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A Model of ZF Set Theory Language

Grzegorz Bancerek
Warsaw University
Białystok

Summary. The goal of this article is to construct a language of the ZF set theory and to develop a notational and conceptual base which facilitates a convenient usage of the language.

MML Identifier: ZF_LANG.

WWW: http://mizar.org/JFM/Vol1/zf_lang.html

The articles [4], [6], [7], [3], [1], [5], and [2] provide the notation and terminology for this paper.

For simplicity, we follow the rules: k, n are natural numbers, a is a set, D is a non empty set, and p, q are finite sequences of elements of $\mathbb{N}$.

The subset VAR of $\mathbb{N}$ is defined by:

(Def. 1) $\mathrm{VAR} = \{k : 5 \leq k\}$.

Let us observe that VAR is non empty.

A variable is an element of VAR.

Let us consider n. The functor x_n yielding a variable is defined by:

(Def. 2) $\mathrm{x}_n = 5 + n$.

In the sequel x, y, z, t denote variables.

Let us consider x. Then $\langle x \rangle$ is a finite sequence of elements of $\mathbb{N}$.

Let us consider x, y. The functor $x{=}y$ yielding a finite sequence of elements of $\mathbb{N}$ is defined by:

(Def. 3) $x{=}y = \langle 0 \rangle \frown \langle x \rangle \frown \langle y \rangle$.

The functor $x \varepsilon y$ yielding a finite sequence of elements of $\mathbb{N}$ is defined by:

(Def. 4) $x \varepsilon y = \langle 1 \rangle \frown \langle x \rangle \frown \langle y \rangle$.

Next we state two propositions:

(6)[1] If $x{=}y = z{=}t$, then $x = z$ and $y = t$.

(7) If $x \varepsilon y = z \varepsilon t$, then $x = z$ and $y = t$.

Let us consider p. The functor $\neg p$ yields a finite sequence of elements of $\mathbb{N}$ and is defined by:

(Def. 5) $\neg p = \langle 2 \rangle \frown p$.

Let us consider q. The functor $p \wedge q$ yielding a finite sequence of elements of $\mathbb{N}$ is defined as follows:

[1] The propositions (1)–(5) have been removed.

(Def. 6) $p \wedge q = \langle 3 \rangle \frown p \frown q$.

One can prove the following proposition

(10)[2] If $\neg p = \neg q$, then $p = q$.

Let us consider x, p. The functor $\forall_x p$ yields a finite sequence of elements of $\mathbb{N}$ and is defined by:

(Def. 7) $\forall_x p = \langle 4 \rangle \frown \langle x \rangle \frown p$.

One can prove the following proposition

(12)[3] If $\forall_x p = \forall_y q$, then $x = y$ and $p = q$.

The non empty set WFF is defined by the conditions (Def. 8).

(Def. 8)(i) For every a such that $a \in$ WFF holds a is a finite sequence of elements of $\mathbb{N}$,

(ii) for all x, y holds $x{=}y \in$ WFF and $x \varepsilon y \in$ WFF,

(iii) for every p such that $p \in$ WFF holds $\neg p \in$ WFF,

(iv) for all p, q such that $p \in$ WFF and $q \in$ WFF holds $p \wedge q \in$ WFF,

(v) for all x, p such that $p \in$ WFF holds $\forall_x p \in$ WFF, and

(vi) for every D such that for every a such that $a \in D$ holds a is a finite sequence of elements of $\mathbb{N}$ and for all x, y holds $x{=}y \in D$ and $x \varepsilon y \in D$ and for every p such that $p \in D$ holds $\neg p \in D$ and for all p, q such that $p \in D$ and $q \in D$ holds $p \wedge q \in D$ and for all x, p such that $p \in D$ holds $\forall_x p \in D$ holds WFF $\subseteq D$.

Let I_1 be a finite sequence of elements of $\mathbb{N}$. We say that I_1 is ZF-formula-like if and only if:

(Def. 9) I_1 is an element of WFF.

One can check that there exists a finite sequence of elements of $\mathbb{N}$ which is ZF-formula-like.
A ZF-formula is a ZF-formula-like finite sequence of elements of $\mathbb{N}$.
One can prove the following proposition

(14)[4] a is a ZF-formula iff $a \in$ WFF.

In the sequel F, F_1, G, G_1, H, H_1 are ZF-formulae.
Let us consider x, y. Observe that $x{=}y$ is ZF-formula-like and $x \varepsilon y$ is ZF-formula-like.
Let us consider H. Note that $\neg H$ is ZF-formula-like. Let us consider G. Observe that $H \wedge G$ is ZF-formula-like.
Let us consider x, H. Note that $\forall_x H$ is ZF-formula-like.
Let us consider H. We say that H is equality if and only if:

(Def. 10) There exist x, y such that $H = x{=}y$.

We introduce H is an equality as a synonym of H is equality. We say that H is membership if and only if:

(Def. 11) There exist x, y such that $H = x \varepsilon y$.

We introduce H is a membership as a synonym of H is membership. We say that H is negative if and only if:

(Def. 12) There exists H_1 such that $H = \neg H_1$.

We say that H is conjunctive if and only if:

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[2] The propositions (8) and (9) have been removed.
[3] The proposition (11) has been removed.
[4] The proposition (13) has been removed.

(Def. 13) There exist F, G such that $H = F \wedge G$.

We say that H is universal if and only if:

(Def. 14) There exist x, H_1 such that $H = \forall_x H_1$.

One can prove the following proposition

(16)[5] (i) H is an equality iff there exist x, y such that $H = x{=}y$,

(ii) H is a membership iff there exist x, y such that $H = x \varepsilon y$,

(iii) H is negative iff there exists H_1 such that $H = \neg H_1$,

(iv) H is conjunctive iff there exist F, G such that $H = F \wedge G$, and

(v) H is universal iff there exist x, H_1 such that $H = \forall_x H_1$.

Let us consider H. We say that H is atomic if and only if:

(Def. 15) H is an equality and a membership.

Let us consider F, G. The functor $F \vee G$ yields a ZF-formula and is defined by:

(Def. 16) $F \vee G = \neg(\neg F \wedge \neg G)$.

The functor $F \Rightarrow G$ yielding a ZF-formula is defined by:

(Def. 17) $F \Rightarrow G = \neg(F \wedge \neg G)$.

Let us consider F, G. The functor $F \Leftrightarrow G$ yielding a ZF-formula is defined as follows:

(Def. 18) $F \Leftrightarrow G = (F \Rightarrow G) \wedge (G \Rightarrow F)$.

Let us consider x, H. The functor $\exists_x H$ yields a ZF-formula and is defined by:

(Def. 19) $\exists_x H = \neg \forall_x \neg H$.

Let us consider H. We say that H is disjunctive if and only if:

(Def. 20) There exist F, G such that $H = F \vee G$.

We say that H is conditional if and only if:

(Def. 21) There exist F, G such that $H = F \Rightarrow G$.

We say that H is biconditional if and only if:

(Def. 22) There exist F, G such that $H = F \Leftrightarrow G$.

We say that H is existential if and only if:

(Def. 23) There exist x, H_1 such that $H = \exists_x H_1$.

The following proposition is true

(22)[6] (i) H is disjunctive iff there exist F, G such that $H = F \vee G$,

(ii) H is conditional iff there exist F, G such that $H = F \Rightarrow G$,

(iii) H is biconditional iff there exist F, G such that $H = F \Leftrightarrow G$, and

(iv) H is existential iff there exist x, H_1 such that $H = \exists_x H_1$.

Let us consider x, y, H. The functor $\forall_{x,y} H$ yielding a ZF-formula is defined as follows:

(Def. 24) $\forall_{x,y} H = \forall_x \forall_y H$.

[5] The proposition (15) has been removed.

[6] The propositions (17)–(21) have been removed.

The functor $\exists_{x,y}H$ yields a ZF-formula and is defined by:

(Def. 25) $\exists_{x,y}H = \exists_x\exists_y H$.

We now state the proposition

(23) $\forall_{x,y}H = \forall_x\forall_y H$ and $\exists_{x,y}H = \exists_x\exists_y H$.

Let us consider x, y, z, H. The functor $\forall_{x,y,z}H$ yielding a ZF-formula is defined by:

(Def. 26) $\forall_{x,y,z}H = \forall_x\forall_{y,z}H$.

The functor $\exists_{x,y,z}H$ yields a ZF-formula and is defined as follows:

(Def. 27) $\exists_{x,y,z}H = \exists_x\exists_{y,z}H$.

We now state a number of propositions:

(24) $\forall_{x,y,z}H = \forall_x\forall_{y,z}H$ and $\exists_{x,y,z}H = \exists_x\exists_{y,z}H$.

(25) H is an equality, a membership, negative, conjunctive, and universal.

(26) H is atomic, negative, conjunctive, and universal.

(27) If H is atomic, then $\operatorname{len} H = 3$.

(28) H is atomic or there exists H_1 such that $\operatorname{len} H_1 + 1 \leq \operatorname{len} H$.

(29) $3 \leq \operatorname{len} H$.

(30) If $\operatorname{len} H = 3$, then H is atomic.

(31) For all x, y holds $(x{=}y)(1) = 0$ and $(x\varepsilon y)(1) = 1$.

(32) For every H holds $(\neg H)(1) = 2$.

(33) For all F, G holds $(F \wedge G)(1) = 3$.

(34) For all x, H holds $(\forall_x H)(1) = 4$.

(35) If H is an equality, then $H(1) = 0$.

(36) If H is a membership, then $H(1) = 1$.

(37) If H is negative, then $H(1) = 2$.

(38) If H is conjunctive, then $H(1) = 3$.

(39) If H is universal, then $H(1) = 4$.

(40)(i) H is an equality and $H(1) = 0$, or
(ii) H is a membership and $H(1) = 1$, or
(iii) H is negative and $H(1) = 2$, or
(iv) H is conjunctive and $H(1) = 3$, or
(v) H is universal and $H(1) = 4$.

(41) If $H(1) = 0$, then H is an equality.

(42) If $H(1) = 1$, then H is a membership.

(43) If $H(1) = 2$, then H is negative.

(44) If $H(1) = 3$, then H is conjunctive.

(45) If $H(1) = 4$, then H is universal.

In the sequel s_1 denotes a finite sequence.
Next we state several propositions:

(46) If $H = F \frown s_1$, then $H = F$.

(47) If $H \wedge G = H_1 \wedge G_1$, then $H = H_1$ and $G = G_1$.

(48) If $F \vee G = F_1 \vee G_1$, then $F = F_1$ and $G = G_1$.

(49) If $F \Rightarrow G = F_1 \Rightarrow G_1$, then $F = F_1$ and $G = G_1$.

(50) If $F \Leftrightarrow G = F_1 \Leftrightarrow G_1$, then $F = F_1$ and $G = G_1$.

(51) If $\exists_x H = \exists_y G$, then $x = y$ and $H = G$.

Let us consider H. Let us assume that H is atomic. The functor $\mathrm{Var}_1(H)$ yields a variable and is defined by:

(Def. 28) $\mathrm{Var}_1(H) = H(2)$.

The functor $\mathrm{Var}_2(H)$ yielding a variable is defined by:

(Def. 29) $\mathrm{Var}_2(H) = H(3)$.

Next we state three propositions:

(52) If H is atomic, then $\mathrm{Var}_1(H) = H(2)$ and $\mathrm{Var}_2(H) = H(3)$.

(53) If H is an equality, then $H = (\mathrm{Var}_1(H))\!=\!\mathrm{Var}_2(H)$.

(54) If H is a membership, then $H = (\mathrm{Var}_1(H))\varepsilon\mathrm{Var}_2(H)$.

Let us consider H. Let us assume that H is negative. The functor $\mathrm{Arg}(H)$ yields a ZF-formula and is defined by:

(Def. 30) $\neg \mathrm{Arg}(H) = H$.

Let us consider H. Let us assume that H is conjunctive and disjunctive. The functor $\mathrm{LeftArg}(H)$ yields a ZF-formula and is defined as follows:

(Def. 31)(i) There exists H_1 such that $\mathrm{LeftArg}(H) \wedge H_1 = H$ if H is conjunctive,

(ii) there exists H_1 such that $\mathrm{LeftArg}(H) \vee H_1 = H$, otherwise.

The functor $\mathrm{RightArg}(H)$ yielding a ZF-formula is defined as follows:

(Def. 32)(i) There exists H_1 such that $H_1 \wedge \mathrm{RightArg}(H) = H$ if H is conjunctive,

(ii) there exists H_1 such that $H_1 \vee \mathrm{RightArg}(H) = H$, otherwise.

One can prove the following propositions:

(56)[7] If H is conjunctive, then $F = \mathrm{LeftArg}(H)$ iff there exists G such that $F \wedge G = H$ and $F = \mathrm{RightArg}(H)$ iff there exists G such that $G \wedge F = H$.

(57) If H is disjunctive, then $F = \mathrm{LeftArg}(H)$ iff there exists G such that $F \vee G = H$ and $F = \mathrm{RightArg}(H)$ iff there exists G such that $G \vee F = H$.

(58) If H is conjunctive, then $H = \mathrm{LeftArg}(H) \wedge \mathrm{RightArg}(H)$.

(59) If H is disjunctive, then $H = \mathrm{LeftArg}(H) \vee \mathrm{RightArg}(H)$.

Let us consider H. Let us assume that H is universal and existential. The functor $\mathrm{Bound}(H)$ yielding a variable is defined by:

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[7] The proposition (55) has been removed.

(Def. 33)(i) There exists H_1 such that $\forall_{\mathrm{Bound}(H)} H_1 = H$ if H is universal,

(ii) there exists H_1 such that $\exists_{\mathrm{Bound}(H)} H_1 = H$, otherwise.

The functor $\mathrm{Scope}(H)$ yields a ZF-formula and is defined by:

(Def. 34)(i) There exists x such that $\forall_x \mathrm{Scope}(H) = H$ if H is universal,

(ii) there exists x such that $\exists_x \mathrm{Scope}(H) = H$, otherwise.

One can prove the following four propositions:

(60) If H is universal, then $x = \mathrm{Bound}(H)$ iff there exists H_1 such that $\forall_x H_1 = H$ and $H_1 = \mathrm{Scope}(H)$ iff there exists x such that $\forall_x H_1 = H$.

(61) If H is existential, then $x = \mathrm{Bound}(H)$ iff there exists H_1 such that $\exists_x H_1 = H$ and $H_1 = \mathrm{Scope}(H)$ iff there exists x such that $\exists_x H_1 = H$.

(62) If H is universal, then $H = \forall_{\mathrm{Bound}(H)} \mathrm{Scope}(H)$.

(63) If H is existential, then $H = \exists_{\mathrm{Bound}(H)} \mathrm{Scope}(H)$.

Let us consider H. Let us assume that H is conditional. The functor $\mathrm{Antecedent}(H)$ yields a ZF-formula and is defined by:

(Def. 35) There exists H_1 such that $H = \mathrm{Antecedent}(H) \Rightarrow H_1$.

The functor $\mathrm{Consequent}(H)$ yields a ZF-formula and is defined as follows:

(Def. 36) There exists H_1 such that $H = H_1 \Rightarrow \mathrm{Consequent}(H)$.

The following propositions are true:

(64) If H is conditional, then $F = \mathrm{Antecedent}(H)$ iff there exists G such that $H = F \Rightarrow G$ and $F = \mathrm{Consequent}(H)$ iff there exists G such that $H = G \Rightarrow F$.

(65) If H is conditional, then $H = \mathrm{Antecedent}(H) \Rightarrow \mathrm{Consequent}(H)$.

Let us consider H. Let us assume that H is biconditional. The functor $\mathrm{LeftSide}(H)$ yields a ZF-formula and is defined by:

(Def. 37) There exists H_1 such that $H = \mathrm{LeftSide}(H) \Leftrightarrow H_1$.

The functor $\mathrm{RightSide}(H)$ yielding a ZF-formula is defined as follows:

(Def. 38) There exists H_1 such that $H = H_1 \Leftrightarrow \mathrm{RightSide}(H)$.

We now state two propositions:

(66) Suppose H is biconditional. Then

(i) $F = \mathrm{LeftSide}(H)$ iff there exists G such that $H = F \Leftrightarrow G$, and

(ii) $F = \mathrm{RightSide}(H)$ iff there exists G such that $H = G \Leftrightarrow F$.

(67) If H is biconditional, then $H = \mathrm{LeftSide}(H) \Leftrightarrow \mathrm{RightSide}(H)$.

Let us consider H, F. We say that H is an immediate constituent of F if and only if:

(Def. 39) $F = \neg H$ or there exists H_1 such that $F = H \wedge H_1$ or $F = H_1 \wedge H$ or there exists x such that $F = \forall_x H$.

One can prove the following propositions:

(69)[8] H is not an immediate constituent of $x{=}y$.

[8] The proposition (68) has been removed.

(70) H is not an immediate constituent of $x\varepsilon y$.

(71) F is an immediate constituent of $\neg H$ iff $F = H$.

(72) F is an immediate constituent of $G \wedge H$ iff $F = G$ or $F = H$.

(73) F is an immediate constituent of $\forall_x H$ iff $F = H$.

(74) If H is atomic, then F is not an immediate constituent of H.

(75) If H is negative, then F is an immediate constituent of H iff $F = \mathrm{Arg}(H)$.

(76) If H is conjunctive, then F is an immediate constituent of H iff $F = \mathrm{LeftArg}(H)$ or $F = \mathrm{RightArg}(H)$.

(77) If H is universal, then F is an immediate constituent of H iff $F = \mathrm{Scope}(H)$.

In the sequel L denotes a finite sequence.

Let us consider H, F. We say that H is a subformula of F if and only if the condition (Def. 40) is satisfied.

(Def. 40) There exist n, L such that

(i) $1 \leq n$,

(ii) $\mathrm{len}\, L = n$,

(iii) $L(1) = H$,

(iv) $L(n) = F$, and

(v) for every k such that $1 \leq k$ and $k < n$ there exist H_1, F_1 such that $L(k) = H_1$ and $L(k+1) = F_1$ and H_1 is an immediate constituent of F_1.

We now state the proposition

(79)[9] H is a subformula of H.

Let us consider H, F. We say that H is a proper subformula of F if and only if:

(Def. 41) H is a subformula of F and $H \neq F$.

One can prove the following propositions:

(81)[10] If H is an immediate constituent of F, then $\mathrm{len}\, H < \mathrm{len}\, F$.

(82) If H is an immediate constituent of F, then H is a proper subformula of F.

(83) If H is a proper subformula of F, then $\mathrm{len}\, H < \mathrm{len}\, F$.

(84) If H is a proper subformula of F, then there exists G which is an immediate constituent of F.

(85) If F is a proper subformula of G and G is a proper subformula of H, then F is a proper subformula of H.

(86) If F is a subformula of G and G is a subformula of H, then F is a subformula of H.

(87) If G is a subformula of H and H is a subformula of G, then $G = H$.

(88) F is not a proper subformula of $x{=}y$.

(89) F is not a proper subformula of $x\varepsilon y$.

(90) If F is a proper subformula of $\neg H$, then F is a subformula of H.

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[9] The proposition (78) has been removed.
[10] The proposition (80) has been removed.

(91) If F is a proper subformula of $G \wedge H$, then F is a subformula of G and a subformula of H.

(92) If F is a proper subformula of $\forall_x H$, then F is a subformula of H.

(93) If H is atomic, then F is not a proper subformula of H.

(94) If H is negative, then $\mathrm{Arg}(H)$ is a proper subformula of H.

(95) If H is conjunctive, then $\mathrm{LeftArg}(H)$ is a proper subformula of H and $\mathrm{RightArg}(H)$ is a proper subformula of H.

(96) If H is universal, then $\mathrm{Scope}(H)$ is a proper subformula of H.

(97) H is a subformula of $x{=}y$ iff $H = x{=}y$.

(98) H is a subformula of $x \varepsilon y$ iff $H = x \varepsilon y$.

Let us consider H. The functor $\mathrm{Subformulae}\, H$ yielding a set is defined as follows:

(Def. 42) $a \in \mathrm{Subformulae}\, H$ iff there exists F such that $F = a$ and F is a subformula of H.

The following propositions are true:

(100)[11] If $G \in \mathrm{Subformulae}\, H$, then G is a subformula of H.

(101) If F is a subformula of H, then $\mathrm{Subformulae}\, F \subseteq \mathrm{Subformulae}\, H$.

(102) $\mathrm{Subformulae}\, x{=}y = \{x{=}y\}$.

(103) $\mathrm{Subformulae}\, x \varepsilon y = \{x \varepsilon y\}$.

(104) $\mathrm{Subformulae}\, \neg H = \mathrm{Subformulae}\, H \cup \{\neg H\}$.

(105) $\mathrm{Subformulae}(H \wedge F) = \mathrm{Subformulae}\, H \cup \mathrm{Subformulae}\, F \cup \{H \wedge F\}$.

(106) $\mathrm{Subformulae}\, \forall_x H = \mathrm{Subformulae}\, H \cup \{\forall_x H\}$.

(107) H is atomic iff $\mathrm{Subformulae}\, H = \{H\}$.

(108) If H is negative, then $\mathrm{Subformulae}\, H = \mathrm{Subformulae}\, \mathrm{Arg}(H) \cup \{H\}$.

(109) If H is conjunctive, then $\mathrm{Subformulae}\, H = \mathrm{Subformulae}\, \mathrm{LeftArg}(H) \cup \mathrm{Subformulae}\, \mathrm{RightArg}(H) \cup \{H\}$.

(110) If H is universal, then $\mathrm{Subformulae}\, H = \mathrm{Subformulae}\, \mathrm{Scope}(H) \cup \{H\}$.

(111) Suppose H is an immediate constituent of G, a proper subformula of G, and a subformula of G and $G \in \mathrm{Subformulae}\, F$. Then $H \in \mathrm{Subformulae}\, F$.

In this article we present several logical schemes. The scheme *ZF Ind* concerns a unary predicate $\mathcal{P}$, and states that:

For every H holds $\mathcal{P}[H]$

provided the following conditions are met:

- For every H such that H is atomic holds $\mathcal{P}[H]$,
- For every H such that H is negative and $\mathcal{P}[\mathrm{Arg}(H)]$ holds $\mathcal{P}[H]$,
- For every H such that H is conjunctive and $\mathcal{P}[\mathrm{LeftArg}(H)]$ and $\mathcal{P}[\mathrm{RightArg}(H)]$ holds $\mathcal{P}[H]$, and
- For every H such that H is universal and $\mathcal{P}[\mathrm{Scope}(H)]$ holds $\mathcal{P}[H]$.

The scheme *ZF CompInd* concerns a unary predicate $\mathcal{P}$, and states that:

For every H holds $\mathcal{P}[H]$

provided the parameters satisfy the following condition:

- For every H such that for every F such that F is a proper subformula of H holds $\mathcal{P}[F]$ holds $\mathcal{P}[H]$.

[11] The proposition (99) has been removed.

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