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SOLUTION OF SYMMETRIC LINEAR COMPLEMENTARITY PROBLEMS BY ITERATIVE METHODS

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ABSTRACT

A unified treatment is given for iterative algorithms for the solution of the symmetric linear complementarity problem: $Mx + q \geqq 0$, $x \geqq 0$, $x^T(Mx+q) = 0$, where M is a given $n \times n$ symmetric real matrix and q is a given $n \times 1$ vector. A general algorithm is proposed in which relaxation may be performed both before and after projection on the nonnegative orthant. The algorithm includes as special cases, extensions of the Jacobi, Gauss-Seidel and nonsymmetric and symmetric successive over-relaxation methods, for solving the symmetric linear complementarity problem. It is shown first that any accumulation point of the iterates generated by the general algorithm solves the linear complementarity problem. It is

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then shown that a class of matrices for which the existence of an accumulation point that solves the linear complementarity problem is guaranteed, is the class of symmetric copositive plus matrices which satisfy a qualification of the type: $Mx + q > 0$ for some x in R^n. This class includes symmetric positive semidefinite matrices satisfying this qualification, symmetric strictly copositive matrices and symmetric positive matrices.

1. INTRODUCTION

We shall be concerned here with <u>iterative</u> methods for solving the symmetric linear complementarity problem of finding an x in R^n such that

(1) $$Mx + q \geqq 0, \quad x \geqq 0, \quad x^T(Mx+q) = 0$$

where M is a given $n \times n$ real symmetric matrix and q is a given $n \times 1$ vector. These iterative methods are infinite in contrast to the <u>direct</u> pivotal methods which are finite [12,2,19,14,15,16]. Iterative methods however are considerably simpler than direct methods and require storage capacity of the order of n whereas direct methods require storage capacity of the order of n^2. Thus iterative methods are well suited for large problems such as those arising from the discretization of free boundary problems [4,6]. Initial attempts at devising iterative methods for solving problems that led to the symmetric linear complementarity problem [21,8] consisted of projecting the Gauss-Seidel iterates [20,p118] for solving $Mx + q = 0$ on the nonnegative orthant $R^n_+ = \{x \mid x \in R^n, x \geqq 0\}$. Subsequently other authors [10,17,4,5,9,18,6,3] proposed projection of the successive over-relaxation (SOR) iterates [20,p121] on the nonnegative orthant. In most of these methods it is assumed that M is at least a symmetric positive definite matrix, however Eckhardt [6] assumed that M is positive semidefinite only, but with a positive diagonal. We propose here a more general method, Algorithm 1, which includes both of these approaches as well as the methods of consisting of projecting the Jacobi iterates [20,p117] and the symmetric SOR (SSOR) iterates [24,Chapter 15] on the nonnegative orthant. In addition, unlike previous methods, the relaxation in Algorithm 1 may be performed both before and after projection on the nonnegative orthant. Our first convergence result Theorem 1, shows, with no assumptions on M, that when the sequence of iterates $\{x^i\}$ generated by Algorithm 1 has an accumulation point, this point solves the linear complementarity problem (1). To show this, use is made of the associated quadratic function

(2) $$f(x) = \frac{1}{2} x^T Mx + q^T x$$

which is shown to be a nonincreasing function on the sequence of iterates. To ensure that at least one accumulation point exists, Theorem 2 imposes the additional condition that the symmetric matrix M is copositive plus, that is

(3)
(a) $x \geqq 0 \Rightarrow x^T Mx \geqq 0$ (Copositive)
(b) $x \geqq 0, \quad x^T Mx = 0 \Rightarrow Mx = 0$ (Plus)

and a qualification of the type: $Mx + q > 0$ for some x in R^n (x not necessarily nonnegative). Under these conditions it is shown that the sequence of iterates of Algorithm 1 has an accumulation point which solves the linear complementarity problem 1. Symmetric copositive plus matrices [2] include symmetric positive semidefinite matrices, because for such matrices $x^T Mx = 0$ is equivalent to $Mx = 0$. They also include symmetric strictly copositive matrices, that is symmetric matrices which satisfy

(4) $\quad 0 \neq x \geqq 0 \Rightarrow x^T Mx > 0$

When M is positive definite, the entire sequence of iterates $\{x^i\}$ converges as shown in Theorem 3, and if in addition the solution $\bar{x}$ is nondegenerate in this case, that is $\bar{x} + M\bar{x} + q > 0$, then the convergence is linear as shown in Theorem 4.

In Section 3 we give some specific realizations of Algorithm 1. Algorithm 2 is closely related to the Jacobi algorithm for solving $Mx + q = 0$. Algorithm 3 is a generalization of the SOR method and Algorithm 4 is a generalization of the symmetric SOR method. A special case of Algorithm 3 is the projected SOR method investigated by Cryer, Eckhardt and others [4,5,6].

We briefly describe now the notation and some of the well known results used in this paper. All matrices and vectors are real. A matrix A with m rows and n columns is said to be in $R^{m\times n}$. Row i of A is denoted by A_i, column j by $A_{\cdot j}$ and the element in row i and column j by A_{ij}. If $I \subset \{1,\ldots,m\}$ and $J \subset \{1,\ldots,n\}$, then A_{IJ} denotes the matrix extracted from A with elements A_{ij}, $i \in I$ and $j \in J$. Similarly if $x \in R^n$, then $x_{j\in J}$ or x_J is the vector extracted from x with elements x_j, $j \in J$. If $A \in R^{m\times n}$ then $|A|$ denotes the matrix obtained from A by replacing each element A_{ij} by its absolute value. An $n\times 1$ vector x is said to be in R^n and its j^{th} element denotes by x_j. The superscript T denotes the transpose. Superscripts such as A^i, x^i, refer to specific matrices and vectors and usually denote iteration number. If $x \in R^n$, x_+ denotes the vector with elements $(x_+)_j = \max\{0, x_j\}$, $j = 1,\ldots,n$.

If $x \in R^n$ and $A \in R^{n\times n}$, then $\|x\|_A^2 = x^T Ax$ and $\|x\| = (x^T x)^{\frac{1}{2}}$ is the 2-norm. If $x \in R^n$, then the projection of x on R_+^n is the unique point in R_+^n which minimizes $\|y-x\|$ over all $y \in R_+^n$. It is easy to verify that the projection of x on R_+^n is x_+. We shall make a passing reference to matrices in the class K (or sometimes called the class M). The class K is the class of square matrices with nonpositive off-diagonal elements and with a nonnegative inverse. These matrices have been extensively studied [7,20,22,23]. The matrix $A \in R^{n\times n}$ is said to be irreducible if no simultaneous permutation of rows and columns can create a block of zeros in its southwest corner [20,pp102-105]. A is strictly diagonally dominant if $|A_{ii}| > \sum_{\substack{j=1 \\ j\neq i}}^{n} |A_{ij}|$, $i=1,\ldots,n$, and irreducibly diagonally dominant if it is irreducible and $|A_{ii}| \geqq \sum_{\substack{j=1 \\ j\neq i}}^{n} |A_{ij}|$, $i=1,\ldots,n$, with strict inequality holding for at least one i. The diagnonal matrix with ones along the diagonal will be denoted by I, and a vector of ones will be denoted by e.

2. THE GENERAL ALGORITHM

We begin by stating and establishing the convergence of a general fundamental iterative algorithm for solving the linear complementarity problem (1). Various special cases of this algorithm will be given in Section 3.

Algorithm 1 Let $x^0 \geqq 0$.

$$x^{i+1} = \lambda(x^i - \omega E^i(Mx^i + q + K^i(x^{i+1} - x^i)))_+ + (1-\lambda)x^i \qquad i=0,1,2,\ldots \tag{5}$$

where $0 < \lambda \leqq 1$, $\omega > 0$, $\{E^i\}$ and $\{K^i\}$ are bounded sequences of matrices in $R^{n\times n}$ with each E^i being a positive diagonal satisfying $E^i > \alpha I$ for some $\alpha > 0$, and such that for some $\gamma > 0$

$$y^T((\lambda\omega E^i)^{-1} + K^i - \frac{M}{2})\, y \geqq \gamma \|y\|^2, \quad \forall i, \quad \forall y \in R^n . \tag{6}$$

Remark 1 In order for algorithm 1 to be a practical one, the matrices K^i must be strictly lower or upper triangular, that is $K^i_{j\ell} = 0$ for $j \leqq \ell$ or $K^i_{j\ell} = 0$ for $j \geqq \ell$, which will be the case for all the specific instances given below. However in obtaining the convergence results, all that is required of the bounded sequence $\{K^i\}$ is that it satisfies (6) above. For computing purposes the following equivalent iteration should be used when K^i is strictly lower triangular

$$x_j^{i+1} = \lambda(x_j^i - \omega E_j^i(M_j x^i + q_j + \sum_{\ell=1}^{\ell=j-1} K^i_{j\ell}(x_\ell^{i+1} - x_\ell^i)))_+ + (1-\lambda)x_j^i \tag{5L}$$

$$j=1,2,\ldots,n, \quad i=0,1,2,\ldots$$

and the following iteration when K^i is strictly upper triangular

$$x_j^{i+1} = \lambda(x_j^i - \omega E_j^i(M_j x^i + q_j + \sum_{\ell=j+1}^{\ell=n} K^i_{j\ell}(x_\ell^{i+1} - x_\ell^i)))_+ + (1-\lambda)x_j^i \tag{5U}$$

$$j=n,n-1,\ldots,1, \quad i=0,1,2,\ldots$$

In (5L) x^{i+1} is computed in the order $x_1^{i+1}, x_2^{i+1}, \ldots, x_n^{i+1}$ and in (5U) in the order $x_n^{i+1}, x_{n-1}^{i+1}, \ldots, x_1^{i+1}$. By definition the summation in (5L) is vacuous when $j = 1$, and the summation in (5U) is vacuous when $j = n$.

Remark 2 The parameters λ and ω in (5) are relaxation factors with ω being a relaxation factor used before projection on R^n_+ and λ after projection. The restriction $\lambda \leqq 1$, which is imposed in order that all $x^i \geqq 0$, can be weakened by making λ be iteration dependent, that is $\lambda = \lambda^i$, and such that

$$0 < \hat{\lambda} \leqq \lambda^i \leqq \max_{\lambda}\{\lambda \mid x^{i+1}(\lambda) \geqq 0, \;\; \lambda \leqq \bar{\lambda}\} \tag{7}$$

where $x^{i+1}(\lambda)$ is x^{i+1} as defined by (5) and $\hat{\lambda}$ and $\bar{\lambda}$ are positive numbers satisfying $\hat{\lambda} \leqq 1 \leqq \bar{\lambda}$. All the results of this paper hold for the above choice of λ^i as well as for a fixed λ as stated in Algorithm 1. For either choice of fixed λ or variable λ , (6) must still be satisfied. For simplicity, the proofs are given for the case of a fixed λ .

Remark 3 The iteration (5) can be motivated by considering the minimization problem

$$(8) \qquad \underset{x \geq 0}{\text{Minimize}} \ \frac{1}{2} x^T Mx + q^T x$$

When M is symmetric, each stationary point of (8) solves the linear complementarity problem (1). Iteration (5) is a type of a gradient projection algorithm for solving (8). When K^i is the strictly lower or upper triangular part of M and when the corresponding (5L) or (5U) iteration is used, each component of the gradient is projected as it is computed instead of projecting the entire gradient. The strictly triangular matrix K^i provides for the use of the latest data in computing each gradient component, as is done in SOR methods. If we take $E^i = I$, $K^i = 0$, $\lambda = 1$, then (5) becomes the Levitin-Poljak [11] gradient projection algorithm for solving (8). If on the other hand we drop the projection operation in (5) by dropping the plus subscript, then this is equivalent to projecting on R^n . Iteration (5) becomes then

$$x^{i+1} = x^i - \lambda\omega E^i(Mx^i+q+K^i(x^{i+1}-x^i))$$

which is an iterative scheme for solving (8) without the restriction $x \geq 0$, or equivalently for solving $Mx + q = 0$. This iteration becomes

the Jacobi method if we take $K^i = 0$, $\lambda\omega = 1$, $E^i = D^{-1}$ where D is the diagonal of M . If we set $E^i = D^{-1}$ and $K^i = L$, where L is the strictly lower triangular part of M , then the above iteration becomes the successive over-relaxation method (SOR) with relaxation factor $\lambda\omega$, and which in turn becomes the Gauss-Seidel method if we take $\lambda\omega = 1$.

Remark 4 If $M = L + D + U$ where L is strictly lower triangular, U is strictly upper triangular, $L = U^T$, D is diagonal, $K^i = L$ or U , and $E^i = E$, where E is a positive diagonal matrix, then condition (6) is equivalent to

$$\lambda\omega < \frac{2}{\max_{j \in J_+} D_{jj}E_{jj}}$$

where $J_+ = \{j | D_{jj} > 0, j=1,\dots,n\}$. If in addition $D = E^{-1} > 0$, condition (6) is equivalent to $\lambda\omega < 2$, the well known SOR relaxation factor condition [20, Theorem 7.1.10].

We state now our first principal convergence result for Algorithm 1.

Theorem 1 Let M be symmetric, then each accumulation point of the sequence $\{x^i\}$ generated by Algorithm 1 solves the linear complementarity problem (1).

It will be convenient to establish two lemmas before proving this theorem. The first lemma shows that any fixed point of the iteration (5) solves the linear complementarity problem.

Lemma 1 Let M $R^{n\times n}$ and let E be a positive diagonal matrix. Then

$$\left.\begin{array}{r} Mx + q \geqq 0 \\ x \geqq 0 \\ x^T(Mx+q) = 0 \end{array}\right\rangle \Leftrightarrow (x-\omega E(Mx+q))_+ - x = 0 \text{ for some or all } \omega > 0$$

Proof ($\Rightarrow$) Let ω be any positive number. When $x_j = 0$ and $(Mx+q)_j \geqq 0$ we have

$$(x_j-\omega E_j(Mx+q)_j)_+ - x_j = (-\omega E_j(Mx+q)_j)_+ = 0$$

When $(Mx+q)_j = 0$ and $x_j \geqq 0$ we have

$$(x_j-\omega E_j(Mx+q)_j)_+ - x_j = x_j - x_j = 0$$

($\Leftarrow$) Let ω be some positive number. Note that $x = (x-\omega E(Mx+q))_+ \geqq 0$ and that $Mx + q \geqq 0$, for if some $(Mx+q)_j < 0$ then

$$0 = (x_j-\omega E_j(Mx+q)_j)_+ - x_j = -\omega E_j(Mx+q)_j > 0$$

which is a contradiction. When $x_j - \omega E_j(Mx+q)_j \geqq 0$

$$0 = (x_j-\omega E_j(Mx+q)_j)_+ - x_j = -\omega E_j(Mx+q)_j$$

and hence $(Mx+q)_j = 0$. When $x_j - \omega E_j(Mx+q)_j < 0$

$$0 = (x_j-\omega E_j(Mx+q)_j)_+ - x_j = - x_j$$

and hence $x_j = 0$. □

Lemma 2 Let E be a positive diagonal matrix in $R^{n\times n}$ and let $x\epsilon R^n$. Then

$$((x)_+ - x)^T E^{-1}((x)_+ - y) \leqq 0 \quad \text{for all} \quad y \geqq 0, \quad y\epsilon R^n .$$

Proof Since

$$\|(x)_+ - x\|^2_{E^{-1}} = \min_{z\geqq 0} \|z-x\|^2_{E^{-1}}$$

it follows by the minimum principle [13, Theorem 9.3.3] that

$$2((x)_+ - x)^T E^{-1}(y-(x)_+) \geqq 0 \quad \text{for all} \quad y \geqq 0 .$$ □

Proof of Theorem 1 Let $f(x)$ be as defined in (2) , and let $\{x^i\}$ be generated by Algorithm 1. Then

$$f(x^{i+1})-f(x^i) = (\omega E^i(Mx^i+q))^T(\omega E^i)^{-1}(x^{i+1}-x^i)+ \|x^{i+1}-x^i\|^2_{\frac{M}{2}}$$

$$= (\frac{x^{i+1}-(1-\lambda)x^i}{\lambda} - x^i+\omega E^i(Mx^i+q+K^i(x^{i+1}-x^i)))^T(\omega E^i)^{-1}(x^{i+1}-x^i)$$

$$+ \|x^{i+1}-x^i\|^2_{\frac{M}{2} - (\lambda\omega E^i)^{-1} - K^i}$$

$$= \lambda\Big(\frac{x^{i+1}-(1-\lambda)x^i}{\lambda} - (x^i - \omega E^i(Mx^i+q+K^i(x^{i+1}-x^i))))^T(\omega E^i)^{-1} \cdot \Big(\frac{x^{i+1}-(1-\lambda)x^i}{\lambda} - x^i\Big) + \|x^{i+1}-x^i\|^2_{\frac{M}{2} - (\lambda\omega E^i)^{-1} - K^i}$$

$$\leq \|x^{i+1} - x^i\|^2_{\frac{M}{2} - (\lambda\omega E^i)^{-1} - K^i} \qquad \text{(By Lemma 2 and (5))}$$

Note that the nonnegativity $x^i \geqq 0$ which follows from (5) and $0<\lambda\leqq 1$ was needed in order to employ Lemma 2 in the last step above. Hence

(9) $$f(x^i) - f(x^{i+1}) \geqq \|x^{i+1} - x^i\|^2_{(\lambda\omega E^i)^{-1} + K^i - \frac{M}{2}} \geqq \gamma \|x^{i+1} - x^i\|^2 \geqq 0$$

where the next-to-the-last inequality follows from (6). Thus the sequence of real numbers $\{f(x^i)\}$ is nonincreasing. Let $\bar{x}$ be an accumulation point of $\{x^i\}$, let $\lim_{k\to\infty} x^{i_k} = \bar{x}$, $\lim_{k\to\infty} E^{i_k} = E$ and $\lim_{k\to\infty} K^{i_k} = K$, where E and K are in $R^{n\times n}$ and E is a positive diagonal. (That the last two limits exist follows from the boundedness of $\{E^i\}$ and $\{K^i\}$ and from $E^i \geqq \alpha I$ for some $\alpha > 0$ and all i.) Because f is continuous we have $\lim_{k\to\infty} f(x^{i_k}) = f(\bar{x})$. It follows that the nonincreasing sequence of real numbers $\{f(x^i)\}$ is bounded below, otherwise there exist $\hat{i}$ and $\hat{k}$ that would give the contradiction $f(x^{i_{\hat{k}}}) \geqq f(\bar{x}) > f(x^{\hat{i}}) \geqq f(x^{i_{\hat{k}}})$. Hence the sequence $\{f(x^i)\}$ converges. From (9) and the convergence of $\{f(x^i)\}$ we get

$$0 = \lim_{k\to\infty} (f(x^{i_k}) - f(x^{i_k+1})) \geqq \lim_{k\to\infty} \gamma \|x^{i_k+1} - x^{i_k}\|^2 \geqq 0 .$$

Hence

$$0 = \lim_{k\to\infty} \|x^{i_k+1} - x^{i_k}\| = \lim_{k\to\infty} \|(x^{i_k+1} - \bar{x}) - (x^{i_k} - \bar{x})\| = \lim_{k\to\infty} \|x^{i_k+1} - \bar{x}\|$$

and consequently $\lim_{k\to\infty} x^{i_k+1} = \bar{x} = \lim_{k\to\infty} x^{i_k}$. Thus

$$0 = \lim_{k\to\infty} \|x^{i_k+1} - x^{i_k}\|$$

$$= \lambda \lim_{k\to\infty} \|(x^{i_k} - \omega E^{i_k}(Mx^{i_k} + q + K^{i_k}(x^{i_k+1} - x^{i_k})))_+ - x^{i_k}\| \qquad \text{(By (5))}$$

$$= \lambda \|(\bar{x} - \omega E(M\bar{x}+q))_+ - \bar{x}\|$$

Hence $(\bar{x}-\omega E(M\bar{x}+q))_+ - \bar{x} = 0$ and by Lemma 1 $\bar{x}$ solves the linear complementarity problem (1). □

Because Theorem 1 does not guarantee the existence of an accumulation point, additional conditions are now imposed in the following theorem which will ensure that the sequence $\{x^i\}$ is bounded and hence will possess an accumulation point. The approach used here is similar to that of Eckhardt [6], however our results are more general and our proofs are different.

Theorem 2 Let either (a) M be a symmetric strictly copositive matrix or (b) let M be a symmetric copositive plus matrix which satisfies one of the two conditions

(10) $\quad Mx + q > 0 \quad$ for some $x \in R^n$

or

(11) $\quad Mx > 0$ for some $x \in R^n$

Then the sequence $\{x^i\}$ of Algorithm 1 is bounded and has an accumulation point $\bar{x}$ which solves the linear complementarity problem (1).

Remark 5 Condition (11) implies condition (10).

We first establish the following lemma. Part (b) of this lemma is a generalization of Eckhardt's Theorem 5.2 [6] to copositive plus matrices.

Lemma 3 Let M be symmetric and copositive and let $\{x^i\}$ be an unbounded sequence in R^n such that for all i, $f(x^i) \leqq c$ and $x^i \geqq 0$, where c is some constant and f is defined in (2).

(a) There exists a subsequence $\{x^{i_k}\}$ such that $\{y^{i_k} = \frac{x^{i_k}}{\|x^{i_k}\|}\}$ converges to a point $\bar{y}$ satisfying

$$0 \neq \bar{y} \geqq 0, \quad \bar{y}^T M\bar{y} = 0, \quad q^T\bar{y} \leqq 0$$

(b) If M is copositive plus then $\bar{y}$ satisfies

$$0 \neq \bar{y} \geqq 0, \quad M\bar{y} = 0, \quad q^T\bar{y} \leqq 0$$

(c) If M is copositive plus then

$Mx + q > 0$ has no solution $x \in R^n$

and

$Mx > 0$ has no solution $x \in R^n$

Proof (a) Let $\{x^{i_k}\}$ be a subsequence of $\{x^i\}$ such that

$$\lim_{k\to\infty} \|x^{i_k}\| = \infty, \quad 0 \neq x^{i_k} \geqq 0 \text{ and}$$

$$\lim_{k\to\infty} \frac{x^{i_k}}{\|x^{i_k}\|} = \lim_{k\to\infty} y^{i_k} = \bar{y} \geqq 0$$

where $\bar{y}$ lies on the compact unit sphere $\{y \mid y \in R^n, \|\bar{y}\| = 1\}$ and hence $0 \neq \bar{y} \geqq 0$.

Since

$$\frac{c}{\|x^{i_k}\|^2} \geqq \frac{f(x^{i_k})}{\|x^{i_k}\|^2} = \frac{1}{2}(y^{i_k})^T My^{i_k} + q^T \frac{y^{i_k}}{\|x^{i_k}\|}$$

it follows that

$$0 \geqq \frac{1}{2}\bar{y}^T M\bar{y} + q^T\bar{y} \lim_{k\to\infty} \frac{1}{\|x^{i_k}\|} = \frac{1}{2}\bar{y}^T M\bar{y} \geqq 0$$

where the last inequality follows from the copositivity of M. Hence

$\bar{y}^T M\bar{y} = 0$. But

$$\frac{c}{\|x^{i_k}\|} \geqq \frac{f(x^{i_k})}{\|x^{i_k}\|} = \frac{\|x^{i_k}\|}{2} (y^{i_k})^T My^{i_k} + q^T y^{i_k} \geqq q^T y^{i_k}$$

Hence $q^T y^{i_k} \leqq \frac{c}{\|x^{i_k}\|}$ and consequently $q^T\bar{y} \leqq 0$.

(b) Since M is symmetric and copositive plus it follows that

$$\bar{y} \geqq 0,\ \bar{y}^T M\bar{y} = 0,\ q^T\bar{y} \leqq 0 \Longleftrightarrow \bar{y} \geqq 0,\ \ M\bar{y} = 0,\ \ q^T\bar{y} \leqq 0 .$$

(c) If there exists an x such that $Mx + q > 0$ we get the contradiction

$$0 \geqq x^T M\bar{y} + q^T\bar{y} = \bar{y}^T(Mx+q) > 0$$

If there exists an x such that $Mx > 0$ we get the contradiction

$$0 < x^T(-M\bar{y}) + \bar{y}^T (Mx) = \bar{y}^T(-Mx+Mx) = 0 .$$ □

Proof of Theorem 2 From (9) we have that $f(x^i) \leqq f(x^0) = c$ for all i , and from $x^0 \geqq 0$, (5) and $0 < \lambda \leqq 1$ we have that $x^i \geqq 0$ for all i . When M is strictly copositive we have by (4) and Lemma 3(a) that the sequence $\{x^i\}$ is bounded. When M is copositive plus and satisfies (10) or (11) we have again by Lemma 3(c) that the sequence $\{x^i\}$ is bounded. In either case the sequence $\{x^i\}$ possesses an accumulation point $\bar{x}$. By Theorem 1 $\bar{x}$ solves the linear complementarity

problem (1). □

Corollary 1 If M is symmetric and has positive elements then the sequence $\{x^i\}$ of Algorithm 1 is bounded and has an accumulation point $\bar{x}$ which solves the linear complementarity problem (1).

Remark 6 Note that Theorem 2 also provides an existence result for the symmetric linear complementarity problem. Thus when M is symmetric and copositive plus and either (10) or (11) is satisfied, then the linear complementarity problem (1) has a solution. Consequently for this case, Lemke's pivotal method cannot terminate in a ray [12,2]. Note also that (11) is satisfied if $q < 0$ and there exists an x satisfying $Mx + q \geqq 0$.

If M is positive definite, assumption (10) of Theorem 2 above is automatically satisfied because M is nonsingular and in addition the entire sequence $\{x^i\}$ converges. In particular we have

Theorem 3 If M is symmetric and positive definite then the sequence $\{x^i\}$ of Algorithm 1 converges to the unique solution of the linear complementarity problem (1).

Proof When M is symmetric and positive definite the minimization problem (3) or equivalently the linear complementarity problem (1) has a unique solution. By Theorem 2 the sequence $\{x^i\}$ generated by Algorithm 1 is bounded and each of its accumulation points is equal to the unique solution $\bar{x}$ of the linear complementarity problem (1). Hence [1, p 47, Problem 8] the sequence $\{x^i\}$ converges to $\bar{x}$. □

When M is positive definite and the solution to the linear complementarity problem is nondegenerate it is also possible to show for Algorithm 1, as Cryer did for the projected SOR method [5], that iteration (5) will, after a finite number of steps, identify which constraints among $Mx + q \geqq 0$, $x \geqq 0$ will be satisfied as equalities at the solution. Thereafter the algorithm behaves like an algorithm for the solution of a system of linear equations and for which a linear convergence rate may be given. More specifically we have

<u>Theorem 4</u> Let M be symmetric and positive definite, and let the solution $\bar{x}$ of the linear complementarity problem (1) be nondegenerate, that is $\bar{x} + M\bar{x} + q > 0$. Then there exists an $\bar{i}$ such that for $i \geqq \bar{i}$, Algorithm 1 becomes

$$(12)\quad \begin{cases} x_R^{i+1} = x_R^i - \lambda\omega E_{RR}^i(M_{RR}\, x_R^i + q_R + K_{RR}^i\,(x_R^{i+1}-x_R^i)) \\ x_S^{i+1} = 0 \end{cases}$$

where $R = \{j|\bar{x}_j > 0\}$ and $S = \{j|M_j\bar{x}+q_j > 0\}$. If in addition $E^i = E$ and $K^i = K$ where E is a positive diagonal and K is the strictly upper or lower triangular part of M, then the sequence $\{x^i\}$ converges to $\bar{x}$ at the linear root rate

$$\limsup_{i\to\infty} \|x^i-\bar{x}\|^{\frac{1}{i}} \leqq \rho(H_{RR}(\lambda\omega)) < 1$$

where $\rho(H_{RR}(\lambda\omega))$ is the spectral radius of

$$H_{RR}(\lambda\omega) = (I+\lambda\omega E_{RR}K_{RR})^{-1}(I-\lambda\omega E_{RR}(M_{RR}-K_{RR}))$$

<u>Proof</u> By Theorem 3 the sequence $\{x^i\}$ converges to $\bar{x}$ and so

$$\lim_{i\to\infty} (Mx^i+q+K^i(x^{i+1}-x^i))_{j\in S} = (M\bar{x}+q)_{j\in S} \geqq 2\delta e_{j\in S}$$

for some $\delta > 0$, and $\lim_{i\to\infty} x^i_{j\in S} = 0$. Since $E^i \geqq \alpha I$, it follows that for some i_1 and all $i \geqq i_1$

$$\omega(E^i(Mx^i+q+K^i(x^{i+1}-x^i)))_{j\in S} \geqq \omega\alpha\delta e_{j\in S} > 0$$

and

$$x^i_{j\in S} < \omega\alpha\delta e_{j\in S}$$

We have then from (5) in Algorithm 1 that

$$x^{i+1}_{j\in S} = (1-\lambda)x^i_{j\in S} \qquad \text{for } i \geqq i_1$$

But since $\lim_{i\to\infty} x^i_{j\in S} = 0$, it follows that

$$x^i_{j\in S} = 0 \qquad \text{for } i \geqq i_1$$

Similarly, since $\lim_{i\to\infty} x^i_{j\in R} = \bar{x}_{j\in R} \geqq 2\varepsilon e_{j\in R}$ for some $\varepsilon > 0$, and $\lim_{i\to\infty} \omega(E^i(Mx^i+q+K^i(x^{i+1}-x^i)))_{j\in R} = 0$, it follows that for some i_2 and all $i \geqq i_2$

$$x^i_{j\in R} \geqq \varepsilon e_{j\in R}$$

and

$$\omega(E^i(Mx^i+q+K^i(x^{i+1}-x^i)))_{j\in R} < \varepsilon e_{j\in R}$$

Since $x^i_S = 0$ for $i \geqq \bar{i} = \max\{i_1, i_2\}$ we get from (5) that for $i \geqq \bar{i}$

$$x^{i+1}_R = x^i_R - \lambda\omega E^i_{RR}(M_{RR}x^i_R + q_R + K^i_{RR}(x^{i+1}_{RR} - x^i_R))$$

$$x^{i+1}_S = 0$$

If in addition $E^i = E$ and $K^i = K$ where E is a positive diagonal and K is the strictly upper or lower triangular part of M, then we have for $i \geqq \bar{i}$

$$x^{i+1}_R = (I+\lambda\omega E_{RR}K_{RR})^{-1}((I-\lambda\omega E_{RR}(M_{RR}-K_{RR}))x^i_R - \lambda\omega E_{RR}q_R)$$

$$= H_{RR}(\lambda\omega)x^i_R - \lambda\omega(I+\lambda\omega E_{RR}K_{RR})^{-1}E_{RR}q_R$$

Since $x^i_R \to \bar{x}_R$, it follows by [20, Theorem 7.1.1] that $\rho(H_{RR}(\lambda\omega)) < 1$

and by [20, Theorem 7.2.2] that $\limsup_{i\to\infty} \|x^i_R - \bar{x}_R\| \leqq \rho(H_{RR}(\lambda\omega))$. But since $x^i_S = \bar{x}_S = 0$ for $i \geqq \bar{i}$, it follows that $\limsup_{i\to\infty} \|x^i - \bar{x}\| \leqq \rho(H_{RR}(\lambda\omega))$.

□

We note that since (12) holds for $i \geqq \bar{i}$, Algorithm 1 can be analyzed in the same manner as standard iterative methods [22] for solving the system of linear equations

$$\begin{bmatrix} M_{RR} & M_{RS} \\ 0 & I_{SS} \end{bmatrix} \begin{bmatrix} x_R \\ x_S \end{bmatrix} + \begin{bmatrix} q_R \\ 0 \end{bmatrix} = 0$$

In particular the optimum choice of the relaxation factors λ, ω can be based on the above equation. For specific cases of M, Cryer [5] has obtained some optimum values of ω for the projected SOR method for solving the linear complementarity problem.

3. SPECIFIC ALGORITHMS

We give in this section some specific realizations of the fundamental Algorithm 1. Throughout this section we shall assume that

$$(13)\quad \begin{cases} M = L + D + U \\ L = U^T \end{cases}$$

where D is diagonal (not necessarily positive), L is strictly lower triangular and U is strictly upper triangular. All the results of Section 2 apply to these specific cases.

By setting $K^i = 0$ and $E^i = E$ in Algorithm 1, where E is a positive diagonal we obtain

<u>Algorithm 2</u> (Projected Jacobi Overrelaxation) Let $x^0 \geqq 0$.

$$(14)\quad x^{i+1} = \lambda(x^i - \omega E(Mx^i + q))_+ + (1-\lambda)x^i \qquad i = 0,1,2,\ldots$$

where $0 < \lambda \leqq 1$, $\omega > 0$ and E is a positive diagonal matrix such that $2(\lambda\omega E)^{-1} - M$ is positive definite.

<u>Remark 7</u> The positive definiteness of $2(\lambda\omega E)^{-1} - M$, which is condition (6) when $K^i = 0$ and $E^i = E$, is implied by either of the two following conditions

(a) $2(\lambda\omega E)^{-1} - D$ is a positive diagonal and $2(\lambda\omega E)^{-1} - M$ is strictly or irreducibly diagonally dominant [22, Theorem 1.8]

(b) $2(\lambda\omega E)^{-1} - D$ is a positive diagonal and $2(\lambda\omega E)^{-1} - D - |L+U| \in K$ [23, Theorem 1 (i), (ii)] where the class K is defined in the Introduction.

Note further that if D is a positive diagonal, E may be set equal to D^{-1} as in the case of the ordinary Jacobi overrelaxation. If in addition $\lambda\omega = 1$, we have the projected ordinary Jacobi iteration.

By setting $K^i = L$ or U , $E^i = E$ in Algorithm 1, where E is a positive diagonal, we obtain

<u>Algorithm 3</u> (Projected SOR) Let $x^0 \geqq 0$.

$$(15)\quad x^{i+1} = \lambda(x^i - \omega E(Mx^i + q + K^i(x^{i+1} - x^i)))_+ + (1-\lambda)x^i \qquad i = 0,1,2,\ldots$$

where $0 < \lambda \leqq 1$, $\omega > 0$, E is a positive diagonal, $K^i = L$ or U , and

$$(16)\quad \lambda\omega < \frac{2}{\max_{j \in J_+} D_{jj} E_{jj}}$$

where $J_+ = \{j | D_{jj} > 0,\ j = 1,\ldots,n\}$.

<u>Remark 8</u> When $\lambda = 1$, $E = D^{-1}$ and $K^i = L$, Algorithm 3 becomes the projected SOR studied by Cryer, Eckhardt and others [4,5,6]. If in addition $\omega = 1$, Algorithm 3 becomes the projected Gauss-Seidel method studied

by, among others, Fridman and Chernina [8].

A symmetric projected SOR version can be obtained from Algorithm 3 if we alternate in the use of L and U for K^i. In particular we have

Algorithm 4 (Projected Symmetric SOR) Let $x^0 \geq 0$.

$$(17)\quad \begin{cases} x^{i+1} = \lambda(x^i - \omega E(Mx^i + q + L(x^{i+1} - x^i)))_+ + (1-\lambda)x^i & i = 0,2,4,\ldots \\ x^{i+1} = \lambda(x^i - \omega E(Mx^i + q + U(x^{i+1} - x^i)))_+ + (1-\lambda)x^i & i = 1,3,5,\ldots \end{cases}$$

where $0 < \lambda \leqq 1$, $\omega > 0$, E is a positive diagonal such that (16) is satisfied.

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