p-ADIC INTERPOLATION OF ITERATES

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ABSTRACT. Extending work of Bell and of Bell, Ghioca, and Tucker, we prove that for a p-adic analytic self-map f on a closed unit polydisk, if every coefficient of $f(\mathbf{x}) - \mathbf{x}$ has valuation greater than that of $p^{1/(p-1)}$, then the iterates of f can be p-adically interpolated; i.e., there exists a function $g(\mathbf{x}, n)$ analytic in both $\mathbf{x}$ and n such that $g(\mathbf{x}, n) = f^n(\mathbf{x})$ whenever $n \in \mathbb{Z}_{\geq 0}$.

Inspired by the work of Skolem [Sko34], Mahler [Mah35], and Lech [Lec53] on linear recursive sequences, Bell [Bel08] proved that for a suitable p-adic analytic function f and starting point $\mathbf{x}$, the iterate-computing map $n \mapsto f^n(\mathbf{x})$ extends to a p-adic analytic function $g(n)$ defined for $n \in \mathbb{Z}_p$. This result, along with its generalization by Bell, Ghioca, and Tucker in [BGT10, §3] and earlier linearization results by Herman and Yoccoz [HY83, Theorem 1] and Rivera-Letelier [RL03, §3.2], has significance beyond its intrinsic interest, because of its applications towards the dynamical Mordell–Lang conjecture [Bel06, GT09, BGT10, BGKT12, BGH$^+$13].

Our main result, Theorem 1, is a variant that is best possible (in a sense explained in Remark 3). Our proof is new even over $\mathbb{Q}_p$, and extends immediately to more general valued fields. It settles an open question about the case $p = 3$. The function g we obtain is analytic in $\mathbf{x}$ as well as n.

We now set the notation for our statement. Let p be a prime number. Let K be a field that is complete with respect to an absolute value $|\ |$ satisfying $|p| = 1/p$. Let R be the valuation ring in K. For $f \in R[\mathbf{x}] := R[x_1, \ldots, x_d]$, let $\|f\|$ be the supremum of the absolute values of the coefficients of f. The Tate algebra $R\langle\mathbf{x}\rangle$ is the completion of $R[\mathbf{x}]$ with respect to $\|\ \|$. More concretely, $R\langle\mathbf{x}\rangle$ is the set of $f = \sum_{\mathbf{i} \in \mathbb{Z}^d_{\geq 0}} f_\mathbf{i} \mathbf{x}^\mathbf{i} \in R[[\mathbf{x}]]$ converging on the closed unit polydisk; convergence is equivalent to $|f_\mathbf{i}| \to 0$ as $\mathbf{i} \to \infty$. For $f, g \in R\langle\mathbf{x}\rangle$ and $c \in \mathbb{R}_{\geq 0}$, the notation $f \in p^c R\langle\mathbf{x}\rangle$ means $\|f\| \leq |p|^c$, and $f \equiv g \pmod{p^c}$ means $\|f - g\| \leq |p|^c$; extend componentwise to $f, g \in R\langle\mathbf{x}\rangle^d$.

Theorem 1. *If $f \in R\langle x_1, \ldots, x_d\rangle^d$ satisfies $f(\mathbf{x}) \equiv \mathbf{x} \pmod{p^c}$ for some $c > \frac{1}{p-1}$, then there exists $g \in R\langle x_1, \ldots, x_d, n\rangle^d$ such that $g(\mathbf{x}, n) = f^n(\mathbf{x})$ in $R\langle\mathbf{x}\rangle^d$ for each $n \in \mathbb{Z}_{\geq 0}$.*

Our proof will check directly that the Mahler series [Mah58] interpolating the sequence

$$\mathbf{x}, f(\mathbf{x}), f(f(\mathbf{x})), \ldots$$

converges to an analytic function. This is the difference operator analogue of proving that a function ϕ is analytic by checking that its Taylor series converges to ϕ.

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Proof. Since $f(\mathbf{x}) \equiv \mathbf{x} \pmod{p^c}$, we have $h(f(\mathbf{x})) \equiv h(\mathbf{x}) \pmod{p^c}$ for any $h \in R[\mathbf{x}]^d$ and (by taking limits) also for any $h \in R\langle\mathbf{x}\rangle^d$. In other words, the linear operator Δ defined by

$$(\Delta h)(\mathbf{x}) := h(f(\mathbf{x})) - h(\mathbf{x})$$

maps $R\langle\mathbf{x}\rangle^d$ into $p^c R\langle\mathbf{x}\rangle^d$. In particular, m applications of Δ to the identity function yields $\Delta^m \mathbf{x} \in p^{mc} R\langle\mathbf{x}\rangle^d$. On the other hand, $|m!| \geq p^{-m/(p-1)}$. Thus the Mahler series

$$g(\mathbf{x}, n) := \sum_{m \geq 0} \binom{n}{m} \Delta^m \mathbf{x} = \sum_{m \geq 0} n(n-1)\cdots(n-m+1) \frac{\Delta^m \mathbf{x}}{m!}$$

converges in $R\langle\mathbf{x}, n\rangle^d$ with respect to $\| \; \|$. Let I be the identity operator. If $n \in \mathbb{Z}_{\geq 0}$, then

$$g(\mathbf{x}, n) = \sum_{m=0}^{n} \binom{n}{m} \Delta^m \mathbf{x} = (\Delta + I)^n \mathbf{x} = f^n(\mathbf{x}). \qquad \square$$

Remark 2. The relation $g(\mathbf{x}, n+1) = f(g(\mathbf{x}, n))$ in $R\langle\mathbf{x}\rangle^d$ holds for each n in the infinite set $\mathbb{Z}_{\geq 0}$, so it is an identity in $R\langle\mathbf{x}, n\rangle^d$.

Remark 3. The hypothesis on f holds for $K = \mathbb{Q}_p$ if $f(\mathbf{x}) \equiv \mathbf{x} \pmod{p}$ and $p \geq 3$; previously the conclusion was known only for $p \geq 5$ [Bel08; BGT10, §3]. On the other hand, $f(x) := -x$ is a counterexample for $p = 2$ [Bel08, §3]. Similarly, the inequality on c in Theorem 1 is best possible for each p: consider $f(x) := \zeta x$ where ζ is a primitive p^{th} root of unity in $\mathbb{C}_p$.

Remark 4. Let $\mathfrak{m}$ be the maximal ideal of R. Let $k := R/\mathfrak{m}$. If $f(\mathbf{x}) \bmod \mathfrak{m} = \mathbf{x}$, so that $f(\mathbf{x}) \equiv \mathbf{x} \pmod{p^c}$ holds for some $c > 0$, then $f^p(\mathbf{x}) \equiv \mathbf{x} \pmod{p^c}$ holds for a larger c, and by iterating we find $r \in \mathbb{Z}_{\geq 0}$ such that Theorem 1 applies to f^{p^r}. More generally, if $f(\mathbf{x}) \bmod \mathfrak{m} = A\mathbf{x}$ for some $A \in \mathrm{GL}_d(k)$ of finite order, then there exists $s \in \mathbb{Z}_{>0}$ such that f^s satisfies the hypothesis of Theorem 1. This finite order hypothesis is automatic if K is $\mathbb{Q}_p$ or $\mathbb{C}_p$ since then k is algebraic over $\mathbb{F}_p$ and *every* element of $\mathrm{GL}_d(k)$ is of finite order. Cf. [BGT10, §2.2].

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