THE NUMBER OF INTERSECTION POINTS MADE BY THE DIAGONALS OF A REGULAR POLYGON

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Abstract. We give a formula for the number of interior intersection points made by the diagonals of a regular n-gon. The answer is a polynomial on each residue class modulo 2520. We also compute the number of regions formed by the diagonals, by using Euler's formula $V - E + F = 2$.

1. Introduction

We will find a formula for the number $I(n)$ of intersection points formed inside a regular n-gon by its diagonals. The case $n = 30$ is depicted in Figure 1. For a *generic* convex n-gon, the answer would be $\binom{n}{4}$, because every four vertices would be the endpoints of a unique pair of intersecting diagonals. But $I(n)$ can be less, because in a regular n-gon it may happen that three or more diagonals meet at an interior point, and then some of the $\binom{n}{4}$ intersection points will coincide. In fact, if n is even and at least 6, $I(n)$ will always be less than $\binom{n}{4}$, because there will be $n/2 \geq 3$ diagonals meeting at the center point. It will result from our analysis that for $n > 4$, the maximum number of diagonals of the regular n-gon that meet at a point other than the center is

2 if n is odd,
3 if n is even but not divisible by 6,
5 if n is divisible by 6 but not 30, and,
7 if n is divisible by 30.

with two exceptions: this number is 2 if $n = 6$, and 4 if $n = 12$. In particular, it is impossible to have 8 or more diagonals of a regular n-gon meeting at a point other than the center. Also, by our earlier remarks, the fact that no three diagonals meet when n is odd will imply that $I(n) = \binom{n}{4}$ for odd n.

A careful analysis of the possible configurations of three diagonals meeting will provide enough information to permit us in theory to deduce a formula for $I(n)$. But because the explicit description of these configurations is so complex, our strategy will be instead to use this information to deduce only the *form* of the answer, and then to compute the answer for enough small n that we can determine the result

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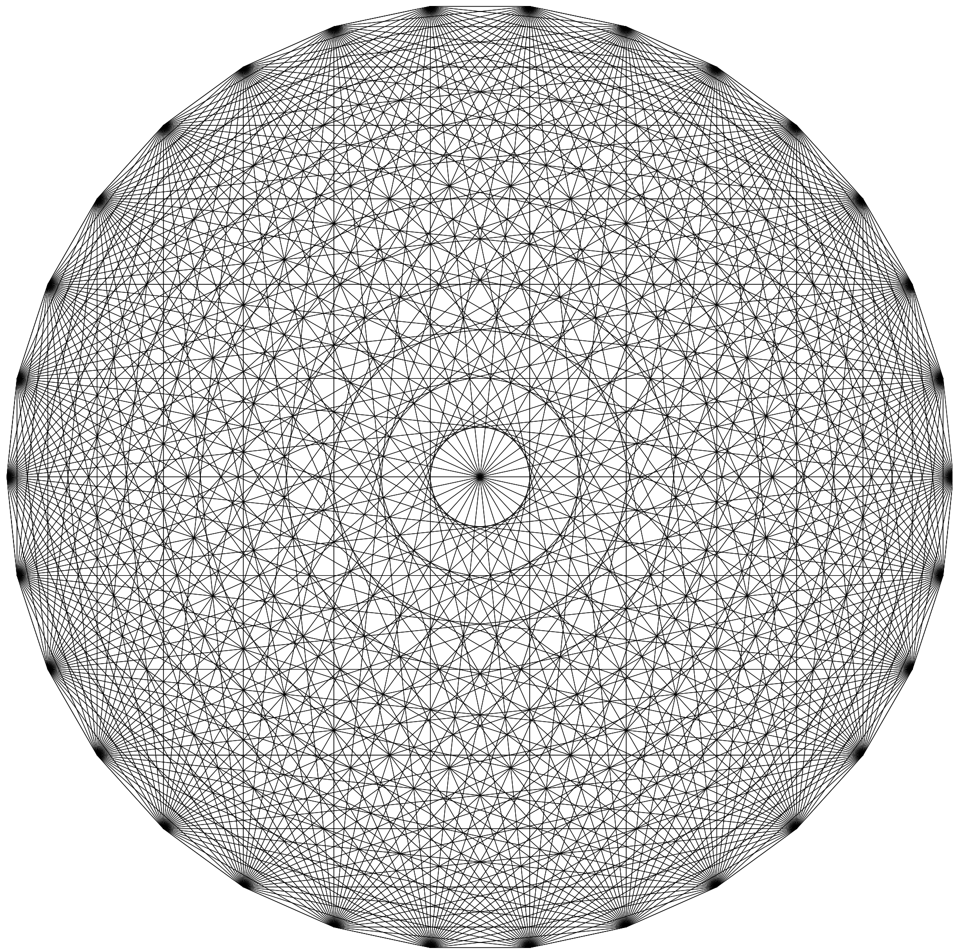

FIGURE 1. The 30-gon with its diagonals. There are 16801 interior intersection points: 13800 two line intersections, 2250 three line intersections, 420 four line intersections, 180 five line intersections, 120 six line intersections, 30 seven line intersections, and 1 fifteen line intersection.

precisely. The computations are done in Mathematica, Maple and C, and annotated source codes can be obtained here: ngon.c, ngon.m.

In order to write the answer in a reasonable form, we define

$$\delta_m(n) = \begin{cases} 1 & \text{if } n \equiv 0 \pmod{m}, \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 1. *For $n \geq 3$,*

$$\begin{aligned} I(n) \quad = \quad & \binom{n}{4} + (-5n^3 + 45n^2 - 70n + 24)/24 \cdot \delta_2(n) - (3n/2) \cdot \delta_4(n) \\ & + (-45n^2 + 262n)/6 \cdot \delta_6(n) + 42n \cdot \delta_{12}(n) + 60n \cdot \delta_{18}(n) \\ & + 35n \cdot \delta_{24}(n) - 38n \cdot \delta_{30}(n) - 82n \cdot \delta_{42}(n) - 330n \cdot \delta_{60}(n) \\ & - 144n \cdot \delta_{84}(n) - 96n \cdot \delta_{90}(n) - 144n \cdot \delta_{120}(n) - 96n \cdot \delta_{210}(n). \end{aligned}$$

Further analysis, involving Euler's formula $V - E + F = 2$, will yield a formula for the number $R(n)$ of regions that the diagonals cut the n-gon into.

Theorem 2. *For $n \geq 3$,*

$$\begin{aligned} R(n) \quad = \quad & (n^4 - 6n^3 + 23n^2 - 42n + 24)/24 \\ & + (-5n^3 + 42n^2 - 40n - 48)/48 \cdot \delta_2(n) - (3n/4) \cdot \delta_4(n) \\ & + (-53n^2 + 310n)/12 \cdot \delta_6(n) + (49n/2) \cdot \delta_{12}(n) + 32n \cdot \delta_{18}(n) \\ & + 19n \cdot \delta_{24}(n) - 36n \cdot \delta_{30}(n) - 50n \cdot \delta_{42}(n) - 190n \cdot \delta_{60}(n) \\ & - 78n \cdot \delta_{84}(n) - 48n \cdot \delta_{90}(n) - 78n \cdot \delta_{120}(n) - 48n \cdot \delta_{210}(n). \end{aligned}$$

These problems have been studied by many authors before, but this is apparently the first time the correct formulas have been obtained. The Dutch mathematician Gerrit Bol [1] gave a complete solution in 1936, except that a few of the coefficients in his formulas are wrong. (A few misprints and omissions in Bol's paper are mentioned in [11].)

The approaches used by us and Bol are similar in many ways. One difference (which is not too substantial) is that we work as much as possible with roots of unity whereas Bol tended to use more trigonometry (integer relations between sines of rational multiples of π). Also, we relegate much of the work to the computer, whereas Bol had to enumerate the many cases by hand. The task is so formidable that it is amazing to us that Bol was able to complete it, and at the same time not so surprising that it would contain a few errors!

Bol's work was largely forgotten. In fact, even we were not aware of his paper until after deriving the formulas ourselves. Many other authors in the interim solved special cases of the problem. Steinhaus [14] posed the problem of showing that no three diagonals meet internally when n is prime, and this was solved by Croft and Fowler [3]. (Steinhaus also mentions this in [13], which includes a picture of the 23-gon and its diagonals.) In the 1960s, Heineken [6] gave a delightful argument which generalized this to all odd n, and later he [7] and Harborth [4] independently enumerated all three-diagonal intersections for n not divisible by 6.

The classification of three-diagonal intersections also solves Colin Tripp's problem [15] of enumerating "adventitious quadrilaterals," those convex quadrilaterals for which the angles formed by sides and diagonals are all rational multiples of π. See Rigby's paper [11] or the summary [10] for details. Rigby, who was aware of

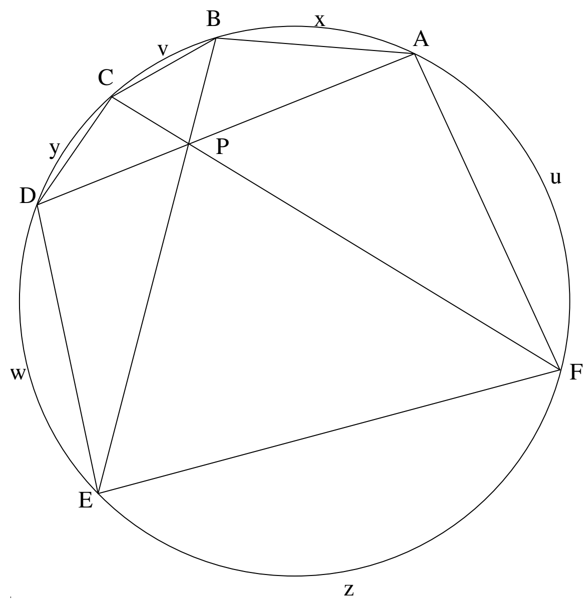

FIGURE 2.

Bol's work, mentions that Monsky and Pleasants also each independently classified all three-diagonal intersections of regular n-gons. Rigby's papers partially solve Tripp's further problem of proving the existence of all adventitious quadrangles using only elementary geometry; i.e., without resorting to trigonometry.

All the questions so far have been in the Euclidean plane. What happens if we count the interior intersections made by the diagonals of a hyperbolic regular n-gon? The answers are exactly the same, as pointed out in [11], because if we use Beltrami's representation of points of the hyperbolic plane by points inside a circle in the Euclidean plane, we can assume that the center of the hyperbolic n-gon corresponds to the center of the circle, and then the hyperbolic n-gon with its diagonals looks in the model exactly like a Euclidean regular n-gon with its diagonals. It is equally easy to see that the answers will be the same in elliptic geometry.

2. When do three diagonals meet?

We now begin our derivations of the formulas for $I(n)$ and $R(n)$. The first step will be to find a criterion for the concurrency of three diagonals. Let A, B, C, D, E, F be six distinct points in order on a unit circle dividing up the circumference into arc lengths u, x, v, y, w, z and assume that the three chords AD, BE, CF meet at P (see Figure 2).

By similar triangles, $AF/CD = PF/PD$, $BC/EF = PB/PF$, $DE/AB = PD/PB$. Multiplying these together yields

$$(AF \cdot BC \cdot DE)/(CD \cdot EF \cdot AB) = 1,$$

and so

$$\sin(u/2)\sin(v/2)\sin(w/2) = \sin(x/2)\sin(y/2)\sin(z/2). \tag{1}$$

Conversely, suppose six distinct points A, B, C, D, E, F partition the circumference of a unit circle into arc lengths u, x, v, y, w, z and suppose that (1) holds. Then the three diagonals AD, BE, CF meet in a single point which we see as follows. Let lines AD and BE intersect at P_0 . Form the line through F and P_0 and let C' be the other intersection point of FP_0 with the circle. This partitions the circumference into arc lengths u, x, v', y', w, z. As shown above, we have

$$\sin(u/2)\sin(v'/2)\sin(w/2) = \sin(x/2)\sin(y'/2)\sin(z/2)$$

and since we are assuming that (1) holds for u, x, v, y, w, z we get

$$\frac{\sin(v'/2)}{\sin(y'/2)} = \frac{\sin(v/2)}{\sin(y/2)}.$$

Let $\alpha = v + y = v' + y'$. Substituting $v = \alpha - y$, $v' = \alpha - y'$ above we get

$$\frac{\sin(\alpha/2)\cos(y'/2) - \cos(\alpha/2)\sin(y'/2)}{\sin(y'/2)} = \frac{\sin(\alpha/2)\cos(y/2) - \cos(\alpha/2)\sin(y/2)}{\sin(y/2)}$$

and so

$$\cot(y'/2) = \cot(y/2).$$

Now $0 < \alpha/2 < \pi$, so $y = y'$ and hence $C = C'$. Thus, the three diagonals AD, BE, CF meet at a single point.

So (1) gives a necessary and sufficient condition (in terms of arc lengths) for the chords AD, BE, CF formed by six distinct points A, B, C, D, E, F on a unit circle to meet at a single point. In other words, to give an explicit answer to the question in the section title, we need to characterize the positive rational solutions to

$$\begin{aligned} \sin(\pi U)\sin(\pi V)\sin(\pi W) &= \sin(\pi X)\sin(\pi Y)\sin(\pi Z) \\ U + V + W + X + Y + Z &= 1. \end{aligned} \tag{2}$$

(Here $U = u/(2\pi)$, etc.) This is a trigonometric diophantine equation in the sense of [2], where it is shown that in theory, there is a finite computation which reduces the solution of such equations to ordinary diophantine equations. The solutions to the analogous equation with only two sines on each side are listed in [9].

If in (2), we substitute $\sin(\theta) = (e^{i\theta} - e^{-i\theta})/(2i)$, multiply both sides by $(2i)^3$, and expand, we get a sum of eight terms on the left equalling a similar sum on the right, but two terms on the left cancel with two terms on the right since $U+V+W = 1 - (X + Y + Z)$, leaving

$$\begin{aligned} &-e^{i\pi(V+W-U)} + e^{-i\pi(V+W-U)} - e^{i\pi(W+U-V)} + e^{-i\pi(W+U-V)} - e^{i\pi(U+V-W)} + e^{-i\pi(U+V-W)} = \\ &\quad -e^{i\pi(Y+Z-X)} + e^{-i\pi(Y+Z-X)} - e^{i\pi(Z+X-Y)} + e^{-i\pi(Z+X-Y)} - e^{i\pi(X+Y-Z)} + e^{-i\pi(X+Y-Z)}. \end{aligned}$$

If we move all terms to the left hand side, convert minus signs into $e^{-i\pi}$, multiply by $i = e^{i\pi/2}$, and let

$$\begin{aligned}
\alpha_1 &= V + W - U - 1/2 \\
\alpha_2 &= W + U - V - 1/2 \\
\alpha_3 &= U + V - W - 1/2 \\
\alpha_4 &= Y + Z - X + 1/2 \\
\alpha_5 &= Z + X - Y + 1/2 \\
\alpha_6 &= X + Y - Z + 1/2,
\end{aligned}$$

we obtain

$$\sum_{j=1}^{6} e^{i\pi\alpha_j} + \sum_{j=1}^{6} e^{-i\pi\alpha_j} = 0, \tag{3}$$

in which $\sum_{j=1}^{6} \alpha_j = U+V+W+X+Y+Z = 1$. Conversely, given rational numbers $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6$ (not necessarily positive) which sum to 1 and satisfy (3), we can recover U, V, W, X, Y, Z, (for example, $U = (\alpha_2 + \alpha_3)/2 + 1/2$), but we must check that they turn out positive.

3. Zero as a sum of 12 roots of unity

In order to enumerate the solutions to (2), we are led, as in the end of the last section, to classify the ways in which 12 roots of unity can sum to zero. More generally, we will study relations of the form

$$\sum_{i=1}^{k} a_i \eta_i = 0, \tag{4}$$

where the a_i are positive integers, and the η_i are distinct roots of unity. (These have been studied previously by Schoenberg [12], Mann [8], Conway and Jones [2], and others.) We call $w(S) = \sum_{i=1}^{k} a_i$ the *weight* of the relation S. (So we shall be particularly interested in relations of weight 12.) We shall say the relation (4) is *minimal* if it has no nontrivial subrelation; i.e., if

$$\sum_{i=1}^{k} b_i \eta_i = 0, \quad a_i \geq b_i \geq 0$$

implies either $b_i = a_i$ for all i or $b_i = 0$ for all i. By induction on the weight, any relation can be represented as a sum of minimal relations (but the representation need not be unique).

Let us give some examples of minimal relations. For each $n \geq 1$, let $\zeta_n = \exp(2\pi i/n)$ be the standard primitive n-th root of unity. For each prime p, let R_p be the relation

$$1 + \zeta_p + \zeta_p^2 + \cdots + \zeta_p^{p-1} = 0.$$

Its minimality follows from the irreducibility of the cyclotomic polynomial. Also we can "rotate" any relation by multiplying through by an arbitrary root of unity to obtain a new relation. In fact, Schoenberg [12] proved that every relation (even those with possibly negative coefficients) can be obtained as a linear combination with positive and negative integral coefficients of the R_p and their rotations. But we are only allowing positive combinations, so it is not clear that these are enough to generate all relations.

In fact it is not even true! In other words, there are other minimal relations. If we subtract R_3 from R_5, cancel the 1's and incorporate the minus signs into the roots of unity, we obtain a new relation

$$\zeta_6 + \zeta_6^{-1} + \zeta_5 + \zeta_5^2 + \zeta_5^3 + \zeta_5^4 = 0, \tag{5}$$

which we will denote $(R_5 : R_3)$. In general, if S and $T_1, T_2, \ldots, T_j$ are relations, we will use the notation $(S : T_1, T_2, \ldots, T_j)$ to denote any relation obtained by rotating the T_i so that each shares exactly one root of unity with S which is different for each i, subtracting them from S, and incorporating the minus signs into the roots of unity. For notational convenience, we will write $(R_5 : 4R_3)$ for $(R_5 : R_3, R_3, R_3, R_3)$, for example. Note that although $(R_5 : R_3)$ denotes unambiguously (up to rotation) the relation listed in (5), in general there will be many relations of type $(S : T_1, T_2, \ldots, T_j)$ up to rotational equivalence. Let us also remark that including R_2's in the list of T's has no effect.

It turns out that recursive use of the construction above is enough to generate all minimal relations of weight up to 12. These are listed in Table 1. The completeness and correctness of the table will be proved in Theorem 3 below. Although there are 107 minimal relations up to rotational equivalence, often the minimal relations within one of our classes are Galois conjugates. For example, the two minimal relations of type $(R_5 : 2R_3)$ are conjugate under $\mathrm{Gal}(\mathbb{Q}(\zeta_{15})/\mathbb{Q})$, as pointed out in [8].

The minimal relations with $k \leq 7$ (k defined as in (4)) had been previously catalogued in [8], and those with $k \leq 9$ in [2]. In fact, the a_i in these never exceed 1, so these also have weight less than or equal to 9.

Theorem 3. *Table 1 is a complete listing of the minimal relations of weight up to 12 (up to rotation).*

The following three lemmas will be needed in the proof.

Lemma 1. *If the relation (4) is minimal, then there are distinct primes $p_1 < p_2 < \cdots < p_s \leq k$ so that each η_i is a $p_1 p_2 \cdots p_s$-th root of unity, after the relation has been suitably rotated.*

Proof. This is a corollary of Theorem 1 in [8]. □

Lemma 2. *The only minimal relations (up to rotation) involving only the $2p$-th roots of unity, for p prime, are R_2 and R_p.*

Proof. Any $2p$-th root of unity is of the form $\pm\zeta^i$. If both $+\zeta^i$ and $-\zeta^i$ occurred in the same relation, then R_2 occurs as a subrelation. So the relation has the form

$$\sum_{i=0}^{p-1} c_i \zeta_p^i = 0$$

By the irreducibility of the cyclotomic polynomial, $\{1, \zeta_p, \ldots, \zeta_p^{p-1}\}$ are independent over $\mathbb{Q}$ save for the relation that their sum is zero, so all the c_i must be equal. If they are all positive, then R_p occurs as a subrelation. If they are all negative, then R_p rotated by -1 (i.e., 180 degrees) occurs as a subrelation. □

Weight	Relation type	Number of relations of that type
2	R_2	1
3	R_3	1
5	R_5	1
6	$(R_5 : R_3)$	1
7	$(R_5 : 2R_3)$	2
	R_7	1
8	$(R_5 : 3R_3)$	2
	$(R_7 : R_3)$	1
9	$(R_5 : 4R_3)$	1
	$(R_7 : 2R_3)$	3
10	$(R_7 : 3R_3)$	5
	$(R_7 : R_5)$	1
11	$(R_7 : 4R_3)$	5
	$(R_7 : R_5, R_3)$	6
	$(R_7 : (R_5 : R_3))$	6
	R_{11}	1
12	$(R_7 : 5R_3)$	3
	$(R_7 : R_5, 2R_3)$	15
	$(R_7 : (R_5 : R_3), R_3)$	36
	$(R_7 : (R_5 : 2R_3))$	14
	$(R_{11} : R_3)$	1

Table 1. The 107 minimal relations of weight up to 12.

Lemma 3. *Suppose S is a minimal relation, and $p_1 < p_2 < \cdots < p_s$ are picked as in Lemma 1 with $p_1 = 2$ and p_s minimal. If $w(S) < 2p_s$, then S (or a rotation) is of the form $(R_{p_s} : T_1, T_2, \dots, T_j)$ where the T_i are minimal relations not equal to R_2 and involving only $p_1p_2 \cdots p_{s-1}$-th roots of unity, such that $j < p_s$ and*

$$\sum_{i=1}^{j} [w(T_i) - 2] = w(S) - p_s.$$

Proof. Since every $p_1p_2 \dots p_s$-th root of unity is uniquely expressible as the product of a $p_1p_2 \dots p_{s-1}$-th root of unity and a p_s-th root of unity, the relation can be rewritten as

$$\sum_{i=0}^{p_s-1} f_i \zeta_{p_s}^i = 0, \tag{6}$$

where each f_i is a sum of $p_1p_2 \dots p_{s-1}$-th roots of unity, which we will think of as a sum (not just its value).

Let K_m be the field obtained by adjoining the $p_1p_2 \dots p_m$-th roots of unity to $\mathbb{Q}$. Since $[K_s : K_{s-1}] = \phi(p_1p_2 \cdots p_s)/\phi(p_1p_2 \cdots p_{s-1}) = \phi(p_s) = p_s - 1$, the only linear relation satisfied by $1, \zeta_{p_s}, \dots, \zeta_{p_s}^{p_s-1}$ over K_{s-1} is that their sum is zero. Hence (6) forces the values of the f_i to be equal.

The total number of roots of unity in all the f_i's is $w(S) < 2p_s$, so by the pigeonhole principle, some f_i is zero or consists of a single root of unity. In the former case, each f_j sums to zero, but at least two of these sums contain at least one root of unity, since otherwise s was not minimal, so one of these sums gives a subrelation of S, contradicting its minimality. So some f_i consists of a single root of unity. By rotation, we may assume $f_0 = 1$. Then each f_i sums to 1, and if it is not simply the single root of unity 1, the negatives of the roots of unity in f_i together with 1 form a relation T_i which is not R_2 and involves only $p_1 p_2 \cdots p_{s-1}$-th roots of unity, and it is clear that S is of type $(R_{p_s} : T_{i_1}, T_{i_2}, \ldots, T_{i_j})$. If one of the T's were not minimal, then it could be decomposed into two nontrivial subrelations, one of which would not share a root of unity with the R_{p_s}, and this would give a nontrivial subrelation of S, contradicting the minimality of S. Finally, $w(S)$ must equal the sum of the weights of R_{p_s} and the T's, minus $2j$ to account for the roots of unity that are cancelled in the construction of $(R_{p_s} : T_{i_1}, T_{i_2}, \ldots, T_{i_j})$. □

Proof of Theorem 3. We will content ourselves with proving that every relation of weight up to 12 can be decomposed into a sum of the ones listed in Table 1, it then being straightforward to check that the entries in the table are distinct, and that none of them can be further decomposed into relations higher up in the table.

Let S be a minimal relation with $w(S) \le 12$. Pick $p_1 < p_2 < \cdots < p_s$ as in Lemma 1 with $p_1 = 2$ and p_s minimal. In particular, $p_s \le 12$, so $p_s =$ 2,3,5,7, or 11.

Case 1: $p_s \le 3$

Here the only minimal relations are R_2 and R_3, by Lemma 2.

Case 2: $p_s = 5$

If $w(S) < 10$, then we may apply Lemma 3 to deduce that S is of type $(R_5 : T_1, T_2, \ldots, T_j)$ Each T must be R_3 (since $p_{s-1} \le 3$), and $j = w(S) - 5$ by the last equation in Lemma 3. The number of relations of type $(R_5 : jR_3)$, up to rotation, is $\binom{5}{j}/5$. (There are $\binom{5}{j}$ ways to place the R_3's, but one must divide by 5 to avoid counting rotations of the same relation.)

If $10 \le w(S) \le 12$, then write S as in (6). If some f_i consists of zero or one roots of unity, then the argument of Lemma 3 applies, and S must be of the form $(R_5 : jR_3)$ with $j \le 4$, which contradicts the last equation in the Lemma. Otherwise the numbers of (sixth) roots of unity occurring in f_0, f_1, f_2, f_3, f_4 must be 2,2,2,2,2 or 2,2,2,2,3 or 2,2,2,3,3 or 2,2,2,2,4 in some order. So the common value of the f_i is a sum of two sixth roots of unity. By rotating by a sixth root of unity, we may assume this value is 0, 1, $1 + \zeta_6$, or 2. If it is 0 or 1, then the arguments in the proof of Lemma 3 apply. Next assume it is $1 + \zeta_6$. The only way two sixth roots of unity can sum to $1 + \zeta_6$ is if they are 1 and ζ_6 in some order. The only ways three sixth roots of unity can sum to $1 + \zeta_6$ is if they are $1, 1, \zeta_6^2$ or $\zeta_6, \zeta_6, \zeta_6^{-1}$. So if the numbers of roots of unity occurring in f_0, f_1, f_2, f_3, f_4 are 2,2,2,2,2 or 2,2,2,2,3, then S will contain R_5 or its rotation by ζ_6, and the same will be true for 2,2,2,3,3 unless the two f_i with three terms are $1 + 1 + \zeta_6^2$ and $\zeta_6 + \zeta_6 + \zeta_6^{-1}$, in which case S contains $(R_5 : R_3)$. It is impossible to write $1 + \zeta_6$ as a sum of sixth roots of unity without using 1 or ζ_6, so if the numbers are 2,2,2,2,4, then again S contains R_5 or its rotation by ζ_6. Thus we get no new relations where the common value of the f_i is $1 + \zeta_6$. Lastly, assume this common value is 2. Any representation of 2 as a sum of four or fewer sixth roots of unity contains 1, unless it is $\zeta_6 + \zeta_6 + \zeta_6^{-1} + \zeta_6^{-1}$, so S will contain R_5 except possibly in the case where f_0, f_1, f_2, f_3, f_4 are 2,2,2,2,4 in some order, and the 4 is as above. But in this final remaining case, S contains $(R_5 : R_3)$. Thus there are no minimal relations S with $p_s = 5$ and $10 \le w(S) \le 12$.

Case 3: $p_s = 7$

Since $w(S) \leq 12 < 2 \cdot 7$, we can apply Lemma 3. Now the sum of $w(T_i) - 2$ is required to be $w(S) - 7$ which is at most 5, so the T's that may be used are R_3, R_5, $(R_5 : R_3)$, and the two of type $(R_5 : 2R_3)$, for which weight minus 2 equals 1, 3, 4, and 5, respectively. So the problem is reduced to listing the partitions of $w(S) - 7$ into parts of size 1, 3, 4, and 5.

If all parts used are 1, then we get $(R_7 : jR_3)$ with $j = w(S) - 7$, and there are $\binom{7}{j}/7$ distinct relations in this class. Otherwise exactly one part of size 3, 4, or 5 is used, and the possibilities are as follows. If a part of size 3 is used, we get $(R_7 : R_5)$, $(R_7 : R_5, R_3)$, or $(R_7 : R_5, 2R_3)$, of weights 10, 11, 12 respectively. By rotation, the R_5 may be assumed to share the 1 in the R_7, and then there are $\binom{6}{i}$ ways to place the R_3's where i is the number of R_3's. If a part of size 4 is used, we get $(R_7 : (R_5 : R_3))$ of weight 11 or $(R_7 : (R_5 : R_3), R_3)$ of weight 12. By rotation, the $(R_5 : R_3)$ may be assumed to share the 1 in the R_7, but any of the six roots of unity in the $(R_5 : R_3)$ may be rotated to be 1. The R_3 can then overlap any of the other 6 seventh roots of unity. Finally, if a part of size 5 is used, we get $(R_7 : (R_5 : 2R_3))$. There are two different relations of type $(R_5 : 2R_3)$ that may be used, and each has seven roots of unity which may be rotated to be the 1 shared by the R_7, so there are 14 of these all together.

Case 4: $p_s = 11$

Applying Lemma 3 shows that the only possibilities are R_{11} of weight 11, and $(R_{11} : R_3)$ of weight 12. ☐

Now a general relation of weight 12 is a sum of the minimal ones of weight up to 12, and we can classify them according to the weights of the minimal relations, which form a partition of 12 with no parts of size 1 or 4. We will use the notation $(R_5 : 2R_3) + 2R_3$, for example, to denote a sum of three minimal relations of type $(R_5 : 2R_3)$, R_3, and R_3. Table 2 lists the possibilities. The parts may be rotated independently, so any category involving more than one minimal relation contains infinitely many relations, even up to rotation (of the entire relation). Also, the categories are not mutually exclusive, because of the non-uniqueness of the decomposition into minimal relations.

4. Solutions to the trigonometric equation

Here we use the classification of the previous section to give a complete listing of the solutions to the trigonometric equation (2). There are some obvious solutions to (2), namely those in which U, V, W are arbitary positive rational numbers with sum 1/2, and X, Y, Z are a permutation of U, V, W. We will call these the trivial solutions, even though the three-diagonal intersections they give rise to can look surprising. See Figure 3 for an example on the 16-gon.

The twelve roots of unity occurring in (3) are not arbitrary; therefore we must go through Table 2 to see which relations are of the correct form, i.e., expressible as a sum of six roots of unity and their inverses, where the product of the six is -1. First let us prove a few lemmas that will greatly reduce the number of cases.

Lemma 4. *Let S be a relation of weight $k \leq 12$. Suppose S is stable under complex conjugation (i.e., under $\zeta \mapsto \zeta^{-1}$). Then S has a complex conjugation-stable decomposition into minimal relations; i.e., each minimal relation occurring*

Partition	Relation type
12	$(R_7 : 5R_3)$
	$(R_7 : R_5, 2R_3)$
	$(R_7 : (R_5 : R_3), R_3)$
	$(R_7 : (R_5 : 2R_3))$
	$(R_{11} : R_3)$
10,2	$(R_7 : 3R_3) + R_2$
	$(R_7 : R_5) + R_2$
9,3	$(R_5 : 4R_3) + R_3$
	$(R_7 : 2R_3) + R_3$
8,2,2	$(R_5 : 3R_3) + 2R_2$
	$(R_7 : R_3) + 2R_2$

Partition	Relation type
7,5	$(R_5 : 2R_3) + R_5$
	$R_7 + R_5$
7,3,2	$(R_5 : 2R_3) + R_3 + R_2$
	$R_7 + R_3 + R_2$
6,6	$2(R_5 : R_3)$
6,3,3	$(R_5 : R_3) + 2R_3$
6,2,2,2	$(R_5 : R_3) + 3R_2$
5,5,2	$2R_5 + R_2$
5,3,2,2	$R_5 + R_3 + 2R_2$
3,3,3,3	$4R_3$
3,3,2,2,2	$2R_3 + 3R_2$
2,2,2,2,2,2	$6R_2$

TABLE 2. The types of relations of weight 12.

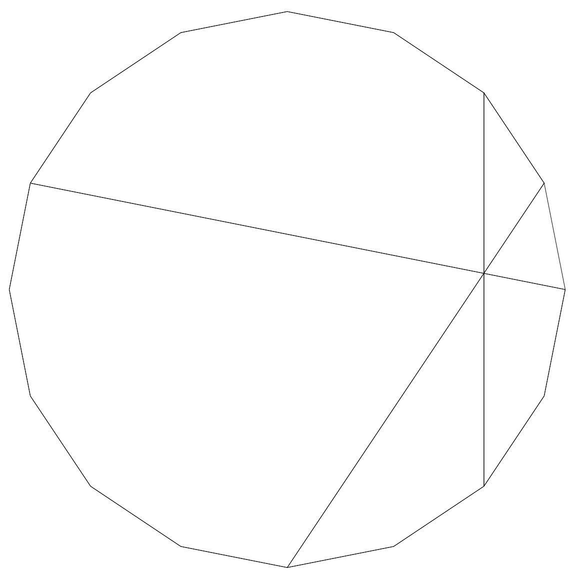

FIGURE 3. A surprising trivial solution for the 16-gon. The intersection point does not lie on any of the 16 lines of symmetry of the 16-gon.

is itself stable under complex conjugation, or can be paired with another minimal relation which is its complex conjugate.

Proof. We will use induction on k. If S is minimal, there is nothing to prove. Otherwise let T be a (minimal) subrelation of S of minimal weight, so T is of weight at most 6. The complex conjugate $\overline{T}$ of T is another minimal relation in S. If they do not intersect, then we take the decomposition of S into T, $\overline{T}$, and a decomposition of $S \setminus (T \cup \overline{T})$ given by the inductive hypothesis. If they do overlap and the weight of T is at most 5, then $T = R_p$ for some prime p, and the fact that T intersects $\overline{T}$ implies that $T = \overline{T}$, and we get the result by applying the inductive hypothesis to $S \setminus T$.

The only remaining case is where S is of type $2(R_5 : R_3)$. If the two $(R_5 : R_3)$'s are not conjugate to each other, then for each there is a root of unity ζ such that ζ and ζ^{-1} occur in that (rotation of) $(R_5 : R_3)$. The quotient ζ^2 is then a 30-th root of unity, so ζ itself is a 60-th root of unity. Thus each $(R_5 : R_3)$ is a rotation of the "standard" $(R_5 : R_3)$ as in (5) by a 60-th root of unity, and we let Mathematica check the 60^2 possibilities. □

We do not know if the preceding lemma holds for relations of weight greater than 12.

Lemma 5. *Let S be a minimal relation of type $(R_p : T_1, \dots, T_j)$, $p \geq 5$, where the T_i involve roots of unity of order prime to p, and $j < p$. If S is stable under complex conjugation, then the particular rotation of R_p from which the T_i were "subtracted" is also stable (and hence so is the collection of the relations subtracted).*

Proof. Let ℓ be the product of the orders of the roots of unity in all the T_i. The elements of S in the original R_p can be characterized as those terms of S that are unique in their coset of μ_ℓ (the ℓ-th roots of unity), and this condition is stable under complex conjugation, so the set of terms of the R_p that were not subtracted is stable. Since $j < p$, we can pick one such term ζ. Then the quotient ζ/ζ^{-1} is a p-th root of unity, so ζ is a $2p$-th root of unity, and hence the R_p containing it is stable. □

Corollary 1. *A relation of type $(R_7 : (R_5 : R_3), R_3)$ cannot be stable under complex conjugation.*

Even with these restrictions, a very large number of cases remain, so we perform the calculation using Mathematica. Each entry of Table 2 represents a finite number of linearly parameterized (in the exponents) families of relations of weight 12. For each parameterized family, we check to see what additional constraints must be put on the parameters for the relation to be of the form of (3). Next, for each parameterized family of solutions to (3), we calculate the corresponding U, V, W, X, Y, Z and throw away solutions in which some of these are nonpositive. Finally, we sort U, V, W and X, Y, Z and interchange the two triples if $U > X$, in order to count the solutions only up to symmetry.

The results of this computation are recorded in the following theorem.

Theorem 4. *The positive rational solutions to (2), up to symmetry, can be classified as follows:*

(1) *The trivial solutions, which arise from relations of type $6R_2$.*

U	V	W	X	Y	Z	Range
$1/6$	t	$1/3-2t$	$1/3+t$	t	$1/6-t$	$0<t<1/6$
$1/6$	$1/2-3t$	t	$1/6-t$	$2t$	$1/6+t$	$0<t<1/6$
$1/6$	$1/6-2t$	$2t$	$1/6-2t$	t	$1/2+t$	$0<t<1/12$
$1/3-4t$	t	$1/3+t$	$1/6-2t$	$3t$	$1/6+t$	$0<t<1/12$

TABLE 3. The nontrivial infinite families of solutions to (2).

(2) *Four one-parameter families of solutions, listed in Table 3. The first arises from relations of type* $4R_3$*, and the other three arise from relations of type* $2R_3+3R_2$*.*

(3) *Sixty-five "sporadic" solutions, listed in Table 4, which arise from the other types of weight 12 relations listed in Table 2.*

The only duplications in this list are that the second family of Table 3 gives a trivial solution for $t=1/12$*, the first and fourth families of Table 3 give the same solution when* $t=1/18$ *in both, and the second and fourth families of Table 3 give the same solution when* $t=1/24$ *in both.*

Some explanation of the tables is in order. The last column of Table 3 gives the allowable range for the rational parameter t. The entries of Table 4 are sorted according to the least common denominator of U,V,W,X,Y,Z, which is also the least n for which diagonals of a regular n-gon can create arcs of the corresponding lengths. The relation type from which each solution derives is also given. The reason 11 does not appear in the least common denominator for any sporadic solution is that the relation $(R_{11}:R_3)$ cannot be put in the form of (3) with the α_j summing to 1, and hence leads to no solutions of (2). (Several other types of relations also give rise to no solutions.)

Tables 3 and 4 are the same as Bol's tables at the bottom of page 40 and on page 41 of [1], in a slightly different format.

The arcs cut by diagonals of a regular n-gon have lengths which are multiples of $2\pi/n$, so U, V, W, X, Y and Z corresponding to any configuration of three diagonals meeting must be multiples of $1/n$. With this additional restriction, trivial solutions to (2) occur only when n is even (and at least 6). Solutions within the infinite families of Table 3 occur when n is a multiple of 6 (and at least 12), and there t must be a multiple of $1/n$. Sporadic solutions with least common denominator d occur if and only if n is a multiple of d.

5. INTERSECTIONS OF MORE THAN THREE DIAGONALS

Now that we know the configurations of three diagonals meeting, we can check how they overlap to produce configurations of more than three diagonals meeting. We will disregard configurations in which the intersection point is the center of the n-gon, since these are easily described: there are exactly $n/2$ diagonals (diameters) through the center when n is even, and none otherwise.

When k diagonals meet, they form $2k$ arcs, whose lengths we will measure as a fraction of the whole circumference (so they will be multiples of $1/n$) and list in counterclockwise order. (Warning: this is different from the order used in Tables 3 and 4.) The least common denominator of the numbers in this list will be called

Denominator	U	V	W	X	Y	Z	Relation type
30	1/10	2/15	3/10	2/15	1/6	1/6	$2(R_5 : R_3)$
	1/15	1/15	7/15	1/15	1/10	7/30	
	1/30	7/30	4/15	1/15	1/10	3/10	
	1/30	1/10	7/15	1/15	1/15	4/15	
	1/30	1/15	19/30	1/15	1/10	1/10	
	1/15	1/6	4/15	1/10	1/10	3/10	$(R_5 : R_3) + 2R_3$
	1/15	2/15	11/30	1/10	1/6	1/6	
	1/30	1/6	13/30	1/10	2/15	2/15	
	1/30	1/30	7/10	1/30	1/15	2/15	
	1/30	7/30	3/10	1/15	2/15	7/30	$R_5 + R_3 + 2R_2$
	1/30	1/6	11/30	1/15	1/10	4/15	
	1/30	1/10	13/30	1/30	2/15	4/15	
	1/30	1/15	8/15	1/30	1/10	7/30	
42	1/14	5/42	5/14	2/21	5/42	5/21	$(R_7 : 5R_3)$
	1/21	4/21	13/42	1/14	1/6	3/14	
	1/42	3/14	5/14	1/21	1/6	4/21	
	1/42	1/6	19/42	1/14	2/21	4/21	
	1/42	1/6	13/42	1/21	1/14	8/21	
	1/42	1/21	13/21	1/42	1/14	3/14	
60	1/20	1/12	29/60	1/15	1/10	13/60	$2(R_5 : R_3)$
	1/20	1/12	9/20	1/15	1/12	4/15	
	1/20	1/12	5/12	1/20	1/10	3/10	
	1/60	4/15	3/10	1/20	1/12	17/60	
	1/60	13/60	9/20	1/12	1/10	2/15	
	1/60	13/60	5/12	1/20	2/15	1/6	
	1/12	1/6	17/60	2/15	3/20	11/60	$(R_5 : 3R_3) + 2R_2$
	1/12	2/15	19/60	1/10	3/20	13/60	
	1/15	11/60	13/60	1/12	1/10	7/20	
	1/20	11/60	3/10	1/12	7/60	4/15	
	1/20	1/10	23/60	1/15	1/12	19/60	
	1/30	7/60	19/60	1/20	1/15	5/12	
	1/30	1/12	7/12	1/15	1/10	2/15	
	1/30	1/20	11/20	1/30	1/15	4/15	
	1/60	3/10	7/20	1/12	7/60	2/15	
	1/60	4/15	23/60	1/12	1/10	3/20	
	1/60	7/30	5/12	1/15	7/60	3/20	
	1/60	13/60	11/30	1/20	1/12	4/15	
	1/60	1/6	31/60	1/15	1/10	2/15	
	1/60	1/6	5/12	1/20	1/15	17/60	
	1/60	2/15	9/20	1/30	1/12	17/60	
	1/60	1/10	31/60	1/30	1/15	4/15	
84	1/12	3/14	19/84	11/84	13/84	4/21	$(R_7 : R_3) + 2R_2$
	1/14	11/84	23/84	1/12	2/21	29/84	
	1/21	13/84	23/84	1/14	1/12	31/84	
	1/42	1/12	7/12	1/21	1/14	4/21	
	1/84	25/84	5/14	5/84	1/12	4/21	
	1/84	5/21	5/12	5/84	1/14	17/84	
	1/84	3/14	37/84	1/21	1/12	17/84	
	1/84	1/6	43/84	1/21	1/14	4/21	
90	1/18	13/90	7/18	11/90	2/15	7/45	$(R_5 : R_3) + 2R_3$
	1/45	19/90	16/45	1/18	1/10	23/90	
	1/90	23/90	31/90	2/45	1/15	5/18	
	1/90	17/90	47/90	1/18	4/45	2/15	
120	13/120	3/20	31/120	2/15	19/120	23/120	$(R_5 : R_3) + 3R_2$
	1/12	19/120	29/120	1/10	13/120	37/120	
	1/20	23/120	29/120	1/15	13/120	41/120	
	1/60	13/120	73/120	1/20	1/12	2/15	
	1/120	7/20	43/120	7/120	11/120	2/15	
	1/120	3/10	49/120	7/120	1/12	17/120	
	1/120	4/15	53/120	1/20	11/120	17/120	
	1/120	13/60	61/120	1/20	1/12	2/15	
210	1/15	41/210	8/35	1/14	31/210	61/210	$(R_7 : (R_5 : 2R_3))$
	13/210	1/10	83/210	1/14	4/35	9/35	
	1/35	2/15	97/210	1/14	17/210	47/210	
	1/210	3/14	121/210	11/210	1/15	3/35	

TABLE 4. The 65 sporadic solutions to (2).

the denominator of the configuration. It is the least n for which the configuration can be realized as diagonals of a regular n-gon.

Lemma 6. *If a configuration of $k \geq 2$ diagonals meeting at an interior point other than the center has denominator dividing d, then any configuration of diagonals meeting at that point has denominator dividing* $\mathrm{LCM}(2d, 3)$.

Proof. We may assume $k = 2$. Any other configuration of diagonals through the intersection point is contained in the union of configurations obtained by adding one diagonal to the original two, so we may assume the final configuration consists of three diagonals, two of which were the original two. Now we need only go through our list of three-diagonal intersections.

It can be checked (using Mathematica) that removing any diagonal from a sporadic configuration of three intersecting diagonals yields a configuration whose denominator is the same or half as much, except that it is possible that removing a diagonal from a three-diagonal configuration of denominator 210 or 60 yields one of denominator 70 or 20, respectively, which proves the desired result for these cases. The additive group generated by $1/6$ and the normalized arc lengths of a configuration obtained by removing a diagonal from a configuration corresponding to one of the families of Table 3 contains $2t$ where t is the parameter, (as can be verified using Mathematica again), which means that adding that third diagonal can at most double the denominator (and throw in a factor of 3, if it isn't already there). Similarly, it is easily checked (even by hand), that the subgroup generated by the normalized arc lengths of a configuration obtained by removing one of the three diagonals of a configuration corresponding to a trivial solution to (2) but with intersection point not the center, contains twice the arc lengths of the original configuration. □

Corollary 2. *If a configuration of three or more diagonals meeting includes three forming a sporadic configuration, then its denominator is 30, 42, 60, 84, 90, 120, 168, 180, 210, 240, or 420.*

Proof. Combine the lemma with the list of denominators of sporadic configurations listed in Table 4. □

For $k \geq 4$, a list of $2k$ positive rational numbers summing to 1 arises this way if and only if the lists of length $2k - 2$ which would arise by removing the first or second diagonal actually correspond to $k - 1$ intersecting diagonals. Suppose $k = 4$. If we specify the sporadic configuration or parameterized family of configurations that arise when we remove the first or second diagonal, we get a set of linear conditions on the eight arc lengths. Corollary 2 tells us that we get a configuration with denominator among 30, 42, 60, 84, 90, 120, 168, 180, 210, 240, and 420, if one of these two is sporadic. Using Mathematica to perform this computation for the rest of possibilities in Theorem 4 shows that the other four-diagonal configurations, up to rotation and reflection, fall into 12 one-parameter families, which are listed in Table 5 by the eight normalized arc lengths and the range for the parameter t, with a finite number of exceptions of denominators among 12, 18, 24, 30, 36, 42, 48, 60, 84, and 120.

We will use a similar argument when $k = 5$. Any five-diagonal configuration containing a sporadic three-diagonal configuration will again have denominator among 30, 42, 60, 84, 90, 120, 168, 180, 210, 240, and 420. Any other five-diagonal

								Range
t	t	t	$1/6-2t$	$1/6$	$1/3+t$	$1/6$	$1/6-2t$	$0<t<1/12$
t	$1/6-t$	$1/6-t$	$1/6-t$	t	$1/6$	$1/6+t$	$1/6$	$0<t<1/6$
$1/6-4t$	$2t$	t	$3t$	$1/6-4t$	$1/6$	$1/6+t$	$1/3+t$	$0<t<1/24$
$2t$	$1/2-t$	$2t$	$1/6-2t$	t	$1/6-t$	t	$1/6-2t$	$0<t<1/12$
$1/3-4t$	$1/6+t$	$1/2-3t$	$-1/6+4t$	$1/6-2t$	t	$1/6-t$	$-1/6+4t$	$1/24<t<1/12$
$2t$	t	$3t$	$1/6-2t$	$1/6$	$1/6-t$	$1/3-t$	$1/6-2t$	$0<t<1/12$
t	t	$2t$	$1/3-t$	$1/6$	$1/6-t$	$1/6-t$	$1/6-t$	$0<t<1/6$
$1/3-4t$	$1/6$	t	t	$1/6-2t$	$1/3-2t$	$3t$	$3t$	$0<t<1/12$
$2t$	$1/3-2t$	$1/6-t$	$1/6-t$	$1/6$	$1/6$	t	t	$0<t<1/6$
$1/3-4t$	$2t$	t	t	$1/6-2t$	$1/6$	$1/6+t$	$1/6+t$	$0<t<1/12$
$1/3-4t$	$2t$	$1/6-t$	t	$1/6-2t$	$2t$	$1/3-t$	$3t$	$0<t<1/12$
$2t$	$1/6-t$	t	$1/6-t$	t	$1/6-t$	$2t$	$1/2-3t$	$0<t<1/6$

TABLE 5. The one-parameter families of four-diagonal configurations.

										Range
t	$2t$	$1/6-2t$	$1/6$	$1/6-t$	$1/6-t$	$1/6$	$1/6-2t$	$2t$	t	$0<t<1/12$
t	$2t$	$1/6-4t$	$1/6$	$1/6+t$	$1/6+t$	$1/6$	$1/6-4t$	$2t$	t	$0<t<1/24$
t	$1/6-2t$	$-1/6+4t$	$1/3-4t$	$1/6+t$	$1/6+t$	$1/3-4t$	$-1/6+4t$	$1/6-2t$	t	$1/24<t<1/12$
t	$1/6-2t$	$2t$	$1/3-4t$	$3t$	$3t$	$1/3-4t$	$2t$	$1/6-2t$	t	$0<t<1/12$

TABLE 6. The one-parameter families of five-diagonal configurations.

configuration containing one of the exceptional four-diagonal configurations will have denominator among 12, 18, 24, 30, 36, 42, 48, 60, 72, 84, 96, 120, 168, and 240, by Lemma 6. Finally, another Mathematica computation shows that the one-parameter families of four-diagonal configurations overlap to produce the one-parameter families listed (up to rotation and reflection) in Table 6, and a finite number of exceptions of denominators among 18, 24, and 30.

For $k = 6$, any six-diagonal configuration containing a sporadic three-diagonal configuration will again have denominator among 30, 42, 60, 84, 90, 120, 168, 180, 210, 240, and 420. Any six-diagonal configuration containing one of the exceptional four-diagonal configurations will have denominator among 12, 18, 24, 30, 36, 42, 48, 60, 72, 84, 96, 120, 168, and 240. Any six-diagonal configuration containing one of the exceptional five-diagonal configurations will have denominator among 18, 24, 30, 36, 48, and 60. Another Mathematica computation shows that the one-parameter families of five-diagonal configurations cannot combine to give a six-diagonal configuration.

Finally for $k \geq 7$, any k-diagonal configuration must contain an exceptional configuration of 3, 4, or 5 diagonals, and hence by Lemma 6 has denominator among 12, 18, 24, 30, 36, 42, 48, 60, 72, 84, 90, 96, 120, 168, 180, 210, 240, and 420.

We summarize the results of this section in the following.

Proposition 1. *The configurations of $k \geq 4$ diagonals meeting at a point not the center, up to rotation and reflection, fall into the one-parameter families listed in*

Tables 5 and 6, with finitely many exceptions (for fixed k) of denominators among 12, 18, 24, 30, 36, 42, 48, 60, 72, 84, 90, 96, 120, 168, 180, 210, 240, and 420.

In fact, many of the numbers listed in the proposition do not actually occur as denominators of exceptional configurations. For example, it will turn out that the only denominator greater than 120 that occurs is 210.

6. The formula for intersection points

Let $a_k(n)$ denote the number of points inside the regular n-gon other than the center where exactly k lines meet. Let $b_k(n)$ denote the number of k-tuples of diagonals which meet at a point inside the n-gon other than the center. Each interior point at which exactly m diagonals meet gives rise to $\binom{m}{k}$ such k-tuples, so we have the relationship

$$b_k(n) = \sum_{m\geq k} \binom{m}{k} a_m(n) \tag{7}$$

Since every four distinct vertices of the n-gon determine one pair of diagonals which intersect inside, the number of such pairs is exactly $\binom{n}{4}$, but if n is even, then $\binom{n/2}{2}$ of these are pairs which meet at the center, so

$$b_2(n) = \binom{n}{4} - \binom{n/2}{2}\delta_2(n). \tag{8}$$

(Recall that $\delta_m(n)$ is defined to be 1 if n is a multiple of m, and 0 otherwise.)

We will use the results of the previous two sections to deduce the form of $b_k(n)$ and then the form of $a_k(n)$. To avoid having to repeat the following, let us make a definition.

Definition . A function on integers $n \geq 3$ will be called *tame* if it is a linear combination (with rational coefficients) of the functions n^3, n^2, n, 1, $n^2\delta_2(n)$, $n\delta_2(n)$, $\delta_2(n)$, $\delta_4(n)$, $n\delta_6(n)$, $\delta_6(n)$, $\delta_{12}(n)$, $\delta_{18}(n)$, $\delta_{24}(n)$, $\delta_{24}(n-6)$, $\delta_{30}(n)$, $\delta_{36}(n)$, $\delta_{42}(n)$, $\delta_{48}(n)$, $\delta_{60}(n)$, $\delta_{72}(n)$, $\delta_{84}(n)$, $\delta_{90}(n)$, $\delta_{96}(n)$, $\delta_{120}(n)$, $\delta_{168}(n)$, $\delta_{180}(n)$, $\delta_{210}(n)$, and $\delta_{420}(n)$.

Proposition 2. *For each $k \geq 2$, the function $b_k(n)/n$ on integers $n \geq 3$ is tame.*

Proof. The case $k = 2$ is handled by (8), so assume $k \geq 3$. Each list of $2k$ normalized arc lengths as in Section 5 corresponding to a configuration of k diagonals meeting at a point other than the center, considered up to rotation (but not reflection), contributes n to $b_k(n)$. (There are n places to start measuring the arcs from, and these n configurations are distinct, because the corresponding intersection points differ by rotations of multiples of $2\pi/n$, and by assumption they are not at the center.) So $b_k(n)/n$ counts such lists.

Suppose $k = 3$. When n is even, the family of trivial solutions to the trigonometric equation (2) has $U = a/n$, $V = b/n$, $W = c/n$, where a, b, and c are positive integers with sum $n/2$, and X, Y, and Z are some permutation of U, V, W. Each permutation gives rise to a two-parameter family of six-long lists of arc lengths, and the number of lists within each family is the number of partitions of $n/2$ into three positive parts, which is a quadratic polynomial in n. Similarly each family of solutions in Table 3 gives rise to a number of one-parameter families of lists, when n is a multiple of 6, each containing $\lceil n/6\rceil - 1$ or $\lceil n/12\rceil - 1$ lists. These functions of n (extended to be 0 when 6 does not divide n) are expressible as a linear combination of $n\delta_6(n)$, $\delta_6(n)$, and $\delta_{12}(n)$. Finally the sporadic solutions to 2 give rise to a finite number of lists, having denominators among 30, 42, 60, 84, 90, 120, and 210, so their contribution to $b_3(n)/n$ is a linear combination of $\delta_{30}(n), \dots, \delta_{210}(n)$.

But these families of lists overlap, so we must use the Principle of Inclusion-Exclusion to count them properly. To show that the result is a tame function, it suffices to show that the number of lists in any intersection of these families is a tame function. When two of the trivial families overlap but do not coincide, they overlap where two of the a, b, and c above are equal, and the corresponding lists lie in one of the one-parameter families $(t, t, t, t, 1/2 - 2t, 1/2 - 2t)$ or $(t, t, t, 1/2 - 2t, t, 1/2 - 2t)$ (with $0 < t < 1/4$), each of which contain $\lceil n/4 \rceil - 1$ lists (for n even). This function of n is a combination of $n\delta_2(n)$, $\delta_2(n)$, and $\delta_4(n)$, hence it is tame. Any other intersection of the infinite families must contain the intersection of two one-parameter families which are among the two above or arise from Table 3, and a Mathematica computation shows that such an intersection consists of at most a single list of denominator among 6, 12, 18, 24, and 30. And, of course, any intersection involving a single sporadic list, can contain at most that sporadic list. Thus the number of lists within any intersection is a tame function of n. Finally we must delete the lists which correspond to configurations of diagonals meeting at the center. These are the lists within the trivial two-parameter family $(t, u, 1/2 - t - u, t, u, 1/2 - t - u)$, so their number is also a tame function of n, by the Principle of Inclusion-Exclusion again. Thus $b_3(n)/n$ is tame.

Next suppose $k = 4$. The number of lists within each family listed in Table 5, or the reflection of such a family, is (when n is divisible by 6) the number of multiples of $1/n$ strictly between α and β, where the range for the parameter t is $\alpha < t < \beta$. This number is $\lceil \beta n \rceil - 1 - \lfloor \alpha n \rfloor$. Since the table shows that α and β are always multiples of $1/24$, this function of n is expressible as a combination of $n\delta_6(n)$ and a function on multiples of 6 depending only on n mod 24, and the latter can be written as a combination of $\delta_6(n)$, $\delta_{12}(n)$, $\delta_{24}(n)$, and $\delta_{24}(n-6)$, so it is tame. Mathematica shows that when two of these families are not the same, they intersect in at most a single list of denominator among 6, 12, 18, and 24. So these and the exceptions of Proposition 1 can be counted by a tame function. Thus, again by the Principle of Inclusion-Exclusion, $b_4(n)/n$ is tame.

The proof for $k = 5$ is identical to that of $k = 4$, using Table 6 instead of Table 5, and using another Mathematica computation which shows that the intersections of two one-parameter families of lists consist of at most a single list of denominator 24.

The proof for $k \geq 6$ is even simpler, because then there are only the exceptional lists. By Proposition 1, $b_k(n)/n$ is a linear combination of $\delta_m(n)$ where m ranges over the possible denominators of exceptional lists listed in the proposition, so it is tame. □

Lemma 7. *A tame function is determined by its values at $n =$ 3, 4, 5, 6, 7, 8, 9, 10, 12, 18, 24, 30, 36, 42, 48, 54, 60, 66, 72, 84, 90, 96, 120, 168, 180, 210, and 420.*

Proof. By linearity, it suffices to show that if a tame function f is zero at those values, then f is the zero linear combination of the functions in the definition of a tame function. The vanishing at $n = 3$, 5, 7, and 9 forces the coefficients of n^3, n^2, n, and 1 to vanish, by Lagrange interpolation. Then comparing the values at $n = 4$ and $n = 10$ shows that the coefficient of $\delta_4(n)$ is zero. The vanishing at $n = 4$, 8, and 10 forces the coefficients of $n^2\delta_2(n)$, $n\delta_2(n)$, and $\delta_2(n)$ to vanish. Comparing the values at $n = 6$ and $n = 54$ shows that the coefficient of $n\delta_6$ is zero. Comparing the values at $n = 6$ and $n = 66$ shows that the coefficient of $\delta_2 4(n-6)$ is zero.

At this point, we know that $f(n)$ is a combination of $\delta_m(n)$, for $m = 6$, 12, 18, 24, 30, 36, 42, 48, 60, 72, 84, 90, 96, 120, 168, 180, 210, and 420. For each m in turn, $f(m) = 0$ now implies that the coefficient of $\delta_m(n)$ is zero. □

Proof of Theorem 1. Computation (see the appendix) shows that the tame function $b_8(n)/n$ vanishes at all the numbers listed in Lemma 7. Hence by that lemma, $b_8(n) = 0$ for all n. Thus by (7), $a_k(n)$ and $b_k(n)$ are identically zero for all $k \geq 8$ as well.

By reverse induction on k, we can invert (7) to express $a_k(n)$ as a linear combination of $b_m(n)$ with $m \geq k$. Hence $a_k(n)/n$ is tame as well for each $k \geq 2$. Computation shows that the equations

$$\begin{aligned}
a_2(n)/n &= (n^3 - 6n^2 + 11n - 6)/24 + (-5n^2 + 46n - 72)/16 \cdot \delta_2(n) \\
&\quad - 9/4 \cdot \delta_4(n) + (-19n + 110)/2 \cdot \delta_6(n) + 54 \cdot \delta_{12}(n) + 84 \cdot \delta_{18}(n) \\
&\quad + 50 \cdot \delta_{24}(n) - 24 \cdot \delta_{30}(n) - 100 \cdot \delta_{42}(n) - 432 \cdot \delta_{60}(n) \\
&\quad - 204 \cdot \delta_{84}(n) - 144 \cdot \delta_{90}(n) - 204 \cdot \delta_{120}(n) - 144 \cdot \delta_{210}(n) \\
a_3(n)/n &= (5n^2 - 48n + 76)/48 \cdot \delta_2(n) + 3/4 \cdot \delta_4(n) + (7n - 38)/6 \cdot \delta_6(n) \\
&\quad - 8 \cdot \delta_{12}(n) - 20 \cdot \delta_{18}(n) - 16 \cdot \delta_{24}(n) - 19 \cdot \delta_{30}(n) + 8 \cdot \delta_{42}(n) \\
&\quad + 68 \cdot \delta_{60}(n) + 60 \cdot \delta_{84}(n) + 48 \cdot \delta_{90}(n) + 60 \cdot \delta_{120}(n) + 48 \cdot \delta_{210}(n) \\
a_4(n)/n &= (7n - 42)/12 \cdot \delta_6(n) - 5/2 \cdot \delta_{12}(n) - 4 \cdot \delta_{18}(n) + 3 \cdot \delta_{24}(n) \\
&\quad + 6 \cdot \delta_{42}(n) + 34 \cdot \delta_{60}(n) - 6 \cdot \delta_{84}(n) - 6 \cdot \delta_{120}(n) \\
a_5(n)/n &= (n - 6)/4 \cdot \delta_6(n) - 3/2 \cdot \delta_{12}(n) - 2 \cdot \delta_{24}(n) + 4 \cdot \delta_{42}(n) \\
&\quad + 6 \cdot \delta_{84}(n) + 6 \cdot \delta_{120}(n) \\
a_6(n)/n &= 4 \cdot \delta_{30}(n) - 4 \cdot \delta_{60}(n) \\
a_7(n)/n &= \delta_{30}(n) + 4 \cdot \delta_{60}(n)
\end{aligned}$$

hold for all the n listed in Lemma 7, so the lemma implies that they hold for all $n \geq 3$. These formulas imply the remarks in the introduction about the maximum number of diagonals meeting at an interior point other than the center. Finally

$$\begin{aligned}
I(n) &= \delta_2(n) + \sum_{k=2}^{\infty} a_k(n) \\
&= \delta_2(n) + \sum_{k=2}^{7} a_k(n),
\end{aligned}$$

which gives the desired formula. (The $\delta_2(n)$ in the expression for $I(n)$ is to account for the center point when n is even, which is the only point not counted by the a_k.) □

7. The formula for regions

We now use the knowledge obtained in the proof of Theorem 1 about the number of interior points through which exactly k diagonals pass to calculate the number of regions formed by the diagonals.

Proof of Theorem 2. Consider the graph formed from the configuration of a regular n-gon with its diagonals, in which the vertices are the vertices of the n-gon together with the interior intersection points, and the edges are the sides of the n-gon together with the segments that the diagonals cut themselves into. As usual, let V denote the number of vertices of the graph, E the number of edges, and F the number of regions formed, including the region outside the n-gon. We will employ Euler's Formula $V - E + F = 2$.

Clearly $V = n + I(n)$. We will count edges by counting their ends, which are $2E$ in number. Each vertex has $n-1$ edge ends, the center (if n is even) has n edge ends, and any other interior point through which exactly k diagonals pass has $2k$ edge ends, so

$$2E = n(n-1) + n\delta_2(n) + \sum_{k=2}^{\infty} 2k a_k(n).$$

So the desired number of regions, not counting the region outside the n-gon, is

$$\begin{aligned} F - 1 &= E - V + 1 \\ &= \left[n(n-1)/2 + n\delta_2(n)/2 + \sum_{k=2}^{\infty} k a_k(n) \right] - [n + I(n)] + 1. \end{aligned}$$

Substitution of the formulas derived in the proof of Theorem 1 for $a_k(n)$ and $I(n)$ yields the desired result. □

Appendix: computations and tables

In Table 7 we list $I(n), R(n), a_2(n), \ldots, a_7(n)$ for $n = 4, 5, \ldots, 30$. To determine the polynomials listed in Theorem 1 more data was needed especially for $n \equiv 0 \bmod 6$. The largest n for which this was required was 420. For speed and memory conservation, we took advantage of the regular n-gon's rotational symmetry and focused our attention on only $2\pi/n$ radians of the n-gon. The data from this computation is found in Table 8. Although we only needed to know the values at those n listed in Lemma 7 of Section 6, we give a list for $n = 6, 12, \ldots, 420$ so that the nice patterns can be seen.

The numbers in these tables were found by numerically computing (using a C program and 64 bit precision) all possible $\binom{n}{4}$ intersections, and sorting them by their x coordinate. We then focused on runs of points with close x coordinates, looking for points with close y coordinates.

Several checks were made to eliminate any fears (arising from round-off errors) of distinct points being mistaken as close. First, the C program sent data to Maple which checked that the coordinates of close points agreed to at least 40 decimal places. Second, we verified for each n that close points came in counts of the form $\binom{k}{2}$ (k diagonals meeting at a point give rise to $\binom{k}{2}$ close points. Hence, any run whose length is not of this form indicates a computational error).

A second program was then written and run on a second machine to make the computations completely rigorous. It also found the intersection points numerically, sorted them and looked for close points, but, to be absolutely sure that a pair of close points p_1 and p_2 were actually the same, it checked that for the two pairs of diagonals (l_1, l_2) and (l_3, l_4) determining p_1 and p_2, respectively, the triples l_1, l_2, l_3 and l_1, l_2, l_4 each divided the circle into arcs of lengths consistent with Theorem 4.

n	$a_2(n)$	$a_3(n)$	$a_4(n)$	$a_5(n)$	$a_6(n)$	$a_7(n)$	$I(n)$	$R(n)$
3							0	1
4							1	4
5	5						5	11
6	12						13	24
7	35						35	50
8	40	8					49	80
9	126						126	154
10	140	20					161	220
11	330						330	375
12	228	60	12				301	444
13	715						715	781
14	644	112					757	952
15	1365						1365	1456
16	1168	208					1377	1696
17	2380						2380	2500
18	1512	216	54	54			1837	2466
19	3876						3876	4029
20	3360	480					3841	4500
21	5985						5985	6175
22	5280	660					5941	6820
23	8855						8855	9086
24	6144	864	264	24			7297	9024
25	12650						12650	12926
26	11284	1196					12481	13988
27	17550						17550	17875
28	15680	1568					17249	19180
29	23751						23751	24129
30	13800	2250	420	180	120	30	16801	21480

TABLE 7. A listing of $I(n)$,$R(n)$ and $a_2(n),\ldots,a_7(n)$, $n = 3,4,\ldots,30$. Note that, when n is even, $I(n)$ also counts the point in the center.

Since this test only involves comparing rational numbers, it could be performed exactly.

A word should also be said concerning limiting the search to $2\pi/n$ radians of the n-gon. Both programs looked at slightly smaller slices of the n-gon to avoid problems caused by points near the boundary. We further subdivided this region into twenty smaller pieces to make the task of sorting the intersection points manageable. More precisely, we limited our search to points whose angle with the origin fell between $[c_1 + 2\pi(m-1)/(20n) + \varepsilon, c_1 + 2\pi m/(20n) - \varepsilon)$, $m = 1, 2, \ldots 20$, and also made sure not to include the origin in the count. Here ε was chosen to be .00000000001 and c_1 was chosen to be .00000123 ($c_1 = 0$ would have led to problems since there are many intersection points with angle 0 or $2\pi/n$). To make sure that no intersection points were omitted, the number of points found (counting multiplicity) was compared with $(\binom{n}{4} - \binom{n/2}{2}\delta_2)/n$.

n	$\frac{a_2(n)}{n}$	$\frac{a_3(n)}{n}$	$\frac{a_4(n)}{n}$	$\frac{a_5(n)}{n}$	$\frac{a_6(n)}{n}$	$\frac{a_7(n)}{n}$	$\frac{I(n)-1}{n}$	n	$\frac{a_2(n)}{n}$	$\frac{a_3(n)}{n}$	$\frac{a_4(n)}{n}$	$\frac{a_5(n)}{n}$	$\frac{a_6(n)}{n}$	$\frac{a_7(n)}{n}$	$\frac{I(n)-1}{n}$
6	2						2	216	392564	4848	119	49			397580
12	19	5	1				25	222	426836	5166	126	54			432182
18	84	12	3	3			102	228	463303	5441	127	54			468925
24	256	36	11	1			304	234	501762	5718	129	57			507666
30	460	75	14	6	4	1	560	240	541612	6121	165	61		5	547964
36	1179	109	11	6			1305	246	584782	6340	140	60			591322
42	1786	194	27	13			2020	252	629399	6693	137	70			636299
48	3168	220	25	7			3420	258	676580	6972	147	63			683762
54	4722	288	24	12			5046	264	725976	7276	151	61			733464
60	6251	422	63	12		5	6753	270	777420	7643	150	66	4	1	785284
66	9172	460	35	15			9682	276	831575	7969	155	66			839765
72	12428	504	35	13			12980	282	887986	8326	161	69			896542
78	15920	642	42	18			16622	288	947132	8640	161	67			956000
84	20007	805	43	28			20883	294	1008358	9056	174	76			1017664
90	25230	863	45	21	4	1	26164	300	1072171	9462	203	72		5	1081913
96	31240	948	53	19			32260	306	1139436	9780	171	75			1149462
102	37786	1096	56	24			38962	312	1208944	10164	179	73			1219360
108	45447	1201	53	24			46725	318	1281100	10582	182	78			1291942
114	53768	1368	63	27			55226	324	1356315	10957	179	78			1367529
120	62652	1601	95	31		5	64384	330	1434110	11375	189	81	4	1	1445760
126	73676	1658	72	34			75440	336	1514816	11856	193	89			1526954
132	85319	1825	71	30			87245	342	1598970	12216	192	84			1611462
138	97990	2002	77	33			100102	348	1685843	12661	197	84			1698785
144	112100	2136	77	31			114344	354	1775788	13108	203	87			1789186
150	127070	2345	84	36	4	1	129540	360	1868312	13669	231	91		5	1882308
156	143635	2549	85	36			146305	366	1965272	14010	210	90			1979582
162	161520	2736	87	39			164382	372	2064919	14465	211	90			2079685
168	180504	3008	95	47			183654	378	2167754	14930	219	97			2183000
174	201448	3178	98	42			204766	384	2274136	15396	221	91			2289844
180	223251	3470	129	42		5	226897	390	2383690	15885	224	96	4	1	2399900
186	247562	3630	105	45			251342	396	2496999	16369	221	96			2513685
192	273144	3844	109	43			277140	402	2613536	16896	231	99			2630762
198	300294	4092	108	48			304542	408	2733888	17380	235	97			2751600
204	329171	4357	113	48			333689	414	2857752	17898	234	102			2875986
210	359556	4661	125	55	4	1	364402	420	2984383	18598	273	112		5	3003371

Table 8. The number of intersection points for one piece of the pie (i.e. $2\pi/n$ radians), $n = 6, 12, \ldots, 420$.

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