Elliptic curves

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Arnold Ross Lecture
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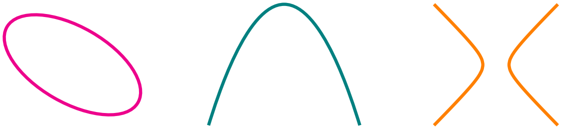
Plane curves

Degree 1 (lines)

$$3x + 7y + 6 = 0$$

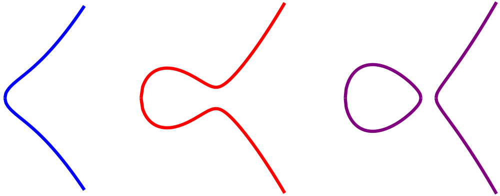
Degree 2 (conics)

$$2x^2 + 9xy + 3y^2 + 3x + 7y + 6 = 0$$



Degree 3 (cubic curves)

$$4x^3 + 5x^2y + xy^2 + 8y^3 + 2x^2 + 9xy + 3y^2 + 3x + 7y + 6 = 0$$



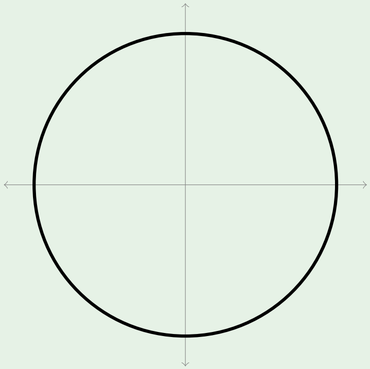
Elliptic curves are special cubic curves.

Rational points on the unit circle

Definition

A rational point on a curve is a point whose coordinates are rational numbers (elements of $\mathbb{Q}$).

Example (The unit circle $x^2 + y^2 = 1$)



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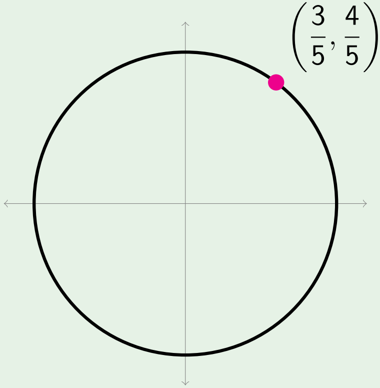

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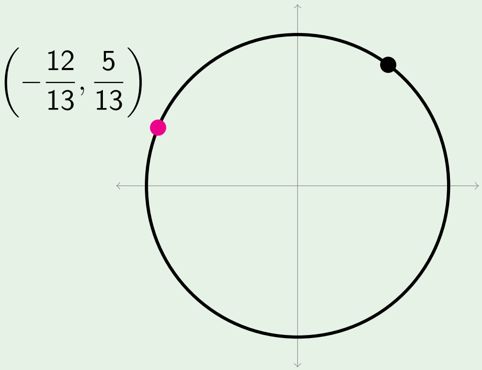

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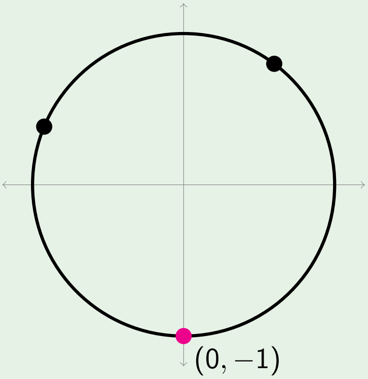

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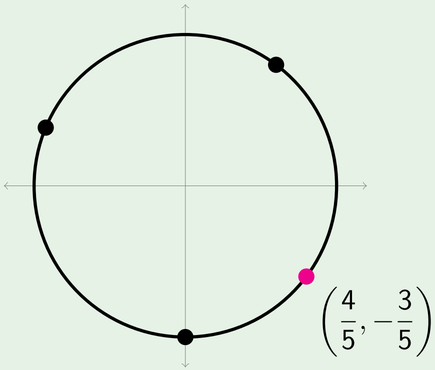

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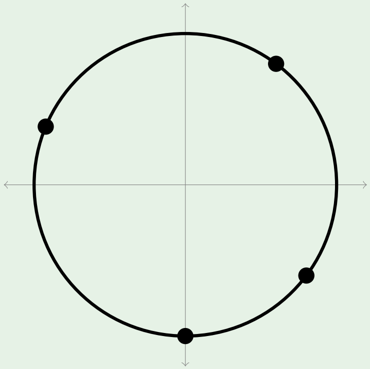

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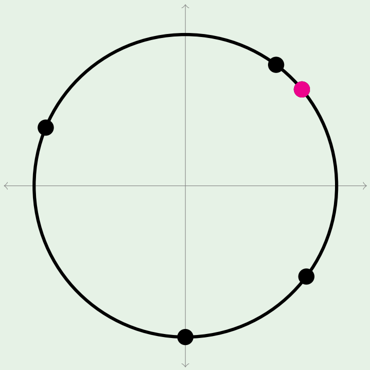

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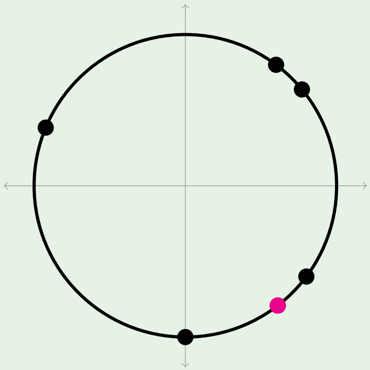

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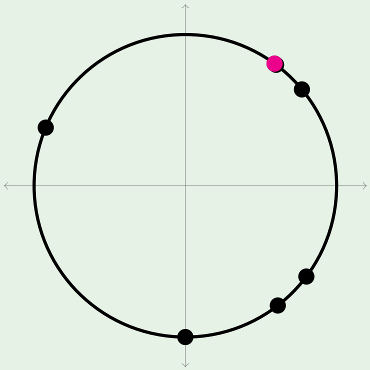

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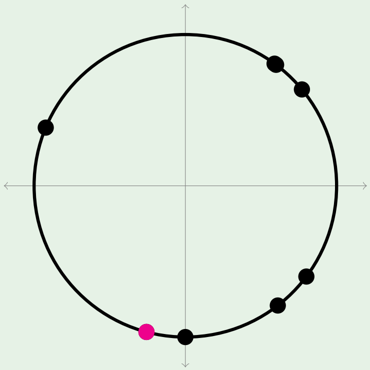

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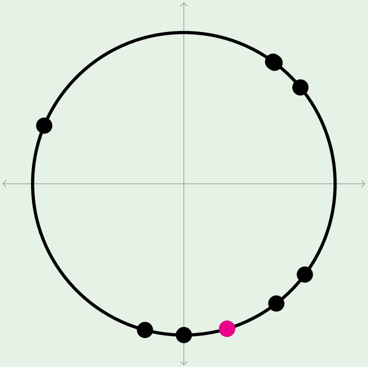

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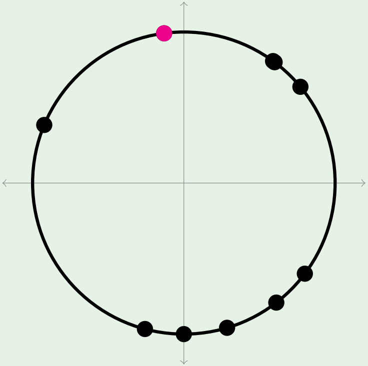

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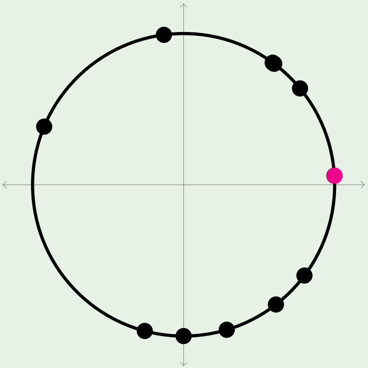

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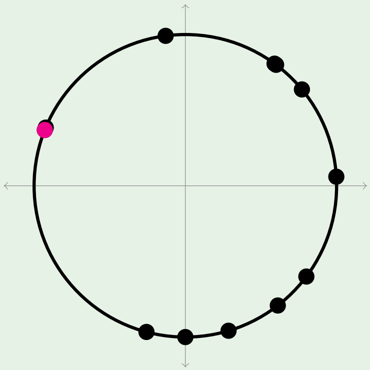

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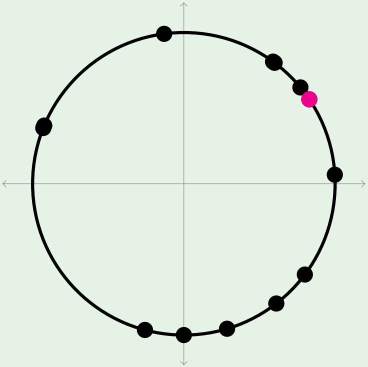

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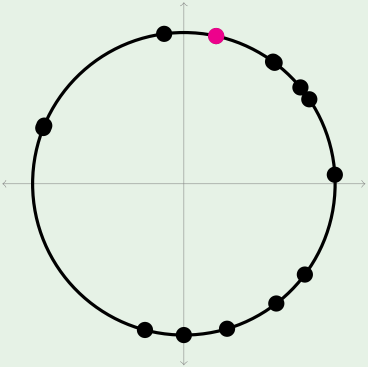

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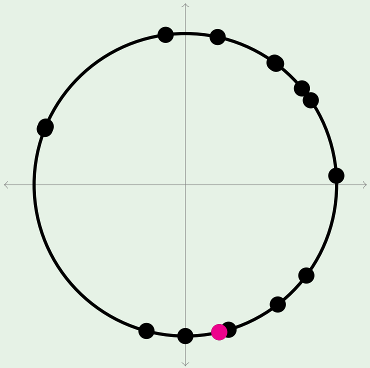

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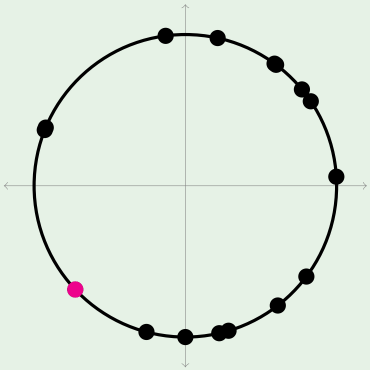

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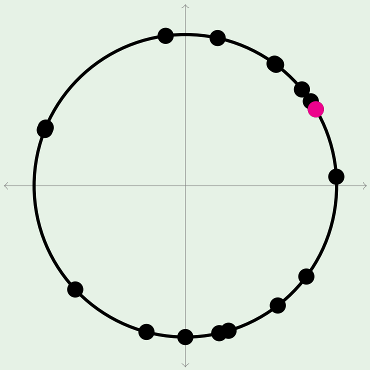

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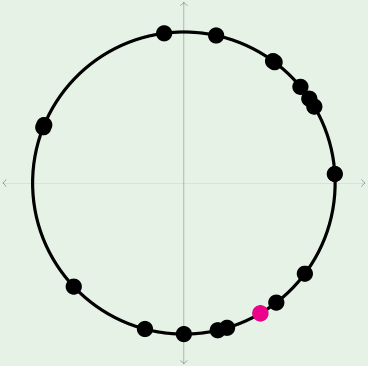

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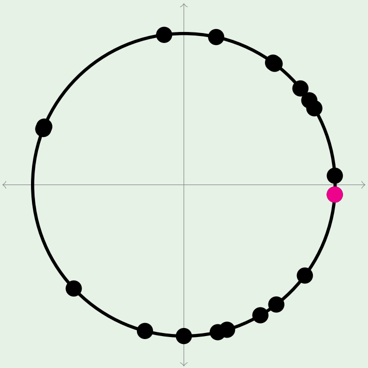

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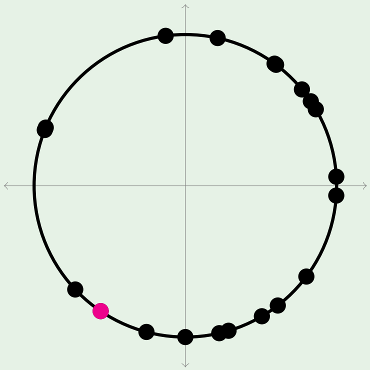

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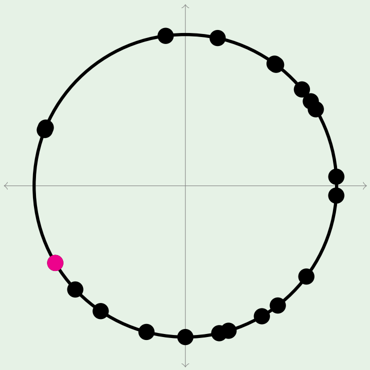

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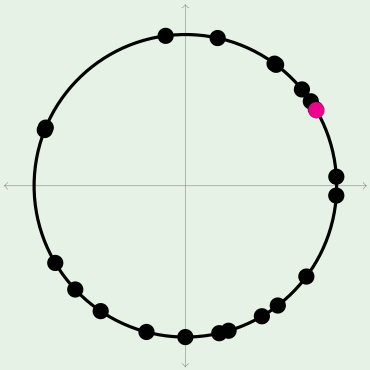

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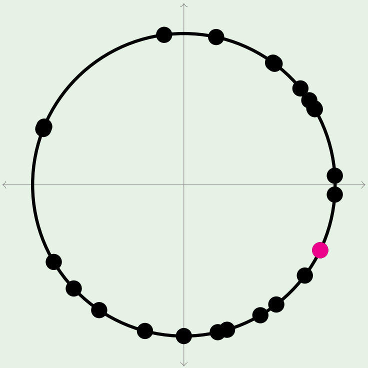

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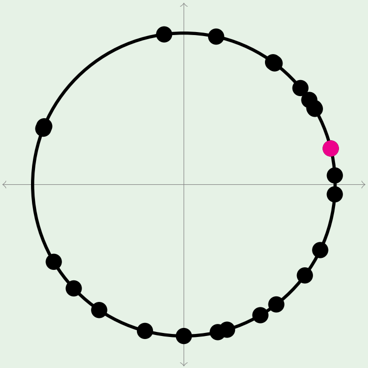

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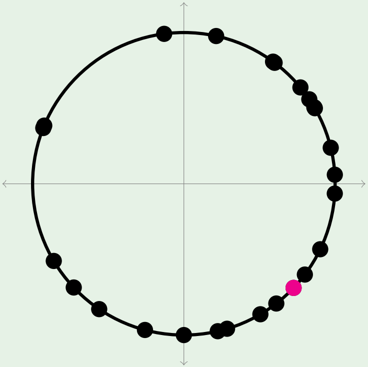

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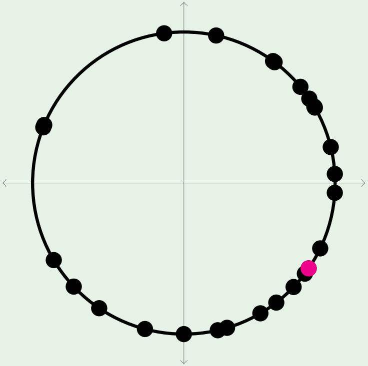

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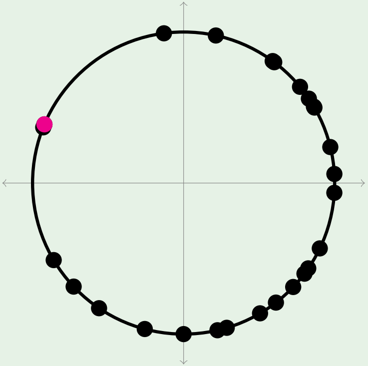

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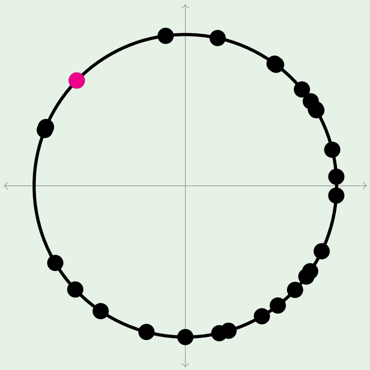

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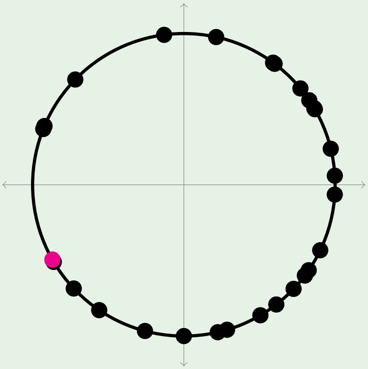

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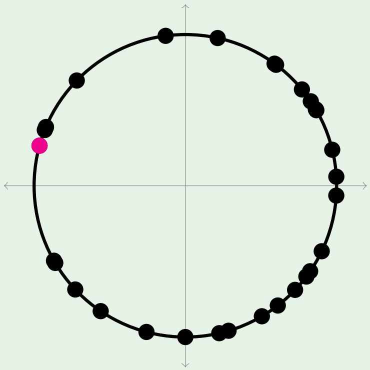

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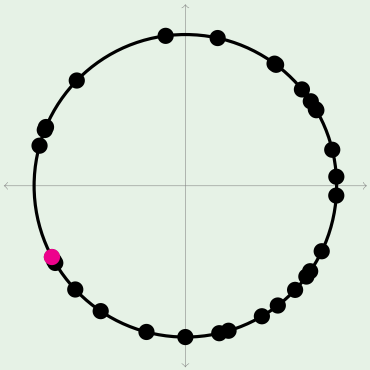

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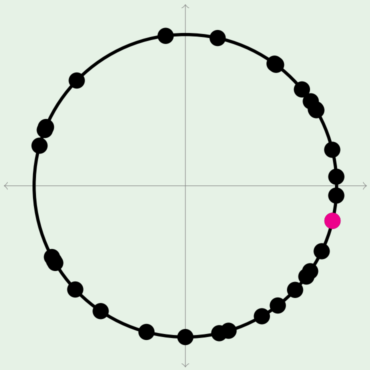

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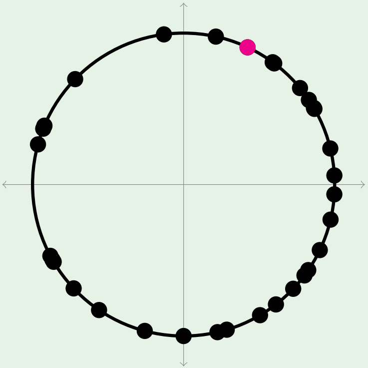

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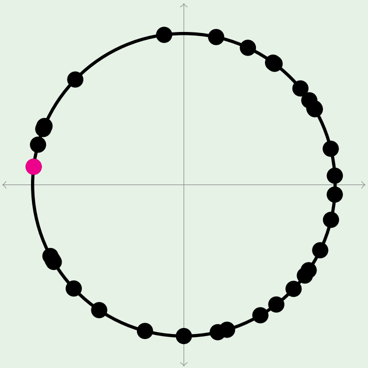

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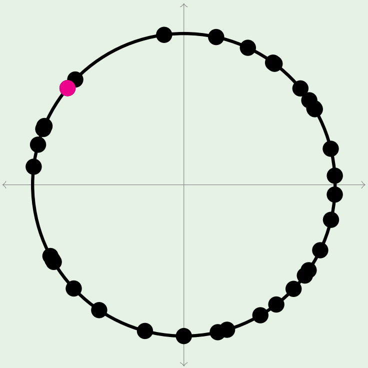

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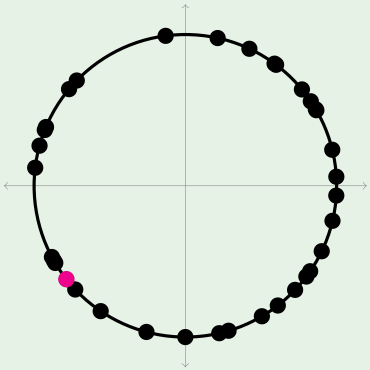

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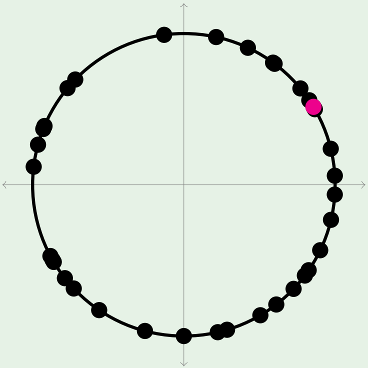

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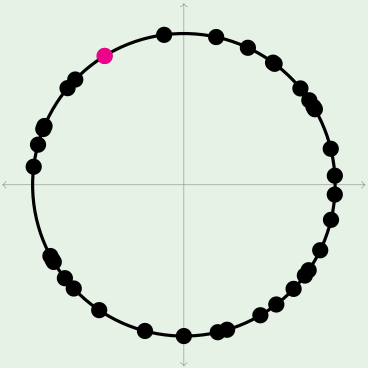

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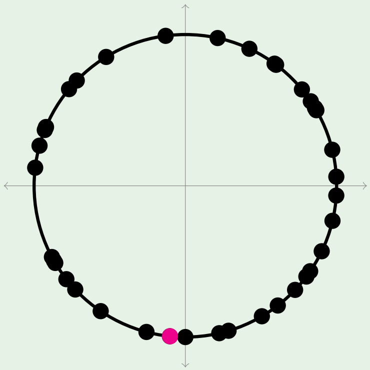

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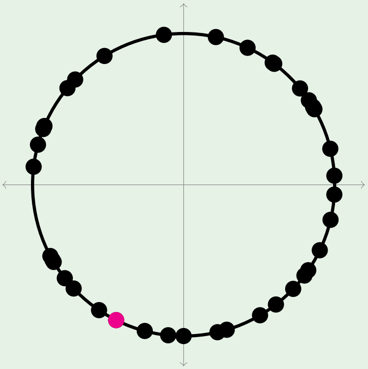

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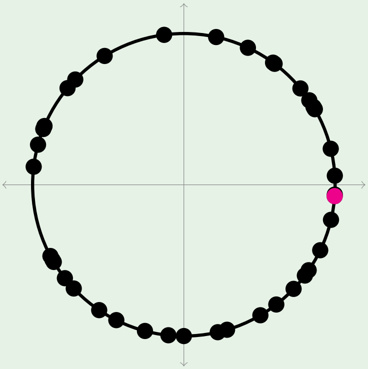

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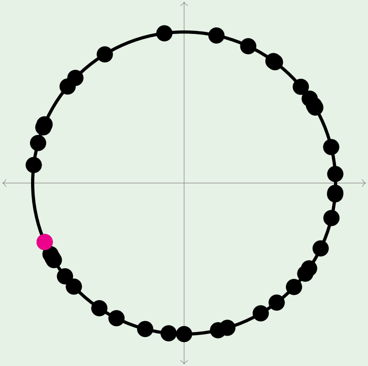

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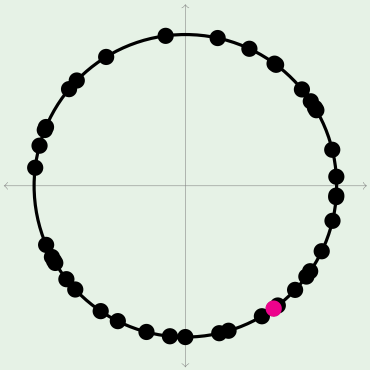

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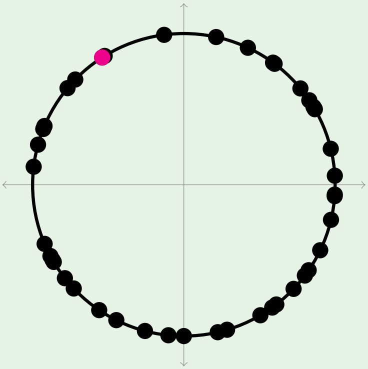

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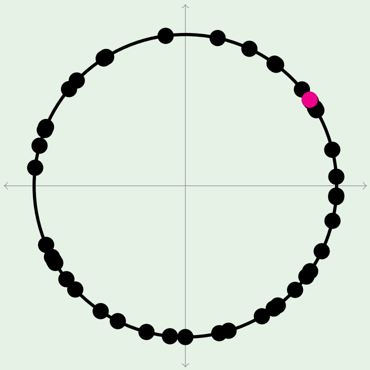

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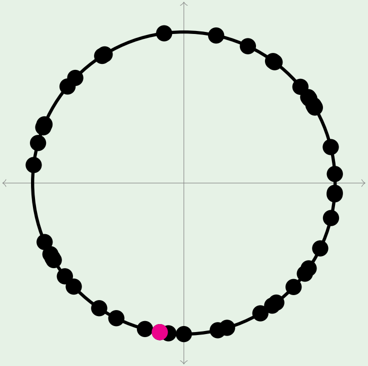

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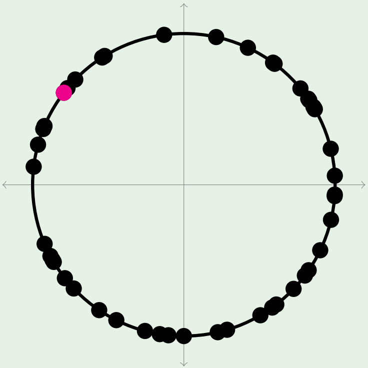

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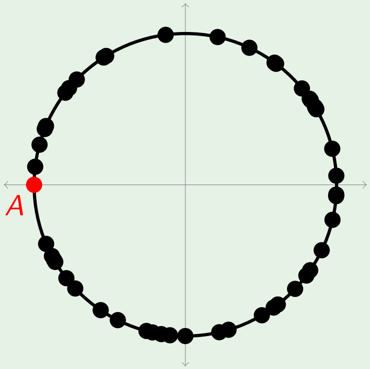
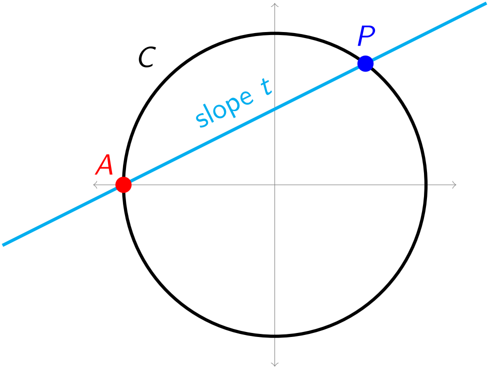
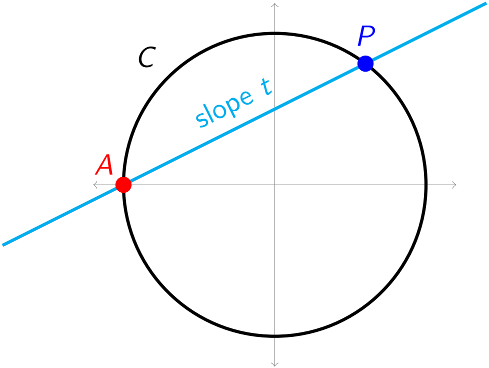




- Given $P \neq A$ on C, form $\overleftrightarrow{AP}$ and take its slope t.
  If P is a rational point, then $t \in \mathbb{Q}$.
- Conversely, given t, draw the line L_t through A with slope t; then L_t intersects C in a second point P.
  If $t \in \mathbb{Q}$, then must P be a rational point?



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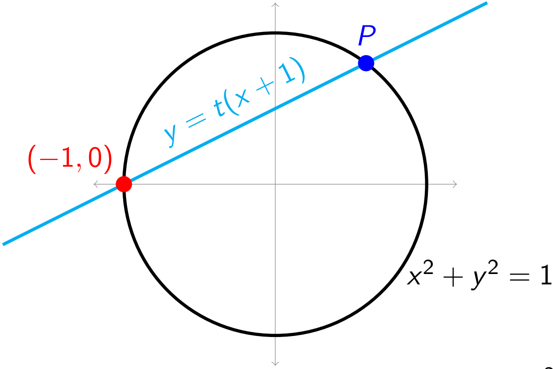
  Yes!



To find the intersection, substitute $y = t(x+1)$ into $x^2 + y^2 = 1$:

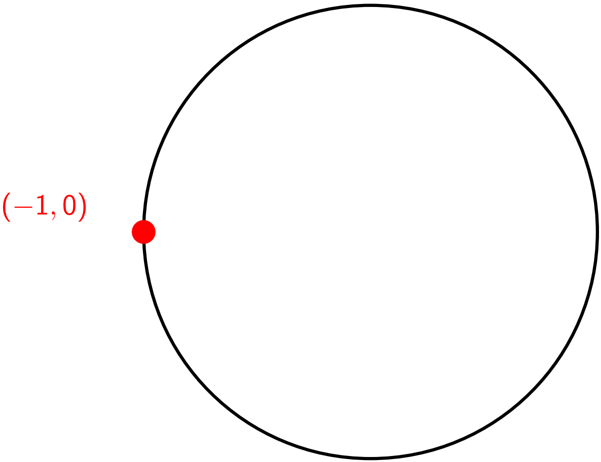
$$\begin{aligned} x^2 + t^2(x+1)^2 &= 1 \\ (x+1)\ \left((1+t^2)x - (1-t^2)\right) &= 0 \\ x = -1 \text{ or } x &= \frac{1-t^2}{1+t^2}. \end{aligned}$$

Then use $y = t(x+1)$ to get the corresponding y-coordinates:

$$(-1, 0) \quad \text{or} \quad \boxed{\left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}\right)}.$$

Theorem

$$\left\{\begin{matrix}\text{rational points on } x^2+y^2=1 \\ \text{other than } (-1,0)\end{matrix}\right\} = \left\{\left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}\right) : t \in \mathbb{Q}\right\}.$$



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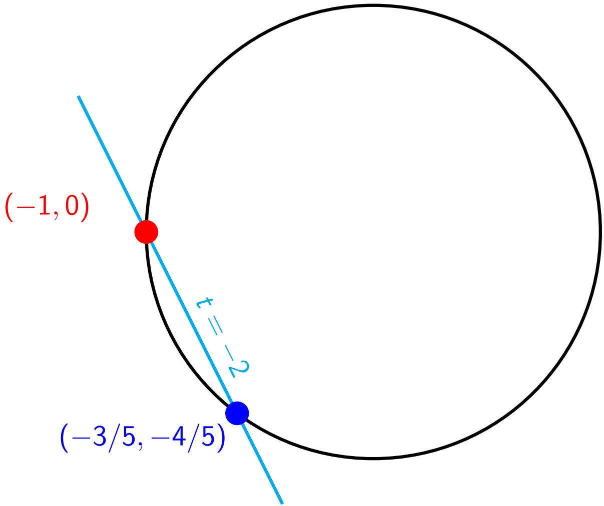

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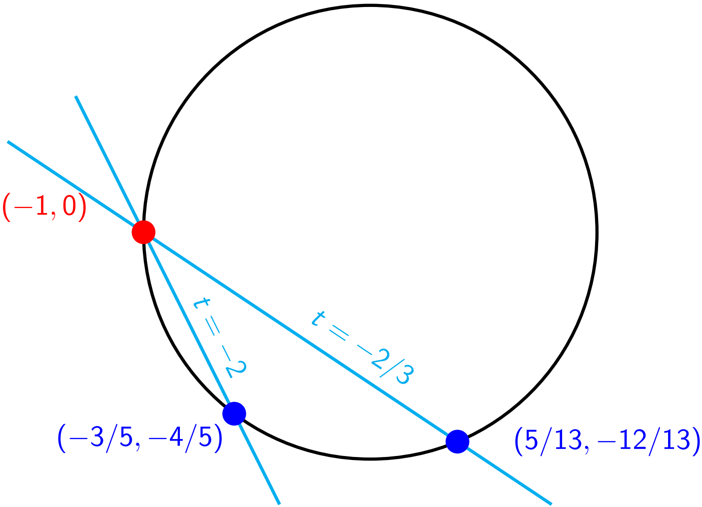

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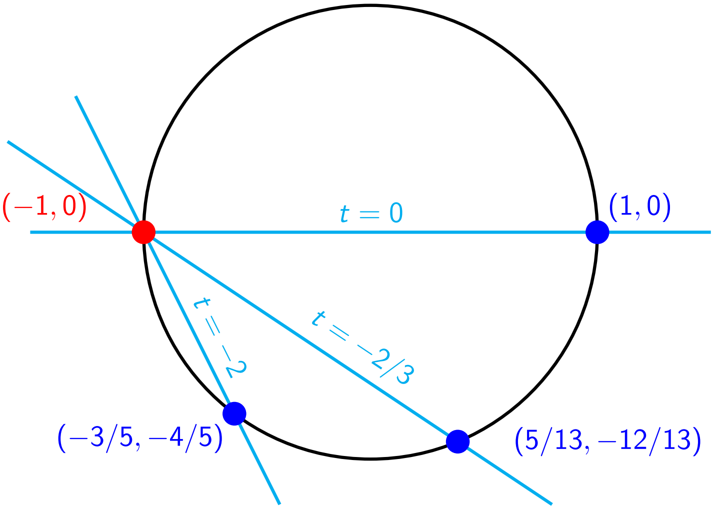

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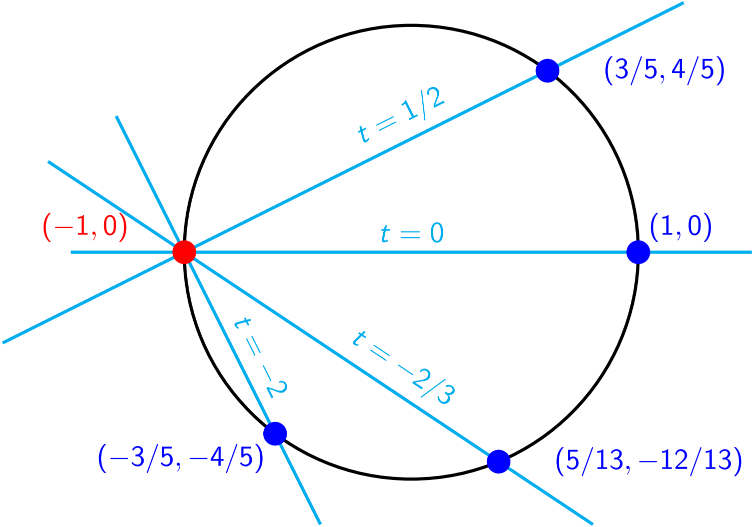

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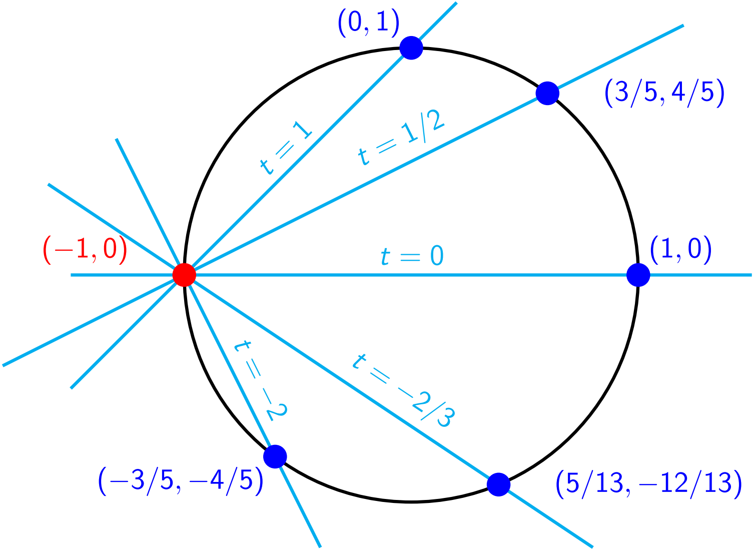
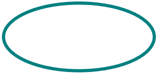


Rational points on other conics (and other shapes)

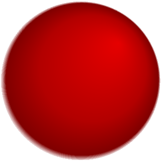
- The same method parametrizes the rational points on any conic, *provided that one rational point is known*.
- There is also a test, based on checking congruences, for deciding whether a conic has a rational point.

Challenge: Parametrize the rational points on

1. the ellipse $x^2 + 5y^2 = 1$



2. the sphere $x^2 + y^2 + z^2 = 1$



Challenge: Use congruences to show that the circle $x^2 + y^2 = 3$ has no rational points.

The projective line $\mathbb{P}^1$

Definition

The projective line $\mathbb{P}^1$ is the set of all lines in $\mathbb{R}^2$ through $(0, 0)$.

Why is this set being called a line?

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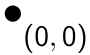

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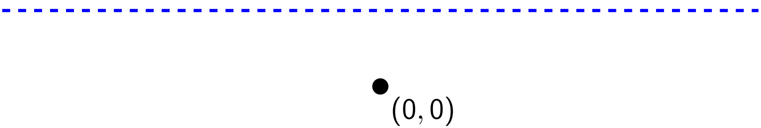

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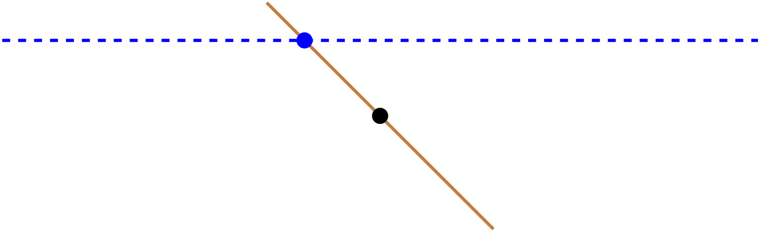

each non-horizontal line $\longleftrightarrow$ **some point on the blue line**

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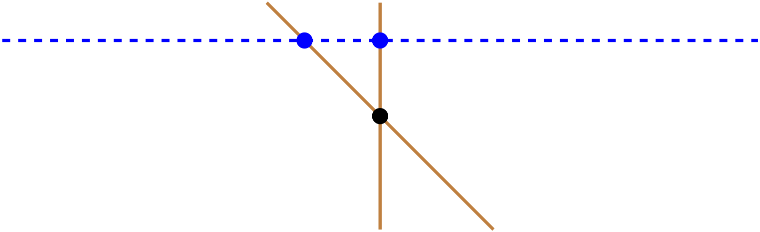

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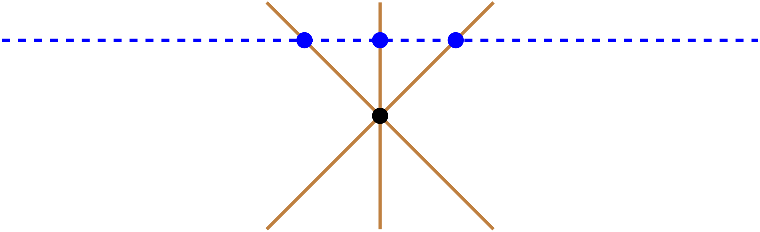

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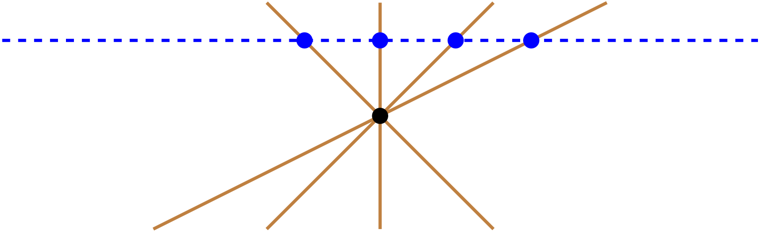

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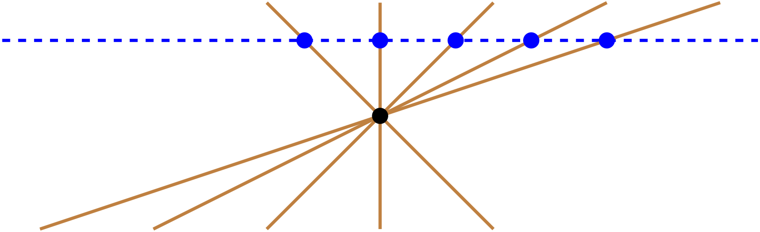

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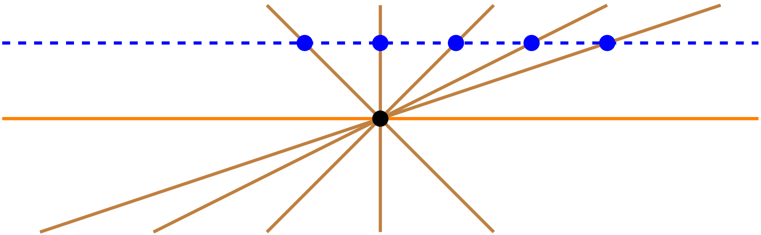

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the horizontal line $\longleftrightarrow$

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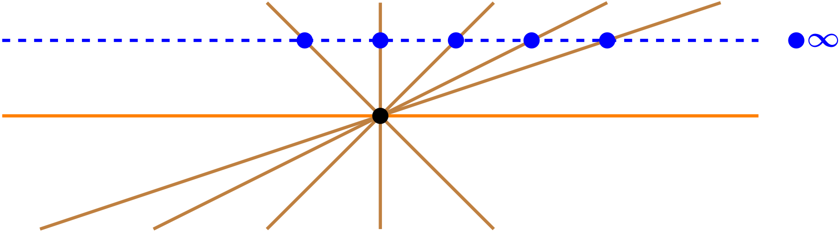


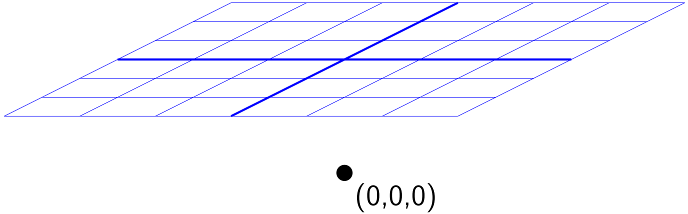
each non-horizontal line $\longleftrightarrow$ **some point on the blue line**

the horizontal line $\longleftrightarrow$ **a new point** ∞

The projective plane $\mathbb{P}^2$

$\mathbb{P}^1 :=$ {lines through $(0,0)$ in $\mathbb{R}^2$} $\longleftrightarrow$ **line** $\cup \{\infty\}$

$\mathbb{P}^2 :=$ {lines through $(0,0,0)$ in $\mathbb{R}^3$} $\longleftrightarrow$ **plane** $\cup$ (   )



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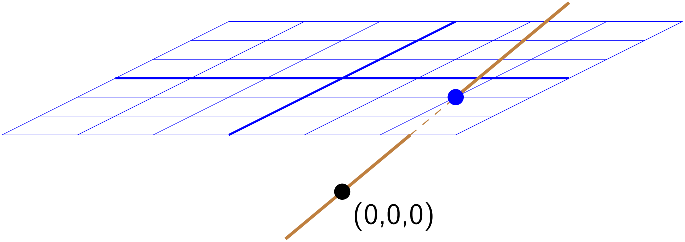

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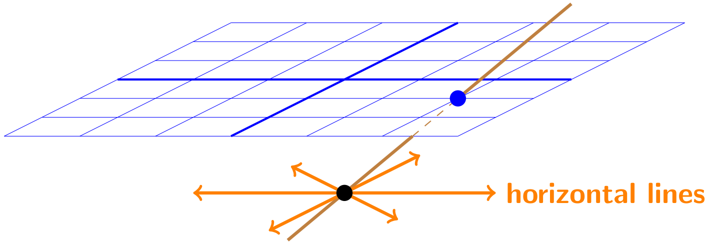

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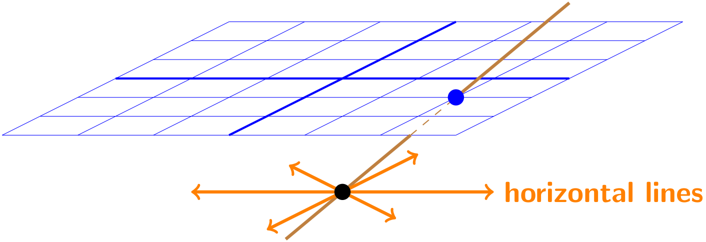
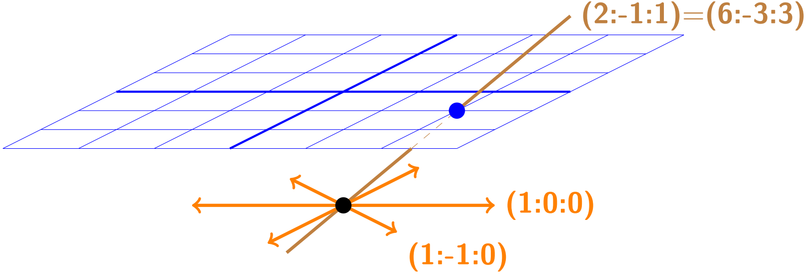


Questions:

- What coordinates can we use on $\mathbb{P}^2$?
- How can we label each line in $\mathbb{R}^3$ through $(0,0,0)$?

Homogeneous coordinates on $\mathbb{P}^2$

Write $(\mathbf{a}:\mathbf{b}:\mathbf{c})$ to mean **the line through $(\mathbf{0},\mathbf{0},\mathbf{0})$ and $(\mathbf{a},\mathbf{b},\mathbf{c})$**.



$$\mathbb{P}^2 = \frac{\mathbb{R}^3 - \{(0,0,0)\}}{\text{scaling}}$$

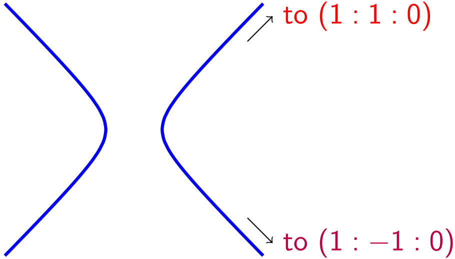
Curves in $\mathbb{P}^2$

Each curve in $\mathbb{R}^2$ can be "completed" to a curve in $\mathbb{P}^2$ by adding points at infinity.

Example

The hyperbola $x^2 - y^2 = 5$ becomes $x^2 - y^2 = 5z^2$.
Points in $\mathbb{R}^2$ like $(3, 2)$ correspond to $(3 : 2 : 1)$ in $\mathbb{P}^2$.

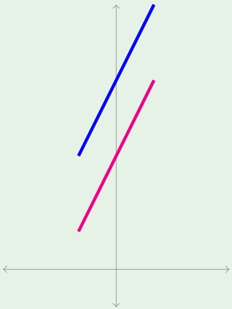
To find the points at infinity, set $z = 0$:
We get $x^2 - y^2 = 0$, which leads to $y = x$ or $y = -x$,
that is, $(x : x : 0) = \boxed{(\mathbf{1} : \mathbf{1} : \mathbf{0})}$ or $(x : -x : 0) = \boxed{(\mathbf{1} : -\mathbf{1} : \mathbf{0})}$.



Intersecting lines in $\mathbb{P}^2$

Example

The lines $y = 2x + 3$ and $y = 2x + 5$ do not intersect.



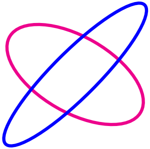
But their projective versions $y = 2x + 3z$ and $y = 2x + 5z$ intersect at the point where $y = 2x$ and $z = 0$, which is $(1 : 2 : 0)$.

Intersecting two curves in $\mathbb{P}^2$

Bézout's theorem

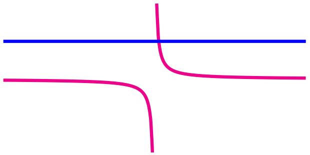
Two plane curves $f(x,y)=0$ and $g(x,y)=0$ of degrees m and n intersect in mn points, **if**

1. *$f(x,y)$ and $g(x,y)$ have no common factor,*
2. *we include intersections at infinity (work in $\mathbb{P}^2$),*
3. *we include intersections with complex number coordinates, and*
4. *points of tangency count as two or more points.*

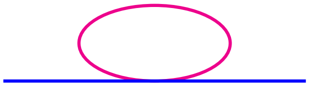


$$2 \cdot 2 = 4$$

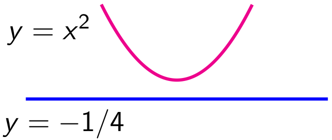
More instances of Bézout's theorem



$2 \cdot 1 = 2$ (one intersection is at infinity)



$2 \cdot 1 = 2$ (tangency counts as 2)



$2 \cdot 1 = 2$ (complex intersection points)

Elliptic curves

Definition

An elliptic curve is the completion in $\mathbb{P}^2$ of the curve

$$y^2 = x^3 + Ax + B,$$

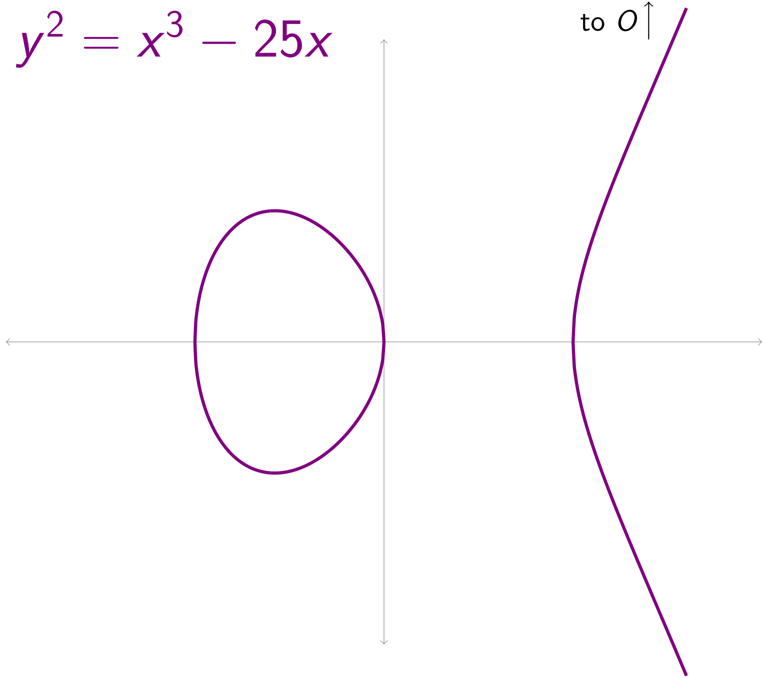
where A and B are numbers
such that $x^3 + Ax + B$ has no double roots.

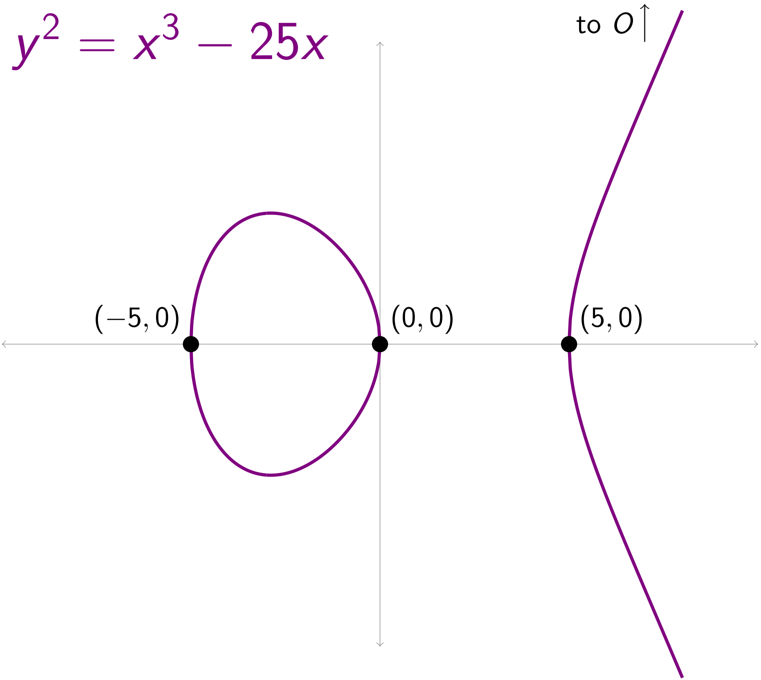
Example

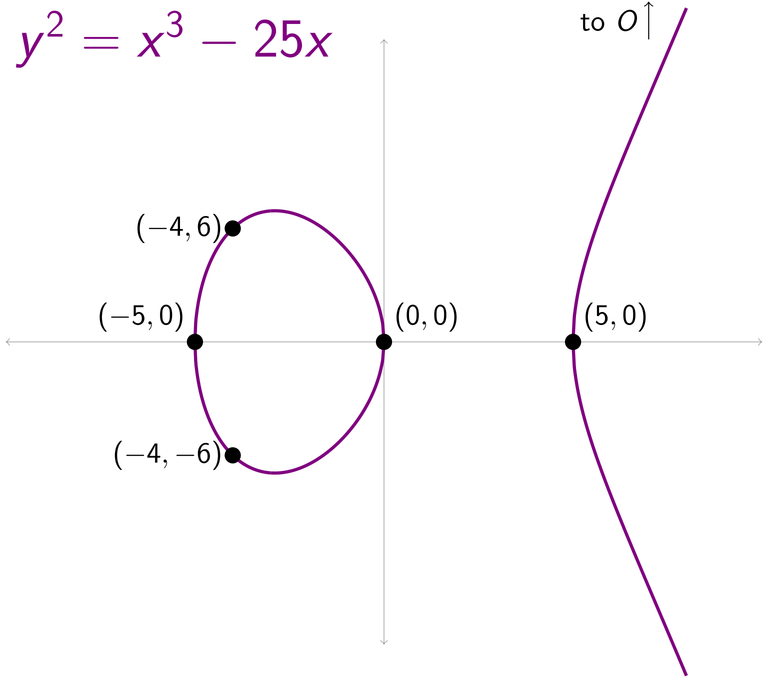
Let E be the completion in $\mathbb{P}^2$ of $y^2 = x^3 - 25x$.
This is an elliptic curve, since the polynomial
$x^3 - 25x = x(x+5)(x-5)$ has no double roots.

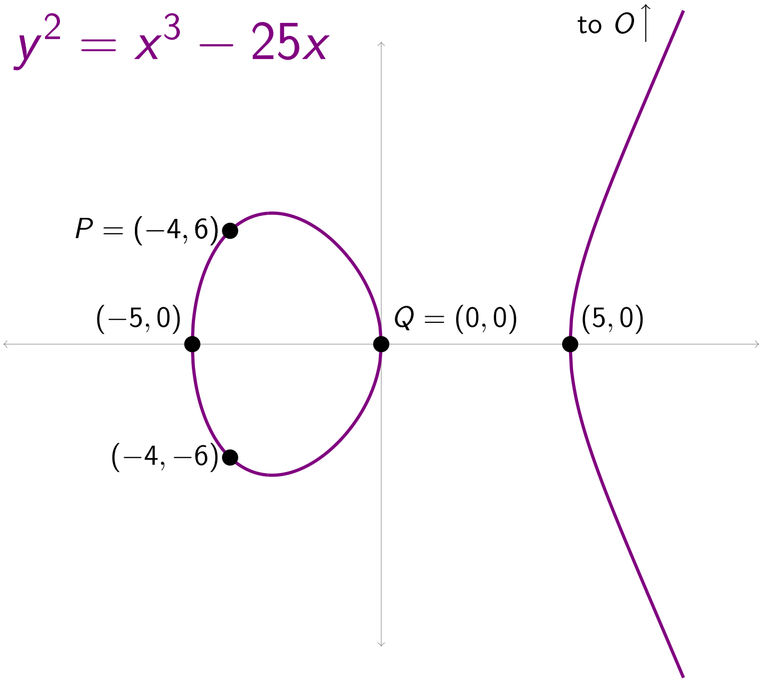
The projective version is $y^2 z = x^3 - 25xz^2$.
To find the points at infinity, set $z = 0$; get $0 = x^3$.

Conclusion: The only point at infinity is $(0 : 1 : 0)$. Call it O.


$y^2 = x^3 - 25x$
to O

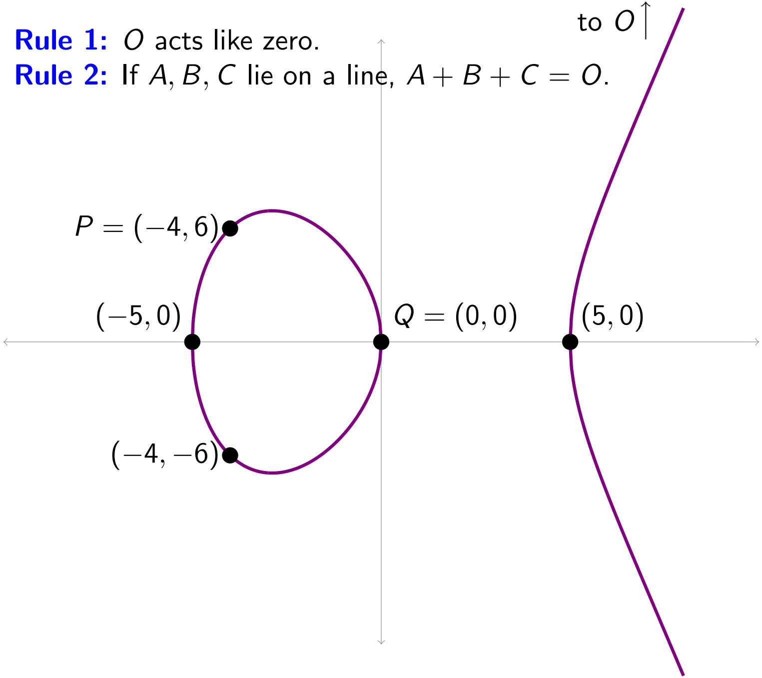

$y^2 = x^3 - 25x$
to O
$(-5, 0)$
$(0, 0)$
$(5, 0)$


$y^2 = x^3 - 25x$
to O ↑
$(-4, 6)$
$(-5, 0)$
$(0, 0)$
$(5, 0)$
$(-4, -6)$


$y^2 = x^3 - 25x$
to O
$P = (-4, 6)$
$(-5, 0)$
$Q = (0, 0)$
$(5, 0)$
$(-4, -6)$

Rule 1: O acts like zero.

Rule 2: If A, B, C lie on a line, $A + B + C = O$.



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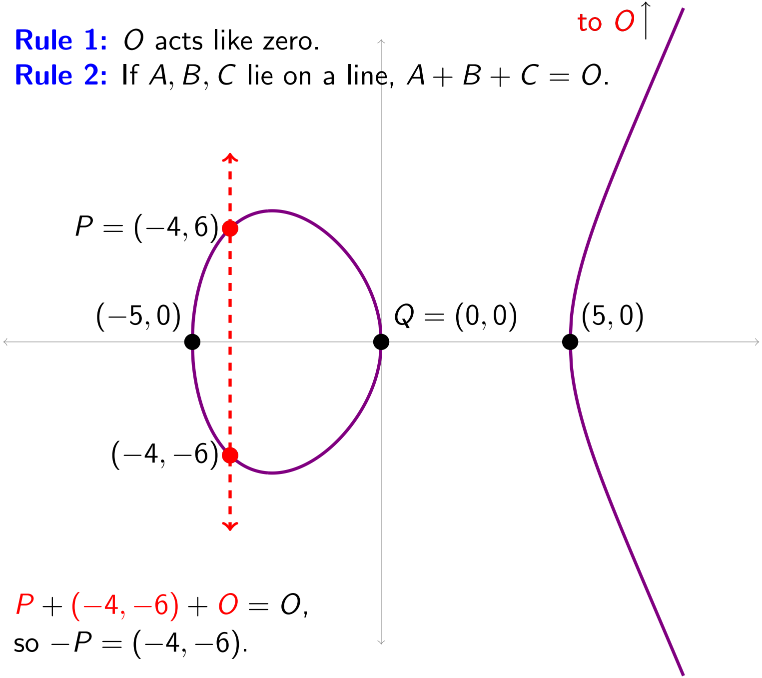

$P + (-4, -6) + O = O$,
so $-P = (-4, -6)$.

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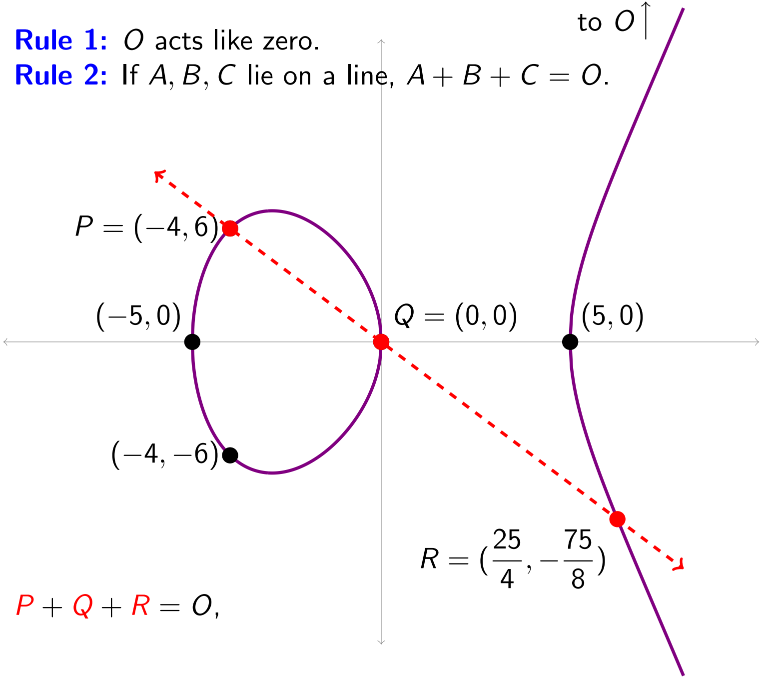

$P + Q + R = O,$

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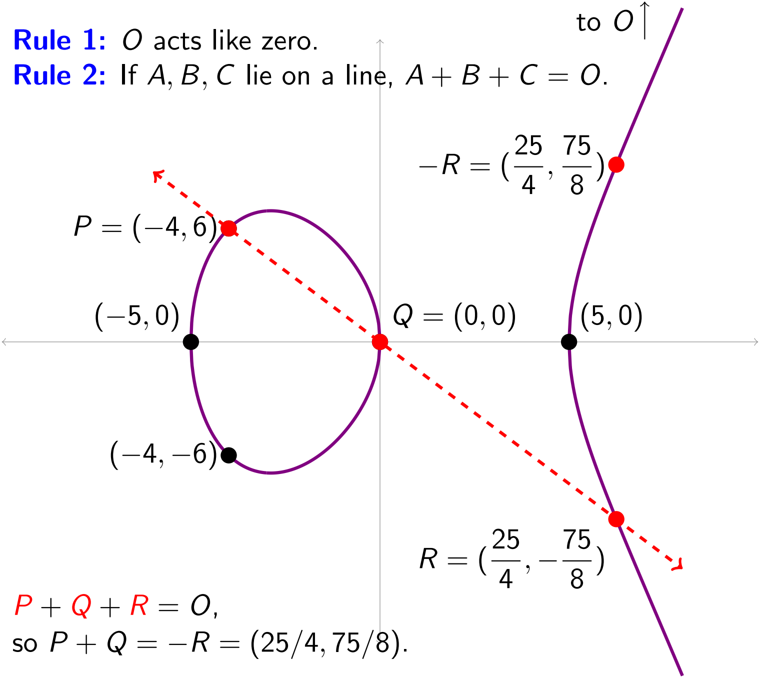

$P + Q + R = O$,

so $P + Q = -R = (25/4, 75/8)$.

Rule 1: O acts like zero.

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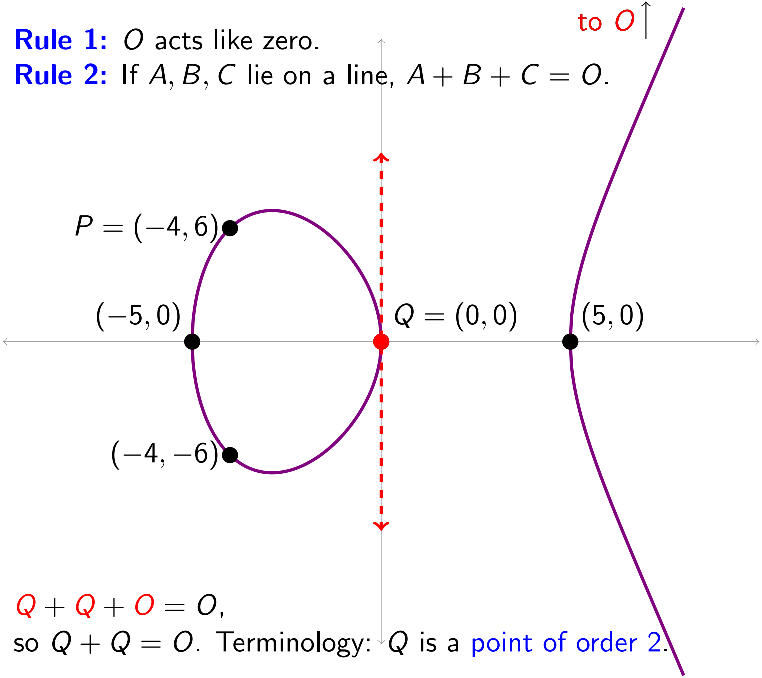


$Q + Q + O = O$,

so $Q + Q = O$. Terminology: Q is a point of order 2.

Generating all rational points from a few starting points

Remark

It turns out that for

$$y^2 = x^3 - 25x,$$

if we start with $P = (-4, 6)$
(*and the points of finite order* $(-5, 0)$, $(0, 0)$, $(5, 0)$),
then all other rational points can be generated from these!

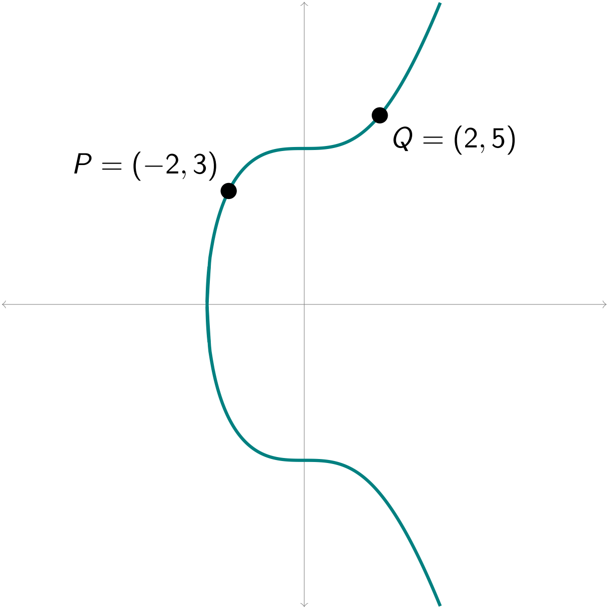
There are infinitely many rational points; in fact,

$$\cdots \quad -3P \quad -2P \quad -P \quad O \quad P \quad 2P \quad 3P \quad \cdots$$

are all distinct.

Because only **one** starting point P was needed
(not counting the points of finite order),
the elliptic curve is said to have rank 1.

The elliptic curve $y^2 = x^3 + 17$



The elliptic curve $y^2 = x^3 + 17$, continued

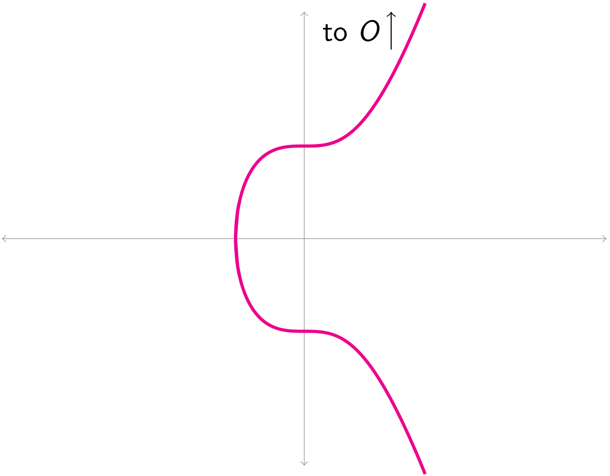
Let $P = (-2, 3)$ and $Q = (2, 5)$. Then the rational points

$$
\begin{array}{ccccccccc}
 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\cdots & -2P+2Q & -P+2Q & 2Q & P+2Q & 2P+2Q & \cdots & \\
\cdots & -2P+Q & -P+Q & Q & P+Q & 2P+Q & \cdots & \\
\cdots & -2P & -P & O & P & 2P & \cdots & \\
\cdots & -2P-Q & -P-Q & -Q & P-Q & 2P-Q & \cdots & \\
\cdots & -2P-2Q & -P-2Q & -2Q & P-2Q & 2P-2Q & \cdots & \\
 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{array}
$$

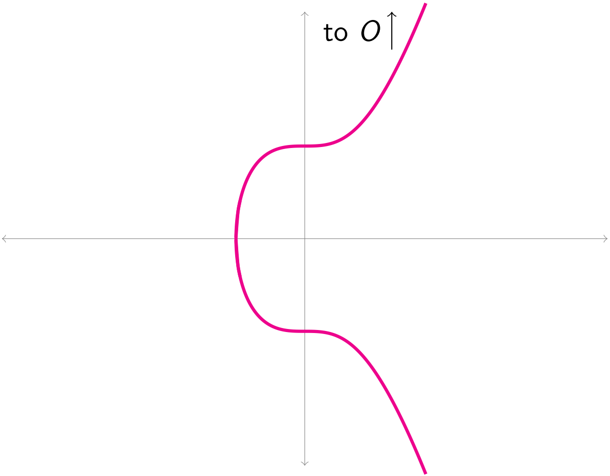
are all distinct, and they are all the rational points on this curve.

Conclusion: $y^2 = x^3 + 17$ has rank 2.

The elliptic curve $y^2 = x^3 + 6$



The elliptic curve $y^2 = x^3 + 6$



The only rational point is O!   So $y^2 = x^3 + 6$ has rank 0.

Mordell's theorem (1922)

For each elliptic curve E, there is a finite list of rational points $P_1, P_2, \ldots, P_n$ such that every other rational point on E can be generated from these.

The number of starting points required

(not including points of finite order, which count as free)

is called the rank of E.

Rank of $y^2 = x^3 + n$

n	rank
1	0
2	1
3	1
4	0
5	1
6	**0**
7	0
8	1
9	1
10	1
11	1
12	1
13	0
14	0
15	2
16	1
17	**2**

Unsolved problems

Problem

Find a method for computing the rank of any given elliptic curve E.

Problem

Find a method for listing points that are guaranteed to generate E.

There is an elliptic curve of rank at least 28 (the record since 2006).

Problem

Is there an elliptic curve whose rank is > 28?

If you want to know more:

Silverman & Tate, *Rational points on elliptic curves*.