Abstract Homotopy Theory
and the Thomason Model Structure

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Abstract Homotopy Theory and the Thomason Model Structure

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Zusammenfassung

Die Kategorie der kleinen Kategorien besitzt eine abgeschlossene Modellstruktur, genannt die Thomason Modellstruktur, welche Quillen äquivalent zur Standard Modellstruktur auf der Kategorie der topologischen Räume ist. Wir geben eine Einführung in die unterschiedlichen Konzepte, die sowohl zum Verständnis, als auch zur Motivation der Definition der Thomason Modellstruktur notwendig sind. Diese Konzepte beinhalten Kategorientheorie, klassische Homotopietheorie auf topologischen Räumen, simpliziale Homotopietheorie auf simplizialen Mengen und abstrakte Homotopietheorie auf Modellkategorien. Wir werden zeigen, dass es eine Modellstruktur auf der Kategorie der kleinen, azyklischen Kategorien gibt, die Quillen äquivalent zur Thomason Modellstruktur ist. Beide Modellstrukturen besitzen die gleichen kofasernden Objekte. Wir werden zeigen, das zu diesen insbesondere endlich Halbverbände, abzählbare Bäume, endliche Zickzacks und Halbordnungen mit fünf oder weniger Elementen gehören.

Abstract

There is a closed model structure on the category of small categories, called Thomason model structure, that is Quillen equivalent to the standard model structure on the category of topological spaces. We will give an introduction to the concepts necessary to understand the definition, as well as the purpose of the Thomason model structure. These concepts include category theory, classical homotopy theory on topological spaces, simplicial homotopy theory on simplicial sets and abstract homotopy theory via the use of model categories. We will show, that there is a model structure on the category of small acyclic categories, that is Quillen equivalent to the Thomason model structure. Both of these model structures share the same cofibrant objects, and we will show that these include finite semilattices, countable trees, finite zigzags and posets with five or less elements.

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Introduction

Homotopy theory has been used for almost a century as an algebraic tool to study topological spaces. The first implicit usage of homotopy goes back as far as 1806, when deformations of paths were used to minimize or maximize certain integrals [Lag06], but it was not until 1934, that a whole theory emerged and *homotopy groups* were formally introduced in [Hur35]. It took another 15 years until in 1949, *CW-complexes* were formally introduced by Whitehead in [Whi49]. They are the first prototype of what is now called a *cofibrant object* in a model category, i. e. a space which can be constructed by "gluing together simpler spaces".

In 1950, Eilenberg and Zilber introduced *simplicial sets* (which back then were called *complete semi-simplicial complexes*) in [EZ50] and in the following years, Kan developed a full-blown homotopy theory for simplicial sets, which was further refined by the work of Quillen in the 1960s (as mentioned in the preface of [GJ99]).

In 1967, Quillen introduced the notion of a *model category* [Qui67] which married the theory of homotopy on cofibrant objects in the category of topological spaces to the theory of homotopy on fibrant objects in the category of simplicial sets, making both settings special cases of homotopy theory in a model category. Quillens work enables us to do homotopy in more abstract settings, where objects are not necessarily "spaces", and homotopies are not necessarily "deformations of paths".

In 1980, Thomason showed that there is a model structure on the category of small categories, that is Quillen equivalent to the standard model structure on topological spaces [Tho80] (to which we will refer to as the *Thomason model structure*), where Quillen equivalent means that the homotopy theories are equivalent, i. e. two categories are homotopy equivalent if and only if the associated topological spaces are homotopy equivalent.

In 2010, Raptis showed that there is a model structure on the category of posets, that is Quillen equivalent to the Thomason model structure on the category of small categories [Rap10].

While there have been many successful attempts to lift the Thomason model structure to other categories, e. g. the category of G-categories [Boh+15], the category of G-posets [MSZ16], the category of strict n-categories [AM14] and the category of small n-fold categories [FP10], very few publications have dealt with the internal structure of the Thomason model structure. In 2014, Meier and Ozornova identified a class of fibrant objects in the Thomason model structure [MO15] and in 2016, May, Stephan and Zakharevich showed that every one-dimensional poset is cofibrant, and gave an example of a poset that is not cofibrant [MSZ16]. To the knowledge of the author, these are the only publications dealing with the internal structure of the Thomason model structure.

In this thesis, we will show that one can lift the Thomason model structure to the category of small acyclic categories, using methods similar to those used in [Rap10]. Furthermore, we will identify various classes of cofibrant objects in the Thomason model structure: we will show that every finite semilattice, every countable tree, every chain, every finite zigzag, and every poset with five or less elements are cofibrant, and that every inclusion of a minimum into one of these objects is a cofibration.

In Chapter 2.1, we will give a brief introduction to category theory and provide the tools necessary to deal with model categories. In Chapter 2.2, we will give a short recapitulation of classical homotopy theory on topological spaces, and show the importance of cofibrant objects to give a motivation for the later chapters. In Chapter 2.3, we introduce simplicial sets and simplicial homotopy theory, and compare it to the classical homotopy theory on topological spaces. In Chapter 3.1, we introduce model categories, develop an abstract homotopy theory and show how it generalizes results from classical and simplicial homotopy theory, as introduced in the previous chapters. In Chapter 3.2 we introduce the Thomason model structure on the category of small categories, and show that one can lift said model structure to the category of small acyclic categories (Theorem 3.2.16). The results of this chapter have been previously published by the author in [Bru15]. In Chapter 4.1 we show that every finite semilattice, every countable tree, every chain and every finite zigzag are cofibrant in the Thomason model structure (Theorem 4.1.5, 4.1.13, 4.1.8, and 4.1.11 respectively), and in Chapter 4.2 we show that every poset with five or less elements is cofibrant (Theorem 4.2.13). The Chapters 4.1 and 4.2 are a joint work with Christoph Pegel and have been published in [BP16].

2

Applied Homotopy Theory

2.1 Introduction to Category Theory

Category theory starts with the observation that many properties of mathematical systems can be unified and simplified with diagrams and arrows.

– Saunders Mac Lane

There are many mathematical constructions that appear in a variety of contexts. For example, given a Cartesian product $A \times B$, A and B may be sets, vector spaces, topological spaces or many other things, but it always means essentially the same thing.

Category theory provides the means to unify these constructions by forgetting the internal structure of A and B, and giving a description using only arrows and diagrams. Moreover, Category theory formalizes "switching" between different mathematical environments by introducing functors and adjunctions, which are ubiquitary throughout mathematics.

2.1.1 Basic Definitions

To give a definition of a category, one will need to talk about the *collection* of all sets, topological spaces, vector spaces, and so forth. This collection cannot be a set, though, since a set of all sets cannot exit (cf. [Bor94a, Proposition 1.1.1]). These problems can be avoided by introducing *classes*, which share some, but not all properties of sets given in ZF. We will not go into further details here, since they do not matter in the context of this thesis, and consider classes as "collections of objects which behave like sets". We refer the interested reader to [Ber42]. A class that is also a set will be called a *small class*, and a class that is not a set will be called *proper class*.

Definition 2.1.1. A *category* $\mathcal{C}$ consists of a class of objects $\mathcal{C}^{(0)}$ and a class of morphisms (also called maps or arrows) $\mathcal{C}^{(1)}$ together with:

(i) A source map $s\colon \mathcal{C}^{(1)} \to \mathcal{C}^{(0)}$.

(ii) A target map $t\colon \mathcal{C}^{(1)} \to \mathcal{C}^{(0)}$.

(iii) A composition

$$\circ\colon \left\{(f,g) \in \mathcal{C}^{(1)} \times \mathcal{C}^{(1)} \,\middle|\, t(f) = s(g)\right\} \to \mathcal{C}^{(1)}$$
$$(f,g) \mapsto g \circ f,$$

satisfying $s(g \circ f) = s(f)$ and $t(g \circ f) = t(g)$, and

(iv) for each $x \in \mathcal{C}^{(0)}$ a morphism $\mathrm{id}_x \in \mathcal{C}^{(1)}$, called *identity*, such that $s(\mathrm{id}_x) = x = t(\mathrm{id}_x)$,

subject to the following axioms:

(i) Associativity: given $f, g, h \in \mathcal{C}^{(1)}$ such that $t(f) = s(g)$ and $t(g) = s(h)$ the following equality holds:

$$h \circ (g \circ f) = (h \circ g) \circ f$$

.

(ii) Unit Law: given $f \in \mathcal{C}^{(1)}$, such that $s(f) = x$ and $t(f) = y$, then

$$f \circ \mathrm{id}_x = f = \mathrm{id}_y \circ f.$$

(iii) Local smallness: given any pair of objects $x, y \in \mathcal{C}^{(0)}$, the subclass

$$\mathcal{C}(x, y) := \left\{ f \in \mathcal{C}^{(1)} \,\middle|\, s(f) = x \text{ and } t(f) = y \right\}$$

is a set.

We will often write $f \in \mathcal{C}$ if it is clear whether f is a morphism or an object in $\mathcal{C}$. We will usually also omit the index of the identity id_x, and call two morphisms $f, g \in \mathcal{C}$ *composable* if $t(f) = s(g)$. Furthermore, a category is called *small* if $\mathcal{C}^{(0)}$ and $\mathcal{C}^{(1)}$ are sets. A morphism $f \in \mathcal{C}(x, y)$ will often be represented diagrammatically as $f\colon x \to y$ or $x \xrightarrow{f} y$, and we will sometimes omit the composition from notation, i. e. given a second morphism $g \in \mathcal{C}$, we will write gf for the composition instead of $g \circ f$.

Definition 2.1.2. Let $\mathcal{C}$ be a category, and $f \in \mathcal{C}$ a morphism. The map f is called *monomorphism* if given any pair of morphisms $g_1, g_2 \in \mathcal{C}$, we have

$$f \circ g_1 = f \circ g_2 \implies g_1 = g_2,$$

and *epimorphism* if for every pair h_1, h_2 of morphisms in $\mathcal{C}$, we have

$$h_1 \circ f = h_2 \circ f \implies h_1 = h_2.$$

In **Set**, monomorphisms are precisely injections, and epimorphisms are precisely surjections. This is true in a lot of categories where objects are "sets with structure", and morphisms are given by "structure preserving maps". There are, however, categories where objects are "sets with structure", and the classes of epi- or monomorphisms are larger than the classes of surjections and injections. For example, let **Haus** be the category of Hausdorff spaces and continuous maps, then epimorphisms are precisely those morphisms, that have dense image.

Definition 2.1.3. A *covariant functor* F from a category $\mathcal{C}$ to a category $\mathcal{D}$ consists of

(i) A mapping

$$\mathcal{C}^{(0)} \to \mathcal{D}^{(0)},$$

associating to each object $x \in \mathcal{C}$ an object $F(x) \in \mathcal{D}$, and

(ii) for each pair of objects $x, y \in \mathcal{C}$ a mapping

$$\mathcal{C}(x, y) \to \mathcal{D}(F(x), F(y)),$$

associating to each morphism $f \in \mathcal{C}$ a morphism $F(f) \in \mathcal{D}$, subject to the following axioms:

(i) Given two composable morphisms $f, g \in \mathcal{C}$, then

$$F(g \circ f) = F(g) \circ F(f),$$

and

(ii) for each object $x \in \mathcal{C}$,

$$F(\mathrm{id}_x) = \mathrm{id}_{F(x)} .$$

A functor is called *contravariant* if given two composable morphisms $f, g \in \mathcal{C}$, instead of $F(g \circ f) = F(g) \circ F(f)$, we have $F(g \circ f) = F(f) \circ F(g)$.

Given a category $\mathcal{C}$, and an object $x \in \mathcal{C}$, there is a contravariant functor

$$\begin{aligned} \mathcal{C}(-, x)\colon \mathcal{C} &\to \mathbf{Set} \\ y &\mapsto \mathcal{C}(y, x) \\ f &\mapsto f^* \end{aligned}$$

where

$$\begin{aligned} f^*\colon \mathcal{C}(t(f), x) &\to \mathcal{C}(s(f), x) \\ g &\mapsto g \circ f \end{aligned}$$

and a covariant functor

$$\begin{aligned} \mathcal{C}(x, -)\colon \mathcal{C} &\to \mathbf{Set} \\ y &\mapsto \mathcal{C}(x, y) \\ f &\mapsto f_* \end{aligned}$$

where

$$\begin{aligned} f_*\colon \mathcal{C}(x, s(f)) &\to \mathcal{C}(x, t(f)) \\ g &\mapsto f \circ g \end{aligned}$$

which we will call the contravariant and covariant Hom-*functor* respectively.

Composition of functors is pointwise and the class of all small categories, together with the class of all functors constitute the *category of small categories*, denoted by **Cat**. Other examples include

(i) the *category of small sets*, denoted by **Set**, with object class small sets, and morphisms given by set maps and

(ii) the *category of topological spaces*, denoted by **Top**, with objects given by topological spaces, and continuous maps as morphisms.

However, categories do not necessarily contain a whole mathematical theory, but are also quite often used as combinatorial objects, examples include:

(i) Given any poset $(P, \leq)$, we can associate a category $\mathcal{P}$ with object set $\mathcal{P}^{(0)} = P$, and given any pair of objects $x, y \in P$, we say there is an arrow $x \to y$ if and only if $x \leq y$.

(ii) Given any group G, we can associate a category $\mathcal{G}$ with set of objects being $\mathcal{G}^{(0)} = \{*\}$ and the set of morphisms being $\mathcal{G}^{(1)} = G$, where composition is given by the group operation.

(iii) Given any directed graph $G = (E, V)$ with edges E and vertices V, one can associate the *free category* $F(G)$ with object set $F(G)^{(0)} = V$, and morphisms being generated by the set of edges E by adding identities and compositions.

We will use (i) and (iii) implicitly throughout this thesis. A poset will always be a category P, satisfying that for each pair of objects $x, y \in P$, $x \neq y$ we have $\#P(x,y) + \#P(y,x) \leq 1$ and $P(x,x) = 1$, and when describing a category $\mathcal{C}$ diagrammatically, we will usually only draw the directed graph G consisting objects and generating morphisms, and omit identities and compositions, so that $\mathcal{C} = F(G)$.

Definition 2.1.4. Let $\mathcal{C}$, $\mathcal{D}$ be categories, $F\colon \mathcal{C} \to \mathcal{D}$ a functor. We say F is *full* if for every $x, y \in \mathcal{C}^{(0)}$ the induced map $\mathcal{C}(x,y) \to \mathcal{D}(F(x), F(y))$ is a surjection, *faithful* if the respective map is an injection, and *fully faithful* if it is a bijection.

A *subcategory* $\mathcal{S}$ of a category $\mathcal{C}$ consists of subcollections $\mathcal{S}^{(0)}$ of objects and $\mathcal{S}^{(1)}$ of morphisms of $\mathcal{C}$, such that $\mathcal{S}$ itself is again a category, having the same identities and compositions. To each subcategory we can associate a faithful functor which is injective on objects, which we will call the associated *embedding* or *inclusion*.

A subcategory $\mathcal{S} \subseteq \mathcal{C}$ is called *full* if given any pair of objects $x, y \in \mathcal{S}$, we have $\mathcal{S}(x,y) = \mathcal{C}(x,y)$ (i. e. if the associated embedding is full) and *wide* if $\mathcal{S}^{(0)} = \mathcal{C}^{(0)}$ (i. e. if $\mathcal{C}$ is small, the associated embedding is a bijection on the classes of objects). Given the description of a poset as a category, the category **Pos** of posets and orderpreserving morphism (i. e. functors) is a full subcategory of **Cat**.

Similar to homotopies between continuous maps, there is a notion of morphisms between functors, called natural transformations:

Definition 2.1.5. Let $\mathcal{C}$, $\mathcal{D}$ be categories, and $F, G\colon \mathcal{C} \to \mathcal{D}$ be functors. A *natural transformation* $\varphi\colon F \Rightarrow G$ is a function which assigns to each object $x \in \mathcal{C}$ a morphism $\varphi_x = \varphi(x)\colon F(x) \to G(x)$ in $\mathcal{D}$, such that for any morphism $f \in \mathcal{C}(x,y)$ the following diagram commutes:

$$\begin{array}{ccccc} x & & F(x) & \xrightarrow{\varphi_x} & G(x) \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle F(f)} & & \downarrow{\scriptstyle G(f)} \\ y & & F(y) & \xrightarrow{\varphi_y} & G(y) \end{array}$$

If every *component* φ_x of φ is invertible, we say φ is a *natural equivalence*.

When dealing with categories, most definitions and theorems require an object only to exist up to isomorphisms. Hence the notion of isomorphisms between categories is usually to strong, since we only require a bijection between isomorphism classes of objects for theorems and definitions to be applicable in both categories. Which leads us to the notion of an *equivalence* between categories.

Definition 2.1.6. Let $\mathcal{C}$, $\mathcal{D}$ be categories, we say that $\mathcal{C}$ is *equivalent* to $\mathcal{D}$, if there is a pair of functors $F\colon \mathcal{C} \leftrightarrows \mathcal{D} : G$, together with a pair of natural equivalences $FG \Rightarrow \mathrm{id}_{\mathcal{D}}$, $\mathrm{id}_{\mathcal{C}} \Rightarrow GF$

Given two categories $\mathcal{C}, \mathcal{D}$, then $\mathbf{Cat}(\mathcal{C}, \mathcal{D})$ is again a category, with objects being functors $\mathcal{C} \to \mathcal{D}$ and morphisms given by natural transformations, where composition of natural transformations is given pointwise. That is, given two natural transformations $\varphi\colon F \Rightarrow G$ and

$\psi\colon G \Rightarrow E$ in $\mathbf{Cat}(\mathcal{C},\mathcal{D})$, we define $\psi\circ\varphi\colon F \Rightarrow E$ to be the natural transformation with components $(\psi\circ\varphi)_x = \psi_x \circ \varphi_x$ for every $x \in \mathcal{C}$. We call this category a *functor category*, and will denote it by $\mathcal{D}^{\mathcal{C}}$.

Definition 2.1.7. Let $\mathcal{C}$, $\mathcal{D}$, $\mathcal{E}$ be categories, $F\colon \mathcal{C}\to\mathcal{D}$, $G\colon \mathcal{E}\to\mathcal{D}$ be functors. We define the *comma category* $F\downarrow G$ to be the category with objects

$$F\downarrow G^{(0)} := \left\{(x,e,f) : x\in\mathcal{C}^{(0)}, e\in\mathcal{E}^{(0)}, f\in\mathcal{D}_{F(x)}^{G(e)}\right\}$$

and morphisms

$$\begin{aligned}&(F\downarrow G)((x,e,f),(x',e',f')) :=\\ &\qquad \{(\varphi,\psi) : \varphi\in\mathcal{C}(x,x'), \psi\in\mathcal{E}(e,e'), G(\psi)\circ f = f'\circ F(\varphi)\}\end{aligned}.$$

If $G = \mathrm{id}_{\mathcal{C}}$, we simply write $F\downarrow\mathcal{C}$. Moreover, given the constant functor

$$\begin{aligned} x\colon * &\to \mathcal{C}\\ * &\mapsto x\\ \mathrm{id}_* &\mapsto \mathrm{id}_x\,,\end{aligned}$$

we call $\mathcal{C}\downarrow x$ the *slice category* over x and $x\downarrow\mathcal{C}$ the *coslice category*.

If we restrict to the subcategory of posets, some types of comma categories yield familiar constructions. Let $f\colon P\to Q$ be a poset map, $x\in Q$, then

(i) $Q\downarrow x = Q_{\le x} := \{y\in P \mid y\le x\}$

(ii) $x\downarrow Q = Q_{\ge x} := \{y\in P \mid y\ge x\}$

(iii) $f\downarrow x = f^{-1}(Q_{\le x})$

(iv) $x\downarrow f = f^{-1}(Q_{\ge x})$

Thus one may view comma categories as generalizations of posets of objects below and above a certain element.

Another important concept in category theory is dualization, which in layman terms can be described as “turning around all arrows”. Formally, we dualize by introducing the *opposite category.*

Definition 2.1.8. Let $\mathcal{C}$ be a category, the *opposite category* $\mathcal{C}^{\mathrm{op}}$ has the same objects as $\mathcal{C}$, but given any $x,y\in\mathcal{C}$, we have $\mathcal{C}^{\mathrm{op}}(x,y) = \mathcal{C}(y,x)$. Given a morphism $f\in\mathcal{C}$, we denote by f^{op} the associated morphisms in the opposite category. Now given a pair of composable morphisms f^{op}, g^{op}, we define composition by $g^{\mathrm{op}}\circ f^{\mathrm{op}} := (g\circ f)^{\mathrm{op}}$.

Note that if a diagram in a category $\mathcal{C}$ commutes, it also commutes in $\mathcal{C}^{\mathrm{op}}$, and we call the construction *dual.* Furthermore, a contravariant functor $F\colon\mathcal{C}\to\mathcal{D}$ induces a covariant functor $F\colon\mathcal{C}^{\mathrm{op}}\to\mathcal{D}$ and vice versa. This property is used in the definition of the Yoneda embedding:

Definition 2.1.9. Let $\mathcal{C}$ be a category. The Yoneda embedding $Y_{\mathcal{C}}$ of $\mathcal{C}$ is the functor

$$\begin{aligned} Y_{\mathcal{C}}\colon \mathcal{C} &\to \mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}\\ x &\mapsto \mathcal{C}(-,x)\\ f &\mapsto \mathcal{C}(-,f)\end{aligned}$$

where $\mathcal{C}(-,f)$ is the obvious natural equivalence $\mathcal{C}(-,s(f))\Rightarrow\mathcal{C}(-,t(f))$.

Furthermore, given a category $\mathcal{C}$, and a class of morphisms $I \subseteq \mathcal{C}$, one can formally add inverses to morphisms in I to obtain a new category. This procedure is called *localization*, and will play an important role when introducing homotopy categories:

Definition 2.1.10. Let $\mathcal{C}$ be a category, and I be a class of maps in $\mathcal{C}$. The *localization* of $\mathcal{C}$ is a category $L_I\mathcal{C}$ and a functor $\gamma\colon \mathcal{C} \to L_I\mathcal{C}$ such that

(i) if $f \in I$, then $\gamma(f)$ is an isomorphism, and

(ii) if $\mathcal{D}$ is a category, and $F\colon \mathcal{C} \to \mathcal{D}$ is a functor such that $F(f)$ is an isomorphism for every $f \in I$, then there is a unique functor $\delta\colon L_I\mathcal{C} \to \mathcal{D}$ such that $\delta\gamma = F$.

2.1.2 Adjunctions

Definition 2.1.11. Let $\mathcal{C}$, $\mathcal{D}$ be categories. An adjunction from $\mathcal{C}$ to $\mathcal{D}$ is a triple (F, G, φ), where F and G are functors

$$F\colon \mathcal{C} \leftrightarrows \mathcal{D} \colon G,$$

and φ is a function which assigns to each pair of objects $x \in \mathcal{C}^{(0)}$, $y \in \mathcal{D}^{(0)}$ a bijection of sets

$$\varphi = \varphi_{x,y}\colon \mathcal{D}(F(x), y) \xrightarrow{\cong} \mathcal{C}(x, G(y))$$

which is *natural* in x and y. That is, for any $f \in \mathcal{D}(y, y')$, $g \in \mathcal{C}(x', x)$, the following diagrams commute:

$$\begin{array}{ccc} \mathcal{D}(F(x),y) & \xrightarrow{\varphi} & \mathcal{C}(x,G(y)) \\ \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle G(f)_*} \\ \mathcal{C}(F(x),y') & \xrightarrow{\varphi} & \mathcal{D}(x,G(y')) \end{array} \qquad \begin{array}{ccc} \mathcal{D}(F(x),y) & \xrightarrow{\varphi} & \mathcal{C}(x,G(y)) \\ \downarrow{\scriptstyle F(g)^*} & & \downarrow{\scriptstyle g^*} \\ \mathcal{C}(F(x'),y) & \xrightarrow{\varphi} & \mathcal{D}(x',G(y)) \end{array}$$

We call F the *left adjoint* to G, and G the *right adjoint* to F.

Note that composition of left adjoints is again a left adjoint, and composition of right adjoints is again a right adjoint. In particular, given two adjunction (F, G, φ) and (L, R, ψ), such that F, L and G, R are composable, then LF is a left adjoint to GR (cf. [Mac98, p.103, Thm. 1]). There is an equivalent definition of an adjunction, which is useful in some contexts:

Theorem 2.1.12. *[Mac98, p.83] Given a pair of functors $F\colon \mathcal{C} \leftrightarrows \mathcal{D} \colon G$, then F is left-adjoint to G if and only if there are natural transformations $\eta\colon \mathrm{id}_{\mathcal{C}} \Rightarrow GF$ and $\varepsilon\colon FG \Rightarrow \mathrm{id}_{\mathcal{D}}$, such that the composites*

$$G \xrightarrow{G(\eta)} GFG \xrightarrow{\varepsilon_{G(-)}} G \qquad \text{and} \qquad F \xrightarrow{\eta_{F(-)}} FGF \xrightarrow{F(\varepsilon)} F$$

are identities. The transformations η and ε are called unit *and* counit*, respectively.*

Given F, G, η, ε as in the previous theorem, and $f \in \mathcal{D}(F(x), y)$, then $\varphi_{x,y}(f)$ is given by

$$\varphi_{x,y}(f)\colon x \xrightarrow{\eta_x} GF(x) \xrightarrow{G(f)} G(y)$$

As usual, we will omit the indices from φ if the domain is either clear, or not relevant.

Proposition 2.1.13. *[GZ67, p.7] Let G be a right adjoint, then G is fully faithful if and only if the counit ε is invertible (that is, ε is a natural equivalence).*

Definition 2.1.14. A full subcategory $\mathcal{S} \subseteq \mathcal{C}$ is called *reflective* if the inclusion $i\colon \mathcal{S} \hookrightarrow \mathcal{C}$ has a left adjoint, which we will call the *reflection* to i.

2.1.3 Limits and Colimits

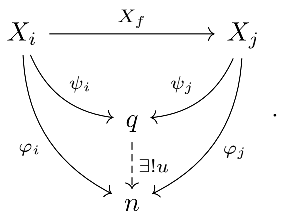
Limits and colimits are ubiquitary in category theory, and many familiar constructions are in fact given by limits or colimits in the ambient category.

Definition 2.1.15. Let $\mathcal{C}, I$ be categories, a *diagram* of shape I is a functor $X\colon I \to \mathcal{C}$. We call I the *index category* or *shape* of the diagram X.

Given a diagram $X\colon I \to \mathcal{C}$, we write X_i for the image $X(i)$ of an object $i \in I$, and $X_{i\to j}$ for the image of a morphism $i \to j$ in I. Moreover, when drawing a diagram, we will usually draw the image of the diagram in the target category.

Definition 2.1.16. A *cocone* $(q, \psi_\bullet)$ of a diagram $X\colon I \to \mathcal{C}$ is an object $q \in \mathcal{C}$ together with a family of morphisms $\psi_i\colon X_i \to q$, such that for every $f \in I(i,j)$, we have $\psi_j \circ X_f = \psi_i$

Definition 2.1.17. A *colimit* of a diagram $X\colon I \to \mathcal{C}$ is a universal cocone. That is a cocone $(q, \psi_\bullet)$, satisfying that given any other cocone $(n, \varphi_\bullet)$, there is a unique morphism $u\colon q \to n$ in $\mathcal{C}$, making the following diagram commutative for every morphism $i \xrightarrow{f} j$ in I:



We write

$$\operatorname*{colim}_I X = q \text{ or } \operatorname{colim} X = q$$

to denote the object of the universal cocone, and usually omit the morphisms from notation.

Note that the colimit of a diagram is only defined up to isomorphisms. We will, however, say *the* colimit of a diagram instead of *a* colimit, as it is common practice in literature. Consequentially, when describing a colimit as *the* unique object satisfying a certain property, we mean *unique up to isomorphisms*. The dual of a cocone is a *cone*, and a universal cone is a *limit*. There are several limits and colimits that go by specific names. We will provide some of those as examples, focusing on the ones that will be important later on.

Definition 2.1.18. Let $X\colon \varnothing \to \mathcal{C}$ be the empty diagram, then the colimit $\operatorname{colim} X =: \emptyset$ is the *initial object* in $\mathcal{C}$, i. e. the unique object satisfying that given any object $y \in \mathcal{C}^{(0)}$, there is exactly one morphism $i\colon \emptyset \to y$.

Dually, the limit $\lim X =: *$ of X is called the *terminal object*. Satisfying that given any object $x \in \mathcal{C}$, there is exactly one morphism $t\colon x \to *$.

In **Set**, the initial object is the empty set, and the terminal object is the one-element set. In $\mathbf{Vect}_{\mathbb{K}}$, the category of vector spaces over a field $\mathbb{K}$ with vector space homomorphisms as morphisms, the initial and terminal objects are identical, given by the zero-dimensional vector space.

Definition 2.1.19. Let $\mathbf{2}$ be the category with two objects $0, 1 \in \mathbf{2}$, and no nonidentity morphisms. Given a category $\mathcal{C}$, the colimit of a diagram $X\colon \mathbf{2} \to \mathcal{C}$ is called *coproduct*, denoted by $\operatorname{colim} X =: X_0 \coprod X_1$, whereas the limit is called *product*, denoted by $\lim X =: X_0 \times X_1$.

As the notation suggests, products and coproducts are—in many cases—what one expects them to be. That is, in **Set** products and coproducts are given by the Cartesian product and disjoint union, respectively. In $\mathbf{Vect}_{\mathbb{K}}$ they are given by the Cartesian product and the direct sum. Note that we can use arbitrary sets as index category to produce arbitrary products and coproducts, instead of binary.

Definition 2.1.20. Let $\mathcal{C}$ be a small category, $X\colon I \to \mathcal{C}$ be the diagram

$$\begin{array}{ccc} x & \xrightarrow{f_2} & y_2 \\ \downarrow{\scriptstyle f_1} & & \\ y_1 & & \end{array} .$$

The colimit of X is called the *pushout* of f_1 along f_2, often denoted by $y_1 \coprod_x y_2$.

Dually, let $Y\colon J \to \mathcal{C}$ be the diagram

$$\begin{array}{ccc} & & x_1 \\ & & \downarrow{\scriptstyle f_1} \\ x_2 & \xrightarrow{f_2} & y \end{array} .$$

The limit of Y is called the *pullback* of f_1 along f_2, often denoted by $x_1 \times_y x_2$.

Let $Y_1 \xleftarrow{f_1} X \xrightarrow{f_2} Y_2$ be a diagram in **Set**. The pushout is given by

$$Y_1 \coprod_X Y_2 = Y_1 \coprod Y_2/\!\sim$$

where $\sim$ is the equivalence relation generated by $f_1(x) \sim f_2(x)$ for every $x \in \mathcal{C}$. If Y_1 and Y_2 are subsets of a larger ambient set, X is their intersection and f_1 and f_2 are inclusions (i. e. injections), then the pushout is simply the union of the subsets. If X is the empty set (or initial object in categories other than **Set**), the pushout is simply the coproduct.

On the other hand, given a diagram $X_1 \xrightarrow{f_1} Y \xleftarrow{f_2} X_2$ in **Set**, the pullback is the subset $X_1 \times_Y X_2 \subseteq X_1 \times X_2$, given by:

$$X_1 \times_Y X_2 = \{(x_1, x_2) \in X_1 \times X_2 \mid f(x_1) = f(x_2)\} .$$

Definition 2.1.21. Let $\mathcal{C}$ be a category, and $X\colon I \to \mathcal{C}$ be the diagram

$$x \underset{g}{\overset{f}{\rightrightarrows}} y \ ,$$

The colimit of X is called *coequalizer*, and the limit is called *equalizer*. Denoted by $\mathrm{Coeq}(f,g)$ and $\mathrm{Eq}(f,g)$ respectively.

In **Set**, the equalizer of two maps $f, g\colon X \to Y$ is the subset

$$\mathrm{Eq}(f,g) = \{x \in X \mid f(x) = g(x)\} \subseteq X,$$

whereas the coequalizer $\mathrm{Coeq}(f,g)$ is the set $Y/\!\sim$, where $\sim$ is the equivalence relation generated by $f(x) \sim g(x)$.

In the category $\mathbf{Vect}_{\mathbb{K}}$, given a vector space V and a subspace U, the coequalizer of the embedding and the zero map yields the quotient space V/U. The same holds in $\mathbf{Top}_*$, the category of pointed topological spaces and basepoint preserving continuous maps.

Note that limits and colimits do not necessarily exist. For example, let **Fields** be the category of fields and field homomorphisms, and let K_1, K_2 be two fields. The definition of a universal cocone requires that the coproduct $K_1 \coprod K_2$ comes with two field homomorphisms $K_1 \to K_1 \coprod K_2$ and $K_2 \to K_1 \coprod K_2$. However, homomorphisms between fields can only exist if both fields have the same characteristic. So if K_1 and K_2 have different characteristic, a coproduct with those properties cannot exist.

We call a category in which all small colimits (i. e. colimits over diagrams with small index category) exist *cocomplete*, a category in which all small limits exist *complete*, and a category which is complete and cocomplete *bicomplete*. The categories **Cat**, **Set** and **Top** are all bicomplete.

Definition 2.1.22. Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor, we say F *preserves colimits*, if given any diagram $X\colon I \to \mathcal{C}$, we have

$$F(\operatorname*{colim}_I X) \cong \operatorname*{colim}_I F(X)$$

and F *preserves limits*, if

$$F(\lim_I X) \cong \lim_I F(X)$$

Preservation of limits and colimits is are important propertis of a functor, and there is a huge class of functors partially satisfying these:

Theorem 2.1.23. *Let (F, G) be an adjunction, then F preserves colimits, and G preserves limits.*

The second half of the theorem is [Bor94a], and the first half is simply the dual statement.

2.1.4 Filtered Colimits

There is a convenient class of index categories called *filtered categories*, which make the calculation of colimits in **Cat** particularly easy. We will make use of these categories later.

Definition 2.1.24. A non-empty category $\mathcal{C}$ is called *filtered* if

(i) for every pair of objects x_1, x_2 in $\mathcal{C}$ there is an object y in $\mathcal{C}$ and morphisms $f_i\colon x_i \to y$, $i = 1, 2$.

(ii) for any pair of parallel morphisms $f_1, f_2\colon x \to y$ there exists an object z and a morphism $h\colon y \to z$, such that $hf_1 = hf_2$.

In **Cat**, there is an explicit method to construct filtered colimits. At first, though, we need to look at filtered colimits in **Set**:

Proposition 2.1.25. *[Bor94a, Proposition 2.13.3] Let $X\colon I \to \mathbf{Set}$ be a filtered diagram, then*

$$\operatorname*{colim}_I X = (C)$$

where the set C is given by

$$C = \coprod_{i\in I} X_i/\sim,$$

and $\sim$ is defined as follows: given $x \in X_i$, $x' \in X_{i'}$, we have $x \sim x'$ if there is an object $j \in I$, together with maps $f\colon X_i \to X_j$ and $g\colon X_{i'} \to X_j$ such that $f(x) = g(x')$.

Lemma 2.1.26. *Let $D\colon I \to \mathbf{Set}$ be a filtered diagram, $x_i \in D_i$, and $D(i \to j)\colon D_i \to D_j$. Then $[x_i] = [D(i \to j)(x_i)]$ in $\operatorname{colim}_I D$.*

Proof. Due to 2.1.24 (i), there exists a D_k, together with morphisms $D(i \to k)\colon D_i \to D_k$ and $D(j \to k)\colon D_j \to D_k$. Moreover, by 2.1.24 (ii), since in general $D(j \to k) \circ D(i \to j) \neq D(i \to k)$, there is an object D_l, and a morphism $D(k \to l)\colon D_k \to D_l$ such that

$$D(k \to l) \circ D(j \to k) \circ D(i \to j) = D(k \to l) \circ D(i \to k)$$

thus

$$D(k \to l) \circ D(j \to k)(D(i \to j)(x_i)) = D(k \to l) \circ D(i \to k)(x_i)$$

and therefore $D(i \to j)(x_i) \sim x_i$ with respect to the equivalence relation from Proposition 2.1.25.

We can use the previous result to give a general construction of filtered colimits in **Cat**:

Proposition 2.1.27. *[Bor94b, 5.2.2.f] Let $D\colon I \to \mathbf{Cat}$ be a filtered diagram. There is an explicit description of $\mathcal{L} = \operatorname{colim}_I D$ given as follows: $\mathcal{L}^{(0)} = \operatorname{colim}_I D(i)^{(0)}$ is just the colimit in* **Set**. *Given a pair of objects L, L' in $\mathcal{L}^{(0)}$, the morphism set $\mathcal{L}(L, L')$ is given by the colimit $\operatorname{colim}_I D_i(L_i, L_i')$ in* **Set**, *where $L = [L_i]$ and $L' = [L_i']$.*

2.1.5 Coequalizers in Cat

Definition 2.1.28. Given a small category $\mathcal{C}$, and an equivalence relation $\sim$ on the set of objects of $\mathcal{C}$. A $\sim$*–composable* sequence in $\mathcal{C}$ is a sequence $(f_0, \ldots, f_n)$ of morphisms in $\mathcal{C}$, satisfying $t(f_i) \sim s(f_{i+1})$.

Definition 2.1.29. Let $\mathcal{C}$ be a small category. A *generalized congruence* on $\mathcal{C}$ is an ordered pair of relations $(\sim_o, \sim_m)$, where $\sim_o$ is an equivalence relation on $\mathcal{C}^{(0)}$, and $\sim_m$ is an equivalencerelation on the set of non-empty, $\sim_o$–composable sequences in $\mathcal{C}$, satisfying the following properties:

(i) if $x \sim_o y$, then $(\mathrm{id}_x) \sim_m (\mathrm{id}_y)$.

(ii) If $(f_0, \ldots, f_n) \sim_m (h_0, \ldots, h_m)$, then $t(f_n) \sim_o t(h_m)$ and $s(f_0) \sim_o s(h_0)$.

(iii) If $s(h) = t(f)$, then $(f, h) \sim_m (h \circ f)$.

(iv) If

$$\begin{aligned}(f_0, \ldots, f_n) &\sim_m (f_0', \ldots, f_{n'}'),\\ (h_0, \ldots, h_m) &\sim_m (h_0', \ldots, h_{m'}'), \text{ and}\\ t(f_n) &\sim_o s(h_0),\end{aligned}$$

then

$$(f_0, \ldots, f_n, h_0, \ldots, h_m) \sim_m (f_0', \ldots, f_{n'}', h_0', \ldots, h_{m'}').$$

Given a generalized congruence on a category $\mathcal{C}$, we can define the quotient of the category with respect to the congruence, thanks to the following proposition (cf. [Hau06, Proposition 1.6]):

Proposition 2.1.30. *Let $(\sim_o, \sim_m)$ be a generalized congruence on a category $\mathcal{C}$, and $\mathbf{F} \subseteq (\mathcal{C} \downarrow \mathbf{Cat})$ be the full subcategory with objects being functors $F\colon \mathcal{C} \to \mathbf{Cat}$, satisfying the following properties:*

(i) for all objects $x, y \in \mathcal{C}$, if $x \sim_o y$, then $F(x) = F(y)$, and

(ii) for all $\sim_o$–composable sequences $(f_0, \ldots, f_n)$ and $(h_0, \ldots, h_m)$ if

$$(f_0, \ldots, f_n) \sim_m (h_0, \ldots, h_m),$$

then

$$F(f_n) \circ \cdots \circ F(f_0) = F(h_m) \circ \cdots \circ F(h_0).$$

Then **F** *has an initial object, which we denote by $Q_\sim \colon \mathcal{C} \to \mathcal{C}/{\sim}$.*

Definition 2.1.31. Given the functor $Q_\sim \colon \mathcal{C} \to \mathcal{C}/{\sim}$ as above, we call $\mathcal{C}/{\sim}$ the *quotient* of $\mathcal{C}$ with respect to $\sim$, and $Q_\sim$ the corresponding *quotient functor*.

There is an explicit construction for the quotient category $\mathcal{C}/{\sim}$, given in [BBP99]: the objects of $\mathcal{C}/{\sim}$ are the equivalence classes of objects of $\mathcal{C}$ with respect to $\sim_o$, whereas the morphisms are given by equivalence classes of $\sim_o$–composable sequences in $\mathcal{C}^{(1)}$ with respect to $\sim_m$. For the sake of readability, we denote equivalence classes with respect to both relations by $[-]$. The category $\mathcal{C}/{\sim}$ is given as follows:

(i) $(\mathcal{C}/{\sim})^{(0)} = \left\{[x] \middle| x \in \mathcal{C}^{(0)}\right\}$

(ii) $(\mathcal{C}/{\sim})^{(1)} = \left\{[(f_0, \ldots, f_n)] \,\middle|\, f_i \in \mathcal{C}^{(1)}, [t(f_i)] = [s(f_{i+1})]\right\}$

(iii) $\mathrm{id}_{[x]} = [\mathrm{id}_x]$

(iv) $s([(f_0, \ldots, f_n)]) = [s(f_0)]$ and $t([(f_0, \ldots, f_n)]) = [t(f_n)]$

(v) $[(h_0, \ldots, h_m)] \circ [(f_0, \ldots, f_n)] = [(f_0, \ldots, f_n, h_0, \ldots, h_m)]$

A *relation* R on a small category $\mathcal{C}$ is a pair $R = (R_o, R_m)$, where R_o is a relation on the set of objects of $\mathcal{C}$, and R_m is a relation on the set of finite, nonempty sequences of morphisms of $\mathcal{C}$. Given a relation R, there is a smallest generalized congruence $(\sim_o, \sim_m)$, such that $R_o \subseteq \sim_o$ and $R_m \subseteq \sim_m$ (cf. [Hau06]). We will call this congruence the *principal congruence* generated by R.

Proposition 2.1.32. *Let $\mathcal{C} \underset{G}{\overset{F}{\rightrightarrows}} \mathcal{D}$ be functors between small categories, let $\sim_{F=G}$ be the relation on $\mathcal{D}$ defined by $F(x) \sim_{F=G} G(x)$ and $F(f) \sim_{F=G} G(f)$ for all $x \in \mathcal{C}^{(0)}$, $f \in \mathcal{C}^{(1)}$. Let $\sim$ be the principal congruence on $\mathcal{D}$ generated by $\sim_{F=G}$. Then the quotient functor $Q_\sim \colon \mathcal{D} \to \mathcal{D}/{\sim}$ is the coequalizer of F and G.*

The ability to calculate coequalizer in **Cat** allows us, in particular, to calculate pushouts due to the following well-known lemma (e.g. [AHS90, Remark 11.31]):

Lemma 2.1.33. *Let $\mathcal{C}$ be a category. If we have a diagram $x \xleftarrow{f} w \xrightarrow{g} y$ in $\mathcal{C}$, and if $x \xrightarrow{\iota_x} x \amalg y \xleftarrow{\iota_y} y$ is a coproduct and $x \amalg y \xrightarrow{h} q$ is the coequalizer of the diagram $w \underset{\iota_y \circ g}{\overset{\iota_x \circ f}{\rightrightarrows}} x \amalg y$, then*

$$\begin{array}{ccc} w & \xrightarrow{g} & y \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle h \circ \iota_y} \\ x & \xrightarrow{h \circ \iota_x} & q \end{array}$$

is a pushout square.

2.1.6 Saturated Classes

Saturated classes play an important role in the construction of cofibrantly generated model structures, which will be introduced later. In fact, every relevant model structure we will encounter in this thesis is cofibrantly generated.

Definition 2.1.34. Let $\mathcal{C}$ be a category. The *arrow category* $\operatorname{Arr}\mathcal{C}$ is the category with objects $\operatorname{Arr}\mathcal{C}^{(0)} = \mathcal{C}^{(1)}$, and morphisms given by commutative squares, i. e. given two arrows $x_1 \xrightarrow{f} y_1$ and $x_2 \xrightarrow{g} y_2$, a morphism in $\operatorname{Arr}\mathcal{C}(f,g)$ is a tuple $(x_1 \xrightarrow{u} x_2, y_1 \xrightarrow{v} y_2)$, such that the following diagram commutes:

$$\begin{array}{ccc} x_1 & \xrightarrow{u} & x_2 \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle v} \\ y_1 & \xrightarrow{v} & y_2 \end{array}$$

Note that the arrow category is just a comma category. That is, given a category $\mathcal{C}$, then $\operatorname{Arr}\mathcal{C} \cong \mathcal{C} \downarrow \mathcal{C}$.

Definition 2.1.35. Let $\mathcal{C}$ be a category $x, y \in \mathcal{C}^{(0)}$, we say y is a *retract* of x if there are morphisms $y \xrightarrow{i} x$, $x \xrightarrow{p} y$, such that $p \circ i = \mathrm{id}$, and we will call the map p the *retraction* of i.

Given a category $\mathcal{C}$, and morphisms $f, g \in \mathcal{C}$, we say g is a retract of f if g is a retract of f in the arrow category $\operatorname{Arr}(\mathcal{C})$, i. e. if there are pairs of morphisms (i_0, p_0) and (i_1, p_1), such that the following diagram commutes:

$$\begin{array}{ccccc} & & \overset{p_0 \circ i_0 = \mathrm{id}}{\frown} & & \\ x_2 & \xrightarrow{i_0} & x_1 & \xrightarrow{p_0} & x_2 \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} \\ y_2 & \xrightarrow{i_1} & y_1 & \xrightarrow{p_1} & y_2 \\ & & \underset{p_1 \circ i_1 = \mathrm{id}}{\smile} & & \end{array}$$

Definition 2.1.36. Let λ be an ordinal, considered as a category, and $\mathcal{C}$ be a cocomplete category. A λ-*sequence* is a colimit-preserving functor $X\colon \lambda \to \mathcal{C}$, often written as

$$X_0 \to X_1 \to \cdots \to X_\beta \to \cdots.$$

Definition 2.1.37. Let $\mathcal{C}$ be a cocomplete category, λ an ordinal and X be a λ-sequence. If I is a class of morphisms in $\mathcal{C}$, such that every map $X_\beta \to X_{\beta+1}$ is in I, we refer to the map $X_0 \to \operatorname{colim} X$ as a *transfinite composition* of maps in I.

Definition 2.1.38. Let $\mathcal{C}$ be a category, and $I \subseteq \mathcal{C}^{(1)}$ a class of morphisms. We say that I is a *saturated* if it is closed under pushouts, retracts and transfinite composition.

Given a class of morphisms I, we call the intersection of all saturated classes containing I as a subclass the *saturation* of I, or *saturated class generated by* I.

2.1.7 Locally Presentable Categories

A poset is called *directed* if every pair of elements has a common upper bound, and we call a colimit over a diagram *directed* if the index category is a directed poset.

Definition 2.1.39. An object x of a category $\mathcal{C}$ is called *locally finitely presentable* if the Hom functor

$$\hom(x, -): \mathcal{C} \to \mathbf{Set}$$

preserves directed colimits. A category $\mathcal{C}$ is called *locally finitely presentable* if it is cocomplete and there is a set $A \subseteq C^{(0)}$ of locally finitely presentable objects such that every object of $\mathcal{C}$ is a directed colimit of objects of A.

There is a useful theorem (cf. [AR94, Theorem 1.39]) that allows us to decide whether a reflective subcategory of a locally presentable category is locally presentable:

Lemma 2.1.40. *Let $\mathcal{C}$ be a locally finitely presentable category and $\mathcal{A} \subseteq \mathcal{C}$. If $\mathcal{A}$ is reflective and the inclusion $i: \mathcal{A} \to \mathcal{C}$ preserves finitely directed colimits, then $\mathcal{A}$ is locally finitely presentable.*

Sometimes it is easier to check whether a functor preserves filtered colimits instead of directed, and the following lemma (cf. [AR94, p. 15]) allows us to do so:

Every poset that is filtered as a category, is directed, since condition (ii) from Definition 2.1.24 is satisfied trivially, and condition (i) is exactly the definition of directedness. Hence, any functor that preserves filtered colimits, also preserves directed colimits. However, the converse holds as well, as shown in [AR94, p. 15]:

Lemma 2.1.41. *A functor $F: \mathcal{C} \to \mathcal{D}$ preserves filtered colimits if and only if it preserves directed colimits.*

The category **Cat** is finitely locally presentable, with the set of locally presentable objects given by the one-element set $\{\mathbf{2}\}$, where $\mathbf{2}$ is defined as in 2.1.19. Also, given a small category $\mathcal{C}$, the functor category $\mathbf{Set}^{\mathcal{C}}$ is finitely locally presentable. These results can be found in [Bor94b, 5.2.2.f] and [Bor94b, 5.2.2.b] respectively.

2.2 Homotopy Theory on Topological Spaces

> *Even though the ultimate goal of topology is to classify various classes of topological spaces up to a homeomorphism, in algebraic topology, homotopy equivalence plays a more important role than homeomorphism, essentially because the basic tools of algebraic topology (homology and homotopy groups) are invariant with respect to homotopy equivalence, and do not distinguish topologically nonequivalent, but homotopic objects.*
>
> – Anatole Katok

In order to understand abstract homotopy theory, one should have a basic knowledge of the concepts abstract homotopy generalizes. In this chapter we will give a brief recapitulation of classical homotopy theory on topological spaces, focusing on the concepts we are going to generalize in the later chapters. This means we will develop an intuition of what cell complexes and cofibrations are in the classical sense, and why they play an important role in classical homotopy theory.

This chapter is by no means an introduction to classical homotopy theory, but rather a collection and reformulation of classical results and definitions the reader should be familiar with in a more categorical, diagrammatical way.

2.2.1 A Convenient Category of Topological Spaces

It is a well known fact, that the category of topological spaces features many pathological counterexamples [SS70]. For that reason, there have been many attempts to replace the category of topological spaces with a more convenient [Ste67] subcategory, excluding some of the pathological counterexamples one might not want to deal with. From now on, we will denote by **Top** the category of compactly generated weakly Hausdorff spaces. There are, however, other equally suited candidates.

2.2.2 Homotopies in Top

Definition 2.2.1. Let $f, g\colon X \to Y$ be maps in **Top**, and $I = [0, 1]$ be the unit interval. We say f is *homotopic* to g if in either of the following diagrams, the dashed arrow exists, making the diagram commutative:

The maps ι_0, ι_1 denote the respective injections at 0 and 1, and π_0, π_1 the respective evaluation maps at 0 and 1. We call the map H a *left homotopy* and the map h a *right homotopy* from f to g and write $H\colon f \Rightarrow g$ and $h\colon f \Rightarrow g$, respectively.

In **Top** the notions of left and right homotopy are equivalent, that is given two maps f, g, there exists a right homotopy $H\colon f \Rightarrow g$ if and only if there exists a left homotopy $h\colon f \Rightarrow g$. Hence it just justifiable to simply say there is a *homotopy* from f to g. We will see later, that this is not necessarily true when dealing with homotopy theories in other categories. Moreover, homotopy is closed under composition of maps. That is, given morphisms $f_0, f_1\colon X \to Y$ and $g_0, g_1\colon Y \to Z$, such that f_0 is homotopic to f_1, and g_0 is homotopic to g_1, then $g_0 \circ f_0$ is homotopic to $g_1 \circ f_1$ (see e. g. [Str11, Ch. 4.1.4]). We will call a map $f\colon X \to Y$ a homotopy equivalence if there is a map $g\colon Y \to X$, such that $f \circ g$ is homotopic to id_Y and $g \circ f$ is homotopic to id_X.

Given any pair of objects $X, Y \in$ **Top**, homotopy induces an equivalence relation on the morphism sets **Top**(X, Y), and due to the previous paragraph, elementwise composition of the respective equivalence classes is well defined. This leads us to the notion of the homotopy category:

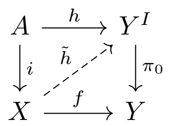
Definition 2.2.2. The *classical homotopy category* of **Top** is the category π**Top**, whose objects are the same as in **Top**, and whose morphisms are homotopy classes of morphisms in **Top**.

Note that identities in π**Top** are given by the equivalence classes of identities in **Top**, hence homotopy equivalent objects in **Top** are isomorphic in π**Top**.

An important class of morphisms when dealing with homotopy theory is the class of Hurewicz cofibrations, which are given by the homotopy extension property:

Definition 2.2.3. Let $i \in$ **Top**(A, X). We say that i has the *homotopy extension property* if for any topological space Y, any $f\colon X \to Y$, and any homotopy $h\colon A \to Y^I$ satisfying $\pi_0 \circ h = f \circ i$,

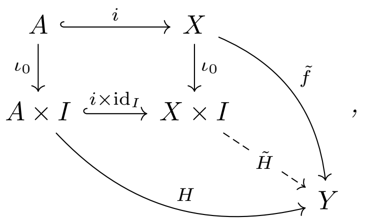
there exists a homotopy $\tilde{h}\colon X \to Y^I$, such that $p_0 \circ \tilde{h} = f$ and $\tilde{h} \circ i = h$.
That is, given the solid arrow diagram



the dashed arrow exists, and both triangles commute. The map i is also called *Hurewicz cofibration*. If additionally $i(A)$ is closed in X, we say i is a *closed cofibration*.

There is a different formulation of the homotopy extension property, based on right homotopies:

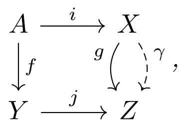
Proposition 2.2.4. *Let $i \in$ **Top**(A, X). We say that i has the* homotopy extension property *if for any topological space Y, any $\tilde{f}\colon X \to Y$, and any homotopy $H\colon A \times I \to Y$ satisfying $H \circ \iota_0 = \tilde{f} \circ i$, there exists a homotopy $\tilde{H}\colon X \times I \to Y$, such that $\tilde{H} \circ \iota_0 = f$ and $\tilde{H} \circ (i \times \mathrm{id}_I) = H$. That is, given the solid arrow diagram*



the dashed arrow exists, making the diagram commutative.

Hurewicz cofibrations satisfy several useful properties with respect to homotopy theory, we will give two of them as propositions:

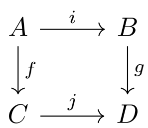
Theorem 2.2.5. *[Str11] Given the solid arrow diagram*



such that $g \circ i$ is homotopic to $j \circ f$. If i is a Hurewicz cofibration, the dashed arrow γ exists, making the diagram commutative, and there is a homotopy from g to γ.

and

Proposition 2.2.6. *[Str11, Cor. 6.50] If in the pushout square*



i is a Hurewicz cofibration, and f is a homotopy equivalence, then g is also a homotopy equivalence.

The class of Hurewicz cofibrations is closed under composition and every homeomorphism is a Hurewicz cofibration, hence the class of Hurewicz cofibrations is also closed under composition with arbitrary homeomorphisms. Moreover, the class of Hurewitz cofibrations is saturated, which follows directly from [Col06, Thm 3.1]:

Theorem 2.2.7. *The class of Hurewicz cofibrations is closed under retracts, pushouts and transfinite composition.*

We will make use of this fact.

2.2.3 CW–Complexes

One of the most important tools for doing homotopy theory in **Top** are CW–complexes. Throughout this chapter, we will discuss several useful properties supporting this statement.

Definition 2.2.8. Consider the diagram

$$\begin{aligned} J\colon \mathbb{N} &\to \mathbf{Top} \\ n &\mapsto X_n \end{aligned}$$

where $\mathbb{N}$ is the poset of all natural numbers, X_0 is a discrete topological space, and X_{n+1} is constructed from X_n via a pushout

$$\begin{array}{ccc} \coprod \partial D^{n+1} & \xrightarrow{\alpha_n} & X_n \\ {\scriptstyle \amalg i}\big\downarrow & & \big\downarrow{\scriptstyle j_n} \\ \coprod D^{n+1} & \longrightarrow & X_{n+1} \end{array},$$

where i denotes the boundary embedding, and j_n is the image of the map $n \to n+1$ in $\mathbb{N}$ under J. Not that the maps j_n are injections, since the space X_{n+1} is obtained from X_n by "glueing" the interiors of the $(n+1)$-discs along the image of their boundary in X_n onto X_n. For that reason, we say that X_n is *obtained from X_{n-1} by attaching n-cells.* A *CW–complex* X is a topological space obtained by taking the colimit $X \cong \operatorname{colim}_{\mathbb{N}} J$. We call the subset $X_n \subseteq X$ the *n-skeleton* of X, and we say that X has *dimension n* if $X_n = X_{n+k}$ for all k in $\mathbb{N}$.

The disks D^n are called (closed) *n-cells* of a complex X, and come with a *characteristic map* $\chi\colon D^n \to X$ defined by the diagram

$$\begin{array}{ccc} \coprod \partial D^{n} & \xrightarrow{\alpha_{n-1}} & X_{n-1} \\ {\scriptstyle \amalg i}\big\downarrow & & \big\downarrow{\scriptstyle j_{n-1}} \\ \coprod D^{n} & \longrightarrow & X_{n} \\ \big\uparrow & & \big\downarrow \\ D^n & \xrightarrow{\chi} & X \end{array}$$

Note that every CW–complex X comes with a sequence

$$X_0 \xhookrightarrow{j_0} X_1 \xhookrightarrow{j_1} \cdots \xhookrightarrow{j_n} X_n \xhookrightarrow{j_{n+1}} \ldots,$$

of subspaces X_k and inclusions j_k. We call this sequence the *CW–structure* of the space X. Since two different sequences may yield (up to homeomorphism) the same space, the CW–structure of a CW–complex is not unique and one may choose different CW–structures for the same complex, depending on the task at hand.

We also want a notion of morphisms between CW–complexes, i. e. maps that preserve the CW–structure.

Definition 2.2.9. Given CW–complexes X, Y and maps $f_i\colon X_i \to Y_i$, such that

$$\begin{array}{ccccccccc} X_0 & \longrightarrow & X_1 & \longrightarrow & \cdots & \longrightarrow & X_n & \longrightarrow & \cdots \\ \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f_1} & & & & \downarrow{\scriptstyle f_n} & & \\ Y_0 & \longrightarrow & Y_1 & \longrightarrow & \cdots & \longrightarrow & Y_n & \longrightarrow & \cdots \end{array}$$

commutes, then the map $f\colon X \to Y$ induced by taking the colimit is called a *cellular map*.

The existence of a CW–structure allows us to prove theorems inductively. For example:

Theorem 2.2.10. *[Str11] Let X be a CW–complex, $Y \in$ **Top** and $f\colon X \to Y$. The map f is continuous if and only if $f_{|X_n}$ is continuous for every n in $\mathbb{N}$.*

Given CW–complexes X, Y, and an inclusion $i\colon X \hookrightarrow Y$, we call i an *inclusion of CW–complexes*, and X a *subcomplex* of Y, if there are CW–structures on X and Y, such that i is a cellular map.

An important feature of CW–complexes is that any inclusion of a subcomplex is a Hurewicz cofibration:

Theorem 2.2.11. *[Whi49, (J)] Let $i\colon X \to Y$ be an inclusion of CW–complexes, then i satisfies the homotopy extension property.*

Recall that a weak homotopy equivalence is a map $f\colon X \to Y$ that induces an isomorphism on the homotopy groups, independently of the choice of a base point. That is, the induced map $f_*\colon \pi_n(X, x) \to \pi_n(Y, y)$ is an isomorphism of groups for $n \geq 1$, and a bijection for $n = 0$ for each choice of a basepoints $x \in X$ and $y \in Y$, respectively. Whitehead proved in [Whi49] that weak homotopy equivalences between CW–complexes are homotopy equivalences:

Theorem 2.2.12 (Whitehead Theorem). *Let X, Y be CW–complexes, and $f\colon X \to Y$ a weak homotopy equivalence. Then f is a homotopy equivalence.*

2.2.4 Hurewicz Fibrations

The dual concept to Hurewicz cofibrations and the homotopy extension property are fibrations and the homotopy lifting property. Since our research focuses on cofibrations and cofibrant objects, we will keep this as brief as possible, and focus on the interplay between fibrations and cofibrations.

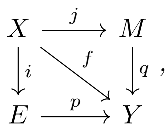
Definition 2.2.13. Let $p \in \mathbf{Top}(E, B)$ and $X \in \mathbf{Top}$, we say the map p has the *homotopy lifting property* with respect to X if for any homotopy $H\colon X \times I \to B$, and any map $f\colon X \to E$, satisfying $p \circ f = H \circ \iota_0$, there is a homotopy $\tilde{H}\colon X \times I \to E$, such that $\tilde{H} \circ \iota_0 = f$ and $p \circ \tilde{H} = H$. That is, given the solid arrow diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & E \\ {\scriptstyle \iota_0}\downarrow & \overset{\tilde{H}}{\nearrow} & \downarrow{\scriptstyle p} \\ X \times I & \xrightarrow{H} & B \end{array} ,$$

the dashed arrow exists and both triangles commute. If p has the homotopy lifting property with respect to every space X, p is called a *Hurewicz fibration*.

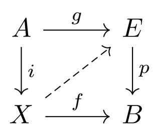
A lot of the theory of cofibrations can be dualized, for example, the duals of Theorem 2.2.5 and Theorem 2.2.6 hold. That is, replacing the Hurewicz cofibrations with Hurewicz fibrations and flipping all arrows (cf. [Str11]). However, as noted before the important results with respect to this thesis are those dealing with the interplay between Hurewicz cofibrations and Hurewicz fibrations. Most prominently the lifting and factorization properties:

Theorem 2.2.14. *[Str11, Thm. 5.42] Every map $f\colon X \to Y$ has factorizations, making every triangle in the following diagram commute:*



where i and j are Hurewicz cofibrations, p and q are Hurewicz fibrations and i and q are homotopy equivalences.

Theorem 2.2.15 (Fundamental lifting property). *[Str11, Thm. 5.64] Given the solid diagram*



such that i is a Hurewicz cofibration and p is a Hurewicz fibration. Then the dashed arrow exists if either i is a homotopy equivalence, or p is a homotopy equivalence.

Later on, we will use both of this properties to define what a homotopy theory is in other categories than **Top**. Generally, we say that a map i has the *left lifting property* with respect to a class of morphisms $\mathcal{M}$ if for every $p \in \mathcal{M}$, given the solid arrow diagram in Theorem 2.2.15, the dashed arrow exists. Dually, a map p has the *right lifting property* with respect to a class of morphisms $\mathcal{M}$ if for every $i \in \mathcal{M}$, given the solid arrow diagram in Theorem 2.2.15, the dashed arrow exists. We will call a Hurewicz cofibration or fibration trivial if it is also homotopy equivalence.

Theorem 2.2.16. *[Str11, Thm. 5.70]*

(i) A map $i\colon A \to X$ is a Hurewicz cofibration if and only if it has the left lifting property with respect to all trivial Hurewicz fibrations.

(ii) A map $p\colon E \to B$ is a Hurewicz fibration if and only if it has the left lifting property with respect to all trivial Hurewicz cofibrations.

(iii) A map $i\colon A \to X$ is a trivial Hurewicz cofibration if and only if it has the left lifting property with respect to all Hurewicz fibrations.

(iv) A map $p\colon E \to B$ is a trivial Hurewicz fibration if and only if it has the left lifting property with respect to all Hurewicz cofibrations.

As a consequence of Theorem 2.2.16, we can define Hurewicz fibrations via their lifting properties with respect to Hurewicz cofibrations, or vice versa. We will use this to define different notions of fibrations and cofibrations in **Top**.

2.2.5 Serre Fibrations and Cofibrations

There are different classes of fibrations and cofibrations, whose significance was first shown by Serre in his thesis [Ser51]. The idea is, that since working with CW–complexes proved to be useful, one does not need fibrations to satisfy the homotopy lifting property for all spaces, but rather only for CW–complexes. Moreover, since every weak homotopy equivalence between CW–complexes is a homotopy equivalence by the Whitehead Theorem (Thm. 2.2.12), we can work with the larger class of weak homotopy equivalences, instead of just homotopy equivalences.

Definition 2.2.17. Let $p\colon E \to B$ be a morphism in **Top**. If p has the homotopy lifting property with respect to every CW–complex, p is called a *Serre fibration*.

There is an equivalent definition of Serre fibration, which will be useful later on:

Theorem 2.2.18. *[Hir15, Prop. 4.4] Let $p\colon E \to B$ be a morphism in* **Top***, then p is* Serre fibration *if and only if for every $n \geq 0$, given any solid diagram*

$$\begin{array}{ccc} D^n & \longrightarrow & E \\ {\scriptstyle (\mathrm{id},\iota_0)}\big\downarrow & \overset{\theta}{\nearrow} & \big\downarrow{\scriptstyle p} \\ D^n \times I & \longrightarrow & B \end{array},$$

the map $\theta\colon D^n \times I \to E$ exists, making the diagram commutative.

Theorem 2.2.19. *[Hov99, Cor. 2.4.14.] Let X be a topological space, then $X \to *$ is a Serre fibration.*

We will call a Serre fibration that is also a weak homotopy equivalence a trivial Serre fibration, and use Theorem 2.2.16 as the defining property for Serre cofibrations:

Definition 2.2.20. Let $i\colon X \to Y$ be a morphism in **Top**, then i is called *Serre cofibration* if it has the left lifting property with respect to all trivial Serre fibration, and i is called a *trivial Serre cofibration* if it has the left lifting property with respect to all Serre fibrations.

Serre fibrations, Serre cofibrations and weak homotopy equivalences enjoy many of the properties that hold for Hurewicz fibrations, Hurewicz cofibrations and homotopy equivalences. In fact, one can take any theorem from this chapter and replace Hurewicz with Serre, and homotopy with weak homotopy and it will still hold.

In particular, there is a different characterization of Serre cofibrations.

Theorem 2.2.21. *Let I be the set of all boundary inclusions $\partial D^n \to D^n$. The class of* Serre cofibrations *is given by the saturation of I.*

Note that a coproduct $X \coprod Y$ is just the pushout of the diagram $X \leftarrow \emptyset \to Y$, hence every cellular map between CW–complexes is a Serre cofibration by definition, and every CW–complex is cofibrant. The class of cofibrant objects is larger than the class of CW–complexes, though:

Definition 2.2.22. A *relative CW–complex* is a pair (X, A) where X is built according to the procedure of Def. 2.2.8 but starting with the "(-1)-skeleton" $X_{-1} = A$, where A is any topological space (thus, $X_0 = A \coprod D$, where D is a discrete space).

A generalized CW–complex X is a topological space obtained in a similar manner as a CW–complex, but with the requirement on the dimension of the attached cells dropped. We can now give a complete classification of Serre cofibrations and Serre cofibrant spaces:

Theorem 2.2.23. *[Str11, Thm. 15.60] A map $i\colon X \to A$ is a Serre cofibration if and only if it is a retract of a generalized relative CW–complex.*

Theorem 2.2.24. *[Str11, Thm. 15.61] The Serre cofibrant spaces are precisely the retracts of generalized CW–complexes, and are homotopy equivalent to CW–complexes.*

Since every homotopy equivalence is also a weak equivalence, when localizing **Top** with respect to the class of weak homotopy equivalences we obtain a category that is equivalent to the homotopy category $\pi\mathbf{Top}_{\mathrm{cf}}$, where $\mathbf{Top}_{\mathrm{cf}}$ is the full subcategory of **Top**, whose class of objects given by the class Serre cofibrant spaces, i. e. retracts of generalized CW–complexes.

2.3 Simplicial Sets

Simplicial sets, an extension of the notion of simplicial complexes, have applications to algebraic topology, where they provide a combinatorial model for the homotopy theory of topological spaces.

– Emily Riehl

Simplicial sets—introduced by Gabriel and Zisman in [GZ67]—are a combinatorial variant of CW–complexes. Instead of gluing together spheres along their boundaries to build a topological space, simplicial sets are obtained by gluing together simplices along their boundaries. To every simplicial set X, one can associate a CW–complex $|X|$ and vice versa, given any topological space Y, one can associate a simplicial set $C(Y)$. Simplicial sets will play an important role when establishing a homotopy theory in **Cat**.

2.3.1 Simplicial Sets as Functors

We will introduce simplicial sets as a specific class of functors. While this definition is the most natural and very useful for a variety of proofs, it is not very intuitive with respect to the combinatorial structure. In particular, it is not quite obvious how this definition encodes "triangles glued together along their boundaries". We will explain this point of view once we have introduced the geometric realization of a simplicial set.

Definition 2.3.1. Let $\mathbf{\Delta}$ be the following category:

(i) The objects of $\mathbf{\Delta}$ are the finite ordinals $[n] := \{0, 1, 2, ..., n\}$, $n \in \mathbb{N}$.

(ii) The morphisms of $\mathbf{\Delta}$ are the non-decreasing maps between finite ordinals.

We call $\mathbf{\Delta}$ the *simplicial category*.

Remark 2.3.2. Note that the epimorphisms $p\colon [m] \to [n]$ in $\mathbf{\Delta}$ are simply the non-decreasing surjections; such an epimorphism has a *section*, i.e. a morphism $s\colon [n] \to [m]$ such that $p \circ s = \mathrm{id}_{[n]}$.

Definition 2.3.3. Consider the category $\mathbf{\Delta}$. We define the following non-decreasing maps:

(i) $\partial_n^i\colon [n-1] \to [n]$ is the increasing injection which does not take the value i.

(ii) $\sigma_n^i\colon [n+1] \to [n]$ is the increasing surjection, which takes the value i twice.

(iii) $\iota_n^i\colon [0] \to [n]$ is the inclusion map, which takes the single element of $[0]$ to i.

We will call ∂_n^i the ith *coface map*, σ_n^i the ith *codegeneracy map*, and ι_n^i the ith vertex map. Furthermore, we will omit the index from notation if domain and codomain do not matter.

Proposition 2.3.4. *[GZ67, p.24] Considering the morphisms ∂_n^i and σ_n^i, the following equations hold:*

$$\begin{aligned} \partial_{n+1}^j \partial_n^i &= \partial_{n+1}^i \partial_n^{j-1} && i < j \\ \sigma_n^j \sigma_{n+1}^i &= \sigma_n^i \sigma_{n+1}^{j+1} && i \leq j \\ \sigma_{n-1}^j \partial_n^i &= \begin{cases} \partial_{n-1}^i \sigma_{n-2}^{j-1} & i < j \\ \mathrm{id}_{[n-1]} & i = j \text{ or } i = j+1 \\ \partial_{n-1}^{i-1} \sigma_{n-2}^j & i > j+1 \end{cases} \end{aligned} \tag{2.1}$$

Lemma 2.3.5. *[GZ67, p.24] Every non-decreasing map $f \in \mathbf{\Delta}([m],[n])$ can be written in one and only one way as*

$$f = \partial_n^{i_s} \circ \partial_{n-1}^{i_{s-1}} \circ \cdots \partial_{n-t+1}^{i_1} \circ \sigma_{m-t}^{j_t} \circ \cdots \sigma_{m-2}^{j_2} \circ \sigma_{m-1}^{j_1} \tag{2.2}$$

with $n \geq i_s > \cdots i_1 \geq 0$, $0 \leq j_t < \cdots < j_1 < m$ and $n = m - t + s$. We call (2.2) *the* canonical decomposition *of f.*

Example 2.3.6. The canonical decomposition of ι_n^i is given by:

$$\iota_n^i = \partial_n^n \circ \partial_{n-1}^{n-1} \circ \cdots \circ \partial_{i+1}^{i+1} \circ \partial_i^{i-1} \circ \partial_{i-1}^{i-2} \circ \cdots \circ \partial_2^1 \circ \partial_1^0$$

Definition 2.3.7. Let **Set** be the category of small sets. A *simplicial set* is a functor $X\colon \mathbf{\Delta}^{\mathrm{op}} \to$ **Set**.

It follows directly from the definition, that there is a category of simplicial sets, denoted by **sSet**, namely the functor category $\mathbf{Set}^{\mathbf{\Delta}^{\mathrm{op}}}$.

Given a simplicial set X, we write X_n for the image of the object $[n]$ under the functor X, and call an element $\sigma \in X_n$ a *simplex* of dimension n, or *n-simplex*, and X_n the set on n-simplices of X. We will write $\tau \in X$ instead of $\tau \in X_n$ if the dimension does not matter. Furthermore we write d_i^n and s_j^n instead of $X(\partial_n^i)$ and $X(\sigma_n^j)$ and call the maps d_i^n the *face operators*, and the maps s_j^n the *degeneracy operators* of X. We say that a simplex τ is a face of a simplex σ if $\tau = d_i(\sigma)$ for some face operator d_i.

A 0-simplex is called a *vertex* of X, and given an n-simplex $\tau \in X_n$ we write τ_i for for the image $X(\iota_n^i)(\tau)$ and say that τ_i is the ith *vertex* of τ. Note that this specifies an order on the set of vertices of a simplex. In particular, we will visualize a 1-simplex σ as an arrow from $d_1^1(\sigma)$ to $d_0^1(\sigma)$.

Remark 2.3.8. Let X, T be two simplicial sets. A morphism $g\colon X \to T$ is a natural transformation $g\colon X \Rightarrow T$, that is a sequence of maps $g_n\colon X_n \to T_n$ in **Set**, satisfying

$${}_T d_i^n \circ g_n = g_{n-1} \circ {}_X d_i^n \qquad \text{and} \qquad {}_T s_i^n \circ g_n = g_{n+1} \circ {}_X s_i^n \qquad \forall i, n,$$

where we use left-indices to illustrate to which simplicial set the respective face and degeneracy operators belong.

Monomorphisms in **sSet** are precisely levelwise injections, i. e. a morphism $f\colon X \to Y$ is a monomorphism if every $f_n\colon X_n \to Y_n$ is an injection. Likewise, epimorphisms are precisely levelwise surjections.

Remark 2.3.9. The maps d_i, s_i satisfy the following identities:

$$\begin{aligned} d_i^n d_j^{n+1} &= d_{j-1}^n d_i^{n+1} && i < j \\ s_i^{n+1} s_j^n &= s_{j+1}^{n+1} s_i^n && i \leq j \\ d_i^n s_j^{n-1} &= \begin{cases} s_{j-1}^{n-2} d_i^{n-1} & i < j \\ \text{id} & i = j \text{ or } i = j+1 \\ s_j^{n-2} d_{i-1}^{n-1} & i > j+1 \end{cases} \end{aligned} \tag{2.3}$$

Example 2.3.10. For each $[n] \in \mathbf{\Delta}$, the Yoneda embedding

$$\begin{aligned} Y_{\mathbf{\Delta}} \colon \mathbf{\Delta} &\to \mathbf{sSet} \\ [n] &\mapsto \mathbf{\Delta}(-, [n]) \end{aligned}$$

yields a simplicial set

$$\Delta^n := Y([n]) = \mathbf{\Delta}(-, [n]).$$

We call this simplicial set the *standard n-simplex.*

We will denote the image of the face operators ∂_i^n under the Yoneda embedding $Y_{\mathbf{\Delta}}$ by $d_n^i = Y_{\mathbf{\Delta}}(\partial_n^i) = \mathbf{\Delta}(-, \partial_n^i)$. Hence, for example, the map $d_1^i \colon \Delta^0 \to \Delta^1$ is the embedding of the single vertex of Δ^1 at the ith vertex of Δ^1

The standard 0-simplex Δ^0 is a terminal object in **sSet**, whereas the initial object is the unique simplicial set, that consists of the empty set $\varnothing$ in each degree. Furthermore, the category **sSet** is bicomplete, so all small limits and colimits exits. The only limit construction we will need are products:

Definition 2.3.11. Let X, Y be simplicial sets, the *product* $X \times Y$ is the simplicial set with n-simplices given by

$$(X \times Y)_n = X_n \times Y_n$$

and face and degeneracy operators

$$d_i := ({}_X d_i, {}_Y d_i)$$

and

$$s_i := ({}_X s_i, {}_Y s_i).$$

Example 2.3.12. Let X, Y be simplicial sets. The *function complex* X^Y is the simplicial set given by

$$(X^Y)_n := \mathbf{sSet}(X \times \Delta^n, Y) \tag{2.4}$$

where the face and degeneracy operators are given by

$$\begin{aligned} d_i f &= f \circ (\text{id}_X \times d^i) \\ s_i f &= f \circ (\text{id}_X \times s^i) \end{aligned}$$

Definition 2.3.13. Let X be a simplicial set, an n-simplex $\tau \in X$ is called *degenerate* if there is m-simplex $\nu \in X$ and an epimorphism $\mu \colon [n] \to [m]$, such that $\tau = X(\mu)(\nu)$.

Proposition 2.3.14. *[GZ67, p.26f] For each m-simplex x of X, there is an epimorphism $s \colon [m] \to [n]$ and a non-degenerate n-simplex y such that $x = X(s)(y)$. Moreover, the pair (s, y) is unique.*

2.3.2 Geometric Realization

The geometric realization is a functor $|-|: \mathbf{sSet} \to \mathbf{Top}$, that associates to each simplicial set X a corresponding CW–complex $|X|$. We will first define the n-dimensional *topological standard simplex* associated to a finite ordinal $[n]$, and then build the geometric realization of a simplicial set by gluing together topological standard simplices.

Definition 2.3.15. The *geometric realization* is a covariant functor $|-|: \mathbf{\Delta} \to \mathbf{Top}$ is given as follows:

(i) Let $[n] \in \mathbf{\Delta}$, define

$$|[n]| := \left\{ (t_0, \cdots, t_n) \subseteq \mathbb{R}^{n+1} \,\middle|\, \sum_{i=0}^{n} t_i = 1 \text{ and } t_i > 0 \right\}$$

(ii) Let $f\colon [n] \to [m]$, define

$$|f|: \ |[n]| \to |[m]| \tag{2.5}$$
$$(t_0, \ldots, t_n) \mapsto (s_0, \ldots, s_m) \tag{2.6}$$

where

$$s_j = \sum_{f(i)=j} t_i. \tag{2.7}$$

We will call the image $|[n]|$ the *topological standard n-simplex*.

We will denote the image of the coface and codegeneracy maps by $D^i_n := \left|\partial^i_n\right|$ and $S^i_n := \left|\sigma^i_n\right|$ respectively.

Definition 2.3.16. The *geometric realization* is a covariant functor $|-|: \mathbf{sSet} \to \mathbf{Top}$. On objects it is defined as follows: given a simplicial set X, let

$$|X| := \left(\coprod_{n=0}^{\infty} X_n \times |[n]| \right) /\sim$$

where the sets X_n are given the discrete topology, and $|X|$ is given the quotient topology with respect to the relation $(x, p) \sim (y, q)$ if either

(i) $(d_i(x), p) = (y, D^i(q))$ or

(ii) $(s_i(x), p) = (y, S^i(q))$.

We will now give some intuition beyond these definitions, and justify why one should really think about simplicial sets as combinatorial data on how to glue together simplices to obtain a topological space. Given a simplicial set X, the space $(\coprod_{n=0}^{\infty} X_n \times |[n]|)$ is the disjoint union of topological standard n-simplices, associating a topological standard n-simplex to each n-simplex in X. The relations (i) and (ii) of Definition 2.3.16 are "gluing instructions", giving a recipe on how to glue these simplices together.

We will discuss Relation (i) first: assume that X is a simplicial set. Let $\sigma \in X_n$, and $\tau = d_i(\sigma)$. Relation (i) says, that we glue the $(n-1)$-dimensional topological standard simplex associated to τ onto the ith face of the n-dimensional standard simplex associated to σ, i. e. the face that is opposite to the geometric realization of the vertex σ_i.

Furthermore, given a degenerate n-simplex $s_i(\sigma) \in X_n$, then a pair $(s_i(\sigma), p)$ is equivalent to $(\sigma, S^i(p))$ where $S^i\colon |[n]| \to |[n-1]|$ is a quotient map, mapping the n-dimensional topological standard simplex to its ith face. Using Lemma 2.3.5, it is easy to see that given any degenerate simplex σ, the pair (σ, p) will be equivalent to the pair $(\tau, |s|(p))$, where (τ, s) is the unique pair given by Proposition 2.3.14. Hence, degenerate simplices should be seen as auxiliary constructions to glue non-degenerate n-simplices along their boundary to non-degenerate $(n-k)$-simplices, $k > 1$.

Example 2.3.17. Consider the simplicial set X, where the non-degenerate simplices are given as follows:

$$\{x, y, z\} \subseteq X_0$$
$$\{\tau_x, \tau_y, \tau_z\} \subseteq X_1$$
$$\{\sigma_1, \sigma_2\} \subseteq X_2,$$

and the face maps with non-degenerate domain are given by

$$d_0^1(\tau_x) = y \qquad d_0^1(\tau_y) = x \qquad d_0^1(\tau_z) = x$$
$$d_1^1(\tau_x) = z \qquad d_1^1(\tau_y) = z \qquad d_1^1(\tau_z) = y$$

and

$$d_0^2(\sigma_1) = \tau_z \qquad d_0^2(\sigma_2) = \tau_z$$
$$d_1^2(\sigma_1) = \tau_y \qquad d_1^2(\sigma_2) = \tau_y$$
$$d_2^2(\sigma_1) = \tau_x \qquad d_2^2(\sigma_2) = \tau_x$$

We leave it up to the reader to convince himself, that these maps satisfy the identities in Remark 2.3.9. Note that X satisfies that every face of a non-degenerate simplex is non-degenerate, hence we can ignore degenerate simplices when dealing with geometric realization. The relevant summands of the codroduct $\coprod_{n=0}^{\infty} X_n \times |[n]|$ are three points, three topological 1-simplices, homeomorphic to the unit interval $[0,1] \cong D^1$, and two 2-simplices, homeomorphic to D^2. The face operators d_i^1 tell us to glue the three unit intervals in between two different points respectively, forming a space homeomorphic to a triangle. The face operators d_i^2 tell us to glue the 2-simplices along their boundaries onto this triangle. Hence the resulting space is homeomorphic to the 2-sphere S^2, and it is easy to see that those gluing instruction provide a canonical CW-structure for our space $|X|$.

Example 2.3.18. Consider the simplicial set X, where the non-degenerate simplices are given as follows:

$$\{x\} \subseteq X_0$$
$$\{\sigma\} \subseteq X_2,$$

and the relevant face maps are given by

$$d_0(\sigma) = s_0(x)$$
$$d_1(\sigma) = s_0(x)$$
$$d_2(\sigma) = s_0(x)$$

The geometric realization $|X|$ consists of two non-degenerate simplices, a 0-simplex and a 2-simplex. The topological 2-simplex associated to σ is glued along its boundary to the 1-simplex associated to $s_0(x)$. Since $s_0(x)$ is degenerate, the associated topological space is equivalent to the point associated to x. Hence the space $|X|$ is obtained by gluing a standard 2-simplex along its boundary to a point. This is exactly the standard procedure for obtaining a CW-structure of a 2-sphere.

From the previous examples it should be obvious, that the geometric realization of a simplicial set X admits a CW-structure, whenever X is locally finite. That is X_n is finite for all n in $\mathbb{N}$.

We will often deliberately confuse a simplicial set with its geometric realization. In particular, when drawing a picture of a simplicial set, we actually draw its geometric realization.

The geometric realization has a right adjoint, called the singular functor:

Definition 2.3.19. Given a topological space X, we define the *singular complex* $C(X)$ to be the simplicial set given by

$$C(X)_n := \mathbf{Top}(|\Delta^n|, X)$$

with face and degeneracy operators being the obvious choice. The construction can be extended to a functor $C\colon \mathbf{Top} \to \mathbf{sSet}$, called the *singular functor*, as follows: given a map $X \xrightarrow{f} Y$ in $\mathbf{Top}$, the nth component of $C(f)$ is given by

$$C(f)_n\colon C(X)_n \to C(Y)_n = \mathbf{Top}(\Delta^n, f)$$

Theorem 2.3.20. *[GJ99, Prop. 2.2] The singular functor $C\colon \mathbf{Top} \to \mathbf{sSet}$ is right adjoint to the geometric realization.*

2.3.3 Skeleton and Coskeleton

Similarly to CW–complexes, one can define the n-skeleton of a simplicial set X. This allows us to introduce the notion of dimension of a simplicial set, which coincides with the dimension of its geometric realization.

Definition 2.3.21. Let $\mathbf{\Delta}_n \subseteq \mathbf{\Delta}$ be the full subcategory of $\mathbf{\Delta}$ with object set

$$\mathbf{\Delta}_n^{(0)} = \left\{ [k] \in \mathbf{\Delta}^{(0)} \,\middle|\, k \leq n \right\}.$$

Given a simplicial set X, we obtain the *n-truncation* of X by precomposition with the embedding $i\colon \mathbf{\Delta}_n \to \mathbf{\Delta}$. In particular, precomposition yields a functor

$$\begin{aligned} \mathrm{tr}_n\colon \mathbf{sSet} = \mathbf{Set}^{\mathbf{\Delta}} &\to \mathbf{Set}^{\mathbf{\Delta}_n} \\ X &\mapsto X \circ i, \end{aligned}$$

which we call the *truncation functor*.

The truncation functor has a left adjoint

$$\mathbf{Sk}_n\colon \mathbf{Set}^{\mathbf{\Delta}_n} \to \mathbf{sSet}$$

and a right adjoint

$$\mathbf{coSk}_n\colon \mathbf{Set}^{\mathbf{\Delta}_n} \to \mathbf{sSet}$$

. We will call the image of a simplicial set X under the the composition $\mathrm{Sk}_n := \mathbf{Sk}_n \circ \mathrm{tr}_n$ the *n-skeleton* of X, and the image under the composition $\mathrm{coSk}_n := \mathbf{coSk}_n \circ \mathrm{tr}_n$ the *n-coskeleton* of X.

Given a simplicial set X, one obtains the n-skeleton by discarding all k-simplices for $k > n$, and then adding degeneracies. In particular, one may view the 1-skeleton of a simplicial set as the free simplicial set on a directed graph, with vertices given by the vertices of the simplicial set, and oriented edges given by 1-simplices, pointing towards the image of d_0^1

The n-coskeleton of a simplicial set X is the smallest simplicial set $\operatorname{coSk}_n X$ with subset $\operatorname{tr}_n X$, satisfying that given any map $\partial\Delta^k \to \operatorname{coSk}_n X$, where $k > n$, there exists a unique map $\phi\colon \Delta^k \to \operatorname{coSk}_n X$, making the following diagram commutative:

$$\begin{array}{ccc} \partial\Delta^k & \longrightarrow & \operatorname{coSk}_n X \\ \big\downarrow & \overset{\exists!\phi}{\nearrow} & \\ \Delta^k & & \end{array} .$$

We say a simplicial set X is n-coskeletal if $X \cong \operatorname{coSk}_n X$.

Definition 2.3.22. Let X be a simplicial set. We say that X has *dimension n* if n is the smallest integer satisfying $X \cong \operatorname{Sk}_k X$ for all $k \geq n$.

The dimension of a simplicial set X coincides with the dimension of its geometric realization $|X|$. Assume that X has dimension n, then there a no non-degenerate simplices with dimension $k > n$. Hence the geometric realization has at most dimension n. On the other hand, the set X_n contains at least one non-degenerate simplex, since otherwise $\operatorname{Sk}_{n-1} X \cong \operatorname{Sk}_n X$. Hence $|X|$ has at least dimension n.

2.3.4 The Nerve and the Fundamental Category

There is an adjunction between the category of simplicial sets and the category of small categories, given by the nerve of a category and the fundamental category of a simplicial set. We will use this adjunction extensively when establishing a homotopy theory on the category of small categories. People familiar with abstract simplicial complexes may notice that the nerve of a category is a generalization of the order complex of a poset, whereas the fundamental category is somehow inverse to the nerve, that is the fundamental category of the nerve of a category is isomorphic to the original category. Or using the words from Chapter 2.1: the counit is an isomorphism.

Definition 2.3.23. Let $\mathcal{C} \in \mathbf{Cat}$. The nerve $N(\mathcal{C})$ is a simplicial set given as follows:

(i) The set of vertices is given by the set of objects, i. e.

$$N(\mathcal{C})_0 = \mathcal{C}^{(0)}.$$

(ii) The set of n-simplices is given by n-tuples of composable morphisms, i. e.

$$N(\mathcal{C})_n = \left\{ (f_0, f_1, \ldots, f_{n-1}) \,\middle|\, f_k \in \mathcal{C}^{(1)} \text{ and } s(f_k) = t(f_{k-1}) \right\}.$$

(iii) The face operators d_i^n are given by either removing the first or the last element if $i = 0$ or $i = n$, or by composing the ith element with its successor. I. e. given a simplex $(f_0, \ldots, f_{n-1}) \in N(\mathcal{C})$, we have

$$d_i^n(f_0, \ldots, f_{n-1}) = \begin{cases} (f_1, \ldots, f_{n-1}) & \text{if } i = 0 \\ (f_0, \ldots, f_i \circ f_{i-1}, \ldots, f_{n-1}) & \text{if } 0 < i < n \\ (f_0, \ldots, f_{n-2}) & \text{if } i = n \end{cases} .$$

(iv) The degeneracy operators s_i^n are given by including an identity after the ith element. I. e. given a simplex $(f_0, \dots, f_n) \in N(\mathcal{C})$, we have

$$s_i^n(f_0, \dots, f_n) = (f_0, \dots, f_{i-1}, \mathrm{id}_{s(f_i)}, f_i, \dots, f_n).$$

The nerve construction is functorial, since given a functor $F\colon \mathcal{C} \to \mathcal{D}$, there is a canonical map

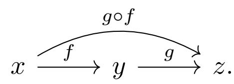
$$\begin{aligned} N(F)\colon N(\mathcal{C}) &\to N(\mathcal{D}) \\ (f_0, \dots, f_n) &\mapsto (F(f_0), \dots, F(f_n)). \end{aligned}$$

It is easy to see, that $N(F)$ is a morphism in **sSet** and that given another functor $G\colon \mathcal{D} \to \mathcal{E}$, we have

$$N(G) \circ N(F) = N(G \circ F)$$

Accordingly, we call the Functor $N\colon \mathbf{Cat} \to \mathbf{sSet}$ the *nerve functor*.

Example 2.3.24. Let $\mathcal{C} \in \mathbf{Cat}$ be the category



The nerve of $\mathcal{C}$ is isomorphic to the standard 2-simplex, as indicated in Figure 2.3.24. In partic-

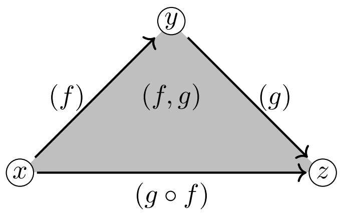


Figure 2.1: Nerve of $x \xrightarrow{f} y \xrightarrow{g} z$

ular, given $[n] \in \mathbf{Cat}$, we have $N([n]) = \Delta^n$ and $|N([n])| \cong |[n]|$.

The nerve construction proved useful in a variety of contexts. For example, let G be a group, and $\mathcal{G}$ the category with one object, and morphism set G. Then the geometric realization $|N(\mathcal{G})|$ yields the classifying space $\mathcal{B}(G)$. Accordingly, the composition $|-| \circ N$ is often called the classifying space functor, and the geometric realization of the nerve the classifying space of a category.

The nerve functor is a right adjoint, and thus commutes with limits. Its left adjoint is given by the fundamental category functor, defined as follows:

Definition 2.3.25. Let X be a simplicial set. The *fundamental category* $\tau_1(X)$ is a quotient of the free category on the 1-truncation of X with respect to the equivalence relation on the morphism set generated by

(i) $g \circ f \sim h \iff \exists \sigma \in X_2 : d_0(\sigma) = f$ and $d_2(\sigma) = g$, and $d_1(\sigma) = h$ and

(ii) $s_0(x) \sim \mathrm{id}_x \, \forall x \in X_0$.

Theorem 2.3.26. Cat *is reflective in* **sSet**

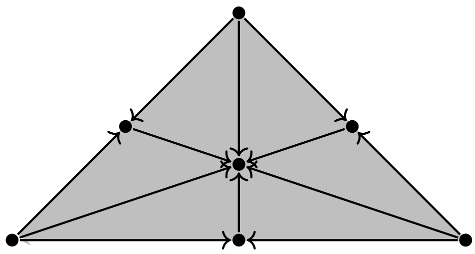


Figure 2.2: The barycentric subdivision of Δ^2

It should be obvious from the definition, that the nerve functor is injective on objects. Moreover, the counit $\varepsilon\colon \tau_0 N \Rightarrow \mathrm{id}_{\mathbf{Cat}}$ is an isomorphism, since given a category $\mathcal{C}$ the 1-trunctation $\mathrm{tr}_1\, N\mathcal{C}$ is exactly the original category $\mathcal{C}$, considered as a directed graph. When taking the fundamental category, the new compositions added in the procedure are equivalent to the old ones, since given any pair of composable morphisms f, g, there is a 2-simplex (f, g) in $N\mathcal{C}$, hence the old compositions are identified with the new ones. Thus by Proposition 2.1.13, N is fully faithful and $N(\mathbf{Cat})$ a full subcategory in $\mathbf{sSet}$.

2.3.5 Barycentric Subdivision

The barycentric subdivision is a functor $\mathrm{Sd}\colon \mathbf{sSet} \to \mathbf{sSet}$ that maps each simplicial set to a (topologically) homeomorphic one consisting of more simplices. On a standard n-simplex it acts as the name suggest, adding a vertex at the barycenter of each k-simplex, and subdividing the simplex accordingly, as shown in Figure 2.2 for the standard 2-simplex.

Similarly to the geometric realization, we will first give a subdivision of the standard n-simplex, and the extend the construction to a functor on the whole category. Before giving the actual definition, though, we need two more auxiliary constructions.

Definition 2.3.27. Let X be a simplicial set, we denote by $\mathbf{P}(X)$ the poset of non-degenerate simplices, ordered by face relation. That is, given non-degenerate simplices $\sigma, \tau \in X$, we say $\sigma \leq \tau$ if there is a composition of face operators $f\colon X_{\dim \tau} \to X_{\dim \sigma}$ satisfying $f(\tau) = \sigma$.

Definition 2.3.28. Let X be a simplicial set. The *category of simplices* is the comma category $(Y_{\boldsymbol{\Delta}} \downarrow X)$, where $Y_{\boldsymbol{\Delta}}$ is the Yoneda embedding. That is objects are maps $\Delta^n \to X$, and morphisms are given by commutative diagrams

$$\begin{array}{ccc} \Delta^n & \xrightarrow{\quad f \quad} & \Delta^k \\ & \searrow \quad \swarrow & \\ & X & \end{array},$$

where f is a morphism in $\boldsymbol{\Delta}$.

Given a simplicial set X, we can identify $\mathbf{P}(X)$ with a subcategory of $(Y_{\boldsymbol{\Delta}} \downarrow X)$ by identifying and object $\tau \in \mathbf{P}(X)$ with a morphism $\phi\colon \Delta^{\dim(X)} \to X$, satisfying $\phi(X) = \tau$. We will use this implicitly in the following lemma:

Lemma 2.3.29. *[Hov99, Lemma 3.1.3 and Lemma 3.1.4] Let X be a simplicial set, and either $I = \mathbf{P}X$ or $I = (Y_{\boldsymbol{\Delta}} \downarrow X)$. The colimit of the diagram $I \to \mathbf{sSet}$, taking $\Delta^n \to X$ to Δ^n is X itself.*

Definition 2.3.30. Let X be a simplicial set, and let $\operatorname{Sd}\Delta^n := N\mathbf{P}(\Delta^n)$. The *barycentric subdivision* $\operatorname{Sd} X$ is given by the colimit

$$\operatorname*{colim}_{\Delta^n \to X} \operatorname{Sd}\Delta^n,$$

indexed over the category of simplices of X.

Note that by Lemma 2.3.29, given a standard n-simplex Δ^n, both definitions of $\operatorname{Sd}\Delta^n$ coincide, which justifies our abuse of notation. Furthermore, there is a particularly simple description of the barycentric subdivision of the standard n-simplex, given in [FP10]: the poset $\mathbf{P}\Delta^n$ of non-degenerate simplices is isomorphic to the poset of non-empty subsets of $[n]$, via identification of a simplex σ with its set of vertices. Thus a k-simplex $\sigma \in \operatorname{Sd}\Delta^n$ is a tuple $(\sigma^0, \sigma^1, \cdots, \sigma^k)$ of non-empty subsets of $[n]$, ordered by inclusion and σ is non-degenerate if all σ^i are distinct. A non-degenerate m-simplex τ in $\operatorname{Sd}\Delta^n$ is a face of σ if and only if

$$\left\{\tau^0, \ldots, \tau^m\right\} \subseteq \left\{\sigma^0, \sigma^1, \cdots, \sigma^k\right\}.$$

In Chapter 4 we will use the double subdivision of the the standard n-simplex extensively, thus it makes sense to have a look at it as well: a k-simplex $\sigma \in \operatorname{Sd}^2\Delta^n$ is a sequence $\sigma = (\sigma_0, \ldots, \sigma_q)$, where each σ_i is a non-degenerate simplex of $\operatorname{Sd}\Delta^n$. As before, σ is non-degenerate if all σ_i are distinct, and a non-degenerate m-simplex τ is a face of σ if and only if

$$\{\tau_0, \ldots, \tau_m\} \subseteq \{\sigma_0, \sigma_1, \cdots, \sigma_k\}.$$

In the following, given the barycentric subdivision of a standard n-simplex, we denote the vertices of $\operatorname{Sd}\Delta^n$ by non-empty subsets of $[n]$, and vertices of $\operatorname{Sd}^2\Delta^n$ by sets of subsets of $[n]$ rather than tuples of vertices of Δ^n, since every vertex is non-degenerate and the order of the elements will not matter. For example,

$$(\operatorname{Sd}^2\Delta^2)_0 = \{\{0\}, \{1\}, \{2\}, \{0,1\}, \{1,2\}, \{0,2\}, \{0,1,2\}\}.$$

Theorem 2.3.31. *Let X be a simplicial set, then $\tau_1 \operatorname{Sd}^2 X$ is a poset.*

Proof. This follows from [LW69, Proposition 8.1] and [FL79, Lemma 3.2].

The barycentric subdivision functor has a right adjoint (cf. [Kan57]), called *extension*, given by

$$\operatorname{Ex}(X)_n := \mathbf{sSet}(\operatorname{Sd}\Delta^n, X).$$

Even though the extension functor will play a crucial role in establishing a homotopy theory on **Cat**, we will not need any of its specific properties, which is why we only give its definition.

2.3.6 Homotopy Theory for Simplicial Sets

A homotopy theory for the category of simplicial sets should be similar to the classical homotopy theory on topological spaces in the sense that if two simplicial maps are homotopic, there geometric realizations should be homotopic in the classical sense. Unfortunately, there are some complications which will occur with regard to homotopies in **sSet**.

Definition 2.3.32. Let $f, g\colon X \to Y$ be simplicial maps, a *simplicial homotopy* $H\colon f \Rightarrow g$ is a map $H\colon X \times \Delta^1 \to Y$, making the following diagram commutative:

$$\begin{array}{ccc} X \times \Delta^0 & \xrightarrow{f} & \\ \downarrow{\scriptstyle \mathrm{id} \times d^0} & & \searrow \\ X \times \Delta^1 & \xrightarrow{H} & Y \\ \uparrow{\scriptstyle \mathrm{id} \times d^1} & & \nearrow \\ X \times \Delta^0 & \xrightarrow{g} & \end{array} .$$

As opposed to **Top**, simplicial homotopy does not necessarily induce an equivalence relation on the morphism set $\mathbf{sSet}(X, Y)$. For example, consider the inclusions $i_0, i_1\colon \Delta^0 \to \Delta^n$, mapping the single vertex of Δ^0 to the zeroth and first vertex of Δ^n respectively. There is a simplicial homotopy $H\colon i_0 \Rightarrow i_1$, given by the inclusion $\iota\colon \Delta^1 \to \Delta^n$ such that $d^0(\iota(\Delta^1)) = i_0(\Delta^0)$ and $d^1(\iota(\Delta^1)) = i_1(\Delta^0)$. There is, however, no 1-simplex that could give a homotopy $i_1 \Rightarrow i_0$.

To remedy the situation, we need to identify a class of objects in **sSet** that is more suited to do homotopy theory. This leads us to the notion of *Kan fibrations* and *Kan complexes*:

Definition 2.3.33. The kth *horn* Λ^n_k is the simplicial subset of the standard n-simplex Δ^n obtained by removing the single n-simplex $\iota_n \in \Delta^n$, its kth face $d_k(\iota_n)$ and their respective degeneracies.

Definition 2.3.34. Let $p\colon X \to Y$ be a morphism in **sSet**. We call p a *Kan fibration* if for each $n \geq 1$, $0 \leq k \leq n$, for each solid diagram

$$\begin{array}{ccc} \Lambda^n_k & \longrightarrow & X \\ {\scriptstyle i}\big\downarrow & \overset{\theta}{\nearrow} & \big\downarrow{\scriptstyle p} \\ \Delta^n & \longrightarrow & Y \end{array}$$

the map θ exists, making the diagram commutative, where i is the canonical inclusion. A simplicial set X is called *Kan complex* if the canonical map $X \to [0]$ is a Kan fibration.

Note that given a map $p\colon E \to B$ in **Top**, we can reformulate the condition of Theorem 2.2.18 for p being a Serre fibration, using inclusions of horns and the geometric realization functor: that is, p is a Serre fribration if for each $n \geq 1$, $0 \leq k \leq n$, for each solid diagram

$$\begin{array}{ccc} |\Lambda^n_k| & \longrightarrow & E \\ {\scriptstyle i}\big\downarrow & \overset{\theta}{\nearrow} & \big\downarrow{\scriptstyle p} \\ |\Delta^n| & \longrightarrow & P \end{array}$$

the map θ exists, making the diagram commutative, where i is the canonical inclusion. By adjointness, all such diagrams may be identified with diagrams

$$\begin{array}{ccc} \Lambda^n_k & \longrightarrow & C(X) \\ {\scriptstyle i}\big\downarrow & \overset{\theta}{\nearrow} & \big\downarrow{\scriptstyle C(p)} \\ \Delta^n & \longrightarrow & C(Y) \end{array} .$$

Thus, p is a Serre fibration if and only if $C(p)$ is a Kan fibration (cf. [GJ99, p. 11]). The benefits of identifying Kan complexes should be obvious from the following theorem:

Theorem 2.3.35. *[Cur71, Cor. 1.16] Let X, Y be simplicial sets, such that Y is a Kan complex. Then simplicial homotopy of maps induces an equivalence relation on* $\mathbf{sSet}(X, Y)$.

Aside from Kan fibrations, there is another class of maps that proved useful in homotopy theory on simplicial sets. The saturated class generated by the set

$$\{\Lambda^n_k \hookrightarrow \Delta^n \,|\, 0 \leq k \leq n, n > 0\}$$

is called the class of *anodyne extensions*. Anodyne extensions are directly linked to Kan fibrations via the following theorem:

Theorem 2.3.36. *[GJ99, Cor. 4.3] A Kan fibration is a map which has the left lifting property with respect to all anodyne extensions*

We say that a map $f \colon X \to Y$ in **sSet** is a *weak equivalence*, precisely if its geometric realization $|f|$ is a weak equivalence in **Top**.

Proposition 2.3.37. *[GJ99, Thm. 11.2] Suppose $f \colon X \to Y$ is a map in* **sSet**. *Then f is a Kan fibration and a weak equivalence if and only if g has the right lifting property with respect to all inclusions $\partial\Delta^n \hookrightarrow \Delta^n$*

Given the set $\{\partial\Delta^n \hookrightarrow \Delta^n \,|\, n \geq 0\}$, its saturation is the class of all monomorphisms in **sSet**. This is essentially [Hov99, Prop. 3.2.2], although he uses a very different language. If we call a morphism a *cofibration* if it is a monomorphism, *fibration* if it is a Kan fibration, trivial cofibration if it is a cofibration and a weak equivalence, and trivial fibration if it is a fibration and a weak equivalence, then Theorem 2.2.14, Theorem 2.2.15 and Theorem 2.2.16 also hold in **sSet**, after removing the word Hurewicz, respectively replacing homotopy equivalence with weak equivalence.

These similarities were the motivation to introduce a general theory of fibrations, cofibrations and weak equivalences for arbitrary categories, enabling us to do homotopy theory without caring about the internal structure of the objects in the respective category.

3

Abstract Homotopy Theory

3.1 Model categories

Model categories, first introduced by Quillen in [Qui67], form the foundation of homotopy theory.

– Mark Hovey

Model categories were introduced by Quillen to unify the theory of cofibrant objects in **Top**, and the theory of fibrant objects in **sSet**. As seen in the previous chapters, once we had established the notions of fibrations, cofibrations and weak equivalences in **Top** and **sSet**, many of the results we established in **Top** carried over to **sSet**. A model structure on a category $\mathcal{C}$ consists of collections of cofibrations, fibrations and weak equivalences subject to certain conditions that ensure these classes satisfy the same properties as the notions in **Top** or **sSet**.

Definition 3.1.1. A *model category* is a category $\mathcal{M}$ together with three classes of morphisms: a class of *weak equivalences* W, a class of *fibrations* F, and a class of *cofibrations* C, satisfying the following properties:

M1 $\mathcal{M}$ is bicomplete.

M2 W satisfies the 2–out–of–3 property, that is given three morphisms $f, g, f \circ g$, if two of them are in W, so is the third.

M3 The classes W, F and C are closed under retracts.

M4 We call a map a *trivial fibration* if it is a fibration and a weak equivalence, and a *trivial cofibration* if it is a cofibration and a weak equivalence. Trivial cofibrations have the left lifting property with respect to fibrations, and cofibrations have the left lifting property with respect to trivial fibrations.

M5 Every morphism f in $\mathcal{M}$ has two functorial factorizations:

a) $f = qi$, where i is a cofibrations and q is a trivial fibration.

b) $f = pj$, where j s a trivial cofibration and j is a fibration.

As before, we call an object X cofibrant if the initial map $\emptyset \to X$ is a cofibration, and fibrant if the terminal map $X \to *$ is a fibration. We say that classes of fibrations, cofibrations and weak equivalences form a *model structure* on a category $\mathcal{M}$.

The original definition of a model category, as given in [Qui67], was strictly weaker than Definition 3.1.1, and our definition is closer to what quillen called originally a *closed model category* in [Qui67]. Since closed model categories are much more useful and common than Quillen expected, people started to dropping the word "closed" from the definition, which is a habbit we adopt here.

We have already seen a few model structures in the previous chapters. The classes of Hurewicz fibrations and cofibrations, and homotopy equivalences form the *Strøm model structure* on **Top**, and the classes of Serre fibrations and cofibrations together with the weak homotopy equivalences form the *Quillen model structure* on **Top**.

In **sSet**, Kan fibrations, monomorphisms and weak equivalences form a model structure called *Kan* or *Quillen model structure*. There are, however, model structures that are not as closely related to traditional homotopy theory as those we have already seen.

Example 3.1.2. Let R be a ring, and $\mathbf{Ch}_R$ be the category of chain complexes of R-modules in non-negative degree. The *projective model structure* on $\mathbf{Ch}_R$ is given as follows: let $f\colon M \to N$ be a morphism in $\mathbf{Ch}_R$, then f is a

(i) weak equivalence if the induced morphism on homologies is an isomorphism,

(ii) cofibration if it is a monomorphism with projective R–module as cokernel in every degree, and a

(iii) fibration if it is an epimorphism in every degree $k \geq 1$.

From now on, we will denote by **Top** and **sSet** the respective model categories carrying the Quillen model structure, and denote by $\mathbf{Top}_{\mathrm{Strom}}$ the model category carrying the Strøm model structure.

Definition 3.1.3. A model category $\mathcal{M}$ is called

(i) *left proper* if weak equivalences are preserved by pushouts along cofibrations,

(ii) *right proper* if weak equivalences are preserved by pullbacks along fibrations, and

(iii) *proper* if it is left and right proper.

There is a useful theorem that helps identifying left and right proper model categories:

Theorem 3.1.4. *[Hir03, Cor. 13.1.3.] Let $\mathcal{M}$ be a model category:*

(i) If every object of $\mathcal{M}$ is cofibrant, $\mathcal{M}$ is left proper.

(ii) If every object of $\mathcal{M}$ is fibrant, $\mathcal{M}$ is right proper.

(iii) If every object of $\mathcal{M}$ is fibrant and cofibrant, $\mathcal{M}$ is proper.

The model structure on $\mathbf{Top}_{\mathrm{Strom}}$ is proper, since every object is fibrant and cofibrant. Moreover, **sSet** is left proper, since every object is cofibrant and **Top** is right proper, since every object is fibrant (cf. Theorem 2.2.19). However, **sSet** and **Top** are even proper, even though there are simplicial sets that are not fibrant, and topological spaces that are not cofibrant with respect to the respective Quillen model structures. These results are shown in [Hir03, Thm. 13.1.13] and [Hir03, Thm. 13.1.10.] respectively.

3.1.1 Homotopy Theory in a Model Category

Given a model category $\mathcal{M}$, we can define path and cylinder objects, that allow us to define homotopies between morphisms in a similar manner to the procedure in **Top** and **sSet**:

Definition 3.1.5. Let $\mathcal{M}$ be a model category, $x, y \in \mathcal{M}$.

(i) A *cylinder object* for x is a factorization

$$x \amalg x \xrightarrow{\iota_0 \amalg \iota_1} \mathrm{Cyl}(x) \xrightarrow{p} x$$

of the fold map $\mathrm{id}_x \amalg \mathrm{id}_x \colon x \amalg x \to x$ such that $\iota_0 \amalg \iota_1$ is a cofibration and p is a weak equivalence.

(ii) A *path object* for y is a factorization

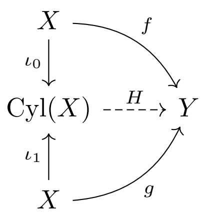
$$y \xrightarrow{\quad s \quad} \mathrm{Path}(y) \xrightarrow{\pi_0 \times \pi_1} y \times y$$

of the diagonal map $\mathrm{id}_y \times \mathrm{id}_y \colon y \to y \times y$ such that s is a weak equivalence and $\pi_0 \times \pi_1$ is a fibration.

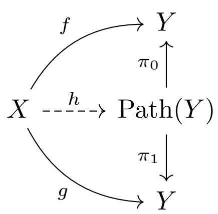
The factorizations from Definition 3.1.5 always exist, due to condition M5 of Definition 3.1.1. In **Top**, given a space X, a cylinder object is the space $X \times [0, 1]$ whereas path object is given by the space $\mathbf{Top}([0, 1], X)$, endowed with the compact open topology. This is true in both model structures we have been looking at. In **sSet**, given a simplicial set X, a cylinder object is the simplicial set $X \times \Delta^1$, whereas a path object is given by the function complex X^{Δ^1}.

Using path and cylinder objects, we can define left- and right homotopy in a similar fashion as before:

Definition 3.1.6. Let $f, g \colon X \to Y$ be maps in a model category $\mathcal{M}$. We say f is *left homotopic* to g if there is cylinder object $\mathrm{Cyl}(X)$, and map $H\colon \mathrm{Cyl}(X) \to Y$, making the following diagram commutative:



and we say that f is *right homotopic* to g if there is a path object $\mathrm{Path}(Y)$, and a map $h\colon X \to \mathrm{Path}(Y)$, making the following diagram commutative:



where $\iota_0, \iota_1, \pi_0, \pi_1$ are given as in Definition 3.1.5. We call the map H a *left homotopy* and the map h a *right homotopy* from f to g and write $H \colon f \Rightarrow g$ and $h \colon f \Rightarrow g$ respectively.

Although left and right homotopy have induced an equivalence relation on morphism sets in **Top**, we have already seen in **sSet**, that this is not necessarily the case in other model structures. There are, however, conditions that ensure that left and right homotopy induce equivalence relations. Moreover, even though left and right homotopy do not necessarily agree, they do so under certain conditions:

Theorem 3.1.7. *[Hir03, Prop. 7.4.8] Let $f, g\colon X \to Y$ be two parallel morphisms in a model category $\mathcal{M}$.*

(i) Assume that X is cofibrant. If there is a left homotopy $H\colon f \Rightarrow g$, then there is also a right homotopy $h\colon f \Rightarrow g$ with respect to any chosen path object.

(ii) Assume that Y is fibrant. If there is a right homotopy $h\colon f \Rightarrow g$, then there is also a left homotopy $H\colon f \Rightarrow g$ with respect to any chosen cylinder object.

Theorem 3.1.8. *[Hir03, Prop. 7.4.9] Let X be a cofibrant object in a model category $\mathcal{M}$, and let $Y \in \mathcal{M}^{(0)}$ be fibrant. Then the relations of left and right homotopy on $\mathcal{M}(X, Y)$ coincide, and are both equivalence relations.*

Note that in **sSet**, every object is cofibrant, since given any simplicial set X, the initial map $\emptyset \to X$ is a monomorphism, and thus a cofibration. Hence, Theorem 2.3.35 follows directly from Theorem 3.1.8. Given the previous theorems, it makes sense to restrict attention to cofibrant-fibrant objects, i. e. objects that are both, fibrant and cofibrant. This allows us to give a definition of homotopy equivalence:

Definition 3.1.9. Given a map $X \xrightarrow{f} Y$ between cofibrant-fibrant objects in a model category $\mathcal{M}$, we say that f is a *homotopy equivalence* if there is a map $Y \xrightarrow{g} X$, such that $f \circ g$ is homotopic to id_Y, and $g \circ f$ is homotopic to id_X.

Using this notion of homotopy equivalence, we can give a generalized version of the Whitehead Theorem:

Theorem 3.1.10 (Whitehead Theorem). *[Hir03, Thm. 7.5.10] Let $\mathcal{M}$ be a model category, $f\colon X \to Y$ be a map between cofibrant-fibrant objects. If f is a weak homotopy equivalence, it is a homotopy equivalence.*

Since in **Top**, every object is fibrant and CW–complexes are cofibrant, we can obtain the classical Whitehead Theorem (cf. 2.2.12) as a corollary of Theorem 3.1.10. Moreover, we can deduce that in **sSet**, every weak equivalence between Kan complexes is a homotopy equivalence.

3.1.2 The Homotopy Category

There are two notions of homotopy categories, and we will start of with the classical definition:

Definition 3.1.11. Let $\mathcal{M}$ be a model category. The *classical homotopy category* $\pi\mathcal{M}_{\mathrm{cf}}$ of $\mathcal{M}$ is the category whose objects are the cofibrant-fibrant objects of $\mathcal{M}$, whose maps are homotopy classes of maps in $\mathcal{M}$, and whose composition is induced by composition of maps in $\mathcal{M}$.

Since in the Strøm model structure every object is cofibrant-fibrant, the classical homotopy category with respect to to the Strøm model structure yields exactly the same category as in Definition 2.2.2. Whereas in the Quillen model structure, we obtain the category $\pi\mathbf{Top}_{\mathrm{cf}}$ as introduced at the end of Chapter 2.2.5.

There is another notion of a homotopy category, which yields a category equivalent to the classical homotopy category:

Definition 3.1.12. Let $\mathcal{M}$ be a model category, the localization $L_W(\mathcal{M})$ of $\mathcal{M}$ with respect to the class of weak equivalences is called the *homotopy category* of $\mathcal{M}$, and will be denoted by $\operatorname{Ho}\mathcal{M}$.

There is an important observation about homotopic maps in the homotopy category:

Lemma 3.1.13. *[Hir03, Lemma 3.8.4] Let $\mathcal{M}$ be a model category, $\mathcal{C}$ be an arbitrary category, and $\delta\colon \mathcal{M} \to \mathcal{C}$ be a functor that takes weak equivalences in $\mathcal{M}$ to isomorphisms in $\mathcal{C}$. If f, g are morphisms in $\mathcal{M}$ such, that f is either left or right homotopic to g, then $\delta(f) = \delta(g)$.*

Thus, in particular, homotopy equivalences in the original model category become identities in the homotopy category.

Theorem 3.1.14. *[Hir03, Thm. 8.3.6] If $\mathcal{M}$ is a model category, then there is a construction of the homotopy category $\gamma\colon \mathcal{M} \to \operatorname{Ho}\mathcal{M}$ such that if X, Y are cofibrant-fibrant objects in $\mathcal{M}$, $\mathcal{M}(\gamma(X), \gamma(y))$ is the set of homotopy classes of maps in $\mathcal{M}(X, Y)$.*

We will not give this construction here, and refer the interested reader to the cited source.

Theorem 3.1.15. *[Hir03, Thm. 8.3.9] If $\mathcal{M}$ is a model category, then the classical homotopy of $\mathcal{M}$ is equivalent to the homotopy category.*

We have already seen an application of this theorem at the end of Chapter 2.2.5. The important lesson here is that given a model structure on a category, on can obtain (up to equivalence of categories) the classical homotopy category without any knowledge about homotopy, simply by localizing with respect to the class of weak equivalences. There is another important theorem:

Theorem 3.1.16. *[Hir03, Thm. 8.3.10] Let $\mathcal{M}$ be a model category, $\gamma\colon \mathcal{M} \to \operatorname{Ho}\mathcal{M}$ be the associated functor to its homotopy category. Given a map $f \in \mathcal{M}$, then f is a weak homotopy equivalence if and only if $\gamma(f)$ is an isomorphism.*

So not only is the homotopy category completely determined by the class of weak equivalences, but vice versa the class of weak equivalences of a model category $\mathcal{M}$ completely determined by the structure of its homotopy category.

3.1.3 Equivalences of Model Categories

Definition 3.1.17. Let $\mathcal{M}, \mathcal{N}$ be model categories,

$$F\colon \mathcal{M} \leftrightarrows \mathcal{N} : G$$

be an adjunction. We call (F, G) a *Quillen pair* if F preserves cofibrations and trivial cofibrations, and G preserves fibrations and trivial fibrations. We will call F a *left Quillen functor* and G a *right Quillen functor*.

Thanks to the following theorem, it is enough to check two of the four conditions for an adjunction to be a Quillen pair:

Proposition 3.1.18. *[Hir03, Prop. 8.5.3] Let $\mathcal{M}, \mathcal{N}$ be model categories,*

$$F\colon \mathcal{M} \leftrightarrows \mathcal{N} : G$$

be an adjunction. Then the following statements are equivalent:

(i) The pair (F, G) is a Quillen pair.

(ii) The left adjoint F preserves cofibrations and trivial cofibrations.

(iii) The right adjoint G preserves fibrations and trivial fibrations.

(iv) The left adjoint F preserves cofibrations and the right adjoint G preserves fibrations.

(v) The left adjoint F preserves trivial cofibrations and the right adjoint G preserves trivial fibrations.

Under certain conditions, Quillen functors also preserve weak equivalences

Proposition 3.1.19. *[Hir03, Prop. 8.5.7] Let $\mathcal{M}, \mathcal{N}$ be model categories,*

$$F\colon \mathcal{M} \leftrightarrows \mathcal{N} : G$$

be a Quillen pair, then:

(i) the left adjoint F preserves weak equivalences between cofibrant objects, and

(ii) the right adjoint G preserves weak equivalences between fibrant objects.

Since every Hurewicz fibration is also a Serre fibration, and every homotopy equivalence is also a weak equivalence in **Top**, the adjunction $\mathrm{id}\colon \mathbf{Top}_{\mathrm{Strom}} \leftrightarrows \mathbf{Top} : \mathrm{id}$ is a Quillen pair by Proposition 3.1.18 (iii).

Definition 3.1.20. Let $\mathcal{M}, \mathcal{N}$ be model categories, $F\colon \mathcal{M} \leftrightarrows \mathcal{N} : G$ be a Quillen pair, then (F, U) is a pair of *Quillen equivalences* if for every cofibrant object $x \in \mathcal{M}$, every fibrant object $y \in \mathcal{N}$, and every map $f\colon x \to G(y)$ in $\mathcal{M}$, the map f is a weak equivalence if and only if the corresponding map $\phi^{-1}(f)\colon F(x) \to y$ is a weak equivalence, where ϕ is the isomorphism corresponding to the adjunction (F, G, ϕ). We will call the functors F and G *left* and *right Quillen equivalence*, respectively.

Theorem 3.1.21. *[Hir03, Thm. 8.5.23] Let $F\colon \mathcal{M} \leftrightarrows \mathcal{N} : G$ be a Quillen equivalence, then there is an induced equivalence of categories $\mathbf{L}F\colon \mathrm{Ho}\,\mathcal{M} \leftrightarrows \mathrm{Ho}\,\mathcal{N} : \mathbf{R}G$.*

Theorem 3.1.22. *[Hov99, Thm. 3.6.7] The geometric realization and singular functor define a Quillen equivalence.*

By the previous theorem, **Top** and **sSet** have equivalent homotopy categories, thus the Quillen model structures on **Top** and **sSet** yield "the same" homotopy theory. In particular, we may view homotopy theory on simplicial sets as a reformulation of the classical homotopy theory on topological spaces.

3.1.4 Cofibrantly Generated Model Categories

Cofibrantly generated model categories are a special class of model categories, where the classes of fibrations and cofibrations are determined by two generating sets of morphisms I, J, respectively. While they have the advantage, that one only needs to find those sets to show that a model structure on a given category exists, they have the disadvantage that one generally does not have an explicit description of fibrations and cofibrations. Furthermore, there is a useful theory that, under certain conditions, allows us to "lift" the model structure on a cofibrantly generated model category along an adjunction, making said adjunction a Quillen pair. We will use that theory to find a model structure on the category of acyclic categories, and dedicate a part of this thesis to the task of identifying cofibrant objects in said model category.

Definition 3.1.23. Let I be a class of maps in a category $\mathcal{C}$. A morphism f in $\mathcal{C}$ is

(i) *I–injective* if it has the *right lifting property* with respect to all morphisms in I. We denote the class of I–injectives by $I-$inj.

(ii) *I–projective* if it has the *left lifting property* with respect to all morphisms in I. We denote the class of I–projectives by $I-$proj.

(iii) an *I–cofibration* if it has the *left lifting property* with respect to every I–injective morphism. We denote the class of I–cofibrations by $I-$cof.

(iv) an *I–fibration* if it has the *right lifting property* with respect to every I–projective morphism. We denote the class of I–fibrations by $I-$fib.

We have already encountered some of these classes in the previous chapter. For example, let

$$I := \left\{ D^n \xrightarrow{(\mathrm{id},\iota_0)} D^n \times I \,\middle|\, n \geq 0 \right\} \subseteq \mathbf{Top}^{(1)},$$

then $I-$inj is the class of Serre fibrations, and $I-$proj is the class of trivial Serre cofibrations. Moreover, let

$$J := \{\Lambda^n_k \hookrightarrow \Delta^n \mid n > 0, 0 \leq k \leq n\} \subseteq \mathbf{sSet}^{(1)},$$

then $J-$inj is exactly the class of Kan fibrations, and $J-$cof is exactly the class of anodyne extensions.

Definition 3.1.24. Let I be a class of morphisms in a category with small colimits. A *relative I–cell complex* is a transfinite composition of pushouts of elements of I. An object $x \in \mathcal{C}$ is an *I–cell complex* if $\emptyset \to x$ is a relative I–cell complex.

In **Top**, consider the set

$$I := \{\partial D^n \to D^n \mid n \in \mathbb{N}\},$$

then an I–cell complex is a generalized CW–complex.

Lemma 3.1.25. *[Hov99, Lemma 2.1.10] Let I be a class of morphisms in a category $\mathcal{C}$ with all small colimits. Then $I-$cell $\subseteq I-$cof.*

Definition 3.1.26. [Hir03] Let $\mathcal{C}$ be a cocomplete category, $\mathcal{D} \subseteq \mathcal{C}$. If κ is a cardinal, then an object $x \in \mathcal{C}$ is *κ–small relative to $\mathcal{D}$* if for every regular cardinal $\lambda \geq \kappa$ and every λ–sequence

$$X_0 \to X_1 \to \cdots \to X_\beta \to \cdots \qquad (\beta < \lambda)$$

in $\mathcal{C}$, such that $X_\beta \to X_{\beta+1}$ is in $\mathcal{D}$ for every β with $\beta + 1 < \lambda$, the map of sets

$$\operatorname*{colim}_{\beta<\lambda} \mathcal{C}(x, X_\beta) \to \mathcal{C}(x, \operatorname*{colim}_{\beta<\lambda} X_\beta)$$

is an isomorphism. We say x is *small relative to $\mathcal{D}$* if it is κ–small for some ordinal κ. And we say x is *small* if it is small relative to $\mathcal{C}$.

In **Cat**, every object is small and there is an easy way to find an ordinal κ such that the conditions from the previous definition are satisfied (cf. [FPP08, Proposition 7.6]).

Proposition 3.1.27. *Every category $\mathcal{C} \in$ **Cat** is κ–small, where*

$$\kappa = |\mathcal{C}^{(0)}| + |\mathcal{C}^{(1)}| + |\mathcal{C}^{(1)}{}_s\!\times_t \mathcal{C}^{(1)}|.$$

Definition 3.1.28. Let $\mathcal{C}$ be a cocomplete category, and $I \subseteq \mathcal{C}^{(1)}$ be a set. An object is *small relative to* I if it is small relative to the category of I–cell complexes and we say that I *permits the small object argument* if the domains of elements of I are small relative to I.

Given a set of morphisms that permits the small object argument, a slightly stronger version of Lemma 3.1.25 holds (cf. [Hir03, Lemma 10.5.23]).

Theorem 3.1.29. *Let $\mathcal{C}$ be a cocomplete category and I be a set of morphisms that permits the small object argument. Then the class of I–cofibrations equals the class of retracts of relative I–cell complexes.*

We have already seen, that $I-$cell complexes in **Top** are generalized CW–complexes, where I is the set of boundary inclusions of n-spheres, hence we obtain Theorem 2.2.24 as a corollary of Theorem 3.1.29

Definition 3.1.30. A *cofibrantly generated model category* is a model category $\mathcal{M}$ such that:

(i) There exists a set $I \subseteq \mathcal{M}^{(1)}$, called the set of *generating cofibrations*, that permits the small object argument and satisfies $F \cap W = I-$inj.

(ii) There exists a set $J \subseteq \mathcal{M}^{(1)}$, called the set of *generating trivial cofibrations*, that permits the small object argument and satisfies $F = J-$inj.

We have already seen two examples of cofibrantly generated model categories: the first one is **Top**, where the sets of generating cofibrations and generating trivial cofibrations are given by

$$I := \{\partial D^n \hookrightarrow D^n \,|\, n \geq 0\}$$

and

$$J := \{|\Lambda_k^n| \hookrightarrow |\Delta^n| \,|\, n > 0, 0 \leq k \leq n\},$$

and the second one is **sSet**, where the sets of generating cofibrations and generating trivial cofibrations are given by

$$I := \{\partial\Delta^n \hookrightarrow \Delta^n \,|\, n \geq 0\}$$

and

$$J := \{\Lambda_k^n \hookrightarrow \Delta^n \,|\, n > 0, 0 \leq k \leq n\}.$$

A model category $\mathcal{M}$ is called *combinatorial* if it is cofibrantly generated and locally presentable. The model categories **sSet** and **Top** are both combinatorial.

The following propositions, which were proved in [Hir03, Proposition 11.2.1] and [Hir03, Proposition 10.5.16], respectively, should give some motivation why a cofibrantly generated model category is defined the way it is.

Proposition 3.1.31. *Let $\mathcal{M}$ be a cofibrantly generated model category with generating cofibrations I and generating trivial cofibrations J. Then:*

(i) The class of cofibrations of $\mathcal{M}$ equals the class of retracts of relative I–cell complexes, which equals the class of I–cofibrations.

(ii) The class of trivial fibrations of $\mathcal{M}$ equals the class of I–injectives.

(iii) The class of trivial cofibrations of $\mathcal{M}$ equals the class of retracts of relative J–cell complexes, which equals the class of J–cofibrations.

(iv) The class of fibrations of $\mathcal{M}$ equals the class of J–injectives.

Proposition 3.1.32 (The small object argument). *Let $\mathcal{C}$ be a small category and $I \subseteq \mathcal{C}^{(1)}$. Assume that I permits the small object argument. Then there is a functorial factorization of every map in $\mathcal{C}$ into a relative I–cell complex followed by an I–injective.*

Note that given a cofibrantly generated model category, by Proposition 3.1.31 the factorizations we obtain by applying the small object argument with respect to the sets I and J yield exactly the factorizations required in M5 of Definition 3.1.1. Although we will not give a full proof of the small object argument, we want to give a short sketch of how the factorization works and fix some notation, which we will need in Chapter 3.2.2. Let $\mathcal{C}$ be a small category, and $I \subseteq \mathcal{C}$ be a set that permits the small object argument. Given any morphism $f\colon x \to y$ in $\mathcal{C}$, we obtain a factorization $x \to E_\infty \to y$ where $x \to E_\infty$ is the transfinite composition of a λ–sequence

$$x = E_0 \to E_1 \to \cdots \to E_\beta \to E_{\beta+1} \to \cdots \qquad (\beta < \lambda),$$

where the E_β are obtained by pushouts of coproducts of elements of I.
We will end the discussion of cofibrantly generated model categories by giving two theorems. The first one gives criteria whether sets of morphisms I, J, together with a class of morphisms W define a model structure on a category $\mathcal{C}$. (cf. [Hir03, Prop. 11.3.1]). The second is commonly known as *Kan's Lemma on Transfer* and allows us to transport a model structure along an adjunction (cf. [Hir03, Theorem 11.3.2]).

Proposition 3.1.33. *Let $\mathcal{C}$ be a bicomplete category. Suppose $\mathcal{W}$ is a subcategory and that $I, J \subseteq \mathcal{C}^{(1)}$ are sets. Then $\mathcal{C}$ is a cofibrantly generated model category with generating cofibrations I and generating trivial cofibrations J and subcategory of weak equivalences $\mathcal{W}$ if and only if the following conditions hold:*

(i) $\mathcal{W}$ satisfies the 2–out–of–3 property and is closed under retracts.

(ii) The domains of I are small relative to I–cell.

(iii) The domains of J are small relative to J–cell.

(iv) $J-\text{cell} \subseteq \mathcal{W} \cap I-\text{cof}$.

(v) $I-\text{inj} \subseteq \mathcal{W} \cap J-\text{inj}$.

(vi) $\mathcal{W} \cap I-\text{cof} \subseteq J-\text{cof}$ *or* $\mathcal{W} \cap J-\text{inj} \subseteq I-\text{inj}$.

Proposition 3.1.34. *Let $\mathcal{M}$ be a cofibrantly generated model category with generating cofibrations I and generating trivial cofibrations J. Let $\mathcal{N}$ be a category that is closed under small limits and colimits and let $F\colon \mathcal{M} \leftrightarrows \mathcal{N} : U$ be a pair of adjoint functors. If we let $FI = \{Fu | u \in I\}$ and $FJ = \{Fv | v \in J\}$ and if*

(i) FI and FJ permit the small object argument and

(ii) U takes relative FJ–cell complexes to weak equivalences

then there is a cofibrantly generated model category structure on $\mathcal{N}$ in which FI is a set of generating cofibrations, FJ is a set of generating trivial cofibrations, and the weak equivalences are the maps that U takes into weak equivalences in $\mathcal{M}$. Furthermore, with respect to this model structure, (F, U) is a Quillen equivalence.

3.2 The Thomason Model Structure

The Thomason model structure on **Cat** is a cofibrantly generated model structure that is Quillen equivalent to the model structure on **sSet**. Thomason first established this model structure in [Tho80] by lifting the model structure on **sSet** along a certain adjunction. In 2010, Raptis managed to establish a Quillen equivalent model structure on **Pos** in [Rap10].

We will give a brief overview of the Thomason model structure on **Cat** in the first section of this chapter, and then establish a Quillen equivalent model structure on the category of small acyclic categories, following ideas published in [Rap10].

3.2.1 On the Category of Small Categories

Theorem 3.2.1. *[Tho80] There is a cofibrantly generated model structure on* **Cat**, *with generating cofibrations and generating trivial cofibrations given by the sets*

$$I = \left\{ \tau_1 \operatorname{Sd}^2 \partial\Delta^n \to \tau_1 \operatorname{Sd}^2 \Delta^n \,\middle|\, n \geq 0 \right\}$$

and

$$J = \left\{ \tau_1 \operatorname{Sd}^2 \Lambda^n_k \to \tau_1 \operatorname{Sd}^2 \Delta^n \,\middle|\, n \geq 1, 0 \leq k \leq n \right\},$$

respectively, such that a morphism is a weak equivalence if and only if $\operatorname{Ex}^2 Nf$ *is a weak equivalence in* **sSet**.

We will call the resulting model structure on **Cat** the *Thomason Model Structure*. While in the previous model structures we have encountered the classes of weak equivalences, cofibrations and fibrations are well-known, in the case of the Thomason Model structure we have only partial results:

Theorem 3.2.2. *[Tho80, Prop. 2.4] Let f be a morphism in* **Cat**. *Then f is a weak equivalence in the Thomason model structure if and only if Nf is a weak equivalence in* **sSet**.

Theorem 3.2.3. *[Tho80, Prop. 2.5] Let f be a morphism in* **Cat**. *Then f is a fibration in the Thomason model structure if and only if* $\operatorname{Ex}^2 Nf$ *is a fibration in* **sSet**.

Using Theorem 3.2.2, it is relatively easy to identify weak equivalences in the Thomason model structure. However, even though Theorem 3.2.3 gives a necessary and sufficiant condition for a morphism to be a fibration, it is generally hard to use, since given a category $\mathcal{C}$, the simplicial set $\operatorname{Ex}^2 N\mathcal{C}$ has a much more complicated, internal structure than the simplicial set $N\mathcal{C}$. Partial results for identifying fibrant objects in **Cat** are given in [MO15]. We will identify several classes of cofibrations and cofibrant objects in Chapter 4. For now, we only need one necessary condition for a map to be a cofibration, using the notion of a Dwyer map:

Definition 3.2.4. Let $\mathcal{C}$ be a category, and $i\colon \mathcal{A} \to \mathcal{C}$ an embedding, i. e. a functor that is faithful and injective on objects. We call i (as well as its image in $\mathcal{C}$) a *sieve* if for every $y \in i(\mathcal{A})$, $f \in \mathcal{C}(x, y)$ implies $x \in i(\mathcal{A})$ and $f \in i(\mathcal{A})$. If i satisfies the dual condition, it is called a *cosieve.*

Definition 3.2.5. Let $i\colon \mathcal{A} \to \mathcal{C}$ be a sieve. We call i a *Dwyer morphism* if there is a decomposition $\mathcal{A} \xrightarrow{f} \mathcal{C}' \xrightarrow{j} \mathcal{C}$ of i, such that j is a cosieve in $\mathcal{C}$ and there is a retraction $r\colon \mathcal{C}' \to \mathcal{A}$ together with a natural transformation $\eta\colon fr \Rightarrow \mathrm{id}_{\mathcal{C}'}$ such that $\eta f = \mathrm{id}_f$.

Note that the original definition of a Dwyer morphism by Thomason [Tho80] was stronger, in the sense that r was supposed to be an adjoint to f. However, the class of Dwyer morphisms as defined by Thomason is not closed under retracts and hence, in particular, not saturated. In [Cis99], Cisinski introduced the notion of a pseudo Dwyer morphisms and showed that the class of pseudo Dwyer morphisms is closed under retracts, and every retract of a Dwyer morphisms is a pseudo Dwyer morphism. Since the properties of pseudo Dwyer morphisms turned out to be more desireable than Thomasons notion of a Dwyer morphism, many authors dropped the word "pseudo" and took Cisinskis notion as the definition of a Dwyer morphism, which is a habit we will adopt here. The importance of Dwyer morphisms is due to the following theorems:

Theorem 3.2.6. *[Rap10, Prop. 2.4 (b)] Let f be a monomorphism in* **sSet***, then $\tau_1 \operatorname{Sd}^2 f$ is a Dwyer morphism.*

Theorem 3.2.7. *[Rap10, Prop. 2.4 (a)] Let $I \subseteq \mathbf{Cat}^{(1)}$ be a class of Dwyer morphisms, then its saturation is also a class of Dwyer morphisms.*

By Theorem 3.2.6 the set I from Theorem 3.2.1 is a set of Dwyer morphisms, and by Theorem 3.2.7, so is its saturation. Hence a cofibration in the Thomason model structure on **Cat** is necessarily a Dwyer morphism. There is another result, giving us a necessary criterion for a category to be cofibrant in the Thomason model structure:

Theorem 3.2.8. *[Tho80, Prop. 5.7] Let $\mathcal{C}$ be a small category. If $\mathcal{C}$ is cofibrant in the Thomason model structure, it is a poset.*

3.2.2 On the Category of Small Acyclic Categories

An acyclic category is a category without inverses and non-identity endomorphisms. Acyclic categories have been known under several names. They were called small categories without loops, or scwols, by Haefliger in [BH99], and loop-free categories by Haucourt [Hau06] and probably several others. In this thesis we adopt the terminology from [Koz08] and call them acyclic categories. Aside from the categorical perspective, we can view acyclic categories as generalized posets, allowing more than one morphism between any ordered pair of objects. We will establish a model structure on the category of small acyclic categories, that is Quillen equivalent to the Thomason model structure on **Cat**. The results of this chapter were previously published as [Bru15].

We will start off with the definition of an acyclic category:

Definition 3.2.9. A category $\mathcal{C}$ is called *acyclic* if it has no inverses and no non-identity endomorphisms.

We denote by **Ac** the category of small acyclic categories, with morphisms the functors between acyclic categories. It is obvious that **Ac** is a full subcategory of **Cat**. Hence there is a fully faithful inclusion $i\colon \mathbf{Ac} \to \mathbf{Cat}$. The inclusion i has a left adjoint, called *acyclic reflection*, which we construct as follows: given a category $\mathcal{C} \in \mathbf{Cat}$, define $p(\mathcal{C})$ to be the acyclic category with object set

$$p(\mathcal{C})^{(0)} = \mathcal{C}^{(0)}/\!\sim_o$$

where $\sim_o$ is the equivalence relation generated by $x \sim_o y$ if $\mathcal{C}(x,y) \neq \emptyset \neq \mathcal{C}(y,x)$ and morphisms

$$p(\mathcal{C})^{(1)} = \mathcal{C}^{(1)}/\!\sim_m$$

where $\sim_m$ is generated by $\mathrm{id}_x \sim_m \mathrm{id}_y$ if $x \sim_o y$, and $f \sim_m \mathrm{id}_x$ if $f \in \mathcal{C}(x,y)$ or $f \in \mathcal{C}(y,x)$ and $\mathcal{C}(x,y) \neq \emptyset \neq \mathcal{C}(y,x)$.

Setting $\mathrm{id}_{[x]} = [\mathrm{id}_x]$, it is easy to see that the composition inherited from $\mathcal{C}$ is well defined on $p(\mathcal{C})$, and hence p is well defined on objects. Given a functor $F\colon \mathcal{C} \to \mathcal{D}$ in **Cat** the components induce well defined maps on $p(\mathcal{C})$ via

$$p(F)([x]) = [p(F(x))] \qquad \text{and} \qquad p(F)([f]) = [p(F(f))].$$

This construction yields a functor $p\colon \mathbf{Cat} \to \mathbf{Ac}$, which is left adjoint to the inclusion i. Hence **Ac** is reflective in **Cat**, and we can calculate colimits in **Ac** by applying the acyclic reflection to the respective colimits in **Cat**. That is:

Lemma 3.2.10. *Given a diagram $D\colon I \to \mathbf{Ac}$, we have*

$$p(\mathrm{colim}_I\, iD) \cong (\mathrm{colim}_I\, D).$$

Proof. This follows directly from p being a right adjoint, and $pi \Rightarrow \mathrm{id}_{\mathbf{Ac}}$ being a natural isomorphism, since then

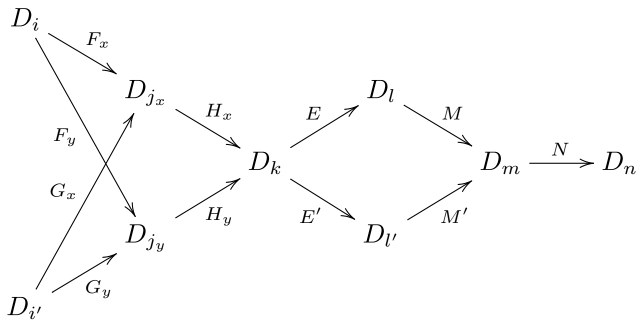
$$p(\mathrm{colim}_I\, iD) \cong (\mathrm{colim}_I\, piD) \cong (\mathrm{colim}_I\, D).$$

We will now establish a model structure on the category **Ac**. For that purpose, we will show that the inclusion $i\colon \mathbf{Ac} \to \mathbf{Cat}$ preserves filtered colimits, and that pushouts of acyclic categories along sieves are again acyclic categories. We will use these features to show that we can lift the Thomason model structure on **Cat** along the adjunction $p\colon \mathbf{Cat} \leftrightarrows \mathbf{Ac} : i$ and obtain a model structure on **Ac**.

Proposition 3.2.11. *The inclusion $i\colon \mathbf{Ac} \to \mathbf{Cat}$ preserves filtered colimits.*

Proof. Let $D\colon I \to \mathbf{Cat}$ be a filtered diagram such that D_i is an acyclic category for every i in I, and let $\mathcal{C} = \mathrm{colim}_I\, D$. At first, we will prove that any endomorphism in $\mathcal{C}$ is necessarily the identity, then we show that now there are antiparallel morphisms in $\mathcal{C}$.
To prove that any endomorphism is an identity, assume that there is an $x \in \mathcal{C}$, and an $[f] \in \mathcal{C}(x,x)$, such that $[f] \neq \mathrm{id}$. Hence, there is a category D_i, with objects $x_i, x_i' \in D_i$, such that $f \in D_i(x_i, x_i')$ and $x_i, x_i' \in x$. From the description of filtered colimits in **Cat**, we know that there is a category D_j and functors $F\colon D_i \to D_j$, $G\colon D_i \to D_j$ such that $F(x_i) = G(x_i')$. Since D is filtered, there is a category D_k and a functor $H\colon D_j \to D_k$ such that $H \circ F = H \circ G$. But D_k is acyclic, and thus $H \circ F(f) = H \circ G(f) = \mathrm{id}_{H\circ F(x_i)}$. By Prop. 2.1.27 this yields $[f] = [\mathrm{id}_{H\circ F(x_i)}] = \mathrm{id}_x$.
We will now show that there are no antiparallel morphisms in $\mathcal{C}$. Therefore we assume that there are objects $x, y \in \mathcal{C}$, together with two morphisms $[f]\colon x \to y$ and $[h]\colon y \to x$. By the construction of filtered colimits in **Cat** there are categories D_i and $D_{i'}$ such that $f \in D_i(x_i, y_i)$, $h \in D_{i'}(y_{i'}, x_{i'})$ and $[x_i] = [x_{i'}] = x$ as well as $[y_i] = [y_{i'}] = y$. We will use filteredness of I and the construction of filtered colimits in **Cat** to construct the following diagram in five consecutive

steps:



First, by Prop. 2.1.27, there are categories D_{j_x} and D_{j_y}, together with pairs of functors $F_x\colon D_i \to D_{j_x}$, $G_x\colon D_i' \to D_{j_x}$ and $F_y\colon D_i \to D_{j_y}$, $G_y\colon D_i' \to D_{j_y}$ satisfying $F_x(x_i) = G_x(x_{i'})$ and $F_y(y_i) = G_y(y_{i'})$. Using Def. 2.1.24 (i), there is a category D_k together with functors $H_x\colon D_{j_x} \to D_k$, $H_y\colon D_{j_y} \to D_k$. In particular, we have

$$H_x \circ F_x \neq H_y \circ F_y\colon D_i \rightrightarrows D_k$$

and

$$H_x \circ G_x \neq H_y \circ G_y\colon D_{i'} \rightrightarrows D_k.$$

Thus, by Def. 2.1.24 (ii), there are categories D_l and $D_{l'}$, together with functors $E\colon D_k \to D_l$ and $E'\colon D_{k'} \to D_{l'}$ satisfying

$$E \circ H_x \circ F_x = E \circ H_y \circ F_y$$

and

$$E' \circ H_x \circ G_x = E' \circ H_y \circ G_y.$$

Again by Def. 2.1.24 (i), there is a category D_m and functors $M\colon D_l \to D_m$, $M'\colon D_{l'} \to D_m$. Yet again by Def. 2.1.24 (ii) there is a category D_n and a functor $N\colon D_m \to D_n$ satisfying

$$N \circ M \circ E = N \circ M' \circ E'.$$

Putting together the previous equations, we have

$$\begin{aligned} &N \circ M \circ E \circ H_x \circ F_x(x_i) \\ =\,&N \circ M' \circ E' \circ H_y \circ G_y(x_{i'}) =: x_n \end{aligned}$$

and

$$\begin{aligned} &N \circ M \circ E \circ H_x \circ F_x(y_i) \\ =\,&N \circ M' \circ E' \circ H_y \circ G_y(y_{i'}) =: y_n. \end{aligned}$$

Hence

$$N \circ M \circ E \circ H_x \circ F_x(f) \in D_n(x_n, y_n)$$

and

$$N \circ M' \circ E' \circ H_y \circ G_y(h) \in D_n(y_n, x_n),$$

which contradicts that D_n is an acyclic category. Thus, the subcategory of acyclic categories is closed under taking filtered colimits, which yields in particular, that the inclusion $i\colon \mathbf{Ac} \to \mathbf{Cat}$ commutes with filtered colimits.

Lemma 2.1.40 in conjunction with Lemma 2.1.41 yields immediately:

Corollary 3.2.12. *The category* **Ac** *is locally finitely presentable.*

The next step is to prove that pushouts of acyclic categories along sieves in **Cat** are again acyclic categories. For that purpose we need a few preparational lemmas. The first of which can be found in [FL79, Proposition 5.2], the second we will prove here.

Lemma 3.2.13. *Given a pushout*

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{F} & \mathcal{C} \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle j} \\ \mathcal{B} & \xrightarrow{G} & \mathcal{B} \amalg_{\mathcal{A}} \mathcal{C} \end{array} \tag{3.1}$$

where $i\colon \mathcal{A} \to \mathcal{B}$ is a sieve, then j is a fully faithful inclusion, i.e. bijective on objects and morphisms.

Lemma 3.2.14. *Given the pushout diagram* (3.1)*, every element $[x] \in \mathcal{B} \amalg_{\mathcal{A}} \mathcal{C}$ satisfies either*

(i) $[x] = \{x\}$ and $x \in \mathcal{B}^{(0)} \setminus i(\mathcal{A}^{(0)})$, or

(ii) there is one and only one $c \in \mathcal{C}$, such that $[x] = [c]$.

Proof. Assumption (i) is obvious, since x has no preimage in $\mathcal{A}$, it is only equivalent to itself. On the other hand, if x is not in $\mathcal{B}^{(0)} \setminus i(\mathcal{A}^{(0)})$, it has a preimage in $\mathcal{A}$, which has an image in $\mathcal{C}$ and then *(ii)* follows directly from Lemma 3.2.13.

Proposition 3.2.15. *Let $\mathcal{B} \xleftarrow{i} \mathcal{A} \xrightarrow{F} \mathcal{C}$ be a diagram of acyclic categories, and assume that i is a sieve. Then the pushout in* **Cat** *is again an acyclic category.*

Proof. By Lemma 2.1.33, the pushout of the diagram $\mathcal{B} \xleftarrow{i} \mathcal{A} \xrightarrow{F} \mathcal{C}$ is given by the coequalizer Q of the diagram $\mathcal{A} \underset{\iota_{\mathcal{C}}\circ F}{\overset{\iota_{\mathcal{B}}\circ i}{\rightrightarrows}} \mathcal{B} \amalg \mathcal{C}$. Where Q is the quotient of $\mathcal{B} \amalg \mathcal{C}$ by the principal general congruence $(\sim_o, \sim_m)$ generated by the relation $\sim_{\iota_{\mathcal{C}}\circ i=\iota_{\mathcal{D}}\circ F}$. For the sake of convenience, we will subsequently ignore the inclusions $\iota_{\mathcal{B}}$ and $\iota_{\mathcal{C}}$ from notation, and simply write $f \in \mathcal{B}$ for a morphism f in the image $\iota_{\mathcal{B}}(\mathcal{B})$.
By Lemma 3.2.14, $Q^{(0)} \cong \left(\mathcal{B}^{(0)} \setminus i\left(\mathcal{A}^{(0)}\right)\right) \amalg \mathcal{C}^{(0)}$. Hence a morphism $f = [(f_0, \dots, f_n)]$ in Q satisfies either

(i) $f_0, \dots, f_n \in \mathcal{B} \setminus i\left(\mathcal{A}^{(0)}\right)$,

(ii) either $f_i \in i(\mathcal{A})$, or $f_i \in \mathcal{C}$ for every $i = 0, \dots, n$, or

(iii) there is a $0 \le k \le n$, such that:

$$\begin{aligned} f_i \in i(\mathcal{A}) \text{ or } f_i \in \mathcal{C} \qquad & \text{for } i < k \\ s(f_k) \in \mathcal{C}^{(0)} \amalg \mathcal{A}^{(0)} \text{ and } t(f_k) \in \mathcal{B}^{(0)} \qquad & \\ f_i \in \mathcal{B} \setminus \mathcal{A}^{(0)} \qquad & \text{for } i > k \end{aligned}$$

In case (i), $(f_0, \dots, f_n) \sim_m f_n \circ \cdots \circ f_0$, since $\mathcal{B} \setminus i\left(\mathcal{A}^{(0)}\right)$ embeds fully into Q. Thus, in particular, $[t(f_n)] \neq [s(f_0)]$ and $Q([t(f_n)], [s(f_0)]) = \emptyset$.
Considering case (ii), we claim that there is a composable sequence of morphisms $(h_0, \dots, h_n)$ in $\mathcal{C}$, such that $(f_0, \dots, f_n) \sim_m (h_0, \dots, h_n)$. Note therefore, that given any $\sim_o$–composable pair of morphisms $f_i,\ f_{i+1}$ in $\mathcal{B} \amalg \mathcal{C}$, satisfying condition (ii), we have $t(f_i) \sim_o s(f_{i+1})$. Hence by Lemma 3.2.14, there is a unique $x \in \mathcal{C}^{(0)}$, such that $x \sim_o t(f_i) \sim_o s(f_{i+1})$. Moreover, since $f_i,\ f_{i+1}$ have preimages in $\mathcal{A}$, by Lemma 3.2.13 there are unique morphisms $h_i = F(i^{-1}(f_i))$ and $h_{i+1} = F(i^{-1}(f_{i+1}))$, such that $t(h_i) \sim_o x$. Since $t(h_{i+1}) \sim_o t(f_{i+1})$, and since $x \sim_o t(h_i) \sim_o s(h_{i+1})$, and x is unique, h_i and h_{i+1} are composable. Thus there is a composable sequence $(h_0, \dots, h_n)$ of morphisms in $\mathcal{C}$, such that $(f_0, \dots, f_n) \sim_m (h_0, \dots, h_n)$. By definition of a generalized congruence, $(h_0, \dots, h_n) \sim_m h_n \circ \cdots \circ h_0 =: h$. Since h is a morphism in $\mathcal{C}$, and $\mathcal{C}$ embeds fully into Q by Lemma 3.2.13, it follows that $s([h]) \neq t([h])$. Furthermore, by the same argument a morphism $[(f'_0, \dots, f'_n)]$ in $Q(t([h]), s([h]))$ would yield a morphism $h' \in \mathcal{C}(t(h), s(h))$, which contradicts $\mathcal{C}$ being acyclic.
In case (iii), if $k = 0$, $(f_0, \dots, f_n) \sim_m f_n \circ \dots \circ f_0 =: f$, since f_k has no preimage in $\mathcal{A}$ for every $k = 0, \dots, n$, hence $[f_k] = \{f_k\}$ and thus composition is well defined. Moreover, $s(f) \neq t(f)$ by construction. And $Q(t(f), s(f)) = \emptyset$ since $i(\mathcal{A})$ is a sieve.
If $k \neq 0$, we can decompose $[(f_0, \dots, f_n)]$ into $[(f_k, \dots, f_n)] \circ [(f_0, \dots, f_{k-1})]$, apply the former arguments to the individual morphisms and use the fact that $s(f_0) \neq t(f_n)$ by construction.

Theorem 3.2.16. *Consider the morphism sets*

$$I = \left\{\tau_1 \operatorname{Sd}^2 \partial\Delta^n \to \tau_1 \operatorname{Sd}^2 \Delta^n \middle| n \geq 0\right\}$$

and

$$J = \left\{\tau_1 \operatorname{Sd}^2 \Lambda^n_k \to \tau_1 \operatorname{Sd}^2 \Delta^n \middle| n > 0, 0 \leq k \leq n\right\}.$$

in **Cat** *and the adjunction* $p\colon \mathbf{Cat} \leftrightarrows \mathbf{Ac} : i$. **Ac** *is a proper combinatorial cofibrantly generated model category with generating cofibrations* pI *and generating trivial cofibrations* pJ*. Moreover, the adjunction* (p, i) *is a Quillen equivalence.*

Proof. Remember that the sets I and J are the generating cofibrations and generating trivial cofibrations for the Thomason model structure on **Cat**. By [Tho80, Lemma 5.1], the domains and codomains of I and J are posets, and by Proposition 3.1.27 κ–small for some finite ordinal κ. Moreover, by 3.2.6 elements of I and J are Dwyer morphisms. Let $f\colon x \to y$ be a morphism in **Ac**. Since **Cat** is a cofibrantly generated model category, the small object argument yields a factorization $i(x) \xrightarrow{j'} E'_\infty \xrightarrow{q'} i(y)$ of $i(f)$ in **Cat**. We know that κ is finite, that i preserves filtered colimits (and by Lemma 2.1.41 also directed colimits) and pushouts along sieves, and that coproducts can be expressed as λ–composable sequences. Thus applying the small object argument to f in **Ac** with respect to to pI or pJ yields a factorization $x \xrightarrow{j} E_\infty \xrightarrow{q} y$ satisfying $i(j) \cong j'$, $i(E_\infty) \cong E'_\infty$, and $i(q) = q'$. Hence, factorizations of morphisms between acyclic categories in **Cat** are identical to the inclusions of the factorizations of the respective morphisms in **Ac**. In particular, the sets pI and pJ permit the small object argument and satisfy condition (i) of Proposition 3.1.34.

Furthermore, since **Cat** is a cofibrantly generated model category, it follows from Lemma 3.1.25 and Proposition 3.1.31 (iii), that every relative J–cell complex is a trivial cofibration in **Cat**. Since analogously to the previous reasoning, i maps pJ–cell complexes to J–cell complexes in **Cat**, condition (ii) of Proposition 3.1.34 is satisfied. Thus pI and pJ are generating cofibrations and generating trivial cofibrations for a cofibrantly generated model structure on **Ac**, and the adjunction (p, i) is a Quillen pair.

The category **Ac** is left proper, because every cofibration is a Dwyer morphism, and pushouts along Dwyer morphisms in **Ac** are the same as in **Cat** by Proposition 3.2.15. The category **Ac** is right proper, because **Cat** is right proper and i is a right adjoint, thus preserves pullbacks.

To show that (p, i) is a Quillen equivalence, note that by Theorem 3.2.8, every cofibrant object $\mathcal{C}$ in **Cat** is a poset, thus (in particular) an acyclic category. Hence the unit component $\eta_{\mathcal{C}} \colon \mathcal{C} \to ip(\mathcal{C})$ is an isomorphism. Let $\phi \colon \mathbf{Ac}(p(\mathcal{C}), \mathcal{D}) \to \mathbf{Cat}(\mathcal{C}, i(\mathcal{D}))$ denote the natural isomorphism associated to the Quillen pair (p, i). Given $f \colon p(\mathcal{C}) \to \mathcal{D}$ in **Ac**, we have $\phi(f) = i(f) \circ \eta_{\mathcal{C}}$. Since W is closed under composition with isomorphisms, the map $\phi(f)$ is a weak equivalence if and only if $i(f)$ is, and by Proposition 3.1.34, the map $i(f)$ is a weak equivalence if and only if f is. Thus (p, i) is a Quillen equivalence.

4

Cofibrancy in the Thomason Model Structure

We have already shown that the Thomason model structure can be lifted to a Quillen equivalent model structure on **Pos** and **Ac**, respectively, and we have already seen that every cofibrant object must be a poset. Since the model structures on **Pos**, **Ac** and **Cat** are cofibrantly generated by the same classes of generating cofibrations and trivial cofibrations, and pushouts and colimits along cofibrations yield the same objects in all of those, this implies that all three categories feature the same class of cofibrant objects.

In [MSZ16, Proposition 6.5] it was proved, that every finite, one-dimensional poset is cofibrant, i. e. every poset that has a 1-dimensional nerve. In this chapter we identify various other classes of cofibrant posets. In Section 4.1, we show that every finite semilattice, every countable tree, every chain and every finite zigzag is cofibrant and in Section 4.2, we show that every poset with five or less elements is cofibrant. Moreover, we prove that every inclusion of a minimum into any of the cofibrant posets we identified is a cofibration.

In the following, when we talk about the Thomason model structure, we mean the Thomason model structure on either **Pos**, **Ac**, or **Cat**. In particular, when we use the term category, we refer to an object in any Thomason model structure.

The results of this chapter are joint work with Christoph Pegel, and were previously published as [BP16]. Moreover, the general idea on how to prove Lemma 4.1.10, as well as the proofs of Lemma 4.2.4 and Lemma 4.2.11 were provided by Viktoriya Ozornova.

We start this chapter by giving two preliminary results that we will use extensively:

Theorem 4.0.1. *The category $\tau_1 \operatorname{Sd} \Delta^n$ is cofibrant, and every inclusion*

$$\begin{aligned} i_k \colon [0] &\to \tau_1 \operatorname{Sd} \Delta^n, \\ 0 &\mapsto \{k\} \end{aligned}$$

is a cofibration.

Proof. This is found in the proof of [Cis99, Lemma 1].

Lemma 4.0.2. *Let $\mathcal{C}$ be a category. If $[0] \to \mathcal{C}$ is a cofibration, $\mathcal{C}$ is cofibrant.*

Proof. Since $[0]$ lies in the image of $\tau_1 \operatorname{Sd}^2$, $[0]$ is cofibrant, and since cofibrations are closed under composition, $\emptyset \to [0] \to \mathcal{C}$ is a cofibration and thus, $\mathcal{C}$ is cofibrant.

4.1 Trees, Zigzags, Chains and Semilattices

In this section, we proof that every countable tree, every finite zigzag, every chain and every finite semilattice is cofibrant in the Thomason model structure.

4.1.1 Finite Semilattices

At first, we show that the semilattices constructed from a Boolean lattice by removing either the top, or the bottom element are cofibrant (Lemma 4.1.3 and 4.1.4, respectively). In Theorem 4.1.5 we use the resulting semilattices to construct arbitrary, finite semilattices as retracts of those. We start by giving some definitions:

Definition 4.1.1. Let $\mathcal{C}$ be a small category, $A \subseteq \mathcal{C}^{(0)}$. We denote by $\mathcal{C} \setminus A$ the full subcategory of $\mathcal{C}$ with object set $\mathcal{C}^{(0)} \setminus A$. In particular, if $A = \{x\}$, we simply write $\mathcal{C} \setminus x$.

Definition 4.1.2. Let $\mathcal{C}$ be a small category. We denote by $\mathcal{P}(\mathcal{C})$ the category which is given by the power set lattice of $\mathcal{C}^{(0)}$.

In the following, we will denote the minimal element of $\mathcal{P}(\mathcal{C})$ by $\varnothing$, as opposed to $\emptyset$ to avoid any confusion on whether we are talking about the minimal element of $\mathcal{P}(\mathcal{C})$, or the initial object in the ambient category.

Lemma 4.1.3. *The category $\mathcal{P}([n]) \setminus [n]$ is cofibrant and the inclusion $[0] \to \mathcal{P}([n]) \setminus [n]$ of the minimum is a cofibration.*

Proof. Let $\xi\colon \mathbf{Pos} \to \mathbf{Cat}$ be the functor which maps a poset P to the lattice of non-empty chains in P, ordered by inclusion. Then $\xi = \tau_1 \operatorname{Sd} N$ and $\xi^2 = \tau_1 \operatorname{Sd}^2 N$ (cf. [Cis99]). Consider the diagram

$$\begin{array}{ccccc} [0] & \longrightarrow & \xi[0] & \longrightarrow & [0] \\ \big\downarrow{\scriptstyle i_\varnothing} & & \big\downarrow{\scriptstyle \iota_\varnothing} & & \big\downarrow{\scriptstyle i_\varnothing} \\ \mathcal{P}([n]) \setminus [n] & \xrightarrow{\ i\ } & \xi(\mathcal{P}([n]) \setminus [n]) & \xrightarrow{\ p\ } & \mathcal{P}([n]) \setminus [n] \end{array},$$

where $i_\varnothing$ and $\iota_\varnothing$ are the minima inclusions, and i and p are given as follows: let

$$A = \{m_1, m_2, \dots, m_k\} \in \mathcal{P}([n]) \setminus [n]$$

, such that $m_1 \leq m_2 \leq \cdots \leq m_k$. We set

$$i(A) = \{\{\varnothing\}, \{m_1\}, \{m_1, m_2\}, \dots, \{m_1, m_2, \dots, m_k\}\}$$

and given $B \in \mathcal{P}([n]) \setminus [n]$, we set $p(B) = \bigcup B$.

It is easy to see, that $p \circ i = \mathrm{id}$, hence $i_\varnothing$ is a retract of $\iota_\varnothing$. If we apply ξ to the whole diagram, we get that $\xi(i_\varnothing) = \iota_\varnothing$ is a retract of $\xi(\iota_\varnothing)$, which is a cofibration since $\xi(\iota_\varnothing) = \tau_1 \operatorname{Sd}^2 N(i_\varnothing)$ and $N(i_\varnothing)$ is a monomorphism in **sSet**. Hence $i_\varnothing$ is a cofibration and $\mathcal{P}([n]) \setminus [n]$ is cofibrant.

Lemma 4.1.4. *The category $\mathcal{P}([n]) \setminus \varnothing$ is cofibrant and the inclusions of the respective minima are cofibrations.*

Proof. Since $\mathcal{P}([n]) \setminus \varnothing \cong \tau_1 \operatorname{Sd} \Delta^n$, this follows from 4.0.1.

Theorem 4.1.5. *Every finite semilattice is cofibrant.*

Proof. Let L be a finite join–semilattice. Define

$$\begin{aligned} i\colon L &\to \mathcal{P}(L) \setminus \varnothing, \\ x &\mapsto \{ y \in L \,|\, y \leq x \} \end{aligned}$$

and

$$\begin{aligned} p\colon \mathcal{P}(L) \setminus \varnothing &\to L, \\ A &\mapsto \bigvee A, \end{aligned}$$

where $\bigvee A$ denotes the join over all elements of A in L. Then $p \circ i = \mathrm{id}$, hence L is a retract of $\mathcal{P}(L) \setminus \varnothing$ and since $\mathcal{P}(L) \setminus \varnothing$ is cofibrant by Lemma 4.1.4, so is L. Thus every join–semilattice is cofibrant.

Let M be a finite meet–semilattice. Since M^{op} is a join–semilattice and $\mathcal{P}(M) = \mathcal{P}(M^{\mathrm{op}})$ we obtain a retract diagram

$$M^{\mathrm{op}} \xrightarrow{\ i\ } \mathcal{P}(M) \setminus \varnothing \xrightarrow{\ p\ } M^{\mathrm{op}} \ ,$$

where i and p are given as before. Dualizing every object we obtain a retract

$$M \xrightarrow{\ i^{\mathrm{op}}\ } (\mathcal{P}(M) \setminus \varnothing)^{\mathrm{op}} \xrightarrow{\ p^{\mathrm{op}}\ } M \ .$$

But $(\mathcal{P}(M) \setminus \varnothing)^{\mathrm{op}}$ is isomorphic to $\mathcal{P}(M) \setminus M$, which is cofibrant by Lemma 4.1.3 and hence so is M.

Corollary 4.1.6. *Let S be a finite semilattice, and $i_m\colon [0] \to S$ an inclusion that maps the single element of $[0]$ to a local minimum m in S, then i_m is a cofibration.*

Proof. Let L be a finite join–semilattice, $m \in L$ be a local minimum. Define

$$\begin{aligned} i_m\colon [0] &\to L, \\ 0 &\mapsto m \end{aligned}$$

and

$$\begin{aligned} \iota_m\colon [0] &\to \mathcal{P}(L) \setminus \varnothing, \\ 0 &\mapsto \{m\} \,. \end{aligned}$$

We obtain a diagram

$$\begin{array}{ccccc} [0] & \longrightarrow & [0] & \longrightarrow & [0] \\ \big\downarrow{\scriptstyle i_m} & & \big\downarrow{\scriptstyle \iota_m} & & \big\downarrow{\scriptstyle i_m} \\ L & \xrightarrow{\ i\ } & \mathcal{P}(L) \setminus \varnothing & \xrightarrow{\ p^{\mathrm{op}}\ } & L \end{array} ,$$

where i and p are given as in the proof of Theorem 4.1.5. It is easy to see that every square commutes and since ι_m is a cofibration by Theorem 4.0.1, so is i_m.

Now let M be a finite meet–semilattice, $m \in M$ be the minimal element. Define

$$\begin{aligned} i_m\colon [0] &\to M, \\ 0 &\mapsto m \end{aligned}$$

as before, and

$$\begin{aligned} \iota_\varnothing \colon [0] &\to \mathcal{P}(M) \setminus M, \\ 0 &\mapsto \varnothing. \end{aligned}$$

Consider the isomorphisms

$$(\mathcal{P}(M) \setminus \varnothing)^{\mathrm{op}} \xrightarrow{\varphi} \mathcal{P}(M) \setminus M \xrightarrow{\psi} (\mathcal{P}(M) \setminus \varnothing)^{\mathrm{op}},$$

both of which are given by mapping a subset $A \subseteq M$ to its complement. We obtain a retract diagram

$$\begin{array}{ccccccccc} [0] & & \longrightarrow & & [0] & & \longrightarrow & & [0] \\ \downarrow{\scriptstyle i_m} & & & & \downarrow{\scriptstyle \iota_\varnothing} & & & & \downarrow{\scriptstyle i_m} \\ M & \xrightarrow{i^{\mathrm{op}}} & (\mathcal{P}(M) \setminus \varnothing)^{\mathrm{op}} & \xrightarrow{\varphi} & \mathcal{P}(M) \setminus M & \xrightarrow{\psi} & (\mathcal{P}(M) \setminus \varnothing)^{\mathrm{op}} & \xrightarrow{p} & M \end{array},$$

and since $\iota_\varnothing$ is a cofibration by Lemma 4.1.3, so is i_m.

4.1.2 Chains

Definition 4.1.7. Let $\mathcal{C}$ be a category. We call $\mathcal{C}$ a *chain* if $\mathcal{C}$ is either isomorphic to a finite ordinal $[n]$, or to the natural numbers $\mathbb{N}$.

Theorem 4.1.8. *Every chain is cofibrant and the inclusion of the minimum is a cofibration.*

Proof. Let $\mathcal{C}$ be a chain. If $\mathcal{C}$ is finite, then it is cofibrant by Theorem 4.1.5 and the inclusion of the minimum is a cofibration by Corollary 4.1.6. Assume that $\mathcal{C}$ is isomorphic to $\mathbb{N}$. We will construct a sequence

$$\begin{aligned} X \colon \mathbb{N} &\to \mathbf{Cat}, \\ i &\mapsto X_i, \\ (i \to i+1) &\mapsto F_i \end{aligned}$$

such that X_0 is cofibrant, $\operatorname{colim}_{\mathbb{N}} X \cong \mathcal{C}$, and the map $X_0 \to \operatorname{colim}_{\mathbb{N}} X$ is a cofibration and the inclusion of the minimum, which yields that $\operatorname{colim}_{\mathbb{N}} X$ is cofibrant. Let $X_0 = x_0$. Assume that

$$X_i = x_0 \xrightarrow{f_0} x_1 \xrightarrow{f_1} x_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{i-1}} x_i$$

and let $\mathcal{D} = x \to y$. We construct X_{i+1} from X_i via the pushout

$$\begin{array}{ccc} [0] & \xrightarrow{f} & \mathcal{D} \\ {\scriptstyle h}\downarrow & & \downarrow \\ X_i & \xrightarrow{F_i} & X_{i+1} \end{array},$$

where f and h are given by $f(0) = x$ and $h(0) = x_i$. Since f is a cofibration by Corollary 4.1.6, so is F_i. Moreover, since the class of cofibrations is closed under transfinite composition, the map $X_0 \to \operatorname{colim}_{\mathbb{N}} X$ is a cofibration and thus $\operatorname{colim}_{\mathbb{N}} X$ is cofibrant. Furthermore $\mathcal{C}$ is a universal co-cone for X by construction and thus $\operatorname{colim}_{\mathbb{N}} X \cong \mathcal{C}$.

4.1.3 Finite Zigzags

We will prove that every finite zigzag is cofibrant by showing that a certain class of zigzags is cofibrant (Lemma 4.1.10) and then glue together arbitrary finite zigzags from elements of this class (Theorem 4.1.11). At first, we should give a definition of what we mean by a zigzag.

Definition 4.1.9. A *zigzag* is a category $\mathcal{Z}$, which is generated by a (possibly infinite) directed graph

$$\cdots \leftrightarrow x_{i-1} \leftrightarrow x_i \leftrightarrow x_{i+1} \leftrightarrow \cdots$$

where $\leftrightarrow$ denotes either an arrow pointing to the left, or to the right.

Alternatively one might say a zigzag is a category generated by a total order

$$\cdots \to x_{i-2} \to x_{i-1} \to x_i \to x_{i+1} \to x_{i+2} \to \cdots$$

where some of the generating arrows are flipped:

$$\cdots \leftarrow x_{i-2} \to x_{i-1} \leftarrow x_i \leftarrow x_{i+1} \to x_{i+2} \to \cdots$$

Lemma 4.1.10. *Let $\mathcal{Z}$ be a finite zigzag with a global maximum, then $\mathcal{Z}$ is cofibrant and the minimum inclusions are cofibrations.*

Proof. The claim is trivial for zigzags with one or less elements. Hence let $\mathcal{Z}$ be a zigzag with $n+2$ elements and a global maximum. If $\mathcal{Z}$ is isomorphic to $[n+1]$, $\mathcal{Z}$ is cofibrant by Theorem 4.1.5 and we are done. Otherwise, we can write $\mathcal{Z}$ as

$$\mathcal{Z} = x_0 \xrightarrow{f_0} \cdots \xrightarrow{f_{i-2}} x_{i-1} \xrightarrow{f_{i-1}} x_i \xleftarrow{f_i} \cdots \xleftarrow{f_n} x_{n+1}.$$

Assume without loss of generality that $i > n/2$. By Theorem 4.0.1, the inclusions

$$\begin{aligned} \iota_k \colon [0] &\hookrightarrow \tau_1 \operatorname{Sd} \Delta^n, \\ 0 &\mapsto \{k\} \end{aligned}$$

are cofibrations. We construct $\mathcal{Z}$ as a retract of $\tau_1 \operatorname{Sd} \Delta^n$ and the inclusions $i_0, i_{n+1} \colon [0] \to \mathcal{Z}$ of the minima as retracts of ι_0 and ι_n, respectively. Define

$$\begin{aligned} i \colon \mathcal{Z} &\to \tau_1 \operatorname{Sd} \Delta^n, \\ x_k &\mapsto \begin{cases} \{0,1,\ldots,k\} & \text{if } k < i \\ \{0,1,\ldots,n\} & \text{if } k = i \\ \{k-1,k,\ldots,n\} & \text{if } k > i \end{cases} \end{aligned}$$

and

$$\begin{aligned} p \colon \tau_1 \operatorname{Sd} \Delta^n &\to \mathcal{Z}, \\ \sigma = \{k_1,k_2,\ldots,k_p\} &\mapsto \begin{cases} x_{|\sigma|-1} & \text{if } 0 \le k_1,\ldots,k_p < i \\ x_{n+2-|\sigma|} & \text{if } i \le k_1,\ldots,k_p \le n \\ x_i & \text{else} \end{cases}. \end{aligned}$$

Then $p \circ i = \mathrm{id}$ and we obtain a retract diagram

$$\begin{array}{ccccc} [0] & \longrightarrow & [0] & \longrightarrow & [0] \\ \downarrow{\scriptstyle i_k} & & \downarrow{\scriptstyle \iota_l} & & \downarrow{\scriptstyle i_k} \\ \mathcal{Z} & \xrightarrow{i} & \tau_1 \operatorname{Sd} N[n] & \xrightarrow{p} & \mathcal{Z} \end{array}$$

where $k = l = 0$ or $k = n+1$, $l = n$. Since ι_0 and ι_n are cofibrations, so are i_0 and i_{n+1}. Hence $\mathcal{Z}$ is cofibrant by Lemma 4.0.2.

Theorem 4.1.11. *Every finite zigzag is cofibrant, and every inclusion of a minimum into a zigzag is a cofibration.*

Proof. Given a zigzag $\mathcal{Z}_n$, we prove the claim by induction over the number n of local minima. If $n = 1$, $\mathcal{Z}_n$ is a either a chain, hence cofibrant and the inclusion of the minimum is a cofibration by Theorem 4.1.8, or can be obtained by gluing together two chains $\mathcal{C}_1$, $\mathcal{C}_2$ along their minima via the pushout

$$\begin{array}{ccc} [0] & \xrightarrow{j} & \mathcal{C}_1 \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle \alpha} \\ \mathcal{C}_2 & \xrightarrow{\beta} & \mathcal{Z}_n \end{array}$$

where i and j are the respective minimum inclusions. Since i and j are cofibrations by Theorem 4.1.8, so are α and β and thus, in particular, the compositions $\alpha \circ j$ or $\beta \circ i$.

Now assume that for every $n < N$, a zigzag with n local minima is cofibrant, and every inclusion of a minimum is a cofibration. Let $\mathcal{Z}_N$ be a zigzag with N local minima. There is a zigzag $\mathcal{Z}_{N-1}$ with $N-1$ local minima, and a zigzag $\mathcal{Z}$ with a global maximum, such that we can construct $\mathcal{Z}_N$ via a pushout

$$\begin{array}{ccc} [0] & \xrightarrow{k_{N-1}} & \mathcal{Z}_{N-1} \\ {\scriptstyle i_0}\downarrow & & \downarrow{\scriptstyle \iota} \\ \mathcal{Z} & \xrightarrow{\kappa} & \mathcal{Z}_N \end{array},$$

where i_0 and k_{N-1} are inclusions of local minima, hence cofibrations. Thus, so are ι and κ and therefore also the compositions

$$\begin{aligned} \emptyset \to \mathcal{Z} &\xrightarrow{\kappa} \mathcal{Z}_N, \\ [0] \xrightarrow{i_0} \mathcal{Z} &\xrightarrow{\kappa} \mathcal{Z}_N, \\ [0] \xrightarrow{i_1} \mathcal{Z} &\xrightarrow{\kappa} \mathcal{Z}_N, \end{aligned}$$

and

$$[0] \xrightarrow{k_j} \mathcal{Z}_{N-1} \xrightarrow{\kappa} \mathcal{Z}_N, \qquad j = 1, \ldots, N-1,$$

where i_1 is the other minimum inclusion of $\mathcal{Z}$, and k_j are the minimum inclusions of $\mathcal{Z}_{N-1}$.

Remark 4.1.12. Since the map ι from the previous proof is a cofibration, we can construct some infinite cofibrant zigzags with a countable number of objects via transfinite composition, similar to the methods used in proof of Lemma 4.1.10. There are, however, zigzags where this will not work.

Assume that $\mathcal{Z}$ is a countable infinite zigzag, with a finite number N of local maxima. If

$$\mathcal{Z} = \cdots \leftrightarrows x_{i-1} \to x_i \leftarrow x_{i+1} \leftarrow \cdots,$$

such that there are no more local extrema to the right of x_i, then $\mathcal{Z}$ is not constructable as a pushout along a cofibration using the methods from the previous proof.

4.1.4 Posets Generated by Directed Trees

By a directed tree, we mean a directed graph which is also a rooted tree, satisfying that every edge points away from the root. There is a natural poset structure on the set of vertices: given two vertices x, y, we say that $x \leq y$ if and only if the unique path from the root to y passes through x. We call the resulting poset the *poset generated by the tree.*

Note that there is a natural grading on the resulting poset structure, where the rank of an element x is given by the length of the minimal chain including x and the root. We will now show, that any such poset is cofibrant in the Thomason model structure, and that the inclusion of the root is a cofibration:

Theorem 4.1.13. *Every countable poset generated by a directed tree is cofibrant, and the inclusion of the root is a cofibration.*

Proof. Let $\mathcal{T}$ be a countable poset generated by a directed tree, and $\mathrm{rk}\colon \mathcal{T} \to \mathbb{N}$ the associated rank function. We define a sequence

$$\begin{aligned} X\colon \mathbb{N} &\to \mathbf{Cat}, \\ i &\mapsto X_i, \\ (i \to i+1) &\mapsto F_i, \end{aligned}$$

such that X_0 is cofibrant, $\operatorname{colim}_{\mathbb{N}} X \cong \mathcal{T}$ and the map $X_0 \to \operatorname{colim}_{\mathbb{N}} X$ is cofibrant and the inclusion of the root, which yields that $\mathcal{T}$ is cofibrant. For that purpose, let X_0 be the category with a single object r, and no non–identity morphisms. Assume that X_i is a subposet of $\mathcal{T}$ that contains all elements with rank lower or equal to i. If the rank of $\mathcal{T}$ is lower than i, we set $X_{i+1} = X_i$. Otherwise we construct X_{i+1} from X_i as follows: Let $J \subseteq \mathcal{T}$ be the set of all elements with rank i. Given $j \in J$, we define

$$K_j := \{\, t \in \mathcal{T} \,|\, \mathrm{rk}(t) = i+1 \text{ and } t \geq j \,\},$$

i. e. the set of all rank $i+1$ elements that are smaller than j. Let $\mathcal{D} = x \to y$. We obtain X_{i+1} via the pushout

$$\begin{array}{ccc} \coprod_{j\in J}\left(\coprod_{k\in K_j}[0]\right) & \xrightarrow{\coprod_{j\in J}\left(\coprod_{k\in K_j} f\right)} & \coprod_{j\in J}\left(\coprod_{k\in K_j}\mathcal{D}\right) \\ \Big\downarrow h & & \Big\downarrow \\ X_i & \xrightarrow{F_i} & X_{i+1} \end{array}$$

where the maps f and h are given by

$$\begin{aligned} f\colon [0] &\to \mathcal{D}, \\ 0 &\mapsto x, \end{aligned}$$

and

$$\begin{aligned} h\colon \coprod_{j\in J}\left(\coprod_{k\in K_j}[0]\right) &\to X_i \\ (0_k)_j &\mapsto j. \end{aligned}$$

Since f is a cofibration, so is $\coprod_{j\in J}\left(\coprod_{k\in K_j} f\right)$ and hence F_i. Thus the transfinite composition $X_0 \to \operatorname{colim}_{\mathbb{N}} X$—which is also the inclusion of the root—is a cofibration. Hence by Lemma 4.0.2, $\operatorname{colim}_{\mathbb{N}} X$ is cofibrant. Furthermore $\mathcal{T}$ is a universal co-cone for X by construction, which finishes the proof.

4.2 Cofibrant Objects with a Fixed Number of Elements

In this section, we proof that every poset with five or less elements is cofibrant and that their respective inclusions of minima are cofibrations. Most of those posets are already covered by previous theorems, so there are only a handful of posets which we have to construct by hand. Before starting with the proofs, we need a small lemma which we use throughout this section.

Lemma 4.2.1. *Let Q be a poset, $A \subseteq Q$ a subset and $i\colon A \to [n]$ an injective map of sets. Let furthermore*

$$\begin{aligned} m\colon N(\coprod_{a\in A}[0]) &\to \Delta^n \\ 0_a &\mapsto i(a) \end{aligned}$$

be a map in **sSet**. *If there is a retract $h\colon \coprod_{a\in A}[0] \to Q$ of $\tau_1 \operatorname{Sd}^2 m$ satisfying $h(0_a) = a$, then the poset P obtained by the pushout*

$$\begin{array}{ccc} \coprod_{a\in A}[0] & \longrightarrow & [0] \\ \big\downarrow h & & \big\downarrow \alpha \\ Q & \longrightarrow & P \end{array}$$

is cofibrant, and the inclusion α is a cofibration.

Proof. Since m is a monomorphism, $\tau_1 \operatorname{Sd}^2 m$ is a cofibration and thus, so is h. Hence α is a cofibration and by Lemma 4.0.2, P is cofibrant.

4.2.1 Posets with Four or Less Elements

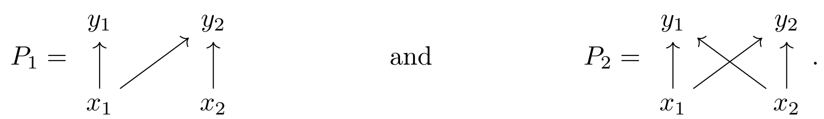
Theorem 4.2.2. *Every poset with three or less elements is cofibrant, and every inclusion of a local minimum into one of those posets is a cofibration.*

Proof. Every connected poset with three or less elements is a semilattice, hence cofibrant by Theorem 4.1.5 and the inclusions are cofibrations by the Corollary 4.1.6. Furthermore, since coproducts of cofibrant objects are cofibrant, every poset with three or less elements is cofibrant.

Theorem 4.2.3. *Every poset with four elements is cofibrant, and the respective inclusions of minima are cofibrations.*

Proof. Every poset with four elements that is a coproduct of cofibrant posets is cofibrant, hence every disconnected poset with four or less elements is cofibrant by Theorem 4.2.2. Up to isomorphism, there are ten connected posets with four elements. Eight of those are semilattices, hence cofibrant by Theorem 4.1.5, and their respective inclusions of minima are cofibrant by Corollary 4.1.6.

The only two posets that are not semilattices are



Let

We construct P_1 from $\mathcal{D}$ and $\mathcal{E}$ via the pushout

$$\begin{array}{ccc} [0] & \xrightarrow{\iota_x} & \mathcal{D} \\ \big\downarrow{\scriptstyle \iota_{b_1}} & & \big\downarrow{\scriptstyle \alpha} \\ \mathcal{E} & \xrightarrow{\beta} & P_1 \end{array},$$

where ι_{b_1} and ι_x are given by $\iota_{b_1}(0) = b_1$ and $\iota_x(0) = x$ respectively. Since ι_{b_1} and ι_x are cofibrations by Theorem 4.2.2, so are α and β and hence—in particular—the compositions with the respective inclusions of minima into $\mathcal{D}$ and $\mathcal{E}$ and every inclusion of a minimum into P_1 can be written as such a composition.

Regarding P_2, consider the pushout

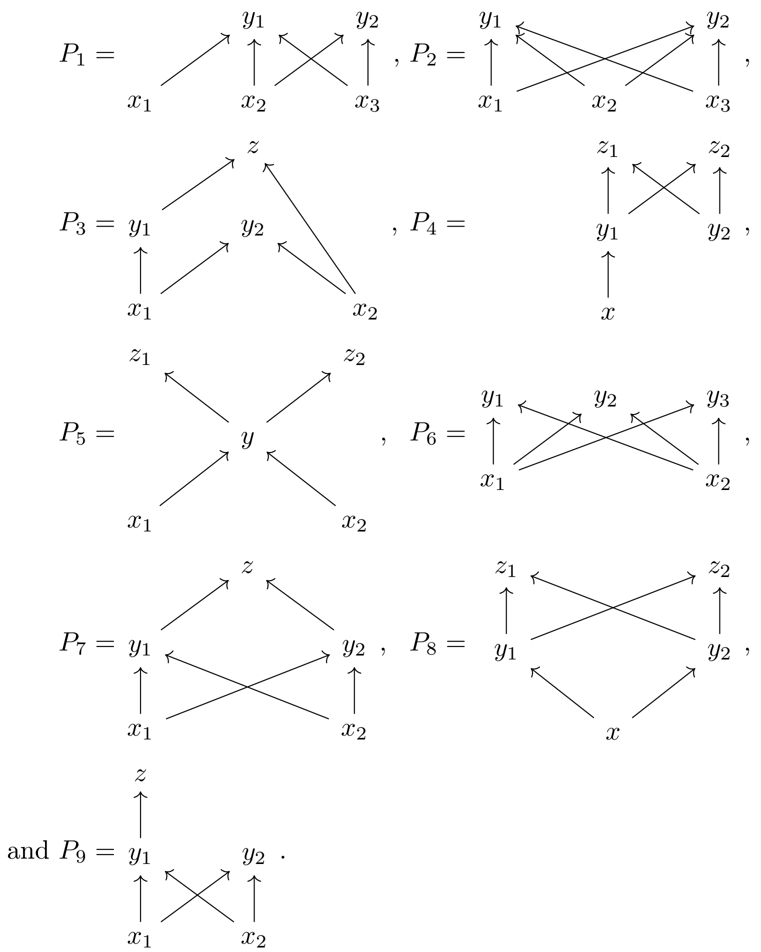
$$\begin{array}{ccc} \tau_1 \operatorname{Sd}^2 \partial\Delta^1 & \longrightarrow & [0] \\ \big\downarrow{\scriptstyle h} & & \big\downarrow{\scriptstyle \alpha} \\ \tau_1 \operatorname{Sd}^2 \Delta^1 & \longrightarrow & P_2 \end{array},$$

where h is the usual boundary inclusion. Since h is a cofibration, so is α. Hence by Lemma 4.0.2, P_2 is cofibrant. Note that α is one of the inclusion $0 \mapsto x_k$, and since P_2 is symmetric, the other inclusion has to be a cofibration as well.

4.2.2 Posets with Five Elements

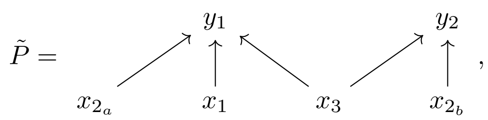
In this section, we will show that every poset with five or less elements is cofibrant, and that every inclusion of a minimum into one of those is a cofibration. As before, we only have to consider posets that are connected. According to the *Chapel Hill Poset Atlas*[1], there are up to isomorphism 44 connected posets with five elements. 25 of those are semilattices. Of the remaining 19, nine can be constructed via simple pushouts in a similar fashion to P_1 in the previous proof, i. e. by gluing a category with two elements and a single non-identity morphism between those two to a cofibrant poset with four elements. Of the remaining ten, one is $\tau_1 \operatorname{Sd}^2 \Delta^1$. To see that the inclusion of $\{\{0,1\}\}$ is a cofibration, we have to construct it by gluing two copies of the poset $\mathcal{E}$ from the previous section together at one of their respective local minima.

There are nine posets left that have to be considered separately. Those are



Lemma 4.2.4. *The poset P_1 is cofibrant, and every inclusion of a minimum into P_1 is a cofibration*

Proof. Let



[1] `http://www.unc.edu/~rap/Posets/index.html`

and

$$\begin{aligned} h\colon \tau_1 \operatorname{Sd}^2 \partial\Delta^1 &\to \tilde{P}, \\ \{\{0\}\} &\mapsto x_{2_a}, \\ \{\{1\}\} &\mapsto x_{2_b}. \end{aligned}$$

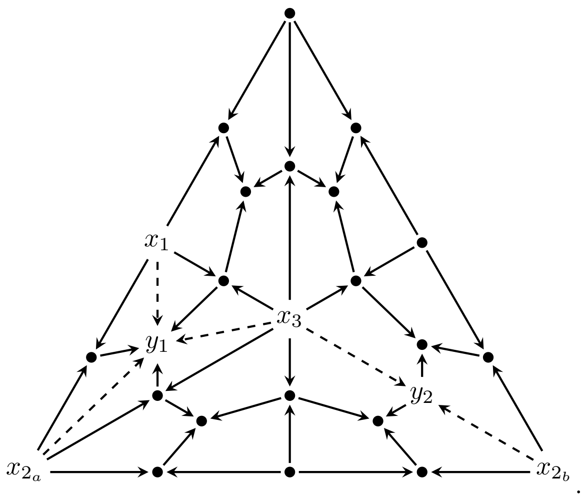
Then P_1 is given by the pushout diagram

$$\begin{array}{ccc} \tau_1 \operatorname{Sd}^2 \partial\Delta^1 & \longrightarrow & [0] \\ \Big\downarrow{\scriptstyle h} & & \Big\downarrow{\scriptstyle i_{x_2}} \\ \tilde{P} & \longrightarrow & P_1 \end{array},$$

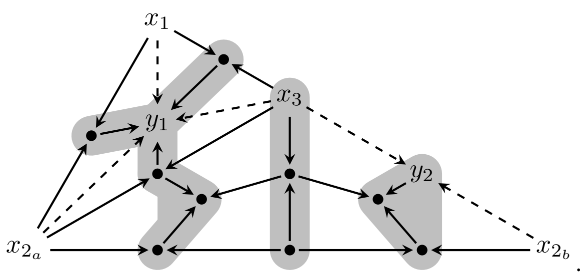
where $i_{x_2}(0) = x_2$. We have to show that h is a cofibration. For that purpose, let

$$\begin{aligned} m\colon \partial\Delta^1 &\to \Delta^2, \\ 0 &\mapsto 0, \\ 1 &\mapsto 2 \end{aligned}$$

in **sSet**. Since m is a monomorphism, $\tau_1 \operatorname{Sd}^2 m$ is a cofibration. We will construct h as a retract of $\tau_1 \operatorname{Sd}^2 m$. Consider the embedding of $\tilde{P}$ into $\tau_1 \operatorname{Sd}^2 \Delta^2$ given as follows:



Folding this along the axis between x_1 and x_{2_b}, we obtain

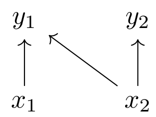


From here, it is easy to see that we can obtain h as a retract of $\tau_1 \operatorname{Sd}^2 m$ as indicated in the diagram above. Hence i_{x_2} is a cofibration and by symmetry, so is $i_{x_3}\colon [0] \to P_1$, given by $i_{x_3}(0) = x_3$. Thus by Lemma 4.0.2, P_1 is cofibrant.

To show that the inclusion

$$\begin{aligned} i_{x_1}\colon [0] &\to P_1, \\ 0 &\mapsto x_1 \end{aligned}$$

is a cofibration, we need to construct P_1 differently. Consider the poset Q given by

$Q =$  ,

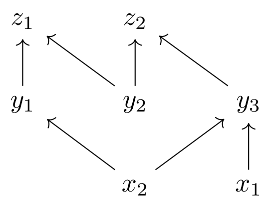
and let

$$\begin{aligned} f\colon \tau_1 \operatorname{Sd}^2 \partial\Delta^1 &\to Q, \\ \{\{0\}\} &\mapsto y_1, \\ \{\{1\}\} &\mapsto y_2. \end{aligned}$$

Taking the pushout

$$\begin{array}{ccc} \tau_1 \operatorname{Sd}^2 \partial\Delta^1 & \xrightarrow{f} & Q \\ \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle \alpha} \\ \tau_1 \operatorname{Sd}^2 \Delta^1 & \longrightarrow & \tilde{P} \end{array},$$

where h is the usual boundary inclusion, we obtain the poset $\tilde{P}$ given by

$\tilde{P} =$ 

The map $\iota_{x_1}\colon [0] \to Q$, given by $\iota_{x_1}(0) = x_1$ is a cofibration by Theorem 4.2.3, and since h is a cofibration, so is α and in particular the composition $\alpha \circ \iota_{x_1}$. We will construct i_{x_1} as a retract of $\alpha \circ \iota_{x_1}$. For that purpose, let

$$\begin{aligned} i\colon P_1 &\to \tilde{P} \\ x_1 &\mapsto x_1, & x_3 &\mapsto y_2, \\ x_2 &\mapsto x_2, & y_k &\mapsto z_k \end{aligned}$$

and

$$\begin{aligned} p\colon \tilde{P} &\to P_1 \\ x_1 &\mapsto x_1, & y_1 &\mapsto y_1, & z_1 &\mapsto y_1, \\ x_2 &\mapsto x_2, & y_2 &\mapsto x_3, & z_2 &\mapsto y_2, \\ & & y_3 &\mapsto y_2. \end{aligned}$$

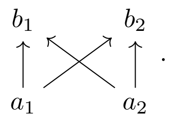
It is easy to see that $p \circ i = \mathrm{id}$ and that this gives i_{x_1} as a retract of $\alpha \circ \iota_{x_1}$.

Lemma 4.2.5. *The poset P_2 is cofibrant, and every inclusion of a minimum into P_2 is a cofibration.*

Proof. Let

$$\begin{aligned} m\colon \partial\Delta^1 &\to \Delta^1, \\ 0 &\mapsto 0, \\ 1 &\mapsto 1 \end{aligned}$$

in **sSet**. Since m is a monomorphism, $\tau_1 \operatorname{Sd}^2 m$ is a cofibration. Let Q be the poset

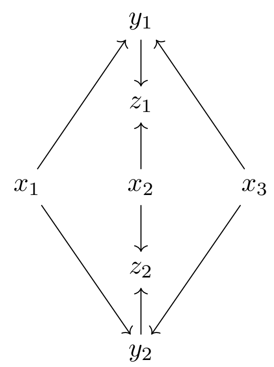


Consider the pushout diagram

$$\begin{array}{ccc} \tau_1 \operatorname{Sd}^2 \partial\Delta^1 & \xrightarrow{i_b} & Q \\ \big\downarrow{\scriptstyle \tau_1 \operatorname{Sd}^2 m} & & \big\downarrow{\scriptstyle \alpha} \\ \tau_1 \operatorname{Sd}^2 \Delta^1 & \xrightarrow{\beta} & \tilde{P} \end{array},$$

where i_b is given by $i_b(\{\{k\}\}) = b_k$. The poset $\tilde{P}$ is given by

$\tilde{P} =$



and the map α by

$$\begin{aligned} \alpha\colon Q &\to \tilde{P}, \\ a_1 &\mapsto x_1, \\ a_2 &\mapsto x_3, \\ b_k &\mapsto y_k \end{aligned}$$

Since $\tau_1 \operatorname{Sd}^2 m$ is a cofibration, so is α and in particular the compositions $\alpha \circ \iota_{a_k}$, where

$$\begin{aligned} \iota_{a_k}\colon [0] &\to Q, \\ 0 &\mapsto a_k \end{aligned}$$

for $k = 1, 3$. Let

$$\begin{aligned} i_{x_k}\colon [0] &\to P_2, \\ 0 &\mapsto x_k, \end{aligned}$$

where $k = 1, 2, 3$. We will construct i_{x_1} as a retract of ι_{b_1} via the retract diagram

$$\begin{array}{ccccc} [0] & \longrightarrow & [0] & \longrightarrow & [0] \\ \downarrow{\scriptstyle i_{x_1}} & & \downarrow{\scriptstyle \iota_{b_1}} & & \downarrow{\scriptstyle i_{x_k}} \\ P_2 & \xrightarrow{i} & \tilde{P} & \xrightarrow{p} & P_2 \end{array},$$

where i and p are given as

$$\begin{aligned} i\colon P_2 &\to \tilde{P}, \\ x_k &\mapsto x_k, \\ y_k &\mapsto y_k, \end{aligned}$$

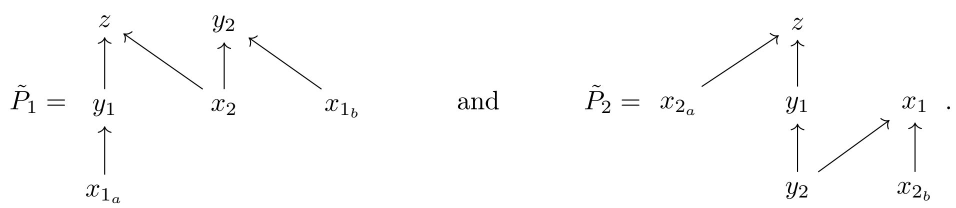
and

$$\begin{aligned} p\colon \tilde{P} &\to P_2, \\ x_k &\mapsto x_k, \\ y_k &\mapsto y_k, \\ z_k &\mapsto y_k. \end{aligned}$$

It is easy to see that $p \circ i = \mathrm{id}$. Hence i_{x_1} is a cofibration and by symmetry, so are i_{x_2} and i_{x_3}.

Lemma 4.2.6. *The poset P_3 is cofibrant, and every inclusion of a minimum into P_3 is a cofibration.*

Proof. We will give two constructions of P_3. One for each of the inclusions of x_1 and x_2 respectively. Let



Similar to the proof of P_1 being cofibrant, we construct cofibrant maps

$$\begin{aligned} h_k\colon \tau_1 \operatorname{Sd}^2 \Delta^1 &\to \tilde{P}_k, \\ \{\{0\}\} &\mapsto x_{k_a}, \\ \{\{1\}\} &\mapsto x_{k_b} \end{aligned}$$

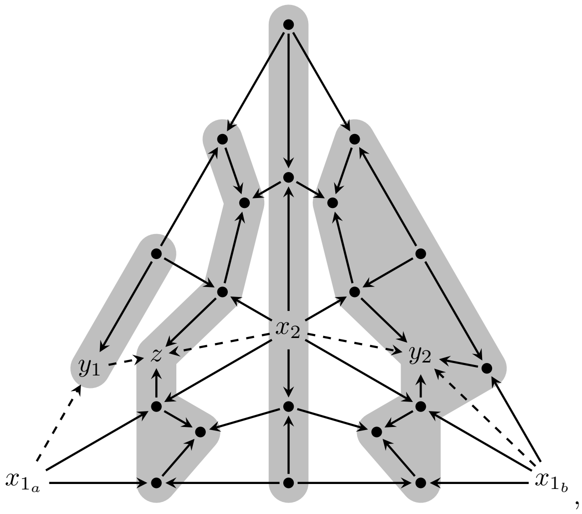
as retracts of an inclusion $\tau_1 \operatorname{Sd}^2 m\colon \tau_1 \operatorname{Sd}^2 \partial\Delta^1 \to \tau_1 \operatorname{Sd}^2 \Delta^2$, and then obtain P_3 via pushouts

$$\begin{array}{ccc} \tau_1 \operatorname{Sd}^2 \partial\Delta^1 & \xrightarrow{f} & [0] \\ \downarrow{\scriptstyle h_k} & & \downarrow{\scriptstyle \alpha} \\ \tilde{P}_k & \xrightarrow{\beta} & P_3 \end{array}.$$

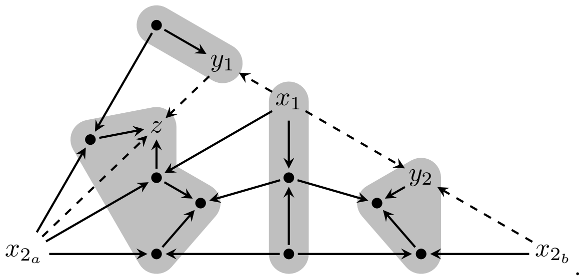
Let

$$m\colon \partial\Delta^1 \to \Delta^2,$$
$$0 \mapsto 0,$$
$$1 \mapsto 2.$$

We obtain h_1 as a retract of $\tau_1 \operatorname{Sd}^2 m$ as indicated in the diagram below



and h_2 as a retract of $\tau_1 \operatorname{Sd}^2 m$ as follows:



Note that we skipped the step where we apply the folding from the proof of Theorem 4.2.4. Since m is a monomorphism in **sSet**, $\tau_1 \operatorname{Sd}^2 m$ is a cofibration and thus, so is h_k for $k = 1, 2$.

Lemma 4.2.7. *The poset P_4 is cofibrant, and every inclusion of a minimum into P_4 is a cofibration.*

Proof. Again, we have to construct P_4 twice, once to show that the inclusion

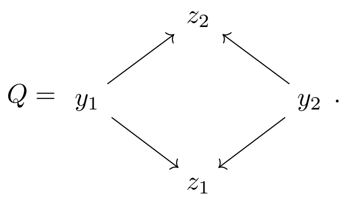
$$i_x\colon [0] \to P_4,$$
$$0 \mapsto x$$

is a cofibration, and once to show that

$$i_{y_2}\colon [0] \to P_4,$$
$$0 \mapsto y_2$$

is a cofibration.

We will start with i_x. Let $\mathcal{D} = x \to y$, and Q be the poset given by

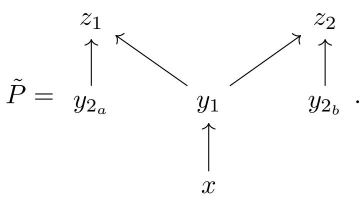


We obtain P_4 via the pushout

$$\begin{array}{ccc} [0] & \xrightarrow{\iota_y} & \mathcal{D} \\ \downarrow{\scriptstyle \iota_{y_1}} & & \downarrow{\scriptstyle \alpha} \\ Q & \xrightarrow{\beta} & P_4 \end{array},$$

where $\iota_y(0) = y$ and $\iota_{y_1}(0) = y_1$. Since ι_{y_1} is a cofibration by Theorem 4.2.3, so is α and hence in particular the composition $i_x = \alpha \circ \iota_x$ where $\iota_x\colon [0] \to \mathcal{D}$ is given by $\iota_x(0) = x$.

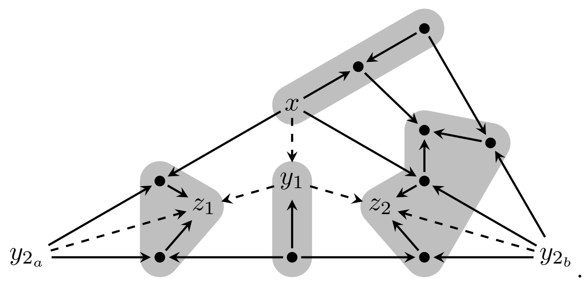
To show that i_{y_2} is a cofibration, let $\tilde{P}$ be the poset



We will use the same procedure as before, i. e. construct the two–point embedding

$$\begin{aligned} h\colon \tau_1 \operatorname{Sd}^2 \Delta^1 &\to \tilde{P}, \\ \{\{0\}\} &\mapsto y_{2_a}, \\ \{\{1\}\} &\mapsto y_{2_b} \end{aligned}$$

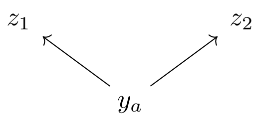
as a retract of a two–point embedding into the folded $\tau_1 \operatorname{Sd}^2 \Delta^2$ as indicated by the following diagram:

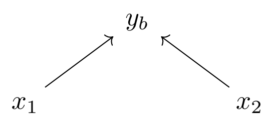


Then apply Lemma 4.2.1 to glue $\tilde{P}$ together at y_{2_a} and y_{2_b}.

Lemma 4.2.8. *The poset P_5 is cofibrant, and every inclusion of a minimum into P_5 is a cofibration.*

Proof. Let

$Q_1 =$ 

and $Q_2 =$ 

and

$$\iota_{x_k}\colon [0] \to Q_2,$$
$$0 \mapsto x_k$$

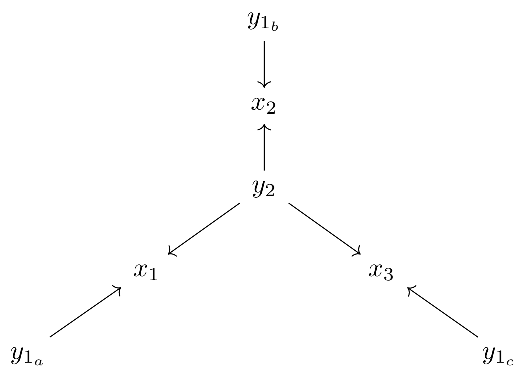
for $k = 1, 2$. Then we can construct P_5 via the pushout

$$\begin{array}{ccc} [0] & \xrightarrow{\iota_{y_b}} & Q_2 \\ \downarrow{\scriptstyle \iota_{y_a}} & & \downarrow{\scriptstyle \alpha} \\ Q_1 & \longrightarrow & P_5 \end{array},$$

where ι_{y_a} and ι_{y_b} are given by $\iota_{y_a}(0) = y_a$ and $\iota_{y_b}(0) = y_b$. Since ι_{y_a} is a cofibration by Corollary 4.1.6, so is α and thus in particular the compositions $\alpha \circ i_{x_k}$ for $k = 1, 2$. By Lemma 4.0.2, P_5 is cofibrant and the minima inclusions are cofibrations.

Lemma 4.2.9. *The poset P_6 is cofibrant, and every inclusion of a minimum into P_6 is a cofibration*

Proof. Let Q be the poset

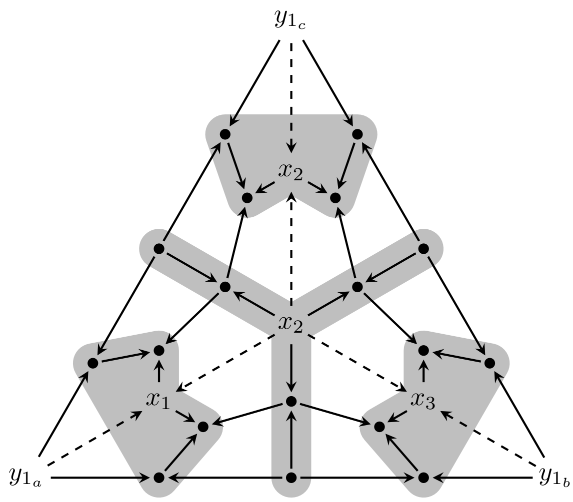
$Q =$ 

,

and let $h\colon \coprod_{i=1}^{3}[0] \to Q$ be the inclusion with image $\{y_{1_a}, y_{1_b}, y_{1_c}\}$. We obtain h as a retract of

$$m\colon N(\coprod_{i=1}^{3})[0] \to \tau_1 \operatorname{Sd}^2 \Delta^2,$$
$$0_k \mapsto \{\{k\}\}$$

as indicated by the following diagram:



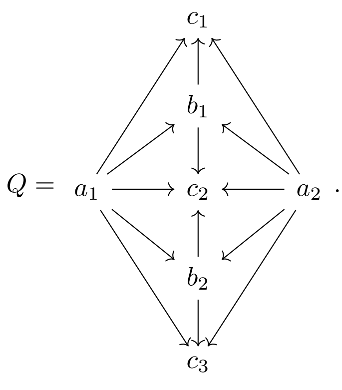
We can construct P_6 via the pushout

$$\begin{array}{ccc} \coprod_{i=1}^{3}[0] & \longrightarrow & [0] \\ \big\downarrow h & & \big\downarrow \alpha \\ Q & \longrightarrow & P_6 \end{array} .$$

Hence by Lemma 4.2.1, P_6 is cofibrant and the inclusion of y_1 into P_6 is a cofibration and by symmetry, so is the inclusion of y_2.

Theorem 4.2.10. *The poset P_7 is cofibrant, and every inclusion of a minimum into P_7 is a cofibration.*

Proof. Let Q be the poset



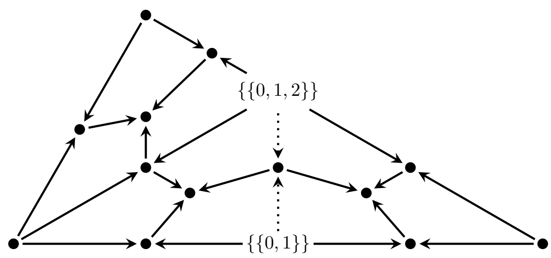
We will obtain a cofibrant embedding

$$\begin{aligned} i_{x_1} \colon [0] &\to P_7, \\ 0 &\mapsto x_1 \end{aligned}$$

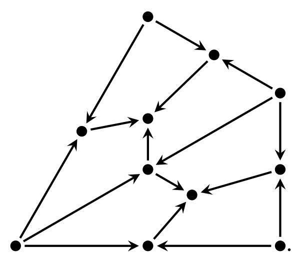
as a retract of the inclusion

$$\begin{aligned}\iota_{a_1}\colon [0] &\to Q,\\ 0 &\mapsto a_1,\end{aligned}$$

and ι_{a_1} as a pushout of a retract of the boundary inclusion $\tau_1 \operatorname{Sd}^2 \partial\Delta^2 \to \tau_1 \operatorname{Sd}^2 \Delta^2$. We will start with the embedding of the folded boundary into the folded $\tau_1 \operatorname{Sd}^2 \Delta^2$:



Folding again at the axis between $\{\{0,1\}\}$ and $\{\{0,1,2\}\}$ as indicated in the diagram above, we obtain



We will denote the resulting inclusion by $m\colon B \to S$. We obtain the poset Q by taking the pushout

$$\begin{array}{ccc} B & \longrightarrow & [0] \\ \downarrow{\scriptstyle m} & & \downarrow{\scriptstyle \iota_{a_1}} \\ S & \longrightarrow & Q \end{array},$$

and since m is a cofibration, so is ι_{a_1}. Let

$$\begin{aligned} i\colon P_7 &\to Q,\\ x_k &\mapsto a_k,\\ y_k &\mapsto b_k,\\ z &\mapsto c_2 \end{aligned}$$

and

$$\begin{aligned} p\colon Q &\to P_7, & & \\ a_k &\mapsto x_k, & c_1 &\mapsto y_1,\\ b_k &\mapsto y_k, & c_2 &\mapsto z,\\ & & c_3 &\mapsto y_2. \end{aligned}$$

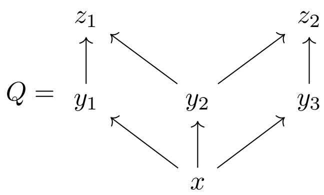
Then $p \circ i = \mathrm{id}$ and the diagram

$$\begin{array}{ccccc} [0] & \longrightarrow & [0] & \longrightarrow & [0] \\ \big\downarrow{\scriptstyle i_{x_1}} & & \big\downarrow{\scriptstyle \iota_{a_1}} & & \big\downarrow{\scriptstyle i_{x_1}} \\ P_7 & \xrightarrow{i} & Q & \xrightarrow{p} & P_7 \end{array}$$

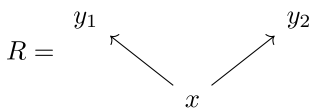
commutes. Thus, i_{x_1} is a cofibration and by symmetry, so is $i_{x_2}\colon [0] \to P_7$, given by $i_{x_2}(0) = x_2$. Applying Lemma 4.0.2 yields that P_7 is cofibrant.

Lemma 4.2.11. *The poset P_8 is cofibrant, and the inclusion of the minimum into P_8 is a cofibration.*

Proof. Let Q and R be the posets given by



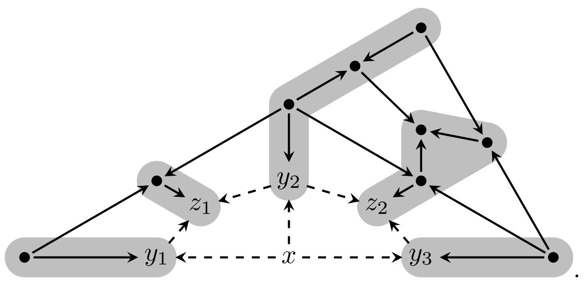
and



and let

$$\begin{aligned} h\colon R &\to Q, \\ y_1 &\mapsto y_1, \\ y_2 &\mapsto y_3, \\ x &\mapsto x. \end{aligned}$$

Consider the embedding $m\colon \Delta^1 \to \Delta^2$ that maps Δ^1 to the 1-simplex between 0 and 1. We obtain h as a retract of $\tau_1 \operatorname{Sd}^2 m$ as indicated by the following diagram (where we skipped the usual folding and the image of $\tau_1 \operatorname{Sd}^2 m$ is located at the bottom of the diagram):



Now let $\mathcal{D} = x \to y$, and

$$\begin{aligned} f\colon R &\to \mathcal{D}, \\ y_k &\mapsto y, \\ x &\mapsto x. \end{aligned}$$

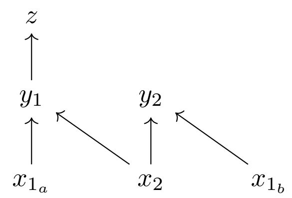
We obtain P_8 via the pushout

$$\begin{array}{ccc} R & \xrightarrow{f} & \mathcal{D} \\ \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle \alpha} \\ Q & \longrightarrow & P_8 \end{array} .$$

Since m is a monomorphism in **sSet**, $\tau_1 \operatorname{Sd}^2 m$ is a cofibration and hence, so is h. Thus α is a cofibration, and since the inclusion $\iota_x \colon [0] \to \mathcal{D}$ given by $\iota_x(0) = x$ is a cofibration, so is the composition $i_x = \alpha \circ \iota_x$, which is the inclusion of the minimum into P_8. Hence by Lemma 4.0.2, P_8 is cofibrant.

Theorem 4.2.12. *The poset P_9 is cofibrant, and every inclusion of a minimum is a cofibration*

Proof. Let

$\tilde{P} =$ 

and

$$\begin{aligned} h \colon \tau_1 \operatorname{Sd}^2 \partial\Delta^1 &\to \tilde{P}, \\ \{\{0\}\} &\mapsto x_{1_a}, \\ \{\{1\}\} &\mapsto x_{1_b}. \end{aligned}$$

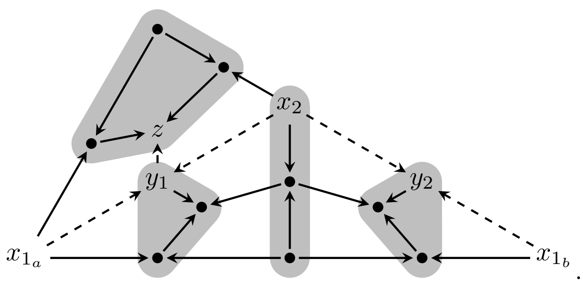
Then P_9 is given by the pushout diagram

$$\begin{array}{ccc} \tau_1 \operatorname{Sd}^2 \partial\Delta^1 & \longrightarrow & [0] \\ \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle i_{x_1}} \\ \tilde{P} & \longrightarrow & P_9 \end{array} ,$$

where i_{x_1} is the inclusion given by $i_{x_1}(0) = x_1$. Let furthermore

$$\begin{aligned} h \colon \partial\Delta^1 &\to \Delta^2, \\ 0 &\mapsto 0, \\ 1 &\mapsto 1 \end{aligned}$$

in **sSet**. We obtain h as a retract of $\tau_1 \operatorname{Sd}^2 m$ as indicated by the following diagram:



Hence h is a cofibration and thus, so is i_{x_1} (and by symmetry, so is $i_{x_2} \colon [0] \to P_9$, given by $i_{x_2}(0) = x_2$). So by Lemma 4.0.2, P_9 is cofibrant.

Combining all of our previous results, we have proved the following theorem:

Theorem 4.2.13. *Every poset with five or less elements is cofibrant, and the respective inclusions of minima are cofibrations.*

Appendix A

Appendix

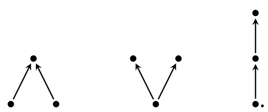
The posets on the following pages are mostly computer generated with the data pulled from *Chapel Hill Poset Atlas*. We denote the posets by their generating graphs. That means that we will only use anonymous nodes, instead of named objects and will only draw the minimal generating set of arrows, i. e. those arrows, that are indecomposable.

Moreover, when arranging the objects, we put our focus on avoiding crossing arrows.

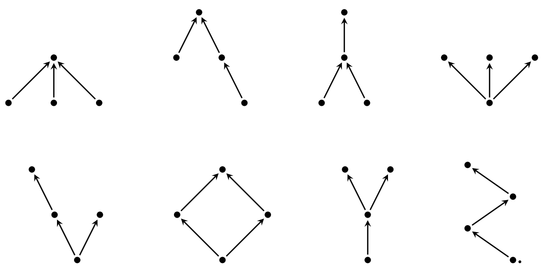
A.1 Posets with Four or Less Elements

There is exactly one poset with one element, and one connected poset with two.

There are three connected posets with three elements, and all three are semilattices:



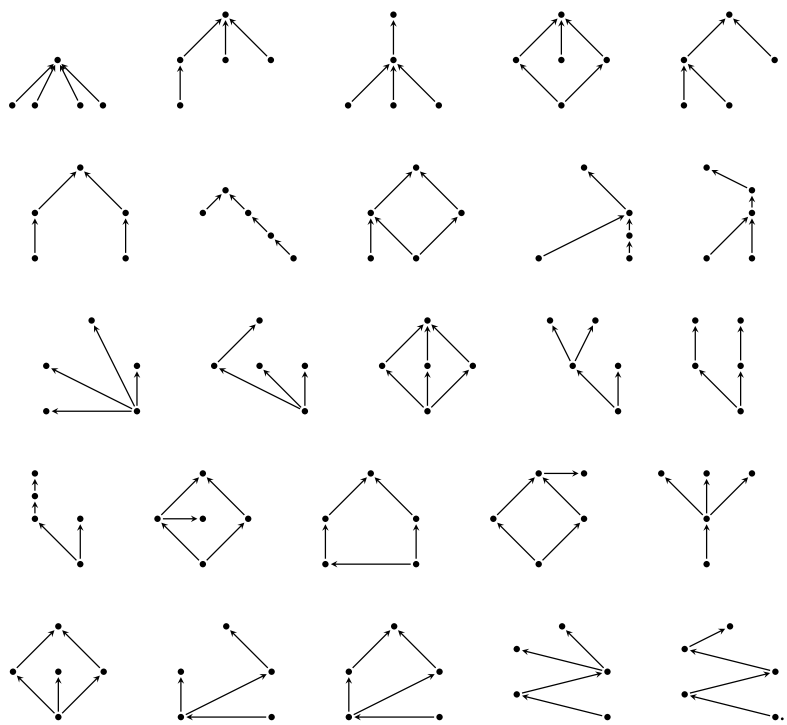
There are ten connected posets with four elements, eight of those are semilattices, namely:



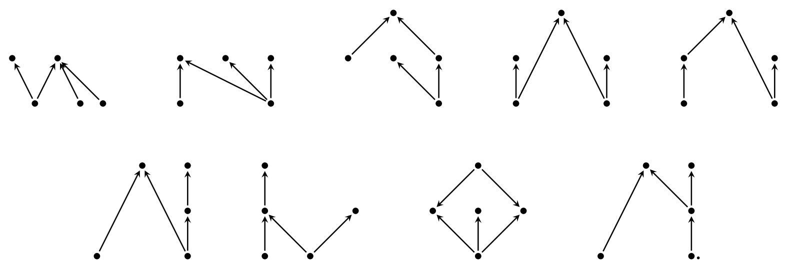
and the remaining two are the posets P_1 and P_2 from Theorem 4.2.3:

A.2 Posets with Five Elements

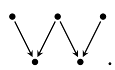
There are 44 connected posets with five elements, 25 of those are semilattices and listed below:



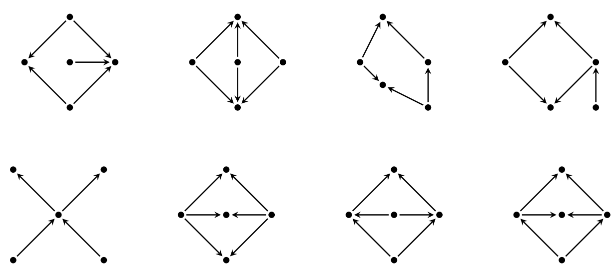
Of the remaining 19, nine can be constructed by gluing the connected poset with two elements along its minimum to a cofibrant poset with four elements. Those are

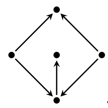


Of the remaining ten, one is $\tau_1 \operatorname{Sd}^2 \Delta^1$:



The remaining nine are the posets P_1 to P_9 from Section 4.2.2.



 .

Bibliography

[AHS90] Jiří Adámek, Horst Herrlich, and George E. Strecker. "Abstract and concrete categories". In: Pure and Applied Mathematics (New York) (1990), pp. xiv+482.

[AM14] Dimitri Ara and Georges Maltsiniotis. "Vers une structure de catégorie de modèles à la Thomason sur la catégorie des n-catégories strictes". In: *Adv. Math.* 259 (2014), pp. 557–654. DOI: 10.1016/j.aim.2014.03.013.

[AR94] Jiří Adámek and Jiří Rosický. *Locally presentable and accessible categories.* Vol. 189. London Mathematical Society Lecture Note Series. Cambridge University Press, Cambridge, 1994, pp. xiv+316. DOI: 10.1017/CBO9780511600579.

[BBP99] Marek A. Bednarczyk, Andrzej M. Borzyszkowski, and Wieslaw Pawlowski. "Generalized congruences—epimorphisms in $\mathscr{C}at$". In: *Theory Appl. Categ.* 5 (1999), No. 11, 266–280 (electronic).

[Ber42] Paul Bernays. "A system of axiomatic set theory". In: *J. Symbolic Logic* 7 (1942), pp. 133–145.

[BH99] Martin R. Bridson and André Haefliger. *Metric spaces of non-positive curvature.* Vol. 319. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Springer-Verlag, Berlin, 1999, pp. xxii+643. DOI: 10.1007/978-3-662-12494-9.

[Boh+15] Anna Marie Bohmann et al. "A model structure on $GCat$". In: *Women in topology: collaborations in homotopy theory.* Vol. 641. Contemp. Math. Amer. Math. Soc., Providence, RI, 2015, pp. 123–134.

[Bor94a] Francis Borceux. *Handbook of categorical algebra. 1.* Vol. 50. Encyclopedia of Mathematics and its Applications. Cambridge University Press, Cambridge, 1994, pp. xvi+345.

[Bor94b] Francis Borceux. *Handbook of categorical algebra. 2.* Vol. 51. Encyclopedia of Mathematics and its Applications. Cambridge University Press, Cambridge, 1994, pp. xviii+443.

[BP16] Roman Bruckner and Christoph Pegel. *Cofibrant objects in the Thomason Model Structure.* 2016. arXiv: 1603.05448.

[Bru15] Roman Bruckner. *A Model Structure On The Category Of Small Acyclic Categories.* 2015. arXiv: 1508.00992.

[Cis99] Denis-Charles Cisinski. "La classe des morphismes de Dwyer n'est pas stable par retractes". In: *Cahiers Topologie Géom. Différentielle Catég.* 40.3 (1999), pp. 227–231.

[Col06] Michael Cole. "Many homotopy categories are homotopy categories". In: *Topology Appl.* 153.7 (2006), pp. 1084–1099. DOI: 10.1016/j.topol.2005.02.006.

[Cur71] Edward B. Curtis. "Simplicial homotopy theory". In: *Advances in Math.* 6 (1971), 107–209 (1971).

[EZ50] Samuel Eilenberg and J. A. Zilber. "Semi-simplicial complexes and singular homology". In: *Ann. of Math. (2)* 51 (1950), pp. 499–513.

[FL79] Rudolf Fritsch and Dana May Latch. "Homotopy inverses for nerve". In: *Bull. Amer. Math. Soc. (N.S.)* 1.1 (1979), pp. 258–262. DOI: 10.1090/S0273-0979-1979-14579-8.

[FP10] Thomas M. Fiore and Simona Paoli. "A Thomason model structure on the category of small n-fold categories". In: *Algebr. Geom. Topol.* 10.4 (2010), pp. 1933–2008. DOI: 10.2140/agt.2010.10.1933.

[FPP08] Thomas M. Fiore, Simona Paoli, and Dorette Pronk. "Model structures on the category of small double categories". In: *Algebr. Geom. Topol.* 8.4 (2008), pp. 1855–1959. DOI: 10.2140/agt.2008.8.1855.

[GJ99] P.G. Goerss and J.F. Jardine. *Simplicial Homotopy Theory.* Progress In Mathematics. Birkhäuser Verlag, 1999.

[GZ67] Peter Gabriel and Michel Zisman. *Calculus of Fractions and Homotopy Theory.* Springer–Verlag, 1967.

[Hau06] Emmanuel Haucourt. "Categories of components and loop-free categories". In: *Theory Appl. Categ.* 16 (2006), No. 27, 736–770.

[Hir03] Philip S. Hirschhorn. *Model categories and their localizations.* Vol. 99. Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 2003, pp. xvi+457.

[Hir15] Philip S. Hirschhorn. *The Quillen model category of topological spaces.* 2015. eprint: arXiv:1508.01942.

[Hov99] Mark Hovey. *Model categories.* Vol. 63. Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 1999, pp. xii+209.

[Hur35] W. Hurewicz. *Beiträge zur Topologie der Deformationen: Höherdimensionale Homotopiegruppen.* Bd. 1. N. V. Noord-Hollandsche Uitgevers Maatschappij, 1935.

[Kan57] Daniel M. Kan. "On c. s. s. complexes". In: *Amer. J. Math.* 79 (1957), pp. 449–476.

[Koz08] Dmitry Kozlov. *Combinatorial algebraic topology.* Vol. 21. Algorithms and Computation in Mathematics. Springer, Berlin, 2008, pp. xx+389. DOI: 10.1007/978-3-540-71962-5.

[Lag06] J.L. Lagrange. *Leçons sur le calcul des fonctions.* Courcier, 1806.

[LW69] A.T. Lundell and S. Weingram. *The topology of CW complexes.* The University series in higher mathematics. Van Nostrand Reinhold Co., 1969.

[MO15] Lennart Meier and Viktoriya Ozornova. "Fibrancy of partial model categories". In: *Homology Homotopy Appl.* 17.2 (2015), pp. 53–80. DOI: 10.4310/HHA.2015.v17.n2.a5.

[MSZ16] Peter May, Marc Stephan, and Inna Zakharevich. "The homotopy theory of equivariant posets". In: *ArXiv e-prints* (Jan. 2016). arXiv: 1601.02521 [math.AT].

[Qui67] Daniel G. Quillen. *Homotopical algebra.* Lecture Notes in Mathematics, No. 43. Springer-Verlag, Berlin-New York, 1967, iv+156 pp. (not consecutively paged).

[Rap10] George Raptis. “Homotopy theory of posets”. In: *Homology, Homotopy Appl.* 12.2 (2010), pp. 211–230.

[Ser51] Jean-Pierre Serre. “Homologie singulière des espaces fibrés. Applications”. In: *Ann. of Math. (2)* 54 (1951), pp. 425–505.

[SS70] Lynn A. Steen and J. Arthur Seebach Jr. *Counterexamples in topology.* New York: Holt, Rinehart and Winston, 1970, pp. xiii+210.

[Ste67] N. E. Steenrod. “A convenient category of topological spaces”. In: *Michigan Math. J.* 14 (1967), pp. 133–152.

[Str11] Jeffrey Strom. *Modern classical homotopy theory.* Vol. 127. Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2011, pp. xxii+835. DOI: 10.1090/gsm/127.

[Tho80] R. W. Thomason. “Cat as a closed model category”. In: *Cahiers Topologie Géom. Différentielle* 21.3 (1980), pp. 305–324.

[Whi49] J. H. C. Whitehead. “Combinatorial homotopy. I”. In: *Bull. Amer. Math. Soc.* 55 (1949), pp. 213–245.

[Mac98] Saunders Mac Lane. *Categories for the Working Mathematician.* 2nd. Graduate Texts in Mathematics 5. Springer–Verlag, 1998.

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