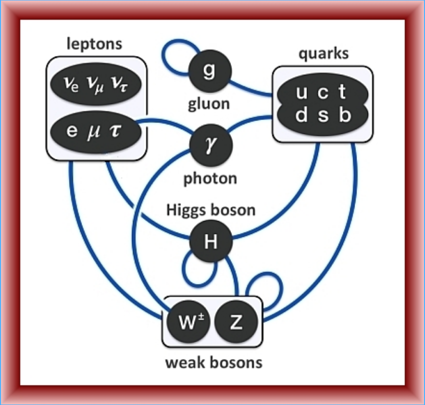
CAN WE UNDERSTAND THE STANDARD MODEL?



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Octonions and the Standard Model
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40 years trying to *go beyond* the Standard Model hasn't yet led to any clear success. Maybe we should try a different game!

For example, we could try to *understand why the Standard Model is the way it is*. This also seems extremely hard, but at least it's different.

In the process we may be forced to go beyond the Standard Model. Or maybe not.

Either way, the *"understanding the Standard Model"* game is not mainly about creating highly symmetrical theories with no clear connection to the world we actually see. What we see is at least *close* to the Standard Model.

There are many things about the Standard Model that would be great to understand. In rough order of difficulty:

1. the Standard Model gauge group
2. its representation on one generation of fermions
3. the Lorentz group action on fermions, especially the chirality
4. the three generations
5. the Higgs and its couplings (22 dimensionless constants)
6. the gauge field coupling constants (3 dimensionless constants)

We may decide that some or all of this is hopeless to understand — but nothing ventured, nothing gained!

Here I will only consider

1. the Standard Model gauge group
2. its representation on one generation of fermions

and attempts to make them seem mathematically natural.

Much of this is a review, but I'll try to make things pretty, and I'll emphasize Lie groups rather than Lie algebras.

For an expository account of *some* of this, see:

- John Baez and John Huerta, The algebra of grand unified theories.

The center of $\mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{SU}(3)$ is $\mathrm{U}(1) \times \mathbb{Z}_2 \times \mathbb{Z}_3$, where:

- $\mathbb{Z}_2 \subset \mathrm{SU}(2)$ consists of square roots of 1 times the identity matrix
- $\mathbb{Z}_3 \subset \mathrm{SU}(3)$ consists of cube roots of 1 times the identity matrix.

Any element in the center of $\mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{SU}(3)$ must act simply as *multiplication by a phase* on any irreducible representation of this group.

The center of $\mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{SU}(3)$ contains an element that acts trivially on all known particles:

$$(\zeta, \zeta^3, \zeta^2) \in \mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{SU}(3)$$

where $\zeta = e^{2\pi i/6}$ is a 6th root of unity. This fact is equivalent to the following requirements on the hypercharges Y:

Case	Requirement on Y	Actual value of Y
Left-handed quark	even integer $+\frac{1}{3}$	$+\frac{1}{3}$
Left-handed lepton	odd integer	-1
Right-handed quarks	odd integer $+\frac{1}{3}$	$+\frac{4}{3}, -\frac{2}{3}$
Right-handed leptons	even integer	$0, -2$

This element $(\zeta, \zeta^3, \zeta^2)$ generates a subgroup

$$\mathbb{Z}_6 \subset \mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{SU}(3)$$

that acts trivially on all known particles.

So, we can call

$$\frac{\mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{SU}(3)}{\mathbb{Z}_6}$$

the **true gauge group** of the Standard Model.

But we need to use the right $\mathbb{Z}_6$ subgroup here!

There are 12 normal subgroups $N \subset \mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{SU}(3)$ isomorphic to $\mathbb{Z}_6$. They are all subgroups of the center. For at least two, $(\mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{SU}(3))/N$ is isomorphic to the true gauge group of the Standard Model.

There is a homomorphism

$$\begin{array}{rccc} \phi\colon & \mathrm{U}(1)\times\mathrm{SU}(2)\times\mathrm{SU}(3) & \to & \mathrm{SU}(5) \\ & (\alpha, g, h) & \longmapsto & \begin{pmatrix} \alpha^3 g & 0 \\ 0 & \alpha^{-2} h \end{pmatrix} \end{array}$$

whose kernel is the $\mathbb{Z}_6$ subgroup that acts trivially on all known particles, and whose image is

$$\mathrm{S}(\mathrm{U}(2)\times\mathrm{U}(3)) = \left\{ x \in \mathrm{SU}(5) : x = \begin{pmatrix} * & * & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 \\ 0 & 0 & * & * & * \\ 0 & 0 & * & * & * \\ 0 & 0 & * & * & * \end{pmatrix} \right\}$$

Thus

$$\mathrm{S}(\mathrm{U}(2)\times\mathrm{U}(3)) \cong \frac{\mathrm{U}(1)\times\mathrm{SU}(2)\times\mathrm{SU}(3)}{\mathbb{Z}_6}$$

is the true gauge group of the Standard Model.

All the fermions and antifermions in one generation, including a right-handed neutrino and its antiparticle, fit into the obvious representation of $\mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(3))$ on

$$\Lambda\mathbb{C}^5 = \bigoplus_{n=0}^{5} \Lambda^n\mathbb{C}^5$$

The left-handed fermions and antifermions live in

$$\Lambda^{\mathrm{even}}\mathbb{C}^5 = \Lambda^0\mathbb{C}^5 \oplus \Lambda^2\mathbb{C}^5 \oplus \Lambda^4\mathbb{C}^5$$

while the right-handed ones live in

$$\Lambda^{\mathrm{odd}}\mathbb{C}^5 = \Lambda^1\mathbb{C}^5 \oplus \Lambda^3\mathbb{C}^5 \oplus \Lambda^5\mathbb{C}^5$$

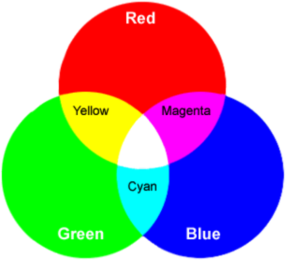
They are interchanged by the conjugate-linear Hodge star operator

$$\star\colon \Lambda\mathbb{C}^5 \to \Lambda\mathbb{C}^5$$

$\Lambda\mathbb{C}^5$ has a basis labelled by 5-bit strings, answering 5 yes-or-no questions:

- is the particle isospin up?
- is it isospin down?
- is it red?
- is it green?
- is it blue?

Here “up & down” means isospin 0, “red & blue” means “magenta” = anti-green, “red, green & blue” means colorless, etc.



How can we *understand* the group $\mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(3))$ and its representation on $\Lambda\mathbb{C}^5$? What's so special about them? It helps to look at grand unified theories, even if we don't believe in them.

$\mathrm{SU}(5)$ is a subgroup of $\mathrm{SO}(10)$. But because it is simply connected, it lifts to become a subgroup of $\mathrm{Spin}(10)$:

$$\begin{array}{ccc} & & \mathrm{Spin}(10) \\ & \nearrow & \big\downarrow {\scriptstyle 2-1} \\ \mathrm{SU}(5) & \hookrightarrow & \mathrm{SO}(10) \end{array}$$

The representation of $\mathrm{SU}(5)$ on $\Lambda\mathbb{C}^5$ extends to a representation of $\mathrm{Spin}(10)$ on $\Lambda\mathbb{C}^5$: the Dirac spinor representation. This contains all all the fermions and antifermions in one generation, including a right-handed neutrino and its antiparticle.

$\mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(3))$ is the intersection of two subgroups of $\mathrm{Spin}(10)$:

$$
\begin{array}{ccc}
\mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(3)) & \hookrightarrow & (\mathrm{Spin}(4) \times \mathrm{Spin}(6))/\mathbb{Z}_2 \\
\downarrow & & \downarrow \\
\mathrm{SU}(5) & \hookrightarrow & \mathrm{Spin}(10)
\end{array}
$$

$$\mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(3)) = \mathrm{SU}(5) \ \cap \ (\mathrm{Spin}(4) \times \mathrm{Spin}(6))/\mathbb{Z}_2$$

- $\mathrm{SU}(5)$ is the gauge group of the Georgi–Glashow model.
- $\mathrm{Spin}(4) \times \mathrm{Spin}(6)$ is the usual gauge group of the Pati–Salam model:

$$\mathrm{Spin}(4) \cong \mathrm{SU}(2) \times \mathrm{SU}(2) \qquad \mathrm{Spin}(6) \cong \mathrm{SU}(4)$$

but there's a subgroup $\mathbb{Z}_2 \subset \mathrm{Spin}(4) \times \mathrm{Spin}(6)$ that acts trivially on all known particles.

Standard Model = Georgi–Glashow ∩ Pati–Salam

$\mathrm{SU}(5) \subset \mathrm{Spin}(10)$ is the subgroup that preserves:

- a **complex structure** $J\colon \mathbb{R}^{10} \to \mathbb{R}^{10}$, $J^2 = -1$, which lets us say $\mathbb{R}^{10} \cong \mathbb{C}^5$.
- a **complex volume form** $\omega \in \Lambda^5\mathbb{C}^5$, $\omega \neq 0$.

$(\mathrm{Spin}(4) \times \mathrm{Spin}(6))/\mathbb{Z}_2 \subset \mathrm{Spin}(10)$ is the subgroup that preserves:

- a **4+6 splitting** $\mathbb{R}^{10} = \mathbb{R}^4 \oplus \mathbb{R}^6$.

The true gauge group of the Standard Model, $\mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(3))$, is the subgroup of $\mathrm{Spin}(10)$ that preserves a complex structure on $\mathbb{R}^{10}$, a complex volume form, and a **compatible** 4+6 splitting: one that gives a splitting $\mathbb{C}^5 \cong \mathbb{C}^2 \oplus \mathbb{C}^3$.

We can dramatize this. A Riemannian manifold M where each tangent space has a complex structure $J\colon T_xM \to T_xM$ preserved by parallel transport is called a **Kähler manifold**. A Kähler manifold where each tangent space has a complex volume form ω preserved by parallel transport is called a **Calabi–Yau manifold**.

Take any 10-dimensional manifold M that's the product of 4d and 6d Calabi–Yau manifolds. Then M is equipped with a principal $\mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(3))$ bundle, and Dirac spinors on M transform in the representation corresponding to the fermions and antifermions in one generation, including a right-handed neutrino and its antiparticle.

- John Baez, Calabi–Yau manifolds and the Standard Model.

So, there is a connection between the Standard Model and 10-dimensional *space*. But it's also connected to 10-dimensional *spacetime!*

- Ivan Todorov and Michel Dubois-Violette, Deducing the symmetry of the standard model from the automorphism and structure groups of the exceptional Jordan algebra.

4d Minkowski spacetime can be seen as the space of 2×2 hermitian complex matrices

$$\mathfrak{h}_2(\mathbb{C}) = \left\{ \begin{pmatrix} t+x & y \\ y^* & t-x \end{pmatrix} : t, x \in \mathbb{R}, y \in \mathbb{C} \right\}$$

with its Minkowski metric:

$$\det \begin{pmatrix} t+x & y \\ y^* & t-x \end{pmatrix} = t^2 - x^2 - |y|^2$$

So, there is a connection between the Standard Model and 10-dimensional *space*. But it's also connected to 10-dimensional *spacetime!*

- Ivan Todorov and Michel Dubois-Violette, Deducing the symmetry of the standard model from the automorphism and structure groups of the exceptional Jordan algebra.

10d Minkowski spacetime can be seen as the space of 2×2 hermitian octonionic matrices

$$\mathfrak{h}_2(\mathbb{O}) = \left\{ \begin{pmatrix} t+x & y \\ y^* & t-x \end{pmatrix} : t, x \in \mathbb{R}, y \in \mathbb{O} \right\}$$

with its Minkowski metric:

$$\det \begin{pmatrix} t+x & y \\ y^* & t-x \end{pmatrix} = t^2 - x^2 - |y|^2$$

Choosing a unit imaginary octonion $i \in \mathbb{O}$ gives an inclusion

$$\mathbb{C} \hookrightarrow \mathbb{O}$$

This gives a splitting

$$\underbrace{\mathbb{O}}_{\text{8 real dimensions}} = \underbrace{\mathbb{C}}_{\text{2 real dimensions}} \oplus \underbrace{\mathbb{C}^{\perp}}_{\text{6 real dimensions}}$$

where $\mathbb{C}^{\perp}$ is the orthogonal complement of $\mathbb{C}$ in $\mathbb{O}$.

Choosing a unit imaginary octonion $i \in \mathbb{O}$ gives an inclusion

$$\mathbb{C} \hookrightarrow \mathbb{O}$$

and thus an inclusion of 4d Minkowski spacetime in 10d Minkowski spacetime:

$$\mathfrak{h}_2(\mathbb{C}) \hookrightarrow \mathfrak{h}_2(\mathbb{O}) = \left\{ \begin{pmatrix} t+x & y \\ y^* & t-x \end{pmatrix} : t, x \in \mathbb{R}, y \in \mathbb{O} \right\}$$

and a splitting

$$\underbrace{\mathfrak{h}_2(\mathbb{O})}_{\text{10d spacetime}} = \underbrace{\mathfrak{h}_2(\mathbb{C})}_{\text{4d spacetime}} \oplus \underbrace{\mathbb{C}^\perp}_{\text{6d space}}$$

The subgroup of $\mathrm{Spin}(9,1)$ preserving the splitting

$$\underbrace{\mathfrak{h}_2(\mathbb{O})}_{\text{10d spacetime}} \quad = \quad \underbrace{\mathfrak{h}_2(\mathbb{C})}_{\text{4d spacetime}} \quad \oplus \quad \underbrace{\mathbb{C}^\perp}_{\text{6d space}}$$

contains a copy of the true gauge group of the Standard Model, $\mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(3))$.

(I'll explain this next time.)

Thus, the true gauge group of the Standard Model again preserves a “4+6 splitting”, but now it's a splitting of 10d Minkowski spacetime into 4d Minkowski spacetime and 6d Euclidean space.

- In the $\mathfrak{h}_2(\mathbb{O})$ approach, $\mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(3))$ is a subgroup of $\mathrm{Spin}(9,1)$.
- In grand unified theories, it is a subgroup of $\mathrm{Spin}(10)$.

How we can reconcile these facts?

$\mathfrak{h}_2(\mathbb{O})$ is a 10d Euclidean space as well as a 10d Minkowski spacetime! In the $\mathfrak{h}_2(\mathbb{O})$ approach we actually have

$$\mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(3)) \subset \mathrm{Spin}(9) = \mathrm{Spin}(9,1) \cap \mathrm{Spin}(10)$$

We can also see directly how $\mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(3)) \subset \mathrm{Spin}(9)$:

- Kirill Krasnov, SO(9) characterisation of the Standard Model gauge group.

Let's see why $\mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(3)) \subset \mathrm{Spin}(9)$.

Start with Pati–Salam:

$$
\begin{array}{ccc}
\mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{SU}(3) & \to & \mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SU}(4) \\
(\alpha, g, h) & \mapsto & \left(g, \begin{pmatrix} \alpha^3 & 0 \\ 0 & \alpha^{-3} \end{pmatrix}, \begin{pmatrix} \alpha h & 0 \\ 0 & \alpha^{-3} \end{pmatrix} \right)
\end{array}
$$

This map is 3-1.

Let's see why $\mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(3)) \subset \mathrm{Spin}(9)$.

Then ignore the right-handed $\mathrm{SU}(2)$:

$$\begin{array}{ccc} \mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{SU}(3) & \stackrel{\phi}{\to} & \mathrm{SU}(2) \times \mathrm{SU}(4) \\ (\alpha, g, h) & \mapsto & \left(g, \left(\begin{array}{cc} \alpha h & 0 \\ 0 & \alpha^{-3} \end{array} \right)\right) \end{array}$$

This map ϕ is 3-1.

Let's see why $\mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(3)) \subset \mathrm{Spin}(9)$.

Remember $\mathrm{SU}(2) \cong \mathrm{Spin}(3)$ and $\mathrm{SU}(4) \cong \mathrm{Spin}(6)$:

$$\mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{SU}(3) \quad \xrightarrow{\phi} \quad \mathrm{Spin}(3) \times \mathrm{Spin}(6)$$

This map ϕ is 3-1.

Let's see why $\mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(3)) \subset \mathrm{Spin}(9)$.

Now compose ϕ with $\mathrm{Spin}(3) \times \mathrm{Spin}(6) \xrightarrow{2-1} \mathrm{Spin}(9)$:

$$\mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{SU}(3) \quad \xrightarrow{\phi} \quad \mathrm{Spin}(3) \times \mathrm{Spin}(6) \quad \xrightarrow{2-1} \quad \mathrm{Spin}(9)$$

The composite is 6-1. Its kernel is exactly the

$$\mathbb{Z}_6 \subset \mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{SU}(3)$$

that acts trivially on all known particles!

Let's see why $\mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(3)) \subset \mathrm{Spin}(9)$.

So, we get a 1-1 map:

$$\frac{\mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{SU}(3)}{\mathbb{Z}_6} \quad \hookrightarrow \quad \mathrm{Spin}(9)$$

But $(\mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{SU}(3))/\mathbb{Z}_6 \cong \mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(3))$.

Let's see why $\mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(3)) \subset \mathrm{Spin}(9)$.

So, we get a 1-1 map:

$$\mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(3)) \quad \hookrightarrow \quad \mathrm{Spin}(9)$$

Summary and Speculations

- The true gauge group of the Standard Model, $\mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(3))$, is precisely the subgroup of $\mathrm{Spin}(10)$ that preserves a complex structure on 10d *Euclidean space*, a complex volume form, and a compatible 4+6 splitting.
- But it's also the subgroup of $\mathrm{Spin}(9, 1)$ that preserves a 4+6 splitting of 10d *Minkowski spacetime* and some extra structure. Next time I'll give an octonionic account of this, and bring $\mathfrak{h}_3(\mathbb{O})$ into the story.
- We should be able to unify these two accounts by complexifying... but do we want to?

- The 10d Minkowski spacetime (or Euclidean space) in this story cannot be *physical* spacetime (or its Euclideanization) since the $\mathrm{SU}(2)$ here:

$$\mathrm{SU}(2) \subset \mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(3)) \subset \mathrm{Spin}(9) = \mathrm{Spin}(9,1) \cap \mathrm{Spin}(10)$$

  is acting on *isospin*, not spin!

- And yet, there is an eerie link between the "left" and "right" $\mathrm{SU}(2)$'s in the Pati–Salam group

$$\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SU}(4) \cong \mathrm{Spin}(4) \times \mathrm{Spin}(6)$$

  and the left-handed and right-handed $\mathrm{SU}(2)$'s acting on the Euclideanization of *physical* spacetime

$$\mathrm{SU}(2) \times \mathrm{SU}(2) \cong \mathrm{Spin}(4)$$

  Somehow the weak force is "more closely tied to 4d spacetime" than the other forces (it grossly violates parity).