Tiling Rectangles with Rectangles

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Rectangles are among the most fundamental geometrical shapes. It is therefore only natural that some of the most common types of tilings involve rectangular pieces (or tiles), arranged in various (nonoverlapping) patterns so as to form some specified geometrical shape, often also a rectangle. In this note we will investigate certain basic kinds of tilings of rectangles by rectangles, namely, the so-called simple tilings. At the end of the paper we list for the interested reader a variety of additional references which treat various rectangular tiling problems, including the famous "squared square."

Definition. A tiling of a rectangle R by rectangles (called the **elements** of the tiling) is called **simple** if no connected set of two or more elements form a rectangle strictly inside of R.

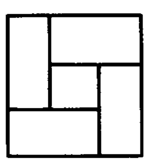


Figure 1

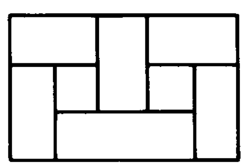


Figure 2

Figure 1 and Figure 2 show simple tilings of rectangles (which will be particularly useful in our later discussion). The fact that even two elements in a simple tiling cannot form a larger rectangle means that two adjacent elements in a simple tiling cannot have as their intersection the (whole) side of each. It is also not hard to see that no generality is lost by assuming (as we will from now on) that all elements have *integral* side lengths. Thus, an $m \times n$ rectangle can have a simple tiling only if $m \geqslant 3$ and $n \geqslant 3$. This shows that Figure 1 actually gives the smallest rectangle which has a simple tiling.

How many elements can be in a simple tiling?

An easy variation of the tiling in Figure 1, shown in Figure 3, (formed by adjusting the side lengths of some of the elements) provides a simple tiling with five elements for any $m \times n$ rectangle with $m \geqslant 3$, $n \geqslant 3$.

A natural problem to pose is:

Problem 1. *For which integers n does there exist a simple tiling with n elements*?

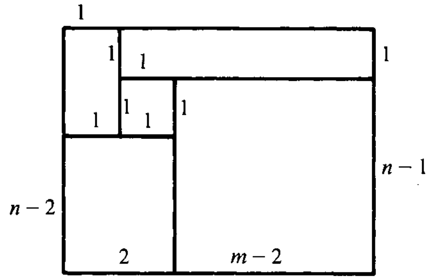


FIGURE 3

At this point the reader is encouraged to experiment in constructing some simple tilings in order to develop some intuition on simple tilings. It is quickly discovered that no simple tilings with n elements exist when $n \leqslant 4$ (no fair counting $n = 1$!). FIGURE 1 gives one with $n = 5$. What about $n = 6$? It turns out (although we will not prove it here) that no simple tiling can have exactly six elements. However, all *larger* values of n admit a tiling with n elements as the following result shows.

THEOREM 1. *For all $n \geqslant 7$, a simple tiling with n elements exists.*

Proof. In FIGURE 4 we show two "left blocks," four "right blocks" and one "middle block" of elements which can be combined to form simple tilings. The general construction pattern is given by taking one "left block," some number m (possibly zero) of copies of the "middle block," and one "right block" to form the tiling.

Since the "left blocks" have five or six elements, the "middle blocks" have seven elements, and the "right blocks" have two, four, six or eight elements, then a general tiling constructed this way can have $\begin{Bmatrix} 5 \\ 6 \end{Bmatrix} + 7m + \begin{pmatrix} 2 \\ 4 \\ 6 \\ 8 \end{pmatrix}$ elements, $m \geqslant 0$. Since these are *all* the integers $\geqslant 7$, the theorem is proved.

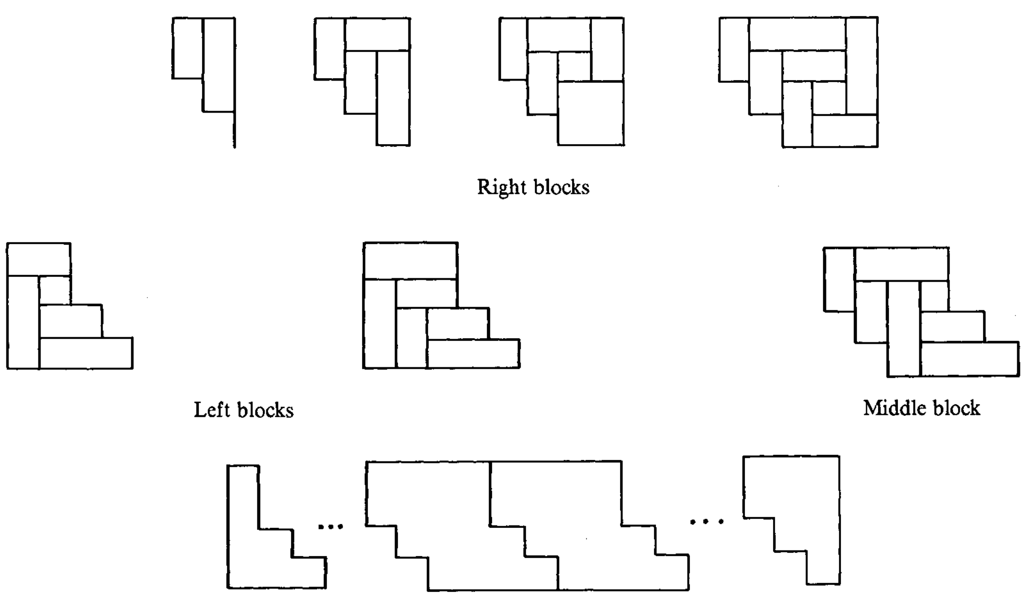


FIGURE 4

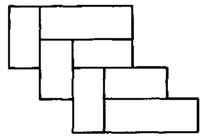


FIGURE 5

We point out here that there is a different middle block of seven elements which would have worked equally well, namely, the one shown in FIGURE 5. Since by forming simple tilings with large values of m middle blocks we could use any (ordered) mixture of the two types, it follows that for a suitable $c > 0$, there are in fact at least $c \cdot 2^{n/7}$ essentially different simple tilings using n elements. Of course, to make precise statements concerning the number of different simple tilings with n elements, we must say just what we mean by two tilings being different. For example, we would not regard the (infinitely many) tilings shown in FIGURE 3 as different. We will not pursue this question any further except to remark that under a rather natural definition of equivalence, it can be shown that there are no more than 20000^n simple tilings with n elements. This bound is not meant to be particularly accurate but rather only to show that the number of tilings is bounded by a simple exponential function of the form C^n as opposed to more rapidly growing functions such as $n!$ or 2^{n^2}.

A related question which suggests itself at this point is the following. What is the *maximum* number of elements possible in a simple tiling of an $m \times n$ rectangle?

This is exactly the problem which motivates the next section.

The average size of elements in a simple tiling

Given a simple tiling of a rectangle R, we define the **average area** of the elements of the tiling to be the ratio $\frac{\text{area } R}{n}$ where n is the number of elements in the tiling. For example, this average area for the tiling in FIGURE 1 is 9/5, while that for the tiling in FIGURE 2 is 15/8. Of course, for a fixed rectangle R, the average area of the elements of a tiling of R is minimized exactly when the number of elements in a tiling of R is made as large as possible.

PROBLEM 2. *Find a lower bound for the average area of the elements in any simple tiling of a rectangle.*

The reader may have discovered by now that almost all simple tilings of rectangles discovered by trial and error have an average area of elements of at least 2. This is true roughly because only a small fraction of the elements can have area 1. Thus, the tilings in FIGURE 1 and FIGURE 2 are especially "tight." The next result answers Problem 2.

THEOREM 2. (a) *The average area of the elements in a simple tiling of a* 3×3 *rectangle is* $9/5$. *For all other simple tilings of rectangles it is strictly greater than* $11/6$. (b) *There is a unique (up to rotation, translation and reflection) simple tiling of the plane for which the average area of the elements is* $11/6$.

Proof. To begin with, we examine what the elements adjacent to a unit square U in a simple tiling of a rectangle R can look like. It is not hard to see that U can never be on the boundary of the tiled rectangle and, in fact, one of the two situations shown in FIGURE 6 must hold.

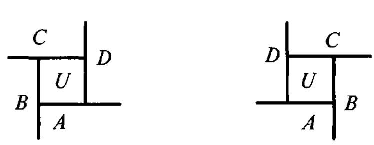


FIGURE 6

In FIGURE 6, elements A, B, C and D are called the **neighbors** of U. Also, we call B the **next neighbor** to A, C the next neighbor to B, etc. We want to define some measure of the "influence" of the various unit squares which may be in the simple tiling of R. To do this, we let A denote a generic element which has a unit square U as a neighbor and an element B as its next neighbor. We assign the value $\nu_U(A)$ to the pair A and U according to the following rules:

(i) If area $A=2$ then $\nu_U(A)=\frac{1}{2}$;

(ii) If area $A>2$ and area $B=2$ then $\nu_U(A)=\frac{7}{2}$;

(iii) If area $A>2$ and area $B>2$ then $\nu_U(A)=\frac{5}{4}$.

Also, we assign to A the value $\nu(A)$, defined to be the sum $\nu_U(A)$ over all unit squares U neighboring A. Since the tiling given in FIGURE 1 cannot be a subconfiguration of our given simple tiling of R, at least one of the neighbors of *any* unit square U must have area greater than 2. Hence, every unit square U with neighbors A, B, C and D satisfies

$$\nu_U(A)+\nu_U(B)+\nu_U(C)+\nu_U(D)\geqslant 5. \tag{1}$$

Before continuing the proof, we need the following Lemma.

LEMMA.

(a) *If area* $A=2$ *then* $\nu(A)\leqslant 1$.

(b) *If area* $A=3$ *then* $\nu(A)\leqslant 7$ *with equality only if* A *has* 1×1 *neighbors as shown in* FIGURE 7(a) *or* 7(b).

(c) *If area* $A=4$ *then* $\nu(A)\leqslant 7$.

(d) *If area* $A>4$ *then* $\nu(A)\leqslant 14$.

Proof (*of Lemma*). (a) This follows from the observation that in this case A can have at most two unit squares as neighbors.

(b) Since in this case A has size 1×3 then A has at most four 1×1 neighbors, i.e., unit squares.

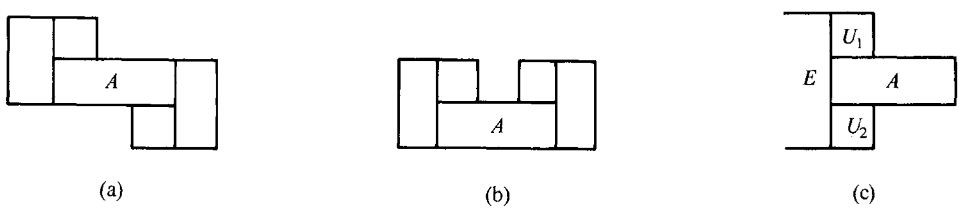


FIGURE 7

If A has two 1×1 neighbors, then they must be placed as shown in FIGURES 7(a), 7(b), or 7(c). In FIGURE 7(c), the neighbor E of A must have area at least 3. Thus, $\nu_{U_1}(A)\leqslant\frac{5}{4}$ and $\nu_{U_2}(A)\leqslant\frac{5}{4}$. It follows that in this situation, $\nu(A)\leqslant\frac{5}{2}$. In FIGURES 7(a) and 7(b), it is easy to see that $\nu(A)=7$. A similar argument shows that if A has three 1×1 neighbors then $\nu(A)\leqslant\frac{5}{2}+\frac{7}{2}=6$ and if A has four 1×1 neighbors then $\nu(A)\leqslant 5$.

(c), (d) The proofs here are nearly the same as in (b). The only thing one needs to observe is that two neighbors of A which are unit squares cannot have a corner of A in common. (The details are left to the reader.)

Proof of Theorem 2 (continued). Let n_i denote the number of elements with area i in a simple tiling of a rectangle R where area $R>9$. From inequality (1) and the Lemma we have

$$
\begin{aligned}
5n_1 &\leqslant \sum_A \nu(A) \leqslant n_2 + 7n_3 + 7n_4 + 14 \sum_{i>4} n_i \\
&\leqslant \sum_{i\geqslant 2} (6i-11)n_i. \qquad (2)
\end{aligned}
$$

From (2) it follows that

$$\text{area } R = \sum_{i\geqslant 1} in_i \geqslant \frac{11}{6} \sum_{i\geqslant 1} n_i. \qquad (3)$$

Since $\Sigma_{i\geqslant 1} n_i$ is just the number of elements of the tiling then (3) implies that the average area of the elements in any simple tiling of R is at least $11/6$.

Let us now focus on the case of equality in (3). This implies that equality must also hold in (2), and in particular,

$$n_2 + 7n_3 + 7n_4 + 14 \sum_{i>4} n_i = \sum_{i\geqslant 2} (6i-11)n_i,$$

i.e.,

$$0 = 6n_4 + 5n_5 + 11n_6 + 17n_7 + \cdots.$$

This can hold with $n_i \geqslant 0$ only if $n_i = 0$ for $i \geqslant 4$.

A further consequence of equality in (3) is that every element A of the tiling with area 2 has two 1×1 neighbors and every element A with area 3 has neighbors as shown in FIGURE 7(a) and 7(b). Since equality in (3) also implies that equality must hold for (1) as well, then any unit square U in the tiling must be surrounded by neighbors as shown in FIGURE 8. It now follows that a

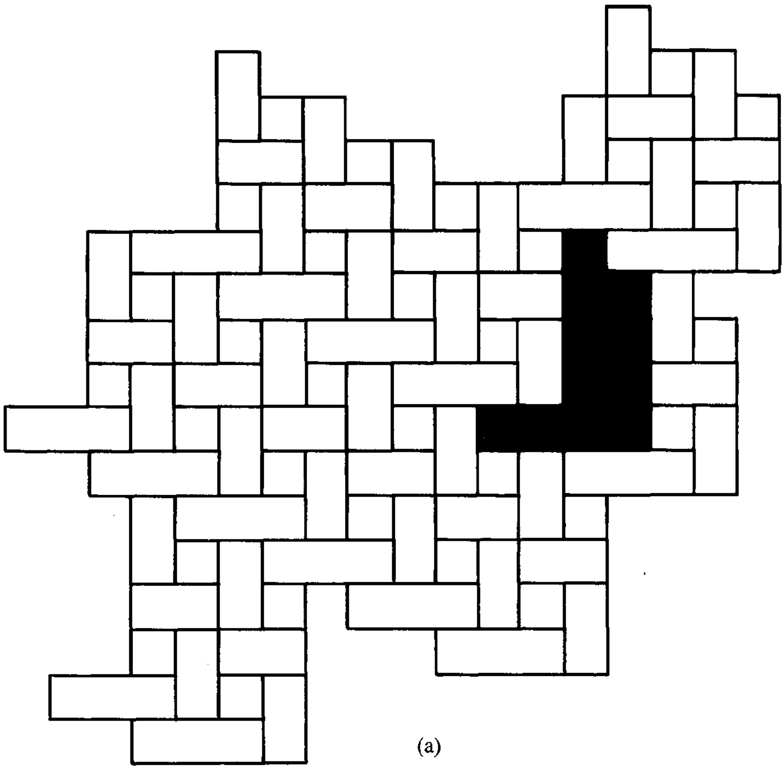
FIGURE 8

1×2 element A of the type shown in FIGURE 7(b) must be part of the configuration shown in FIGURE 2. However, for a *simple* tiling, this is impossible. Consequently, for elements A of area 3, all must occur as shown in FIGURE 7(a). From this we deduce that every 1×2 element must have its two 1×1 neighbors diagonally opposite, i.e., as shown in FIGURE 7(a) (or the mirror image of this).

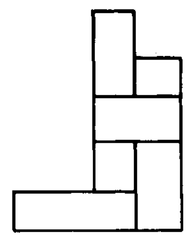
Finally, if we start with the configuration shown in FIGURE 7(a), apply FIGURE 8 for the two unit squares and use the preceding remark for all 1×2 elements, a unique (up to reflection) tiling of the plane is forced. A portion of it is shown in FIGURE 9(a). Note that it can be generated by tiling with translates of the pattern B shown in FIGURE 9(b). It is also not difficult to check that this process can never terminate with an exact rectangle being simply tiled. Thus, any simply tiled *rectangle* must have the area of its elements strictly greater than $11/6$. However, if we appropriately tile a large nearly rectangular region S of the plane with translates of the pattern B in FIGURE 9(b), then it is not too hard to show that the jagged border of S can be completed by relatively few elements (possibly with areas greater than 3) to form a simply tiled rectangle which has an average element area as close to $11/6$ as desired (depending on the size of S; see the cover of this MAGAZINE). We leave the details of this construction to the energetic reader. This completes the proof of Theorem 2.

There is an active literature on the topic of tilings of rectangles by rectangles. The interested reader should consult some of the references ([**1**], [**14**], and especially [**6**]) for a sample.

As is natural with many geometrical problems, one might ask the analogous questions in three (or more) dimensions. To the best of our knowledge, almost nothing is known in these cases although it would certainly be an interesting area for exploration.



(a)



(b)

FIGURE 9

The authors gratefully acknowledge the incisive comments of Doris Schattschneider in preparing the final form of this note.

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