THE p-ADIC CLOSURE OF A SUBGROUP OF RATIONAL POINTS ON A COMMUTATIVE ALGEBRAIC GROUP

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ABSTRACT. Let G be a commutative algebraic group over $\mathbb{Q}$. Let Γ be a subgroup of $G(\mathbb{Q})$ contained in the union of the compact subgroups of $G(\mathbb{Q}_p)$. We formulate a guess for the dimension of the closure of Γ in $G(\mathbb{Q}_p)$, and show that its correctness for certain tori is equivalent to Leopoldt's conjecture.

1. INTRODUCTION

1.1. **Notation.** Let $\mathbb{Q}$ be the field of rational numbers. Let p be a prime, and let $\mathbb{Q}_p$ be the corresponding completion of $\mathbb{Q}$. Let $\mathbb{Z}_p$ be the completion of $\mathbb{Z}$ at p. If K is a number field, then $\mathcal{O}_K$ is the ring of the integers, and for any finite set S of places, $\mathcal{O}_{K,S}$ is the ring of S-integers. If G is a group (or group scheme), then $H \leq G$ means that H is a subgroup (or subgroup scheme) of G. If A is an integral domain with fraction field K, and M is an A-module, then $\operatorname{rk}_A M$ is the dimension of the K-vector space $M \otimes_A K$; we write rk for $\operatorname{rk}_{\mathbb{Z}}$.

1.2. **The logarithm map for a p-adic Lie group.** Let G be a finite-dimensional commutative Lie group over $\mathbb{Q}_p$ (see [Bou98, III.§1] for terminology). The Lie algebra $\operatorname{Lie} G$ is the tangent space of G at the identity. So $\operatorname{Lie} G$ is a $\mathbb{Q}_p$-vector space of dimension $\dim G$. Let G_f be the union of the compact subgroups of G. By [Bou98, III.§7.6], G_f is an open subgroup of G, and there is a canonical homomorphism

$$\log\colon G_f \to \operatorname{Lie} G,$$

defined first on a sufficiently small compact open subgroup by formally integrating translation-invariant 1-forms, and then extended by linearity. Moreover, log is a local diffeomorphism, and its kernel is the torsion subgroup of G_f. It behaves functorially in G.

Examples 1.1.

(i) If $G = \mathbb{Q}_p$ (the additive group), then $G_f = \mathbb{Q}_p$, and log is an isomorphism. In this example, G_f is not compact.
(ii) If $G = \mathbb{Q}_p^\times$, then $G_f = \mathbb{Z}_p^\times$.
(iii) If $G = A(\mathbb{Q}_p)$ for an abelian variety A over $\mathbb{Q}_p$, then $G_f = G$.

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1.3. **Dimension of an analytic subgroup.** Let Γ be a finitely generated subgroup of G_f. Then $\log \Gamma \subseteq \operatorname{Lie} G$ is a finitely generated abelian group of the same rank. The closure $\overline{\log \Gamma} = \log \overline{\Gamma}$ with respect to the p-adic topology equals the $\mathbb{Z}_p$-submodule of $\operatorname{Lie} G$ spanned by $\log \Gamma$, so it is a finitely generated $\mathbb{Z}_p$-module. Define

$$\dim \overline{\Gamma} := \operatorname{rk}_{\mathbb{Z}_p} \overline{\log \Gamma}.$$

This agrees with the dimension of $\overline{\Gamma}$ viewed as a Lie group over $\mathbb{Q}_p$.

1.4. **Rational points.** Now let G be a commutative group scheme of finite type over $\mathbb{Q}$. Fix a prime p. Define $G(\mathbb{Q})_f := G(\mathbb{Q}) \cap G(\mathbb{Q}_p)_f$. We specialize the previous sections to the Lie group $G(\mathbb{Q}_p)$ and to a finitely generated subgroup Γ of $G(\mathbb{Q})_f$. Our goal is to predict the value of $\dim \overline{\Gamma}$.

1.5. **Applications.** The value of $\dim \overline{\Gamma}$ is important for a few reasons:

(i) If C is a curve of genus $g \geq 2$ over $\mathbb{Q}$ embedded in its Jacobian J, then the condition $\dim \overline{J(\mathbb{Q})} < g$ is necessary for the application of Chabauty's method, which attempts to calculate $C(k)$ or at least bound its size [Cha41, Col85].

(ii) Leopoldt's conjecture on p-adic independence of units in a number field predicts $\dim \overline{\Gamma}$ in a special case: see Corollary 5.3 in Section 5. Leopoldt's conjecture is important because it governs the abelian extensions of K of p-power degree.

1.6. **Outline of the paper.** Section 2 axiomatizes some of the properties of $\dim \overline{\Gamma}$ in order to identify possible candidates for its value. Section 3 defines a "maximal" function $d(\Gamma)$ satisfying the same axioms and Question 3.3 asks whether it always equals $\dim \overline{\Gamma}$. Section 4 shows that $\dim \overline{\Gamma}$ and $d(\Gamma)$ share many other properties. Section 5 computes $d(\Gamma)$ for subgroups of integer points on tori, and shows that a positive answer to Question 3.3 for certain tori would imply Leopoldt's conjectures. We end with further open questions.

2. Dimension functions

Let $\mathcal{G}$ be the set of pairs (G, Γ) where G is a commutative group scheme of finite type over $\mathbb{Q}$ and Γ is a finitely generated subgroup of $G(\mathbb{Q})_f$.

Definition 2.1. A dimension function is a function $\partial \colon \mathcal{G} \to \mathbb{Z}_{\geq 0}$ satisfying

(1) If $\Gamma \leq H(\mathbb{Q})_f$ for some subgroup scheme $H \leq G$, then $\partial(H, \Gamma) = \partial(G, \Gamma)$. (Because of this, we generally write $\partial(\Gamma)$ instead of $\partial(G, \Gamma)$.)

(2) $\partial(\Gamma) \leq \operatorname{rk} \Gamma$.

(3) If $H \leq G$ and Γ'' is the image of Γ in $(G/H)(\mathbb{Q})_f$, then $\partial(\Gamma) \leq \dim H + \partial(\Gamma'')$.

Proposition 2.2. *The expression* $\dim \overline{\Gamma}$ *is a dimension function.*

Proof. Since $H(\mathbb{Q}_p)$ is closed in $G(\mathbb{Q}_p)$, the closure of Γ in $H(\mathbb{Q}_p)$ equals the closure of Γ in $G(\mathbb{Q}_p)$; therefore (1) holds. The fact that $\overline{\log \Gamma}$ is the $\mathbb{Z}_p$-submodule spanned by $\log \Gamma$ gives the middle step in

$$\dim \overline{\Gamma} = \operatorname{rk}_{\mathbb{Z}_p} \overline{\log \Gamma} \leq \operatorname{rk}(\log \Gamma) \leq \operatorname{rk} \Gamma,$$

so (2) holds. Finally, by continuity, the image of $\overline{\Gamma}$ in $(G/H)(\mathbb{Q}_p)$ equals $\overline{\Gamma''}$, so we have an exact sequence

$$0 \to \overline{\Gamma} \cap H \to \overline{\Gamma} \to \overline{\Gamma''} \to 0.$$

Taking dimensions of p-adic Lie groups and observing that the group on the left has dimension at most $\dim H$ yields (3). $\square$

3. The guess

With notation as before, define

$$d(G,\Gamma) := \inf_{H \leq G} \left(\dim H + \operatorname{rk}\Gamma - \operatorname{rk}(\Gamma \cap H)\right),$$

where the infimum is over all subgroup schemes $H \leq G$.

Proposition 3.1. *The function d is a dimension function, and any dimension function ∂ satisfies $\partial \leq d$.*

Proof. First we check that d is a dimension function:

(1) Suppose $G' \leq G$ and $\Gamma \leq G'(\mathbb{Q})_f$. If $H \leq G'$ then the subgroup $H' := H \cap G'$ satisfies $\dim H' \leq \dim H$ and $\Gamma \cap H' = \Gamma \cap H$, so

$$\dim H' + \operatorname{rk}\Gamma - \operatorname{rk}(\Gamma \cap H') \leq \dim H + \operatorname{rk}\Gamma - \operatorname{rk}(\Gamma \cap H).$$

Therefore the infimum in the definition of $d(G,\Gamma)$ is attained for some $H \leq G'$, so $d(G,\Gamma) = d(G',\Gamma)$.

(2) The $H = \{0\}$ term in the infimum is $0 + \operatorname{rk}\Gamma - 0$, so $d(\Gamma) \leq \operatorname{rk}\Gamma$.

(3) Let K'' be the subgroup of G/H realizing the infimum defining $d(\Gamma'')$. Let K be the inverse image of K'' under $G \to G/H$. Then $\Gamma'' \cap K''$ is a homomorphic image of $\Gamma \cap K$, so $\operatorname{rk}(\Gamma'' \cap K'') \leq \operatorname{rk}(\Gamma \cap K)$ and

$$\begin{aligned} d(\Gamma) &\leq \dim K + \operatorname{rk}\Gamma - \operatorname{rk}(\Gamma \cap K) \\ &= \dim H + \dim K'' + \operatorname{rk}\Gamma - \operatorname{rk}(\Gamma \cap K) \\ &\leq \dim H + \dim K'' + \operatorname{rk}\Gamma - \operatorname{rk}(\Gamma'' \cap K'') \\ &= \dim H + d(\Gamma''). \end{aligned}$$

Now we check that any dimension function ∂ satisfies $\partial \leq d$. If $H \leq G$ and Γ'' is the image of Γ in $(G/H)(\mathbb{Q})$, then properties (3) and (2) for ∂ and the isomorphism $\Gamma'' \simeq \Gamma/(\Gamma \cap H)$ yield

$$\partial(\Gamma) \leq \dim H + \partial(\Gamma'') \leq \dim H + \operatorname{rk}\Gamma'' = \dim H + \operatorname{rk}\Gamma - \operatorname{rk}(\Gamma \cap H).$$

This holds for all H, so $\partial(\Gamma) \leq d(\Gamma)$. $\square$

Corollary 3.2. *We have $\dim \overline{\Gamma} \leq d(\Gamma)$.*

Proposition 3.1 shows that the function $d(\Gamma)$ gives the largest guess for $\dim \overline{\Gamma}$ compatible with the elementary inequalities based on rank and the dimension of the group. Therefore we ask:

Question 3.3. Does $\dim \overline{\Gamma} = d(\Gamma)$ always hold?

In other words, are rational points p-adically independent whenever dependencies are not forced by having a subgroup of too high rank inside an algebraic subgroup?

4. Further properties shared by both dimension functions

Proposition 4.1. *Let $\partial(\Gamma)$ denote either $\dim \overline{\Gamma}$ or $d(\Gamma)$. Then*

(i) *If $\Gamma' \leq \Gamma \leq G(\mathbb{Q})_f$, then $\partial(\Gamma') \leq \partial(\Gamma)$.*

(ii) *If $G \to G''$ is a homomorphism and $\Gamma \leq G(\mathbb{Q})_f$, then the image Γ'' in $G''(\mathbb{Q})$ is contained in $G''(\mathbb{Q})_f$ and $\partial(\Gamma'') \leq \partial(\Gamma)$.*

(iii) *If $G \to G''$ has finite kernel and $\Gamma \leq G(\mathbb{Q})$, then Γ is contained in $G(\mathbb{Q})_f$ if and only if its image Γ'' in $G''(\mathbb{Q})$ belongs to $G''(\mathbb{Q})_f$; in this case, $\partial(\Gamma) = \partial(\Gamma'')$.*

(iv) *Suppose that Γ_1, Γ_2 are* commensurable *subgroups of $G(\mathbb{Q})$; i.e., $\Gamma_1 \cap \Gamma_2$ has finite index in both Γ_1 and Γ_2. Then $\Gamma_1 \leq G(\mathbb{Q})_f$ if and only if $\Gamma_2 \leq G(\mathbb{Q})_f$; in this case, $\partial(\Gamma_1) = \partial(\Gamma_2)$.*

(v) *If $\Gamma_i \leq G_i(\mathbb{Q})_f$ for $i = 1, 2$, then $\Gamma_1 \times \Gamma_2 \leq (G_1 \times G_2)(\mathbb{Q})_f$ and $\partial(\Gamma_1 \times \Gamma_2) = \partial(\Gamma_1) + \partial(\Gamma_2)$.*

(vi) *If $\Gamma_1, \Gamma_2 \leq G(\mathbb{Q})_f$, then $\partial(\Gamma_1 + \Gamma_2) \leq \partial(\Gamma_1) + \partial(\Gamma_2)$.*

(vii) *If $\operatorname{rk} \Gamma = 1$, then $\partial(\Gamma) = 1$.*

(viii) *If $G \simeq \mathbb{G}_a^n$, then $\partial(\Gamma) = \operatorname{rk} \Gamma$.*

Proof.

(i) For $\partial(\Gamma) := \dim \overline{\Gamma}$ the result is obvious. For $d(\Gamma)$ it follows since $\operatorname{rk} \Gamma - \operatorname{rk}(\Gamma \cap H)$ equals the rank of the image of Γ in G/H.

(ii) We have $\Gamma'' \leq G''(\mathbb{Q})_f$ by functoriality. For $\dim \overline{\Gamma}$, the inequality follows since $\overline{\log \Gamma}$ surjects onto $\overline{\log \Gamma''}$. For $d(\Gamma)$, if $H \leq G$ and H'' is its image in G'', then the subgroup $\Gamma/(\Gamma \cap H)$ of G/H surjects onto the subgroup $\Gamma''/(\Gamma'' \cap H'')$ of G''/H'', and this implies the second inequality in

$$\begin{aligned} d(\Gamma'') &\leq \dim H'' + \operatorname{rk} \Gamma'' - \operatorname{rk}(\Gamma'' \cap H'') \\ &\leq \dim H'' + \operatorname{rk} \Gamma - \operatorname{rk}(\Gamma \cap H) \\ &\leq \dim H + \operatorname{rk} \Gamma - \operatorname{rk}(\Gamma \cap H). \end{aligned}$$

This holds for all $H \leq G$, so $d(\Gamma'') \leq d(\Gamma)$.

(iii) The map of topological spaces $G(\mathbb{Q}_p) \to G''(\mathbb{Q}_p)$ is proper, so the inverse image of $G''(\mathbb{Q}_p)_f$ is contained in $G(\mathbb{Q}_p)_f$; this gives the first statement. To prove $\partial(\Gamma) = \partial(\Gamma'')$, first use (1) to assume that $G \to G''$ is surjective, so $G'' = G/H$ for some finite $H \leq G$. By (ii), $\partial(\Gamma'') \leq \partial(\Gamma)$. By (3), $\partial(\Gamma) \leq \dim H + \partial(\Gamma'') = \partial(\Gamma'')$. Thus $\partial(\Gamma) = \partial(\Gamma'')$.

(iv) We may reduce to the case in which Γ_1 is a finite-index subgroup of Γ_2. Let $n = (\Gamma_2 : \Gamma_1)$, so $n\Gamma_1 \leq \Gamma_2 \leq \Gamma_1$. If $\Gamma_1 \leq G(\mathbb{Q})_f$, then $\Gamma_2 \leq G(\mathbb{Q})_f$. Conversely, if $\Gamma_2 \leq G(\mathbb{Q})_f$, then $n\Gamma_1 \leq G(\mathbb{Q})_f$, so $\Gamma_1 \leq G(\mathbb{Q})_f$ by (iii) applied to $G \xrightarrow{n} G$. In this case, (iii) gives $\partial(n\Gamma_1) = \partial(\Gamma_1)$, and (ii) implies that both equal $\partial(\Gamma_2)$.

(v) Let $\Gamma := \Gamma_1 \times \Gamma_2$. Since a product of compact open subgroups is a compact open subgroup, we have $G_1(\mathbb{Q}_p)_f \times G_2(\mathbb{Q}_p)_f \leq (G_1 \times G_2)(\mathbb{Q}_p)_f$. (In fact, equality holds.) Thus $\Gamma \leq (G_1 \times G_2)(\mathbb{Q})_f$. The equality $\dim \overline{\Gamma} = \dim \overline{\Gamma_1} \times \dim \overline{\Gamma_2}$ follows from the definitions. To prove the corresponding equality for d, we must show that the infimum in the definition of $d(\Gamma)$ is realized for an H of the form $H_1 \times H_2$ with $H_i \leq G_i$. Suppose instead that $K \leq G_1 \times G_2$ realizes the infimum. Let $\pi_1 \colon G_1 \times G_2 \to G_1$ be the first projection. Let $H_1 = \pi_1(K)$. Let $H_2 = \ker(\pi_1|_K)$; view H_2 as a subgroup scheme of G_2. Let $H = H_1 \times H_2$. Thus $\dim K = \dim H$. The exact sequence

$$0 \to \Gamma_2 \cap H_2 \to \Gamma \cap K \xrightarrow{\pi_1} \Gamma_1 \cap H_1$$

shows that $\operatorname{rk}(\Gamma \cap K) \le \operatorname{rk}(\Gamma \cap H)$, so

$$\dim H + \operatorname{rk}\Gamma - \operatorname{rk}(\Gamma \cap H) \le \dim K + \operatorname{rk}\Gamma - \operatorname{rk}(\Gamma \cap K).$$

Thus H too realizes the infimum in the definition of $d(\Gamma)$, as desired.

(vi) Apply (ii) to the addition homomorphism $G \times G \to G$ and $\Gamma := \Gamma_1 \times \Gamma_2$, and use (v).

(vii) We have $\partial(\Gamma) \le 1$ by (2). If $\dim \overline{\Gamma} = 0$, then the finitely generated torsion-free $\mathbb{Z}_p$-module $\overline{\log \Gamma}$ is of rank 0, so it is 0; therefore $\Gamma \subseteq \ker \log$, so Γ is torsion, contradicting the hypothesis $\operatorname{rk}\Gamma = 1$. If $d(\Gamma) = 0$, then there exists $H \le G$ with $\dim H = 0$ and $\operatorname{rk}\Gamma = \operatorname{rk}(\Gamma \cap H)$; then H is finite, so $\operatorname{rk}(\Gamma \cap H) = 0$ and $\operatorname{rk}(\Gamma) = 0$, contradicting the hypothesis.

(viii) By applying an element of $\mathrm{GL}_n(\mathbb{Q}) = \operatorname{Aut}\mathbb{G}_a^n$, we may assume that $\Gamma = \mathbb{Z}^r \times \{0\}^{n-r} \le \mathbb{Q}^n = \mathbb{G}_a^n(\mathbb{Q})$, where $r := \operatorname{rk}\Gamma$. Using (v), we reduce to the case $n = 1$. If $r = 0$, then the result is trivial. If $r = 1$, use (vii).

□

5. Tori

Lemma 5.1. *Let K be a Galois extension of $\mathbb{Q}$. Let $\mathcal{G} := \operatorname{Gal}(K/\mathbb{Q})$. Then the representation $\mathcal{O}_K^\times \otimes \mathbb{C}$ of $\mathcal{G}$ is a subquotient of the regular representation.*

Proof. Define the $\mathcal{G}$-set $E := \operatorname{Hom}_{\mathbb{Q}\text{-algebras}}(K, \mathbb{C})$ of embeddings and the $\mathcal{G}$-set P of archimedean places of K, the difference being that conjugate complex embeddings are identified in P. Then E is a principal homogeneous space of $\mathcal{G}$, and there is a natural surjection $E \to P$. Therefore $\mathbb{C}^E$ is the regular representation and the permutation representation $\mathbb{C}^P$ is a quotient of $\mathbb{C}^E$. The proof of the Dirichlet unit theorem gives a $\mathcal{G}$-equivariant exact sequence

$$0 \to \mathcal{O}_K^\times \otimes \mathbb{R} \xrightarrow{\log} \mathbb{R}^P \to \mathbb{R} \to 0$$

so $\mathcal{O}_K^\times \otimes \mathbb{C}$ is a subrepresentation of $\mathbb{C}^P$. □

Proposition 5.2. *Let $\mathcal{T}$ be a group scheme of finite type over $\mathbb{Z}$ whose generic fiber $T := \mathcal{T} \times \mathbb{Q}$ is a torus. Then*

(a) *$\mathcal{T}(\mathbb{Z})$ is a finitely generated abelian group.*
(b) $\operatorname{rk}\mathcal{T}(\mathbb{Z}) \le \dim T$.
(c) *If $\Gamma \le \mathcal{T}(\mathbb{Z})$, then $d(T, \Gamma) = \operatorname{rk}\Gamma$.*

Proof.

(a) For some number field K and set S of places of K, we have $\mathcal{T} \times \mathcal{O}_{K,S} \simeq (\mathbb{G}_m)^n_{\mathcal{O}_{K,S}}$, so $\mathcal{T}(\mathcal{O}_{K,S})$ is finitely generated by the Dirichlet S-unit theorem. Therefore the subgroup $\mathcal{T}(\mathbb{Z})$ is finitely generated.

(b) We may assume that K is Galois over $\mathbb{Q}$. Let $\mathcal{G} = \operatorname{Gal}(K/\mathbb{Q})$. Let X be the character group $\operatorname{Hom}(T_K, (\mathbb{G}_m)_K)$ of T. Let χ_X be the character of the representation $X \otimes \mathbb{C}$ of $\mathcal{G}$. Let χ_K be the character of the representation $\mathcal{O}_K^\times \otimes \mathbb{C}$ of $\mathcal{G}$. By Theorem 6.7 and Corollary 6.9 of [Eis03], $\operatorname{rk}\mathcal{T}(\mathbb{Z}) = (\chi_X, \chi_K)$. On the other hand, $\dim T = \operatorname{rk} X = (\chi_X, \chi_{\text{reg}})$, where χ_{reg} is the character of the regular representation of $\mathcal{G}$. The result now follows from Lemma 5.1.

(c) First, $\mathcal{T}(\mathbb{Z})$ is contained in the compact open subgroup $\mathcal{T}(\mathbb{Z}_p)$ of $T(\mathbb{Q}_p)$, so $d(T,\Gamma)$ is defined. By (2), $d(T,\Gamma) \leq \operatorname{rk}\Gamma$. To prove the opposite inequality, we must show that for every subgroup scheme $H \leq T$, we have $\operatorname{rk}(\Gamma \cap H) \leq \dim H$. By replacing H by its connected component of the identity, we may assume that H is a subtorus of T. Let $\mathcal{H}$ be the Zariski closure of H in $\mathcal{T}$. Then $\Gamma \cap H \leq \mathcal{H}(\mathbb{Z})$, so $\operatorname{rk}(\Gamma \cap H) \leq \operatorname{rk}\mathcal{H}(\mathbb{Z}) \leq \dim H$ by (b).

□

Corollary 5.3. *Let K be a number field. Let T be the restriction of scalars $\operatorname{Res}_{K/\mathbb{Q}} \mathbb{G}_m$. Let $\Gamma \leq T(\mathbb{Q}) \simeq K^\times$ correspond to $\mathcal{O}_K^\times$. Then Leopoldt's conjecture is equivalent to a positive answer to Question 3.3 for Γ.*

Proof. Leopoldt's conjecture is the statement $\dim \overline{\Gamma} = \operatorname{rk}\Gamma$. Let $\mathcal{T} = \operatorname{Res}_{\mathcal{O}_K/\mathbb{Z}} \mathbb{G}_m$. By Proposition 5.2(c) applied to $\mathcal{T}$, $d(T,\Gamma) = \operatorname{rk}\Gamma$. So Leopoldt's conjecture is equivalent to $\dim \overline{\Gamma} = d(T,\Gamma)$. □

Remark 5.4. In effect, we have shown that Leopoldt's conjecture cannot be disproved simply by finding a subtorus H of $\operatorname{Res}_{K/\mathbb{Q}} \mathbb{G}_m$ containing a subgroup of integer points of rank greater than $\dim H$. This seems to have been known to experts, but we could not find a published proof.

6. Further questions

Question 6.1. Is $d(\Gamma)$ computable in terms of G and generators for Γ?

Question 6.2. If the answer to Question 6.1 is positive, can $\dim \overline{\Gamma} = d(\Gamma)$ be verified in each instance where it is true?

Question 6.3. Can one define a plausible generalization of $d(\Gamma)$ for the analogous situation where $\mathbb{Q}$ and $\mathbb{Q}_p$ are replaced a number field k and some nonarchimedean completion k_v?

Remark 6.4. Applying restriction of scalars from k to $\mathbb{Q}$ and then applying d does not answer Question 6.3: it would instead predict the dimension of the closure of Γ in the product $\prod_{v|p} G(k_v)$ instead of in a single $G(k_v)$.

Remark 6.5. If G be a commutative group scheme of finite type over $\mathbb{Q}$, we can consider also $G(\mathbb{R})$, and define $G(\mathbb{R})_f$ and $G(\mathbb{Q})_f$. The closure $\overline{\Gamma}$ of any subgroup $\Gamma \leq G(\mathbb{Q})_f$ in $G(\mathbb{R})$ is a real Lie group. The natural guess for $\dim \overline{\Gamma}$ seems now to be that it equals the dimension of the Zariski closure H of Γ in G; in other words, $\overline{\Gamma}$ should be open in $H(\mathbb{R})$. See [Maz92, §7] for a discussion of the abelian variety case.

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