ENUMERATION OF PERFECT MATCHINGS IN GRAPHS WITH REFLECTIVE SYMMETRY

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Abstract. A plane graph is called symmetric if it is invariant under the reflection across some straight line. We prove a result that expresses the number of perfect matchings of a large class of symmetric graphs in terms of the product of the number of matchings of two subgraphs. When the graph is also centrally symmetric, the two subgraphs are isomorphic and we obtain a counterpart of Jockusch's squarishness theorem. As applications of our result, we enumerate the perfect matchings of several families of graphs and we obtain new solutions for the enumeration of two of the ten symmetry classes of plane partitions (namely, transposed complementary and cyclically symmetric, transposed complementary) contained in a given box. Finally, we consider symmetry classes of perfect matchings of the Aztec diamond graph and we solve the previously open problem of enumerating the matchings that are invariant under a rotation by 90 degrees.

0. Introduction

The starting point of this paper is a result [**18**, Theorem 1] concerning domino tilings of the Aztec diamond compatible with certain barriers. This result has also been generalized and proved bijectively by Propp [**17**]. We present (see Lemma 1.1) a further generalization, which allows us to prove a basic factorization theorem for the number of perfect matchings of plane bipartite graphs with a certain type of symmetry.

As a direct consequence, we obtain a counterpart of Jockusch's squarishness theorem [**8**, Theorem 1]. We then use the factorization theorem to enumerate the perfect matchings of several families of graphs that either generalize or are concerned with the Aztec diamond. Furthermore, we obtain new solutions for the enumeration of two of the ten symmetry classes of plane partitions contained in a given box.

Motivated by the example of plane partitions, in the last section we consider the enumeration of perfect matchings of the Aztec diamond graph that are invariant under certain symmetries. There are a total of five enumerative problems that arise in this way. Two of them have been previously considered (one of which corresponds to matchings invariant

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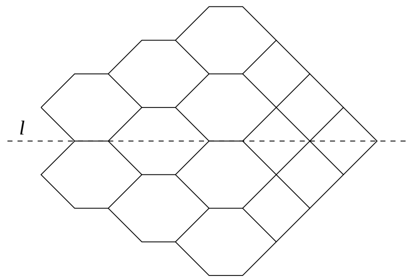

FIGURE 1.1

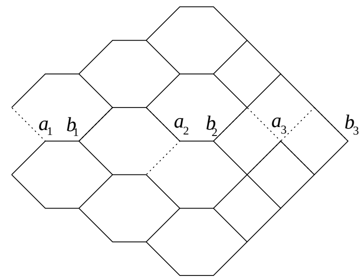

FIGURE 1.2

under the trivial group). We present a solution for a previously open case and a new proof for the previously solved non-trivial case.

1. A Factorization Theorem

A *perfect matching* of a graph is a collection of vertex-disjoint edges that are collectively incident to all vertices. We will often refer to a perfect matching simply as a matching.

Let G be a plane graph. We say that G is *symmetric* if it is invariant under the reflection across some straight line. Figure 1.1 shows an example of a symmetric graph. Clearly, a symmetric graph has no perfect matching unless the axis of symmetry contains an even number of vertices (otherwise, the total number of vertices is odd); we will assume this throughout the paper.

A *weighted* symmetric graph is a symmetric graph equipped with a weight function on the edges that is constant on the orbits of the reflection. The *width* of a symmetric graph G, denoted $\mathrm{w}(G)$, is defined to be half the number of vertices of G lying on the symmetry axis.

Let G be a weighted symmetric graph with symmetry axis l, which we consider to be horizontal. Let $a_1, b_1, a_2, b_2, \dots, a_{\mathrm{w}(G)}, b_{\mathrm{w}(G)}$ be the vertices lying on l, as they occur from left to right. A *reduced* subgraph of G is a graph obtained from G by deleting at each vertex a_i either all incident edges above l (we refer to this operation for short as "cutting above a_i") or all incident edges below l ("cutting below l," for short). Figure 1.2 shows a reduced subgraph of the graph presented in Figure 1.1 (the deleted edges of the original graph are represented by dotted lines).

The weight of a matching μ is defined to be the product of the weights of the edges contained in μ. The matching generating function of a weighted graph G, denoted $M(G)$, is the sum of the weights of all matchings of G. The matching generating function is clearly multiplicative with respect to disjoint unions of graphs. We will henceforth assume that all graphs under consideration are connected.

LEMMA 1.1. *All $2^{\mathrm{w}(G)}$ reduced subgraphs of a weighted symmetric graph G have the same matching generating function.*

Proof. It is enough to prove the statement of the Lemma for two reduced subgraphs that differ only around a single vertex a_i. Let G_1 and G_2 be two reduced subgraphs obtained by identical cutting operations except that for the former we made a cut above a_i, while for the latter we cut below a_i (for some $i \in \{1, 2, \dots, \mathrm{w}(G)\}$). Let μ be a matching of G_1 and let μ' be the matching of G obtained from μ by reflection across l. Then $\nu = \mu \cup \mu'$ (where

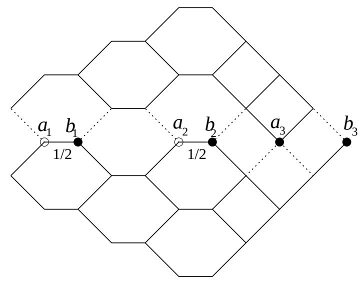

FIGURE 1.3

the union is a multi-set union) is a 2-factor of G that is symmetric about l. Therefore, ν is a disjoint union of even-length cycles. Consider the cycle C containing a_i, and let C' be the reflection of C across l. Since ν is symmetric about l, C' is a cycle of ν. Note that $C' \neq C$ would imply that C is disjoint from C', contradicting $a_i \in C \cap C'$. Therefore C' coincides with C and C is symmetric with respect to l. Thus, since all vertices of C have degree two, C has only one vertex on l besides a_i. We claim that this vertex is one of $b_1, b_2, \ldots, b_{\mathrm{w}(G)}$.

Otherwise, the set of vertices encircled by C has an odd number of elements on l. Since this set is symmetric about l, it follows that it has an odd number of elements, contradicting the fact that the 2-factor ν is a disjoint union of even-length cycles.

Define μ'' to be the matching of G obtained from μ by replacing $\mu \cap C$ by $\mu' \cap C$. Then clearly μ'' is a matching of G_2 and the correspondence $\mu \mapsto \mu''$ is a weight-preserving involution between the matchings of G_1 and those of G_2. $\square$

Let G be a weighted symmetric graph that is also bipartite. Suppose that the set of vertices lying on l is a cut set (i.e., removing these vertices disconnects the graph). In such a case we say that l *separates* G. Let us color the vertices in the two bipartition classes black and white. For definiteness, choose the leftmost vertex on the symmetry axis l to be white. We define two subgraphs G^+ and G^- as follows. Perform cutting operations above all white a_i's and black b_i's and below all black a_i's and white b_i's. Note that this procedure yields cuts of the same kind at the endpoints of each edge lying on l. Reduce the weight of each such edge by half; leave all other weights unchanged. Since l separates G, the graph produced by the above procedure is disconnected into one component lying above l, which we denote by G^+, and one below l, denoted by G^-. Figure 1.3 illustrates this procedure for the graph pictured in Figure 1.1 (the edges whose weight has been reduced by half are marked by 1/2).

THEOREM 1.2 (FACTORIZATION THEOREM). *Let G be a bipartite weighted symmetric graph separated by its symmetry axis. Then*

$$M(G) = 2^{\mathrm{w}(G)} M(G^+) M(G^-).$$

In our proof we make use of the following preliminary result. Let v be a vertex of the weighted graph G and let $X \cup Y \cup Z$ be a partition of the edges incident to v. For an edge e incident to v, let $\varphi(e)$ be the other endpoint of e. Construct a new graph G' from G by changing the neighborhood of v as follows (see Figure 1.4):

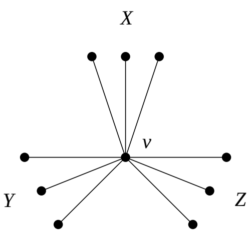

FIGURE 1.4(a)

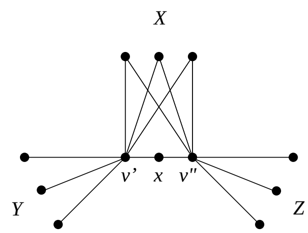

FIGURE 1.4(b)

(i) remove v and the incident edges
(ii) insert three new vertices x, v' and v''
(iii) connect x to v' and v'' by edges of weight 1
(iv) for $e \in X$, connect v' and v'' to $\varphi(e)$ by edges of weight $(1/2)\,\mathrm{wt}(e)$
(v) for $e \in Y$, connect v' to $\varphi(e)$ by an edge of weight $\mathrm{wt}(e)$
(vi) for $e \in Z$, connect v'' to $\varphi(e)$ by an edge of weight $\mathrm{wt}(e)$.

LEMMA 1.3. *The graphs G and G' have the same matching generating function.*

Proof. The matchings of G can be partitioned into $|X|+|Y|+|Z|$ classes according to the possible ways of matching v. Note that since $\{x, v'\}$ and $\{x, v''\}$ are the only edges incident to x, in any matching of G' there is exactly one edge with one endpoint in $\{v', v''\}$ and the other in $\{\varphi(e) : e \in X \cup Y \cup Z\}$. This determines a partition of the set of matchings of G' into $|X|+|Y|+|Z|$ classes.

Let μ' be a matching of G' and let e be the unique edge incident to v such that $\{v', \varphi(e)\}$ or $\{v'', \varphi(e)\}$ is contained in μ'. Define μ to be the matching of G obtained from μ' by removing the edges incident to $\{v', v'', x\}$ and including edge e. Then the mapping $\mu' \mapsto \mu$ is onto. Moreover, the weight of the preimage of μ is equal to $\mathrm{wt}(\mu)$, for all matchings μ of G (the weight of a set of matchings is defined to be the sum of the weights of the elements).

Indeed, this mapping is one to one and weight-preserving on the classes corresponding to $e \in X \cup Y$. On the classes corresponding to $e \in X$, the mapping is two to one. However, by the choice of weights is step (iv) of the construction of G' we have that the sum of the weights of the two preimages equals the weight of the image. This proves the statement of the Lemma. $\square$

Proof of Theorem 1.2. First, we show that we can reduce to the case when the vertices of G lying on l form an independent set. To see this, we construct a new graph $\tilde{G}$ as follows. Cut the graph G along l so that we obtain two copies of each vertex lying on l, and two copies of each edge contained in l. Assign half the weight of the original edge to each copy; keep the original weights for all other edges. Finally, insert a new vertex between the two copies of each vertex formerly on l, and join it to both copies by an edge weighted 1. It is clear that we can carry out this construction such that the resulting graph is symmetric (this is illustrated in Figure 1.5 in the case of the graph G shown in Figure 1.1). Denote it by $\tilde{G}$, and let $\tilde{l}$ be its symmetry axis. We claim that G and $\tilde{G}$ have the same matching generating function.

Indeed, let $G^{(1)}$ be the graph obtained from G by performing the operation involved in Lemma 1.3 around a_1, with X, Y and Z being taken to be the set of edges incident

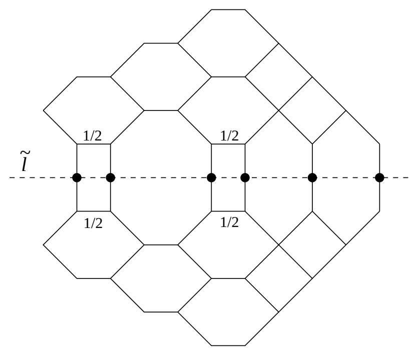

FIGURE 1.5

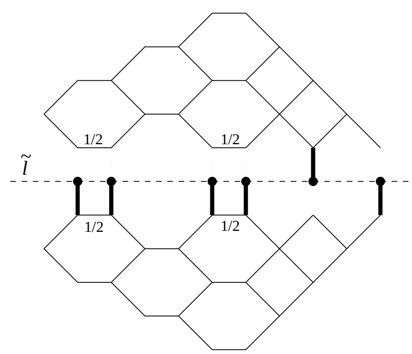

FIGURE 1.6

to a_1 that lie on, above and below l, respectively. Let $G^{(2)}$ be the graph obtained from $G^{(1)}$ by modifying the neighborhood of b_1, with a similar partition of the incident edges. Continue in this manner, obtaining graphs $G^{(i)}$, $i = 3, 4, \ldots, 2\,\mathrm{w}(G)$. By Lemma 1.3, $M(G^{(i-1)}) = M(G^{(i)})$ for all $i \geq 2$. Since the last graph in this sequence is isomorphic to $\tilde{G}$, this proves our claim.

Note that each vertex of $\tilde{G}^{+}$ lying on $\tilde{l}$ has degree 1, hence any matching of $\tilde{G}^{+}$ must contain the edge incident to this vertex (see Figure 1.6). Also, by construction, all edges of $\tilde{G}^{+}$ incident to such vertices have weight equal to 1. Therefore, $M(\tilde{G}^{+})$ is equal to the matching generating function of the subgraph of $\tilde{G}^{+}$ obtained by deleting the vertices matched by the forced edges of weight 1. However, this subgraph is isomorphic to G^{-}, and we obtain that $M(\tilde{G}^{+}) = M(G^{-})$. Similarly, we deduce that $M(\tilde{G}^{-}) = M(G^{+})$.

Thus, it is enough to prove the statement of the theorem for a graph G whose vertices lying on l form an independent set. According to Lemma 1.1, it is enough to show that $M(G^{+})M(G^{-})$ is the matching generating function of some (hence any) of the $2^{\mathrm{w}(G)}$ reduced subgraphs of G. We prove this for the reduced graph H obtained by cutting above the white a_i's and below the black a_i's. For this, it suffices to show that every matching of H is also a matching of $G^{+} \cup G^{-}$, i.e., that in every matching μ of H the white b_i's are matched upward and the black b_i's downward. Let x and y be the number of white and black vertices of G lying above l, respectively. Let x_1 and y_1 (resp., x_2 and y_2)

be the number of white and black a_i's (resp., b_i's). We then clearly have

$$2x + x_1 + x_2 = 2y + y_1 + y_2 \tag{1.1}$$

and

$$x_1 + y_1 = \mathrm{w}(G) = x_2 + y_2. \tag{1.2}$$

Let α and β be the number of white and black b_i's matched upward in μ, respectively. We need to show that $\alpha = x_2$ and $\beta = 0$.

Consider the set of edges of μ that lie above l. Among their endpoints, $x + \alpha$ are white and $y + y_1 + \beta$ are black, so $x + \alpha = y + y_1 + \beta$. We therefore obtain

$$x_2 \geq \alpha \geq \alpha - \beta = y - x + y_1. \tag{1.3}$$

However, by relations (1.2) and (1.1) we have

$$x_2 - y_1 = \frac{1}{2}((x_2 - y_1) + (x_1 - y_2)) = \frac{1}{2}(2y - 2x) = y - x,$$

so we actually have equality in (1.3). This implies $\alpha = x_2$ and $\beta = 0$, as desired. □

2. Perfect Matchings and Perfect Squares

Let G be a symmetric bipartite graph separated by the symmetry axis l. We say that G is *Klein-symmetric* if G is in addition invariant under the rotation ρ by 180°. The graph is said to be *Klein-even-symmetric* if there is an even number of edges in a path connecting a vertex to its image under ρ.

Theorem 2.1. *For any Klein-even-symmetric graph G we have*

$$M(G) = 2^{\mathrm{w}(G)} M(G^+)^2.$$

Proof. Since G is invariant under ρ, we must have $\rho(a_i) = b_{\mathrm{w}(G)-i+1}$, for $i = 1, \dots, \mathrm{w}(G)$. Therefore, a_i and $b_{\mathrm{w}(G)-i+1}$ have the same color. Given our algorithm for constructing G^+ and G^-, this implies that G^- is the image of G^+ under ρ. The factorization theorem thus yields the stated result. □

Remark 2.2. The above theorem gives a combinatorial explanation for the fact, first proved by Montroll using linear algebra (see [**11**, Problem 4.29] for an exposition), that the number of perfect matchings of the $2n \times 2n$ grid graph is either a perfect square or twice a perfect square; thus Theorem 2.1 answers the last question of [**8**, p.114]. In the equivalent language of domino tilings, we obtain that the number of tilings of the $2n \times 2n$ chessboard equals 2^n times the square of the number of tilings of the portion lying below the zig-zag line in Figure 2.1.

Remark 2.3. Let us consider the tiling of the plane by unit squares with vertices having integer coordinates. Define a *planar region* to be the union of finitely many such unit squares. In case the symmetric graph G is the dual of a planar region R and the line of symmetry has slope ± 1, the factorization theorem can be stated more directly in terms

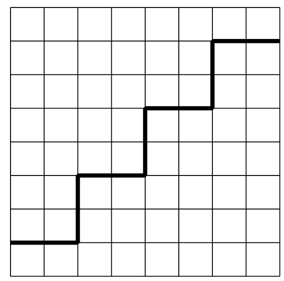

FIGURE 2.1

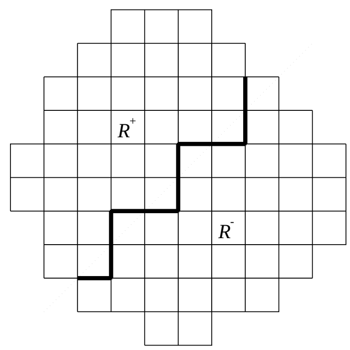

FIGURE 2.2

of domino tilings. Indeed, let P be a zig-zag lattice path that leaves the unit squares that intersect the line of symmetry alternately on one side of P and the other (see Figure 2.2).

Let R^+ and R^- be the subregions of R determined by P. Then if $D(R)$ denotes the number of domino tilings of R, we obtain that

$$D(R) = 2^{\mathrm{w}(R)} D(R^+) D(R^-), \tag{2.1}$$

where $\mathrm{w}(R)$ is half the number of unit squares of R intersected by the line of symmetry.

3. The Holey Aztec Diamond

Consider a $(2n+1) \times (2n+1)$ chessboard with black corners. The graph whose vertices are the white squares and whose edges connect precisely those pairs of white squares that are diagonally adjacent is called the *Aztec diamond* of order n. (Technically speaking, this is the dual of the region dubbed the Aztec diamond in [**6**].) In [**6**] it is shown that it has $2^{n(n+1)/2}$ perfect matchings (see [**3**] for an alternate proof).

The *holey* Aztec diamond is obtained by removing the vertices of the central 4-cycle of this graph. Jockusch conjectured formulas for the number of perfect matchings of the holey Aztec diamond, formulas that have been recently proved [**17**]. We give a short proof in the case when the order of the diamond is congruent to 2 or 3 modulo 4.

THEOREM 3.1. *The holey Aztec diamonds of order* $4n+2$ *and* $4n+3$ *have* 2^{8n^2+10n} *and* $2^{8n^2+14n+3}$ *perfect matchings, respectively.*

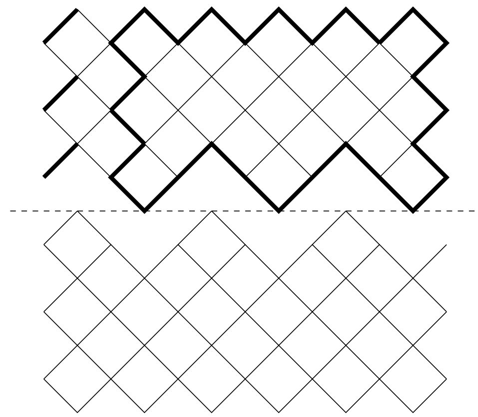

FIGURE 3.1

Proof. Let G be the (plain) Aztec diamond of order $4n+2$. Apply Theorem 2.1 to G with respect to the horizontal symmetry axis l (see Figure 3.1). The vertices lying on l are in the same bipartition class, thus by our convention they are all white. Therefore, the cuts producing the two subgraphs involved in the factorization theorem occur alternately above and below the vertices lying on l, as viewed from left to right.

Note that any matching of G^+ has $2n+1$ forced edges along the western part of its boundary. Let H be the graph obtained from G^+ by deleting the $4n+2$ endpoints of these forced edges. Then $M(G) = 2^{2n+1}M(H)^2$, and since $M(G) = 2^{(2n+1)(4n+3)}$ we obtain that $M(H) = 2^{(2n+1)^2}$. Note that H has a vertical symmetry axis; denote it by l'.

Let us now delete the four central vertices of G and denote the remaining graph by G_1. Let x be rightmost of the two deleted vertices lying on l (see Figure 3.2). Note that as in the case of G^+, any matching of G_1^+ has $2n+1$ forced edges. Let H_1 be the graph obtained from G_1^+ by deleting the endpoints of these edges. Then H_1 is obtained from H by deleting vertex x and its northwestern neighbor. Also, the fact that the order of the diamond is congruent to 2 modulo 4 implies that x lies on l'. A similar statement is true for orders congruent to $3 \pmod 4$, but not for the other two cases.

The matchings of H can be partitioned in two classes: those in which x is matched to the northwest and those in which it is matched to the northeast. Note that the matchings of the first class are in bijection with the matchings of H_1. Similarly, the matchings of the second class are in bijection with the matchings of the subgraph of H obtained by removing vertex x and its northeastern neighbor. However, this subgraph is just the reflection of H_1 across l', hence the two classes have the same size. Therefore, $2M(H_1) = M(H)$, so $M(H_1) = 2^{4n^2+4n}$. Thus, by Theorem 2.1 we obtain

$$M(G_1) = 2^{2n}M({G_1}^+)^2 = 2^{2n}M(H_1)^2 = 2^{8n^2+10n}.$$

The case when the order is congruent to $3 \pmod 4$ is treated similarly. The key fact is that we can deduce again the relation $2M(H_1) = M(H)$ (see the observation at the end of the third paragraph of the proof). $\square$

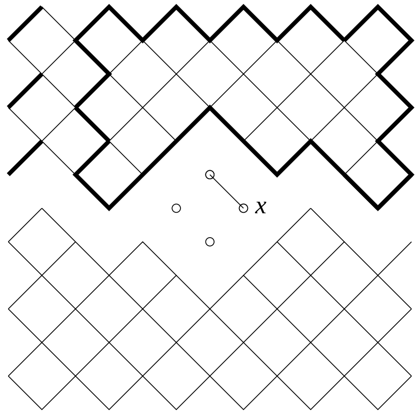

FIGURE 3.2

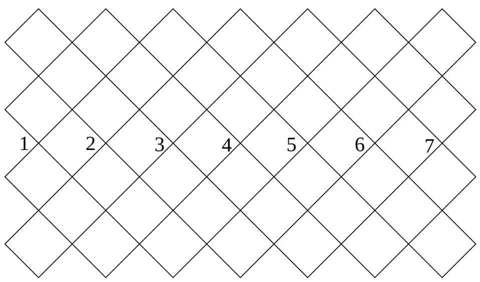

FIGURE 4.1(a)

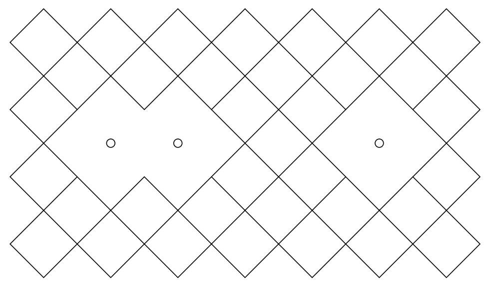

FIGURE 4.1(b)

4. Holey Aztec Rectangles

A natural generalization of the Aztec diamonds is the following. Consider a $(2m+1)\times(2n+1)$ *rectangular* chessboard and suppose the corners are black. Define the *Aztec rectangle* $R(m,n)$ to be the graph whose vertices are the white squares and whose edges connect precisely those pairs of white squares that are diagonally adjacent.

However, the graphs $R(m,n)$ have no perfect matchings unless $m=n$. Indeed, the sizes of the two bipartition classes differ by $|m-n|$. One is therefore naturally lead to consider subgraphs of $R(m,n)$ obtained by deleting $|m-n|$ vertices from the larger bipartition class.

Let m be even and suppose $m \le n$. The vertices of $R(m,n)$ lying on the horizontal symmetry axis l are then contained in the larger bipartition class. Label them consecutively by 1 through n (see Figure 4.1(a)). For any subset S of $[n] := \{1,\ldots,n\}$ of size $n-m$ define $R(m,n;S)$ to be the graph obtained from $R(m,n)$ by deleting the vertices with labels in S; an example is shown in Figure 4.1(b).

Note that for odd m, the vertices lying on l are contained in the smaller bipartition class, and therefore the graphs obtained by the above procedure have no perfect matchings.

$$
\begin{array}{ccccccc}
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & -1 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & -1 & 1 & 0 \\
1 & 0 & -1 & 1 & 0 & -1 & 1
\end{array}
\qquad
\begin{array}{ccccccc}
 & & & 4 & & & \\
 & & 2 & & 5 & & \\
 & 2 & & 3 & & 6 & \\
1 & & 2 & & 4 & & 7
\end{array}
$$

$$
\begin{array}{ccccccc}
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 1 & 0 \\
1 & 1 & 0 & 1 & 0 & 0 & 1
\end{array}
$$

FIGURE 4.2

THEOREM 4.1. *For even m, the graph $R(m,n;S)$ has*

$$\frac{2^{m(m+4)/4}}{(0!\,1!\cdots(m/2-1)!)^2}\prod_{1\le i<j\le m/2}(t_{2j-1}-t_{2i-1})(t_{2j}-t_{2i})$$

perfect matchings, where $[n]\setminus S=\{t_1,\dots,t_m\}$, $t_1<\cdots<t_m$.

Before giving the proof we need some preliminary results. Let $m\le n$ and let A be an $m\times n$ matrix. We say that A is an *alternating sign matrix* if

(i) all entries are 1, 0 or -1
(ii) every row sum equals 1
(iii) in reading every row from left to right and every column from top to bottom the nonzero entries alternate in sign, starting with a $+1$.

(Note that this is a generalization of the standard notion of an alternating sign matrix introduced in [**13**], which assumes the matrix is square.)

Let $ASM(m,n;S)$ be the set of $m\times n$ alternating sign matrices whose column sums are zero precisely for the column indices belonging to S (note that $|S|=n-m$). We denote by $N_+(A)$ and $N_-(A)$ the number of 1's and -1's in A, respectively.

A *monotone triangle* of size n is an n-rowed triangular array of non-negative integers such that

($T1$) all rows are strictly increasing
($T2$) the numbers are non-decreasing in the polar directions $+60^\circ$ and -60°.

Let us weight every monotone triangle T by $2^{s(T)}$, where $s(T)$ is the number of elements of T that are strictly between their neighbors in the row below, and let $f(t_1,\dots,t_n)$ be the generating function of monotone triangles with bottom row $t_1,\dots,t_n$. Then by [**13**, Theorem 2] we have

$$f(t_1,\dots,t_n)=\frac{2^{n(n-1)/2}}{0!\,1!\cdots(n-1)!}\prod_{1\le i<j\le n}(t_j-t_i). \tag{4.1}$$

Proof of Theorem 4.1. In [**13**] a bijection is given between $ASM(n,n;\emptyset)$ and the set of monotone triangles with bottom row $1,2,\dots,n$; the -1 entries of a matrix correspond to

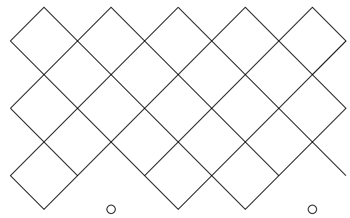

FIGURE 4.3

entries of the monotone triangle that are strictly between their neighbors in the row below. This construction generalizes immediately to give a bijection between $ASM(m,n;S)$ and the set of monotone triangles with bottom row $t_1,\dots,t_m$, where $\{t_1,\dots,t_m\}=[n]\setminus S$ (see Figure 4.2 for an illustration of this bijection; the second matrix is obtained by considering partial column sums in the alternating sign matrix, while the triangular array records the position of the 1's in the rows of the second matrix). We obtain therefore

$$\sum_{A\in ASM(m,n;S)} 2^{N_-(A)} = f(t_1,\dots,t_m). \tag{4.2}$$

Suppose $m\leq n$ and label the bottom n vertices of $R(m,n)$ consecutively by 1 through n. Let S be an $(n-m)$-element subset of $[n]$ and denote by $\bar{R}(m,n;S)$ the graph obtained from $R(m,n)$ by deleting the vertices with labels in S (see Figure 4.3 for an example).

Shade the faces of $R(m,n)$ in a chessboard fashion so that the edges on the boundary belong to shaded faces. By a *cell* we mean a 4-cycle of $R(m,n)$ with shaded interior. Let μ be a matching of $\bar{R}(m,n;S)$. Write in each cell one of the numbers $1,0$ or -1, corresponding to the cases when the cell contains $2,1$ or 0 edges of μ. Let A be the $m\times n$ matrix generated in this fashion.

Although the graph $\bar{R}(m,n;S)$ is not "cellular" in the sense of [**3**], the proof of Lemma 2.1 of [**3**] shows that $A\in ASM(m,n;S)$. Furthermore, the proof of Lemma 2.2 of [**3**] shows that there are exactly $2^{N_+(A)}$ matchings giving rise to the alternating sign matrix A. We therefore obtain that

$$M(\bar{R}(m,n;S)) = \sum_{A\in ASM(m,n;S)} 2^{N_+(A)}. \tag{4.3}$$

Therefore, since $N_+(A)-N_-(A)=m$, we have by relations (4.1)–(4.3) that

$$M(\bar{R}(m,n;S)) = \frac{2^{m(m+1)/2}}{0!\,1!\cdots(m-1)!}\prod_{1\leq i<j\leq m}(t_j-t_i), \tag{4.4}$$

where $[n]\setminus S=\{t_1,\dots,t_m\}$, $t_1<\cdots<t_m$.

Apply the factorization theorem to $R(m,n;S)$, with l chosen to be the horizontal symmetry axis. From the definition of the two subgraphs involved in the factorization theorem, it follows that $R(m,n;S)^+$ is isomorphic to $\bar{R}(m/2,n;S\cup\{t_1,t_3,\dots,t_{2n-1}\})$ and $R(m,n;S)^-$ is isomorphic to $\bar{R}(m/2,n;S\cup\{t_2,t_4,\dots,t_{2n}\})$ (see Figure 4.4). The result follows then from relation (4.4). □

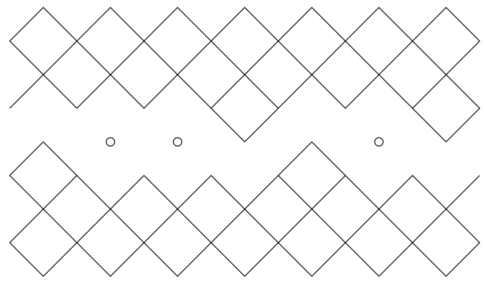

FIGURE 4.4

REMARK 4.2. Okada has obtained [**15**, Theorem 4.4] a deformation of Weyl's denominator formula for the root system of type C_n that yields by specialization a product formula for the number of *centrally symmetric* matchings of $R(2n, 2n+1; \{n+1\})$. It turns out that the methods of [**3**] can be used to prove this as well; details will appear elsewhere.

REMARK 4.3. The number of matchings of $R(2m, n; S)$ can be interpreted as follows. Let μ be a matching of $R(2m, n; S)$. Write in each cell one of the numbers $1, 0$ or -1 according to the instances when the cell contains $2, 1$ or 0 edges of μ, respectively. Denote by A and B the $m \times n$ matrices thus formed above and below l, respectively. Using the ideas of [**3**, Section 2] one can show that A and the matrix obtained by turning B upside down are alternating sign matrices. Moreover, a pair of $m \times n$ alternating sign matrices arises in this way if and only if the two sets of column indices corresponding to columns of sum 1 form a partition of $[n] \setminus S$. Furthermore, any such pair (A, B) corresponds to exactly $2^{N_+(A)} 2^{N_+(B)}$ matchings μ. Therefore, denoting by $c(A)$ the set of column indices of A corresponding to columns of sum 1, we obtain that

$$\begin{aligned} M(R(2m, n; S)) &= \sum_{c(A) \cup c(B) = [n] \setminus S} 2^{N_+(A)} 2^{N_+(B)} \\ &= 2^{2m} \sum_{c(A) \cup c(B) = [n] \setminus S} 2^{N_-(A)} 2^{N_-(B)} \\ &= 2^{2m} \sum_{\substack{|U|=|V|=m \\ U \cup V = [n] \setminus S}} \sum_{c(A)=U} 2^{N_-(A)} \sum_{c(B)=V} 2^{N_-(B)}, \end{aligned} \tag{4.5}$$

where in the summation ranges involving A and B it is assumed that they are $m \times n$ alternating sign matrices. Note that the expression on the right hand side of (4.1) may be written in terms of Schur functions as $2^{n(n-1)/2} s_\lambda(1^n)$, where λ is the partition having parts $t_1 \leq t_2 - 1 \leq \cdots \leq t_n - n + 1$ and $s_\lambda(1^n)$ is the Schur function indexed by λ with n variables specialized to 1 and the rest to 0 (see Example I.3.4 of [**12**]). Therefore, according to (4.2), we can rewrite the right hand side of (4.5) as

$$2^{2m+m(m-1)} \sum s_{\lambda(U)}(1^m) s_{\lambda(V)}(1^m),$$

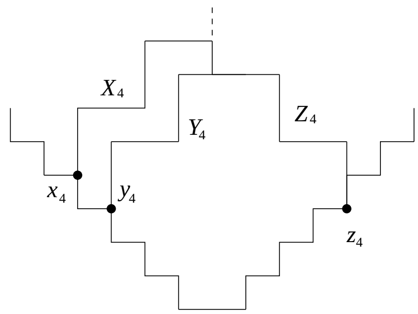

FIGURE 5.1

where for a set $Z = \{z_1 < z_2 < \cdots < z_k\}$, $\lambda(Z)$ denotes the partition with parts $z_1 \le z_2 - 1 \le \cdots \le z_k - k + 1$. Furthermore, it follows from the proof of Theorem 4.1 that we can write the left hand side of (4.5) as

$$2^{3m+m(m-1)} s_{\lambda(\{t_1,t_3,\dots,t_{2m-1}\})}(1^m) s_{\lambda(\{t_2,t_4,\dots,t_{2m}\})}(1^m).$$

(recall that $[n] \setminus S = \{t_1 < \cdots < t_{2m}\}$). Therefore, we obtain the following

COROLLARY 4.4. *Let $T = \{t_1 < t_2 < \cdots < t_{2m}\}$ be a set of positive integers. Then*

$$\sum s_{\lambda(U)}(1^m) s_{\lambda(V)}(1^m) = 2^m s_{\lambda(\{t_1,t_3,\dots,t_{2m-1}\})}(1^m) s_{\lambda(\{t_2,t_4,\dots,t_{2m}\})}(1^m),$$

where the sum ranges over all partitions of T into two classes U and V of equal size.

Let s_λ be the generic (unspecialized) Schur function indexed by the partition λ. Based on the previous corollary and numerical evidence, we propose the following

CONJECTURE 4.5. *Let T and the summation range be as above. Then*

$$\sum s_{\lambda(U)} s_{\lambda(V)} = 2^m s_{\lambda(\{t_1,t_3,\dots,t_{2m-1}\})} s_{\lambda(\{t_2,t_4,\dots,t_{2m}\})}.$$

5. Quasi-Quartered Aztec Diamonds

Consider the infinite planar region R shown in Figure 5.1, having rows consisting of $2, 4, 6, \dots$ unit squares. Let x_n and y_n be the NW and SE corners of the leftmost unit square in the n-th row from the bottom, respectively, and let z_n be the SE corner of the rightmost square of the n-th row from the bottom. Let X_n (resp., Y_n) be the infinite lattice path starting at x_n (resp., y_n) and taking alternately two steps North and two steps East. Let Z_n be the infinite lattice path starting at z_n and taking alternately two steps North and two steps West. The paths X_n and Z_n divide R into four regions, exactly one of which is bounded; denote it by A_n. Define similarly the region B_n, using the paths Y_n and Z_n.

The bounded region determined by Z_n and the translation of X_n one unit downward is an example of what is called a quartered Aztec diamond in [**9**]. We call A_n and B_n *quasi-quartered Aztec diamonds*. Recall that $D(R)$ denotes the number of domino tilings of the planar region R.

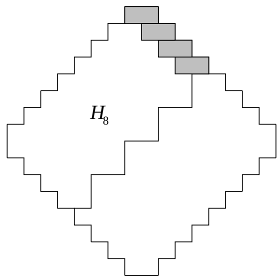

FIGURE 5.2(a)

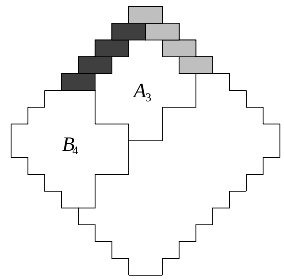

FIGURE 5.2(b)

THEOREM 5.1.
(a) For all n, we have $D(A_n) = 2^{n(n+1)/2}$.
(b) For even n we have $D(B_n) = 2^{n^2/2}$, while for odd n $D(B_n) = 2^{(n^2-1)/2}$.

Proof. Let AD_n be the planar region whose dual is the Aztec diamond graph of order n. Apply the domino-tiling version of the factorization theorem to AD_n (see Remark 2.3). Since $D(AD_n) = 2^{n(n+1)/2}$ and AD_n^+ is isometric to AD_n^-, we obtain that

$$D(AD_n^+) = 2^{n^2/4}, \quad \text{for even } n \tag{5.1}$$

and

$$D(AD_n^+) = 2^{(n^2-1)/4}, \quad \text{for odd } n \tag{5.2}$$

(these formulas also follow from more general formulas given in [**EKLP**]).

Part of the tiling of AD_n^+ along the boundary is forced; let H_n be the portion of AD_n^+ left uncovered after placing the forced dominoes (see Figure 5.2(a)). Note that H_n is symmetric, so we can apply the factorization theorem to H_n. Delete the unit squares covered by forced dominoes from the two resulting regions H_n^+ and H_n^-. It turns out that the remaining regions are either A_i's or B_i's, for suitable indices i.

Indeed, let us assume for example that $n = 4k$, $k \in \mathbf{Z}$ (this case is illustrated in Figure 5.2(b)). Then one of the regions H_{4k}^+ and H_{4k}^- has no forced dominoes and is congruent to B_{2k}, while the portion of the other not covered by forced dominoes is congruent to A_{2k-1}. Since H_{4k} has width k, we obtain

$$D(H_{4k}) = 2^k D(A_{2k-1}) D(B_{2k}). \tag{5.3}$$

Similarly, we deduce that

$$D(H_{4k+1}) = 2^k D(A_{2k}) D(B_{2k}), \tag{5.4}$$

$$D(H_{4k+2}) = 2^{k+1} D(A_{2k}) D(B_{2k+1}), \tag{5.5}$$

and

$$D(H_{4k+3}) = 2^{k+1} D(A_{2k+1}) D(B_{2k+1}). \tag{5.6}$$

From relations (5.3) and (5.4) we deduce that $D(A_{2k})/D(A_{2k-1}) = D(H_{4k+1})/D(H_{4k})$, and since $D(H_n) = D(AD_n^+)$ we obtain by (5.1) and (5.2) that $D(A_{2k})/D(A_{2k-1}) = 2^{2k}$. Similarly, equations (5.5) and (5.6) yield $D(A_{2k+1})/D(A_{2k}) = 2^{2k+1}$. Therefore, we have in fact that $D(A_n) = 2^n D(A_{n-1})$ for all n, and since $D(A_1) = 2$, we obtain part (a) of the theorem. Solving for $D(B_n)$ in equations (5.3) and (5.5) we obtain part (b). □

Remark 5.2. It is remarkable that A_n has exactly as many domino tilings as AD_n, and that the number of tilings of B_n is the square of the number of tilings of AD_n^+.

6. Plane Partitions

A plane partition π is a rectangular array of non-negative integers with non-increasing rows and columns and finitely many nonzero entries. We can also regard π as an order ideal of $\mathbf{N}^3$, i.e., a finite subset of $\mathbf{N}^3$ such that $(i,j,k) \in \pi$ implies $(i',j',k') \in \pi$, whenever $i \geq i'$, $j \geq j'$ and $k \geq k'$.

By permuting the coordinate axes, one obtains an action of S_3 on the set of plane partitions. Let $\pi \mapsto \pi^t$ and $\pi \mapsto \pi^r$ denote the symmetries corresponding to interchanging the x- and y-axes and to cyclically permuting the coordinate axes, respectively. For the set of plane partitions π contained in the box $B(a,b,c) := \{(i,j,k) \in \mathbf{N}^3 : i < a, j < b, k < c\}$, there is an additional symmetry

$$\pi \mapsto \pi^c := \{(i,j,k) \in \mathbf{N}^3 : (a-i-1, b-j-1, c-k-1) \notin \pi\},$$

called complementation.

These three symmetries generate a group isomorphic to the dihedral group of order 12, which has 10 conjugacy classes of subgroups. These lead to 10 enumeration problems: determine the number of plane partitions contained in a given box that are invariant under the action of one of these subgroups. The program of solving these problems was formulated by Stanley [**21**] and has been recently completed (see [**1**], [**10**] and [**22**]). We present a short way of enumerating transposed complementary plane partitions (i.e, plane partitions π with $\pi^t = \pi^c$) contained in a given box. This case was first solved by Proctor [**16**]. We then relate the number of cyclically symmetric ($\pi^r = \pi$), transposed complementary plane partitions to the number of cyclically symmetric plane partitions that fit in a given box, thus obtaining a new proof of the cyclically symmetric, transposed complementary case, first solved by Mills, Robbins and Rumsey [**14**].

Let $TC(a,a,2b)$ be the number of transposed complementary plane partitions contained in $B(a,a,2b)$. Let $P(a,b,c)$ denote the number of plane partitions contained in $B(a,b,c)$.

Theorem 6.1.

$$TC(a,a,2b) = 2^b \prod_{i=0}^{b-1} \frac{P(a+2i, a+2i, 2b-2i)}{P(a+2i+1, a+2i+1, 2b-2i-1)}.$$

Proof. We employ the following idea illustrated in [**5**], [**19**] and [**10**]. Define the $a \times b \times c$ *honeycomb graph*, denoted $H(a,b,c)$, to be the graph obtained by gluing congruent regular hexagons along edges so that their centers form an $a \times b \times c$ hexagonal array (an example is shown in Figure 6.1).

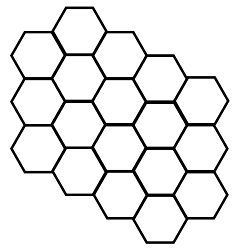

FIGURE 6.1. $H(2,3,4)$.

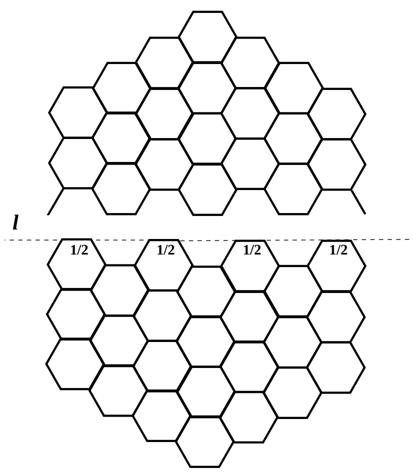

FIGURE 6.2(a)

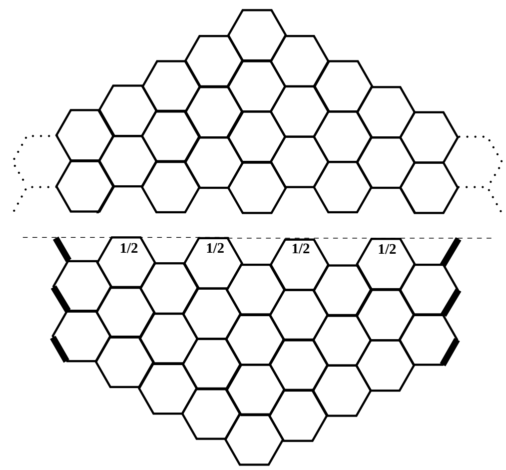

FIGURE 6.2(b)

Then there is a bijection between plane partitions contained in $B(a,b,c)$ and perfect matchings of $H(a,b,c)$. Moreover, the three symmetries of plane partitions translate into symmetries of the honeycomb graph: t becomes reflection in a symmetry axis of the honeycomb not containing any vertex, r rotation by 120° and c rotation by 180°. Therefore, $TC(a,a,2b)$ is the number of matchings of $H(a,a,2b)$ that are invariant under reflection across the symmetry axis l of the honeycomb perpendicular to the sides containing $2b$ hexagons (we consider l to be horizontal; see Figure 6.2(a)).

Note that every such matching has a forced edges on l, so $TC(a,a,2b)$ is actually the number of matchings of the subgraph of $H(a,a,2b)$ induced by the vertices lying above l.

Let us apply the factorization theorem to $H(a,a,2b)$ with respect to l. Note that our algorithm for performing cutting operations yields cuts above all the vertices on l. By the previous paragraph, the number of matchings of $H(a,a,2b)^+$ is just $TC(a,a,2b)$ and we obtain

$$P(a,a,2b) = 2^a TC(a,a,2b) M(H(a,a,2b)^-). \tag{6.1}$$

Let us now also consider the honeycomb $H(a+1,a+1,2b-1)$ (this is illustrated in Figure 6.2(b)). Observe that the graph $H(a+1,a+1,2b-1)^+$ has the same number of matchings as $H(a+2,a+2,2b-2)^+$. Indeed, $H(a+1,a+1,2b-1)^+$ can be embedded in a natural way in $H(a+2,a+2,2b-2)^+$. (See Figure 6.2(b); the edges not contained in the smaller graph are represented by dotted lines.) However, any matching of the latter has $2b-2$ forced edges, and the subgraph induced by the set of vertices not matched by forced edges is isomorphic to $H(a+1,a+1,2b-1)^+$.

Similarly, because of forced edges in the matchings of $H(a+1,a+1,2b-1)^-$, this graph and $H(a,a,2b)^-$ have the same number of matchings. Hence by the factorization theorem we obtain

$$P(a+1,a+1,2b-1) = 2^{a+1}TC(a+2,a+2,2b-2)M(H(a,a,2b)^-). \tag{6.2}$$

Relations (6.1) and (6.2) imply

$$TC(a,a,2b) = 2\cdot TC(a+2,a+2,2b-2)\frac{P(a,a,2b)}{P(a+1,a+1,2b-1)}.$$

By applying this relation repeatedly we obtain the statement of the theorem. □

It is routine to verify that the formula given by the above theorem agrees with the one originally obtained by Proctor in [**16**], which may be written

$$TC(a,a,2b) = \binom{a+b-1}{a-1}\prod_{i=1}^{a-2}\prod_{j=i}^{a-2}\frac{2b+i+j+1}{i+j+1}.$$

Let $CS(a)$ and $CSTC(a)$ be the number of cyclically symmetric and cyclically symmetric, transposed complementary plane partitions contained in $B(a,a,a)$, respectively. Note that $CSTC(a)$ is nonzero only for even a.

Theorem 6.2.

$$2\cdot CSTC(2a+2) = CSTC(2a)\frac{CS(2a+1)}{CS(2a)}.$$

Proof. The honeycomb $H(a) := H(a,a,a)$ has three symmetry axes that contain vertices of the graph; we refer to them as short symmetry axes. Suppose the x-axis is one of them and let the origin be at the center of the honeycomb. Let $G(a)$ be the subgraph of $H(a)$ induced by the vertices in the right half-plane contained in the (closed) 120° angle determined by the other two short symmetry axes. Let S be the set of vertices of $G(a)$ lying on the sides of this angle. Label the vertices in S according to their distance from the origin as follows: label the two vertices closest to the origin by 1, the two next closest by 2, and so on, ending with two vertices labeled a (Figure 6.3(a) illustrates this for $a=4$). Denote by $\tilde{G}(a)$ the graph obtained from $G(a)$ by identifying identically labeled vertices (edges with both endpoints identified are considered to be identified). It is easy to see that the r-invariant matchings of $H(a)$ may be identified with the matchings of $\tilde{G}(a)$.

Clearly, we can embed $\tilde{G}(a)$ symmetrically in the plane (see Figure 6.3(b)). Since $\tilde{G}(a)$ is also bipartite, we can apply to it the factorization theorem. Note that our algorithm for performing the cutting operations yields cuts above all vertices on the symmetry axis.

We now consider separately the cases of even and odd a. Apply the factorization theorem to $\tilde{G}(2a)$ with respect to the x-axis. As illustrated in Figure 6.4(a), the graph

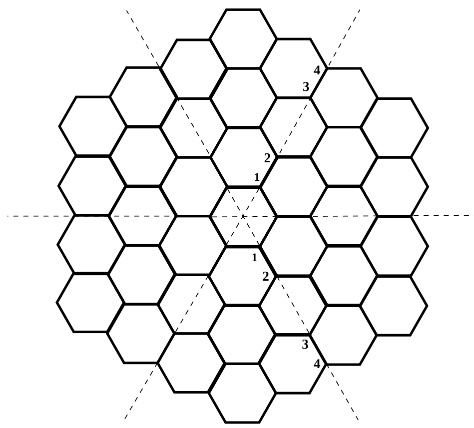

FIGURE 6.3(a)

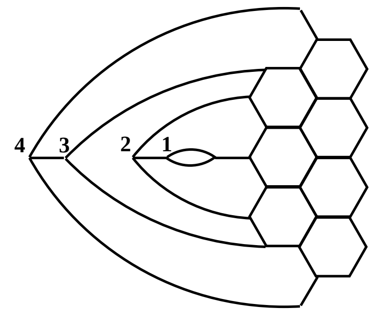

FIGURE 6.3(b)

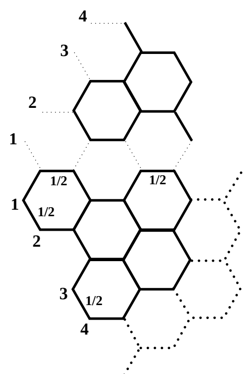

FIGURE 6.4(a)

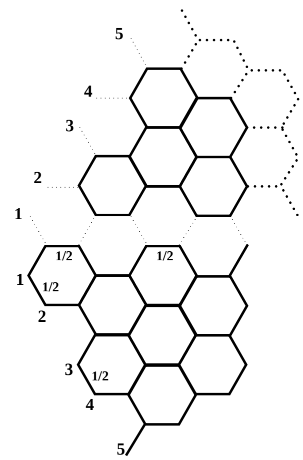

FIGURE 6.4(b)

$\tilde{G}(2a)^+$ is isomorphic to the subgraph $K(2a)$ of $H(2a)$ spanned by the vertices lying strictly inside a $60°$ angle determined by two short symmetry axes of $H(2a)$.

Since the number of matchings of $K(2a)$ is precisely $CSTC(2a)$, we obtain that

$$CS(2a) = 2^{2a} CSTC(2a) M(\tilde{G}(2a)^-). \tag{6.3}$$

Apply now the factorization theorem to $\tilde{G}(2a+1)$ with respect to the x-axis. The graph $\tilde{G}(2a+1)^+$ is in a natural way an induced subgraph of $K(2a+2)$ (see Figure 6.4(b)). Moreover, any matching of $K(2a+2)$ is forced on the vertices not belonging to $\tilde{G}(2a+1)^+$. Therefore, the number of matchings of $\tilde{G}(2a+1)^+$ is just $CSTC(2a+2)$ and we obtain

$$CS(2a+1) = 2^{2a+1} CSTC(2a+2) M(\tilde{G}(2a+1)^-). \tag{6.4}$$

However, due to forced edges in any matching of $\tilde{G}(2a+1)^-$, this graph is seen to have the same number of matchings as $\tilde{G}(2a)^-$ (see Figure 6.4(a)). Then relations (6.3) and (6.4) imply the statement of the theorem. □

The cyclically symmetric case was first solved by Andrews [**2**]. The cyclically symmetric, self-complementary case was first solved by Mills, Robbins and Rumsey [**14**]. Based on the formula for $CS(a)$ and by repeated use of Theorem 6.2 we obtain a new proof of the latter case.

Remark 6.3. It is interesting that Theorem 6.2 is also a consequence of relations (0.2) and (0.3) of [**22**], relations that may be used to solve the totally symmetric case.

7. Symmetries of Matchings of the Aztec Diamond

The honeycomb graphs can be alternatively described as follows. Consider the tiling of the plane by congruent regular hexagons. Let H_1 be one of these hexagonal tiles, and define H_n for $n \geq 2$ to be the union of the set of tiles sharing at least one edge with some tile contained in H_{n-1}. Then H_n is the $n \times n \times n$ honeycomb graph.

Motivated by the simple product formulas that enumerate the symmetry classes of matchings of honeycombs, it is natural to investigate symmetry classes of the graphs arising by applying the inductive construction in the previous paragraph to the tiling of the plane by squares. Remarkably, these graphs are just the Aztec diamonds.

The symmetry group of the Aztec diamond is isomorphic to the dihedral group of order 8. Let r and t be the symmetries corresponding to rotation by 90° and reflection across a diagonal, respectively. Then r and t generate the symmetry group of the Aztec diamond. Since the elements rt and r^3t act as reflections across lines containing independent vertices of the Aztec diamond, there are no rt- or r^3t-invariant matchings. Up to conjugacy, there are five distinct subgroups of $\langle r, t\rangle$ not containing any of these two elements. Imposing the condition that a matching is invariant under the action of one of these subgroups G leads to five different enumeration problems.

Two of these problems have been previously considered: the case when G is the trivial group (treated in [**6**]) and the case $G = \langle r^2\rangle$ (i.e., centrally symmetric matchings), which was solved by Yang [**23**]. The latter case is also implicit in the unpublished work of Robbins [**20**]; see Remark 7.3.

In this section, we solve the case $G = \langle r\rangle$ and we present a new solution for the centrally symmetric case.

The two problems that remain open are the enumeration of matchings invariant under reflection across one or both diagonals (i.e., invariant under $\langle t\rangle$ or $\langle r^2, t\rangle$). For the first few orders of the Aztec diamond the corresponding numbers and their factorizations are shown in Table 7.1. Apparently these numbers do not all factor into small primes, so a simple product formula seems unlikely in these two cases.

Our proofs involve the planar regions known as quartered Aztec diamonds defined in [**9**], which can be described as follows. Let us consider the planar region whose dual is the Aztec diamond graph of order n. This region can be divided into two congruent parts by a zig-zag lattice path that changes direction after every two steps, as shown in Figure 7.1.

By superimposing two such paths that intersect at the center of the region we divide it into four pieces that are called quartered Aztec diamonds. Up to symmetry, there are two

n	$\langle t \rangle -$ invariant	factorization
1	2	2
2	6	$2 \cdot 3$
3	24	$2^3 \cdot 3$
4	132	$2^2 \cdot 3 \cdot 11$
5	1048	$2^3 \cdot 131$
6	11960	$2^3 \cdot 5 \cdot 13 \cdot 23$

n	$\langle r^2, t \rangle -$ invariant	factorization
1	2	2
2	4	2^2
3	10	$2 \cdot 5$
4	28	$2^2 \cdot 7$
5	96	$2^5 \cdot 3$
6	384	$2^7 \cdot 3$
7	1848	$2^3 \cdot 3 \cdot 7 \cdot 11$
8	10432	$2^6 \cdot 163$
9	70560	$2^5 \cdot 3^2 \cdot 5 \cdot 7^2$
10	564224	$2^{10} \cdot 19 \cdot 29$
11	5386080	$2^5 \cdot 3 \cdot 5 \cdot 7^2 \cdot 229$

TABLE 7.1. $\langle t \rangle$-invariant and $\langle r^2, t \rangle$-invariant matchings of the Aztec diamond.

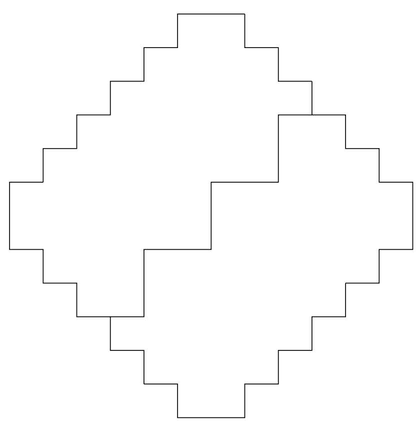

FIGURE 7.1

different ways we can superimpose the two cuts. For one of them, the obtained pattern has fourfold rotational symmetry and the four pieces are congruent (see Figure 7.2(a)); denote the number of their domino tilings by $R(n)$. For the other, the resulting pattern has Klein 4-group reflection symmetry and there are two different kinds of pieces (see Figure 7.2(b)); they are called abutting and non-abutting quartered Aztec diamonds and the numbers of their domino tilings are denoted by $K_a(n)$ and $K_{na}(n)$, respectively.

We are now able to state and prove our results. Denote by $Q(n)$ and $H(n)$ the number of $\langle r \rangle$-invariant and $\langle r^2 \rangle$-invariant matchings of the Aztec diamond of order n, respectively (this notation is motivated by the words "quarter-turn" and "half-turn," which describe

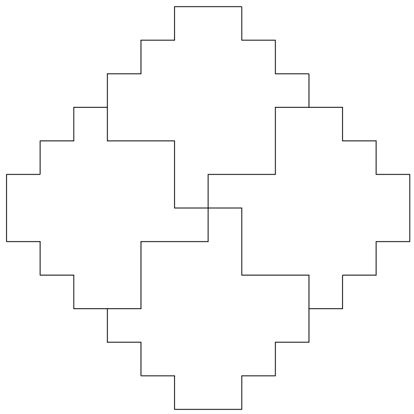

FIGURE 7.2(a)

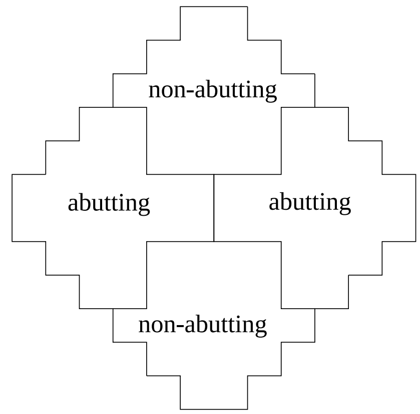

FIGURE 7.2(b)

the corresponding symmetries).

THEOREM 7.1. *For all $n \geq 1$, we have*

$$Q(n) = 2^{\lfloor (n+1)/4 \rfloor} R(n).$$

*Therefore, by Theorem 1 of [**9**] we obtain that $Q(n) = 0$ for n congruent to 1 or 2 (mod 4), and*

$$Q(4k) = 2^k Q(4k-1) = 2^{k(3k+1)/2} \prod_{1 \leq i < j \leq k} \frac{2i+2j-1}{i+j-1}.$$

Proof. Consider the Aztec diamond of order n and suppose it is centered at the origin. Let G be the subgraph induced by the vertices lying in the closed first quadrant. Let S be the set of vertices of G lying on the coordinate axes. Label the two vertices in S that are closest to the origin by 1, label the two next closest by 2 and so on, ending with two vertices labeled $\lfloor (n+1)/2 \rfloor$ (see Figure 7.3). Then the r-invariant matchings of the Aztec diamond are in bijection with the matchings of the graph $\tilde{G}$ obtained from G by identifying all pairs of vertices with the same label. Note that since no r-invariant matching of the Aztec diamond contains the edge with both endpoints labeled 1 in Figure 7.3, we may discard from $\tilde{G}$ the loop arising from this edge.

As illustrated by Figure 7.4(a), we can embed $\tilde{G}$ symmetrically in the plane. By our observation at the beginning of the paper, $\tilde{G}$ has no perfect matching unless there are an even number of vertices on its symmetry axis; i.e., $\lfloor (n+1)/2 \rfloor$ has to be even. Therefore, an Aztec diamond has no r-invariant perfect matching unless its order is congruent to 0 or 3 (mod 4). Since this is precisely the condition for $R(n)$ to be nonzero, the theorem is verified for n congruent to 1 or 2 (mod 4).

Let us therefore assume that $\lfloor (n+1)/2 \rfloor = 2k$, $k \in \mathbf{Z}$. By Lemma 1.1, all 2^k reduced subgraphs of $\tilde{G}$ have the same number of matchings. Consider the reduced graph H obtained by performing cutting operations above all even-labeled vertices of $\tilde{G}$. We claim that in any matching of H the odd-labeled vertices are matched upward.

Indeed, let us consider the collection $\mathcal{C}$ of the 2^k subgraphs of H obtained by making all possible combinations of cuts at the odd-labeled vertices. The set of matchings of H

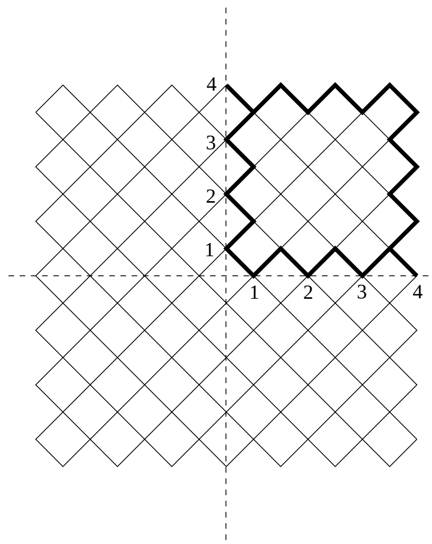

Figure 7.3

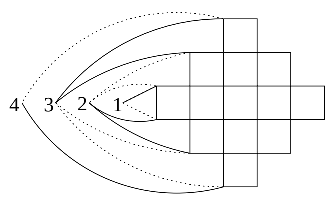

Figure 7.4(a)

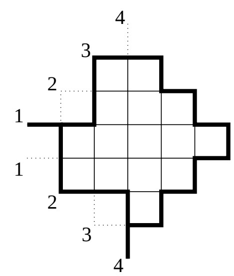

Figure 7.4(b)

is in bijection with the disjoint union of the sets of matchings of the members of $\mathcal{C}$: the instance of a particular odd-labeled vertex being matched upward (downward) corresponds to making a cut below (above) that vertex. However, every member K of $\mathcal{C}$ is a bipartite graph, and it is easy to see that the two bipartition classes have the same size only if K was obtained by cutting below all odd-labeled vertices. This proves our claim.

Therefore, $M(\tilde{G}) = 2^k M(K)$, where K is the subgraph of $\tilde{G}$ obtained by cutting above all even-labeled vertices and below all odd-labeled ones. However, the graph K is easily seen to be isomorphic to the dual of the quartered Aztec diamond of order n, quartered with fourfold rotational symmetry (see Figure 7.4(b)). Also, it is straightforward to check that $k = \lfloor (n+1)/4 \rfloor$ for both $n \equiv 0$ and $n \equiv 3 \pmod 4$. This completes the proof. □

Theorem 7.2. *For all $n \geq 1$, we have*

$$H(n) = 2^{\lfloor (n+1)/2 \rfloor} K_a(n) K_{na}(n).$$

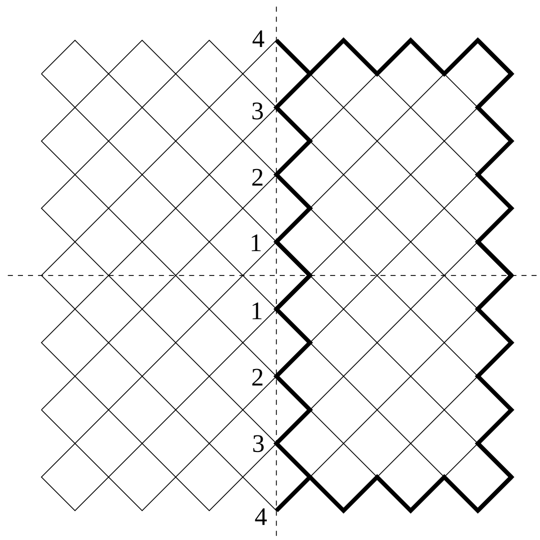

FIGURE 7.5

*Therefore, by Theorem 1 of [**9**] we obtain that*

$$H(4k) = 2^{2k} H(4k-1) = 2^{k(3k-1)} \prod_{1 \le i < j \le k} \frac{2i+2j-3}{i+j-1} \prod_{1 \le i \le j \le k} \frac{2i+2j-1}{i+j-1},$$

and

$$H(4k-2) = 2^{2k-1} H(4k-3) = 2^{3k^2-4k+2} \prod_{1 \le i < j \le k} \frac{2i+2j-3}{i+j-1} \prod_{1 \le i \le j \le k-1} \frac{2i+2j-1}{i+j-1}.$$

Proof. Consider the Aztec diamond of order n and choose the origin to be at its center. Let G be the subgraph induced by the vertices having non-negative x-coordinates. Let S be the set of vertices of G lying on the y-axis. Label the two vertices of S lying closest to the origin by 1, label the two next closest by 2 and so on, ending with the two vertices furthest away from the origin being labeled $\lfloor (n+1)/2 \rfloor$ (see Figure 7.5). Then the centrally symmetric matchings of the Aztec diamond are equinumerous with the matchings of the graph $\tilde{G}$ obtained from G by identifying all pairs of identically labeled vertices (note that there is a pair of vertices of $\tilde{G}$ connected by two parallel edges).

As shown in Figure 7.6(a), we can embed $\tilde{G}$ symmetrically in the plane. Since $\tilde{G}$ is also bipartite, we can apply the factorization theorem. However, as indicated in Figure 7.6(b), the graphs $\tilde{G}^+$ and $\tilde{G}^-$ are isomorphic to the duals of the non-abutting and abutting quartered Aztec diamonds, respectively. This proves the statement of the theorem. □

REMARK 7.3. The symmetry classes of matchings of the Aztec diamond are related to the symmetry classes of square alternating sign matrices considered by Robbins [**20**]. The subgroups G act on this set of matrices (in fact, in this context G can be any subgroup of $\langle r, t \rangle$). Give each alternating sign matrix A the weight $x^{N_-(A/G)}$, where $N_-(A/G)$ is the

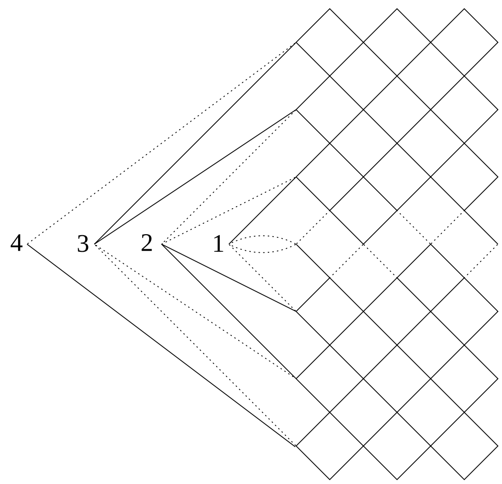

FIGURE 7.6(a)

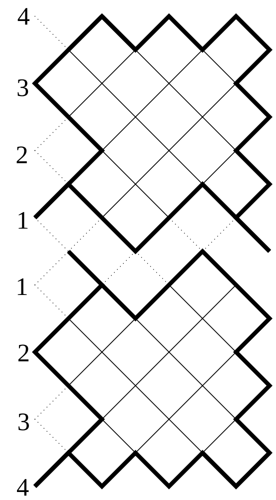

FIGURE 7.6(b)

number of orbits of -1's in the action of G on A. Let $f_{n,G}(x)$ be the generating function of G-invariant alternating sign matrices of order n.

Denote by $a_{n,G}$ the number of G-invariant matchings of the Aztec diamond of order n.

The connection mentioned at the beginning of this Remark lies in a construction (related to the one described in Remark 4.3) that expresses $a_{n,\langle 1\rangle}$ as $f_{n+1,\langle 1\rangle}(2)$ (see Lemmas 2.3 and 2.4 of [**3**]).

There are similar formulas for each symmetry class of matchings. Indeed, using the methods in [**3**], one can prove that for $G \neq \langle r\rangle$ we have

$$a_{n,G} = f_{n+1,G}(2). \tag{7.1}$$

In the special case $G = \langle r\rangle$, the above formula is true for odd n, but for even n a potential problem can occur: for even n, in case the central entry of an $\langle r\rangle$-invariant alternating sign matrix is -1, the corresponding matchings are not $\langle r\rangle$-invariant (the reason for this exception is the fact that the set consisting of two opposite edges of the central 4-cycle in the Aztec diamond is invariant under the action of all subgroups under consideration, except for $\langle r\rangle$). However, one can prove that the central entry of an $\langle r\rangle$-invariant alternating sign matrix of order $2k+1$ depends only on the parity of k: it is always 1 for even k and -1 for odd k.

This shows that the potential problem mentioned above never actually occurs for orders of the form $4k+1$, and hence $f_{4k+1,\langle r\rangle}(2) = a_{4k,\langle r\rangle}$. It also follows that $f_{4k-1,\langle r\rangle}(2) = 2a^*_{4k-2,\langle r\rangle}$, where $a^*_{4k-2,\langle r\rangle}$ denotes the number of r-invariant matchings of the *holey* Aztec diamond of order $4k-2$. However, by [**4**, Theorem 1.4.2] this latter number is equal to $R(4k-1)$. Since there are no r-invariant alternating sign matrices of order of the form $4k-2$ (see e.g. [**20**]), this completes the evaluation of $f_{n,\langle r\rangle}(2)$ for all n.

Using the product formula for $R(n)$ given in [**9**] we obtain the following result, which appears to be new.

THEOREM 7.4. *For all $k \geq 1$ we have*

$$2^{k-1} f_{4k-1,\langle r\rangle}(2) = f_{4k,\langle r\rangle}(2) = 2^{-k} f_{4k+1,\langle r\rangle}(2) = 2^{k(3k-1)/2} \prod_{1\leq i<j\leq k} \frac{2i+2j-1}{i+j-1}.$$

On the other hand, using relation (7.1) we can regard the statement of Theorem 7.2 as providing a formula for $f_{n,\langle r^2\rangle}(2)$. Thus, Theorem 7.2 is equivalent to a result (Theorem 4) stated (but not proved) in [**20**].

Note. Conjecture 4.5 has been proved independently by Glenn Tesler (private communication) and Markus Fulmek [**7**].

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