $\underline{\underline{Gro}}(H^{n+1})$. Theorem (5.4) is due to Mac Lane-Whitehead [TC] for $n = 2$ and Eilenberg-Mac Lane [CW] for $n \geq 3$. It is also a consequence of the 'Postnikov tower' of a space, see for example Baues [OT]. The theorem yields a special solution of Whitehead's realization problem (3.7):

(5.5) Corollary. *Let X, Y be objects in $\underline{\underline{K}}_1^n$, then $\phi_* : \pi_* X \to \pi_* Y$ has a geometrical realization $X \to Y$ if and only if (ϕ_1, ϕ_n) is a morphism in $\underline{\underline{Mod}}$ and the equation*

$$(\phi_1)^* k(Y) = (\phi_N)_* k(X)$$

holds where $k(X)$, $k(Y)$ are the k-invariants.

In view of theorem (5.4) elements in the cohomology of groups can be considered as representatives of special n-types. We now recall the following notation which partially already was used in the theorem above.

(5.6) Notation. Let $F : \underline{\underline{C}} \to \underline{\underline{K}}$ be a functor. We say that F is full, resp. faithful if the induced map on morphism sets $F : \underline{\underline{C}}(X, Y) \to \underline{\underline{K}}(FX, FY)$ is surjective, resp. injective for all objects X, Y in $\underline{\underline{C}}$. Moreover F reflects isomorphisms if f in $\underline{\underline{C}}$ is an isomorphism if and only if $F(f)$ in $\underline{\underline{K}}$ is an isomorphism. The functor F is representative if for each object Y in $\underline{\underline{K}}$ there is an object X in $\underline{\underline{C}}$ and an isomorphism $F(X) \cong Y$. We call X a 'realization' of Y. We say that F is a detecting functor if F reflects isomorphisms, is full and representative. A detecting functor which is faithful is the same as an equivalence of categories.

The properties of the functor T^n in (5.4) just say that T^n is a detecting functor. One readily checks that every detecting functor $F : \underline{\underline{C}} \to \underline{\underline{K}}$ induces a 1-1 correspondence between isomorphism types of objects in $\underline{\underline{C}}$ and isomorphism types of objects in $\underline{\underline{K}}$. The functor T^n has actually a further nice property which is less well known, namely T^n is a 'linear extension' of categories. To this end we recall from Baues [AH] the following concept of a linear extension which plays a crucial role in topology and algebra.

(5.7) Notation. Let $\underline{\underline{C}}$ be a category. The category of factorizations in $\underline{\underline{C}}$, denoted by $F\underline{\underline{C}}$, is given as follows: Objects are the morphisms $f, g, \ldots$ in $\underline{\underline{C}}$ and morphisms $f \to g$ are pairs (α, β) for which

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & A' \\ {\scriptstyle f}\uparrow & & \uparrow{\scriptstyle g} \\ B & \xleftarrow{\beta} & B' \end{array}$$

commutes in $\underline{\underline{C}}$. Hence $\alpha f \beta = g$ is a factorization of g. Composition is defined by $(\alpha', \beta')(\alpha, \beta) = (\alpha'\alpha, \beta\beta')$. We clearly have $(\alpha, \beta) = (\alpha, 1)(1, \beta) = (1, \beta)(\alpha, 1)$. A natural system (of abelian groups) on $\underline{\underline{C}}$ is a functor

$$D : F\underline{\underline{C}} \to \underline{\underline{Ab}}$$

from the category of factorizations to the category of abelian groups. The functor D carries the object f to $D_f = D(f)$ and carries the morphism $(\alpha, \beta) : f \to g$ to the induced homomorphism

$$D(\alpha, \beta) = \alpha_* \beta^* : D_f \to D_{\alpha f \beta} = D_g$$

where $D(\alpha, 1) = \alpha_*$, $D(1, \beta) = \beta^*$. We say that

$$D \xrightarrow{+} \underline{\underline{E}} \xrightarrow{p} \underline{\underline{C}}$$

is a linear extension of $\underline{\underline{C}}$ by the natural system D if the following properties hold. The categories $\underline{\underline{E}}$ and $\underline{\underline{C}}$ have the same objects and p is a full functor which is the identity on objects. For each morphism $f : B \to A$ in $\underline{\underline{C}}$ the abelian group D_f acts transitively and effectively on the subset $p^{-1}(f)$ of morphisms in $\underline{\underline{E}}$ with $p^{-1}(f) \subset \underline{\underline{E}}(B, A)$. We write $f_0 + \alpha$ for the action of $\alpha \in D_f$ on $f_0 \in p^{-1}(f)$. Moreover, the action satisfies the linear distributivity law:

$$(f_0 + \alpha)(g_0 + \beta) = f_0 g_0 + f_* \beta + g^* \alpha$$

Two linear extensions $\underline{\underline{E}}, \underline{\underline{E}}'$ are equivalent if there is an isomorphism $\varepsilon : \underline{\underline{E}} \cong \underline{\underline{E}}'$ of categories with $p'\varepsilon = p$ and $\varepsilon(f_0 + \alpha) = \varepsilon(f_0) + \alpha$. The extension $\underline{\underline{E}}$ is split if there is a functor $s : \underline{\underline{C}} \to \underline{\underline{E}}$ with $ps = 1$.

As an example we obtain the natural system

(5.8) $$H^n : \underline{Mod} \to \underline{\underline{Ab}}$$

which carries the object $(a, f) : (\pi, M) \to (\pi', M')$ to the abelian group

$$H^n_{(a,f)} = H^n(\pi, a^* M')$$

which is the cohomology of π with coefficients in $a^* M'$. Hence $H^n_{(a,f)}$ depends on a and not on f. Induced maps are given by $(a', f')_*(x) = (f')_*(x)$ and $(a'', f'')^*(x) =$

$(a'')^*(x)$ for $x \in H^n_{(a,f)}$. The natural system H^n on $\underline{Mod}$ yields also a natural system H^n on $\underline{\underline{Gro}}(H^{n+1})$ via the forgetful functor in (5.3). Using the functor T^n in (5.4) we identify isomorphism types in $\underline{\underline{K}}^n_1$ and in $\underline{\underline{Gro}}(H^{n+1})$ so that this way T^n is the identity on objects. The next result is a consequence of (VIII.2.5) in Baues [AH].

(5.9) Theorem. *The category $\underline{\underline{K}}^n_1$ is part of a linear extension of categories*

$$H^n \xrightarrow{+} \underline{\underline{K}}^n_1 \xrightarrow{T^n} \underline{\underline{Gro}}(H^{n+1})$$

which is not split.

The result classifies maps in $\underline{\underline{K}}^n_1$ completely in terms of the cohomology of groups. Since the functor T^n is not split the extension, however, is non-trivial. We now introduce the cohomology of categories which classifies linear extensions. In analogy to the category $\underline{\underline{mod}}$ in (5.2) we obtain the category $\underline{\underline{nat}}$ of natural systems: Objects are pairs $(\underline{\underline{C}}, D)$ where D is a natural system of the small category $\underline{\underline{C}}$. Morphisms are pairs

$$(5.10) \qquad (\phi^{op}, \tau) : (\underline{\underline{C}}, D) \to (\underline{\underline{C}}', D')$$

where $\phi : \underline{\underline{C}}' \to \underline{\underline{C}}$ is a functor and where $\tau : \phi^* D \to D'$ is a natural transformation. Here $\phi^* D : F\underline{\underline{C}}' \to Ab$ is given by $(\phi^* D)_f = D_{\phi f}$ and $\alpha_* = \phi(\alpha)_*$, $\beta^* = \phi(\beta)^*$. A natural transformation $T : D \to \tilde{D}$ yields as well the natural transformation $\phi^* t : \phi^* D \to \phi^* \tilde{D}$. Now morphisms in $\underline{\underline{nat}}$ are composed by the formula

$$(\psi^{op}, \sigma)(\phi^{op}, \tau) = (\phi\psi)^{op}, \sigma \circ \psi^* \tau)$$

The <u>cohomology of categories</u> (introduced in Baues-Wirsching [CS] and Baues [AH]) is the functor

$$(5.11) \qquad H^n : \underline{\underline{nat}} \to \underline{\underline{Ab}}$$

defined in (5.13) below. One has the full inclusion of categories

$$\underline{\underline{mod}} \subset \underline{\underline{nat}}$$

which carries (π, M) to $(\underline{\underline{C}}, D)$ where $\underline{\underline{C}} = \pi$ is the category given by the group π and where D is the natural system on $\underline{\underline{C}}$ with $D_f = M$ for $f \in \pi$ and $\alpha^* =$ identity and $\beta_*(x) = x^\beta$ for $x \in M$, $\beta \in \pi$. Then the composition of functors

$$(5.12) \qquad \underline{\underline{mod}} \subset \underline{\underline{nat}} \xrightarrow{H^n} \underline{\underline{Ab}}$$

coincides with the cohomology of groups in (5.2). In fact, we may consider the cohomology of categories as a canonical generalization of the cohomology of groups.

(5.13)Definition. Let $\underline{\underline{X}}$ be a small category and let D be a natural system on $\underline{\underline{X}}$. The n-th cochain group F^n is the abelian group of all functions

$$f : \mathrm{Nerve}(\underline{\underline{X}})[n] \to \bigcup_{g \in \mathrm{Mor}(\underline{\underline{X}})} D_g$$

with $f(\lambda_1, \dots, \lambda_n) \in D_{\lambda_1 \circ \dots \circ \lambda_n}$ and $f(A) \in D_{1_A}$ for $n = 0$. The right hand side denotes the disjoint union of all abelian groups D_g with g a morphism in $\underline{\underline{X}}$. Additions in F^n is given by adding pointwise in the abelian goups D_g. The coboundary $\delta = \delta^{n-1} : F^{n-1} \to F^n$ is defined by the formula

$$\begin{aligned}
(\delta f)(\lambda) &= \lambda_* f(A) - \lambda^* f(B) \quad \text{for} \quad \lambda : A \to B,\ n = 1, \\
(\delta f)(\lambda_1, \dots, \lambda_n) &= (\lambda_1)_* f(\lambda_2, \dots, \lambda_n) \\
&+ \sum_{i=1}^{n-1} (-1)^i f(\lambda_1, \dots, \lambda_i \lambda_{i+1}, \dots, \lambda_n) \\
&+ (-1)^n (\lambda_n)^* f(\lambda_1, \dots, \lambda_{n-1})
\end{aligned}$$

One can check that $\delta\delta = 0$ so that the <u>cohomology</u>

$$H^n(\underline{\underline{X}}, D) = \mathrm{kernel}\, \delta^n / \mathrm{image}\, \delta^{n-1}$$

is defined. Induced maps $(\phi^{op}, \tau)_* = \phi^* \tau_*$ for the functor H_n in (5.11) are given by

$$(\phi^* \tau_* f)(\lambda_1', \dots, \lambda_n') = \tau_f \circ f(\phi\lambda_1', \dots, \phi\lambda_n')$$

This completes the definition of the functor H^n in (5.11).

It is proved in Baues-Wirsching [CS] that an equivalence of categories ϕ induces an isomorphism ϕ^* for cohomology groups as above. Moreover a crucial property of this cohomology is the next result:

(5.14) Theorem. *Let $M(\underline{\underline{X}}, D)$ be the set of equivalence classes of linear extensions $D \to \underline{\underline{E}} \to \underline{\underline{X}}$ where $\underline{\underline{X}}$ is a small category. Then there is a natural bijection*

$$\phi : M(\underline{\underline{X}}, D) = H^2(\underline{\underline{X}}, D)$$

which carries the split extension to the trivial element.

If $\underline{\underline{X}} = G$ is a group this is the classical result on the classification of extensions of G. We define the bijection ϕ as follows. Let $s : \mathrm{Mor}(\underline{\underline{X}}) \to \mathrm{Mor}(\underline{\underline{E}})$ be a function with $ps(f) = f$. For $(\lambda_1, \lambda_2) \in \mathrm{Nerve}(\underline{\underline{X}})[2]$ there is a unique element $c(\lambda_1, \lambda_2) \in D_{\lambda_1 \lambda_2}$ satisfying

$$s(\lambda_1 \lambda_2) = s(\lambda_1)s(\lambda_2) + c(\lambda_1, \lambda_2)$$

This defines a cocycle $c \in F^2$ which represents the cohomology class $\phi\{\underline{\underline{E}}\} = \{c\}$. By 'change of universe' we can define the cohomology above also in case $\underline{\underline{X}}$ is not small so that (5.14) remains true.

As an example we now consider the linear extension (5.9) which represents a non-trivial cohomology class $\phi\{\underline{\underline{K}}_1^n\}$; in fact, the functors H^n, H^{n+1} and this cohomology class determine the category $\underline{\underline{K}}_1^n$ up to equivalence. Pirashvili [CE] computed the following restrictions of the class $\phi\{K_1^n\}$.

(5.15) Theorem. *Let π be a finite group and let $\underline{\underline{Gro}}(H^{n+1})_\pi$ be the subcategory of $\underline{\underline{Gro}}(H^{n+1})$ consisting of objects (π, M, k) and morphisms $(1_\pi, f)$. Moreover let $\underline{\underline{K}}_\pi^n$ be the corresponding subcategory of $\underline{\underline{K}}_1^n$. Then one has the linear extension*

$$H^n \xrightarrow{+} \underline{\underline{K}}_\pi^n \to \underline{\underline{Gro}}(H^{n+1})_\pi$$

which is a restriction of the linear extension (5.9). This extension represents the generator

$$\phi\{\underline{\underline{K}}_\pi^n\} \in H^2(\underline{\underline{Gro}}(H^{n+1})_\pi, H^n) = \mathbb{Z}/|\pi|$$

where the right hand side is a cyclic group of order $|\pi|$ = number of elements of π. Moreover the cohomology groups

$$0 = H^i(Gro(H^{n+1})_\pi, H^n),\ i \neq 2$$

are trivial otherwise.

These examples may suffice to show that cohomology of groups and cohomology of categories are both important ingredients of the homotopy classification problem. Further applications of the cohomolohy of categories above can for example be found in Jibladze-Pirashvili [CA], Dwyer-Kan [HM], Moerdijk-Svensson [SL], Pavesic [DC]. Basic properties are described in Baues-Wirsching [CS], Baues [AH], Baues [CH] and Baues-Dreckmann [CH].

§ 6 Simply connected homotopy types and $H\pi$-duality

Any group can be obtained as the fundamental group of a polyhedron. This yields a multifacted relationship between homotopy theory and group theory. There are natural restrictions to avoid the full complexity of homotopy theory. For example one can restrict to homotopy types which are determined by the fundamental group; such homotopy types are the acyclic spaces for which the universal covering space is contractible. Many basic examples in geometry deal with acyclic spaces in which the complexities of higher homotopy theory do not arise. From the point of view of homotopy theory acyclic spaces are extremely special since they are just 1-types or Eilenberg-Mac Lane spaces $K(G,1)$, $G \in \underline{\underline{Gr}}$.

In contrast to acyclic spaces it is natural to consider simply connected spaces which avoid the complexities of group theory arising from the fundamental group. Indeed, for spaces with fundamental group π one has to use the theory of group rings $\mathbb{Z}[\pi]$ and $\mathbb{Z}[\pi]$-modules which are highly intricate algebraic objects. For simply connected spaces only the ring $\mathbb{Z}$ and abelian groups are needed. From now on we deal with simply connected homotopy types.

An important feature of the theory of simply connected homotopy types is an $H\pi$-duality between homology groups and homotopy groups. Though the definitions of these groups are completely different in nature it turned out that they have many properties which are "dual" to each other. This kind of duality is different from Eckmann-Hilton duality discussed in Hilton [DH]. We shall describe various examples of $H\pi$-dual properties though a complete axiomatic characterization is not known. The starting point is again the theorem of Whitehead which yields $H\pi$-dual properties as follows: A simply connected CW-space X is contractible if and only if homology groups, or equivalently homotopy groups vanish so that

$$H_*(X) = 0 \Longleftrightarrow \pi_*(X) = 0. \tag{6.1}$$

Here H_* denotes the reduced homology. A map $f : X \to Y$ between simply connected CW-spaces is a homotopy equivalence if and only if f induces an isomorphism for homology groups, or equivalently homotopy groups, hence

$$H_*(f) \quad \text{is iso} \quad \Longleftrightarrow \pi_*(f) \quad \text{is iso.} \tag{6.2}$$

Moreover for any abelian group A and $n \geq 2$ there are simply connected CW-spaces X, Y with

$$\begin{cases} H_n(X) \cong A \quad \text{and} \quad H_iX = 0 \quad \text{for} \quad i \neq n \\ \pi_n(Y) \cong A \quad \text{and} \quad \pi_iX = 0 \quad \text{for} \quad i \neq n \end{cases} \tag{6.3}$$

The homotopy types of X, Y are well defined by (A, n) and $X = M(A, n)$ is called a Moore space and $Y = K(A, n)$ is called an Eilenberg-Mac Lane space. The next result shows that these spaces are the important building blocks for simply connected homotopy types. First we observe by (6.3) the following realizability result. Let A_i be a sequence of abelian groups, $i \in \mathbb{Z}$, with $A_i = 0$ for $i \leq 1$. Then there exist simply connected CW-spaces X, Y with

$$(6.4) \qquad \begin{cases} H_i(X) = A_i & \text{for} \quad i \geq 0 \\ \pi_i(Y) = A_i & \text{for} \quad i \geq 0 \end{cases}$$

For this we take

$$X = \bigvee_{n \geq 2} M(A_n, n)$$

to be the one point union of Moore spaces and we take

$$Y = \underset{n \geq 2}{\times} K(A_n, n)$$

to be the product of Eilenberg-Mac Lane space (with the CW-topology). All simply connected homotopy types can be obtained by 'twisting' these constructions, see (6.7) below.

In the category $\underline{\underline{Top}}^*$ of pointed spaces one has the notions of <u>fibration</u> and <u>cofibration</u> which are Eckmann-Hilton dual to each other. Compare for example Baues [AH]. We consider pull backs and push outs in $\underline{\underline{Top}}^*$ respectively,

$$\begin{tikzcd} X' \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{pull}"] & X \arrow[d, "a"] \\ Y' \arrow[r, "f"'] & Y \end{tikzcd}$$

$$\begin{tikzcd} X \arrow[r] \arrow[dr, phantom, "\text{push}"] & X'' \\ Y \arrow[u, "b"] \arrow[r, "g"'] & Y'' \arrow[u] \end{tikzcd}$$

where a is a fibration and b is a cofibration. If X is contractible we call $X' \to Y' \to Y$ a <u>fiber sequence</u> and $Y \to Y'' \to X''$ a <u>cofiber sequence</u>. If also Y', Y'' are contractible we write

$$\begin{aligned} X' &= \Omega(Y) = \underline{\text{loop space}} \text{ of } \quad Y, \\ X'' &= \Sigma(Y) = \underline{\text{suspension}} \text{ of } \quad Y. \end{aligned}$$

We have the $H\pi$-dual properties

$$(6.5) \qquad \begin{cases} \Sigma M(A,n) = M(A,n+1), \\ \Omega K(A,n) = K(A,n-1) \end{cases}$$

of Moore spaces and Eilenberg-Mac Lane spaces respectively. Moreover if f and g are null-homotopic we get

$$X' \simeq Y' \times \Omega(Y)$$
$$X'' \simeq Y'' \vee \Sigma(Y)$$

where the right hand side is a product and a one point union respectively. If f and g are not null homotopic we consider X' and X'' as 'twisted' via f and g. Then f is called a classifying map for X' and g is called a coclassifying map for X''.

(6.6) Definition. Let $A_* = (A_n, n \geq 2)$ be a sequence of abelian groups. A homotopy decomposition associated to A_* is a system of fiber sequences $(n \geq 3)$

$$Y_n \to Y_{n-1} \xrightarrow{k_n} K(A_n, n+1)$$

with $Y_2 = K(A_2, 2)$. This implies that Y_n is an n-type and therefore k_n induces the trivial homomorphism on homotopy groups. A homology decomposition associated to A_* is a system of cofiber sequences $(n \geq 3)$

$$X_n \leftarrow X_{n-1} \xleftarrow{k'_n} M(A_n, n-1)$$

with $X_2 = M(A_2, 2)$ where k'_n is required to induce the trivial homomorphism on homology groups.

Homology and homotopy decompositions are $H\pi$-dual constructions for which the following classical result holds (due to Postnikov [HT] and Eckmann-Hilton [HH], Brown-Copeland [HA]). Let $\varinjlim$ and $\varprojlim$ be the direct and inverse limits in $\underline{\underline{Top}}$.

(6.7) Theorem. *Let X be a simply connected CW-space. Then there exists a homology decomposition associated to H_*X and a map*

$$\varinjlim X_n \to X$$

which induces an isomorphism of homology groups. Moreover there exist a homotopy decomposition associated to π_*X and a map

$$X \longrightarrow \varprojlim Y_n$$

which induces isomorphisms of homotopy groups.

Hence each simply connected homotopy type X can be constructed in two ways, either by a homology decomposition or by a homotopy decomposition. The space

$Y_n \simeq P_n X$ may also be obtained by the Postnikov functor in (4.3). Using the Postnikov decomposition Schön [EC] showed that an 'effective' classification of homotopy types of simply connected compact polyhedra is possible. The Whitehead theorem (6.2) and theorem (6.7) somehow show that a simply connected homotopy type is 'generated' by homology groups and in a dual way also by homotopy groups. For this compare also the minimal models in (12.9), (12.11) below. Theorem (6.7), however, does not tell us how to compare two homology decompositions or two homotopy decompositions respectively, that is, we do not know under which condition two such decompositions represent the same homotopy type. For this one has to solve Whitehead's realization problem.

(6.8) Remark. Dwyer-Kan-Smith [TF] construct for a graded abelian group A_* (with $A_i = 0$ for $i \leq 1$) a space $B(A_*)$ which parameterizes all homotopy decompositions associated to A_*. More precisely the set of path components, $\pi_0 B(AS_*)$, coincides with the set of all homotopy types X for which there exists an isomorphism $A_* \cong \pi_*(X)$. The fundamental group of the path component $B_X \subset B(A_*)$, corresponding to X, is the same as the group of homotopy equivalences $\pi_0 E(X)$ of X. In fact, the path component B_X has the homotopy type of the classifying space $B(E(X))$ where $E(X)$ is the topological monoid of homotopy equivalences of X, i.e. $B_X \simeq B(E(X))$.

We now consider the functorial properties of Moore spaces and Eilenberg-Mac Lane spaces respectively. Let $\underline{\underline{Ab}}$ be the category of abelian groups and for $n \geq 2$ let

$$\underline{\underline{K}}^n, \underline{\underline{M}}^n \subset \underline{\underline{Top}}/\simeq \tag{6.9}$$

be the full homotopy categories consisting of spaces $K(A,n)$ and $M(A,n)$ respectively with $A \in \underline{\underline{Ab}}$.

(6.10) Lemma. *The $n-th$ homotopy group functor*

$$\pi_n : \underline{\underline{K}}^n \xrightarrow{\sim} \underline{\underline{Ab}}$$

is an equivalence of categories. The $n-th$ homology group functor

$$H_n : \underline{\underline{M}}^n \longrightarrow \underline{\underline{Ab}}$$

is not an equivalence but a detecting functor, see (5.6).

(6.11) Remark. In fact there is a functor $\underline{\underline{Ab}} \to \underline{\underline{Top}}$ which carries an abelian group A to a space $K(A,n)$. For this we observe that the classifying space $B(H)$ of an abelian topological monoid H is again an abelian topological monoid in a canonical way. Hence we can iterate the classifying space construction and obtain the n-fold classifying space

$$K(A,n) = B \dots B(A)$$

Compare Segal [CC]. On the other hand there is no functor $\underline{\underline{Ab}} \to \underline{\underline{Top}}/\simeq$ which carries A to $M(A,n)$ and which is compatible with the homology H_n. For this we observe that there is actually a linear extension of categories

$$E^n \to \underline{\underline{M}}^n \to \underline{\underline{Ab}}$$

which represents a non trivial class in $H^2(\underline{\underline{Ab}}, E^n)$, see Baues [AH]. The bifunctor E^n on $\underline{\underline{Ab}}$ is given by

$$E^n(A,B) = Ext(A, \Gamma_1^n B)$$

where $\Gamma_1^n B = B \oplus \mathbb{Z}/2$ for $n \geq 2$ and $\Gamma_1^2(B) = \Gamma(B)$ is the quadratic construction of J.H.C. Whitehead [CE].

The lemma and the remark describe a lack of $H\pi$ -duality. We shall describe many further examples of $H\pi$ -dual properties; yet this duality does not cover all important features of homotopy groups and homology groups respectively. In particular homology is often computable while there is still no simply connected (non contractible) finite polyhedron known for which all homotopy groups are computed. The homotopy groups $\pi_* M(A,n)$ of a Moore space are $H\pi$ -dual to the homology groups $H_* K(A,n)$. If A is finitely generated it is a fundamental unsolved problem to compute $\pi_* M(A,n)$. The computation of $H_* K(A,n)$, however, was achieved in the work of Eilenberg-Mac Lane [CW] and Cartan [HC]. For example we have

$$(6.12) \qquad H_{n+2}K(A,n) = \pi_{n+1}M(A,n) = \Gamma_1^n(A)$$

where we use Γ_1^n in (6.11). Recall that $[X,Y]$ denotes the set of homotopy classes of pointed maps $X \to Y$. The homology $H_* K(A,n)$ is used for the computation of the groups

$$[K(A,n),\ K(B,m)]$$

whose elements are also called <u>cohomology operations</u>. In particular the first non trivial classifying map in a homotopy decomposition is such an operation. Applications of cohomology operations are discussed by Steenrod [CO]. On the other hand the groups

$$[M(A,n),\ M(B,m)]$$

are not at all understood; for $A = B = \mathbb{Z}$ these are the homotopy groups of spheres.

A further lack of $H\pi$ -duality is the following result on decompositions in (6.7).

(6.13) <u>Proposition</u>. *The homotopy decomposition of X can be chosen in $\underline{\underline{Top}}$ to be functorial in X. The homology decomposition of X cannot be chosen to be functorial, neither in $\underline{\underline{Top}}$ nor in the homotopy category $\underline{\underline{Top}}/\simeq$.*

Using Eilenberg-Mac Lane spaces and Moore spaces we obtain the groups ($n \geq 2$)

$$\begin{aligned} H^n(X,A) &= [X, K(A,n)], \\ \pi_n(A,X) &= [M(A,n), X] \end{aligned} \tag{6.14}$$

which are called the cohomology of X, resp. the homotopy group of X with coefficients in the abelian group A. Hence the decompositions of X in (6.7) yield elements

$$\begin{aligned} k_nX = k_n &\in H^{n+1}(P_{n-1}X, \pi_nX) \\ k'_nX = k'_n &\in \pi_{n-1}(H_nX, X_{n-1}) \end{aligned} \tag{6.15}$$

Here k_nX is actually an invariant of the homotopy type of X in the sense that a map $f : X \to Y$ satisfies

$$(P_{n-1}f)^*k_nY = (\pi_nf)_*k_nX \tag{6.16}$$

in $H^{n+1}(P_{n-1}X, \pi_nY)$. Here we use the Postnikov functor P_{n-1} and the naturality of the Postnikov decomposition in (6.13). The element k_nX in (6.15) is called the $n-th$ k-invariant or Postnikov invariant of X. The element k'_nX given by a homology decomposition of X is not an invariant of X since the homotopy type of X_n is not well defined by the homotopy type of X. We shall describe below new invariants of X which are $H\pi$ -dual to Postnikov invariants and which we call boundary invariants of X. They are given by the 'invariant portion' of the elements k'_nX; see (8.10) below.

The groups in (6.14) are part of natural short exact sequences which are $H\pi$ -dual to each other:

$$\begin{aligned} &Ext(H_{n-1}X, A) \overset{\Delta}{\rightarrowtail} H^n(X,A) \overset{\mu}{\twoheadrightarrow} Hom(H_nX, A) \\ &Ext(A, \pi_{n+1}X) \overset{\Delta}{\rightarrowtail} \pi_n(A,X) \overset{\mu}{\twoheadrightarrow} Hom(A, \pi_nX) \end{aligned} \tag{6.17}$$

Here the surjection μ carries $\varphi : X \to K(A,n)$, resp. $\psi : M(A,n) \to X$, to the induced map

$$\begin{aligned} &H_n\varphi : H_nXC \to H_nK(A,n) = A, \quad \text{resp.} \\ &\pi_n\psi : A = \pi_nM(A,n) \to \pi_nX. \end{aligned}$$

The exact sequence for $H^n(X,A)$ is always split (unnaturally) while the exact sequence for $\pi_n(A,X)$ needs not to be split. We point out that the cohomology $H^n(X,A)$ may also be defined by

$$H^n(X,A) = [C_*X, C_*M(A,n)] \tag{6.18}$$

Here C_* is the singular chain complex and the right hand side denotes the set of homotopy classes of chain maps. Dually we define the pseudo-homology

$$H_n(A, X) = [C_* M(A, n), C_* X] \tag{6.19}$$

which yields a well defined bifunctor $\underline{\underline{Ab}}^{op} \times \underline{\underline{Top}} \to \underline{\underline{Ab}}$. This is the analogue of $\pi_n(A, X)$ in the category of chain complexes. As in (6.17) one has the natural short exact sequence

$$Ext(A, H_{n+1}X) \overset{\Delta}{\rightarrowtail} H_n(A, X) \overset{\mu}{\twoheadrightarrow} Hom(A, H_n X) \tag{6.20}$$

which is always split (unnaturally).

§ 7 The Hurewicz homomorphism

Homology groups and homotopy groups are connected by the Hurewicz homomorphism

$$(7.1) \qquad h = h_n X : \pi_n X \to H_n X$$

This is the special case $A = \mathbb{Z}$ of the homomorphism

$$h^A = h_n(A, X) : \pi_n(A, X) \to H_n(A, X)$$

which carries $\psi : M(A,n) \to X$ to the induced chain map $C_*\psi$. These homomorphisms are compatible with the short exact $\Delta - \mu$ -sequences in (6.17) and (6.7), and they are natural in X and hence invariants of the homotopy type of X. In fact, the next result shows that the Hurewicz homomorphism has a strong impact on homotopy types.

(7.1) Proposition. *Let X be a simply connected CW-space. Then (A) and (B) hold.*

(A) *The Hurewicz homomorphism $h_n X$ is split injective for all n if and only if X has the homotopy type of a product of Eilenberg-Mac Lane spaces.*

(B) *Moreover $h_n X$ is split surjective for all n if and only if X has the homotopy type of a one point union of Moore spaces.*

Properties (A) and (B) form a further nice example of $H\pi$ -duality.

Proof. (A) Let r_n be a retraction of $h_n X$ and let $f_n \in H^n(X, \pi_n X)$ be a map with $\mu(f_n) = r_n$, see (6.17). Then the collection $\{f_n\}$ defines a map

$$f : X \longrightarrow \underset{n \geq 2}{\times} K(\pi_n X, n)$$

which is a homotopy equivalence by the Whitehead theorem.
(B) Let s_n be a splitting of $h_n X$ and let $g_n \in \pi_n(H_n X, X)$ be a map with $\mu(g_n) = s_n$. Then the collection $\{g_n\}$ defines a map

$$g : \bigvee_{n \geq 2} M(H_n X, n) \longrightarrow X$$

which is a homotopy equivalence by the Whitehead theorem. q.e.d.

We now discuss topological analogues of the Hurewicz homomorphism. We consider for a simply connected CW-complex X the infinite symmetric product $SP_\infty = \lim SP_n X$ where

$$(7.3) \qquad SP_n X = X^n / S_n$$

is the space of orbits of the action of the symmetric group S_n on the n-fold product $X^n = X \times \ldots \times X$ obtained by permuting coordinates. The map $SP_{n-1}X \to SP_n X$

is induced by the inclusion $X^{n-1} = X^{n-1} \times * \subset X^n$ where $*$ is the base point of X. The inclusion

$$X = SP_1X \to SP_\infty X$$

induces the Hurewicz homomorphism

$$h_nX : \pi_nX \to \pi_nSP_\infty X = H_nX$$

where the right hand side is the Dold-Thom isomorphism [QU]. Let ΓX be the homotopy fiber of $X \subset SP_\infty X$ so that

$$\Gamma X \to X \to SP_\infty X \tag{7.4}$$

is a fiber sequence. Using the simplicial group GX there is an alternative way to obtain this fiber sequence by the short exact sequence

$$\Gamma_2X \rightarrowtail GX \twoheadrightarrow AX \tag{7.4'}$$

where AX is the abelianization and where Γ_2X is the commutator subgroup of GX. Then $\Gamma X \simeq B|\Gamma_2X|$ is the classifying space of the realization of Γ_2X and the functor $B|\ \ |$ applied to (7.4') yields (7.4) up to homotopy equivalence. For this compare Kan [HC] who as well proved that $GX \to AX$ induces the Hurewicz homomorphism. Using the skeleta X^n of a CW-complex J.H.C. Whitehead introduced the Γ -groups of X given by

$$\Gamma_nX = \operatorname{image}(\pi_n X^{n-1} \to \pi_nX^n) \tag{7.5}$$

where the homomorphism is induced by the inclusion $X^{n-1} \subset X^n$. Moreover we introduce in Baues [HT] the Γ -groups with coefficients $\Gamma_n(A, X)$ by the following push-pull diagram derived from the $\Delta - \mu$ -sequence (6.7)

$$\begin{tikzcd}
Ext(A,\Gamma_{n+1}X) \arrow[d, "\Delta"'] \arrow[dr, phantom, "\text{push}"] & Ext(A,\pi_{n+1}X^n) \arrow[l, "p_*"'] \arrow[d, tail] \arrow[dr, tail, "\Delta"] & \\
\Gamma_n(A,X) \arrow[dr, "\mu"'] & P \arrow[l] \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\text{pull}"] & \pi_n(A;X^n) \arrow[d, two heads, "\mu"] \\
 & Hom(A,\Gamma_nX) \arrow[r, "i_*"'] & Hom(A,\pi_nX^n)
\end{tikzcd}$$

Here $i : \Gamma_nX \subset \pi_nX^n$ is the inclusion and $\varphi : \pi_{n+1}X^n \twoheadrightarrow \Gamma_{n+1}X$ is the projection defined by (7.5).

(7.6) Proposition. *let X be a simply connected CW-complex. Then there are natural isomorphisms*

(a) $\Gamma_n X = \pi_n \Gamma X$
(b) $\Gamma_n(A,X) = \pi_n(A, \Gamma X)$
(c) $H_n X = \pi_n SP_\infty X$
(d) $H_n(A,X) = \pi_n(A, SP_\infty X)$

The isomorphisms which we shall use as identifications are compatible with $\Delta - \mu$ exact sequences above.

Here (a) and (c) are due to Kan [HC] and Dold-Thom [QU] respectively. Hence the long exact sequence of homotopy groups for the fiber sequence (7.4) yields by identification, as in (7.6), the exact sequences

$$\text{(7.7)}\qquad \begin{aligned} &\ldots \longrightarrow H_{n+1}X \xrightarrow{b} \Gamma_n X \xrightarrow{i} \pi_n X \xrightarrow{h} H_n X \xrightarrow{b} \ldots \\ &\ldots \longrightarrow H_{n+1}(A,X) \xrightarrow{b^A} \Gamma_n(A,X) \xrightarrow{i^A} \pi_n(A,X) \xrightarrow{h^A} H_n(A,X) \xrightarrow{b^A} \ldots \end{aligned}$$

in which all operators are compatible with the $\Delta - \mu$ exact sequences. We call these the <u>Γ-sequence</u> and the <u>Γ-sequence with coefficients in</u> A respectively. Hence kernel and cokernel of the Hurewicz homomorphisms can be determined by the operators i, b in these sequences. Here i and i^A are induced by $X^n \subset X$ and b is the <u>secondary boundary operator</u> of J.H.C. Whitehead. In Baues [HT] (II.3.5) we give also an explicit description of the operator b^A. The Γ -sequence coincides with the classical certain exact sequence of J.H.C. Whitehead which is the special case, $A = \mathbb{Z}$, of the second exact sequence. Clearly both exact sequences are invariants of the homotopy type of X. In fact, J.H.C. Whitehead [CE] used part of the Γ -sequence as a classifying invariant of a simply connected 4-dimensional homotopy type.

The definition of $\Gamma_n X$ in (7.5) shows that this group depends only on the $(n-1)$ -type of X, in fact we have the natural isomorphism

$$\text{(7.8)}\qquad \Gamma_k(X) = \Gamma_k(P_{n-1}X),\ k \leq n,$$

induced by a map $p_{n-1} : X \to P_{n-1}X$ which extends the inclusion $X^n \subset P_{n-1}X$, see (4.3). Moreover the map p_{n-1} applied to the Γ -sequence of X yields natural isomorphisms

$$\text{(7.9)}\qquad \begin{aligned} H_n P_{n-1} X &= \Gamma''_{n-1} P_{n-1} X = \Gamma''_{n-1}(X) \\ H_{n+1} P_{n-1} X &= \Gamma_n P_{n-1} X = \Gamma_n(X) \end{aligned}$$

where $\Gamma''_{n-1}X = kernel(\Gamma_{n-1}X \to \pi_{n-1}X)$. These groups are used in the following result on the <u>'realizability of Hurewicz homomorphisms'</u>, proved in III.4.7 of Baues [HT].

(7.10) Theorem. *Let Y be a simply connected $(n-1)$ -type and let*

$$(*) \qquad H_1 \longrightarrow \Gamma_n Y \longrightarrow \pi \longrightarrow H_0 \longrightarrow \Gamma''_{n-1} Y \longrightarrow 0$$

be an exact sequence of abelian groups where H_1 is free abelian. Then there exists an $(n+1)$ -dimensional complex X and a map $p: X \to Y$ inducing isomorphisms $\pi_k X \cong \pi_k Y$ for $k \leq n-1$ together with a commutative diagram

$$\begin{tikzcd}
H_{n+1}X \arrow[r] \arrow[d, "\cong"'] & \Gamma_n X \arrow[r] \arrow[d, "\cong"', "p_*"] & \pi_n X \arrow[r, "h"] \arrow[d, "\cong"'] & H_n X \arrow[r] \arrow[d, "\cong"'] & \Gamma''_{n-1} X \arrow[r] \arrow[d, "\cong"', "p_*"] & 0 \\
H_1 \arrow[r] & \Gamma_n Y \arrow[r] & \pi \arrow[r] & H_0 \arrow[r] & \Gamma''_{n-1} Y \arrow[r] & 0
\end{tikzcd}$$

in which all vertical arrows are isomorphisms. The top row is part of the Γ -sequence of X. The space Y together with the sequence (*) in general does not determine the homotopy type of X.

The result shows exactly what kind of abstract homomorphisms $\pi \to H_0$ between abelian groups can be realized as a Hurewicz homomorphism $\pi_n \to H_n$ of a space with a given $(n-1)$ -type. This also demonstrates to what extend homotopy groups and homology groups depend on each other.

(7.11) __Example__. We may choose for Y in (7.10) an Eilenberg-Mac Lane space

$$Y = K(A,k) \quad \text{with} \quad 2 \leq k \leq n-1$$

Then the groups, see (7.9),

$$\Gamma_n Y = H_{n+1}\, K(A,k)$$
$$\Gamma''_{n-1} Y = H_n\, K(A,k)$$

are known by the work of Eilenberg-Mac Lane and Cartan. Hence any exact sequence

$$H_1 \longrightarrow H_{n+1}\, K(A,k) \longrightarrow \pi \longrightarrow H_0 \longrightarrow H_n K(A,k) \longrightarrow 0$$

with H_1 free abelian can be realized as a Γ -sequence of an $(n+1)$ -dimensional CW-complex X with $P_{n-1}X = K(A,k)$. For example for $k=5$, $n=9$ we have

$$H_{10}\, K(A,5) = \Lambda^2(A) \oplus A * \mathbb{Z}/6$$
$$H_9\, K(A,5) = A \otimes \mathbb{Z}/6$$

where Λ^2 is the exterior square and $\mathbb{Z}/6$ is the cyclic group of order 6. Hence for any exact sequence

$$H_{10} \longrightarrow \Lambda^2(A) \oplus A * \mathbb{Z}/6 \longrightarrow \pi_9 \longrightarrow H_9 \longrightarrow A \otimes \mathbb{Z}/6 \longrightarrow 0$$

of abelian groups with H_{10} free abelian there exists a 10-dimensional CW-complex X with $\pi_5 X = A$, $\pi_i = 0$ for $i < 5$ and $5 < i < 9$, such that this sequence is part of the Γ -sequence of X.

§ 8 Postnikov invariants and boundary invariants

Recall that the $n-th$ Postnikov invariant of a simply connected space X is an element

$$(8.1) \qquad k_nX \in H^{n+1}(P_{n-1}X, \pi_nX)$$

This element is highly related to the Γ -sequence of X. For this we observe that by (6.17) and (7.9) we obtain the natural short exact sequence

$$(8.2) \qquad Ext(\Gamma''_{n-1}X, A) \overset{\Delta}{\rightarrowtail} H^{n+1}(P_{n-1}X, A) \overset{\mu}{\twoheadrightarrow} Hom(\Gamma_nX, A)$$

Each element $k \in H^{n+1}(P_{n-1}X, A)$ yields elements

$$\begin{aligned} k_* &= \mu(k) \in Hom(\Gamma_nX, A) \\ k_\dagger &= \Delta^{-1}q_*(k) \in Ext(\Gamma''_{n-1}X, cok\, k_*) \end{aligned}$$

where $q: A \twoheadrightarrow cok(k_*)$ is the projection of the cokernel of k_*. We have by $(X, A) \mapsto H^{n+1}(P_{n-1}X, A)$ a bifunctor in X and A.

(8.3) Theorem on Postnikov invariants. *To each 1-connected CW-space X there is canonically associated a sequence of elements $(k_3, k_4, \dots)$ with*

$$k_n = k_nX \in H^{n+1}(P_{n-1}X, \pi_nX)$$

such that the following properties are satisfied:

(a) *Naturality: For a map $F: X \to Y$ we have*

$$(\pi_nF)_*(k_nX) = F^*(k_nY) \in H^{n+1}(P_{n-1}X, \pi_nY)$$

(b) *Compatibility with i_nX in the Γ -sequence:*

$$(k_nX)_* = i_nX \in Hom(\Gamma_nX, \pi_nX)$$

(c) *Compatibility with the extension H_nX in the Γ -sequence:*

$$(k_nX)_\dagger = \{H_nX\} \in Ext(\Gamma''_{n-1}X, cok\, i_nX)$$

Here the extension element $\{H_nX\}$ is given by the exact Γ -sequence of X,

$$\Gamma_nX \xrightarrow{i_nX} \pi_nX \to H_nX \to \Gamma''_{n-1}X \to 0$$

(d) *Vanishing condition: All Postnikov invariants are trivial if and only if X has the homotopy type of a product of Eilenberg-Mac Lane spaces.*

This result which partially seems to be unknown is proved in (II.5.10) of Baues [HT]. We now introduce new invariants which are $H\pi$ -dual to the Postnikov invariants above. For this we first define the subgroup

$$\Gamma''_{n-1}(A,X) \subset \Gamma_{n-1}(A,X) \tag{8.4}$$

obtained by all elements $\alpha \in \Gamma_{n-1}(A,X)$ for which $\mu(\alpha)(A) \subset \Gamma''_{n-1}X$. Hence one has the short exact sequence

$$Ext(A,\Gamma_n X) \overset{\Delta}{\rightarrowtail} \Gamma''_{n-1}(A,X) \overset{\mu}{\twoheadrightarrow} Hom(A,\Gamma''_{n-1}X) \tag{8.5}$$

Here Γ''_{n-1} is actually a bifunctor in $A \in \underline{\underline{Ab}}$ and $X \in \underline{\underline{Top}}^*/\simeq$. To see this we observe that the map $p_{n-1} : X \to P_{n-1}X$ induces a binatural isomorphism

$$\Gamma''_{n-1}(A,X) = H_n(A,P_{n-1}X) \tag{8.6}$$

Here the right hand side is the pseudo homology and we use the Γ -sequence with coefficients in A and (7.9). Since $b_nX : H_nX \to \Gamma_{n-1}X$ yields a surjection $b_nX : H_nX \to \Gamma''_{n-1}X$ we see that the boundary operator b^A in the Γ -sequence with coefficients maps to $\Gamma''_{n-1}(A,X)$. Hence we obtain the following commutative diagram which is natural in $A \in \underline{\underline{Ab}}$ and simply connected spaces X.

$$\begin{array}{ccccc}
Ext(A,H_{n+1}X) & \xrightarrow{\Delta} & H_n(A,X) & \xrightarrow{\mu} & Hom(A,H_nX) \\
\big\downarrow{\scriptstyle (b_{n+1}X)_*} & & \big\downarrow{\scriptstyle b^A} & & \big\downarrow{\scriptstyle (b_nX)_*} \\
Ext(A,\Gamma_nX) & \xrightarrow{\Delta} & \Gamma''_{n-1}(A,X) & \longrightarrow & Hom(A,\Gamma''_{n-1}X)
\end{array}$$

(8.7) Definition. Consider this diagram for $A = H_nX$ and let $\bar{1} \in H_n(H_nX,X)$ be an element with $\mu(\bar{1})$ = identity of H_nX. Then the coset of $b^A(\bar{1})$ modulo the image of $\Delta(b_{n+1}X)_*$ is the <u>boundary invariant</u> β_nX of X, that is

$$\beta_nX = \{b^A(\bar{1})\} \in \frac{\Gamma''_{n-1}(H_nX,X)}{im\,(\Delta(b_{n+1}X)_*)}$$

We have the short exact sequence

$$Ext(A, cok\, b_{n+1}X) \overset{\Delta}{\rightarrowtail} \frac{\Gamma''_{n-1}(A,X)}{im\,(\Delta(b_{n+1}X)_*)} \overset{\mu}{\twoheadrightarrow} Hom(A,\Gamma''_{n-1}X) \tag{8.8}$$

which is natural in $A \in \underline{\underline{Ab}}$ and simply connected spaces X. This sequence is the $H\pi$ -dual of the sequence in (8.2) above. Each element

$$\beta \in \frac{\Gamma''_{n-1}(A,X)}{im\,(\Delta(b_{n+1}X)_*)}$$

yields elements

$$\beta_* = \mu(\beta) \in Hom(A, \Gamma''_{n-1}X)$$
$$\beta_\dagger = \Delta^{-1} j^*(\beta) \in Ext(ker\,\beta_*, cok\, b_{n+1}X)$$

where $j : ker\,\beta_* \subset A$ is the inclusion of the kernel of β_*. The next result is the $H\pi$ -dual of the 'theorem on Postnikov invariants' in (8.3).

(8.9) <u>Theorem on boundary invariants</u>. *To each 1-connected CW-space X there is canonically associated a sequence of elements $(\beta_3, \beta_4, \dots)$ with*

$$\beta_n = \beta_n X \in \frac{\Gamma''_{n-1}(H_n X, X)}{im\,\Delta(b_{n+1}X)_*}$$

such that the following properties are satisfied:

(a) *<u>Naturality</u>: For a map $F : X \to Y$ we have*

$$(H_n F)^*(\beta_n Y) = F_*(\beta_n X) \in \frac{\Gamma''_{n-1}(H_n X, Y)}{im\,\Delta(b_{n+1}X)_*}$$

(b) *<u>Compatibility with</u> $b_n X$ <u>in the</u> Γ <u>-sequence</u>:*

$$(\beta_n X)_* = b_n X \in Hom(H_n X, \Gamma''_{n-1}X)$$

(c) *<u>Compatibility with the extension</u> $\pi_n X$ <u>in the</u> Γ <u>-sequence</u>:*

$$(\beta_n X)_\dagger = \{\pi_n X\} \in Ext(ker\, b_n X, cok\, b_{n+1}X)$$

Here the extension element $\{\pi_n X\}$ is determined by the exact Γ -sequence of X,

$$H_{n+1} \xrightarrow{b_{n+1}X} \Gamma_n X \to \pi_n X \to H_n X \xrightarrow{b_n X} \Gamma''_{n-1}X$$

(d) *<u>Vanishing condition</u>: All boundary invariants are trivial if and only if X has the homotopy type of a one point union of Moore spaces.*

This result is proved in II.6.9 of Baues [HT].

(8.10) <u>Remark</u>. The boundary invariants have the following connection with the coclassifying maps k'_n in a homology decomposition of X. For this let $X = \lim X_n$ be given by a homology decomposition. Then X is a CW-complex with skeleta X^n and there are inclusions

$$X^{n-1} \subset X_n \subset X^n.$$

Moreover the classifying map k'_n can be chosen such that the following diagram commutes, $H_n = H_n X$.

$$\begin{tikzcd}
 & X_{n-1} \\
M(H_n, n-1) \arrow[r, "\beta"] \arrow[ur, "k_n'"] & X^{n-1} \arrow[u] \\
M(H_n, n-1)^{n-2} \arrow[r] \arrow[u] & X^{n-2} \arrow[u]
\end{tikzcd}$$

Hence β represents an element in $\Gamma_{n-1}(H_n, X)$ by using the definition in (7.5) and this element represents the boundary invariant $\beta_n X$. Therefore (8.9) (c) yields an explicit formula how to derive $\pi_n X$ from k_n'.

(8.11) ***Remark***. Let $\underline{\underline{C}}$ be a homotopy category of simply connected spaces. Then we have the functors

$$\Gamma_n, \Gamma''_{n-1} : \underline{\underline{C}} \to \underline{\underline{Ab}} \tag{1}$$

which both appear in a dual fashion in the natural exact sequences (8.2) and (8.5). There is an obstruction $\mathcal{O}$ for the existence of a splitting of (8.2) which is natural in $X \in \underline{\underline{C}}$ and $A \in \underline{\underline{Ab}}$. This obstruction is an element in the cohomology of $\underline{\underline{C}}$,

$$\mathcal{O} \in H^1(\underline{\underline{C}}, Ext(\Gamma''_{n-1}, \Gamma_m). \tag{2}$$

Here $Ext(\Gamma''_{n-1}, \Gamma_n)$ is the natural system which carries $f : X \to Y$ in $\underline{\underline{C}}$ to the abelian group $Ext(\Gamma''_{n-1}X, \Gamma_n Y)$. The element $\mathcal{O}$ determines the extension (8.2) as a bifunctor in $X \in \underline{\underline{C}}$, $A \in \underline{\underline{Ab}}$ up to equivalence. On the other hand there is an obstruction $\mathcal{O}'$ for the existence of a splitting of (8.5) which is natural in $X \in \underline{\underline{C}}$ and $A \in \underline{\underline{Ab}}$. This obstruction turns out to be as well an element in the cohomology (2),

$$\mathcal{O}' \in H^1(\underline{\underline{C}}, Ext(\Gamma''_{n-1}, \Gamma_n)) \tag{3}$$

Again $\mathcal{O}'$ determines the extension (8.5) as a functor in $X \in \underline{\underline{C}}$, $A \in \underline{\underline{Ab}}$ up to equivalence. Now the extension (8.2) and (8.5) are dual in the explicit sense that the elements (2) and (3) actually coincide; that is $\mathcal{O} = \mathcal{O}'$. This is proved in III. § 3 of Baues [HT]. In the next section we use the extensions (8.2) and (8.5) in a crucial way to obtain models of homotopy types which are $H\pi$ -dual to each other.

§ 9 The classification theorems

We now show that k-invariants and boundary invariants both can be used to classify homotopy types. For this we choose a full subcategory

$$(9.1) \qquad \underline{\underline{C}} \subset (n-1) - \underline{\underline{types}}$$

consisting of simply connected $(n-1)$ -types. For example we can take for $1 < k < n$ the category $\underline{\underline{C}} = \underline{\underline{K}}^k \cong \underline{\underline{Ab}}$ consisting of Eilenberg-Mac Lane spaces $K(A,k)$ with $A \in \underline{\underline{Ab}}$. We consider the functor

$$(9.2) \qquad P_n : \underline{\underline{spaces}}^{n+1}(\underline{\underline{C}}) \to n - \underline{\underline{types}}(\underline{\underline{C}})$$

where the left hand side is the full homotopy catgeory of $(n+1)$ -dimensional CW-spaces U for which the $(n-1)$ -Postnikov section $P_{n-1}U$ is in $\underline{\underline{C}}$, similarly the right hand side is the full homotopy catgeory of n-types $\mathcal{V}$ for which $P_{n-1}\mathcal{V}$ is in $\underline{\underline{C}}$. The functor P_n is the restriction of the Postnikov functor in (4.3). In the next definition we use the new word 'kype' which is an amalgamation of k-invariant and type.

(9.3)Definition. Let $\underline{\underline{C}}$ be a category as in (9.1). A $\underline{\underline{C}}$ -kype

$$\bar{X} = (X, \pi, k, H, b)$$

is a tuple consisting of an object X in $\underline{\underline{C}}$, abelian groups π, H and elements

$$k \in H^{n+1}(X, \pi)$$
$$b \in Hom(H, \Gamma_n X)$$

such that the sequence

$$H \xrightarrow{b} \Gamma_n X \xrightarrow{k_*} \pi$$

is exact, see (8.2). A morphism between $\underline{\underline{C}}$ -kypes

$$(f, \varphi, \psi) : (X, \pi, k, H, b) \to (X', \pi', k', H', b')$$

is given by a map $f : X \to X'$ in $\underline{\underline{C}}$ and homomorphisms $\varphi : \pi \to \pi'$, $\psi : H \to H'$ between abelian groups such that

$$f^*(k') = \varphi_*(k)$$
$$(\Gamma_n f) b = b' \psi$$

The $\underline{\underline{C}}$ -kype $\bar{X}$ is free, resp. injective, if H is free abelian, resp. b is an injective homomorphism. Let $\underline{\underline{Kypes}}(\underline{\underline{C}})$, resp. $\underline{\underline{kypes}}(\underline{\underline{C}})$ be the categories of free, resp. injective $\underline{\underline{C}}$ -kypes with morphisms as above. We have the forgetful functor

$$\phi : \underline{\underline{Kypes(\underline{\underline{C}})}} \to \underline{\underline{kypes(\underline{\underline{C}})}}$$

which carries (X, π, k, H, b) to $(X, \pi, k.H', b')$ where H' is the image of b and where b' is the inclusion of this image. The functor ϕ is easily seen to be full and representative.

Recall that a 'detecting' functor is a functor which reflects isomorphisms and is full and representative.

(9.4) **<u>Classification by Postnikov invariants</u>.** *There are detecting functors Λ, λ for which the following diagram of functors commutes up to natural isomorphism.*

$$\begin{tikzcd}
\underline{\underline{spaces}}^{n+1}(\underline{\underline{C}}) \arrow[r, "\Lambda"] \arrow[d, "P_n"] & \underline{\underline{Kypes}}(\underline{\underline{C}}) \arrow[d, "\phi"] \\
n - \underline{\underline{types}}(\underline{\underline{C}}) \arrow[r, "\lambda"] & \underline{\underline{kypes}}(\underline{\underline{C}})
\end{tikzcd}$$

Here the functor Λ carries the space X to the free $\underline{\underline{C}}$ -kype

$$\Lambda(X) = (P_{n-1}X, \pi_n X, k_n X, H_{n+1}X, b_{n+1}X) \tag{9.5}$$

given by the Postnikov invariant (8.1), see (8.3). We point out that only the detecting functor λ is a classical result of Postnikov, the existence of the detecting functor Λ seems to be a new property of k-invariants which did not appear in the literature. Theorem (9.4) is proved in III.4.4 of Baues [HT].

Using boundary invariants we obtain the $H\pi$ -dual of the classification theorem above. We are now going to use a new word 'bype' which is an amalgamation of boundary invariant and type.

(9.6) Definition. Let $\underline{\underline{C}}$ be a category as in (9.1). A <u>$\underline{\underline{C}}$ -bype</u>

$$\bar{X} = (X, H_0, H_1, b, \beta)$$

is a tuple consisting of an object X in $\underline{\underline{C}}$, abelian groups H_0, H_1 and elements

$$b \in Hom(H_1, \Gamma_n X)$$
$$\beta \in \frac{\Gamma''_{n-1}(H_0, X)}{im\,(\Delta\, b_*)}$$

Here we use Δ in (8.5) and $b_* : Ext(H_0, H_1) \to Ext(H_0, \Gamma_n X)$. Moreover the induced homomorphism

$$\beta_* = \mu(\beta) : H_0 \twoheadrightarrow \Gamma''_{n-1}X$$

is surjective, see (8.5). A <u>morphism</u> between $\underline{\underline{C}}$ -bypes

$$(f,\varphi_0,\varphi_1):(X,H_0,H_1,b,\beta)\to(X',H_0',H_1',b',\beta')$$

is given by a morphism $f:X\to X'$ in $\underline{\underline{C}}$ and by homomorphisms $\varphi_0:H_0\to H_0'$, $\varphi_1:H_1\to H_1'$ such that

$$(\Gamma_n f)b = b'\varphi_0$$
$$f_*(\beta)=\varphi_0^*(\beta')$$

The $\underline{\underline{C}}$ -bype $\bar{X}$ is free, resp. injective, if H_1 is a free abelian group, resp. b is an injective homomorphism. Let $\underline{\underline{Bypes}}(\underline{\underline{C}})$, resp. $\underline{\underline{bypes}}(\underline{\underline{C}})$, be the categories of free, resp. injective, $\underline{\underline{C}}$ -bypes with morphisms as above. We have the forgetful functor

$$\phi:\underline{\underline{Bypes}}(\underline{\underline{C}})\to\underline{\underline{bypes}}(\underline{\underline{C}})$$

which carries (X,H_0,H_1,b,β) to (H,H_0,H_0',b',β) where H_1' is the image of b and where b' is the inclusion of this image. The functor ϕ is full and representative.

(9.7) Classification by boundary invariants. *There are detecting functors Λ', λ' for which the following diagram of functors commutes up to natural isomorphism.*

$$\begin{array}{ccc}\underline{\underline{spaces}}^{n+1}(\underline{\underline{C}}) & \xrightarrow{\Lambda'} & \underline{\underline{Bypes}}(\underline{\underline{C}})\\ \downarrow{\scriptstyle P_n} & & \downarrow{\scriptstyle \phi}\\ n-\underline{\underline{types}}(\underline{\underline{C}}) & \xrightarrow{\lambda'} & \underline{\underline{bypes}}(\underline{\underline{C}})\end{array}$$

Here the functor Λ' carries the space U to the free $\underline{\underline{C}}$ -bype

$$\Lambda'(U)=(P_{n-1}X,H_nX,H_{n+1}X,b_{n+1}X,\beta_nX) \tag{9.8}$$

given by the boundary invariant β_nX in (8.7), see (8.9). The classification theorem (9.7) is proved in III.4.4 of Baues [HT]. It shows that boundary invariants can be used in the same way as Postnikov invariants for the classification of homotopy types. In the book Baues [HT] we give many explicit examples of applications for the classification theorems above.

(9.9) Remark. J.H.C. Whitehead [CE] obtained for the homotopy category of simply connected 4-dimensinal CW-spaces two detecting functors. These coincide exactly with Λ and Λ' above if we take $n=3$ and $\underline{\underline{C}}=\underline{\underline{K}}^2$. This is, in fact, a very simple case of the classification theorems above for which we use

$$\Gamma_3K(A,2)=\Gamma(A)$$
$$\Gamma_2''K(A,2)=0$$

We leave this as an exercise to the reader, see also (10.8) below. In Baues [HT] we use (9.7) for the classification of simply connected 5-dimensional homotopy types.

§ 10 Stable homotopy types

The suspension Σ is an endofunctor of the homotopy category $\underline{\underline{Top}}^*/\simeq$ given by the quotient space

$$\Sigma X = I \times X/(\{0\} \times X \cup I \times * \cup \{1\} \times X) \tag{10.1}$$

where $I = [0,1]$ is the unit interval, see also (6.5). The functor Σ carries a map $f : X \to Y$ to $\Sigma f : \Sigma X \to \Sigma Y$ with $(\Sigma f)(t,x) = (t, fx)$ for $t \in I$, $x \in X$. It is easy to see that Σ carries homotopic maps to homotopic maps. We say that two finite dimensional CW-complexes X, Y are stably homotopy equivalent if there is $k \geq 0$ and a homotopy equivalence $\Sigma^k X \simeq \Sigma^k Y$ where Σ^k is the k-fold suspension. A stable homotopy type is a class of stably homotopy equivalent CW-complexes. Since the classification of homotopy types is so hard Spanier-Whitehead [FA] supposed that stable homotopy types might give a first approximation of the homotopy classification problem which is easier to understand. For this the 'stable homotopy theory of spectra' was invented which, however, turned out to be still an extremely complicated world, see G.W. Whitehead [RA].

The impact of the suspension operator Σ comes from a classical result of Freudenthal which we state in the following form.

(10.2) Freudenthal suspension theorem. *Let $\underline{\underline{spaces}}_n^k$ be the full homotopy category in $\underline{\underline{Top}}^*/\simeq$ consisting of $(n-1)$ -connected $(n+k)$ -dimensional CW-complexes, $n \geq 1$, $k \geq 0$. Then the suspension yields a functor*

$$\Sigma : \underline{\underline{spaces}}_n^k \to \underline{\underline{spaces}}_{n+1}^k$$

which is an equivalence of (additive) categories for $k+1 < n$ and which is a detecting functor for $k+1 = n$. Moreover for $k = n$ this functor is representative.

Compare for example Gray [HT].

For $n \geq 2$ the functor Σ in the theorem reflects isomorphisms. This follows from the Whitehead theorem (6.2) since the (reduced) homology of a suspension satisfies

$$H_n \Sigma X = H_{n-1} X \quad \text{for all} \quad n. \tag{10.3}$$

As pointed out in § 3 the main numerical invariants of a homotopy type are dimension and degree of connectedness. These invariants are of particular importance in the theory of manifolds. Therefore it is natural to consider for given n, k the properties of $(n-1)$ -conneted $(n+k)$ -dimensional CW-complexes which J.H.C. Whitehead [HT] called A_n^k -polyhedra. The A_n^k -polyhedra, $n \geq 1$, are the objects in the homotopy categories of the sequence

$$\underline{\underline{spaces}}_1^k \xrightarrow{\Sigma} \underline{\underline{spaces}}_2^k \to \dots \underline{\underline{spaces}}_n^k \xrightarrow{\Sigma} \underline{\underline{spaces}}_{n+1}^k \to$$

which by Freudenthal's theorem above 'stabilizes' for $n \geq k+2$. Hence there are only $k+2$ different categories in this sequence. This also shows that the stable homotopy types of A_n^k -polyhedra ($n \geq 0$) can be identified with the homotopy types in the category $\underline{\underline{spaces}}_n^k$, $n \geq k+1$. We say that A_n^k -polyhedra are stable if $n \geq k+1$.

Each homotopy type of an A_n^k -polyhedron can be represented by a (reduced) CW-complex X with $X^{n-1} = *$ and $dim(X) = n+k$. Hence $X - \{*\}$ has only cells in dimension $n, n+1, \ldots, n+k$. For $k=0$ the CW-complex X is thus a one point union of n-spheres. This also shows that one has equivalences of categories

$$(10.5) \qquad \begin{aligned} \underline{\underline{spaces}}_1^0 &= \text{ category of free groups} \\ \underline{\underline{spaces}}_2^0 &= \text{ category of free abelian groups} \end{aligned}$$

where Σ coincides with the abelianization functor for groups. For $k > 0$ the algebraic models of the categories in (10.4) get more complicated. J.H.C. Whitehead [CE], [HT], [CHII] studied the case $k=2$ and we study the case $k=3$ in Baues [CH], [HT], see (10.8) and (10.11) below. Moreover Unsöld [AP] considers for $k=4$, $n \geq 3$ the subcategory of $\underline{\underline{spaces}}_n^4$ consisting of CW-complexes with finitely generated torsion free homology. We do not think that it is reasonable to investigate the case, say $k=10$, completely. It will, however, increase our knowledge on the nature of homotopy types considerably if we are able to discuss in detail homotopy types of A_n^k -polyhedra for small k, say $k \leq 5$. This for example includes, for $n=2$, simply connected 7-dimensional homotopy types.

(10.6) Remark. M.J. Hopkins [GM] discusses new global methods to study stable homotopy types. For this a fundamental filtration of the stable homotopy category $\underline{\underline{C}}_0$ of 'p-local finite spectra':

$$\underline{\underline{C}}_0 \supset \underline{\underline{C}}_1 \supset \ldots \supset \underline{\underline{C}}_n \supset \underline{\underline{C}}_{n+1} \supset \ldots$$

is considered where C_n contains all objects which are acyclic with respect to the Morawa K-theory $K(n-1)$.

The classical dimension filtration of the stable homotopy category, coming from the sequence (10.4), is more related to problems like the classification of manifolds in a particular dimension. J.H.C. Whitehead [CE] obtained the following algebraic models of stable A_n^2 -polyhedra, $n \geq 3$.

(10.7) Definition. An A^2 -system

$$S = (H_0, H_2, \pi_1, b_2, \eta)$$

is a tuple consisting of abelian groups H_0, H_2, π_1 and elements

$$\begin{aligned} b_2 &\in Hom(H_2, H_0 \otimes \mathbb{Z}/2) \\ \eta &\in Hom(H_0 \otimes \mathbb{Z}/2, \pi_1) \end{aligned}$$

such that the sequence

$$H_2 \xrightarrow{b_2} H_0 \otimes \mathbb{Z}/2 \xrightarrow{\eta} \pi_1$$

is exact. A morphism

$$(\varphi_0, \varphi_2, \varphi_{pi}) : (H_0, H_2, \pi_1, b_2, \eta) \to (H_0', H_2', \pi_1', b_2', \eta')$$

is given by homomorphisms $\varphi_i : H_i \to H_i'$ for $i = 0, 2$ and $\varphi_\pi : \pi_1 \to \pi_1'$ such that the diagram

$$\begin{tikzcd}
H_2 \arrow[r, "b_2"] \arrow[d, "\varphi_2"'] & H_0 \otimes \mathbb{Z}/2 \arrow[r, "\eta"] \arrow[d, "\varphi_0 \otimes \mathbb{Z}/2"] & \pi_1 \arrow[d, "\varphi_\pi"] \\
H_2' \arrow[r, "b_2'"] & H_0' \otimes \mathbb{Z}/2 \arrow[r, "\eta'"] & \pi_1'
\end{tikzcd}$$

commutes. The A^2 -system S is free, resp. injective, if H_2 is free abelian resp. b_2 is injective. Let $A^2 - \underline{\underline{Systems}}$, resp. $A^2 - \underline{\underline{systems}}$ be the categories of free, resp. injective A^2 -systems with morphisms as above. We have a forgetful functor

$$\phi : A^2 - \underline{\underline{Systems}} \to A^2 - \underline{\underline{systems}}$$

which carries $(H_0, H_2, \pi_1, b_2, \eta)$ to $(H_0, H_2', \pi_1, b_2', \eta)$ where H_2' is the image of b_2 and b_2' is the inclusion of this image.

Let $\underline{\underline{types}}^k_n$ be the full homotopy category of $(n-1)$ -connected $(n+k)$ -types and let

$$P^k_n : \underline{\underline{spaces}}^k_n \to \underline{\underline{types}}^{k-1}_n$$

be the restriction of the Postnikov functor.

(10.8) <u>Classification of J.H.C. Whitehead</u>. *For $n \geq 3$ there exist detecting functors Λ, λ for which the following diagram of functors commutes up to natural isomorphism.*

$$\begin{tikzcd}
\underline{\underline{spaces}}^2_n \arrow[r, "\Lambda"] \arrow[d, "P^2_n"'] & A^2 - \underline{\underline{Systems}} \arrow[d, "\phi"] \\
\underline{\underline{types}}^1_n \arrow[r, "\lambda"] & A^2 - \underline{\underline{systems}}
\end{tikzcd}$$

This result is an easy application of (9.4), compare (9.9) and (6.12). The functor Λ carries a space X to part of the Γ -sequence of X,

$$H_{n+2}X \xrightarrow{b} \Gamma_{n+1}X \xrightarrow{\eta} \pi_{n+1}X$$

where $\Gamma_{n+1}X = H_nX \otimes \mathbb{Z}/2$. Here η can be identified with the Postnikov invariant $\eta = k_{n+1}X$.

Next we describe algebraic models of stable A_n^3 -polyhedra, $n \geq 4$. For this let $\mathbb{Z}/2$ be the cyclic group of two elements and let

$$Hom(\otimes\mathbb{Z}/2, -) : \underline{\underline{Ab}}^{op} \times \underline{\underline{Ab}} \to \underline{\underline{Ab}}$$

be the functor which carries H, L to $Hom(H \otimes \mathbb{Z}/2, L)$. Moreover let

$$\text{(10.9)} \qquad \underline{\underline{Gro}}(Hom(\otimes\mathbb{Z}/2, -))$$

be the Grothendieck construction of this functor. Objects in the category (10.9) are triple $\eta = (H, L, \eta)$ with $\eta \in Hom(H \otimes \mathbb{Z}/2, L)$ and morphisms $(\psi_1, \psi_0) : \eta \to \eta'$ are homomorphisms $\psi_1 : L \to L'$, $\psi_0 : H \to H'$ with $\psi_1\eta = \eta'(\psi_0 \otimes \mathbb{Z}/2)$. We point out that there is an obvious equivalence of categories

$$\underline{\underline{Gro}}(Hom(\otimes\mathbb{Z}/2, -)) \xrightarrow{i} A^2 - systems \tag{1}$$

For each abelian group A we have the short exact sequence

$$A \otimes \mathbb{Z}/2 \overset{\Delta}{\rightarrowtail} G(A) \overset{\mu}{\twoheadrightarrow} A * \mathbb{Z}/2 \tag{2}$$

associated to the natural homomorphisms

$$\tau_A : A * \mathbb{Z}/2 = \{x \in A,\ 2x = 0\} \subset A \twoheadrightarrow A/2A = A \otimes \mathbb{Z}/2$$

The abelian extension (2) is determined up to equivalence by $\Delta^{-1}(2\mu^{-1}(x)) = \tau_A(x)$ for $x \in A * \mathbb{Z}/2$. Let

$$\underline{\underline{G}} \subset \underline{\underline{G}}' \subset \underline{\underline{Gro}}(Hom(\otimes\mathbb{Z}/2, -)) \tag{3}$$

be the following subcategories. Objects in $\underline{\underline{G}}$ are the triple $A = (A, G(A), \Delta)$ given by (2) and morphisms are pairs $(\varphi, \bar{\varphi})$ which are compatible with (2), that is $\mu\bar{\varphi} = (\varphi * \mathbb{Z}/2)\mu$. There is a full forgetful functor

$$\underline{\underline{G}} \to \underline{\underline{Ab}} \tag{4}$$

which carries $(A, G(A), \Delta)$ to A and there is an equivalence $\underline{\underline{G}} = \underline{\underline{M}}^n$, $n \geq 3$, where $\underline{\underline{M}}^n$ is the homotopy category of Moore spaces in degree n; see (6.9). Moreover $\underline{\underline{G}}'$ in (3) is the full subcategory consisting of objects $\eta = (H, L, \eta)$ for which there exists a factorization $\eta : H \otimes \mathbb{Z}/2 \overset{\Delta}{\rightarrowtail} G(H) \to L$. We shall need the group $G(\eta)$ defined by the puh out diagram

$$\begin{array}{ccccc} L \otimes \mathbb{Z}/2 & \xrightarrow{\Delta} & G(\eta) & \xrightarrow{\mu} & H * \mathbb{Z}/2 \\ {\scriptstyle \eta\otimes 1}\uparrow & & \uparrow & & \| \\ H \otimes \mathbb{Z}/2 & \longrightarrow & G(H) & \longrightarrow & H * \mathbb{Z}/2 \end{array} \tag{5}$$

Moreover we shall use a canonical bifunctor

$$\bar{G} : \underline{\underline{G}}^{op} \times \underline{\underline{G}}' \to \underline{\underline{Ab}} \tag{6}$$

which carries the pair of objects (A, η) to an abelian group $\bar{G}(A, \eta)$. We here only define this group if A or H is finitely generated; for a complete definition of $\bar{G}$ see IIIV.1.3 (B) in Baues [HT]. Using (2) we have the dual extension

$$\begin{tikzcd}
Ext(A, \mathbb{Z}/2) \arrow[r, tail, "\Delta"] \arrow[d, equal] & Hom(G(A), \mathbb{Z}/4) \arrow[r, two heads, "\mu"] \arrow[d, equal] & Hom(A, \mathbb{Z}/2) \arrow[d, equal] \\
Hom(A * \mathbb{Z}/2, \mathbb{Z}/4) \arrow[r, tail] & Hom(G(A), \mathbb{Z}/4) \arrow[r, two heads] & Hom(A \otimes \mathbb{Z}/2, \mathbb{Z}/4)
\end{tikzcd} \tag{7}$$

which we use in the following push out diagram for the definition of $\bar{G}(A, \eta)$.

$$\begin{tikzcd}
Ext(A, L) \arrow[r, tail, "\Delta"] & \bar{G}(A, \eta) \arrow[r, two heads, "\mu"] & Hom(A, H \otimes \mathbb{Z}/2) \\
Ext(A, H \otimes \mathbb{Z}/2) \arrow[u, "\eta_*"] \arrow[d, equal] \arrow[r, phantom, "push"] & & \\
Ext(A, \mathbb{Z}/2) \otimes H \arrow[r] & Hom(G(A), \mathbb{Z}/4) \otimes H \arrow[r, two heads] \arrow[uu] & Hom(A, \mathbb{Z}/2) \otimes H \arrow[uu, equal]
\end{tikzcd} \tag{8}$$

The bottom row is obtained by applying the functor $- \otimes H$ to (7). The top row is short exact. Induced homomorphisms for the functor $\bar{G}$ are defined by

$$\begin{aligned}
(\varphi, \bar{\varphi})^* &= Ext(\varphi, L) \oplus Hom(\bar{\varphi}, \mathbb{Z}/4) \otimes H \\
(\psi_1, \psi)_* &= Ext(A, \psi_1) \oplus Hom(G(A), \mathbb{Z}/4) \otimes \psi_0
\end{aligned} \tag{9}$$

Using these constructions of $G(\eta)$ and $\bar{G}(A, \eta)$ we are now ready to define the following algebraic models of stable A_n^3 -polyhedra.

(10.10) Definition. An A^3 -system

$$S = (H_0, H_2, H_3, \pi_1, b_2, \eta, b_3, \beta) \tag{1}$$

is a tuple consisting of abelian groups H_0, H_2, H_3, π_1 and elements

$$\begin{aligned}
& b_2 \in Hom(H_2, H_0 \otimes \mathbb{Z}/2), \\
& \eta \in Hom(H_0 \otimes \mathbb{Z}/2, \pi_1) \\
& b_3 \in Hom(H_3, G(\eta)), \\
& \beta \in \bar{G}(H_2, \eta_\sharp).
\end{aligned} \tag{2}$$

Here $\eta_\sharp = q\Delta(\eta \otimes 1)$ is the composition

$$\eta_\sharp : H_0 \otimes \mathbb{Z}/2 \xrightarrow{\eta\otimes 1} \pi_1 \otimes \mathbb{Z}/2 \xrightarrow{\Delta} G(\eta) \overset{q}{\twoheadrightarrow} cok\,(b_3) \tag{3}$$

where q is the quotient map for the cokernel of b_3. These elements satisfy the following conditions (4) and (5). The sequence

$$H_2 \xrightarrow{b_2} H_0 \otimes \mathbb{Z}/2 \xrightarrow{\eta} \pi_1 \tag{4}$$

is exact and β satisfies

$$\mu(\beta) = b_2 \tag{5}$$

where μ is the operator on $\bar{G}$ in (10.9) (8). A morphism

$$(\varphi_0, \varphi_2, \varphi_3, \varphi_\pi, \varphi_\Gamma) : S \to S' \tag{6}$$

between A^3 -systems is a tuple of homomorphisms

$$\begin{cases} \varphi_i : H_i \to H_i' & (i = 0, 2, 3) \\ \varphi_\pi : \pi_1 \to \pi_1' \\ \varphi_\Gamma : G(\eta) \to G(\eta') \end{cases}$$

such that the following diagrams (7), (8), (9) commute and such that the equation (10) holds.

$$\begin{array}{ccccc} H_2 & \xrightarrow{b_2} & H_0 \otimes \mathbb{Z}/2 & \xrightarrow{\eta} & \pi_1 \\ \downarrow{\scriptstyle \varphi_2} & & \downarrow{\scriptstyle \varphi_0\otimes 1} & & \downarrow{\scriptstyle \varphi_\pi} \\ H_2' & \xrightarrow[b_2']{} & H_0' \otimes \mathbb{Z}/2 & \xrightarrow[\eta']{} & \pi_1' \end{array} \tag{7}$$

$$\begin{array}{ccccc} \pi_1 \otimes \mathbb{Z}/2 & \xrightarrow{\Delta} & G(\eta) & \xrightarrow{\mu} & H_) * \mathbb{Z}/2 \\ \downarrow{\scriptstyle \varphi_\pi\otimes 1} & & \downarrow{\scriptstyle \varphi_\Gamma} & & \downarrow{\scriptstyle \varphi_0 * 1} \\ \pi_1' \otimes \mathbb{Z}/2 & \xrightarrow[\Delta]{} & G(\eta') & \xrightarrow[\mu]{} & H_0' * \mathbb{Z}/2 \end{array} \tag{8}$$

$$\begin{array}{ccc} H_3 & \xrightarrow{b_3} & G(\eta) \\ \downarrow{\scriptstyle \varphi_3} & & \downarrow{\scriptstyle \varphi_\Gamma} \\ H_3' & \xrightarrow[b_3']{} & G(\eta') \end{array} \tag{9}$$

Hence φ_Γ induces $\varphi_\Gamma : cok(b_3) \to cok(b_3')$ such that $(\varphi_0, \varphi_\Gamma) : q\Delta(\eta\otimes 1) \to q\Delta(\eta'\otimes 1)$ is a morphism in $\underline{\underline{G}}'$ which induces $(\varphi_0, \varphi_\Gamma)_*$ as in (10.9) (9). We have

$$(\varphi_0, \varphi_\Gamma)_*(\beta) = (\varphi_2, \bar{\varphi}_2)^*(\beta') \tag{10}$$

in $\bar{G}(H_2, q\Delta(\eta' \otimes 1))$. In (10) we choose $\bar{\varphi}_2$ for φ_2. The right hand side of (10) does not depend on the choice of $\bar{\varphi}_2$.

An A^3 -system S as above is <u>free</u> if H_3 is free abelian, and S is <u>injective</u> if $b_3 : H_3 \to G(\eta)$ is injective. Let $A^3 - \underline{\underline{Systems}}$ (resp. $A^3 - \underline{\underline{systems}}$ be the full category of free, resp. injective, A^3 -systems. We have the canonical forgetful functor

$$\phi : A^3 - \underline{\underline{Systems}} \to A^3 - \underline{\underline{systems}} \tag{11}$$

which replaces $b_3 : H_3 \to G(\eta)$ by the inclusion $b_3(H_3) \subset G(\eta)$ of the image of b_3. One readily checks that this forgetful functor ϕ is full and representative. We associate with an A^3 -system S the exact $\Gamma - \underline{\underline{sequence}}$

$$H_3 \xrightarrow{b_3} G(\eta) \to \pi_2 \to H_2 \xrightarrow{b_2} H_0 \otimes \mathbb{Z}/2 \xrightarrow{\eta} \pi_1 \to H_1 \to 0 \tag{12}$$

Here $H_1 = cok\,(\eta)$ is the cokernel of η and the extension

$$cok\,(b_3) \rightarrowtail \pi_2 \twoheadrightarrow ker\,(b_2)$$

is obtained by the element β, that is, the group π_2 is given by the extension element $\beta_\dagger \in Ext\,(ker\,(b_2), cok\,(b_3))$ defined by

$$\beta_\dagger = \Delta^{-1}(j, \bar{j})^*(\beta).$$

Here $j : ker\,(b_2) \subset H_2$ is the inclusion.

(10.11) <u>Classification theorem</u>. *For $n \geq 4$ there exist detecting functors Λ', λ' for which the following diagram of functors commutes up to natural isomorphism*

$$\begin{array}{ccc}
\underline{\underline{spaces}}^3_n & \xrightarrow{\Lambda'} & A^3 - \underline{\underline{Systems}} \\
{\scriptstyle P^3_n}\downarrow & & \downarrow{\scriptstyle \phi} \\
\underline{\underline{types}}^2_n & \xrightarrow{\lambda'} & A^3 - \underline{\underline{systems}}
\end{array}$$

Moreover for $S = \Lambda'(X)$, $X \in \underline{\underline{spaces}}^3_n$, the Γ -sequence of S describes part of the Γ -sequence of X, that is $H_0 = H_nX$ and

$$\begin{array}{ccccccc}
H_3 \to & G(\eta) \to & \pi_2 \to & H_2 \to & H_0 \otimes \mathbb{Z}/2 \xrightarrow{\eta} & \pi_1 \twoheadrightarrow & H_1 \\
\| & \| & \| & \| & \| & \| & \| \\
H_{n+3}X \to & \Gamma_{n+2}X \to & \pi_{n+2}X \to & H_{n+2}X \to & \Gamma_{n+1}X \to & \pi_{n+1}X \twoheadrightarrow & H_{n+1}X
\end{array}$$

In addition $\bar{G}(A, \eta) = \Gamma_{n+1}(A, X)$.

In Baues [HT] we prove similar theorems also for $n = 2, 3$. We point out that the functor λ' classifies all homotopy types Y for which at most the homotopy groups π_nY, $\pi_{n+1}Y$, $\pi_{n+2}Y$ are non trivial, i.e. $Y \in \underline{\underline{types}}^2_n$. The functor Λ' carries X to the A^3 -system $(H_nX,\ H_{n+2}X,\ H_{n+3}X,\ \pi_{n+1}X,\ b_{n+2}X,\ \eta = k_{n+1}X,\ b_{n+3}X,\ \beta_{n+2}X)$ given by the Γ -sequence of X and the boundary invariant $\beta_{n+2}X$. In fact, the classification theorem (10.11) is an application of (8.9); see VIII.1.6 in Baues [HT] and (4.10) in Baues-Hennes [HC].

(10.12) Example. Let $\mathbb{R}P_4$ be the real projective space of dimension 4. Then the iterated suspension $\Sigma^{n-1}\mathbb{R}P_4$ is an object in $\underline{\underline{spaces}}^3_n$ which satisfies

$$\Lambda'(\Sigma^{n-1}\mathbb{R}P_4) = (\mathbb{Z}/2; \mathbb{Z}/2, 0, \mathbb{Z}/2, 0, 1, 0, \Delta(1)).$$

Therefore $G(\eta) = \mathbb{Z}/4$ and the extension

$$G(\eta) = \mathbb{Z}/4 \rightarrowtail \pi_2 \twoheadrightarrow H_2 = \mathbb{Z}/2$$

is non trivial so that $\pi_2 = \mathbb{Z}/8$. This yields a new proof that $\pi_{n+2}\Sigma^{n-1}\mathbb{R}P_4 = \mathbb{Z}/8$; see for example G.W. Whitehead [RA].

The classification theorem (10.11) shows exactly what homology homomorphisms are realizable by maps between stable A^3_n -polyhedra. Hence (10.11) yields a partial solution of Whitehead's realization problem described in (3.7).

(10.13) Remark. One of the deepest problems of homotopy theory is the computation of homotopy groups of spheres $\pi_{n+k}(S^n)$. Ravenel [LP] writes

"The study of the homotopy groups of spheres can be compared with astronomy. The groups themselves are like distant stars waiting to be discovered by the determined observer, who is constantly building better telescopes to see further into the distant sky. The telescopes are spectral sequences and other algebraic constructions of various sorts. Each time a better instrument is built new discoveries are made and our perspective changes. The more we find the more we see how complicated the problem really is."

For us elements of homotopy groups of spheres, $\alpha \in \pi_{n+k-1}(S^n)$, yield very special elementary A^k_n -polyhedra

$$X = S^n \cup_\alpha e^{n+k}$$

obtained by attaching via α an $(n+k)$ -cell to the sphere S^n. Such A^k_n -polyhedra with $k \geq 2$ are determined by the homological condition

$$\begin{aligned} H_i(X) &= 0 \quad \text{for} \quad i \neq n, n+k \\ H_i(X) &= \mathbb{Z} \quad \text{for} \quad i = n, n+k \end{aligned}$$

and the homotopy type of X essentially can be identified with the homotopy class α. Hence the 'telescopes' above are directed to only a very small but distinguished section of the universe of homotopy types. In view of 'Freyd's generating hypothesis' [SH] one might speculate that the classification of finite stable homotopy types is of similar complexity as the computation of all stable homotopy groups of spheres.

§ 11 Decomposition of stable homotopy types

Given a class of objects with certain properties one would like to furnish a complete list of isomorphism types of such objects. This is an ultimate objective of classification. In mathematics indeed many classification problems arise but complete solutions are extremely rare. We here describe a complete list of homotopy types of $(n-1)$ -connected $(n+k)$ -dimensional polyhedra which are finite and stable with $k \leq 3$. This also yields a list of all $(n-1)$ -connected $(n+k)$ -types with finitely generated homotopy groups and $k \leq 2$, $n \geq k+2$.

Let $\underline{\underline{C}}$ be a category with an initial object $*$ and assume sums or products, denoted by $A \vee B$, exist in $\underline{\underline{C}}$. An object X in $\underline{\underline{C}}$ is decomposable if there exists an isomorphism $X \cong A \vee B$ in $\underline{\underline{C}}$ where A and B are not isomorphic to $*$. Hence an object X is indecomposable if $X \cong A \vee B$ implies $A \cong *$ or $B \cong *$. A decomposition of X is an isomorphism

$$X \cong A_1 \vee \ldots \vee A_n,\ n < \infty, \tag{11.1}$$

in $\underline{\underline{C}}$ where A_i is indecomposable for all $i \in \{1, \ldots, n\}$. The decomposition of X is unique up to permutation if $B_1 \vee \ldots \vee B_m \cong X \cong A_1 \vee \ldots \vee A_n$ implies that $m = n$ and that there is a permutation σ with $B_{\sigma_i} \cong A_i$ for all i. A morphism f in $\underline{\underline{C}}$ is indecomposable if the object f is indecomposable in the catgeory $\underline{\underline{Pair}}(\underline{\underline{C}})$. The objects of $\underline{\underline{Pair}}(\underline{\underline{C}})$ are the morphisms of $\underline{\underline{C}}$ and the morphisms of $\underline{\underline{C}}$ and the morphisms $f \to g$ in $\underline{\underline{Pair}}(\underline{\underline{C}})$ are the pairs (α, β) of morphism in $\underline{\underline{C}}$ with $g\alpha = \beta f$. The sum of f and g is the morphism $f \vee g = (i_1 f, i_2 g)$. Below we consider decompositions of CW-spaces in the homotopy category $\underline{\underline{C}} = \underline{\underline{Top}}^* / \simeq$ where the operation $\vee$ is either the one point union or the product of spaces. The main (and perhaps hopeless) purpose of representation theory is the determination of indecomposable objects in the category of R-modules satisfying some finiteness restraint.

(11.2) Theorem. *Let $k \leq 3$ and $n \geq k+1$ and let X be an $(n-1)$ -connected $(n+k)$ -dimensional finite CW-complex. Then there exists a decomposition*

$$X \simeq X_1 \vee \ldots \vee X_r, \quad r < \infty,$$

where the one point union of CW-complexes X_i on the right hand side is unique up to permutation.

Hence homotopy types in the theorem admit a unique prime factorization with respect to the operation of 'one point union'. The prime factors are called indecomposable A_n^k -polyhedra, $k \leq 3$. For $k \geq 4$ a unique prime factorization as in the theorem does not exist. For this we describe the following example. Let α be the generator of the cyclic group $\pi_{n+3} S^n = \mathbb{Z}/24$ where $n \geq 5$. Then the spaces

$$X_{t\alpha} = S^n \cup_{t\alpha} e^{n+4}$$

are indecomposable for $0 < t < 24$ but there is a homotopy equivalence

$$X_{2\alpha} \vee X_{3\alpha} \simeq S^n \vee S^{n+4} \vee X_{5\alpha}$$

which shows that in this case the decomposition is not unique. The homotopy equivalence is obtained in 4.25 of Cohen [SH]. In the presence of only one prime such decompositions are unique; see (12.12). Below we give a complete list of all indecomposable stable A_n^k -polyhedra, $k \leq 3$, which are the prime factors in (11.1).

General aspects on stable indecomposable polyhedra can be found in chapter 4 of Cohen [SH].

(11.3) Theorem. *Let $k \leq 2$ and $n \geq k+2$ and let Y be an $(n-1)$ -connected $(n+k)$ -type with finitely generated homotopy groups. Then there exists a decomposition*

$$Y \simeq K_1 \times \ldots \times K_r, \quad r < \infty,$$

where the product of CW-spaces K_i on the right hand side is unique up to permutation.

Thus homotopy types in this theorem admit a unique prime factorization with respect to the product operation. We call the prime factors indecomposable a_n^k -types, $k \leq 2$. For $k \geq 3$ a unique prime factorization as in the theorem does not exist. The next result shows that the prime factors in (11.2) correspond exactly to the prime factors in (11.3); this is a consequence of (4.4).

(11.4) Theorem. *Let $k \leq 3$ and $n \geq k+1$. Then the Postnikov functor P_{n+k-1} yields a bijection*

$$Ind(A_n^k) - \{S^{n+k}\} \approx Ind(a_n^{k-1})$$

where the left hand side is the set of all indecomposable A_n^k -homotopy types different from the sphere S^{n+k} and the right hand side is the set of all indecomposable a_n^{k-1} -homotopy types.

These results are proved in chapter X of Baues [HT].

The elementary Moore spaces are the spheres S^m and the Moore spaces $M(\mathbb{Z}/p^i, m)$ where p^i is a power of a prime p. The elementary Eilenberg-Mac Lane spaces are $K(\mathbb{Z}, m)$ and $K(\mathbb{Z}/p^i, m)$. The following result is easy to prove.

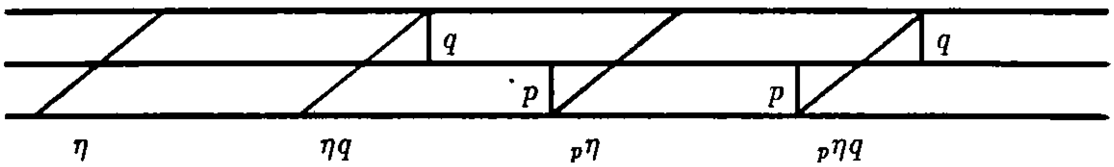
(11.5) Proposition. *For $k = 0$, $n \geq 1$ there is only one indecomposable A_n^0 -polyhedron namely the sphere S^n. For $k = 1$, $n \geq 2$ the indecomposable A_n^1 -polyhedra are exactly the elementary Moore spaces. For $k = 0$, $n \geq 2$ the indecomposable a_n^0 -types are the elementary Eilenberg-Mac Lane spaces.*

The first non trivial case is described in the next result due to J.H.C. Whitehead [CE] and Chang [AS]. For this we define the

(11.6) Elementary Chang complexes. Let η_n be the Hopf map in $\pi_{n+1}S^n$ and let p and q be powers of 2. The elementary Chang complex X in the list below is the mapping cone of the corresponding attaching map where i_1, i_2 denote the inclusion of S^n, S^{n+1} in $S^n \vee S^{n+1}$.

X	attaching map
$X(\eta) = S^n \cup e^{n+2}$	$\eta_n : S^{n+1} \to S^n$
$X(\eta q) = S^n \vee S^{n+1} \cup e^{n+2}$	$qi_1 + i_2\eta_n : S^{n+1} \to S^{n+1} \vee S^n$
$X({}_p\eta) = S^n \cup e^{n+1} \cup e^{n+2}$	$(\eta_n, p) : S^{n+1} \vee S^n \to S^n$
$X({}_p\eta q) = S^n \vee S^{n+1} \cup e^{n+1} \cup e^{n+2}$	$(qi_1 + i_2\eta_n, pi_2) : S^{n+1} \vee S^n \to S^{n+1} \vee S^n$

These complexes are also discussed in the books of Hilton [IH], [HT]. Our notation of the elementary Chang complexes above in terms of the "words" η, ηq, ${}_p\eta$, ${}_p\eta q$ is compatible with the notation on elementary A_n^3 -complexes below. These words can also be visualized by the following graphs where vertical edges are associated with numbers p, q and where the edge, connecting level 0 and 2, is denoted by η.



Hence the elementary Chang complexes correspond to all subgraphs (or subwords) of ${}_p\eta q$ which contain η. We shall describe the elementary A_n^3 -polyhedra by subgraphs (or subwords) of more complicated graphs.

(11.7) Theorem. *Let $n \geq 3$. The elementary Moore spaces and the elementary Chang complexes furnish a complete list of all indecomposable A_n^2 -polyhedra.*

(11.8) Elementary Chang types. Let p, q be powers of 2 and let $\eta : \mathbb{Z} \to \mathbb{Z}/2 \to \mathbb{Z}/q$ and $\eta' : \mathbb{Z}/p \to \mathbb{Z}/2 \to \mathbb{Z}/q$ be the unique non trivial homomorphisms. The elementary Chang types $K(\mathbb{Z}, \mathbb{Z}/q, n)$ and $K(\mathbb{Z}/p, \mathbb{Z}/q, n)$ are the $(n-1)$ -connected $(n+1)$ -types with k-invariant η and η' respectively.

Using (11.4) we get the following application of Chang's theorem.

(11.9) Corollary. *Let $n \geq 3$. The elementary Eilenberg-Mac Lane spaces and the elementary Chang types furnish a complete list of indecomposable a_n^1 -types. Moreover the bijection in (11.4) is given by the following list.*

X	$P_{n+1}X$
S^n	$K(\mathbb{Z}, \mathbb{Z}/2, n)$
S^{n+1}	$K(\mathbb{Z}, n+1)$
$M(\mathbb{Z}/p, n)$	$K(\mathbb{Z}/p, \mathbb{Z}/2, n)$
$M(\mathbb{Z}/q, n+1)$	$K(\mathbb{Z}/q, n+1)$
$X(\eta)$	$K(\mathbb{Z}, n)$
$X({}_p\eta)$	$K(\mathbb{Z}/p, n)$
$X(\eta q)$	$K(\mathbb{Z}, \mathbb{Z}/2q, n)$
$X({}_p\eta q)$	$K(\mathbb{Z}/p, \mathbb{Z}/2q, n)$

Moreover P_{n+1} carries an elementary Moore space of odd primes in $\underline{\underline{A}}_n^2$ to the corresponding elementary Eilenberg-Mac Lane space.

We say that a CW-space X is finite if there is a finite CW-complex homotopy equivalent to X. Let $\underline{\underline{spaces}}_n^k$ (finite) be the full homotopy catgeory of finite $(n-1)$ -connected $(n+k)$ -dimensional CW-spaces X. Then Spanier-Whitehead duality [DH] is an endofunctor D of this category with $n \geq k+2$ satisfying $DD =$ identity. We say that the space X is self-dual if there is a homotopy equivalence $DX \simeq X$.

(11.9) Example. Let $X = M - *$ be obtained by deleting a point in an $(n-1)$ - connected closed differential manifold M of dimension $2n+k$ with $n \geq k+2$. Then X is self dual. Compare Baues [GH] and Stöcker [TP]. Hence self-dual CW-spaces play an important role in the classification of highly connected manifolds.

Spanier-Whitehead duality carries a one point union to a one point union, i.e. $D(X \vee Y) = D(X) \vee D(Y)$, and hence D carries indecomposable polyhedra to indecomposable polyhedra. In particular we have the following properties of elementary Chang complexes.

(11.10) Proposition. *The Spanier-Whitehead duality functor $D : \underline{\underline{A}}_n^2 \cong \underline{\underline{A}}_n^2$ satisfies $DX(\eta) = X(\eta)$, $DX(\eta q) = X({}_q\eta)$, $DX({}_p\eta) = X(\eta p)$, $DX({}_p\eta q) = X({}_q\eta p)$. Hence the Spanier-Whitehead duality turns the graphs in (11.6) around by 180 degrees. For example $X({}_p\eta p)$, $X(\eta)$ and $X({}_p\eta) \vee X(\eta p)$ are self-dual. While clearly $X({}_p\eta)$ is not self-dual.*

For the description of the indecomposable objects in $\underline{\underline{A}}_n^3$, $n \geq 4$, we use certain words. Let L be a set, the elements of which are called "letters". A word with letters in L is an element in the free monoid generated by L. Such a word a is written $a = a_1 a_2 \ldots a_n$ with $a_i \in L$, $n \geq 0$; for $n = 0$ this is the empty word ϕ. Let $b = b_1 \ldots b_k$ be a word. We write $w = \ldots b$ if there is a word a with $w = ab$, similarly we write $w = b \ldots$ if there is a word c with $w = bc$ and we write $w = \ldots b \ldots$ if there exists words a and c with $w = abc$. A subword of an infinite sequence $\ldots a_{-2} a_{-1} a_0 a_1 a_2 \ldots$ with $a_i \in L$, $i \in \mathbb{Z}$, is a finite connected subsequence $a_n a_{n+1} \ldots a_{n+k}$, $n \in \mathbb{Z}$. For the word $a = a_1 \ldots a_n$ we define the word $-a = a_n a_{n-1} \ldots a_1$ by reversing the order in a.

(11.11) Definition. We define a collection of finite words $w = w_1 w_2 \ldots w_k$. The letters w_i of w are symbols ξ, η, ϵ or natural numbers $t, s_i, r_i, i \in \mathbb{Z}$, which are powers of 2. We write the letters s_i as upper indices, the letters r_i as lower indices, and the letter t in the middle of the line since we have to distinguish between these numbers. For example $\eta 4 \xi^2 \eta_8$ is such a word with $t = 4, r_1 = 8, s_i = 2$. A basic sequence is defined by

$$\xi^{s_1} \eta_{r_1} \xi^{s_2} \eta_{r_2} \ldots \tag{1}$$

This is the infinite product $a(1)a(2)\ldots$ of words $a(i) = \xi^{s_i} \eta_{r_i}$, $i \geq 1$. A basic word is any subword of (1). A central sequence is defined by

(2) $$\ldots{}^{s_{-2}}\xi_{r_{-2}}\eta^{s_{-1}}\xi_{r_{-1}}\eta t\xi^{s_1}\eta_{r_1}\xi^{s_2}\eta_{r_2}\ldots$$

A <u>central word</u> w is any subword of (2) containing the number t. Hence a central word w is of the form $w = atb$ where $-a$ and b are basic words. An ϵ -sequence is defined by

(3) $$\ldots{}^{s_{-2}}\xi_{r_{-2}}\eta^{s_{-1}}\xi_{r_{-1}}\epsilon^{s_1}\eta_{r_1}\xi^{s_2}\eta_{r_2}\ldots$$

An ϵ <u>-word</u> w is any subword of (3) containing the letter ϵ; again we can write $w = a\epsilon b$ where $-a$ and b are basic words.

A <u>general word</u> is a basic word, a central word or an ϵ -word.

A general word w is called <u>special</u> if w contains at least one of the letters ξ, η or ϵ and if the following conditions (i), D(i), (ii) and D(ii) are satisfied in case $w = a\epsilon b$ is an ϵ -word. We associate with b the tuple

$$s(b) = (s_1^b, s_2^b, \ldots) = \begin{cases} (s_1, \ldots, s_m, \infty, 1, 1, \ldots) & \text{if} \quad b = \ldots\xi \\ (s_1, \ldots, s_m, 1, 1, 1, \ldots) & \text{otherwise} \end{cases}$$

$$r(b) = (r_1^b, r_2^b, \ldots) = \begin{cases} (r_1, \ldots, r_\ell, \infty, 1, 1, \ldots) & \text{if} \quad b = \ldots\eta \\ (r_1, \ldots, r_\ell, 1, 1, 1, \ldots) & \text{otherwise} \end{cases}$$

where $s_1 \ldots s_m$ and $r_1 \ldots r_\ell$ are the words of upper indices and lower indices respectively given by b. In the same way we get $s(-a) = (s_1^{-a}, s_2^{-a}, \ldots)$ and $r(-a) = (r_1^{-a}, r_2^{-a}, \ldots)$ with $s_i^{-a} \in \{s_{-i}, \infty, 1\}$ and $r_i^{-a} \in \{r_{-i}, \infty, 1\}$, $i \in \mathbb{N}$. The conditions in question on the ϵ -word $w = a\epsilon b$ are:

(i) $$b = \phi \Longrightarrow a \neq \xi_2$$

(D(i)) $$a = \phi \Longrightarrow b \neq {}^2\eta$$

Moreover if $a \neq \phi$ and $b \neq \phi$ we have:

(ii) $$s_1 = 2 \Longrightarrow r_{-1} \geq 4 \quad \text{and}$$

$$(2r_1^b, -s_2^b, r_2^b, -s_3^b, r_3^b, \ldots, -s_i^b, r_i^b, \ldots) < (r_1^{-a}, -s_1^{-a}, r_2^{-a}, -s_2^{-a}, r_3^{-a}, -s_3^{-a}, \ldots, r_i^{-a}, -s_i^{-a}, \ldots)$$

(D(ii)) $$r_{-1} = 2 \Longrightarrow s_1 \geq 4 \quad \text{and}$$

$$(-s_1^b, r_1^b, -s_2^b, r_2^b, -s_3^b, r_3^b, \dots, -s_i^b, r_i^b, \dots) < (-2\cdot s_1^{-a}, r_2^{-a}, -s_2^{-a}, r_3^{-a}, -s_3^{-a}, \dots, r_i^{-a}, -s_i^{-a}, \dots)$$

The index i runs through $i = 2, 3, \dots$ as indicated. In (ii) and D(ii) we use the lexicographical ordering from the left, that is $(n_1, n_2, \dots) < (m_1, m_2, \dots)$ if and only if there is $t \geq 1$ with $n_j = m_j$ for $j < t$ and $n_t < m_t$.

Finally we define a cyclic word by a pair (w, φ) where w is a basic word of the form $(p \geq 1)$

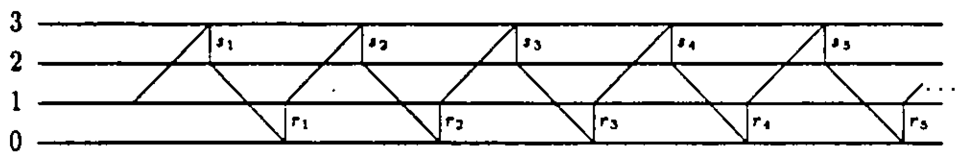
$$w = \xi^{s_1}\eta_{r_1}\xi^{s_2}\eta_{r_2}\dots\xi^{s_p}\eta_{r_p} \tag{4}$$

and where φ is an automorphism of a finite dimensional $\mathbb{Z}/2$ -vector space $V = V(\varphi)$. Two cyclic words (w, φ) and (w', φ') are equivalent if w' is a cyclic permutation of w, that is

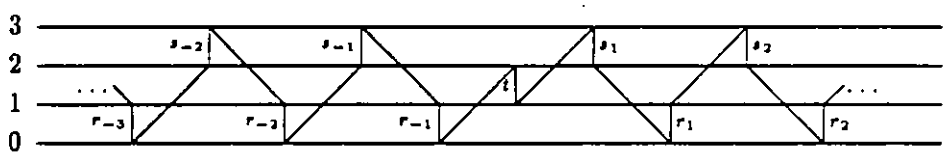
$$w' = \xi^{s_i}\eta_{r_i}\dots\xi^{s_p}\eta_{r_p}\xi^{s_1}\eta_{r_1}\dots\xi^{s_{i-1}}\eta_{r_{i-1}},$$

and if there is an isomorphism $\Psi : V(\varphi) \cong V(\varphi')$ with $\varphi = \Psi^{-1}\varphi'\Psi$. A cyclic word (w, φ) is a special cyclic word if φ is an indecomposable automorphism and if w is not of the form $w = w'w'\dots w'$ where the right hand side is a j-fold power of a word $w^{|prime}$ with $j > 1$.

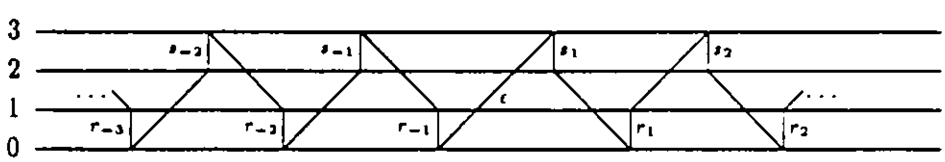
The sequences (1), (2), (3) can be visualized by the infinite graphs sketched below. The letters s_i, resp. r_i, correspond to vertical edges connecting the levels 2 and 3, resp. the levels 0,1. The letters η, resp. ξ, correspond to diagonal edges connecting the levels 0 and 2, resp. the levels 1 and 3. Moreover ϵ connects the levels 0 and 3 and t the levels 1 and 2. We identify a general word with the connected finite subgraph of the infinite graphs below. Therefore the vertices of level i of a general word are defined by the vertices of level i of the corresponding graph, $i \in \{0, 1, 2, 3\}$. We also write $|x| = i$ if x is a vertex of level i.



basic sequence



central sequence



ϵ-sequence

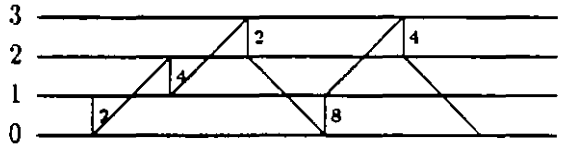
Remark. There is a simple rule which creates exactly all graphs corresponding to general words. Draw in the plane $\mathbb{R}^2$ a connected finite graph of total height at most 3 that alternatingly consists of vertical edges of height one and diagonal edges of height 2 or 3. Moreover endow each vertical edge with a power of 2. An <u>equivalence relation</u> on such graphs is generated by reflection at a vertical line. One readily checks that the equivalence classes of such graphs are in $1-1$ correspondence to all general words.

(11.12) Definition. Let w be a basic word, a central word or an ϵ -word. We obtain the <u>dual word</u> $D(w)$ by reflection of the graph w at a horizontal line and by using the equivalence defined in (2.2). Then $D(w)$ is again a basic word, a central word, or an ϵ -word respectively. Clearly the reflection replaces each letter ξ in w by the letter η and vice versa, moreover it turns a lower index into an upper index and vice versa. We define the <u>dual cyclic word</u> $D(w,\varphi)$ as follows. For the cyclic word (w,φ) in (3.1) (4) let $D(w,\varphi) = (w',(\varphi^*)^{-1})$. Here we set

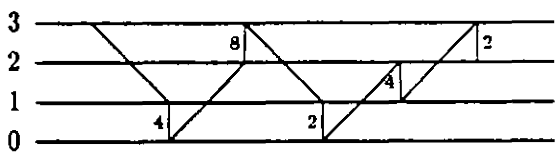
$$w' = \xi^{r_1}\eta_{s_2}\xi^{r_2}\dots\eta_{s_p}\xi^{r_p}\eta_{s_1}$$

and we set $\varphi^* = Hom(\varphi,\mathbb{Z}/2)$ with $V(\varphi^*) = Hom(V(\varphi),\mathbb{Z}/2)$. Up to cyclic permutation w' is just $D(w)$ defined above. We point out that the dual words $D(w)$ and $D(w,\varphi)$ are special if and only if w and (w,φ) are special.

As an example we have the special words $w = {}_2\eta 4\xi^2\eta_8\xi^4\eta$ and $D(w) = \xi_4\eta^8\xi_2\eta 4\xi^2$ which are dual to each other, they correspond to the graphs



$\omega = {}_2\eta 4\xi^2\eta_8\xi^4\eta$



$D(\omega) = \xi_4\eta^8\xi_2\eta 4\xi^2$

Hence the dual graph $D(w)$ is obtained by turning around the graph of w.

We are going to construct certain A_n^3 -polyhedra, $n \geq 4$, associated to the words in (2.1). To this end we first define the homology of a word.

(11.13) Definition. Let w be a general word and let $r_\alpha \dots r_\beta$ and $s_\mu \dots s_\nu$ be the words of lower indices and of upper indices respectively given by w. We define the torsion groups of w by

$$T_0(w) = \mathbb{Z}/r_\alpha \oplus \dots \oplus \mathbb{Z}/r_\beta, \tag{1}$$

$$T_1(w) = \mathbb{Z}/t \quad \text{if} \quad w \quad \text{is a central word,} \tag{2}$$

$$T_2(w) = \mathbb{Z}/s_\mu \oplus \dots \oplus \mathbb{Z}/s_\nu, \tag{3}$$

and we set $T_i(w) = 0$ otherwise. We define the integral homology of w by

$$H_i(w) = \mathbb{Z}^{L_i(w)} \oplus T_i(w) \oplus \mathbb{Z}^{R_i(w)}. \tag{4}$$

Here $\beta_i(w) = L_i(w) + R_i(w)$ is the Betti number of w; this is the number of end points of the graph w which are vertices of level i and which are not vertices of vertical edges; we call such vertices x spherical vertices of w. Let $L(w)$, resp. $R(w)$, be the left, resp. right, spherical vertex of w in case they occur. Now we set $L_i(w) = 1$ if $|L(w)| = i$ and $R_i(w) = 1$ if $|R(w)| = i$, moreover $L_i(w) = 0$ and $R_i(w) = 0$ otherwise.

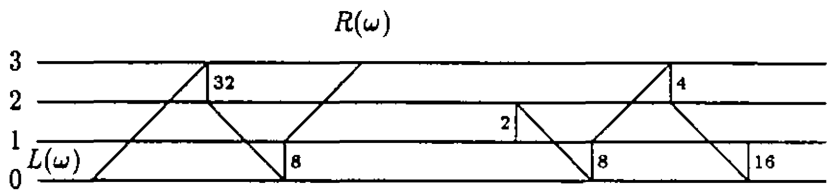
Using the equation (4) we have specified an ordered basis B_i of $H_i(w)$. We point out that

$$\beta_0(w) + \beta_1(w) + \beta_2(w) + \beta_3(w) \leq 2. \tag{5}$$

For a cyclic word (w, φ) we set

$$H_i(w, \varphi) = \bigoplus_v T_i(w) \tag{6}$$

where $v = dim\, V(\varphi)$ and where the right hand side is the v-fold direct sum of $T_i(w)$. As an example we consider the special words



$\omega = \epsilon^{32}\eta_8\xi$ $\qquad$ $\omega' = 2\eta_8\xi^4\eta_{16}$

The homology of these words is:

	$w = \epsilon^{32}\eta_8\xi$	$w' = 2\eta_8\xi^4\eta_{16}$
H_3	$\mathbb{Z}$	0
H_2	$\mathbb{Z}/32$	$\mathbb{Z}/4$
H_1	0	$\mathbb{Z}/2$
H_0	$\mathbb{Z}\oplus\mathbb{Z}/8$	$\mathbb{Z}/8\oplus\mathbb{Z}/16$

Here w has 2 spherical vertices while w' has no spherical vertex. We point out that the numbers 2^k attached to vertical edges correspond to cyclic groups $\mathbb{Z}/2^k$ in the homology. We describe many further examples below.

For the construction of polyhedra $X(w)$ associated to words w we use the following generators.

(11.14) <u>*Generators of homotopy groups*</u>. Let r, s be powers of 2. We have the Hopf maps

$$\eta = \eta_n : S^{n+1} \to S^n,\ \xi = \eta_{n+1} : S^{n+2} \to S^{n+1},\ \epsilon = \eta_n^2 : S^{n+2} \to S^n$$

We use the compositions

$$\eta = i\eta_n : S^{n+1} \to M(\mathbb{Z}/r, n),\ \xi = \eta_{n+1}q : M(\mathbb{Z}/r, n+1) \to S^{n+1}$$

which are $(2n+1)$ -dual. Moreover we have the $(2n+2)$ -dual groups, $n \geq 4$

$$[S^{n+2}, M(\mathbb{Z}/r,n)] = \begin{cases} \mathbb{Z}/4\,\xi_2 & \text{for} \quad r=2 \\ \mathbb{Z}/2\,\xi_r + \mathbb{Z}/2\,\epsilon_r & \text{for} \quad r\geq 4 \end{cases}$$

$$[M(\mathbb{Z}/s,n+1), S^n] = \begin{cases} \mathbb{Z}/4\,\eta^2 & \text{for} \quad r=2 \\ \mathbb{Z}/2\,\eta^s + \mathbb{Z}/2\,\epsilon^s & \text{for} \quad s\geq 4 \end{cases}$$

where $\epsilon_r = i\eta_n^2$ and $\epsilon^s = \eta_n^2 q$ and $\xi_r = \chi_r^2\xi_2$ and $\eta^s = \eta^2\,\chi_2^s$. Next we use

$$[M(\mathbb{Z}/s,n+1), M(\mathbb{Z}/r,n)] = \begin{cases} \mathbb{Z}/2\,\xi_2^2 \oplus \mathbb{Z}/2\,\eta_2^2 & \text{for} \quad s=r=2 \\ \mathbb{Z}/4\,\xi_2^s \oplus \mathbb{Z}/2\,\eta_2^s & \text{for} \quad s\geq 4,\ r=2 \\ \mathbb{Z}/2\,\xi_r^2 \oplus \mathbb{Z}/4\,\eta_r^2 & \text{for} \quad s=2,\ r\geq 4 \\ \mathbb{Z}/2\,\xi_r^s \oplus \mathbb{Z}/2\eta_r^s \oplus \mathbb{Z}/2\,\epsilon_r^s & \text{otherwise} \end{cases}$$

Here we have $\xi_r^s = \chi_r^2\,\xi_2 q$, $\eta_r^s = i\eta^2\chi_2^s$ and $\epsilon_r^s = i\eta_n^2 q$. We have the $(2n+2)$ -dualities $D(\xi_r^s) = \eta_s^r$ and $D(\epsilon_r^s) = \epsilon_s^r$.

(11.15) Definition. Let $n \geq 4$ and let w be a general word. We define the A_n^3 -polyhedron $X(w) = C_f$ by the mapping cone C_f of a map $f = f(w) : A \to B$ where

$$(1)\qquad \begin{cases} A = M(H_3, n+2) \vee M(H_2, n+1) \vee S_c^{n+1} \\ B = M(H_0, n) \vee S_c^{n+1} \vee S_b^{n+1} \end{cases}$$

Here $H_i = H_i(w)$ is the homology group above. We set $S_c^{n+1} = S^{n=1}$ if w is a central word and we set $S_c^{n+1} = *$ otherwise, moreover we set $S_b^{n+1} = S^{n+1}$ if w is a basic word of the form $w = \xi \dots$ and we set $S_b^{n+1} = *$ otherwise. The attaching map

$$(2)\quad f = f(w) : M(H_3, n+2) \vee M(H_2, n+1) \vee S_c^{n+1} \to M(H_0, n) \vee S_c^{n+1} \vee S_b^{n+1}$$

is constructed exactly via the pattern defined by the word w or the associated graph w. For this we subdivide the graph of w by a horizontal line between level 1 and 2; all edges crossing this line are summands in the attaching map $f(w)$. For example consider the graphs $\epsilon^{32}\eta_8\xi$, $2\eta_8\xi^4\eta_{16}$ and ${}_2\eta 4\xi^2\eta_8\xi^4\eta$ above. Then we get

$$f(\epsilon^{32}\eta_8\xi) = \begin{tikzcd} M(\mathbb{Z}/32, n+1) \arrow[d, "\epsilon"] \arrow[drr, "\eta"] & \vee & S^n + 2 \arrow[d, "\xi"] \\ S^n & \vee & M(\mathbb{Z}/8, n) \end{tikzcd}$$

$$f(2\eta_8\xi^4\eta_{16}) = \begin{tikzcd} S^{n+1} \arrow[d, "2"] \arrow[drr, "i\eta_n"] & \vee & & M(\mathbb{Z}/4, n+1) \arrow[dl, "\xi"] \arrow[dr, "\eta"] & \\ S^{n+1} & \vee & M(\mathbb{Z}/8, n) & \vee & M(\mathbb{Z}/16, n) \end{tikzcd}$$

$$f({}_2\eta 4\xi^2\eta_8\xi^4\eta) = \begin{tikzcd} S^{n+1} \arrow[d, "i\eta_n"] \arrow[drr, "4"] & \vee & M(\mathbb{Z}/2, n+1) \arrow[d, "\xi"] \arrow[drr, "\eta"] & \vee & M(\mathbb{Z}/4, n+1) \arrow[d, "\xi"] \arrow[dr, "\eta"] & \\ M(\mathbb{Z}/2, n) & \vee & S^{n+1} & \vee & M(\mathbb{Z}/8, n) \vee & S^n \end{tikzcd}$$

Here ξ, η, ϵ are the corresponding generators in (11.14). For a cyclic word (w, φ) the construction of $X(w, \varphi)$ is slightly different; see Baues-Hennes [HC]. Clearly the homology of $X(w)$ or $X(w, \varphi)$ is the homology in (11.13).

(11.16) Theorem. *Let $n \geq 4$. The elementary Moore spaces, the complexes $X(w)$ where w is a special word, and the complexes $X(w, \varphi)$ where (w, φ) is a special cyclic word furnish a complete list of all indecomposable A_n^3 -polyhedra. For two complexes X, X' in this list there is a homotopy equivalence $X \simeq X'$ if and only if there are equivalent special cyclic words $(w, \varphi) \sim (w', \varphi')$ with $X = X(w, \varphi)$ and $X' = X(w', \varphi')$. Moreover Spanier-Whitehead duality D satisfies*

$$D(X(w)) = X(Dw)$$
$$D(X(w, \varphi)) = X(D(w, \varphi))$$

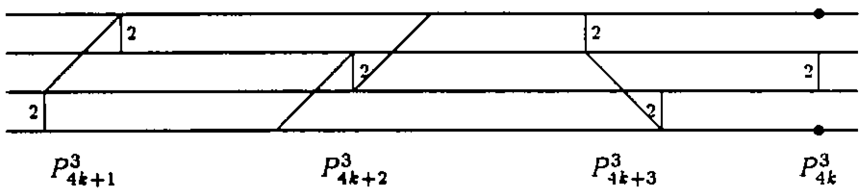
where the right hand side is given by the dual words in (11.12).

The proof of this theorem relies on the classification by A^3 -systems in (10.11). The result is then obtained by classifying the indecomposable A^3 -systems with finitely generated homology; this being a purely algebraic question can be considered as a problem of representation theory. For a complete proof see Baues-Hennes [HC].

(11.17) Example. Let $P_n^3 = \mathbb{R}P_{n+3}/\mathbb{R}P_{n-1}$ be the truncated real projective space. Then one has stable equivalences, $n \geq 1$,

$$P_n^3 \sim \begin{cases} X({}_2\xi^2) & \text{for} \quad n \equiv 1(4) \\ X(\eta 2\xi) & \text{for} \quad n \equiv 2(4) \\ X({}^2\eta_2) & \text{for} \quad n \equiv 3(4) \\ S^n \vee S^{n+3} \vee M(\mathbb{Z}/2, n+1) & \text{for} \quad n \equiv 0(4) \end{cases}$$

Hence the graphs of these stable spaces are $(k \geq 0)$



where P^3_{4k} with $k \geq 1$ is a one point union of Moore spaces.

We now give an application of the classification theorem (11.16). We describe explicitely all indecomposable $(n-1)$ -connected $(n+3)$ -dimensional homotopy types X, $n \geq 4$, for which all homology groups H_iX are cyclic, $i \geq 0$.

Let $H_* = (H_0, H_1, H_2, H_3)$ be a tuple of finitely generated abelian groups with H_3 free abelian and let $N(H_*)$ be the number of all indecomposable homotopy types X as above with homology groups $H_{n+i}(X) \cong H_i$ for $i \in \{0,1,2,3\}$.

(11.18) Corollary. *Let $n \geq 4$. The indcomposable $(n-1)$ -connected $(n+3)$ -dimensional homotopy types X, for which all homology groups $H_i(X)$ are cyclic, are exactly the elementary Moore spaces, the elementary Chang complexes, and the spaces $X(w)$ where w is one of the words in the following list.*

The list describes all w ordered by the homology $H_* \cong H_*(X(w))$. The attaching map for $X(w)$ is obtained by (11.15). Let (r,t,s) be powers of 2.

$H_* = (H_0$	H_1	H_2	$H_3)$	$N(H_*)$	w with $H_*X(w) \cong H_*$
$\mathbb{Z}/r$	$\mathbb{Z}/t$	$\mathbb{Z}/s$	$\mathbb{Z}$	3	$\xi_r\eta t\xi^s$, $t\xi^s\eta_r\xi$, ${}^s\xi_r\eta t\xi$
$\mathbb{Z}/r$	$\mathbb{Z}/t$	$\mathbb{Z}/s$	0	3	${}_r\eta t\xi^s$, $t\xi^s\eta_r$, ${}^s\xi_r\eta t$
$\mathbb{Z}/r$	$\mathbb{Z}/t$	0	$\mathbb{Z}$	2	${}_r\eta t\xi$, $\xi_r\eta t$
$\mathbb{Z}/r$	$\mathbb{Z}$	$\mathbb{Z}/s$	$\mathbb{Z}$	1	$\xi^s\eta_r\xi$
$\mathbb{Z}/r$	$\mathbb{Z}$	$\mathbb{Z}/s$	0	1	$\xi^s\eta_r$
$\mathbb{Z}/r$	0	$\mathbb{Z}/s$	$\mathbb{Z}$	$\begin{cases} 2, & r=s=2 \\ 3, & rs \geq 8 \end{cases}$	${}^s\eta_r\xi$, ${}^s\xi_r\varepsilon$ and $\xi_r\varepsilon^s$ for $rs \geq 8$,
$\mathbb{Z}/r$	0	$\mathbb{Z}/s$	0	$\begin{cases} 3, & r=s=2 \\ 4, & rs \geq 8 \end{cases}$	${}_r\xi^s$, ${}^s\eta_r$, $(\eta^s\xi_r, 1)$, and ${}_r\varepsilon^s$ for $rs \geq 8$
$\mathbb{Z}/r$	0	$\mathbb{Z}$	$\mathbb{Z}$	1	$\eta_r\xi$
$\mathbb{Z}/r$	0	0	$\mathbb{Z}$	2	${}_r\xi$, ${}_r\varepsilon$
$\mathbb{Z}$	$\mathbb{Z}/t$	$\mathbb{Z}/s$	0	2	$\eta t\xi^s$, $t\xi^s\eta$
$\mathbb{Z}$	$\mathbb{Z}/t$	0	$\mathbb{Z}$	1	$\eta t\xi$
$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}/s$	0	1	$\xi^s\eta$
$\mathbb{Z}$	0	$\mathbb{Z}/s$	0	2	η^r, ε^r
$\mathbb{Z}$	0	0	$\mathbb{Z}$	1	ε

All words in the list are special words, except the word $(\eta^s\xi_r, 1)$ which is a special cyclic word associated to the automorphism 1 of $\mathbb{Z}/2$.

Example. Let $n \geq 4$ and let $H_* = (H_0, H_1, H_2, H_3)$ be a tuple of cyclic groups with $H_3 \in \mathbb{Z}, 0$. Then it is easy to describe (by use of (11.18)) all simply connected homotopy types X with $H_{n+1}(X) = H_i$ for $0 \leq i \leq 3$ and $i > n+3$. In fact all such homotopy types are in a canonical way one point unions of the indecomposable homotopy types in the list above. For example for $H_* = (\mathbb{Z}/6, \mathbb{Z}/2, \mathbb{Z}/2, 0)$ there exist exactly 9 such homotopy types X which are:

$$M(\mathbf{Z}/6,n) \vee M(\mathbf{Z}/2,n+1) \vee M(\mathbf{Z}/2,n+2)$$
$$M(\mathbf{Z}/6,n) \vee X(2\xi^2)$$
$$M(\mathbf{Z}/3,n) \vee X({}_2\eta 2) \vee M(\mathbf{Z}/2,n+2)$$

$$M(\mathbf{Z}/3,n) \vee X({}_2\xi^2) \vee M(\mathbf{Z}/2,n+1)$$
$$M(\mathbf{Z}/3,n) \vee X({}^2\eta_2) \vee M(\mathbf{Z}/2,n+1)$$
$$M(\mathbf{Z}/3,n) \vee X(\eta^2\xi_2,1) \vee M(\mathbf{Z}/2,n+1)$$

$$M(\mathbf{Z}/3,n) \vee X({}_2\eta 2\xi^2)$$
$$M(\mathbf{Z}/3,n) \vee X(2\xi^2\eta_2)$$
$$M(\mathbf{Z}/3,n) \vee X({}^2\xi_2\eta 2)$$

Similarly we see that there are 24 homotopy types X for $H_* = (\mathbb{Z}/2,\mathbb{Z}/2,\mathbb{Z}/2,\mathbb{Z})$; we leave this as an exercise.

Next we describe explicitly all indecomposable $(n-1)$ -connected $(n+2)$ -types X, $n \geq 4$, for which all homotopy groups are cyclic. For this we use the bijection (11.4) and the computation of $\pi_{n+2}X$, $\pi_{n+1}X$, $\pi_n X$ in (10.11). Let $\pi_* = (\pi_0,\pi_1,\pi_2)$ be a tuple of finitely generated abelian groups and let $N(\pi_8)$ be the number of all indecomposable homotopy types X with homotopy groups $\pi_{n+i}(X) \cong \pi_i$ for $i = 0,1,2$ and $\pi_j(X) = 0$ otherwise, $n \geq 4$.

(11.19) Corollary. *Let $n \geq 4$. The indecomposable $(n-1)$ -connected $(n+2)$ -types X for which all homotopy groups $\pi_i(X)$ are cyclic are exactly the elementary Eilenberg-Mac Lane spaces, the elementary Chang types, and the spaces $P_{n+2}X(w)$ where w is one of the words in the following list.*

The list describes all w of the theorem ordered by the homotopy groups $\pi_* \cong \pi_* X(w)$. Let $r,t,s \geq 2$ be powers of 2 and for $t,s \geq 4$ let $2t' = t$ and $2s' = s$.

$\pi_* = (\pi_0$	π_1	$\pi_2)$	$N(\pi_*)$	w with $\pi_* X(w) \cong \pi_*$
$\mathbb{Z}$	0	$\mathbb{Z}$	1	η
$\mathbb{Z}/r$	0	$\mathbb{Z}$	1	$\eta_r \xi$
$\mathbb{Z}$	0	$\mathbb{Z}/s$	1	${}^{2s}\eta$
$\mathbb{Z}/r$	0	$\mathbb{Z}/s$	3	$\begin{cases} {}^{2}\eta_r \text{ for } s = 2, \quad {}^{2}\eta_r \xi^{s'} \text{ for } s = 2s' \geq 4 \\ {}^{2s}\eta_r \xi, \quad (\eta^s \xi_r, 1) \end{cases}$
$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}/s$	1	$\xi^s \eta$
$\mathbb{Z}/r$	$\mathbb{Z}$	$\mathbb{Z}/s$	1	$\xi^s \eta_r \xi$
$\mathbb{Z}$	$\mathbb{Z}/t$	$\mathbb{Z}/s$	1	$\begin{cases} P_{n+2} S^n, \ t = s = 2 \\ \eta t', \ t = 2t' \geq 4, \ s = 2 \\ \varepsilon^{s'}, \ t = 2, \ s = 2s' \geq 4 \\ \eta t' \xi^{s'}, \ t = 2t' \geq 4, \ s = 2s' \geq 4 \end{cases}$
$\mathbb{Z}/r$	$\mathbb{Z}/t$ $t \geq 4$	$\mathbb{Z}/s$ $s \geq 4$	2	$\begin{cases} \xi_r \eta t' \xi^{s'}, \ {}^{s'}\xi_r \eta t' \xi \\ \text{with } t = 2t', \ s = 2s' \end{cases}$
$\mathbb{Z}/r$	$\mathbb{Z}/t$ $t \geq 4$	$\mathbb{Z}/2$	1	$\xi_r \eta t', \ t = 2t'$
$\mathbb{Z}/r$ $r \geq 4$	$\mathbb{Z}/2$	$\mathbb{Z}/s$ $s \geq 4$	2	$\begin{cases} {}^{s'}\xi_r \varepsilon \text{ and} \\ \xi_r \varepsilon^{s'}, \ s = 2s' \end{cases}$
$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/s$ $s \geq 4$	2	$\begin{cases} P_{n+2} M(\mathbb{Z}/2, n) \text{ for } s = 4 \text{ and} \\ {}^{s'}\xi_2 \varepsilon \text{ for } s = 2s' \geq 4, \text{ and} \\ {}_2\xi^{s''} \text{ for } s = 4s'' \geq 8 \end{cases}$
$\mathbb{Z}/r$ $r \geq 4$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	2	$\begin{cases} {}_r\varepsilon \text{ and} \\ {}_r\xi \end{cases}$
$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	1	${}_2\varepsilon$

For all tuple of cyclic groups $\pi_* = (\pi_0, \pi_1, \pi_2)$, $\pi_0 \neq 0$, $\pi_2 \neq 0$ which are not in the list we have $N(\pi_*) = 0$. All words in the list are special words, except the word $(\eta^s \xi_r, 1)$ which is a special cyclic word associated to the identity automorphism 1 of $\mathbb{Z}/2$.

Example. Let $n \geq 4$ and let $\pi_* = (\pi_0, \pi_1, \pi_2)$ be a tuple of cyclic groups. Then it is easy to describe all homotopy types X with $\pi_{n+i}(X) \cong \pi_i$ for $i = 0, 1, 2$ and $\pi_j X = 0$ for $j < n$ and $j > n+2$. In fact all such homotopy types are in a canonical way products of the indecomposable homotopy types in (11.19). For example for $\pi_* = (\mathbb{Z}/6, \mathbb{Z}/2, \mathbb{Z}/2)$ there exist exactly 7 such homotopy types X which are

$$
\begin{aligned}
&K(\mathbb{Z}/6,n)\times K(\mathbb{Z}/2,n+1)\times K(\mathbb{Z}/2,n+2)\\
&K(\mathbb{Z}/6,n)\times K(\mathbb{Z}/2,\mathbb{Z}/2,n+1)\\
&K(\mathbb{Z}/3,n)\times K(\mathbb{Z}/2,\mathbb{Z}/2,n)\times K(\mathbb{Z}/2,n+1)\\
&K(\mathbb{Z}/3,n)\times K(\mathbb{Z}/2,n+1)\times P_{n+2}X({}^2\eta_2)\\
&K(\mathbb{Z}/3,n)\times K(\mathbb{Z}/2,n+1)\times P_{n+2}X({}^4\eta_2\xi)\\
&K(\mathbb{Z}/3,n)\times K(\mathbb{Z}/2,n+1)\times P_{n+2}X(\eta^2\xi_2,1)\\
&K(\mathbb{Z}/3,n)\times P_{n+2}X({}_2\varepsilon)
\end{aligned}
$$

It is clear how to compute the homology H_n, H_{n+1} and H_{n+2} of these spaces and, in fact, we can easily describe the A^3 -system of these spaces. We leave it to the reader to consider other cases, for example for $\pi_* = (\mathbb{Z}_4, \mathbb{Z}_{10}, \mathbb{Z})$ there exist exactly 3 homotopy types X with $\pi_* \cong \pi_* X$.

Finally we have the following applications of the classification theorem (11.16) which single out spaces which are highly desuspendable.

(11.20) **Theorem.** *The stable homotopy types of connected compact 4-dimensional polyhedra coincide with finite one point unions $X_1 \vee \ldots \vee X_r$ where the X_i are elementary Moore spaces in A^3_n or the spaces $X(t\xi^s)$, $X(t\xi)$, $X(\xi^s)$, $X(\xi)$, and $X({}_r\xi^s)$. Here r, s, t are powers of 2 and $r \geq s$.*

For this compare V Appendix A in Baues [HT]. The theorem shows that only a few spaces arise as prime factors in the stabilization of 4-dimensional polyhedra. This, for example, has the practical effect that the computation of generalized homology and cohomology groups of 4-dimensional polyhedra can be easily achieved by computing these groups only for the elementary spaces in (11.20).

(11.21) **Theorem.** *The stable homotopy types of simply connected compact 5-dimensional polyhedra coincide with finite one point unions $X_1 \vee \ldots \vee X_r$ where the X_i are elementary Moore spaces in A^3_n or the elementary spaces $X(w)$, $X(w, \varphi)$. Here the special words satisfy the following conditions (1), (2),*

(1) $w \neq \eta^s \ldots$ *and* $w \neq \ldots {}^s\eta$,
(2) *for each subword of the form* ${}_r\eta^s$ *or* ${}^s\eta_r$ *of* w *(that is* $w = \ldots {}_r\eta^s \ldots$ *or* $w = \ldots {}^s\eta_r \ldots$*) we have* $r > s$.

See X.7.3 in Baues [HT].

§ 12 Localization

A generalized homology theory k_* (as for example defined in Gray [HT]) can be used to define equivalence classes of spaces which are called ' k_* -local homotopy types'. We assume that k_* satisfies the limit axiom, namely that for all CW-complexes X the map $\varinjlim k_*(X_\alpha) \to k_*X$ is an isomorphism where the X_α run over all finite subcomplexes of X. We consider mainly the classical homology theory

$$k_*(X) = H_*(X, R) = H_*(SX \otimes_{\mathbf{Z}} R) \tag{12.1}$$

given by the homology of X with coefficients in a ring R; compare (3.4).

(12.2) Definition. Let $\underline{\underline{spaces}}$ be the full subcategory of $\underline{\underline{Top}}$ consisting of CW-spaces. A CW-pair (X, A) is a cofibration $A \rightarrowtail X$ in $\underline{\underline{Top}}$ for which A and X are CW-spaces. For example a CW-complex X together with a subcomplex A is a CW-pair. A map $f : X \to Y$ between CW-spaces is a k_* -equivalence if f induces an isomorphism

$$f_* : k_*(X) \cong k_*(Y)$$

A CW-space A is k_* -local if each CW-pair (X, A) for which $A \rightarrowtail X$ is an k_* -equivalence admits a retraction $A \to X$. A map $g : Y \to A$ is called a k_* -localization if A is k_* -local and g is a k_* -equivalence.

Recall that we introduced the localized catgeory $Ho(\underline{\underline{C}})$ in (3.12). The next result is due to Bousfield [LS].

(12.3) Theorem. *For all CW-spaces there exist k_* -localizations. Moreover there is an equivalence of categories*

$$Ho_{k_*}(\underline{\underline{spaces}}) \xrightarrow{\sim} \underline{\underline{spaces}}_{k_*} / \simeq$$

where the left hand side is the localization with respect to k_* -equivalences and the right hand side in the full homotopy category in $\underline{\underline{Top}}/\simeq$ consisting of k_* -local CW-complexes. The equivalence carries a CW-space to its k_* -localization.

We refer the reader also to I.5.10 in Baues [AH] where we consider k_* -equivalences as weak equivalences in a 'cofibration category'. The k_* -equivalences generate an equivalence relation for CW-spaces as follows. We say that CW-spaces X, Y are k_* -equivalent if there exist finitely many CW-spaces X_i, $i = 1, \dots, n$ together with k_* -equivalences α_i,

$$X = X_1 \xleftarrow{\alpha_1} X_2 \xrightarrow{\alpha_2} X_3 \xleftarrow{\alpha_3} \dots X_n = Y,$$

where α_i and α_{i+1} have opposite directions. The theorem shows that the corresponding k_* -equivalence classes can be identified with the homotopy types of k_* -local CW-spaces which are called k_* -local homotopy types. The k_* -local homotopy type of a CW-space X singles out the k_* -specific properties of X. This turned out to be a very successful technique of homotopy theory.

(12.4) Theorem. Let R be a subring of $\mathbb{Q}$ and let X be a simply connected CW-space. Then X is $H_*(-,R)$ -local if and only if (a) or equivalently (b) is satisfied:

(a) The homotopy groups $\pi_n X$ are R-modules.

(b) The homology groups $H_n X$ are R-modules.

Moreover an $H_*(-,R)$ -localization $\ell : X \to X_R$ induces isomorphisms

$$\begin{aligned}\pi_n(X) \otimes_{\mathbb{Z}} R &\cong \pi_n(X_R)\\ H_n(X) \otimes_{\mathbb{Z}} k &\cong H_n(X_R)\end{aligned}$$

which carries $\xi \otimes 1$ to $\ell_*(\xi)$.

A proof can be found for example in Hilton-Mislin-Roitberg [LN]. Spaces as in the theorem are also called R -local, these are the rational spaces if $R = \mathbb{Q}$. Moreover for a prime p these are the p -local spaces if $R = \mathbb{Z}_p$ is the subring of $\mathbb{Q}$ generated by $1/q$ where q runs over all primes different from p. The classification theorems in § 9 are actually compatible with R-localization, $R \subset \mathbb{Q}$. For this we define for the category $\underline{\underline{C}}$ in (9.1) the full subcategory

$$\underline{\underline{C}}_R \subset (n-1) - \text{types} \tag{12.5}$$

consisting of R-localizations X_R of objects X in $\underline{\underline{C}}$. Let

$$\ell_R : \underline{\underline{C}} \to \underline{\underline{C}}_R$$

be the localization functor. A $\underline{\underline{C}}_R$ -kype $\bar{X}_R = (X_R, \pi, k, H, b)$ is R -local if π and H are R-modules, and $\bar{X}_R$ is R -free if H is a free R-module. Similarly a $\underline{\underline{C}}_R$ -bype $\bar{Y} = (Y_R, H_0, H_1, b, \beta)$ in R -local if H_0, H_1 are R-modules, and $\bar{Y}_R$ is R -free if H_1 is a free R-module. Let

$$\underline{\underline{spaces}}_R^{n+1}(\underline{\underline{C}}_R)$$

be the full homotopy category of R-local CW-spaces X with $P_{n-1}X \in \underline{\underline{C}}_R$ and with $H_i(X_R) = 0$ for $i > n+1$ and $H_{n+1}(X_R)$ a free R-module.

(12.6) Classification theorem. *There are detecting functors Λ_R, Λ'_R for which the following diagrams of functors commute up to natural isomorphism.*

$$\begin{tikzcd} \underline{\underline{spaces}}^{n+1}(\underline{\underline{C}}) \arrow[r, "\Lambda"] \arrow[d, "\ell_R"'] & \underline{\underline{Kypes}}(\underline{\underline{C}}) \arrow[d, "\ell_R"] \\ \underline{\underline{spaces}}_R^{n+1}(\underline{\underline{C}}_R) \arrow[r, "\Lambda_R"] & \underline{\underline{Kypes}}_R(\underline{\underline{C}}_R) \end{tikzcd}$$

Here $Kypes_R(\underline{\underline{C}}_R)$ is the category of free R-kypes and ℓ_R denotes the obvious localization functors.

$$\begin{array}{ccc}
\underline{\underline{spaces}}^{n+1}(\underline{\underline{C}}) & \xrightarrow{\Lambda'} & \underline{\underline{Bypes}}(\underline{\underline{C}}) \\
{\scriptstyle \ell_R}\downarrow & & \downarrow{\scriptstyle \ell_R} \\
\underline{\underline{spaces}}_R^{n+1}(\underline{\underline{C}}_R) & \xrightarrow{\Lambda'_R} & \underline{\underline{Bypes}}_R(\underline{\underline{C}}_R)
\end{array}$$

Here $\underline{\underline{Bypes}}_R(\underline{\underline{C}}_R)$ *is the category of free* R*-bypes and* ℓ_R *denotes again the localization functors.*

For the definition of Λ_R, Λ'_R we use the Γ -sequence of X_R which coincides with (Γ -sequence of X) $\otimes R$. The theorem shows:

(12.7) Corollary. *The Postnikov invariants of the localization* X_R *are obtained by* R*-localizing the Postnikov invariants of* X. *The boundary invariants of the localization* X_R *are obtained by* R*-localizing the boundary invariants of* X.

If $R = \mathbb{Q}$ is the ring of rational numbers the theory of Postnikov invariants and boundary invariants is completely understood. In fact Postnikov invariants correspond to the differential in the 'minimal model of Sullivan' and boundary invariants correspond to the differential in the 'Quillen minimal model' constructed in Baues-Lemaire [MM]. Compare Quillen [RH], Sullivan [IC] and chapter I in Baues [AH].

(12.8) Definition. Let V be a graded $\mathbb{Q}$ -vector space with $V_i = 0$ for $i \leq 0$. Let $T(V) = \oplus\{V^{\otimes n}, n \geq 0\}$ be the tensor algebra of V which is a Lie algebra by

$$[x,y] = xy - (-1)^{|x||y|}yx.$$

The free Lie algebra $L(V)$ is the Lie subalgebra of $(T(V), [\ \ ,\ \ ])$ generated by V. Let $[L(V), L(V)] \subset L(V)$ be the subset of all brackets $[x,y]$ with $x, y \in L(V)$ and let

$$d : L(V) \to [L(V), L(V)] \subset L(V)$$

be a $\mathbb{Q}$ -linear map of degree -1 satisfying $dd = 0$ and $d[x,y] = [dx,y] + (-1)^{|x|}[x,dy]$. Then $(L(V), d)$ is called a Quillen minimal model with differential d. A morphism between such models in a $\mathbb{Q}$ -linear map of degree 0 compatible with brackets and differentials.

(12.9) Theorem. *Homotopy types of 1-connected rational spaces* X *are in* $1-1$ *correspondence with isomorphism types of Quillen minimal models* $(L(V), d)$ *where* $V_i = H_{i+1}(X, \mathbb{Q})$ *and* $H_i(L(V), d) = \pi_{i+1}X$ *for* $i \geq 1$.

(12.10) Definition. Let V be a graded $\mathbb{Q}$ -vector space such that V^i is finitely generated and $V^2 = 0$ for $i \leq 1$. Let $\Lambda(V)$ be the free graded-commutative algebra generated by V, that is

$$\Lambda(V) = \text{ Exterior algebra}\,(V^{\text{odd}}) \otimes \text{Symmetric algebra}\,(V^{\text{even}})$$

Let $\tilde{\Lambda}(V) \cdot \tilde{\Lambda}(V)$ be the subset of products $x \cdot y$ with $x, y \in \Lambda(V)$, $|x|, |y| \geq 1$ and let

$$d : \Lambda(V) \to \tilde{\Lambda}(V) \cdot \tilde{\Lambda}(V) \subset \Lambda(V)$$

be a $\mathbb{Q}$ -linear map of dgree $+1$ satisfying $dd = 0$ and $d(xy) = (dx)y + (-1)^{|x|}x(dy)$. Then $(\Lambda(V), d)$ is called a Sullivan minimal model with differential d. A morphism between such models is a $\mathbb{Q}$ -linear map of degree 0 compatible with multiplications and differentials.

(12.11) Theorem. *Homotopy types of 1-connected rational spaces X for which H_nX is a finitely generated $\mathbb{Q}$ -vector space, $n \in \mathbb{Z}$, are in $1-1$ correspondence with isomorphism types of Sullivan minimal models $(\Lambda(V), d)$ where $V_i = Hom(\pi_i(X), \mathbb{Q})$ and $H^i(\Lambda(V), d) = Hom(H_i(X), \mathbb{Q}$ for $i \geq 1$.*

These minimal models yield solutions of Whitehead's realization problem for rational spaces, see (3.7). They illustrate again that homology groups and homotopy groups respectively both 'generate' a homotopy type in a mutually $H\pi$ -dual way. The Baues-Lemaire conjecture [MM] (recently proved by Majewski [BL]) describes the algebraic nature of this $H\pi$ -duality. The minimal models allow a deep analysis of the rational properties of a simply connected space. For example we refer the reader to the wonderful torsion gap result of Halperin [TG] or to the alternative 'hyperbolic-elliptic' for rational spaces in Felix [DE].

There are p-local analogues of A_n^k -polyhedra as follows. We say that a p-local CW-space X is a pA_n^k polyhedron if X is $(n-1)$ -connected, $n \geq 2$, and the homology H_iX is trivial for $i > n+k$ and is a free $\mathbb{Z}_p$ -module for $i = n+k$. Moreover X is a finite pA_n^k -polyhedron if in addition all H_iX are finitely generated $\mathbb{Z}_p$ -modules. In the stable range we have by 3.6 (2) in Wilkerson [GC] unique decompositions as follows.

(12.12) Theorem. *Let p be a prime and $n \geq k+1 \geq 2$. Then each finite pA_n^k -polyhedron X admits a homotopy equivalence*

$$X \simeq X_1 \vee \ldots \vee X_r$$

where the one point union of p-local indecomposable CW-spaces on the right hand side is unique up to permutation.

(12.13) Remark. Generalizing the result of Chang (11.7) Henn [CL] furnished a complete list of indecomposable pA_n^k -polyhedra for $k = 4p-5$ and p odd. Such spaces are detected by primary cohomology operations while the A_n^3 -polyhedra in (11.16) are not detected by primary cohomology operations. The classifications of Henn uses implicitely the boundary invariants of X.

(12.14) Remark. For the ring $R = \mathbb{Z}/p$ where p is a prime the $H_*(-, \mathbb{Z}/p)$ -localization X_p of a simply connected space X is the p -completion of Bousfield-Kan [HL]. If in addition X has finite type then X_p is the p-profinite completion for which $\pi_n X_p$ is given by the p-profinite completion of $\pi_n X$; compare Sullivan [GT] and Quillen [AS]. Recently Goerss [SC] considers simplicial coalgebras as models of $H_*(-, \mathbb{F})$ -local spaces where $\mathbb{F}$ is an algebraicly closed field; see also Kriz [AH]. Moreover Bousfield [HT] and Franke [UT] consider algebraic models of k_* -local spaces with $k_* = K$ -theory; they restrict, however, to the stable range.

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